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Shadow Wave Solutions for Some Balance Law Systems

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Contents

1

Introduction

In the first part we introduce the systems of conservation laws. We will take the pressureless gas dynamics model (PGD for short). That model can be derived from the well known isentropic gas dynamics of momentum conservation law

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) &= 0 \end{aligned} \tag{1.1}$$

by letting $p(\rho) \in 0$. This system is a part of almost any textbook concerning conservation laws, see [8], [11], [12], [44], for example.

It a model of gas dynamics in a gravitational field together with an entropy is assumed to be a constant. The energy conservation law is now used as a selection criteria for admissible solutions: For all continuous solutions energy is conserved, while it should decrease for discontinuous ones. It is known that admits a non–classical solution that contains the Dirac delta function (contrary the isentropic one). More precisely, we can uniquely solve its Riemann problem only by using such singular solutions, and assuming that they are overcompressive. Also, it can be used as a model of sticky particles. There are a lot of nice, classical by now papers about the pressureless conservation laws system. We can look in [4] for definition of measure valued solutions, in [6] for sticky particles method, in [17] for variational method, and in [14] for weak asymptotic method. All of them have detailed different methods of solving Riemann problem for PGD conservation law systems. We can look in [21] for a result about generalized pressureless system. In the paper [9] we can find a proof that passing from the isentropic to pressureless system by letting the pressure to vanish also transform weak solution of one system to weak solutions of another one. The same was done for generalized pressureless system in [35].

In the third part we introduce shadow waves [36] (SDW for short), entropies and interactions for delta and singular shock solutions to systems of conservation laws. We can look in ([29], [37], [48], [49], [50], etc) for delta shock solutions and in ([23], [38], etc) for singular shock solutions. The Shadow Waves are represented by nets of piecewise constant functions for time variable t fixed parameterized by some small parameter $\varepsilon > 0$ and bounded in $L^1_{loc}(\mathbb{R})$. A use of such parameter enable us to include the Dirac delta function as a part of solution. A definition of a shadow wave is made to be as a simple and robust as possible. Roughly speaking, we perturb a speed c of a wave from both sides by some small parameter ε so that left– and right handed states are connected by a state that can be of order $1/\varepsilon$ for same components. The main advantage of their use is that we use only Rankine–Hugoniot conditions for each ε. So we obtaining a net of classical weak solutions that satisfy the system in a distributional limit $\varepsilon \to 0$. Also, the usual entropy inequality can be easily checked regardless of the form of entropy and entropy–flux functions.

In the forth part we will take the pressureless gas model with added a force term on the right–hand side of momentum conservation law. The body force source term is present if there is some external force acting on the fluid. The force assumed here is the gravity with b being the gravitational constant. By letting $p(\rho) \equiv 0$ we get the PGD conservation law system.

Among a lot of different approaches in explaining such a type of solutions, we will use the above mentioned Shadow Waves in order to solve the balance law of pressureless gas with body force source term. We can look in the book [12] for explanation and origin of balance law in different physical situations. After that we will use a simpler condition–so called overcompressibility: All characteristics should run into the shock curve. Also, it is proved that entropy condition is not enough to exclude non– admissible waves for pressureless conservation law system in paper [17]. We will introduce the advantage is a simplicity of treating an interaction problem involving a shadow wave and explane it detailed in the part 4.

In the fifth part we will look on some numeric procedures supporting such a solution and to see how interactions could be handled in the presence of gravity. In practice, the interaction region is best solved numerically. However, we introduce some examples that are solved by using Godunov method (see [1], [2], [30], [31] for efficient numerical methods for conservation and balance laws).

2

Systems of conservation law

The important class of homogeneous hyperbolic equations called conservation laws. The simplest case of conservation law in one space dimension is the partial differential equation (PDE) of the form

$$\partial_t u + \partial_x \Big(f(u) \Big) = 0$$

where $f(u)$ is a sufficiently regular flux function and $u = u(t, x)$ is called the conserved quantity or density. One can find a lot of examples from literature given in the introduction. Let us note that one dimensional case is much better understood than more dimensional cases.

2.1 Rankine–Hugoniot conditions

Due to nonlinearity of the flux function we expect that a solution, even with a smooth initial data would explode. The most usual explosion is so called gradient catastrophe when solution breaks into two disjoint pieces. One can look in [8], [12], [44] or [47], for example, for more description of the breakdown. We will use piecewise smooth functions as a possible weak solutions, so we start with the following fundamental result.

Let $u \in C^1(\mathbb{R} \times [0,\infty))$ be a solution to the following partial differential equation.

$$\begin{aligned} \partial_t u + \partial_x \Big(f(u) \Big) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{2.1}$$

where u is called the conserved quantity, while f is the flux, t denotes to the time and x is the one dimensional space variable. We take $\phi \in C_0^1(\mathbb{R} \times [0,\infty))$, i.e smooth function such that support intersected by $\mathbb{R} \times [0,\infty)$ is compact.
Then

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty \partial_t u(x,t) + \partial_x \Big(f(u) \Big) \phi(x,t) dx dt \\ &= - \int_0^\infty \int_{-\infty}^\infty f(u) \partial_x \phi dt dx + \int_{-\infty}^\infty u(x,t)\phi(x,t) dx|_{t=0}^{t=\infty} \\ &\quad - \int_0^\infty \int_{-\infty}^\infty u \partial_t \phi dx dt \\ &= - \int_0^\infty \int_{-\infty}^\infty (u\partial_t \phi + f(u)\partial_x \phi) dx dt - \int_{-\infty}^\infty u_0(x)\phi(x,0) dx \end{aligned}$$

Then the above calculation inspired the following definition of weak solution for (2.1)

Definition 1. *A function $u \in L^\infty(\mathbb{R} \times (0,\infty))$ (where u is bounded function up to a set of lebesque measure zero) will be called a weak solution of (2.1) provided that*

$$\int_0^\infty \int_{-\infty}^\infty (u\partial_t \phi + f(u)\partial_x \phi) dx dt + \int_{-\infty}^\infty u_0(x)\phi(x,0) dx = 0,$$

for every compact support $\phi \in C_0^1(\mathbb{R} \times [0,\infty))$.

Remark 1. 1. All classical solutions are also weak.

2. If u is a weak solution, then u is also a distributive solution.

3. If $u \in C^1(\mathbb{R} \times [0, \infty))$ is a weak solution, then it is classical, too.

Theorem 1. *Let a necessary and sufficient condition be*

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t), \qquad t \geq 0 \\ u_r(x,t), & x > \gamma(t), \qquad t \geq 0 \end{cases}$$

where u_l and u_r are in C^1 solutions on their domains, be a weak solution of the system of conservation laws (2.1) is

$$\dot{\gamma} = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{[f(u)]_\gamma}{[u]_\gamma} \tag{2.2}$$

where condition (2.2) is called Rankine–Hugoniot (RH) condition.

2.2 Rarefection waves for single conservation law

The solution of the system of conservation law (2.1) of the form $u(x,t) = \widetilde{u}(\frac{x}{t})$ is called self–similar solution.
Now, we will try to find a solution of the system of conservation law (2.1) in simple way, we substitute the function of this form into the equation of the system (2.1)

$$\partial_t \left(\widetilde{u} \left(\frac{x}{t} \right) \right) + \partial_x \left(f \left(\widetilde{u} \left(\frac{x}{t} \right) \right) \right) = 0$$

After the differentiation, we obtain

$$-\frac{x}{t^2}\, \widetilde{u}' \left(\frac{x}{t} \right) + f' \left(\widetilde{u} \left(\frac{x}{t} \right) \right) \frac{1}{t}\, \widetilde{u}' \left(\frac{x}{t} \right) = 0$$

Now we multiple above equation by t and by substituing $\frac{x}{t} \longmapsto y$, we get the ODE

$$\widetilde{u}'(y) \Big(f'(\widetilde{u}(y)) - y \Big) = 0$$

After omitting constant value, it will be called trivial solutions ($\widetilde{u}' \neq 0$), we can see that solution is given by the following relation

$$f'(\widetilde{u}) = y \qquad \text{i.e} \qquad \widetilde{u}(y) = f'^{-1}(y),$$

if f' is bijection.

We can interpret the initial data in the following way:

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases} \Longrightarrow \widetilde{u}(+\infty) = u_r,\ \widetilde{u}(-\infty) = u_l. \tag{2.3}$$

If $f'' > 0$ where f is convex, then f' is increasing function and the solution $\widetilde{u}$ with the equation (2.1) satisfying (2.3) exists if $u_l < u_d$. Such solution is called centred rarefaction wave. Here the initial data has a singularity at zero.

Example 1. We will present solutions to a well known example of a single conservation law, the simplest nonlinear conservation law, so called inviscid Burgers equation

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0$$
$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0. \end{cases}$$

In the case $u_l = 1$, $u_r = 0.5$, there is only shock wave solution (see Fig. 2.1)

$$u(x,t) = \begin{cases} u_l, & x < 0.75t \\ u_r, & x > 0.75t. \end{cases}$$

(wave speed is given by $c = \dfrac{u_l + u_r}{2}$.)

Now, let $u_l = 0.5$, $u_r = 1$. There are more that one solution. Let us mention a shock wave solution (see Fig. 2.2)

$$u(x,t) = \begin{cases} u_l, & x < 0.75t \\ u_r, & x > 0.75t, \end{cases}$$

and a rarefaction wave solution (see Fig. 2.3)

$$u(x,t) = \begin{cases} u_l, & x < 0.5t \\ \dfrac{x}{t}, & 0.5t < x < t \\ u_r, & x > t. \end{cases}$$

Let us note that the rarefaction wave is the proper solution (entropic one). We will explain how to chose an entropic solution later on. See Fig. 2.4 for 3D plot of entropic solutions in both cases.

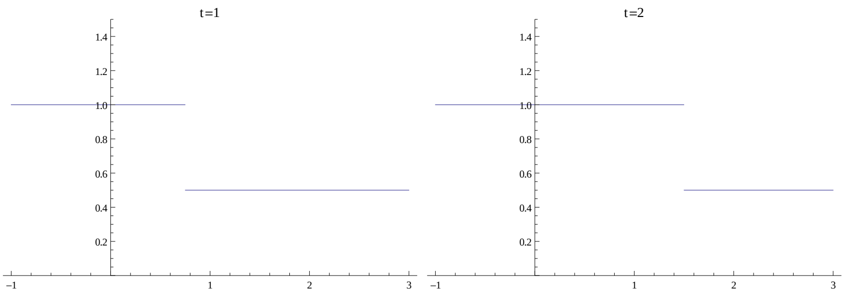


Figure 2.1: "Proper" shock wave solution

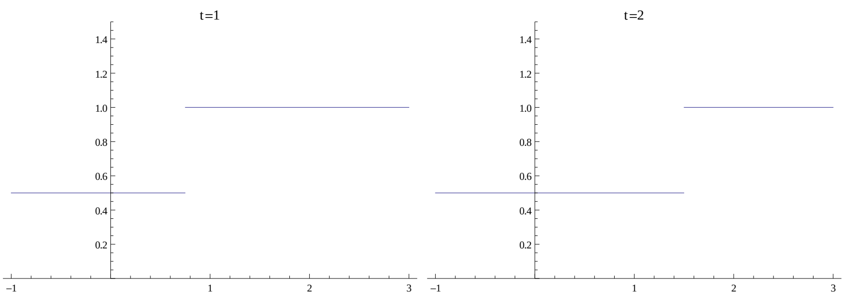


Figure 2.2: "Wrong" shock wave solution

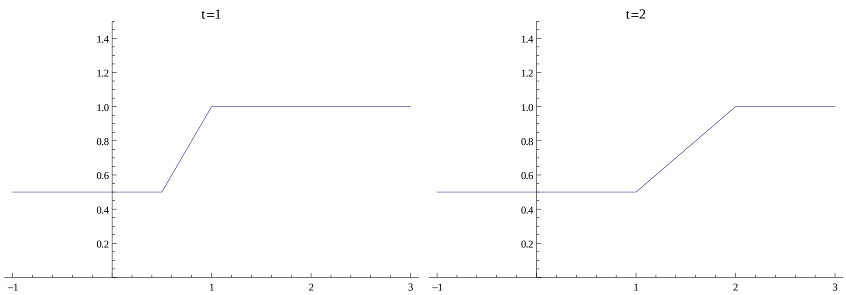


Figure 2.3: "Proper" rarefaction wave solution

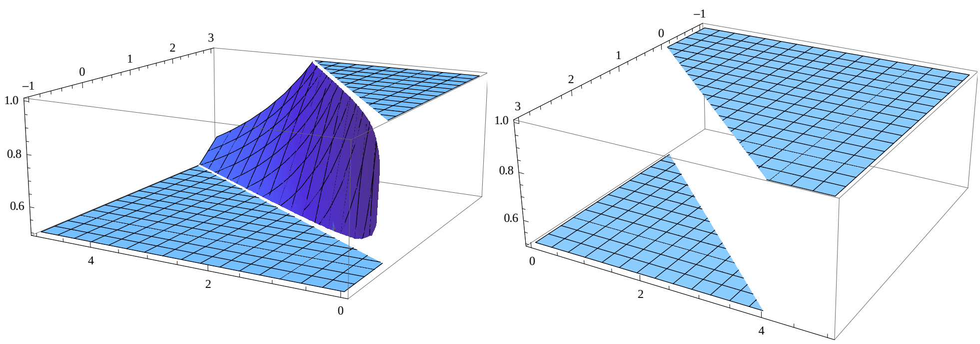


Figure 2.4: Entropic solutions in both cases

2.3 Linear hyperbolic systems

In order to deal with systems of conservation laws, we will repeat some facts about linear hyperbolic systems in this section.

One of the first actions one could do when investigating nonlinear problem is to linearise it around some state assuming small variations. So, we present some basic facts about linear hyperbolic systems here.

Consider homogeneous linear scalar Cauchy problem with constant coefficients

$$\begin{aligned} \partial_t u + \lambda \partial_x u = 0 \\ u(x,0) = \widetilde{u}(x), \end{aligned} \tag{2.4}$$

with

$$\lambda \in \mathbb{R},\ \widetilde{u} \in C^1\Big([0,\infty) \times \mathbb{R}\Big).$$

It has a simple solution in a travelling wave form

$$u(x,t) = \widetilde{u}(x - \lambda t). \tag{2.5}$$

If $\widetilde{u} \in L^1_{loc}$, then the above function (2.5) is a weak solution to (2.4), what we can show easily.

Let a homogeneous system with constant coefficients

$$\begin{aligned} \partial_t u + A\partial_x u =& 0 \\ u(x,0) =& \widetilde{u}(x) \end{aligned} \tag{2.6}$$

be given, where A is $n \times n$ a strictly hyperbolic matrix with real eigenvalues $\lambda_1 < ... < \lambda_n$. Let right-hand side and left-hand side eigenvectors $r_i, l_i,\ i = 1, ..., n$, are chosen such that $l_i r_j = \delta_{i,j}, i,\ j = 1, ..., n$, holds. Here

$$\delta_{ij} = \begin{cases} 1\ if & i = j \\ 0\ if & i \neq j \end{cases}$$

as usual. Recall from the linear algebra, $u_i = l_i{\cdot}u$ coordinates of a vector $u \in \mathbb{R}^n$ with respect to the basis of right eigenvectors $\{r_1, ..., r_n\}$. Multiplying

(2.6) on the left-hand side by $l_1, ..., l_n$, we obtain

$$\begin{aligned} \partial_t u_i + \lambda_i \partial_x u_i = \partial_t (l_i u) + \lambda_i \partial_x (l_i u) = l_i \partial_t u + l_i A \partial_x u =& 0 \\ u_i(x,0) = l_i \overline{u}(x) =& \overline{u}_1(x). \end{aligned}$$

Therefore, (2.6) decouples into n scalar Cauchy problems, which can be solved separately like (2.4). By using (2.5) we can see that

$$u(x,t) = \sum_{i=1}^{n} \overline{u}_i(x - \lambda_i t) r_i \tag{2.7}$$

is a solution to (2.6) because

$$\partial_t u(x,t) = \sum_{i=1}^{n} -\lambda_i \Big(l_i \partial_x \overline{u}(x - \lambda_i t) \Big) r_i = -A \partial_x u(x,t).$$

$$u_t(x,t) = \sum_{i=1}^{n} -\lambda_i \Big(l_i \overline{u}_x(x - \lambda_i t) \Big) r_i = -A u_x(x,t).$$

Thus, the initial profile decompose into a sum of n waves, each travelling with one of the characteristic speeds $\lambda_1, ..., \lambda_n$.

As a special case, we take the Riemann problem

$$\overline{u} = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0 \end{cases}$$

We write down a solution of (2.7) by using the following representation:

$$u_d - u_l = \sum_{j=1}^{n} c_j r_j.$$

Let interstates be given by

$$w_i = u^l + \sum_{j \leq i} c_j r_j, \qquad i = 0, ..., n$$

such that $w_i - w_{i-1}$ is $(i-n)$–th eigenvector of A. Then the solution is

$$u(x,t) = \begin{cases} w_o = u_l, & \frac{x}{t} < \lambda_1 \\ ...., & \\ w_i \quad , & \lambda_i < \frac{x}{t} < \lambda_{i+1} \\ ..., & \\ w_n = u_d, & \frac{x}{t} > \lambda_n. \end{cases}$$

We will expect solutions to a nonlinear problem to resemble the above form of solution in some way (small perturbations, for example).

2.4 Basic definitions

In this section one can find very useful class of function for one-dimensional problems, so called function with finite total variation.

Definition 2. *Total variation of a function v is defined by*

$$TV(v) = sup \sum_{j=1}^{N} |v(\xi_j) - v(\xi_{j-1})|, \tag{2.8}$$

where the supremum is taken by all partitions of real line

$$-\infty = \xi_0 < \xi_1 < ... < \xi_N = \infty.$$

Then we can write (2.8) in the form

$$TV(v) = \lim_{\varepsilon \to 0} sup \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |v(x) - v(x-\varepsilon)|.$$

Let

$$\begin{cases} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, ..., u_n) = 0 \\ \qquad\qquad . \\ \qquad\qquad . \\ \qquad\qquad . \\ \frac{\partial}{\partial t} u_n + \frac{\partial}{\partial x} f_n(u_1, ..., u_n) = 0 \end{cases} \tag{2.9}$$

be $n \times n$ one–dimensional conservation laws system, where

$$u = (u_1, ..., u_n) \in \mathbb{R}^n, \qquad f = (f_1, ..., f_n) : \mathbb{R}^n \to \mathbb{R}^n.$$

We call $A(u) = Df(u)$ the $n \times n$ Jacobi matrix of the map f at a point u. The system (2.9) reads as

$$\partial_t u + \partial_x f(u) = 0 \tag{2.10}$$

If a solution is smooth enough (C^1 at least), then the system (2.9) can be written in the quasilinear form

$$\partial_t u + A(u)\partial_x u = 0 \tag{2.11}$$

and these systems are equivalent.

Definition 3. *(Strictly hyperbolic system). The system of conservation laws (2.9) is called strictly hyperbolic if all eigenvalues of Jacobian matrix $A(u)$ are real and distinct. The eigenvalues are ordered in the following way:*

$$\lambda_1(u) < \lambda_2(u) < ... < \lambda_n(u).$$

If there exist n linearly independent eigenvectors, then the system (2.9) is called hyperbolic. Note that we will always assume that eigenvalues are ordered in non-decreasing way:

$$\lambda_1(u) \leq \ldots \leq \lambda_n(u).$$

In the sequel, left–hand side $l_1(u), ..., l_n(u)$ and right–hand side $r_1(u), ..., r_n(u)$ eigenvectors are normalized, i.e. determined in a way that it holds

$$l_i(u) r_j(u) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Note that now (strict) hyperbolicity depends on solution to, contrary to linear systems.

2.5 Elementary waves – an introduction

2.5.1 Shock waves

Let $u_l, u_d : \mathbb{R}^2 \to \mathbb{R}^n$ be any two continuous function and suppose that $x = \gamma(t)$ defines a discontinuity curve of piecewise smooth solutions $u_l(x,t)$ and $u_r(x,t)$, i.e

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t) \\ u_r(x,t), & x > \gamma(t) \end{cases}$$

where u defines a weak solution which has to find a speed γ, satisfy the Rankine–Hugoniot equations for system

$$\dot{\gamma} \cdot (u_r - u_l) = f(u_r) - f(u_l) \tag{2.12}$$

Where $u_r,\ u_l,\ f(u_r)$ and $f(u_l)$ are n-dim vectors, and $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$. The sign $[\cdot]$ denotes a jump over γ (or $[\cdot]_\gamma$ if there is a chance for misunderstanding). The formula (2.12) looks like

$$\dot{\gamma}[u] = [f(u)].$$

That means that a discontinuity curve $x = \gamma(t)$ can not by found in a direct way like in the case of single equation. Also, it is not true for each pair of constant initial vectors u_l, u_r where there exists a shock wave solution (like in the case of a single equation).

Let us call

$$A(u,v) := \int_0^1 A\Big(\theta u + (1-\theta)v\Big) d\theta$$

averaged matrix, where $\lambda_i(u,v), \quad i = 1, ..., n$ are its eigenvalues. Then the equation (2.12) can be written in the equivalent form

$$\dot{\gamma} \cdot (u_r - u_l) = f(u_r) - f(u_l) = A(u_l, u_r)(u_r - u_l), \tag{2.13}$$

or

$$\dot{\gamma}[u] = [f(u)] = A(u_l, u_r)[u].$$

Moreover, the Rankine–Hugoniot conditions hold if and only if (u_r, u_l) is an eigenvector of the averaged matrix $A(u_l, u_r)$ and the speed $\dot{\gamma}$ equals its eigenvalue.

2.5.2 Rarefaction waves

We will try to find solutions of the form $u = u(\frac{x}{t})$ (self similar solutions) for the system (2.11):

$$\partial_t u + A(u)\partial_x u = -\frac{x}{t^2}u'(y) + \frac{1}{t}A\Big(u(y)\Big)u' = 0$$

where $y = \frac{x}{t}$. From the last equation it follows

$$A(u)u' = yu',$$

which means that u' is equal to the right–hand side eigenvector r_i and $y = \lambda_i, \qquad i = 1, ..., n.$

2.6 Admissibiliry conditions

2.6.1 Vanishing viscosity

A weak solution u of (2.9) is admissible if there exists a sequence of smooth solutions u_ε to

$$\partial_t u_\varepsilon + A(u_\varepsilon)\partial_x u_\varepsilon = \varepsilon\partial_x^2 u_\varepsilon$$

which converges to u in L^1 as $\varepsilon \to 0$. Such a solution is called vanishing viscosity solution.

That condition arises naturally from the various applications. A viscosity is almost always present in a real life problems (like a friction in the solid mechanics, for example). Here, we are using so called artificial viscosity, i.e. the term on the right-hand side is $\varepsilon I\partial_x^2 u_\varepsilon$. In the real-life problems we usually have some other matrix B instead of I that could be of rank less than the dimension of the system. For example, the isentropic gas dynamics model with viscosity is given by $m = \rho u$,

$$\partial_t \rho + \partial_x m = 0$$
$$\partial_t m + \partial_x\left(\frac{m^2}{\rho} + p(\rho)\right) = \varepsilon u_{xx}.$$

i.e.

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recently, Bressan and Biancini made a complete analysis of vanishing viscosity solutions to conservation laws in the famous paper [3].

2.6.2 Entropy inequality

A continously differentiable function $\eta : \mathbb{R}^n \to \mathbb{R}$ is called entropy for the system of conservation laws (2.9) with entropy flux $q : \mathbb{R}^n \to \mathbb{R}$, if

$$D\eta(u)Df(u) = Dq(u), \qquad u : \mathbb{R}^n \to \mathbb{R}^n \tag{2.14}$$

We note that (2.14) implies

$$\partial_t \eta(u) + \partial_x q(u) = 0,$$

if $u \in C^1$ is a solution to (2.9). If we substitutes $\partial_t u = -Df(u)\partial_x u$ into the above equation then we obtain

$$D_\eta(u)\partial_t u + D_q(u)\partial_x u = D_\eta(u)(-Df(u)\partial_x u) + D_q(u)\partial_x u = 0.$$

Also, a weak solution u of (2.9) is admissible if

$$\partial_t \eta(u) + \partial_x q(u) \leq 0$$

in a distributional sence, i.e

$$-\int \eta(u)\partial_t \varphi + q(u)\partial_x \varphi \geq 0,$$

for every $\varphi \geq 0, \varphi \in C_0^\infty\Big(\mathbb{R} \times [0, \infty)\Big)$.

Thus,

$$D\eta(u)\partial_t u + Dq(u)\partial_x u = 0$$

outside a discontinuity, and

$$\dot{x}_\alpha\Big(\eta(u(x_\alpha+)) - \eta(u(x_\alpha-)\Big) \geq q\Big(u(x_\alpha+)\Big) - q\Big(u(x_\alpha-)\Big)$$

on the discontinuity curve $x = \dot{x}_\alpha(t)$

For (2.9), the following relation about two admissibility conditions can be given.

Suppose that η is a convex entropy for a system

$$\partial_t u + \partial_x f(u) = 0$$

and let $A := Df$ as usual. Take a viscosity perturbed system

$$\partial_t u + A(u)\partial_x u = \varepsilon \partial_{xx} u$$

and multiply it with $D\eta$ to get

$$\varepsilon_t \eta(u) + \partial_x q(u) = \varepsilon D\eta(u)\varepsilon_{xx} u.$$

We know that

$$\partial_{xx}\eta(u) = D\partial_{xx}\eta(u)u + \sum_{i,j=1}^{n} \frac{\partial^{2\eta(u)}}{\partial u_i \partial u_j} u_{i,x} u_{j,x}.$$

If η is convex, then the last term is positive. That means

$$\partial_t \eta(u) + \partial_x q(u) < \varepsilon \partial_{xx} \eta(u).$$

So, we have proved that vanishing viscosity solution satisfies entropy condition.

Remark 2. Let us note that physical entropy for the gas dynamics system is concave, and athematical entropy used above can be construcded as minus the physical one.

2.6.3 Lax shock condition

A shock wave connecting states u_l and u_r and travelling with speed $\dot{\gamma} = \lambda_i(u_l, u_r)$ is admissible if

$$\lambda_i(u_l) \geq \lambda(u_l, u_r) = \dot{\gamma} \geq \lambda_i(u_r). \tag{2.15}$$

Because of the ordering of eigenvalues

$$\lambda_j(u_l) > \dot{\gamma}, \qquad j > i$$
$$\lambda_j(u_r) < \dot{\gamma}, \qquad j < i$$

where this wave is called i-th shock wave.

Remark 3. Let us note that for genuinely nonlinear systems, solution that satisfies Lax shock condition and vanishing viscosity solution coincide.

2.6.4 Liu admissibility E-condition

Let u_l be fixed and let $(W_i(s, u_l), c_i(s, u_l))$ be i-th shock curve and speed parametrized by $s > 0$, respectively. We say that an i-shock connecting u_l and $u_r = W_i(s_r, u_l)$ satisfies Liu E-condition if

$$c = c_i(s_r) \leq s_i(s, u_l), \ s \in (0, s_r).$$

We know that Liu E-condition implies the Lax shock condition. In general, it is finer than the Lax one and well adapted to other admissibility conditions.

2.7 Rarefaction and shock wave curves

Fix a state $u_0 \in \mathbb{R}^n$ and $i \in \{1, ..., n\}$ let $r_i(u)$ be an i-eigenvector of the Jacobian matrix Df. The integral curve of the vector field r_i through the point u_0 is called the i-rarefaction curve through u_0. We can get it explicitly by solving the Cauchy problem

$$\frac{du}{d\sigma} = r_i(u), \qquad u(0) = u_0 \tag{2.16}$$

We denote that curve by

$$\sigma \longmapsto R_i(\sigma)(u_0).$$

By above definition, u_0 can be joined with $u \in RW_i(u_0)$ by a single rarefaction wave.

Note that a curve parametrization depends on a choice of r_i. If $|r_i| \equiv 1$ then the curve will be parametrizated by its length.

Next, for a fixed $u_l \in \mathbb{R}^n$ and $i \in \{1, ..., n\}$. Let u be a right–hand side which can be connected to u_l with i-th shock wave. (We uses RH conditions and Lax condition (2.15)). Values of u lies on a curve $W_i(s, u_l)$ for some s. A shock speed is then $c = c_i(s, u_l)$. So, the vector $u - u_l$ is a right–hand side i-th eigenvector of the averaged matrix $A(u, u_l)$. By the theorem of linear algebra that is true if and only if $u - u_l$ is orthogonal to all left eigenvectors l_j for every $j \neq i$. This means

$$l_j(u_l)(u - u_l) = 0, \qquad \forall j \neq i, \qquad \dot{\gamma} = \lambda_i(u, u_l). \tag{2.17}$$

We can see that (2.17) is the system of $n - 1$ scalar equation with n variables $(u_1, ..., u_n)$. Linearizing (2.17) in a neighborhood of u_0 we get the linear system

$$l_j(u_l) \cdot (u - u_l) = 0, \qquad j \neq i.$$

which it has a solution $w = u_l + Cr_i(u_l)$, $C \in \mathbb{R}$. By Implicit Function Theorem, a set of solutions forms a regular curve ($C^1 -$ class) which can be connected to u_l with a tangent vector r_i in the point u_l. That curve is called the curve of i-th shock wave and denoted by S_i.

Both of the above curves exist in neighbourhood of u_l (if f is smooth enough), and it can be proved that they have the same tangent in the point u_l parallel to $r_i(u_l)$.

2.8 Riemann problem

Definition 4. *We say that the i-th characteristic field is genuinely nonlinear if*

$$D\lambda_i(u)r_i(u) \neq 0.$$

If

$$D\lambda_i(u)r_i(u) \equiv 0,$$

then the i-th field is called linearly degenerate.

Note that in the case when the i-th field is genuinely nonlinear, we can chose the orientation of r_i (by changing its sign, if needed) such that

$$D\lambda_i(u)r_i(u) > 0.$$

Note that a field could be neither genuinely nonlinear not linearly degenerate.

2.8.1 Centered rarefaction wave

Let the i-th field be genuinely nonlinear, and assume that u_r lies on the a positive part of rarefaction curve starting from u_l, i.e $u_r = R(\sigma)(u_l)$ for some $\sigma > 0$

Theorem 2. *Let us define the characteristic speed by*

$$\lambda_i(s) = \lambda_i\Big(R_i(s)(u_l)\Big)$$

for each $s \in [0, \sigma)$

By genuine nonlinearity, the map $s \to \lambda_i(s)$ is strictly increasing. Let $t \geq 0$, the function

$$u(x,t) = \begin{cases} u_l, & \frac{x}{t} < \lambda_i(u_l) \\ R_i(s)(u_l), & \frac{x}{t} = \lambda_i(s) \\ u_r = R_i(\sigma)(u_l), & \frac{x}{t} > \lambda_i(u_r) \end{cases} \tag{2.18}$$

where $\frac{x}{t} = y = \lambda_i(s) \in \Big[\lambda_i(u_l), \lambda_i(u_r)\Big]$, $s \in [0, \sigma]$, is a piecewise smooth solution to Riemann problem

$$\partial_t u + \partial_x f(u) = 0$$

$$u|_{t=0} = u_0 = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

Proof. We can easily see that

$$\lim_{t\to 0} \| u(x,t) - u_0 \|_{L^1} = 0$$

so, the initial data are satisfied. Moreover, the equation (2.9) trivially holds true for $x < t\lambda_i(u_l)$ and $x > t\lambda_i(u_r)$, because $\partial_t u = \partial_x u = 0$. Assume that $x = t\lambda_i(s)$, for some $[0,\sigma]$. Since u is constant along each halfline $\{(x,t) : x = t\lambda_i(s)\}$, then we have

$$\partial_t u(x,t) + \lambda_i(s)\partial_x u(x,t) = 0. \tag{2.19}$$

Since

$$\begin{aligned}\partial_x u =& \frac{\partial u}{\partial x} = \frac{dR_i(s)(u_l)}{ds}\left(\frac{d\lambda_i(s)}{ds}\right)^{-1}\frac{d\lambda_i}{dx}\\ =& r_i(u)\left(\frac{d_i(s)}{ds}\right)^{-1}\cdot\frac{1}{t},\end{aligned}$$

where $\partial_x u$ is eigenvector for the Jacobian matrix $A(u)$ when $\lambda_i(s) = \lambda_i\Big(u(t,x)\Big)$, i.e

$$A(u)\partial_x u = \lambda_i \partial_x u.$$

Note that the assumption $\sigma > 0$ is crucial for the above construction of a solution. If $\sigma < 0$, (2.18) would define a triple valued function in the area $\frac{x}{t} \in \Big[\lambda_i(u_d), \lambda_i(u_l)\Big]$. □

2.8.2 Shock waves

Let the i-th characteristic field be genuinely nonlinear and let u_r be connected with u_l by i-shock wave, $u_r = S_i(\sigma)(u_l)$. Then $\lambda = \lambda_i(u_r, u_l)$ is the speed of that wave and

$$u(x,t) = \begin{cases} u_l, & x < \lambda t \\ u_r, & x > \lambda t \end{cases} \tag{2.20}$$

is a piecewise constant solution to the above Riemann problem. Note that

$$\lambda = \frac{[f_i(u)]}{[u_i]}, \ i = 1, \dots, n$$

and a special value of γ from (2.12) is $\gamma = \lambda t$ now. That is, a discontinuity curve is a straight line.

Note that if $\sigma < 0$, then this solution is entropy admissible in the Lax sence, because

$$\lambda_i(u_r) < \lambda_i(u_l, u_r) < \lambda_i(u_l).$$

In the case $\sigma > 0$, we would have

$$\lambda_i(u_l) < \lambda_i(u_r)$$

and Lax condition could not be satisfied.

2.8.3 Contact discontinuities

Assume that the i-th characteristic field is linearly degenerate and $u_r = R_i(\sigma)$ for some σ. By the assumption, the i-th characteristic speed λ_i is constant along that curve, i.e. $D\lambda_i r_i = 0$. Choosing $\lambda = \lambda_i(u_l)$, we can see that piecewise constant function given by (2.20) solves the above problem, because the Rankine–Hugonniot conditions is satisfied at discontinuity curve.

$$\begin{aligned} f(u_r) - f(u_l) &= \int_0^\sigma Df\Big(R_i(s)(u_l)\Big) r_i\Big(R_i(s)(u_l)\Big) ds \\ &= \int_0^\sigma \lambda_i\Big(R_i(s)(u_l)\Big) r_i\Big(R_i(s)(u_l)\Big) ds \\ &= \lambda_i(u_l) \int_0^\sigma \frac{dR_i(s)(u_l)}{ds} ds = \lambda_i(u_l)\Big[R_i(\sigma)(u_l) - u_l\Big]. \end{aligned}$$

We have used here that

$$\begin{aligned}\frac{d}{ds}\lambda_i\Big(R_i(s)(u_l)\Big) =& D\lambda_i\Big(R(s)(u_l)\Big)\frac{d\Big(R_i(s)(u_l)\Big)}{ds}\\ =&\Big(D\lambda_i r_i\Big)\Big(R_i(s)(u_l)\Big) = 0\end{aligned}$$

as well as the definition of linear degeneracy.

In that case, the Lax condition holds thus regardless to the sign of σ, because

$$\lambda_i(u_r) = \lambda_i(u_l, u_r) = \lambda_i(u_l).$$

Then from above calculation, we can deduce that

$$R_i(\sigma)(u_0) = S_i(\sigma)(u_0)$$

for every σ.

2.8.4 Solution to Riemann problem

As we have seen before, a set of points $\{u_r : u \in \mathbb{R}^n\}$ which could be connected with a left-hand side state of Riemann problem makes a curve in $\mathcal{R}^2$. In order to connect two arbitrary points u_l, $u_r \in \mathbb{R}^n$ with an entropic solution of Riemann problem one can insert at most $n-1$ vectors

$$u_l =: u_0, u_1, u_2, \dots, u_{n-1}, u_n := u_r$$

such that between each pair (u_l, u_1), (u_1, u_2), ..., (u_{n-1}, u_d) there is one of elementary waves: rarefaction, shock waves or contact discontinuities. That is possible for sure if the total variation of the initial data is small enough. One can see the illustration in Fig. 2.5. Here Ψ denotes any kind of elementary wave.

Remark 4. For bounded initial data, one can approximate it by piecewise constant function. So there are Riemann problems which have to be simultaneously solved. One by one solution in the form of elementary waves can be easily find, but the main problem is how to deal with a huge number of mutual wave interactions. There are two famous methods to do it.

(1) **Glimm scheme** ([18]). Before the first interaction of the initial elementary waves, one approximates a solution with new piecewise constant function by choosing finite number of points in a random way. That becomes a new initial data and procedure is repeated as many times as needed. Rarefaction wave is approximated by a fan of non-admissible shock waves in this procedure. The procedure will converge for small enough variation of initial states, i.e. when the total variation of the initial data is small enough. One can also be sure that each approximation is independent of the previous ones.

There are a lot of technical problems concerning the above scheme, so a lot of effort was given to find a new procedure. Later on, randomness was excluded from the assumptions above, see [33].

The following scheme is the best choice both for prooving solution existence and numerical approximation of a solution.

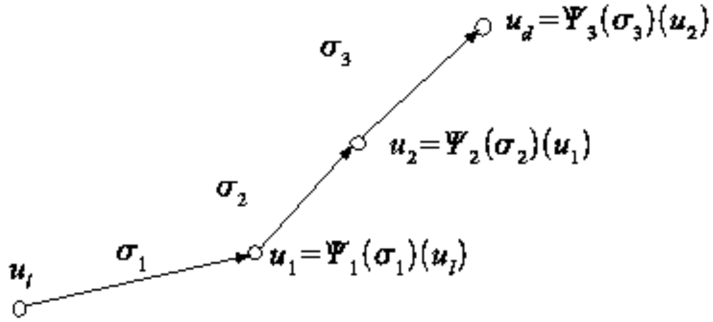


Figure 2.5: Sketch of a solution to Riemann problem

(2) **Front-tracking method** ([7]). Again, rarefaction wave is approximated with a fan of non-entropic shock waves. But now waves are permitted to interact. In a point of interaction there is a new Riemann problem. One can solve it accurately or approximatively. In the later case, one constructs non-physical shock wave with small amplitude, but with the larger speed of all possible waves in order to prevent blow-up effect. After that one can again use the same method for later interactions. Again, this procedure will converge when total variation of the initial data is small enough.

Let us note that this procedure is the main idea of Shadow Waves definition given in the next section.

3

Shadow waves

In this part we are introducing the solution of shadow waves (SDW for short) to systems of conservation laws

$$\partial_t f(u) + \partial_x g(U) = 0,$$

where f and g are as regular as needed.

A definition of shadow waves include delta and singular shocks (see [36]). Generally speaking, we perturb a speed c of a wave from both sides by some small parameter ε so that the states U_0 and U_1 of a solution candidate U_ε are connected by three shocks. Two of shocks have perturbed speed and the third shock, in the middle, has a speed c. The intermediate values, $U_{1,\varepsilon}$ on the left and $U_{2,\varepsilon}$ on the right-hand side of the shock front, can tend to infinity as $\varepsilon \to 0$. Also, we have the following types of shadow waves:

- *Constant shadow wave* has constant $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ for each ε. If its speed is constant, it is called *simple.*
- *Weighted shadow wave* has $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ depending on t.

Then, U_ε is still a piecewise constant function for each ε. Let us briefly explain why we are using such waves.

Until now, there are a lot of conservation law cases having non-classical solutions that contain the delta function. Such solutions are usually called delta shocks or singular shocks. The main problem in using such solutions is to find a way how to perform nonlinear operations. There are different methods dealing with that problem. We will use Shadow Waves for the delta function approximation. They are piecewise constant states depending on a small parameter ε. So, we can use standard Rankine-Hugoniot conditions for a fixed ε in order to perform nonlinear operations. After that, we let $\varepsilon \to 0$ and look at a distributional limit.

Beside that, the usual entropy inequality can be easily checked regardless of the form of entropy and entropy–flux functions.

And finally, we can use ideas and procedures from Wave Front Tracking algorithm (see [7] or [8]) to deal with wave interactions. That makes Shadow Waves extremely useful in problems where we expect unbounded solution.

3.1 Basic formulas

A parameter ε belongs to some interval $(0, \varepsilon_0)$, with ε_0 being as small as needed. Let a_ε be a net of real numbers and u_ε be a net of locally integrable functions over some domain $\omega \subset \mathbb{R}^m$. We have

$a_e \sim \varepsilon$ if there exists $A \in \big(0, \infty\big)$ such that $\lim\limits_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon} = A$

also

$u_\varepsilon \approx g \in \mathcal{D}'\big(\omega\big)$ if $\int_\omega u_\varepsilon \phi \to \big(g, \phi\big)$ as $\varepsilon \to 0$ for every function $\phi \in \mathcal{C}_0^\infty$

The relation $u_\varepsilon \approx v_\varepsilon$ means $u_\varepsilon - v_\varepsilon \approx 0$, and we call it distributional equality or simply equality if there is no chance for misunderstanding. In what follows, relations $\approx, \sim$, a growth order, Landau symbols $\mathcal{O}(\cdot)$ and $o(\cdot)$ will always be used assuming $\varepsilon \to 0$. The half-space $\big\{\big(x, t\big) \in \mathbb{R} \times \mathbb{R}_+\big\}$ is denoted by $\mathbb{R}^2_+$.

All calculations in this part are based on exploitation of the Rankine–Hugoniot conditions. We will obtain all results by the following basic lemma:

Lemma 1. *Let* $f, g \in \mathcal{C}\big(\Omega : \mathbb{R}^n\big)$ *and* $U : \mathbb{R}^2_+ \to \Omega \subseteq \mathbb{R}^n$ *be a piecewise constant function given by*

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < c(t) - a_\varepsilon(t) - x_{1,\varepsilon} \\ U_{1,\varepsilon}, & c(t) - a_\varepsilon(t) - x_{1,\varepsilon} < x < c(t) \\ U_{2,\varepsilon}, & c(t) < x < c(t) + b_\varepsilon(t) + x_{2,\varepsilon} \\ U_1, & x > c(t) + b_\varepsilon(t) + x_{2,\varepsilon} \end{cases} \tag{3.1}$$

Where $x_{1,\varepsilon}$, $x_{2,\varepsilon} \sim \varepsilon$*, while* a_ε, b_ε *are smooth functions equal to zero at* $t = 0$ *with growth order less than or equal to* ε*. Assume*

$$\max_{i=1,2}\{\| f(U_{i,\varepsilon}) \|_{L\infty}, \| g(U_{i,\varepsilon}) \|_{L\infty}\} = \mathcal{O}(\varepsilon^{-1}). \tag{3.2}$$

Then

$$\begin{aligned} \partial_t f(U_\varepsilon) \approx & - c'(t)\Big(f(U_1) - f(U_0)\Big)\delta + \Big(a'_\varepsilon(t) f(U_{1,\varepsilon}) + b'_\varepsilon(t) f(U_{2,\varepsilon})\Big)\delta \\ & - c'(t)\Big(a_\varepsilon(t) + x_{1,\varepsilon}) f(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_{2,\varepsilon}) f(U_{2,\varepsilon})\Big)\delta' \\ \partial_t f(U_\varepsilon) \approx & \Big(g(U_1) - g(U_0)\Big)\delta \\ & + \Big((a_\varepsilon(t) + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_{2,\varepsilon}) g(U_{2,\varepsilon})\Big)\delta', \end{aligned} \tag{3.3}$$

where δ *and* δ' *are both supported by the line* $x = ct$.

Proof. We shall use the Taylor expansion formula in the space variable for a test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2_+)$:

$$\begin{aligned} \phi(c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t) = & \phi(c(t), t) + \sum_{j=1}^{m} \partial_x^j \phi(c(t), t) \frac{(-a_\varepsilon(t) - x_{1,\varepsilon})}{j!} \\ & + \mathcal{O}(\varepsilon^{m+1}) \end{aligned}$$

and

$$\begin{aligned} \phi(c(t) + b_\varepsilon(t) + x_{2,\varepsilon}, t) = & \phi(c(t), t) + \sum_{j=1}^{m} \partial_x^j \phi(c(t), t) \frac{(-a_\varepsilon(t) - x_{2,\varepsilon})}{j!} \\ & + \mathcal{O}(\varepsilon^{m+1}) \end{aligned}$$

In view of the above growth assumptions on a_ε, b_ε, $f(U_{i,\varepsilon})$ and $g(U_{i,\varepsilon})$, $i = 1, 2$ we see that it is enough to take m = 1 in the above expansion, so

$$\phi(c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t) = \phi(c(t), t) - \partial_x \phi(c(t), t)(a_\varepsilon(t) + x_{1,\varepsilon}) + \mathcal{O}(\varepsilon^2)$$
$$\phi(c(t) + b_\varepsilon(t) + x_{2,\varepsilon}, t) = \phi(c(t), t) + \partial_x \phi(c(t), t)(b_\varepsilon(t) + x_{2,\varepsilon}) + \mathcal{O}(\varepsilon^2)$$

By using the standard Rankine–Hugoniot shock conditions and the above approximations, we have, up to terms less than or equal to ε^2 as $\varepsilon \to 0$.

$$\begin{aligned}
\langle \partial_t f(U_\varepsilon), \phi) \rangle \approx &- \int_0^\infty \int_{-\infty}^{c(t)-a_\varepsilon(t)-x_{1,\varepsilon}} f(U_0) \partial_t \phi dx dt \\
&- \int_0^\infty \int_{c(t)-a_\varepsilon(t)-x_{1,\varepsilon}}^{c(t)} f(U_{1,\varepsilon}) \partial_t \phi dx dt \\
&- \int_0^\infty \int_{c(t)}^{c(t)+b_\varepsilon(t)+x_{2,\varepsilon}} f(U_{2,\varepsilon}) \partial_t \phi dx dt \\
&- \int_0^\infty \int_{-\infty}^{c(t)+b_\varepsilon(t)+x_{2,\varepsilon}} f(U_1) \partial_t \phi dx dt
\end{aligned}$$

Integration by parts gives now:

$$\begin{aligned}
\approx &- \int_0^\infty f(U_0) \phi(c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t)(c'(t) - a'_\varepsilon(t)) dt \\
&- \int_0^\infty f(U_{1,\varepsilon}) \Big(\phi(c(t), t) c'(t) - (c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t)(c'(t) - a'_\varepsilon(t)) \Big) dt \\
&- \int_0^\infty f(U_{2,\varepsilon}) \Big(\phi(c(t) + b(t) + x_{2,\varepsilon}, t)(c'(t) + b'(t)) - \phi(c(t), t) c'(t) \Big) dt \\
&- \int_0^\infty f(U_1) \phi(c(t) + b_\varepsilon(t) + x_{2,\varepsilon}, t) \Big(c'(t) + b'_\varepsilon(t) \Big) dt
\end{aligned}$$

Then we have

$$
\begin{aligned}
&\langle \partial_t f(U_\varepsilon), \phi \rangle \approx -\int_0^\infty (c'(t) - a'_\varepsilon(t))(f(U_{1,\varepsilon}) - f(U_0)) \\
&\cdot \phi(c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t)dt, \\
&- \int_0^\infty c'(t)(f(U_{2,\varepsilon}) - f(U_{1,\varepsilon}))\phi(c(t), t)dt \\
&- \int_0^\infty (c'(t) + b'_\varepsilon(t))(f(U_1) - f(U_{2,\varepsilon}))\phi(c(t) + b_\varepsilon(t) + x_{2,\varepsilon}, t)dt \\
\approx &- (f(U_{1,\varepsilon}) - f(U_0)) \int_0^\infty (c'(t) - a'_\varepsilon(t))\Big(\phi(c(t), t) - \partial_x \phi(c(t), t) \\
&\cdot (a_\varepsilon(t) + x_{1,\varepsilon}\Big)dt \\
&- (f(U_{2,\varepsilon}) - f(U_{1,\varepsilon})) \int_0^\infty c'(t)\phi(c(t), t)dt - \Big(f(U_1) - f(U_{2,\varepsilon})\Big) \\
&\int_0^\infty (c'(t) + b'_\varepsilon(t))\Big(\phi(c(t), t) + \partial_x \phi(c(t), t)(b_\varepsilon(t) + x_{2,\varepsilon})\Big)dt.
\end{aligned}
$$

$$
\begin{aligned}
\approx & - f(U_{1,\varepsilon}) \int_0^\infty c'(t) \cdot \phi(c(t),t)dt + f(U_{1,\varepsilon}) \int_0^\infty c'(t)\partial_x \phi(c(t),t) \\
& \cdot (a_\varepsilon(t) + x_{1,\varepsilon})dt \\
& + f(U_{1,\varepsilon}) \int_0^\infty a'_\varepsilon(t)\phi(c(t),t)dt + f(U_{1,\varepsilon}) \int_0^\infty a'_\varepsilon(t)\partial_x \phi(c(t),t) \\
& \cdot (a_\varepsilon(t) + x_{1,\varepsilon})dt \\
& + f(U_0) \int_0^\infty a'_\varepsilon(t)\phi(c(t),t)dt + f(U_0) \int_0^\infty a_\varepsilon(t)\partial_x \phi(c(t),t) \\
& \cdot (a_\varepsilon(t) + x_{1,\varepsilon})dt \\
& + f(U_0) \int_0^\infty c'(t) \cdot \phi(c(t),t)dt - f(U_0) \int_0^\infty c'(t)\partial_x \phi(c(t),t) \\
& \cdot (a_\varepsilon(t) + x_{1,\varepsilon})dt \\
& - f(U_{2,\varepsilon}) \int_0^\infty c'(t)\phi(c(t),t)dt + f(U_{1,\varepsilon}) \int_0^\infty c'(t)\phi(c(t),t)dt \\
& - f(U_1) \int_0^\infty c'(t)\phi(c(t),t)dt - f(U_1) \int_0^\infty c'(t)\partial_x \phi(c(t),t) \\
& \cdot (b_\varepsilon(t) + x_{1,\varepsilon})dt \\
& - f(U_1) \int_0^\infty b'_\varepsilon(t)\phi(c(t),t)dt - f(U_1) \int_0^\infty b'_\varepsilon(t)\partial_x \phi(c(t),t) \\
& \cdot (b_\varepsilon(t) + x_{1,\varepsilon})dt \\
& + f(U_{2,\varepsilon}) \int_0^\infty c'(t) \cdot \phi(c(t),t)dt + f(U_{2,\varepsilon}) \int_0^\infty c'(t)\partial_x \phi(c(t),t) \\
& \cdot (b_\varepsilon(t) + x_{1,\varepsilon})dt \\
& + f(U_{2,\varepsilon}) \int_0^\infty b'_\varepsilon(t) \cdot \phi(c(t),t)dt + f(U_{2,\varepsilon}) \int_0^\infty b'_\varepsilon(t)\partial_x \phi(c(t),t) \\
& \cdot (b_\varepsilon(t) + x_{1,\varepsilon})dt
\end{aligned}
$$

□

Remark 5. In above step we neglect all terms of growth rate $\varepsilon^\alpha, \alpha > 1$.

The assumptions from Lemma 1 imply

$$\begin{aligned}
&\langle \partial_t f(U_\varepsilon), \phi\rangle \approx -\Big(f(U_1) - f(U_0)\Big) \int_0^\infty c'(t)\phi(c(t), t)dt \\
&+ \int_0^\infty \Big(a'_\varepsilon(t) f(U_{1,\varepsilon}) + b'_\varepsilon(t) f(U_{2,\varepsilon})\Big)\phi(c(t), t)dt \\
&+ \int_0^\infty c'(t)\Big((a_\varepsilon(t) + x_{1,\varepsilon}) f(U_{1,\varepsilon}) + (b'_\varepsilon(t) + x_{2,\varepsilon}) f(U_{2,\varepsilon})\Big) \\
&\cdot \partial_x \phi(c(t), t)dt \\
\approx&\langle\Big(- c'(t)\Big(f(U_1) - f(U_0)\Big) + a'(t) f(U_{1,\varepsilon}) + b'(t) f(U_{2,\varepsilon})\Big) \\
&\cdot \delta(x - c(t)), \phi(x, t)\rangle \\
\approx&\langle -c'(t)\Big((a_\varepsilon(t) + x_1, \varepsilon) f(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_2, \varepsilon) f(U_{2,\varepsilon})\Big) \\
&\cdot \delta'(x - c(t)), \phi(x, t)\rangle
\end{aligned}$$

With the same type of reasoning, we see that the space derivative is given by

$$\begin{aligned}
\langle \partial g(U_\varepsilon), \phi\rangle \approx &- \int_0^\infty \int_{-\infty}^{c(t)-a_\varepsilon(t)-x_{1,\varepsilon}} g(U_0)\partial_x \phi dx dt \\
&- \int_0^\infty \int_{c(t)-a_\varepsilon(t)-x_{1,\varepsilon}}^{c(t)} g(U_{1,\varepsilon})\partial_x \phi dx dt \\
&- \int_0^\infty \int_{c(t)}^{c(t)+b_\varepsilon(t)+x_{2,\varepsilon}} g(U_{2,\varepsilon})\partial_x \phi dx dt \\
&- \int_0^\infty \int_{-\infty}^{c(t)+b_\varepsilon(t)+x_{2,\varepsilon}} g(U_1)\partial_x \phi dx dt
\end{aligned}$$

Integration by parts gives

$$
\begin{aligned}
\langle \partial_x g(U_\varepsilon), \phi \rangle \approx &- \int_0^\infty g(U_0) \cdot \phi(c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t) dt \\
&- \int_0^\infty g(U_{1,\varepsilon}) \big[\phi(c(t), t) - (c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t)\big] dt \\
&- \int_0^\infty g(U_{2,\varepsilon}) \big[\phi(c(t) + b_\varepsilon(t) + x_{2,\varepsilon}, t) - \phi(c(t), t)\big] dt \\
&- \int_0^\infty g(U_1) \phi(c(t) + b_\varepsilon(t) + x_{2,\varepsilon}, t) dt
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\langle \partial_x g(U_\varepsilon), \phi \rangle \approx &\int_0^\infty (g(U_{1,\varepsilon}) - g(U_0)) \cdot \phi(c(t) - a_\varepsilon(t) - x_{1,\varepsilon}, t) dt \\
&+ \int_0^\infty (g(U_{2,\varepsilon}) - g(U_{1,\varepsilon})) \phi(c(t), t) dt \\
&+ \int_0^\infty (g(U_1) - g(U_{2,\varepsilon})) \cdot \phi(c(t) + b_\varepsilon(t) + x_{2,\varepsilon}, t) dt
\end{aligned}
$$

$$
\begin{aligned}
&\langle \partial_x g(U_\varepsilon), \phi\rangle \approx (g(U_{1,\varepsilon}) - g(U_0)) \int_0^\infty \phi(c(t),t) - \partial_x \phi(c(t),t) \\
&\cdot (a_\varepsilon(t) + x_{1,\varepsilon}) dt \\
&+ (g(U_{2,\varepsilon}) - g(U_{1,\varepsilon})) \int_0^\infty \phi(c(t),t) dt \\
&+ (g(U_1) - g(U_{2,\varepsilon})) \int_0^\infty \phi(c(t),t) + \partial_x \phi(c(t),t)(b_\varepsilon(t) + x_{2,\varepsilon}) dt \\
\approx & g(U_1) - g(U_0) \int_0^\infty \phi(c(t),t) dt \\
&- \int_0^\infty \Big((a_\varepsilon(t) + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_{2,\varepsilon}) g(U_{2,\varepsilon}) \Big) \partial_x \phi(c(t),t) dt \\
\approx & \langle \Big(g(U_1 - g(U_0) \delta(x - c(t)), \phi(x,t) \Big) \rangle \\
&+ \langle \Big((a_\varepsilon(t) + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_{2,\varepsilon}) g(U_{2,\varepsilon}) \Big) \\
&\cdot \delta'(x - c(t)), \phi(x,t) \rangle.
\end{aligned}
$$

Definition 5. *The functions of the form (3.1) are called constant shadow waves or constant SDWs for short. The value*

$$\sigma_\varepsilon(t) = (a_\varepsilon(t) + x_{1,\varepsilon}) U_{1,\varepsilon} + (b_\varepsilon(t) + x_{2,\varepsilon}) U_{2,\varepsilon}$$

is called the strength of shadow wave and $c'(t)$ is called the speed of shadow wave. We suppose that $\lim_{\varepsilon\to 0} \sigma_\varepsilon(t) = \sigma(t) \in \mathbb{R}^n$ exists for every $t \geq 0$ and

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \int U_\varepsilon(x,t)\phi(x,t) dx\ dt =& \langle \sigma(t)\delta(x - c(t) + U_0 + [U]\theta(x - c(t)), \phi(x,t)\rangle \\
=& \int \sigma(t)\phi(c(t),t) dt + \int (U_0 \\
&+ [U]\theta(x - c(t))\phi(x,t) dx\ dt,
\end{aligned}
$$

where $[U] = U_1 - U_0$ and θ is the Heaviside function. The SDW central line ("front of the SDW') is given by $x = c(t)$, while $x = c(t) - a_\varepsilon(t) - x_{1,\varepsilon}$ and $x = c(t) + b_\varepsilon(t) + x_{2,\varepsilon}$ are called the external SDW lines. The functions $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are called the intermediate states while the values $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ are called the shifts of given SDW.

3.2 General formula for Riemann problem

The following special case of (3.1)

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < (c - a_\varepsilon)t \\ U_{1,\varepsilon}, & (c - a_\varepsilon)t < x < ct \\ U_{2,\varepsilon}, & ct < x < (c + b_\varepsilon)t \\ U_1, & x > (c + b_\varepsilon)t \end{cases}$$

We will call it the simple SDW. And it is general enough for solving Riemann problem as we could see bellow.

Then we get simple form from formula (3.3)

$$\begin{aligned} \partial_t f(U_\varepsilon) \approx & -c(f(U_1) - f(U_0))\delta - c(a_\varepsilon f(U_{1,\varepsilon}) + b_\varepsilon f(U_{2,\varepsilon}))t\delta' \\ & + (a_\varepsilon f(U_{1,\varepsilon}) + b_\varepsilon f(U_{2,\varepsilon}))\delta \\ \partial_x g(U_\varepsilon) \approx & \ (g(U_1) - g(U_0))\delta + (a_\varepsilon g(U_{1,\varepsilon}) + b_\varepsilon g(U_{2,\varepsilon}))t\delta'. \end{aligned}$$

The support of δ and (δ' consequently) is the line $x = ct$.

Remark 6. Consider a following conservation law system

$$\partial_t f(U) + \partial_x g(U) = 0, U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n \tag{3.4}$$

Where $f = (f^1, ..., f^n)$ and $g = (g^1, ..., g^n)$ are continuous mapping from Ω in $\mathbb{R}^n$, where f is called density function and g is called a flux function. The functions f and g are continuous mappings from a physical domain Ω in $\mathbb{R}^n$ into $\mathbb{R}^n$.

We can find a shadow wave solutions to a system of conservation laws (3.4) immediately follows assumption to keep our discussion on general level.

Assumption 1. *All the components* $U_\varepsilon^i, i = 1, ..., n$ *of an SDW (3.1) satisfy*
$\| U_\varepsilon^i \|_{L\infty} = \mathcal{O}(\varepsilon^{-1})$ *if* f *and* g *are at most linear with respect to* i *-th variable or*
$\| U_\varepsilon^i \|_{L\infty}$ *has a growth order small enough for (3.2) to hold, otherwise.*

Definition 6. *A major components or* ε^{-1} *-components are the components satisfying the first criteria, while all other are called the minor ones.*

A delta shock is an SDW associated with a δ *distribution with all minor components having finite limits as* $\varepsilon \to 0$*. The wave is called singular shock if some of the minor components are unbounded as* $\varepsilon \to 0$*.*

The following definition contains analogous notion to Hugoniot locus for shocks.

Definition 7. *Let* U_0 *be fixed. The set of all* $U_1 \in \Omega$ *such that there exists an SDW solution to (3.4) with the initial data*

$$U \mid_{t=0} = \begin{cases} U_0, & x < 0 \\ U_1, & x > 0 \end{cases}$$

is called the shadow locus. Also the above set called delta (singular delta) locus when the SDW is delta (singular) shock.

To find SDW solutions of (3.4), we substitution the function U from (3.1) into the i-th equation implies

$$\begin{aligned}
&\Big(- c(f^i(U_1) - f^i(U_0)) + a_\varepsilon f^i(U_{1,\varepsilon}) + b_\varepsilon f^i(U_{2,\varepsilon})\Big)\delta(x - ct) \\
&\quad - ct\Big(a_\varepsilon f^i(U_{1,\varepsilon}) + b_\varepsilon f^i(U_{2,\varepsilon})\Big)\delta'(x - ct) + \Big(g^i(U_1) - g^i(U_0)\Big)\delta(x - ct) \\
&\quad + t\Big(a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon})\Big)\delta'(x - ct) \approx 0.
\end{aligned}$$

which yields

$$-c(f^i(U_1) - f^i(U_0)) + a_\varepsilon f^i(U_{1,\varepsilon}) + b_\varepsilon f^i(U_{2,\varepsilon}) + g^i(U_1) - g^i(U_0) \approx 0.$$
$$-c(a_\varepsilon f^i(U_{1,\varepsilon}) + b_\varepsilon f^i(U_{2,\varepsilon})) + a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon}) \approx 0,$$
$$i = 1, ..., n.$$

We define

$$\kappa^i := c(f^i(U_1) - f^i(U_0) - (g^i(U_1) - g^i(U_0))$$

to be called Rankine–Hugoniot deficit (RH deficit for short) in the i -th equation. Also, (3.3) can be written as

$$a_\varepsilon f^i(U_{1,\varepsilon}) + b_\varepsilon f^i(U_{2,\varepsilon}) \approx \kappa^i$$
$$a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon}) \approx c\kappa^i, i = 1, ..., n.$$

That was the most general case with Assumption (3.1). Let us start our investigation of the above system for the the simplest evolutionary case.

3.2.1 Evolutionary systems

If the system of conservation laws (3.4) is given in the evolutionary form $f^i(y) \equiv y^i, i = 1, ..., n$, then the relation (3.3) implies that

$$-c(U_1^i - U_0^i) + a_\varepsilon U_{1,\varepsilon}^i + b_\varepsilon U_{2,\varepsilon}^i + g^i(U_1) - g^i(U_0) \approx 0$$
$$-c(a_\varepsilon U_{1,\varepsilon}^i + b_\varepsilon U_{2,\varepsilon}^i) + a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon}) \approx 0, i = 1, ..., n.$$

And

$$\kappa^i = c(U_1^i - U_0^i) - (g^i(U_1) - g^i(U_0)), i = 1, ..., n.$$

Also the system (3.4) has a simpler form

$$a_\varepsilon U_{1,\varepsilon}^i + b_\varepsilon U_{2,\varepsilon}^i \approx \kappa^i$$
$$a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon}) \approx c\kappa^i, i = 1, ..., n.$$

The following proposition bellow explains what is difference between a ”general result” and a ”concrete solution”.

Proposition 1. *Suppose that all the flux-functions in (3.4) are of at most linear growth with respect to k components, we say that* $U^1, ..., U^k$ *and superlinear with respect to others. Then a shadow locus to the system of conservation laws (3.4) with* $f(y) = y$ *is contained in a* $(k+1)$ *–dimensional manifold.*

In the case when the SDW is delta shocks the situation is simpler because we can assume that $U_j^{i,\varepsilon} \to U_{s,j}^i \in \mathbb{R}, i = k+1, ..., n, j = 1, 2.$ and that the limits $\xi_1^i = \lim\limits_{\varepsilon\to 0} a_\varepsilon U_{1,\varepsilon}^i$ and $\xi_2^i = \lim_{\varepsilon\to 0} b_\varepsilon U_{2,\varepsilon}^i, i = 1, ..., n$ exist. If $g_i(U) = \sum_{j=1}^k g_j^i(U^{k+1}, ..., U^n)U^j$, then the system (3.6) reduces to

$$\xi_1^i + \xi_2^i = \kappa^i,\ i = 1, ..., k.$$

$$\sum_{j=1}^k g_j^i(U_{s,1}^{k+1}, ..., U_{s,1}^n)\xi_1^j + \sum_{j=1}^k g_j^i(U_{s,2}^{k+1}, ..., U_{s,2}^n)\xi_2^j = c\kappa^i,\ i = 1, ..., n,$$

$$\text{where } \kappa^i = 0,\ i = k+1, ..., n,$$

We note that the above system has 2k major intermediate states $U_{j,\varepsilon}^i, i = 1, ..., k, j = 1, 2.$ and $2(n-k)$ minor ones with limits $U_{s,j}^i, i = k+1, ..., n, j = 1, 2$ as $\varepsilon \to 0$. The general idea for solving the system is treating these limits as real parameters which are chosen such that the system has a solution $\xi_j^i, i = 1, ..., k, j = 1, 2.$

3.3 Entropy conditions for Riemann problem

Let $\eta(U)$ be a (strictly) convex or semi-convex entropy function for the systems of conservation law (3.4) with entropy-flux function $q(U)$. We shall use entropy conditions in the following definition.

Definition 8. *A solution* U_ε *to the system of conservation law (3.4) with initial data* $U\mid_{t=0} = U_{0,\varepsilon}$ *is admissible if for every* $T > 0$, *we have*

$$\varliminf_{\varepsilon\to 0} \int_{\mathbb{R}} \int_0^T \eta(U_\varepsilon)\partial_t\phi + q(U_\varepsilon)\partial_x\phi dtdx + \int_{\mathbb{R}} \eta(U_{0,\varepsilon}(x,0))\phi(x,0)dx \geqq 0,$$

for all non-negative test functions $\phi \in C_0^\infty(\mathbb{R} \times (-\infty, T))$.

Now, we take a simple SDW U_ε *from (3.1) and use the equality (2.3) from Lemma 2.1 with* f *replaced by* η *and* g *by* q. *Here the delta function is a non–negative distribution, then the first condition becomes*

$$\overline{\lim_{\varepsilon \to 0}} -c(\eta(U_1) - \eta(U_0)) + a_\varepsilon \eta(U_{1,\varepsilon}) + b_\varepsilon \eta(U_{2,\varepsilon}) + q(U_1) - q(U_0) \leqq 0$$

But a derivative of the delta function has no constant sign and the second condition becomes

$$\lim_{\varepsilon \to 0} -c(a_\varepsilon \eta(U_{1,\varepsilon}) + b_\varepsilon \eta(U_{2,\varepsilon})) + a_\varepsilon g(U_{1,\varepsilon}) + b_\varepsilon g(U_{2,\varepsilon}) = 0$$

where $U_0, U_1, U_{1,\varepsilon}$ *and* $U_{2,\varepsilon}$ *are constants.*

In the most of this part with delta or singular shock solution, we use overcompressibility as the admissibility condition: A wave is called overcompressive if all characteristics from both sides of the SDW line run into a shock curve, that is,

$$\lambda_i(U_0) \geqq c^{'}(t) \geqq \lambda_i(U_1), i = 1, ..., n,$$

where c is a shock speed and $x = \lambda_i(U)t, i = 1, ..., n$ are the characteristics of the system.

Definition 9. *The SDW solution is called weakly unique if its distributional image is unique. More precisely, the speed* c *of the wave has to be unique as well as the limit*

$$\lim_{\varepsilon \to 0} a_\varepsilon U_{1,\varepsilon} + b_\varepsilon U_{2,\varepsilon}$$

If the limit $\lim_{\varepsilon \to 0} a_\varepsilon U^i_{1,\varepsilon} + b_\varepsilon U^i_{2,\varepsilon}, i \in \{1, ..., n\}$ *is unique, then the* i *-th component is also unique.*

By above definition, we note that all minor components of U_ε are unique. The following proposition is a direct consequence of the SDW definition.

Proposition 2. *Suppose that the system (3.4) has an SDW solution*

(a) If there exists an equation of the system, say the i -th one, such that a density function $f^i(U)$ is independent of major components of U, then the speed of the SDW is uniquely determined by the equation

$$-c[f^i(U)] + [g^i(U)] = 0$$

(b) If there is an equation in the system, say the i -th one, such that $f^i(U) = U^j$, where U^j is a major component, then it is uniquely determined by

$$a_\varepsilon U^j_{1,\varepsilon} + b_\varepsilon U^j_{2,\varepsilon} = \kappa_i \in \mathbb{R}.$$

Consequently, if (a) and (b) holds for all major components, then a distributional limit of an SDW solution to the system (3.4) is unique. Specially, if the case for the system (3.4) is given in evolutionary form.

Definition 10. *The solution to the system (3.4) is called weakly unique if it consists of a unique combination of standard admissible elementary waves (shocks, rarefactions and contact discontinuities) and an admissible SDW.*

4

Shadow waves for pressureless gas balance laws

This is completely original part. The aim of this part is to solve the pressureless gas dynamics model (PGD for short) with added the body force. That model can be derived from (1.1) with added a force term on the right-hand side of momentum conservation law, then we have the following

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = b\rho$$

It is a model of gas dynamics in a gravitational field with entropy assumed to be a constant. The energy conservation law is now used as a selection criteria for admissible solutions: For all continuous solutions energy is conserved, while is should decrease for discontinuous ones. The body force source term is present if there is some external force acting on the fluid. The force assumed here is the gravity with b being the gravitational constant. By letting $p(\rho) \equiv 0$ we get the PGD conservation law system.

Among a lot of different approaches in explaining such type of solutions, we will use shadow waves in order to solve the balance law of pressureless gas with body force source term. We will use here a simpler condition - so called overcompressibility: All characteristics should run into the shock curve. Also, it is proved that entropy condition is not enough to exclude non-admissible waves for pressureless conservation law system in paper [17]. The next advantage is a simplicity of treating an interaction problem involving a shadow wave. Our primary goal is to solve the following Riemann problem

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) &= b\rho, \end{aligned} \tag{4.1}$$

$$(\rho, u)(x, 0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0. \end{cases}$$

It seems that the generalized pressure system can be treated in the same way.

4.1 Elementary waves

Let us first state some known fact about elementary waves of the given system. We can look in [8] or [12] for more details. Writing the system (4.1) into the evolutionary form by taking the new variable $m = \rho u$,

$$\begin{aligned} \partial_t \rho + \partial_x m &= 0 \\ \partial_t m + \partial_x \Big(\frac{m^2}{\rho}\Big) &= b\rho, \end{aligned}$$

we can easily see that it is a weakly hyperbolic with the double eigenvalue $\lambda_{1,2} = \frac{m}{\rho} = u$. Let us first look for a solution to (4.1) when initial data are constants, $(\rho(x,0), u(x,0)) = (\rho_0, u_0)$. For smooth solutions, we can substitute ρ_t from the first equation of (4.1) into the second one and eliminate ρ from it by division (provided that we are away from a vacuum state). So, we have now the following equation

$$\partial_t u + u \partial_x u = b,$$

that can be solved by a method of characteristics,

$$\frac{du}{dt} = 0,\ \frac{dx}{dt} = u,\ x(0) = x_0,\ u(0) = u_0.$$

A solution for constant initial data is given by

$$u = bt + u_0,\ x = x_0 + \frac{1}{2}bt^2 + u_0 t.$$

Then the first equation becomes

$$\partial_t \rho + (bt + u_0)\partial_x \rho = 0$$

with a solution $\rho = \rho_0$ on each curve $x = x_0 + \frac{1}{2}bt^2 + u_0 t$. So, a "constant state" solution is given by

$$(\rho, u) = (\rho_0, bt + u_0).$$

It will be used in the rest of the part.

Let us now look at a Riemann problem

$$(\rho, u)(x, 0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0. \end{cases}$$

In the case $u_0 = u_1$ there is a contact discontinuity solution (CD) given by

$$(\rho, u)(x, t) = \begin{cases} (\rho_0, u_0 + bt), & x < \frac{1}{2}bt^2 + u_0 t \\ (\rho_1, u_0 + bt), & x > \frac{1}{2}bt^2 + u_0 t. \end{cases}$$

Also , we can see that the vacuum state is always a solution. Thus, in a general case when $u_0 < u_1$ we have a solution of the form CD+Vacuum+CD, two contact discontinuities connected by the vacuum:

$$(\rho, u)(x, t) = \begin{cases} (\rho_0, u_0 + bt), & x < \frac{1}{2}bt^2 + u_0 t \\ (0, u), & \frac{1}{2}bt^2 + u_0 t < x < \frac{1}{2}bt^2 + u_1 t \\ (\rho_1, u_1 + bt), & x > \frac{1}{2}bt^2 + u_1 t \end{cases} \tag{4.2}$$

where u is an arbitrary function satisfying

$$u(\frac{1}{2}bt^2 + u_0 t, t) = bt + u_0 \text{ and } u(\frac{1}{2}bt^2 + u_1 t, t) = bt + u_1.$$

4.2 Shadow waves solution

In the case $u_0 > u_1$, there is not elementary wave solutions to the Riemann problem. We can try to substitute a SDW solution (see [36])

$$(\rho, u)(x,t) = \begin{cases} (\rho_0, u_0 + bt), & x < c(t) - \varepsilon t \\ (\rho_\varepsilon(t), u_\varepsilon(t)), & c(t) - \varepsilon t < x < c(t) + \varepsilon t \\ (\rho_1, u_1 + bt), & x > c(t) + \varepsilon t \end{cases} \tag{4.3}$$

in both equations of the system. The classical solution in the case $u_0 \leq u_1$ satisfies all the usual admissibility criteria (entropy inequalities). As an admissibility criteria for SDWs we will use the overcompressibility condition. That is the most frequent admissibility condition for all delta shock type nonstandard solutions of conservation law systems in the literature.

Definition 11. *A shadow wave of the form (4.3) is called overcompressive if*

$$\lambda_2(\rho_0, u_0 + bt) \geq \lambda_1(\rho_0, u_0 + bt) \geq c'(t) \geq \lambda_2(\rho_1, u_1 + bt) \geq \lambda_1(\rho_1, u_1 + bt), \tag{4.4}$$

i.e. all characteristics run into a shock. We can look in [6] or [17] for a detailed explanation of that admissibility condition.

In a few figures we would like to illustrate how Shadow Waves look like for a fixed ε.

Let us first start with the simple one.

Its fronts are described at Figure 4.1. All lines are straight and with angles $\approx \varepsilon$.

Its shape for a fixed ε and four different times is shown on Figure 4.2.

Weighted Shadow Waves, needed for out solution to Riemann problem with a source term are in the next two figures. In the first of, Figure 4.3 one can see that fronts are now not straight lines but curves. 4.1. All lines are straight and with angles $\approx \varepsilon$.

At Figure 4.4 one can see a variable speed of delta function movement with four time slices.

Now we can formulate the following theorem.

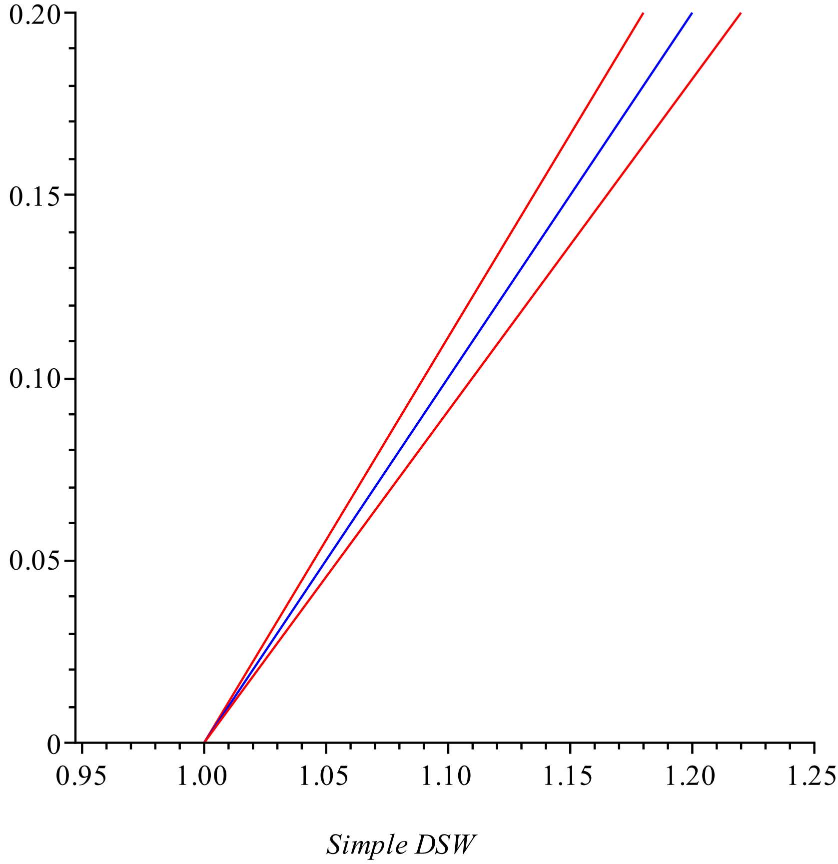


Figure 4.1: SDW fronts

Theorem 3. *The Riemann problem (4.1) has a unique solution in a set of elementary and shadow waves. If $u_0 \leq u_1$ a solution consists of two contact discontinuities connected with the vacuum state (4.2). In the case $u_0 > u_1$, there exists an overcompressive SDW solution of the form (4.3).*

Proof. The first, elementary waves case, $u_0 \leq u_1$ is explained in the previous

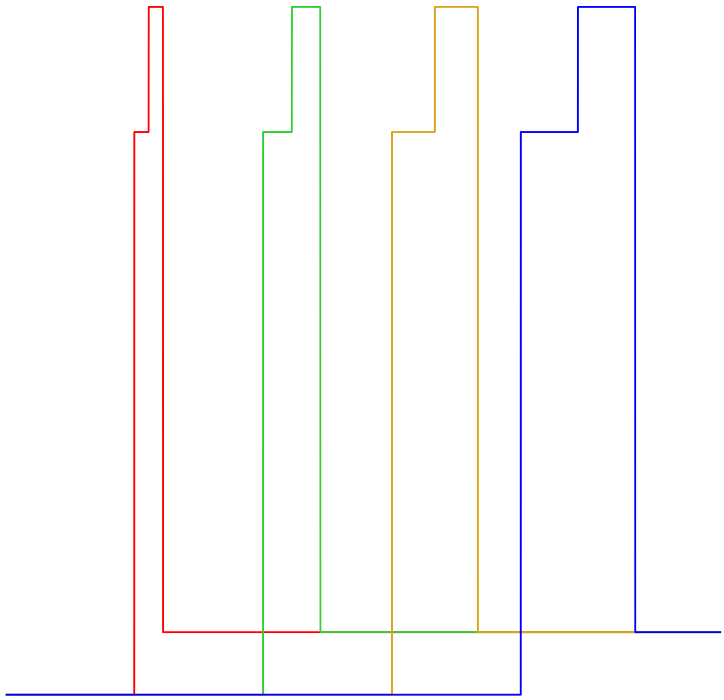


Figure 4.2: SDW shape

section. Suppose $u_0 > u_1$ and substitute a function of the form (4.3) into system (4.1). For the first equation we have

$$
\begin{aligned}
I_1 := &- \int_0^\infty \int_{-\infty}^{c(t)-\varepsilon t} \rho_0 \partial_t \varphi(x,t) + \rho_0 (u_0 + bt) \partial_x \varphi(x,t) dx dt \\
&- \int_0^\infty \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \rho_\varepsilon \partial_t \varphi(x,t) + \rho_\varepsilon u_\varepsilon \partial_x \varphi(x,t) dx dt \\
&- \int_0^\infty \int_{c(t)+\varepsilon t}^{\infty} \rho_1 \partial_t \varphi(x,t) + \rho_1 (u_1 + bt) \partial_x \varphi(x,t) dx dt = 0.
\end{aligned}
$$

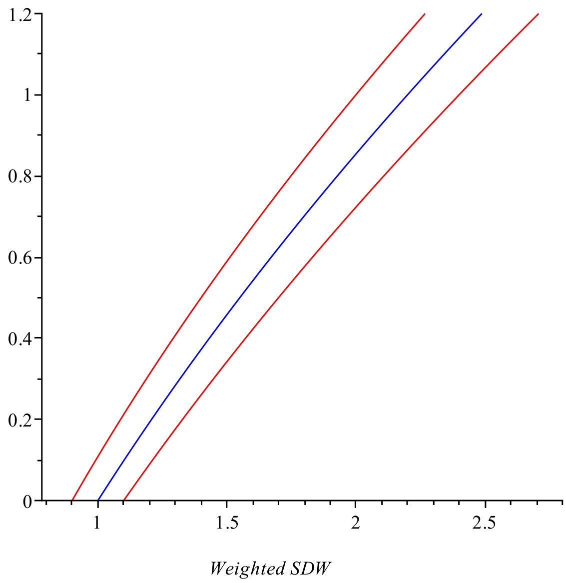


Figure 4.3: Non-constant speed SDW

where $\varphi \in C_0^\infty(\mathbb{R}^2)$. The first relation is obtained from δ terms and the

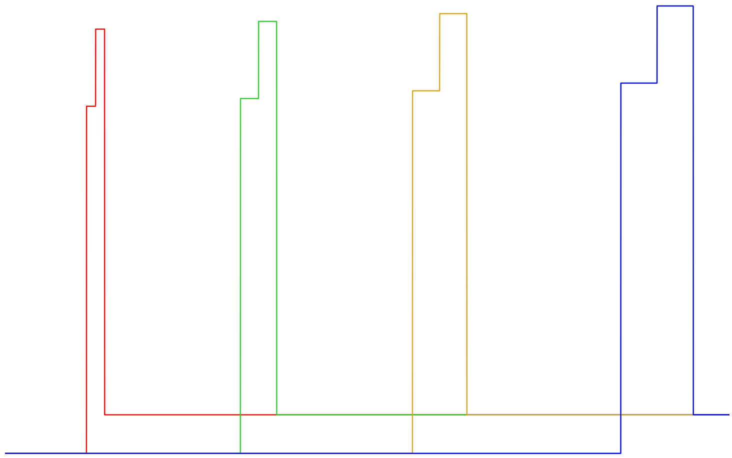


Figure 4.4: Weighted SDW shape

other one is from δ' terms. Integration by parts gives

$$
\begin{aligned}
I_1 \approx &\int_0^\infty \rho_0(c'(t)-\varepsilon)\Big(\varphi(c(t),t)-\partial_x\varphi(c(t),t)\varepsilon t\Big)dt + \int_{-\infty}^0 \rho_0\varphi(x,0)dx \\
&- \int_0^\infty \rho_0(u_0+bt)\Big(\varphi(c(t),t)-\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&+ \int_0^\infty \rho_\varepsilon(t)(c'(t)+\varepsilon)\Big(\varphi(c(t),t)+\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&- \int_0^\infty \rho_\varepsilon(t)(c'(t)-\varepsilon)\Big(\varphi(c(t),t)-\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&+ \int_0^\infty \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \partial_t\rho_\varepsilon(t)\varphi(x,t)dxdt \\
&- \int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)\Big(\varphi(c(t),t)+\partial_x\varphi(c(t),t)\varepsilon t-\varphi(c(t),t)-\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&- \int_0^\infty \rho_1(c'(t)+\varepsilon)\Big(\varphi(c(t),t)+\partial_x\varphi(c(t),t)\varepsilon t\Big)dt + \int_0^\infty \rho_1\varphi(x,0)dx \\
&+ \int_0^\infty \rho_1(u_1+bt)\Big(\varphi(c(t),t)+\partial_x\varphi(c(t),t)\varepsilon t\Big)dt
\end{aligned}
$$

Then we obtain,

$$\begin{aligned}
I_1 \approx & \int_0^\infty \rho_0(c'(t)-\varepsilon)\varphi(c(t)-\varepsilon t,t)dt \\
& + \int_{-\infty}^0 \rho_0\varphi(x,0)dx - \int_0^\infty \rho_0(u_0+bt)\varphi(c(t)-\varepsilon t,t)dt \\
& + \int_0^\infty \rho_\varepsilon(t)\varphi(c(t)+\varepsilon t,t)(c'(t)+\varepsilon)dt - \int_0^\infty \rho_\varepsilon(t)\varphi(c(t)-\varepsilon t,t) \\
& \cdot (c'(t)-\varepsilon)dt + \int_0^\infty \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \partial_t\rho_\varepsilon(t)\varphi(x,t)dxdt \\
& - \int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)\Big(\varphi(c(t)+\varepsilon t,t)-\varphi(c(t)-\varepsilon t,t)\Big)dt \\
& - \int_0^\infty \rho_1(c'(t)+\varepsilon)\varphi(c(t)+\varepsilon t,t)dt + \int_0^\infty \rho_1\varphi(x,0)dx \\
& + \int_0^\infty \rho_1(u_1+bt)\varphi(c(t)+\varepsilon t,t)dt.
\end{aligned}$$

The sign "$\approx$" simply means a convergence to zero as $\varepsilon \to 0$. Note that

$$\int_{-\infty}^0 \rho_0\varphi(x,0)dx + \int_0^\infty \rho_1\varphi(x,0)dx = \langle \rho|_{t=0}, \varphi\rangle$$

that cancels with the initial data and we will drop it in the rest of calculations. We will use the fact that

$$\varphi(c(t)\pm\varepsilon t,t) = \varphi(c(t),t) \pm \partial_x\varphi(c(t),t)\varepsilon t + \mathcal{O}(\varepsilon^2),$$

and

$$\rho_\varepsilon \sim \frac{1}{\varepsilon},\ u_\varepsilon \sim \text{const.}$$

Then (assuming that the initial conditions are satisfied) we get the following equation

$$\begin{aligned}
& - \int_0^\infty ([\rho]c'(t) - [\rho(u+bt)] - 2(t\partial_t\rho_\varepsilon(t)+\rho_\varepsilon(t))\varepsilon\varphi(c(t),t)dt \\
& + 2\int_0^\infty (\rho_\varepsilon(t)c'(t) - \rho_\varepsilon(t)u_\varepsilon(t))\varepsilon t\partial_x\varphi(c(t),t)dt \approx 0,
\end{aligned}$$

where $[x] := x_1 - x_0$. Note that we have abused the usual notation since here $[\rho u]$ means $\rho_1 u_1 - \rho_0 u_0$ and not the real jump $\rho_1(u_1+bt) - \rho_0(u_0+bt)$, that is denoted by $[\rho(u+bt)]$. We could see that the above relation is true if and only if

$$\lim_{\varepsilon\to 0} 2\varepsilon(\rho_\varepsilon + t\partial_t\rho_\varepsilon) = k_1 := c'(t)[\rho] - [\rho(u+bt)] \tag{4.5}$$

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(c'(t) - u_\varepsilon)\varepsilon = 0. \tag{4.6}$$

We see immediately that

$$u_s(t) := \lim_{\varepsilon\to 0} u_\varepsilon(t) = c'(t)$$

Using the notation $\xi = \xi(t) := \lim_{\varepsilon\to 0} 2\varepsilon\rho_\varepsilon$ equation (4.5) becomes

$$t\xi'(t) + \xi(t) = k_1(t) = [\rho]c'(t) - [\rho u] - b[\rho]t \tag{4.7}$$

with $\xi(0) = 0$ because we do not have a delta function in the initial data. With the same method, and with the substitution

$$\rho_\varepsilon \to \rho_\varepsilon u_\varepsilon,\ \rho_\varepsilon u_\varepsilon \to \rho_\varepsilon u_\varepsilon^2.$$

from the second equation

$$\partial_t(\rho u) + \partial_x(\rho u^2) = b\rho$$

we have

$$
\begin{aligned}
I_2 = &- \int_0^\infty \int_{-\infty}^{c(t)-\varepsilon t} \rho_0(u_0+bt)\partial_t\varphi(x,t) + \rho_0(u_0+bt)^2\partial_x\varphi(x,t)dxdt \\
&- \int_0^\infty \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \rho_\varepsilon u_\varepsilon \partial_t\varphi(x,t) + \rho_\varepsilon u_\varepsilon^2 \partial_x\varphi(x,t)dxdt \\
&- \int_0^\infty \int_{c(t)+\varepsilon t}^{\infty} \rho_1(u_1+bt)\partial_t\varphi(x,t) + \rho_1(u_t+bt)^2\partial_x\varphi(x,t)dxdt \\
=&b\int_0^\infty \Big(\int_{-\infty}^{c(t)-\varepsilon t} \rho_0\varphi(x,t)dx + \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \rho_\varepsilon\varphi(x,t)dx \\
&+ \int_{c(t)+\varepsilon t}^{\infty} \rho_1\varphi(x,t)dx \Big) dt
\end{aligned}
$$

Integration by part gives:

$$
\begin{aligned}
I_2 &= \int_0^\infty \rho_0(u_0+bt)(\varphi(c(t),t)-\partial_x\rho(c(t),t)\varepsilon t)(c'(t)-\varepsilon)dt \\
&+\int_0^\infty\int_{-\infty}^{c(t)-\varepsilon t}\rho_0 b\varphi(x,t)dxdt+\int_0^\infty \rho_0 u_0\varphi(x,0)dx \\
&-\int_0^\infty \rho_0(u_0+bt)^2\Big(\varphi(c(t),t)-\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&+\int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)(c'(t)-\varepsilon)\Big(\varphi(c(t),t)-\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&+\int_0^\infty\int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t}\partial_t\Big(\rho_\varepsilon(t)u_\varepsilon(t)\Big)\varphi(x,t)dxdt \\
&-\int_0^\infty \rho_\varepsilon(t)u_\varepsilon^2(t)\Big(\varphi(c(t),t)+\partial_x\varphi(c(t),t)\varepsilon t-\varphi(c(t),t)-\partial_x(c(t),t)\varepsilon t\Big)dt \\
&-\int_0^\infty \rho_1(u_1+bt)(c'(t)+\varepsilon)\Big(\varphi(c(t),t)+\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&-\int_0^\infty\int_{c(t)-\varepsilon t}^{\infty}\rho_1 b\varphi(x,t)dxdt+\int_0^\infty \rho_1 u_1\varphi(x,0)dx \\
&+\int_0^\infty \rho_1(u_1+bt)^2\Big(\varphi(c(t),t)+\partial_x\varphi(c(t),t)\varepsilon t\Big)dt \\
&-b\int_0^\infty\Bigg(\int_{-\infty}^{c(t)-\varepsilon t}\rho_0\varphi(x,t)dx+\int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t}\rho_\varepsilon\varphi(x,t)dx \\
&\quad+\int_{c(t)-\varepsilon t}^{\infty}\rho_1\varphi(x,t)dx\Bigg)dt\approx 0
\end{aligned}
$$

Like in the previous case, we know that

$$\varphi(c(t)\pm\varepsilon t,t)=\varphi(c(t),t)\pm\partial_x\varphi(c(t),t)\varepsilon t+\mathcal{O}(\varepsilon^2)$$

Then we get

$$\begin{aligned}
I_2 = &\int_0^\infty \rho_0(u_0+bt)(c'(t)-\varepsilon)\varphi(c(t)-\varepsilon t,t)dt \\
&+\int_0^\infty\int_{-\infty}^{c(t)-\varepsilon t} \rho_0 b\varphi(x,t)dxdt \\
&+\int_{-\infty}^0 \rho_0 u_0\varphi(x,0)dx - \int_0^\infty \rho_0(u_0+bt)^2\varphi(c(t)-\varepsilon t,t)dt \\
&+\int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)\varphi(c(t)+\varepsilon t,t)(c'(t)+\varepsilon)dt \\
&-\int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)\varphi(c(t)-\varepsilon t,t)(c'(t)-\varepsilon)dt \\
&+\int_0^\infty\int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \partial_t(\rho_\varepsilon(t)u_\varepsilon(t))\varphi(x,t)dxdt \\
&-\int_0^\infty \rho_\varepsilon(t)u_\varepsilon^2(t)(\varphi(c(t)+\varepsilon t,t)-\varphi(c(t)-\varepsilon t,t))dt \\
&-\int_0^\infty \rho_1(u_1+bt)(c'(t)+\varepsilon)\varphi(c(t)+\varepsilon t,t)dt \\
&+\int_0^\infty\int_{c(t)-\varepsilon t}^{\infty} \rho_1 b\varphi(x,t)dxdt \\
&+\int_0^\infty \rho_1 u_1\varphi(x,0)dx + \int_0^\infty \rho_1(u_1+bt)^2\varphi(c(t)+\varepsilon t,t)dt \\
&-b\int_0^\infty \Big(\int_{-\infty}^{c(t)-\varepsilon t}\rho_0\varphi(x,t)dx + \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t}\rho_\varepsilon\varphi(x,t)dx. \\
&.+\int_{c(t)+\varepsilon t}^{\infty}\rho_1\varphi(x,t)dx\Big)dt \approx 0.
\end{aligned}$$

Like in the previous case, we can see that the above relation holds if the following relations are satisfied,

$$\begin{aligned}
&t(\xi(t)u_s(t))' + \xi(t)u_s(t) - bt\xi(t) = c'(t)(\rho_1(u_1+bt)-\rho_0(u_0+bt)) \\
&-(\rho_1(u_1+bt)^2-\rho_0(u_0+bt)^2) =: k_2(t)
\end{aligned} \tag{4.8}$$

$$\xi(t)u_s(t)c'(t) = \xi(t)u_s(t)^2. \tag{4.9}$$

We can see that (4.6) and (4.9) are equivalent and satisfied if and only if $c'(t) = u_s(t)$.

Next, let us write (4.7) in the following form:

$$(t\xi(t))' = \Big(c(t) - \frac{b}{2}t^2\Big)'[\rho] - [\rho u].$$

Then

$$t\xi(t) = (c(t) - \frac{b}{2}t^2)[\rho] - [\rho u]t,$$

due to the initial data. Its substitution into (4.8) gives:

$$\begin{aligned}
&\Big(\big((c(t) - \frac{b}{2}t^2)[\rho] - [\rho u]t\big)(c'(t) - bt)\Big)' \\
&+ \underbrace{\Big(\big((c(t) - \frac{b}{2}t^2)[\rho] - [\rho u]t\big)bt\Big)'}_{=\big((c'(t)-bt)[\rho]-[\rho u]\big)bt+\big((c(t)-\frac{b}{2}t^2)[\rho]-[\rho u]t\big)b} \quad -b\Big(\big(c(t) - \frac{b}{2}t^2\big)[\rho] - [\rho u]t\Big) \\
&=(c'(t) - bt)([\rho u] + b[\rho]t) + bt([\rho u] + b[\rho]t) - [\rho u^2] - 2b[\rho u]t - b^2[\rho]t^2
\end{aligned}$$

With the change of variables $s(t) = c(t) - \frac{b}{2}t^2$ we have the equation

$$((s(t)[\rho] - [\rho u]t)s'(t))' - s'(t)[\rho u] + [\rho u^2] = 0$$

that can be integrated again, so we get

$$(s(t)[\rho] - [\rho u]t)s'(t) - s(t)[\rho u] + [\rho u^2]t = \text{const} = 0,$$

using $s(0) = 0$.

The above equation can be written as

$$\frac{1}{2}[\rho](s(t)^2)' - [\rho u](ts(t))' + [\rho u^2]t = 0,$$

and integrated, so

$$\frac{1}{2}[\rho]s(t)^2 - [\rho u]ts(t) + \frac{1}{2}[\rho u^2]t^2 = 0, \tag{4.10}$$

where we again have used that $s(0) = 0$. Suppose that $\rho_0 \neq \rho_1$. Thus, we can find an explicit formula for s

$$s(t) = \frac{[\rho u]t \pm t\sqrt{([\rho u]^2 - [\rho][\rho u^2])}}{[\rho]}.$$

Then

$$c(t) = \frac{[\rho u]t \pm t\sqrt{([\rho u]^2 - [\rho][\rho u^2])}}{[\rho]} + \frac{b}{2}t^2 \text{and}$$
$$c'(t) = bt + \frac{\rho_1 u_1 - \rho_0 u_0 \pm (u_0 - u_1)\sqrt{\rho_0 \rho_1}}{\rho_1 - \rho_0}$$

We have to find which sign is to be used such that the obtained SDW satisfies the overcompressibility condition: $u_0 + bt \geq c'(t) \geq u_1 + bt$. It will suffice to prove that

$$u_0 \geq c'(0) = \frac{[\rho u] \pm \sqrt{[\rho u]^2 - [\rho][\rho u^2]}}{[\rho]} \geq u_1,$$

since $c'(t) = c(0) + bt$. Thus,

$$\begin{aligned} c'(0) =& \frac{u_0(\sqrt{\rho_0\rho_1} - \rho_0) \pm u_1(\rho_1 - \sqrt{\rho_0\rho_1})}{\rho_1 - \rho_0} \\ =& \frac{\sqrt{\rho_0\rho_1} - \rho_0}{\rho_1 - \rho_0} u_0 \pm \frac{\rho_1 - \sqrt{\rho_0\rho_1}}{\rho_1 - \rho_0} u_1, \end{aligned}$$

i.e. $c'(0) = \alpha u_0 \pm \beta u_1$,with $\alpha + \beta = 1$. We can check that in both cases $\rho_0 < \rho_1$ or $\rho_0 > \rho_1$, we have $\alpha, \beta \geq 0$. That implies

$$u_0 \geq \alpha u_0 + \beta u_1 = c'(0) \geq u_1$$

if we use the plus sign above. Thus, if $u_0 > u_1$ the weak solution (4.3) to (4.1) is always admissible. It is unique with respect to a limit in the distributional sense. One can easily see that there are no unwanted SDWs in the case $u_0 \leq u_1$ since it contradicts (4.4).

Let us now check the case $\rho_0 = \rho_1$. Then (4.10) reduces to

$$s(t) = \frac{1}{2}(u_0 + u_1)t \text{ and } c(t) = s'(t) + bt = \frac{1}{2}(u_0 + u_1) + bt.$$

Then the solution is always overcompressive since $u_0 > \frac{1}{2}(u_0 + u_1) > u_1$. That concludes the proof. □

Remark 7. We could say that we have proved that shadow waves follow the physical intuitions as well as all other elementary waves in the given balance law system: If b denotes the gravity acceleration, an SDW speed is increased exactly by bt (or decreased if $b < 0$) as expected.

Remark 8. In the above proof, we have exploited a special form of pressureless system. In general, system (4.7,4.8) is a singular ODE system, since the second equation is of the form $\xi(t)u_s'(t) = ...$ with the initial data $\xi(0) = 0$ end the usual existence-uniqueness theorems are not applicable immediately.

4.3 Further possibilities

Suppose that an interaction involving a split delta shock happens at a time $t = T$ in a point $x = X$. Then we have to solve a new initial data that contains a delta function, say

$$(\rho, u)|_{t=T} = \begin{cases} (\rho_0, u_0), & x - X < 0 \\ (\rho_1, u_1), & x - X > 0 \end{cases} + \gamma_0 \delta(X, T).$$

Note that (ρ_i, u_i). $i = 0, 1$ are not necessary the initial values for the above Riemann problem. Solution of any Riemann problem found above has values of (ρ, u) that depends only on t, so (ρ_i, u_i). $i = 0, 1$ are obtained by freezing $t = T$.

We will try to find a solution to (4.1) and the above initial data in the form of SDW,

$$(\rho, u)|_{t=T} = \begin{cases} (\rho_0, u_0 + b(t-T)), & x - X < c(t) - \varepsilon(t-T) - x_0\varepsilon \\ (\rho_\varepsilon(t), u_\varepsilon(t)), & c(t) - \varepsilon(t-T) - x_0\varepsilon < x - X \\ & < c(t) + \varepsilon(t-T) + x_0\varepsilon \\ (\rho_1, u_1 + b(t-T)), & c(t) + \varepsilon(t-T) + x_0\varepsilon < x - X \end{cases}$$

Due to Theorem 7.1 in [36] about infraction of SDW's we can see that the value of x_0 has to be chosen in a way that we have a continuity of delta function across the interaction line $t = T$. We shall see that bellow.

Using the same change of variables and arguments as in the Riemann case, we get the same equations (4.7) and (4.8),

$$\begin{aligned} t\xi'(t) + \xi(t) &= [\rho]u_s'(t) - [\rho u] - b[\rho]t \\ t(\xi(t)u_s(t))' - b\xi(t) &= u_s(t)([\rho u](1-2bt) + bt[\rho](1-bt) - [\rho u^2]) \end{aligned}$$

but now with the initial data

$$\xi(T) = 2x_0\varepsilon\rho_\varepsilon = \gamma_0 > 0,\ u_s(T) = \zeta_0.$$

The values for initial data are chosen in order to preserve mass of delta functions before and rather the interaction (see Theorem 7.1 in [36]), Thus, if there is one incoming SDW with $\overline{\xi}(t), \overline{u}_s(t)$ determined from appropriate equations (4.5–4.9), then $\gamma_0 = \overline{\xi}(T), \zeta_0 = \overline{u}_s(T)$. If there are two of them, with $\overline{\xi}_1(t), \overline{u}_{s,1}(t)$ and $\overline{\xi}_2(t), \overline{u}_{s,2}(t)$ determined, then $\gamma_0 = \overline{\xi}_0(T) + \overline{\xi}_1(T)$ and ζ_0 can be found from the relation

$$\zeta_0\gamma_0 = \overline{u}_{s,1}(T)\overline{\xi}_1(T) + \overline{u}_{s,2}(T)\overline{\xi}_2(T).$$

Concerning a solution to an interaction problem, it can be solved like in [36]. We just have to check overcompressibility conditions once when a solution to the above problem is found. Note that the system is not singular (the initial data are not given at zero anymore):

$$\begin{aligned}
\xi'(t) =& \frac{[\rho]u_s(t) - b[\rho]t - [\rho u]}{t\xi(t)}, \ \xi(T) = \gamma_0 \\
u_s'(t) =& \frac{1}{t\xi^2(t)}((u_s(t) - b)\xi^2(t) + (b^2[\rho]t^2 + b[\rho]t + [\rho u^2] - [\rho u])u_s(t)\xi(t) \\
& + [\rho]u_s^2(t) - (b[\rho]t + [\rho u])u_s(t)), \ u_s(T) = \zeta_0.
\end{aligned}$$

Contrary to the Riemann case, we do not have to use manipulation using special properties of (4.7) and (4.8). Now, at least in some small enough time interval after $t > T$, the above initial data problem always has a solution due to the usual existence-uniqueness theorems for ordinary differential equations (Picard-Lindelöf Theorem, for example). That is possible since $\xi(t) > 0$, at least for some small time interval $t > T$ since $\gamma_0 > 0$,

5

Numerical methods

In this part we will introduce some numeric procedures supporting a solution type we have discovered analytically above. Also, one could see how interactions could be handled in the presence of gravity.

let us note that we need a vary robust and precise scheme due to the fact that we have delta function approximation in a solution. That is, we are dealing with functions that are very steep, i.e. with huge variation. We will basically use modified procedures from well known software package CLAWPACK (see [32]). Such procedures are made taking into account numerical procedures from [25]. In the end of the chapter we present some original results concerning numerical procedures for pressureless gas dynamics model with a source.

5.1 Conservative shemes

In general, we note that the weak solutions of conservation law systems are not unique and the produces a lot of numerical problems.

Now, we will introduce the following simple example to explain how the situation for nonlinear problems could be even worse.

Example 2. Consider the Burgers' equation

$$\partial_t u + u\partial_x u = 0$$

with initial condition

$$U_j^0 = \begin{cases} 1, & j < 0 \\ 0, & j \geq 0. \end{cases}$$

where, under the hypothesis $U_j^n \geq 0.$, for every j, n, we can take simple scheme for the above equation:

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n \big(U_j^n - U_{j-1}^n\big)$$

Then, that gives $U_j^1 = U_j^0$ for every j. Thus $U_j^n = U_j^0$ for every j, n, and the approximate solution is not even solution to the given equation, which is converges to $u(x,t) = u_0(x)$.

Definition 12. *The numerical procedure is conservative if we can write it by the following form*

$$\begin{aligned} U_j^{n+1} =& U_j^n - \frac{k}{h}\Big[F\big(U_{j-p}^n, U_{j-p+1}^n, ..., U_{j+q}^n\big) \\ &- F\big(U_{j-p-1}, U_{j-p}^n, ..., U_{j+q-1}^n\big)\Big]. \end{aligned} \tag{5.1}$$

where the function F is called the numerical flux function.
For the simplest case, when $p = 0$ and $q = 1$, the relation (5.1) becomes

$$U_j^{n+1} = U_j^n - \frac{k}{h}\Big[F\big(U_j^n, U_{j+1}^n\big) - F\big(U_{j-1}^n, U_j^n\big)\Big]. \tag{5.2}$$

Let U_j^n be an average value of u in the interval $\big[x_{j-1/2}, x_{j+1/2}\big]$ defined by

$$\overline{u}_j^n = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx$$

Since the integral for of conservation law is satisfied by the weak solution $u(x,t)$, then we have

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})dx =& \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)dx - \Big[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))dt \\ &- \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))dt\Big]. \end{aligned}$$

we divide the above integral by h, becomes

$$\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{1}{h}\Big[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))dt\Big]$$

.

we can note that

$$F\Big(U_j, U_{j+1}\Big) \sim \frac{1}{k}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))dt$$

Then for the simpicity we can use the following formula

$$F\Big(U^n, j\Big) = F\Big(U_{j-p}^n, U_{j-p+1}^n, ..., U_{j+q}^n\Big)$$

Thus, the formula (5.1) can be written by

$$U_j^{n+1} = U_j^n - \frac{k}{h}\Big[F\Big(U^n, j\Big) - F\Big(U^n, j-1\Big)\Big].content... \tag{5.3}$$

Definition 13. *If $u(x,t) \equiv \overline{u}$ holds*

$$F(\overline{u}, \overline{u}) = f(\overline{u})$$

for every $\overline{u} \in \mathbb{R}$. Then the numerical procedure (5.2) is consistent with the original conservation law.

In general, if F is a function of more than two variables and it is a Lipschitz continuous consistency condition reads

$$F\Big(\overline{u}, \overline{u}, ..., \overline{u}\Big) = f(\overline{u}),$$

and for Lipschitz condition we have to exist a constant K such that

$$\mid F\Big(U_{j-p}, ..., U_{j+q}\Big) - f(\overline{u}) \mid \leq K \max_{-p\leq i\leq q} \mid U_{j+i} - \overline{u} \mid,$$

holds for each U_{j+i} is close enough to $\overline{u}$.

Now, we will introduce the importante theorem for numerical solving of conservation law systems.

Theorem 4. *Theorem (Lax-Wendorff)([31]). Let a sequence of schemes indexed by $l = 1, 2, ...$ with parameters $k_1, h_1 \to 0$ as $l \to \infty$. Let $U_l(x,t)$ be a numeric approximation obtained by a consistent and conservative procedure at l-th scheme. Suppose that $U_l \to u$ as $l \to \infty$. Then, a function $u(x,t)$ is a weak solution to conservation law system.*

To prove that a weak solution $u(x,t)$ obtained by a conservative procedure, satisfy entropy condition, it is enough to prove that it satisfies so called discrete entropy condition (see [31]).

$$\eta\Big(U_j^{n+1}\Big) \leq \eta\Big(U_j^n\Big) - \frac{k}{h}\Big[\Psi\Big(U^n, j\Big) - \Psi\Big(U^n, j-1\Big)\Big],$$

when Ψ is appropriate numerical entropy flux consistent with a entropy flux Ψ in the same sence as F with f is.

5.2 The Godunov method

One of the best numerical procedures for conservation law system is so called Godunov method. There are a lot of its variations sharing the name. it is known to be robust and gives entropic solutions usually. Let us note that this is not a case with Lax-Wendorff scheme given above, for example. Let us give some basic facts about it.

To define piecewise constant function $\tilde{u}^n(x, t_n)$ which equals U_j^n in the interval $(x_{j-1/2}, x_{j+1/2})$, by using the numerical solution U^n. Let function is not a constant in the interval (t_n, t_{n+1}). For this Rescon we use the function $\tilde{u}^n(x, t_n)$ as an initial data for conservation law, which we analytically solve it in order to get $\tilde{u}^n(x,t)$ for $[t_n, t_{n+1}]$. Next, we define approximate solution U^{n+1} at time t_{n+1} as a mean value of the exact solution at time t_{n+1}, then we have

$$U_j^{n+1} = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x, t_{n+1})dx, \tag{5.4}$$

where $h = x_j - x_{j-1}$.

Then, we get values for a piecewise constant function $\tilde{u}^{n+1}(x, t_{n+1})$, and the procedure continues. Also, by the integral form of the conservation law, we can get (5.4). Since $\tilde{u}$ is a weak solution of the conservation law. then holds

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x, t_{n+1})dx = \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x, t_n)dx + \int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j-1/2}, t))dt - \int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j+1/2}, t))dt \tag{5.5}$$

Dividing the above expression by h. After that we use (5.4) and the fact $\tilde{u}^n(x, t_n) \equiv U_j^n$ in the interval $(x_{j-1/2}, x_{j+1/2})$ to transform (5.5) to

$$U_j^{n+1} = U_j^n - \frac{k}{h}\Big[F\Big(U_j^n, U_{j+1}^n\Big) - F\Big(U_{j-1}^n, U_j^n\Big)\Big]$$

where $h = x_j - x_{j-1}, \qquad k = t_{n+1} - t_n$.

Also, the numerical flux function F is given by

$$F\Big(U_j^n, U_{j+1}^n\Big) = \frac{1}{k}\int_{t_n}^{t_{n+1}} f\Big(\tilde{u}^n(x_{j+1/2}, t)\Big)dt. \tag{5.6}$$

We note that (5.6) it can be written in the form (5.2), then that proves that the Godunov procedure is conservative. Also, the calculation of the integral (5.6) is very easy, since the function $\tilde{u}^n$ is a constant in the interval (t_n, t_{n+1}) at the point $x_{j+1/2}$. Because of that we get the fact that the solution of Riemann problem will be a constant along a characteristic curve

$$\Big(x - x_{j+1/2}\Big)/t = const.$$

Here, $\tilde{u}^n$ depends only on U_j^n and U_{j+1}^n along the line $x = x_{j+1/2}$, then $\tilde{u}^n$ can be denoted by $u^*\Big(U_j^n, U_{j+1}^n\Big)$. That follows that the numerical flux function (5.6) becomes

$$F\Big(U_j^n, U_{j+1}^n\Big) = f\Big(u^*\Big(U_j^n, U_{j+1}^n\Big)\Big), content... \tag{5.7}$$

and Godunov procedure is now becomes

$$U_{j+1}^n = U_j^n - \frac{k}{h}\Big[f\Big(u^*\Big(U_j^n, U_{j+1}^n\Big)\Big) - f\Big(u^*\Big(U_{j-1}^n, U_j^n\Big)\Big)\Big]$$

We note that the numerical flux function (5.7) is consistent with f because

$$U_j^n = U_{j+1}^n \equiv \overline{u}$$

which implies

$$u^*\Big(U_j^n, U_{j+1}^n\Big) = \overline{u}$$

We know that $\tilde{u}^n$ is a constant in the interval (t_n, t_{n+1}) at the point $x_{j+1/2}$, so that constancy of $\tilde{u}^n$ depends on a lenght of a time interval. If a time interval is so long, then the interaction of waves obtained by solving the closest Riemann problems may occur. Since the sonsecutive discontinuity points (origins of appropriate Riemann problems) are separated by h, $\tilde{u}^n(x_{j+1/2}, t)$ is a constant in the interval $[t_n, t_{n+1}]$ for k small enough and since the speeds of these waves are bounded by characteristics values of the matrix $f'(u)$. So, we can avoid that interaction of wave by the following condition:

$$\mid \frac{k}{h}\lambda_p\Big(U_j^n\Big) \mid \leq 1$$

for every λ_p and U_j^n.

Definition 14. *The condition*

$$CFL = \max_{j,p} \mid \frac{k}{h}\lambda_p\Big(U_j^n\Big) \mid$$

is called Courant number or CFL (Courant–Friedrichs–Levy) for short. If the condition

$$CFL \leq 1$$

then it is called CFL condition.

5.3 Numerical results

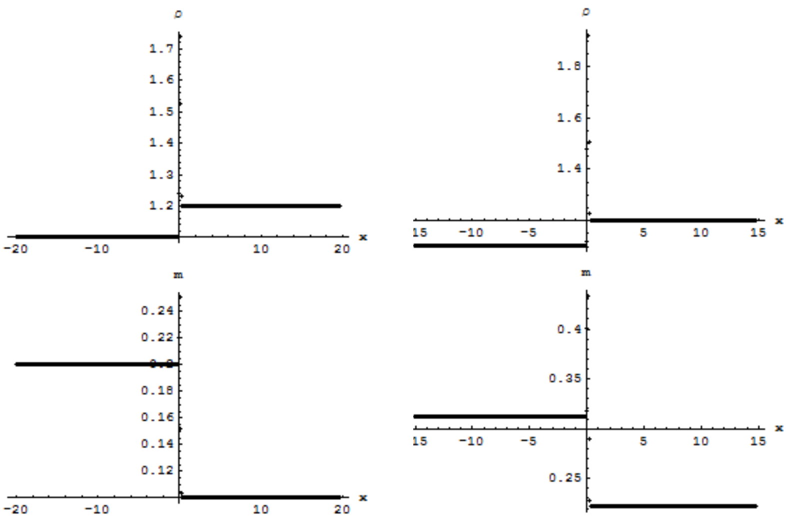
In the present section we shall compare numerical two systems: the pressureless gas conservation law system and the same system with added a source term $b\rho$ in the first equation. Our aim is to find a numerical evidence of the Shadow Wave shift analytically explained above. Here, we will use $b = 0.1$ due to practical reasons - to make nice visual comparation. Any positive value of b would give the similar result.

We have used a variation of the famous CLAWPACK free software. It is based on high-resolution Godunov type methods. We have to modify it to suit our purposes since any approximation of the delta function (that is a part of our solution) has a very big total variation.

Here, we will present our numerical results for pressureless gas dynamic model with a source and compare it with the same model with the same initial data, but without the source term.

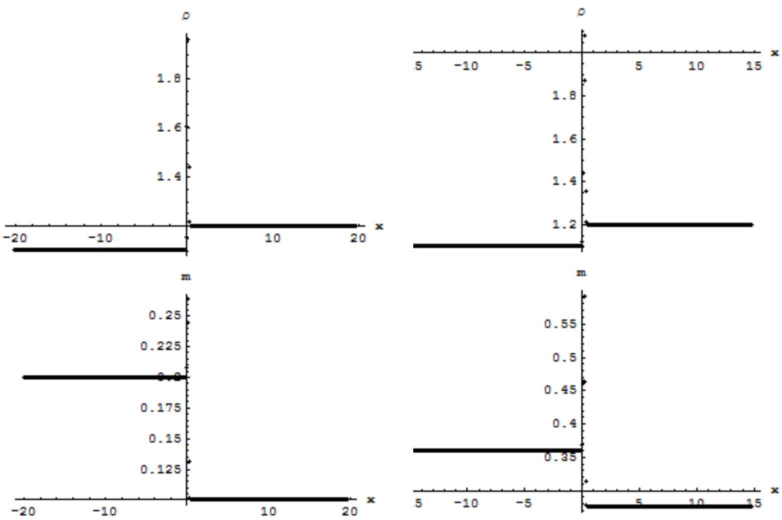
One can clearly see the effect of shifting in ρ-variable due to the external force (gravity, for example) as well as a shift and increased momentum when source is present. In all these examples we have used $b = 0.1$. Such scaling is appropriate: if we use $b = 1$, for example, a value of the delta part in m is to high to be numerically captured for a long time ($t < 1$). We should use much more points in a mesh and a computation time would increase very much.

In order to give illustration of solution numerical approximation we will present also some 3D plots of computed components ρ and m bellow.



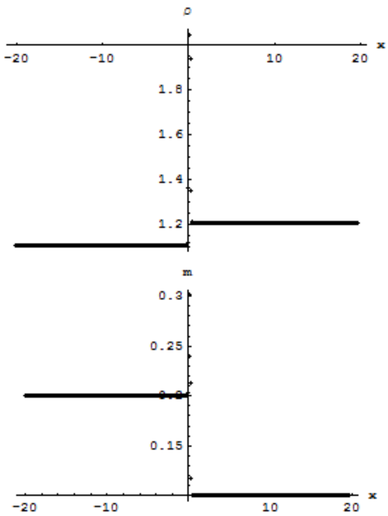
(a) no source (b) source

Figure 5.1: t=1

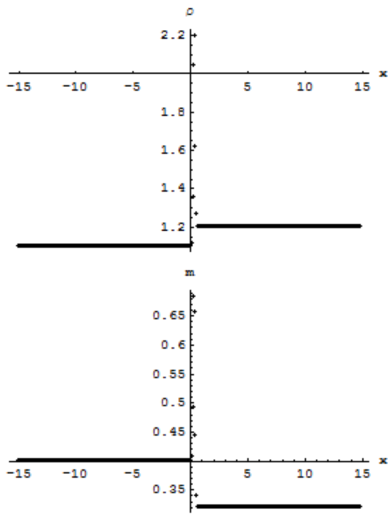


(a) no source (b) source

Figure 5.2: t=1.4

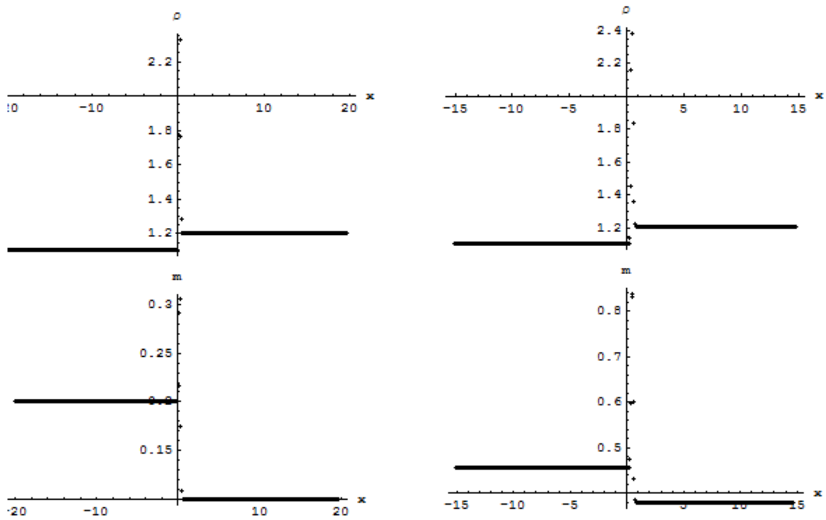


(a) no source



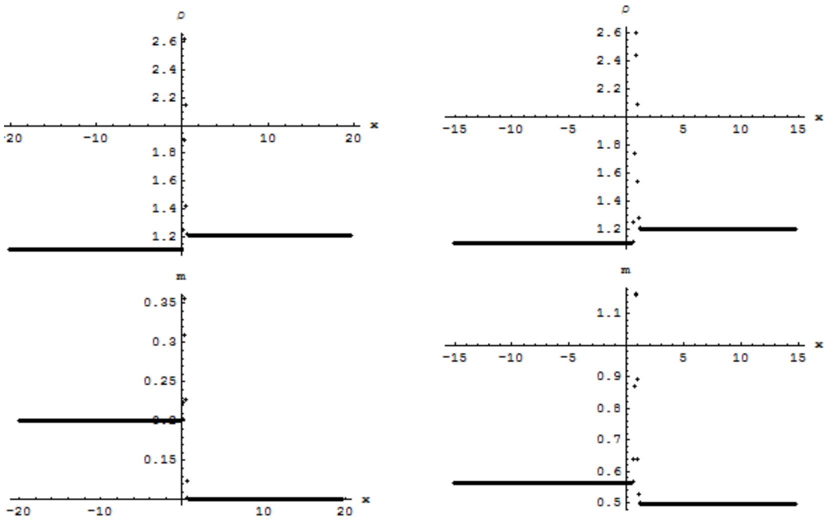
(b) source

Figure 5.3: t=1.85



(a) no source (b) source

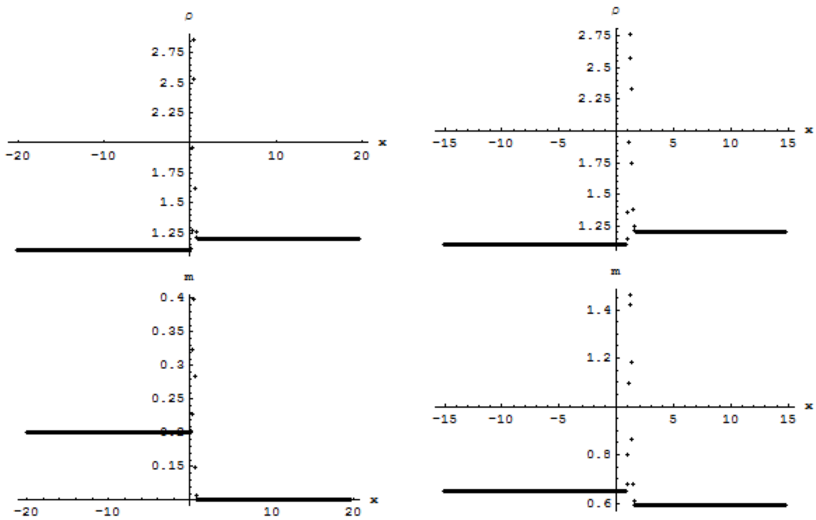
Figure 5.4: t=2.3



(a) no source

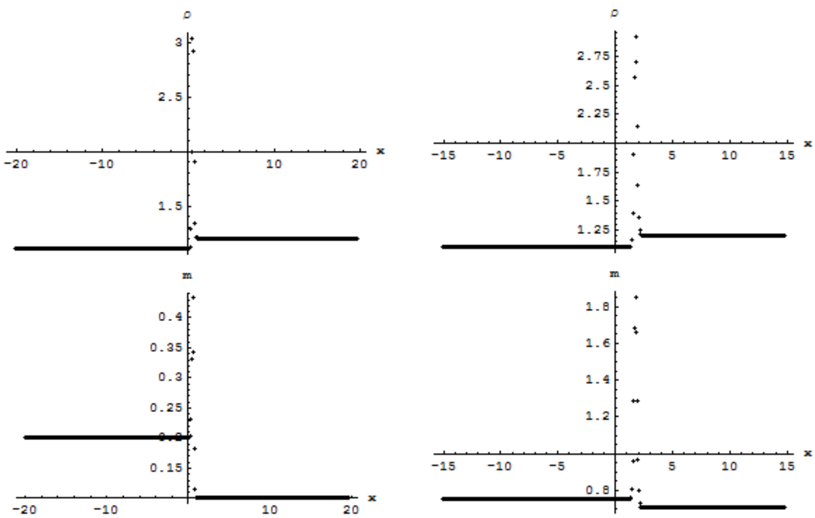
(b) source

Figure 5.5: t=3.3



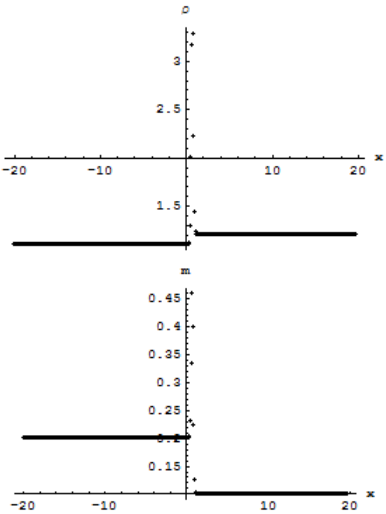
(a) no source (b) source

Figure 5.6: t=4

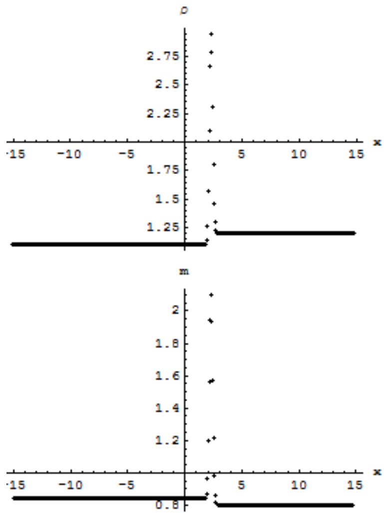


(a) no source (b) source

Figure 5.7: t=5



(a) no source



(b) source

Figure 5.8: t=5.8

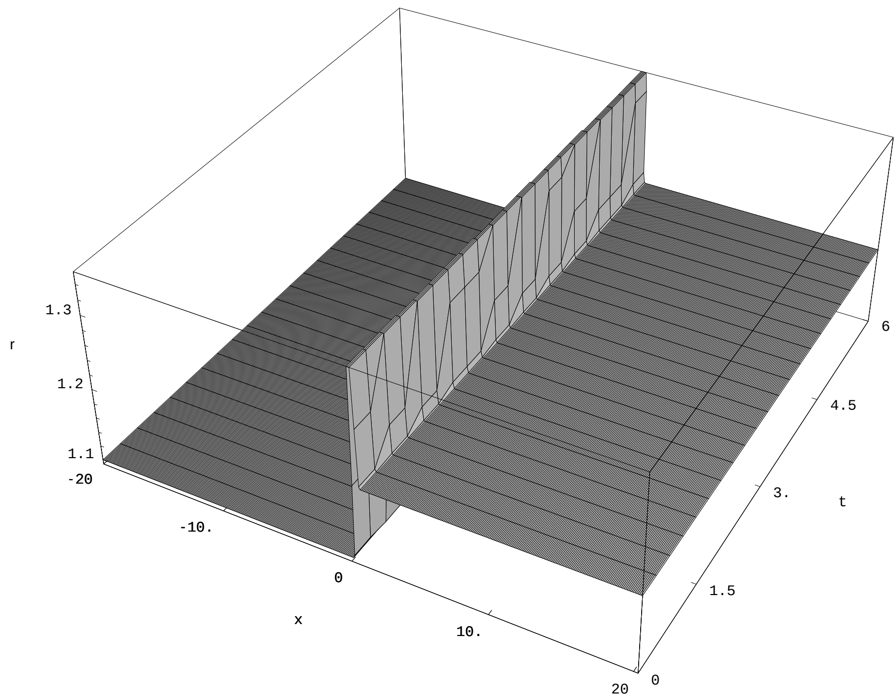


Figure 5.9: b=0, ρ

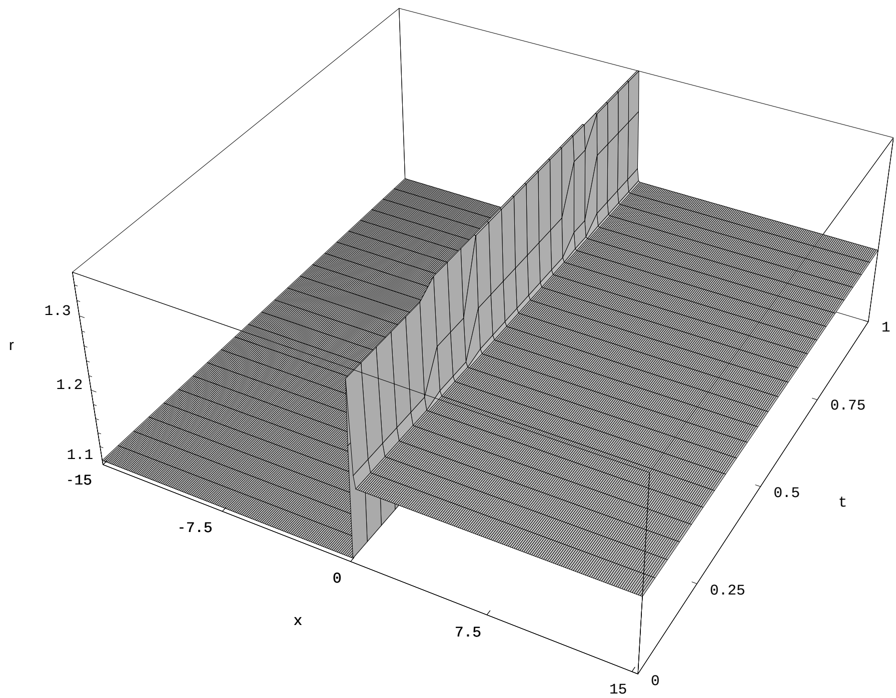


Figure 5.10: b=0.5, ρ

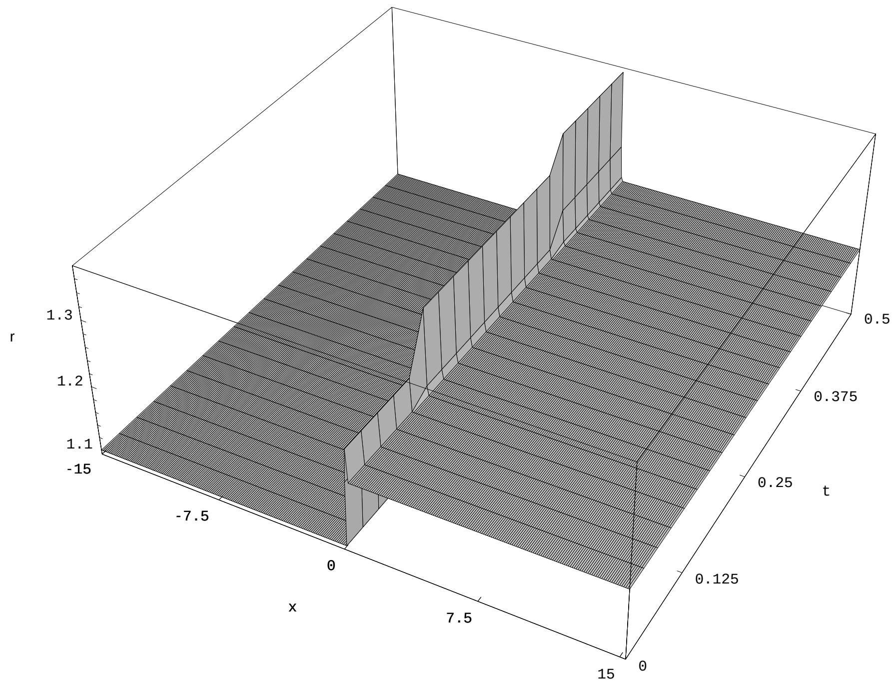


Figure 5.11: b=0.8, ρ

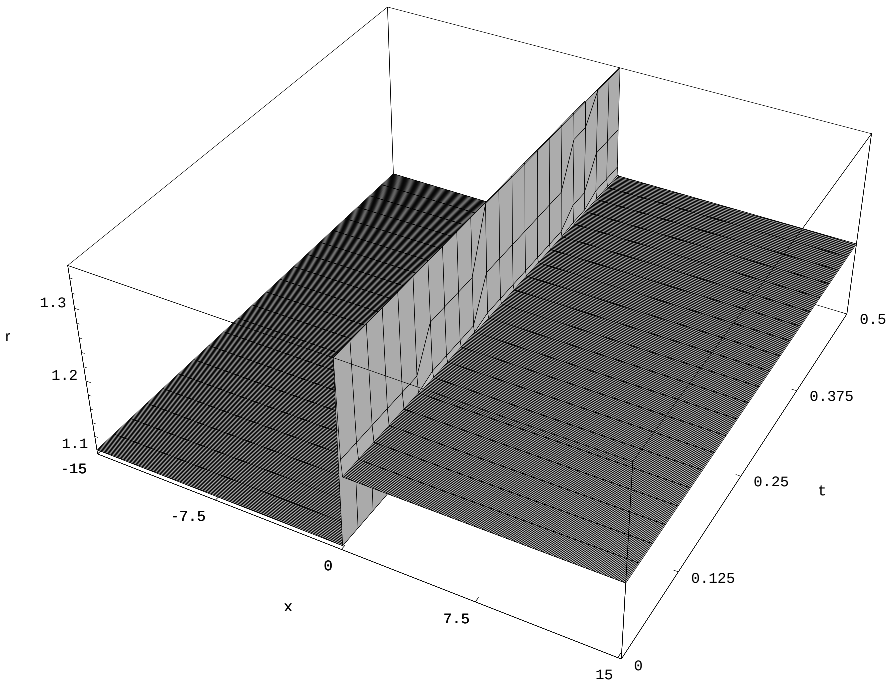


Figure 5.12: b=1, ρ

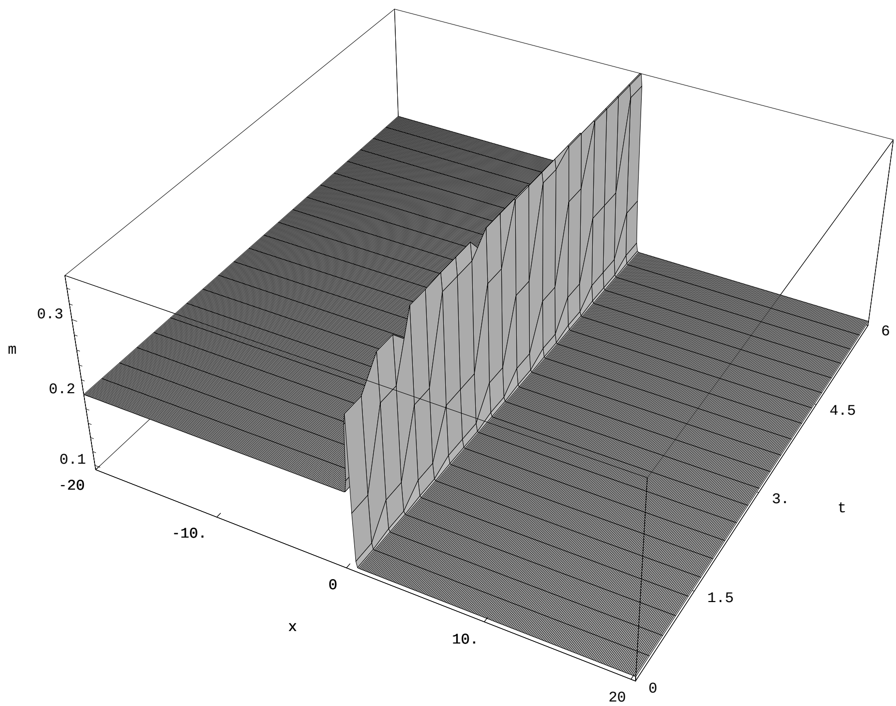


Figure 5.13: b=0, m

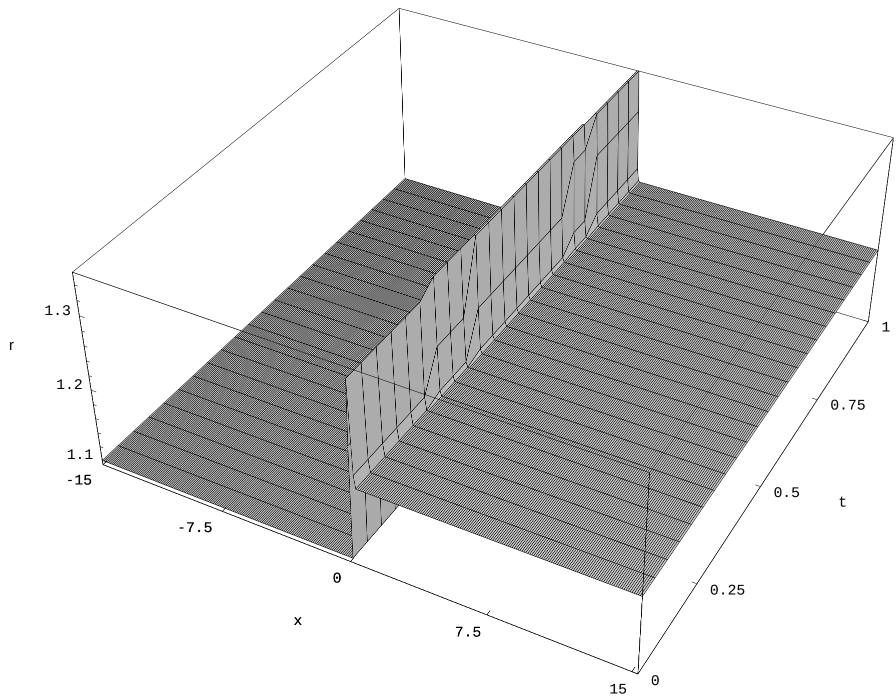


Figure 5.14: b=0.5, m

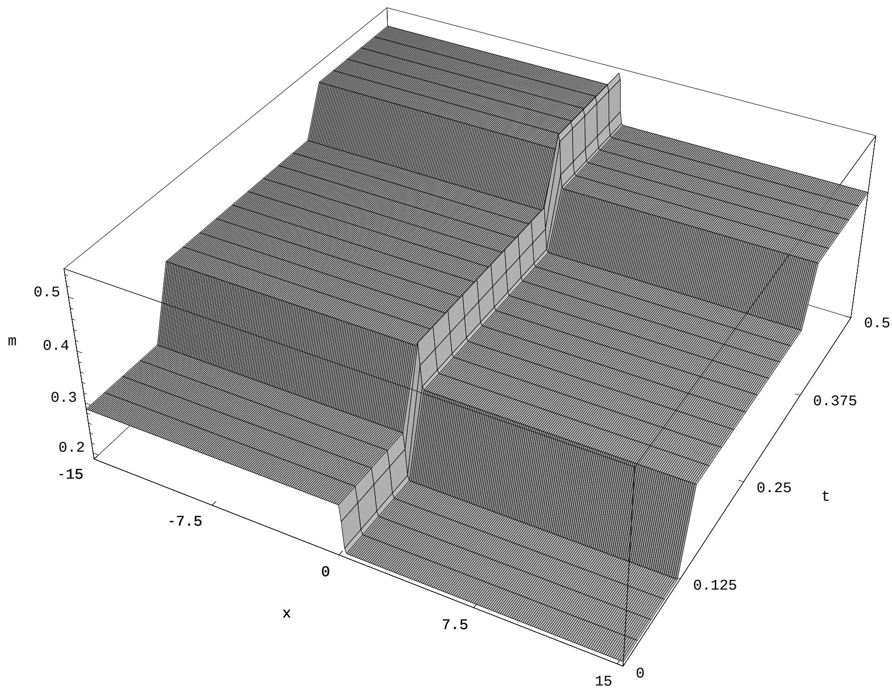


Figure 5.15: b=0.8, m

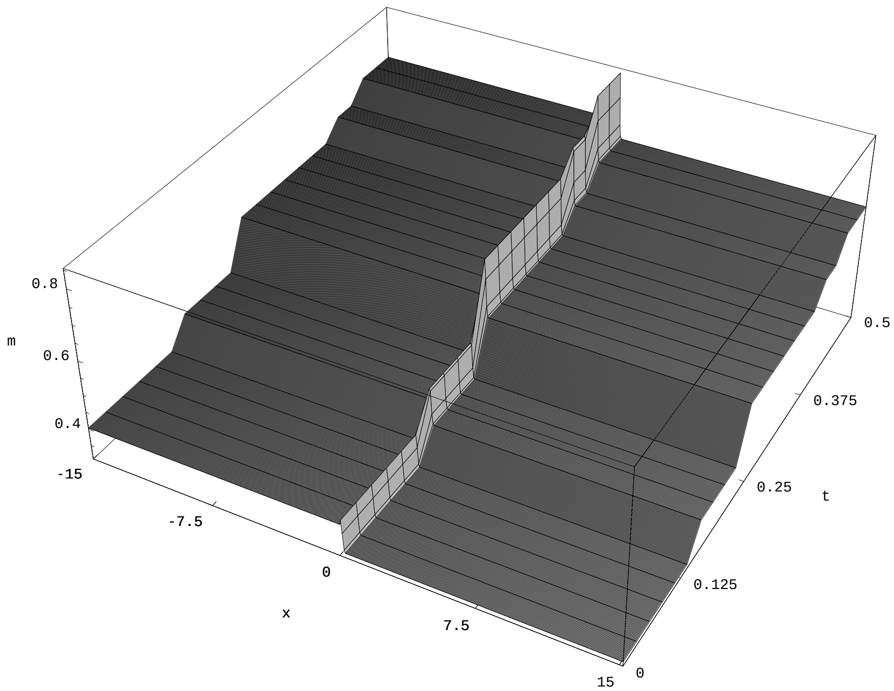


Figure 5.16: b=1, m

5.4 Listing of the mathematica code

In our calculations we were using the following Mathematica code. Below is the listing of the program. It is based on the well known open source CLAWPACK made by R. LeVeque.

The explanation of the names of the graphics:

- 3D graphics have names of the form: 3Dm0.1 (This means 'the solution of variable m for $b = 0.1$') or 3DRo0.5 (This means 'the solutions of variable ρ for $b = 0.5$').

-2D graphics have names of the form: $b\_0.1\_t\_3.3$ (this means 'the solution of both variables ρ and m at $t = 3.3$ in the case $b = 0.1$') or $b\_0\_t\_5.8$ (this means 'the solution of both variables ρ and m at $t = 5.8$ in the case $b = 0$')

```
sistem[mx_, t0_, tfinal_, ts_, x1_, x2_, qlevo_,
      qdesno_, nout_, out_, alfa1_, $\varepsilon$_,
      grav_, granica1_, granica2_] :=
      Module[{dx, deltax, cflv, dt, cflmax, dtmin,
      dtmax, told, t, kol, xstaro, xpret, e,
  estar, qcopy, dtv, mon, ostatak, numt,
  sistem, mreza, gr, resenje, s, wave,
  leviflux, desniflux, dexstar, tstart, tend,
  dtout, xizvod, norma, xcell, qr, ql, u, r1, r2,
  $\alpha_1$, $\alpha_2$, s0, s1, s2, s3, df, tk, fac},
  cflv = {1, 0.8, 0, 0};
  t = t0;
  trid = {};
  trid1 = {};
  trodimu = {};
  trodimw = {};
  numt = 0;
  dtv = {0.1, 1, 0, 0, 0};
  dt = dtv[[1]]; (*start value of dt*)
  dx = (x2 - x1)/mx // N;
```

```
x = Table[x1 + (i - 1)*dx // N, {i, 1, mx + 1}];
xcell = Table[x1 + (i - 0.5)*dx, {i, 1, mx}];
xstaro = x;
q = Table[qlevo[[j]] +
0.5*(qdesno[[j]] - qlevo[[j]])*(1 + Tanh[(x[[
i]] - 0)/$\varepsilon$]), {i, 1, mx}, {j, 1, 2}];
u = Table[{x[[i]], q[[i, 1]]}, {i, 1, mx}];
v = Table[{x[[i]], q[[i, 2]]}, {i, 1, mx}];
ListPlot[u, AxesLabel -> {"x", "$\rho$"},
PlotStyle -> PointSize[0.015]];
ListPlot[v, AxesLabel -> {"x", "m"},
PlotStyle -> PointSize[0.015]];
q1 = Table[0, {i, mx}];
levi = {}; desni = {}; raz = {};
deltax = Table[dx, {i, 1, mx}];
dexstar = deltax;
kol = Table[dt/dx, {i, 1, mx + 4}];
cflmax = 0; dtmin = dt; dtmax = dt;
cfl = 0;
kontrolniz = {}; maxq = 0;
xizvod = Table[0, {i, 1, mx + 1}];
e = Table[0, {i, 1, mx - 3}];
mreza = Table[{}, {i, 1, mx + 1}];
wave = Table[{{0, 0}, {0, 0}}, {i, 1, mx + 4}];
s = Table[{0, 0}, {i, 1, mx + 4}];
leviflux = s; desniflux = s;
tend = t0;
dtout = (tfinal - t0)/nout // N;
Do[AppendTo[q, {0, 0}]; PrependTo[q, {0, 0}],
{i, 1, 2}];
ostatak = Table[0, {i, mx - 4}];
alfa = alfa1; nizalfa = {};

tk = (tfinal - t0)/ts;
brojac = 0; korekcija = 0;
maxi = 2;
While[tend<=tfinal, tstart = tend;
tend = tstart + dtout;

  While[t <= tend,
    numt = numt + 1;
```

```
dx = Min[deltax];
AppendTo[mininterval, {t, dx}];
xstaro = x;
dt = dtv[[1]];
told = t;
t = t + dt;
(*interval movement*)
If[t >=ts, index = 0;
  While[x[[maxi - index]] > (First[x] +
  Last[x])/2 // N,
    AppendTo[x, Last[x] + (x[[2]] - x[[1]])];
    x = Drop[x, 1];
    q = Drop[q, 1];
    AppendTo[q, qdesno]
    ]];
(*end of interval movement*)
Do[deltax[[i]] = x[[i + 1]] - x[[i]],
{i, 1, mx - 1}];
br = 0;
Do[kol[[i]] = dt/deltax[[i]], {i, 1, mx}];
Do[
kol[[i]] = kol[[3]], {i, 1, 2}]; Do[kol[[i]]
= kol[[mx + 2]], {
  i, mx + 3, mx + 4}];

qcopy = q;
cflstar = cfl; Goto[nastavak];
 Label[vreme];
dt = dt/2; Print["dt=", dt];
t = told + dt;
Do[kol[[i]] = dt/deltax[[i]], {i, 1, mx}];
Do[kol[[i]] = kol[[3]], {i, 1, 2}]; Do[
kol[[i]] = kol[[mx + 2]], {i, mx + 3, mx + 4}];
q = qcopy;
cfl = 0;

(* a step of solving the homogenous
conservation law, step1 *)

Label[nastavak];
```

```
Do[Do[q[[i, m]] = q[[3, m]], {i, 1, 2}],
{m, 1, 2}];
Do[Do[q[[i, m]] = q[[mx + 2, m]],
{i, mx + 3, mx + 4}], {m, 1, 2}];

(*clawpack*)
ql = q;
qr = q;
Do[
 $\lambda$1 = (qr[[i, 2]]/qr[[i, 1]] +
 ql[[i - 1, 2]]/ql[[i - 1, 1]] - Abs[
  qr[[i, 2]]/qr[[i, 1]] - ql[[i - 1, 2]]
  /ql[[i - 1, 1]]])/2;
 $\lambda$2 = (qr[[i,
  2]]/qr[[i, 1]] + ql[[i - 1, 2]]/ql[[
    i - 1, 1]] + Abs[qr[[i,
          2]]/qr[[i, 1]] - ql[[i - 1, 2]]
          /ql[[i - 1, 1]]])/2;
  If[$\lambda$1 ==$\lambda$2, $\alpha_1$ = 0;
  $\alpha_2$ = qr[[i, 1]] - ql[[i - 1, 1]],
    $\alpha_1$ = ($\lambda$2*(qr[[i,
  1]] - ql[[i - 1, 1]]) + ql[[i - 1, 2]]
  - qr[[i, 2]])/($\lambda$2 -$\lambda$1);
    $\alpha_2$ = ($\lambda$1*(qr[[i, 1]]
    - ql[[i - 1, 1]]) + ql[[i - 1, 2]] -
    qr[[i, 2]])/($\lambda$1 -$\lambda$2)];

  (*wave calculating*)

  wave[[i, 1, 1]] = $\alpha_1$;
  wave[[i, 2, 1]] = $\alpha_1$*$\lambda$1;
  s[[i, 1]] =$\lambda$1;

  wave[[i, 1, 2]] = $\alpha_2$;
  wave[[i, 2, 2]] = $\alpha_2$*$\lambda$2;
  s[[i, 2]] =$\lambda$2, {i, 2, mx + 4}];

Do[
  Do[
    leviflux[[i, m]] = 0;
    desniflux[[i, m]] = 0;
```

```
    Do[
      If[s[[i, mw]] < 0,
        leviflux[[i, m]] = leviflux[[i, m]]
        + s[[i, mw]]*wave[[i, m,
  mw]],
        desniflux[[i, m]] = desniflux[[i, m]]
        + s[[i, mw]]*wave[[i, m, mw]]],
        {mw, 1, 2}], {i, 1, mx + 4}], {m, 1, 2}];

(* calculating the source term*)

source = Table[{
0, grav*dt*0.5*(
  q[[i, 1]] + q[[i - 1,
      1]])}, {i, 2, mx + 4}];
        source = PrependTo[source, First[source]];

Do[
  Do[
    q[[i, m]] = q[[i, m]] - kol[[i]]
    *desniflux[[i, m]];
    q[[i - 1, m]] = q[[i - 1, m]]
    - kol[[i - 1]]*leviflux[[i, m]],
    {m, 1, 2}], {i, 2, mx + 3}];

q = q + source; (* updating q after
calculating the source*)

cfl = 0;
Do[
  Do[
    cfl = Max[cfl, kol[[i]]*(s[[i,
        mw]](*-xizvod[[i]]*)), -kol[[
          i - 1]]*(s[[i, mw]](*-xizvod[[i - 1]]*))],
    {mw, 1, 2}],
  {i, 2, mx + 1}];

If[cfl > 0, dt = Min[dtv[[2]], dt*(cflv[[2]]/cfl)];
  dtmin = Min[dt, dtmin]; dtmax = Max[dt, dtmax],
  dt = dtv[[2]]
  ];
```

```
  (* CFL number checking *)
  If[cfl<=cflv[[1]],(* accept this step *)
    cflmax = Max[cfl, cflmax],(* deny this step*)
    Print["The CFL number is to high"];
    Print["cfl=", cfl];
    Goto[vreme]
    (*endif*)];

  cflv[[3]] = cflmax; cflv[[4]] = cfl;
  dtv[[3]] = dtmin; dtv[[4]] = dtmax; dtv[[5]] = dt;

   dtv[[1]] = dtv[[5]]; (* use the last value of dt
  in the new time step *)

  (*end clawpack*)


  If[t >=ts,
    Do[AppendTo[mreza[[i + brojac]], {x[[i]], t}],
    {i, 1, Length[x] - index}];
    mreza = Join[mreza, Table[{{x[[i + Length[x]
    - index]], t}}, {i, index}]];
     If[korekcija == 0, korekcija = index]
     ];

  (* take a new time step *)
  ];(*while*)
(*output*)
If[t > granica1 && t < granica2, AppendTo[trid,
Transpose[q][[2]]];
AppendTo[trid1, Transpose[q][[1]]]];
Do[AppendTo[trodimu, {x[[i]], q[[i, 1]], t}],
{i, mx}];
Do[AppendTo[trodimw, {x[[i]], q[[i, 2]], t}],
{i, mx}];
Print["t=", t]; Print["alfa=", alfa];
Print["cfl=", cfl];

resu = Table[{x[[i]], q[[i + 2, 1]]},
{i, 1, mx - 2}];
ListPlot[resu, AxesLabel -> {"x", "$\rho$"},
```

```
    PlotRange -> All,
    PlotStyle -> PointSize[0.015]];
    resv = Table[{x[[i]], q[[i + 2, 2]]}, {i, 1, mx - 2}];
    ListPlot[resv, AxesLabel -> {"x", "m"},
        PlotRange -> All, PlotStyle -> PointSize[0.015]];
    Print["maxu=", maxq]; Print["maxw=", maxw]
    ];

  (* Blow are the 3D plots.
  They are called TRID and TRID1 *)


fig1 = ListPlot3D[trid1, Mesh -> True, Mesh -> False,
LightSources -> {{{1`,
  0, 1}, GrayLevel[0.1]}, {{1, 1`, 1},
  GrayLevel[0.2]}, {{0, 1, 1},
    GrayLevel[0.3]}}, AmbientLight ->
    GrayLevel[0.3], Ticks -> {{{
      1, ToString[x1]}, {(mx +
          1)*0.25, ToString[0.25*(x1 - x2)]},
          {(mx + 1)*0.5, ToString[
          0]}, {(mx + 1)*0.75, ToString[0.25*(
        x2 - x1)]}, {mx + 1, ToString[x2]}},
        {{1, ToString[0]}, {0.25*Length[trid1],
        ToString[(0.25*tfinal)]}, {0.5*Length[trid1],
    ToString[0.5*
      tfinal]}, {0.75Length[trid1], ToString[0.75 tfinal]},
      {Length[trid1],
              ToString[
            tfinal]}}, Automatic},
            AxesLabel -> {" x ", " t ", " $\rho$ "}];
fig2 = ListPlot3D[trid,
            Mesh -> True, LightSources -> {{{1`, 0, 1},
                GrayLevel[0.1]}, {{1`, 1`, 1`},
                GrayLevel[0.2]}, {{0, 1, 1},
                GrayLevel[0.3]}},
                AmbientLight -> GrayLevel[0.3],
                Ticks -> {{{1, \
ToString[x1]}, {(mx + 1)*0.25,
          ToString[0.25*(x1 - x2)]},
          {(mx + 1)*0.5,
```

```
        ToString[0]}, {(mx + 1)*0.75,
        ToString[0.25*(x2 - x1)]}, {mx + 1,
                   ToString[x2]}}, {{1, ToString[0]},
                   {0.25*Length[trid], \
ToString[0.25*tfinal]}, {0.5*Length[trid],
ToString[0.5*tfinal]}, {0.75Length[
             trid], ToString[0.75*tfinal]},
             {Length[trid], ToString[
                 tfinal]}}, Automatic},
                AxesLabel -> {" x ", " t ", " m" }];
Print[figureName];
Export["D:\delal\\" <> "3DRo" <> ToString[grav]
<> ".pdf", fig1];
Export["D:\delal\\" <> "3Dm" <> ToString[grav]
<> ".pdf", fig2];
Print["dtmax= ", dtmax];
Print["Number of time steps:", numt];

mreza = Drop[mreza, korekcija];

]
```

Below is the explanation of the input parameters

mx - numer of points at the x - axes, mx=400 in our example,

t0 - start time, it's value was 0 in all examples,

tfinal - end time, you can see the end time from every 3D graphic. For example, at the graphic called 3Dm0.1, we show the solution for m, and b=0.1, the end time shown on the graphic is t=6.

$[x1, x2]$ - interval at the x - axes, (look at the 2D graphics. In some examples we have $[-15, 15]$, and in others $[-20, 20]$).

qlevo, qdesno - left and right initial data, i.e. $qlevo = \{\rho_l, m_l\}$, $qdesno =$

$\{\rho_r, m_r\}$. In all our examples we have: $qlevo = \{1.1, 0.2\}$, $qdesno = \{1.2, 0.1\}$.

nout - number of output graphics,

out - number or outputs (not significant for the result)

μ - not significant for the results (enables a moving mesh procedure and is not used for obtaining this results)

alfa1 -not significant for the results (enables a moving mesh procedure and is not used for obtaining this results)

ε - a parameter for smoothing the initial data. Its value is 0.005 in all cases.

ts - enables movement of the interval if the case of high speed delta (not used here)

grav - is the parameter b (0 and 0.1 in our 2D plots and 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1 in the 3D plots)

- granica1, granica2 - some parameters connected with the interval of the plots

Below is the explanation of the program

Lines 1-19: initialization

lines 20-23: smoothing of initial data

lines 24-25: plotting graphics of initial data lines 26-48: initialization

line 51: start of numerical procedure

line 58: start of a new time step

line 60-66: interval movement - not used for this examples

lines 86-128: solving the homogeneous part of the conservation law

lines 130-144: calculating the source term for the current time step and updating the solution obtained by solving the homogeneous part

lines 147-170: calculating and checking the CFL value (checking the scheme stability) lines 173-180: used for mesh movement - not used in our case

lines 184-187: creating the 3D graphics of the solution

lines 191-197: creating and plotting the 2D graphics

lines 203-227: plotting the 3D graphics

lines 228-231: Exporting the 3D plots for each case into pdf files

6

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