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KATO TYPE DECOMPOSITIONS AND GENERALIZATIONS OF DRAZIN INVERTIBILITY

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ДЕКОМПОЗИЦИЈЕ КАТООВОГ ТИПА И УОПШТЕЊА ДРАЗИНОВЕ ИНВЕРТИБИЛНОСТИ

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Abstract:	The main objective of this dissertation is to give necessary and sufficient conditions under which a bounded linear operator T can be represented as the direct sum of a nilpotent (quasinilpotent, Riesz) operator T_N and an operator T_M which belongs to any of the following classes: upper (lower) semi-Fredholm operators, Fredholm operators, upper (lower) semi-Weyl operators, Weyl operators, upper (lower) semi-Browder operators, Browder operators, bounded below operators, surjective operators and invertible operators. These results are applied to different types of spectra. In addition, we introduce the notions of the generalized Kato-Riesz decomposition and generalized Drazin-Riesz invertible operators. Moreover, we study the generalized Drazin spectrum of an upper triangular operator matrix acting on the product of Banach or separable Hilbert spaces. Further, motivated by the Atkinson type theorem for B-Fredholm operators, we introduce the notion of a B-Fredholm Banach algebra element. These objects are characterized and their main properties are studied. We also extend some results from the Fredholm theory to unbounded closed operators..
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Наслов:	Декомпозиције Катоовог типа и уопштења Дразинове инвертибилности
Резиме:	Централна тема дисертације је успостављање потребних и довољних услова под којима ограничен линеаран оператор T може бити представљен као директна сума једног нилпотентног (квазинилпотентног, Рисовог) оператора T_N и једног оператора T_M који припада било којој од следећих класа: горњи (доњи) семи-Фредхолмови оператори, Фредхолмови оператори, горњи (доњи) семи-Вејлови оператори, Вејлови оператори, горњи (доњи) семи-Браудерови оператори, Браудерови оператори, оператори ограничени одоздо, сурјективни оператори и инвертибилни оператори. Добијени резултати се примењују на изучавање различитих типова спектара. Такође, уводе се појмови уопштене Като-Рисове декомпозиције и уопштених Дразин-Рис инвертибилних оператора. Проучава се уопштени Дразинов спектар горње троугаоних операторских матрица које делују на производ Банахових или сепарабилних Хилбертових простора. Даље, мотивисани теоремом Аткинсоновог типа за Б-Фредхолмове операторе, уводимо појам Б-Фредхолмовог елемента Банахове алгебре. Ови елементи се карактеришу и изучавају се њихове главне особине. Проширујемо и неке резултате из Фредхолмове теорије на неограничене затворене операторе.
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To all of you, who fought for the liberation of Serbia in the 1912-1918 wars

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Preface

A major event in mathematics at the beginning of the 20[th] century was the appearance of Fredholm's theory of integral equations. In a preliminary report in 1900 and in an article in *Acta Mathematica* in 1903, Fredholm gave a complete analysis of integral equations of the second type, now known as Fredholm equations. The classical Fredholm integral equation is

$$\lambda f(s) - \int_a^b K(s,t)f(t)dt = g(s),\ a \leq s \leq b,$$

where g is given in $C[a,b]$, $K(s,t)$ is a continuous complex-valued function defined on $[a,b] \times [a,b]$, λ is a parameter and $f \in C[a,b]$ is the unknown. This equation can be rewritten as $(\lambda I - T)f = g$, where T is a compact operator defined by the rule $(Tf)(s) = \int_a^b K(s,t)f(t)dt$. Naturally, we are led to the study of operators of the form $\lambda I - T$ on any Banach space, where $\lambda \neq 0$ and T is compact; this idea goes back to the work by F. Riesz [63]. The operators $\lambda I - T$ are special cases of a class of operators called Fredholm operators (and also special cases of semi-Fredholm operators), so it seems that the birth of Fredholm operators is closely related to the problem of solving integral equations.

Semi-Fredholm operators were studied by a number of authors. The best general references here are the books by T. Kato [46], V. Müller [57], P. Aiena [1, 2], S. Caradus, W. Pfaffenberger and B. Yood [17], M. Schechter [65], I. Gohberg, S. Goldberg and M. A. Kaashoek [29], and others. In his famous paper [45], T. Kato showed that semi-Fredholm operators possess an important decomposition property. Namely, let $T \in L(\mathcal{X})$ and suppose that there exists a decomposition $\mathcal{X} = M \oplus N$, where M and N are closed subspaces of $\mathcal{X}$ such that $T(M) \subset M$ and $T(N) \subset N$. It is said that T admits a Kato decomposition if:

(i) $\dim N < \infty$, T_M is a Kato operator and T_N is nilpotent, where T_M and T_N are respectively reductions of T on M and N.

According to [45], every semi-Fredholm operator admits a Kato decomposition. Furthermore, it is possible to consider the following more general cases:

(ii) T_M is a Kato operator and T_N is nilpotent;

(iii) T_M is a Kato operator and T_N is quasinilpotent;

(iv) T_M is a Kato operator and T_N is Riesz.

It is said that T is of Kato type (T admits a generalized Kato decomposition) if T satisfies (ii) ((iii)). Decompositions (i) - (iii) have been already studied; for a survey of the results we refer to [1] and [57]. Decomposition (iv) is introduced in this dissertation. We say that $T \in L(\mathcal{X})$ admits a Kato type decomposition if T satisfies any condition (i) - (iv). It is obvious that decompositions (i) - (iii) are special cases of (iv).

As an additional point, semi-Browder operators may be characterized by means of the Kato decomposition [57, Theorem 20.10]: an operator $T \in L(\mathcal{X})$ is upper semi-Browder (lower semi-Browder, Browder) if and only if there exists a decomposition $\mathcal{X} = M \oplus N$, where $\dim N < \infty$, $T(M) \subset M$ and $T(N) \subset N$, with T_M bounded below (onto, invertible) and T_N nilpotent. In addition, M. Berkani introduced the concept of B-Fredholm operators and proved that $T \in L(\mathcal{X})$ is B-Bredholm if and only if $T = T_M \oplus T_N$, where T_M is Fredholm and T_N is nilpotent [7, Theorem 2.7]. It is easy to see that B-Fredholm operators satisfy (ii). Further, operators of the form $T = T_M \oplus T_N$ with T_M Fredholm (Weyl, bounded below, surjective) and T_N quasinilpotent were studied recently [15, 40, 70]. Finally, an operator $T \in L(\mathcal{X})$ is Drazin (generalized Drazin) invertible exactly when T possesses decomposition (ii) (decomposition (iii)) with T_M invertible.

All these observations strongly motivated us to consider a situation of an operator $T \in L(\mathcal{X})$ which admits a decomposition $T = T_M \oplus T_N$, with T_N nilpotent (quasinilpotent, Riesz) and $T_M \in \mathbf{R}$, where $\mathbf{R}$ is any of the following classes: upper (lower) semi-Fredholm operators, Fredholm operators, upper (lower) semi-Weyl operators, Weyl operators, upper (lower) semi-Browder operators, Browder operators, bounded below operators, surjective operators, and invertible operators. The dissertation is organized in six chapters and its central theme is to give sufficient and necessary conditions under which $T = T_M \oplus T_N$, where $T_M \in \mathbf{R}$ and T_N is nilpotent (quasinilpotent, Riesz). The results of this dissertation were published in international mathematical journals included in Science Citation Index Expanded (SCIe); see [20, 21, 22, 80, 81]. The article [19] was published in a national journal.

Chapter 1 presents some preliminaries. According to the available literature, the results given here are known except the statements (ii) and (iv) of Proposition 1.4.9. For many items we document their sources, but they are not always original sources. Also, the proofs of some selected statements are included since we find that it improves our presentation.

Chapter 2 deals with the generalized Kato decomposition and it is based on [21]. In our main results we prove that $T \in L(\mathcal{X})$ may be represented by $T = T_M \oplus T_N$ with $T_M \in \mathbf{R}$ and T_N quasinilpotent (T_N nilpotent) if and only if T admits a generalized Kato decomposition (T is of Kato type) and 0 is not an interior point of $\sigma_{\mathbf{R}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{R}\}$. In addition, we show that if $T - \lambda_0 I$ admits a generalized Kato decomposition, then $\sigma_{\mathbf{R}}(T)$

does not cluster at λ_0 if and only if λ_0 is not an interior point of $\sigma_{\mathbf{R}}(T)$. Also, this chapter contains several examples that supplement the presentation. In Section 2.4 our results are applied to different types of spectra.

In Chapter 3 we present the results from the paper [20]. The goal is to describe the set

$$\bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gD}(M_C),$$

where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, $A \in L(\mathcal{H})$, $B \in L(\mathcal{K})$, $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is an upper triangular operator matrix which is acting on the product space $\mathcal{H}\oplus\mathcal{K}$, and $\sigma_{gD}(M_C)$ is the generalized Drazin spectrum of M_C. This is done using the result where we give sufficient conditions under which M_C is generalized Drazin invertible. More precisely, consider the following conditions:

(i) A and B each admits a generalized Kato decomposition;
(ii) The approximate point spectrum of A does not cluster at 0;
(iii) The surjective spectrum of B does not cluster at 0;
(iv) There exists $\delta > 0$ such that $\beta(A-\lambda I) = \alpha(B-\lambda I)$ for $0 < |\lambda| < \delta$.

Theorem 3.2.6 states that if the conditions (i)-(iv) are satisfied then there exists $C \in L(\mathcal{K},\mathcal{H})$ such that M_C is generalized Drazin invertible. It is worth pointing out that we use the results in Chapter 2 to prove this theorem. In addition, we give sufficient and necessary conditions for $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma_{gD}(M_C) = \emptyset$ and for $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma_{gD}(M_C) = \bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma(M_C)$.

The results presented in Chapter 4 are taken from [81]. We introduce the notion of the generalized Kato-Riesz decomposition (abbreviated as GKRD). It is said that $T \in L(\mathcal{X})$ admits a GKRD if $T = T_M \oplus T_N$ with T_M Kato and T_N Riesz. We give sufficient and necessary conditions for the existence of a decomposition $T = T_M \oplus T_N$, where $T_M \in \mathbf{R}$ and T_N is Riesz. Moreover, the concept of generalized Drazin-Riesz invertible operators is introduced and studied. It is said that an operator $T \in L(\mathcal{X})$ is generalized Drazin-Riesz invertible if there exists $S \in L(\mathcal{X})$ commuting with T such that $STS = S$ and $TST - T$ is Riesz. We show that $T \in L(\mathcal{X})$ is generalized Drazin-Riesz invertible if and only if T admits a GKRD and 0 is not an interior point of the spectrum of T, and it is also equivalent to the assertion that $T = T_M \oplus T_N$, where T_M is invertible and T_N is Riesz. Evidently, every generalized Drazin invertible operator is generalized Drazin-Riesz invertible. On the other hand, if $T \in L(\mathcal{X})$ is a Riesz operator with infinite spectrum, then the point 0 is an accumulation point of the spectrum of T, so T is not generalized Drazin invertible. Obviously, T is generalized Drazin-Riesz invertible ($S = 0$). It follows that the class of generalized Drazin invertible operators is a proper subset of the class of generalized Drazin-Riesz invertible operators. By this conclusion, the concept of generalized Drazin invertible operators is extended.

Chapter 5 is an attempt to generalize the theory of B-Fredholm operators. The results of this chapter are from [22]. The Atkinson-type theorem for B-Fredholm operators [12] leads to the following definition. Let $\mathcal{A}$ and $\mathcal{B}$ be two complex unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. The element $a \in \mathcal{A}$ will be said to be B-Fredholm relative to $\mathcal{T}$, if $\mathcal{T}(a)$ is Drazin invertible. We also introduce other classes of objects such as B-Weyl and generalized B-Fredholm elements. In this chapter the aforementioned elements will be characterized and their main properties will be studied. In addition, the perturbation properties will be also considered.

In the sixth chapter some results from the Fredholm theory will be extended to unbounded closed operators. We give sufficient and necessary conditions such that a closed operator is upper or lower semi-Browder. Consequently, the corresponding spectra are described. The work done in this chapter comes from the paper [80].

Chapter 1

Introduction

Let $\mathbb{N}$ ($\mathbb{N}_0$) denote the set of all positive (non-negative) integers, and let $\mathbb{C}$ ($\mathbb{R}$) denote the set of all complex (real) numbers. The modulus of a complex number λ will be denoted by $|\lambda|$ and its conjugate by $\bar{\lambda}$. Throughout this thesis $\mathcal{X}$ and $\mathcal{Y}$ will be infinite dimensional Banach spaces over the field of complex numbers. We use the symbol ($\|\cdot\|$) for the norm in any space and also for the norm of operators.

1.1 Sets of the complex plane

For $S \subset \mathbb{C}$, the set of accumulation points of S, the set of isolated points of S, the interior of S, the boundary of S, the closure of S, and the complement of S are denoted by $\operatorname{acc} S$, $\operatorname{iso} S$, $\operatorname{int} S$, ∂S, $\overline{S}$ and S^c, respectively. In the following proposition we collect some basic facts concerning the sets of the complex plane.

Proposition 1.1.1. *Let S and L be sets of the complex plane. The following statements hold:*

(i) $\operatorname{int} S \subset \operatorname{acc} S$;

(ii) *If $S \subset L$ then $\operatorname{acc} S \subset \operatorname{acc} L$ and $\operatorname{int} S \subset \operatorname{int} L$;*

(iii) *If S is closed then $S = \operatorname{int} S \cup \partial S$ and $\operatorname{int} S \cap \partial S = \emptyset$;*

(iv) *If S is closed and $\lambda \in \mathbb{C}$, then $\lambda \notin \operatorname{int} S$ if and only if $\lambda \in \operatorname{acc} S^c$;*

(v) *If S is bounded then S is finite if and only if $\operatorname{acc} S = \emptyset$.*

(vi) *If S is closed then S is at most countable if and only if $\operatorname{acc} S$ is at most countable.*

Proposition 1.1.2. *Let $K \subset \mathbb{C}$ be a closed set and let $\lambda \in \partial K$. Then:*

$$\lambda \in \operatorname{acc} K \quad \textit{if and only if} \quad \lambda \in \operatorname{acc} \partial K. \tag{1.1}$$

Proof. Since $\operatorname{acc}\partial K \subset \operatorname{acc} K$, it is sufficient to prove the opposite implication. Let $\lambda \in \partial K \cap \operatorname{acc} K$, but $\lambda \notin \operatorname{acc}\partial K$. It means that $\lambda \in \operatorname{iso}\partial K$, so there exists an $\epsilon > 0$ such that

$$\partial K \cap \overset{\circ}{U} = \emptyset, \tag{1.2}$$

where $\overset{\circ}{U} = D(\lambda, \epsilon) \setminus \{\lambda\}$ and $D(\lambda, \epsilon) = \{\mu \in \mathbb{C} : |\mu - \lambda| < \epsilon\}$. Clearly,

$$\overset{\circ}{U} = (K \cap \overset{\circ}{U}) \cup (K^c \cap \overset{\circ}{U}). \tag{1.3}$$

The set $K^c \cap \overset{\circ}{U}$ is open as an intersection of two open sets. Using (1.2) we obtain

$$K \cap \overset{\circ}{U} = (\partial K \cup \operatorname{int} K) \cap \overset{\circ}{U} = (\partial K \cap \overset{\circ}{U}) \cup (\operatorname{int} K \cap \overset{\circ}{U}) = \operatorname{int} K \cap \overset{\circ}{U},$$

so $K \cap \overset{\circ}{U}$ is also open since $\operatorname{int} K$ is open. According to (1.3), $\overset{\circ}{U}$ is a union of two open disjoint sets, and since $\overset{\circ}{U}$ is connected it follows that either $K \cap \overset{\circ}{U} = \emptyset$ or $K^c \cap \overset{\circ}{U} = \emptyset$. Suppose that $K^c \cap \overset{\circ}{U} = \emptyset$, i.e. $\overset{\circ}{U} \subset K$. Now, $D(\lambda, \epsilon) \subset \overline{\overset{\circ}{U}} \subset \overline{K} = K$. Consequently, $\lambda \in \operatorname{int} K$, what is not possible. It follows that $K \cap \overset{\circ}{U} = \emptyset$, so $\lambda \in \operatorname{iso} K$, a contradiction. □

The claim of Proposition 1.1.2 is not true in the context of the metric space $\mathbb{R}$! Indeed, let $K = [0, 1]$. Then, $\partial K = \{0, 1\}$, $0 \in \operatorname{acc} K$, but $0 \notin \operatorname{acc}\partial K$. The key reason is the fact that in this case the set $\overset{\circ}{U} = (-\epsilon, +\epsilon) \setminus \{0\}$, $0 < \epsilon < 1$, is not connected:

$$\overset{\circ}{U} = (-\epsilon, 0) \cup (0, +\epsilon).$$

We recall that a set $K \subset \mathbb{C}$ is compact if it is closed and bounded. In that case the set K^c is open and unbounded. The connected components of K^c are open and, obviously, one of them is unbounded. The bounded components of K^c are called holes in K. The connected hull of K is denoted by ηK and it is known that ηK is the union of K and its holes (for example see [60, Lema 5.7.4]).

Proposition 1.1.3. *Let K and H be compact sets of the complex plane. If $\partial H \subset K \subset H$, then the following statements hold:*

(i) $\partial H \subset \partial K \subset K \subset H$*;*

(ii) $\eta K = \eta H$*.*

Lemma 1.1.4. *Let K and H be compact sets of the complex plane. Then the following statements hold:*

(i) $\eta K = K$ *if K is at most countable;*

(ii) *If $\eta K = \eta H$ then K is at most countable if and only if H is at most countable. In that case $H = K$.*

1.2 Bounded linear operators in Banach spaces

The vector spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ are said to be isomorphic whenever there exists a one-one linear mapping from $\mathcal{X}_1$ onto $\mathcal{X}_2$. A vector space V is finite dimensional if its Hamel basis contains finitely many elements. Otherwise, V is infinite dimensional space. The dimension of V, denoted by $\dim V$, is the cardinal number of its Hamel basis if V is finite dimensional. If V is infinite dimensional, we simply take $\dim V = \infty$ (we do not distinguish different infinite cardinalities). Obviously, two isomorphic vector spaces have the same dimension. Further, it is said that the normed spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic, denoted by $\mathcal{X}_1 \cong \mathcal{X}_2$, if there exists a linear bijective operator $J : \mathcal{X}_1 \to X_2$ which preserves the norm. Let $L(\mathcal{X}, \mathcal{Y})$ denote the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. For simplicity, we write $L(\mathcal{X})$ for $L(\mathcal{X}, \mathcal{X})$. Given $T \in L(\mathcal{X}, \mathcal{Y})$, the kernel and the range of T are defined respectively as $N(T) = \{x \in \mathcal{X} : Tx = 0\}$ and $R(T) = \{Tx : x \in \mathcal{X}\}$. The numbers $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim \mathcal{X}/R(T) = \operatorname{codim} R(T)$ are nullity and deficiency of T, respectively. The space of all bounded linear functionals defined on $\mathcal{X}$ is denoted by $\mathcal{X}'$. Given $M \subset \mathcal{X}$, the annihilator of M is defined by $M^{\perp} = \{f \in \mathcal{X}' : f(x) = 0 \text{ for every } x \in M\}$. If $R(T)$ is closed, then $\alpha(T') = \beta(T)$ and $\beta(T') = \alpha(T)$, where $T' \in L(\mathcal{Y}', \mathcal{X}')$ is the adjoint operator of T. An operator $T \in L(\mathcal{X}, \mathcal{Y})$ is injective if $N(T) = \{0\}$, and surjective if $R(T) = \mathcal{Y}$. We say that $T \in L(\mathcal{X})$ is invertible if there exists $S \in L(\mathcal{X})$ such that $TS = ST = I$, where I is the identity operator on $\mathcal{X}$, and in that case we write $S = T^{-1}$. It is well known that $T \in L(\mathcal{X})$ is invertible if and only if it is both injective and surjective. The group of all invertible operators on $\mathcal{X}$ is denoted by $L(\mathcal{X})^{-1}$, and the sets

$$\begin{aligned} &\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin L(\mathcal{X})^{-1}\}, \\ &\rho(T) = \mathbb{C} \setminus \sigma(T), \end{aligned}$$

are the spectrum and resolvent set of $T \in L(\mathcal{X})$, respectively. The set of all compact operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $K(\mathcal{X}, \mathcal{Y})$; as usual $K(\mathcal{X}) = K(\mathcal{X}, \mathcal{X})$. The set $K(\mathcal{X}, \mathcal{Y})$ is a closed subspace in $L(\mathcal{X}, \mathcal{Y})$ and $K(\mathcal{X})$ is a closed two-sided ideal in $L(\mathcal{X})$. This fact enables us to define the Calkin algebra over $\mathcal{X}$ as the quotient algebra $L(\mathcal{X})/K(\mathcal{X})$. $L(\mathcal{X})/K(\mathcal{X})$ is itself a Banach algebra with the quotient algebra norm

$$\|T + K(\mathcal{X})\| = \inf_{U \in K(\mathcal{X})} \|T + U\|.$$

We will use π to denote the natural homomorphism of $L(\mathcal{X})$ onto $L(\mathcal{X})/K(\mathcal{X})$: $\pi(T) = T + K(\mathcal{X})$. An operator $T \in L(\mathcal{X}, \mathcal{Y})$ is of finite rank if $\dim R(T) < \infty$. We will denote by $F(\mathcal{X}, \mathcal{Y})$ the set of all finite rank operators from $\mathcal{X}$ to $\mathcal{Y}$; if $\mathcal{X} = \mathcal{Y}$, then $F(\mathcal{X}, \mathcal{X}) = F(\mathcal{X})$. Since $F(\mathcal{X})$ is not necessarily closed two-sided ideal, $L(\mathcal{X})/F(\mathcal{X})$ is not a Banach algebra.

Definition 1.2.1. Let $T \in L(\mathcal{X})$ and $d \in \mathbb{N}$.
(i) T is nilpotent of degree d if $T^{d-1} \neq 0$ and $T^d = 0$;
(ii) T is quasinilpotent if $T - \lambda I \in L(\mathcal{X})^{-1}$ for $0 \neq \lambda \in \mathbb{C}$.

Let $T \in L(\mathcal{X})$. Is is well known that if $N(T^n) = N(T^{n+1})$, then $N(T^k) = N(T^n)$ when $k \geq n$. In this case, the ascent of T, denoted by $\operatorname{asc}(T)$, is the smallest $n \in \mathbb{N}_0$ such that $N(T^n) = N(T^{n+1})$. If such n does not exist, then $\operatorname{asc}(T) = \infty$. Similarly, if $R(T^{n+1}) = R(T^n)$, then $R(T^k) = R(T^n)$ for $k \geq n$. In this case, the descent of T, denoted by $\operatorname{dsc}(T)$, is the smallest $n \in \mathbb{N}_0$ such that $R(T^{n+1}) = R(T^n)$. If such an n does not exist, then $\operatorname{dsc}(T) = \infty$.

The injectivity modulus (minimum modulus) of $T \in L(\mathcal{X}, \mathcal{Y})$ is defined as

$$j(T) = \inf_{\|x\|=1} \|Tx\|.$$

Immediately from this definition it follows that $j(T)\|x\| \leq \|Tx\|$ for every $x \in \mathcal{X}$.

Definition 1.2.2. An operator $T \in L(\mathcal{X}, \mathcal{Y})$ is bounded below if there exists some $c > 0$ such that

$$c\|x\| \leq \|Tx\| \ \text{ for all } \ x \in \mathcal{X}.$$

Clearly, $T \in L(\mathcal{X}, \mathcal{Y})$ is bounded below if and only if $j(T) > 0$. Also, [57, Theorem 9.4] asserts that $T \in L(\mathcal{X}, \mathcal{Y})$ is bounded below if and only if it is injective with closed range. We will use the following notation:

$$\mathcal{M}(\mathcal{X}) = \{T \in L(\mathcal{X}) : T \text{ is bounded below}\},$$
$$\mathcal{Q}(\mathcal{X}) = \{T \in L(\mathcal{X}) : T \text{ is surjective}\}.$$

The approximate point and surjective spectrum of $T \in L(\mathcal{X})$ are defined by

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{M}(\mathcal{X})\},$$
$$\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{Q}(\mathcal{X})\},$$

respectively. The spectra $\sigma(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ are non-empty and compact subsets of $\mathbb{C}$. The sets $\rho_{ap}(T) = \mathbb{C} \setminus \sigma_{ap}(T)$ and $\rho_{su}(T) = \mathbb{C} \setminus \sigma_{su}(T)$ are corresponding resolvent sets.

Let M and N be subspaces of $\mathcal{X}$. The sum of M and N is defined as

$$M + N = \{z \in \mathcal{X} : z = x + y, \ x \in M, \ y \in N\}.$$

If $M \cap N = \{0\}$ we say that the sum $M + N$ is direct and write $M \oplus N$ instead of $M + N$. It is evident that every vector $z \in M \oplus N$ can be represented in a unique way as $z = x + y$, where $x \in M$ and $y \in N$.

Given $T \in L(\mathcal{X})$ and a subspace $M \subset \mathcal{X}$, it is said that M is T-invariant if $T(M) \subset M$. We define $T_M : M \to M$ as $T_M x = Tx$, $x \in M$. In addition, if M is closed, then it is a Banach space, $T_M \in L(M)$ and $\|T_M\| \leq \|T\|$. If M and N are two closed T-invariant subspaces of $\mathcal{X}$ such that $\mathcal{X} = M \oplus N$, we say that T is completely reduced by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T_M \oplus T_N$ and say that T is a direct sum of T_M and T_N.

An operator $P \in \text{Ł}(\mathcal{X})$ with the property that $P^2 = P$ is called projection. It is easy to see that both $N(P)$ and $R(P)$ are closed and $\mathcal{X} = N(P) \oplus R(P)$. On the other hand, if M and N are closed subspaces of $\mathcal{X}$ such that $\mathcal{X} = M \oplus N$ then there exists $P^2 = P \in L(\mathcal{X})$ such that $R(P) = M$ and $N(P) = N$. In addition, operators $P^2 = P \in L(\mathcal{X})$ and $T \in L(\mathcal{X})$ commute if and only if $R(P)$ and $N(P)$ are T-invariant.

Definition 1.2.3. Let $T \in L(\mathcal{X})$. The operator-valued function $R(\lambda, T) : \rho(T) \to L(\mathcal{X})$ defined by

$$R(\lambda, T) = (\lambda I - T)^{-1}$$

is called the resolvent function of T.

The function $R(\lambda, T)$ is analytic on $\rho(T)$, and an isolated point λ_0 of $\sigma(T)$ is an isolated singular point of $R(\lambda, T)$. It follows that there exists $\delta > 0$ such that $R(\lambda, T)$ admits a Laurent expansion on the punctured open disc centered at λ_0 with radius δ:

$$R(\lambda, T) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n, \quad 0 < |\lambda - \lambda_0| < \delta,$$

where the coefficients A_n and B_n belong to $L(\mathcal{X})$. These coefficient operators are given by the standard formulas:

$$\begin{aligned} A_n &= \frac{1}{2\pi i} \int_C (\lambda - \lambda_0)^{-n-1} R(\lambda, T) d\lambda, \\ B_n &= \frac{1}{2\pi i} \int_C (\lambda - \lambda_0)^{n-1} R(\lambda, T) d\lambda, \end{aligned}$$

where C is a circle centered at λ_0, separating λ_0 from the remaining spectrum of T. In particular, B_1 is the projection and it is called the spectral projection of T corresponding to λ_0. We shall say that λ_0 is a pole of $R(\lambda, T)$ of order m if $B_m \neq 0$ and $B_n = 0$ when $n > m$. The set of poles of $R(\lambda, T)$ will be denoted by $\Pi(T)$. Sufficient and necessary for $\lambda_0 \in \sigma(T)$ to be a pole of the resolvent function is that ascent and descent of $T - \lambda_0 I$ are both finite. It is worth mentioning that the classical references on this topic are [67, Section 5.8] and [53].

1.3 Kato operators, Kato type decompositions and SVEP

Let $u \in \mathcal{X}$ and let $S \subset \mathcal{X}$ be a subset. By

$$\mathrm{dist}(u, S) = \inf_{v \in S} \|u - v\|$$

we define the distance of a vector u from a subset S. If S is closed and $u \notin S$, then $\mathrm{dist}(u, S) > 0$.

In the following definition we introduce the gap function which measures the "distance" between two closed subspaces.

Definition 1.3.1. Let M and N be two closed subspaces of $\mathcal{X}$. We set

$$\delta(M, N) = \sup_{\substack{u \in M \\ \|u\| = 1}} \mathrm{dist}(u, N).$$

The gap between M and N, denoted by $\hat{\delta}(M, N)$, is defined as

$$\hat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}.$$

We define $\delta(\{0\}, N) = 0$ for any N. On the other hand, $\delta(M, \{0\}) = 1$ if $M \neq \{0\}$, as is seen from the definition.

For more details concerning the gap we refer the reader to [46, 57]. We only mention the result that is relevant for our work.

Lemma 1.3.2. ([46, Corollary IV-2.6]) *Let M and N be closed subspaces of $\mathcal{X}$. If $\hat{\delta}(M, N) < 1$ then* $\mathrm{dim}M = \mathrm{dim}N$.

Definition 1.3.3. An operator $T \in L(\mathcal{X})$ is Kato if $R(T)$ is closed and $N(T) \subset R(T^n)$ for all $n \in \mathbb{N}_0$.

It is evident that any operator that is either bounded below or surjective is Kato. Let $T \in L(\mathcal{X}, \mathcal{Y})$. The reduced minimum modulus of T is defined by

$$\gamma(T) = \inf\{\|Tx\| : x \in X, \mathrm{dist}(x, N(T)) = 1\}.$$

If $T = 0$ we set $\gamma(T) = \infty$. An operator $T \in L(\mathcal{X}, \mathcal{Y})$ has closed range if and only if $\gamma(T) > 0$ [57, Theorem 10.2], and the notion of reduced minimum modulus is motivated by this characterization. Obviously, if $T \in L(\mathcal{X})$ is Kato then $\gamma(T) > 0$.

Theorem 1.3.4. ([1, Theorem 1.38]) *Let $T \in L(\mathcal{X})$ and $\lambda_0 \in \mathbb{C}$. The following statements are equivalent:*

(i) $T - \lambda_0 I$ *is Kato;*

(ii) $\gamma(T - \lambda_0 I) > 0$ *and the mapping $\lambda \to N(T - \lambda I)$ is continuous at the point λ_0 in the gap metric.*

Lemma 1.3.5. *Let $T \in L(\mathcal{X})$ be Kato. Then there exists $\epsilon > 0$ such that $T - \lambda I$ is Kato, $\alpha(T) = \alpha(T - \lambda I)$ and $\beta(T) = \beta(T - \lambda I)$ for all $|\lambda| < \epsilon$.*

Proof. Let $T \in L(\mathcal{X})$ be Kato. There exists $\epsilon_1 > 0$ such that $T - \lambda I$ is Kato for $|\lambda| < \epsilon_1$ [57, Corollary 12.4]. By Theorem 1.3.4, there exists $\epsilon_2 > 0$ such that $|\lambda| < \epsilon_2$ implies $\hat{\delta}(N(T), N(T - \lambda I)) < 1$. From Lemma 1.3.2 we obtain $\dim N(T) = \dim N(T - \lambda I)$, i.e. $\alpha(T) = \alpha(T - \lambda I)$ for $|\lambda| < \epsilon_2$.

Further, using the fact that T' is also Kato [57, Corollary 12.4] and from what has already been proved we see that there exists $\epsilon_3 > 0$ such that

$$\alpha(T') = \alpha(T' - \lambda I') \text{ and } R(T - \lambda I) \text{ is closed for } |\lambda| < \epsilon_3. \tag{1.4}$$

Now, from (1.4) we conclude $\beta(T) = \alpha(T') = \alpha(T' - \lambda I') = \beta(T - \lambda I)$ for $|\lambda| < \epsilon_3$. We set $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$, and the lemma follows. □

Definition 1.3.6. Let $T \in L(\mathcal{X})$. Then:
(i) T admits a generalized Kato decomposition (GKD for short) if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent;
(ii) T is of Kato type if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is nilpotent;
(iii) T is essentially Kato if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato, T_N is nilpotent and $\dim N < \infty$.

We have the following implications:

$$\begin{aligned} T \text{ is Kato } &\Longrightarrow T \text{ is essentailly Kato } \Longrightarrow T \text{ is of Kato type} \\ &\Longrightarrow T\text{admits a GKD.} \end{aligned}$$

The Kato spectrum, the essentially Kato spectrum, the Kato type spectrum and the generalized Kato spectrum of $T \in L(\mathcal{X})$ are defined respectively by

$$\begin{aligned} \sigma_K(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Kato}\}, \\ \sigma_{eK}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not essentially Kato}\}, \\ \sigma_{Kt}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not of Kato type}\}, \\ \sigma_{gK}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a GKD}\}. \end{aligned}$$

Obviously,

$$\sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_K(T) \subset \sigma_{ap}(T) \cap \sigma_{su}(T). \tag{1.5}$$

The Kato spectrum and essentially Kato spectrum are non-empty and compact subsets of the complex plane [57, Theorem 12.11 and Theorem 21.11]. The Kato type spectrum and generalized Kato spectrum are also compact (see [1, Corollary 1.45] and [41, Corollary 2.3]), but they may be non-empty. For example, if $P \in L(\mathcal{X})$ is a projection and $Q \in L(\mathcal{X})$ is a quasinilpotent operator, then $\sigma_{Kt}(P)$ and $\sigma_{gK}(Q)$ are empty sets.

An operator $T \in L(\mathcal{X})$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T - \lambda I)f(\lambda) = 0 \text{ for all } \lambda \in D_{\lambda_0}, \tag{1.6}$$

is the function $f \equiv 0$. The set of all $\lambda \in \mathbb{C}$ where T does not have the SVEP is denoted by $\mathcal{S}(T)$; it is said that T has the SVEP if $\mathcal{S}(T) = \emptyset$. Let $\lambda_0 \notin \operatorname{int} \sigma_{ap}(T)$ and let $f : D_{\lambda_0} \to \mathcal{X}$ be an analytic function satisfying (1.6), where D_{λ_0} is arbitrary. Then, $\lambda_0 \in \partial\sigma_{ap}(T)$ or $\lambda_0 \in \rho_{ap}(T)$, but in both cases there exists a sequence (λ_n) in $\rho_{ap}(T) \cap D_{\lambda_0}$, $\lambda_n \neq \lambda_0$ for all $n \in \mathbb{N}$, such that $\lim \lambda_n = \lambda_0$. Using (1.6) we have

$$(T - \lambda_n I)f(\lambda_n) = 0 \text{ for all } n \in \mathbb{N}.$$

Since $T - \lambda_n I$ is injective, then $f(\lambda_n) = 0$ for all $n \in \mathbb{N}$. It follows that $f \equiv 0$ on D_{λ_0} by the identity theorem for analytical functions. We have just proved the implication:

$$\lambda_0 \notin \operatorname{int} \sigma_{ap}(T) \implies T \text{ has the SVEP at } \lambda_0. \tag{1.7}$$

If $\lambda_0 \notin \operatorname{int} \sigma_{su}(T)$, then $\lambda_0 \notin \operatorname{int} \sigma_{ap}(T')$ since $\sigma_{su}(T) = \sigma_{ap}(T')$. We now apply (1.7), with T replaced by T', to obtain the following implication:

$$\lambda_0 \notin \operatorname{int} \sigma_{su}(T) \implies T' \text{ has the SVEP at } \lambda_0. \tag{1.8}$$

Clearly, (1.7) and (1.8) give the implication:

$$\lambda_0 \notin \operatorname{int} \sigma(T) \implies T \text{ and } T' \text{ have the SVEP at } \lambda_0. \tag{1.9}$$

The following two results will be needed in this work. For more comprehensive study of the SVEP see [1, 52].

Proposition 1.3.7. ([1, Theorem 2.9]) *Suppose that $T \in L(\mathcal{X})$ and that $(M, N) \in Red(T)$. Then, T has the SVEP at λ_0 if and only if both T_M and T_N have the SVEP at λ_0.*

Theorem 1.3.8. ([1, Theorem 2.49]) *Let $T - \lambda_0 I \in L(\mathcal{X})$ be a Kato operator. Then the following equivalences hold:*

(i) *T has the SVEP at λ_0 if and only if $T - \lambda_0 I$ is bounded below.*

(ii) *T' has the SVEP at λ_0 if and only if $T - \lambda_0 I$ is surjective.*

1.4 Fredholm theory

Definition 1.4.1. Let $T \in L(\mathcal{X}, \mathcal{Y})$. We say that:

(i) T is upper semi-Fredholm if $R(T)$ is closed and $\alpha(T) < \infty$;

(ii) T is lower semi-Fredholm if $\beta(T) < \infty$;

(iii) T is Fredholm if $\alpha(T)$ and $\beta(T) < \infty$.

The set of all upper semi-Fredholm, lower semi-Fredholm and Fredholm operators will be denoted by $\Phi_+(\mathcal{X}, \mathcal{Y})$, $\Phi_-(\mathcal{X}, \mathcal{Y})$ and $\Phi(\mathcal{X}, \mathcal{Y})$, respectively. We recall that the condition $\beta(T) < \infty$ implies that the range of T is closed [45, Lemma 332]. According to this observation, it is obvious that $\Phi(\mathcal{X}, \mathcal{Y}) = \Phi_+(\mathcal{X}, \mathcal{Y}) \cap \Phi_-(\mathcal{X}, \mathcal{Y})$. The set of all semi-Fredholm operators is defined by $\Phi_\pm(\mathcal{X}, \mathcal{Y}) = \Phi_+(\mathcal{X}, \mathcal{Y}) \cup \Phi_-(\mathcal{X}, \mathcal{Y})$. We shall set $\Phi_+(\mathcal{X}) = \Phi_+(\mathcal{X}, \mathcal{X})$, $\Phi_-(\mathcal{X}) = \Phi_-(\mathcal{X}, \mathcal{X})$, $\Phi(\mathcal{X}) = \Phi(\mathcal{X}, \mathcal{X})$, and $\Phi_\pm(\mathcal{X}) = \Phi_\pm(\mathcal{X}, \mathcal{X})$. The class of semi-Fredholm operators belongs to the class of essentially Kato operators [57, Theorem 16.21].

Probably one of the most important results concerning Fredholm operators is the Atkinson theorem; see for example [17, Theorem 3.2.8].

Theorem 1.4.2. (Atkinson theorem) *An operator $T \in L(\mathcal{X})$ is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra $L(\mathcal{X})/K(\mathcal{X})$.*

If $T \in \Phi_\pm(\mathcal{X}, \mathcal{Y})$ then it is possible to define the index of T, denoted by $\operatorname{ind}(T)$, as $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. Using the notion of index we introduce the following classes of operators.

Definition 1.4.3. Let $T \in L(\mathcal{X}, \mathcal{Y})$. We say that:

(i) T is upper semi-Weyl if $T \in \Phi_+(\mathcal{X}, \mathcal{Y})$ and $\operatorname{ind}(T) \leq 0$;

(ii) T is lower semi-Weyl if $T \in \Phi_-(\mathcal{X}, \mathcal{Y})$ and $\operatorname{ind}(T) \geq 0$;

(iii) T is Weyl if $T \in \Phi(\mathcal{X}, \mathcal{Y})$ and $\operatorname{ind}(T) = 0$.

The set of all upper semi-Weyl, lower semi-Weyl and Weyl operators will be denoted by $\mathcal{W}_+(\mathcal{X}, \mathcal{Y})$, $\mathcal{W}_-(\mathcal{X}, \mathcal{Y})$ and $\mathcal{W}(\mathcal{X}, \mathcal{Y})$, respectively. The meaning of $\mathcal{W}_+(\mathcal{X})$, $\mathcal{W}_-(\mathcal{X})$ and $\mathcal{W}(\mathcal{X})$ is clear.

Theorem 1.4.4. [1, Theorem 3.39] *Let $T \in L(\mathcal{X})$. The following assertions are equivalent:*

(i) *T is a Weyl operator;*

(ii) *There exists a finite rank operator $F \in L(\mathcal{X})$ and $A \in L(\mathcal{X})^{-1}$ such that $T = A + F$;*

(iii) *There exists a compact operator $K \in L(\mathcal{X})$ and $A \in L(\mathcal{X})^{-1}$ such that $T = A + K$.*

Definition 1.4.5. Let $T \in L(\mathcal{X})$. We say that:
(i) T is upper semi-Browder if $T \in \Phi_+(\mathcal{X})$ and $\operatorname{asc}(T) < \infty$;
(ii) T is lower semi-Browder if $T \in \Phi_-(\mathcal{X})$ and $\operatorname{dsc}(T) < \infty$;
(iii) T is Browder if $T \in \Phi(\mathcal{X})$, $\operatorname{asc}(T) < \infty$ and $\operatorname{dsc}(T) < \infty$.
The set of all upper semi-Browder, lower semi-Browder and Browder operators will be denoted by $\mathcal{B}_+(\mathcal{X})$, $\mathcal{B}_-(\mathcal{X})$ and $\mathcal{B}(\mathcal{X})$, respectively. It is clear that T is Browder if it is both lower and upper semi-Browder.

Theorem 1.4.6. [17, Theorem 1.4.5] *Let $T \in L(\mathcal{X})$. The following assertions are equivalent:*
(i) *T is a Browder operator;*
(ii) *T can be written as $T = A + F$, where $A \in L(\mathcal{X})^{-1}$, $F \in L(\mathcal{X})$ is a finite rank operator and $AF = FA$;*
(iii) *T can be written as $T = A + K$, where $A \in L(\mathcal{X})^{-1}$, $K \in L(\mathcal{X})$ is a compact operator and $AK = KA$.*

We will use the following notation:

$\mathbf{R_1} = \mathcal{M}(\mathcal{X})$	$\mathbf{R_2} = \mathcal{Q}(\mathcal{X})$	$\mathbf{R_3} = L(\mathcal{X})^{-1}$
$\mathbf{R_4} = \mathcal{B}_+(\mathcal{X})$	$\mathbf{R_5} = \mathcal{B}_-(\mathcal{X})$	$\mathbf{R_6} = \mathcal{B}(\mathcal{X})$
$\mathbf{R_7} = \Phi_+(\mathcal{X})$	$\mathbf{R_8} = \Phi_-(\mathcal{X})$	$\mathbf{R_9} = \Phi(\mathcal{X})$
$\mathbf{R_{10}} = \mathcal{W}_+(\mathcal{X})$	$\mathbf{R_{11}} = \mathcal{W}_-(\mathcal{X})$	$\mathbf{R_{12}} = \mathcal{W}(\mathcal{X})$

The sets $\mathbf{R}_i$, $1 \le i \le 12$, are open in $L(\mathcal{X})$ and contain $L(\mathcal{X})^{-1}$ (for the openness of the set of upper (lower) semi-Browder operators see [50, Satz 4]). The spectra with respect to the sets $\mathbf{R}_i$, $1 \le i \le 12$, are defined by

$$\sigma_{\mathbf{R}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{R}_i\}, \ \ 1 \le i \le 12.$$

Obviously, $\sigma_{\mathbf{R}_1}(T) = \sigma_{ap}(T)$, $\sigma_{\mathbf{R}_2}(T) = \sigma_{su}(T)$ and $\sigma_{\mathbf{R}_3}(T) = \sigma(T)$. The set $\sigma_{\mathbf{R}_7}(T) = \sigma_{\Phi_+}(T)$ is the upper semi-Fredholm spectrum of T, the set $\sigma_{\mathbf{R}_5}(T) = \sigma_{\mathcal{B}_-}(T)$ is the lower semi-Browder spectrum of T, etc. All spectra $\sigma_{\mathbf{R}_i}(T)$, $4 \le i \le 12$, are also non-empty and compact subsets of $\mathbb{C}$, and common name for them is essential spectra. By $\rho_{\mathbf{R}_i}(T) = \mathbb{C} \setminus \sigma_{\mathbf{R}_i}(T)$, $4 \le i \le 12$, we define the corresponding resolvent sets.

Lemma 1.4.7. *Let $T \in L(\mathcal{X})$ and $(M, N) \in Red(T)$. The following statements hold:*
(i) *$T \in \mathbf{R}_i$ if and only if $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, $1 \le i \le 9$, and in that case* $\operatorname{ind}(T) = \operatorname{ind}(T_M) + \operatorname{ind}(T_N)$;
(ii) *If $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, then $T \in \mathbf{R}_i$, $10 \le i \le 12$;*
(iii) *If $T \in \mathbf{R}_i$ and T_N is Weyl, then $T_M \in \mathbf{R}_i$, $10 \le i \le 12$.*

Proof. (i). From the equalities $N(T) = N(T_M) \oplus N(T_N)$ and $R(T) = R(T_M) \oplus R(T_N)$ it follows that

$$\alpha(T) = \alpha(T_M) + \alpha(T_N) \text{ and } \beta(T) = \beta(T_M) + \beta(T_N). \tag{1.10}$$

Consequently,

$$\alpha(T) < \infty \text{ if and only if } \alpha(T_M) < \infty \text{ and } \alpha(T_N) < \infty; \tag{1.11}$$
$$\beta(T) < \infty \text{ if and only if } \beta(T_M) < \infty \text{ and } \beta(T_N) < \infty. \tag{1.12}$$

Further, by [41, Lemma 3.3],

$$R(T) \text{ is closed if and only if } R(T_M) \text{ and } R(T_N) \text{ are closed.} \tag{1.13}$$

Let $\text{asc}(T) = p < \infty$. Then $N(T^p) = N(T^{p+1})$. Since for all $n \in \mathbb{N}_0$ we have $N((T_M)^n) = N(T^n) \cap M$, then $N((T_M)^p) = N(T^p) \cap M = N(T^{p+1}) \cap M = N((T_M)^{p+1})$, so $\text{asc}(T_M) < \infty$. Similarly, $\text{asc}(T_N) < \infty$. On the other hand, suppose that $\text{asc}(T_M) = p_1 < \infty$ and $\text{asc}(T_N) = p_2 < \infty$, and let $p = \max\{p_1, p_2\}$. We have $N(T^p) = N((T_M)^p) \oplus N((T_N)^p) = N((T_M)^{p+1}) \oplus N((T_N)^{p+1}) = N(T^{p+1})$, hence $\text{asc}(T) < \infty$. We have just proved

$$\text{asc}(T) < \infty \text{ if and only if } \text{asc}(T_M) < \infty \text{ and } \text{asc}(T_N) < \infty. \tag{1.14}$$

It is not difficult to show that $R((T_M)^n) = R(T^n) \cap M$ and $R((T_N)^n) = R(T^n) \cap N$ for all $n \in \mathbb{N}_0$. Using these facts and applying a similar method as above we obtain that

$$\text{dsc}(T) < \infty \text{ if and only if } \text{dsc}(T_M) < \infty \text{ and } \text{dsc}(T_N) < \infty. \tag{1.15}$$

Now, the result follows from (1.10), (1.11), (1.12), (1.13), (1.14) and (1.15). Moreover, $\text{ind}(T) = \alpha(T) - \beta(T) = (\alpha(T_M) + \alpha(T_N)) - (\beta(T_M) + \beta(T_N)) = \text{ind}(T_M) + \text{ind}(T_N)$.

(ii). Follows from (i).

(iii). Suppose that $T \in \mathcal{W}_+(\mathcal{X})$ and that T_N is Weyl. According to (i) it follows that $T_M \in \Phi_+(\mathcal{X})$ and $\text{ind}(T_M) = \text{ind}(T_M) + \text{ind}(T_N) = \text{ind}(T) \leq 0$. Thus T_M is upper semi-Weyl. The cases $i = 11$ and $i = 12$ can be proved similarly. □

Lemma 1.4.8. ([57, Lemma 20.9]) *Let $T \in L(\mathcal{X})$ be upper semi-Browder and Kato. Then T is bounded below. If T is lower semi-Browder and Kato, then T is surjective.*

The following proposition plays an important role in this dissertation.

Proposition 1.4.9. *Let $T \in L(\mathcal{X})$. Then the following implications hold:*
(i) *If T is Kato and $0 \in \operatorname{acc} \rho_{\Phi_+}(T)$ ($0 \in \operatorname{acc} \rho_{\Phi_-}(T)$), then T is upper (lower) semi-Fredholm;*
(ii) *If T is Kato and $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$ ($0 \in \operatorname{acc} \rho_{\mathcal{W}_-}(T)$), then T is upper (lower) semi-Weyl;*
(iii) *If T is Kato and $0 \in \operatorname{acc} \rho_{ap}(T)$ ($0 \in \operatorname{acc} \rho_{su}(T)$), then T is bounded below (surjective);*
(iv) *If T is Kato and $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$ ($0 \in \operatorname{acc} \rho_{\mathcal{B}_-}(T)$), then T is bounded below (surjective).*

Proof. (i). Suppose that T is Kato and $0 \in \operatorname{acc} \rho_{\Phi_+}(T)$ ($0 \in \operatorname{acc} \rho_{\Phi_-}(T)$). According to Lemma 1.3.5, there exists $\epsilon > 0$ such that $\alpha(T) = \alpha(T - \lambda I)$ and $\beta(T) = \beta(T - \lambda I)$ for $|\lambda| < \epsilon$. Also, there exists $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $T - \mu I$ is upper semi-Fredholm (lower semi-Fredholm), so $\alpha(T) = \alpha(T - \mu I) < +\infty$ ($\beta(T) = \beta(T - \mu I) < +\infty$). Since $R(T)$ is closed, T is upper semi-Fredholm (lower semi-Fredholm).
(ii). Suppose that T is Kato and $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$. Again Lemma 1.3.5 implies that there exists $\epsilon > 0$ such that $\alpha(T) = \alpha(T - \lambda I)$ and $\beta(T) = \beta(T - \lambda I)$ for $|\lambda| < \epsilon$. Since $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$, there exists $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $T - \mu I$ is upper semi-Weyl. Then, $\alpha(T) = \alpha(T - \mu I) < \infty$ and $\beta(T) = \beta(T - \mu I)$. In addition, $\operatorname{ind}(T) = \alpha(T) - \beta(T) = \alpha(T - \mu I) - \beta(T - \mu I) = \operatorname{ind}(T - \mu I) \leq 0$ and so T is upper semi-Weyl.

The second statement can be obtained similarly.
(iii). As above we conclude that there exists $\epsilon > 0$ such that $\alpha(T) = \alpha(T - \lambda I)$ ($\beta(T) = \beta(T - \lambda I)$) for $|\lambda| < \epsilon$. Also, there exists $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and that $T - \mu I$ is bounded below (surjective). Consequently, $\alpha(T) = \alpha(T - \mu I) = 0$ ($\beta(T) = \beta(T - \mu I) = 0$), so T is bounded below (surjective).
(iv). From Lemma 1.3.5 we see that there exists $\epsilon > 0$ such that $T - \lambda I$ is Kato and $\alpha(T) = \alpha(T - \lambda I)$ ($\beta(T) = \beta(T - \lambda I)$) for $|\lambda| < \epsilon$. Since $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$ ($0 \in \operatorname{acc} \rho_{\mathcal{B}_-}(T)$) it follows that there exists $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $T - \mu I$ is upper semi-Browder (lower semi-Browder). Lemma 1.4.8 implies that $T - \mu I$ is bounded below (surjective). Now, $\alpha(T) = \alpha(T - \mu I) = 0$ ($\beta(T) = \beta(T - \mu I) = 0$), and hence T is bounded below (surjective). $\square$

Corollary 1.4.10. *Let $T \in L(\mathcal{X})$. Then the following implications hold:*
(i) *If T is Kato and $0 \notin \operatorname{acc} \sigma_{\Phi_+}(T)$ ($0 \notin \operatorname{acc} \sigma_{\Phi_-}(T)$), then T is upper semi-Fredholm (lower semi-Fredholm);*
(ii) *If T is Kato and $0 \notin \operatorname{acc} \sigma_{\mathcal{W}_+}(T)$ ($0 \notin \operatorname{acc} \sigma_{\mathcal{W}_-}(T)$), then T is upper semi-Weyl (lower semi-Weyl);*
(iii) *If T is Kato and $0 \notin \operatorname{acc} \sigma_{ap}(T)$ ($0 \notin \operatorname{acc} \sigma_{su}(T)$), then T is bounded below (surjective);*
(iv) *If T is Kato and $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_+}(T)$ ($0 \notin \operatorname{acc} \sigma_{\mathcal{B}_-}(T)$), then T is bounded below (surjective).*

Proof. (i). By Proposition 1.1.1, $0 \notin \operatorname{acc} \sigma_{\Phi_+}(T)$ implies $0 \in \operatorname{acc} \rho_{\Phi_+}(T)$. The assertion follows from Proposition 1.4.9.

The remaining statements can be proved analogously. □

For $T \in L(\mathcal{X})$ using Proposition 1.4.9 we can easily proved that $\partial\sigma(T) \subset \sigma_K(T)$; see also [57, Theorem 12.11] and [78, Theorem 2.1]. In the following proposition $\sigma_l(T)$ and $\sigma_r(T)$ are respectively the left and right spectrum of T, see Section 1.5.

Proposition 1.4.11. *Let $T \in L(\mathcal{X})$. Then:*
(i) *$\partial\sigma(T) \subset \sigma_K(T) \subset \sigma(T)$;*
(ii) *$\sigma_K(T)$ is finite if and only if $\sigma(T)$ is finite, and in that case $\sigma_K(T) = \sigma(T)$;*
(iii) *$\sigma_l(T)$ is finite if and only if $\sigma(T)$ is finite, and in that case $\sigma_l(T) = \sigma(T)$;*
(iv) *$\sigma_r(T)$ is finite if and only if $\sigma(T)$ is finite, and in that case $\sigma_r(T) = \sigma(T)$.*

Proof. (i). It is sufficient to prove $\partial\sigma(T) \subset \sigma_K(T)$. Suppose that there exists $\lambda_0 \in \partial\sigma(T)$ but such that $\lambda_0 \notin \sigma_K(T)$. It means that $T - \lambda_0 I$ is Kato, and $\lambda_0 \in \operatorname{acc} \rho(T)$. Consequently, $\lambda_0 \in \operatorname{acc} \rho_{ap}(T)$ and $\lambda_0 \in \operatorname{acc} \rho_{su}(T)$. According to Proposition 1.4.9, $T - \lambda_0 I$ is both bounded below and surjective, i.e. $T - \lambda_0 I$ is invertible. But this contradicts our assumption since $\lambda_0 \in \partial\sigma(T) \subset \sigma(T)$.
(ii). Apply Proposition 1.1.3 and Lemma 1.1.4.
(iii) and (iv). $\sigma_K(T) \subset \sigma_l(T) \cap \sigma_r(T)$ and (i) imply $\partial\sigma(T) \subset \sigma_l(T) \subset \sigma(T)$ and $\partial\sigma(T) \subset \sigma_r(T) \subset \sigma(T)$. As above, we apply Proposition 1.1.3 and Lemma 1.1.4, and obtain the desired conclusions. □

A point $\lambda_0 \in \sigma(T)$ is a Riesz point of $T \in L(\mathcal{X})$ if $\lambda_0 \in \operatorname{iso} \sigma(T)$ and if the spectral projection corresponding to λ_0 has finite-dimensional range.

Definition 1.4.12. An operator $T \in L(\mathcal{X})$ is Riesz if every nonzero point of $\sigma(T)$ is a Riesz point of T.

Lemma 1.4.13. ([79, Lemma 2.11]) *Let $T \in L(\mathcal{X})$ and let $(M, N) \in Red(T)$. Then T is Riesz if and only if T_M and T_N are Riesz.*

For $T \in L(\mathcal{X})$ and $n \in \mathbb{N}$, define $T_n : R(T^n) \to R(T^n)$ by $T_n x = Tx$, $x \in R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm operator, then T is called a B-Fredholm operator. In addition, if $\operatorname{ind}(T_n) = 0$ we say that T is a B-Weyl operator. The classes of B-Fredholm and B-Weyl operators are denoted by $\mathbf{B\Phi}(\mathcal{X})$ and $\mathbf{B}\mathcal{W}(\mathcal{X})$ respectively. M. Berkani introduced and characterized B-Fredholm and B-Weyl operators [7, 9].

Theorem 1.4.14. ([7, Theorem 2.7]) *Let $T \in L(\mathcal{X})$. Then T is a B-Fredholm operator if and only if there exist two closed subspaces M and N such that $\mathcal{X} = M \oplus N$ and:*
(i) *$T(N) \subset N$ and T_N is a nilpotent operator;*
(ii) *$T(M) \subset M$ is a Fredholm operator.*

Theorem 1.4.15. ([9, Lemma 4.1]) *Let $T \in L(\mathcal{X})$. Then T is a B-Weyl operator if and only if $T = T_0 \oplus T_1$, where T_0 is a Weyl operator and T_1 is a nilpotent operator.*

1.5 Drazin and generalized Drazin inverse

Let $\mathcal{A}$ denote a complex unital Banach algebra with identity 1. Many notions concerning the bounded linear operators can be extended in the context of Banach algebras. We say that $a \in \mathcal{A}$ is left (right) invertible if there exists $b \in \mathcal{A}$ such that $ba = 1$ ($ab = 1$). An element $a \in \mathcal{A}$ is invertible if it is both left and right invertible. It is easy to see that in this case there exists a unique $b \in \mathcal{A}$ such that $ab = ba = 1$. Let $\mathcal{A}^{-1}$, $\mathcal{A}^{-1}_{left}$ and $\mathcal{A}^{-1}_{right}$ denote the set of all invertible elements, the set of all left invertible elements and the set of all right invertible elements, respectively. The spectrum, the left spectrum and the right spectrum of $a \in \mathcal{A}$ are respectively the following sets

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin \mathcal{A}^{-1}\},$$
$$\sigma_l(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin \mathcal{A}^{-1}_{left}\},$$
$$\sigma_r(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin \mathcal{A}^{-1}_{right}\}.$$

The spectra $\sigma(a)$, $\sigma_l(a)$ and $\sigma_r(a)$ are non-empty and compact subsets of $\mathbb{C}$. It is said that $a \in \mathcal{A}$ is nilpotent (quasinilpotent) if $a^n = 0$ for some $n \in \mathbb{N}$ (if $\sigma(a) = \{0\}$). Every nilpotent element is quasinilpotent. An element $p \in \mathcal{A}$ is an idempotent if $p^2 = p$. An idempotent p is nontrivial if $p \notin \{0, 1\}$. The set of all nilpotent elements, the set of all quasinilpotent elements and the set of all idempotents on $\mathcal{A}$ is denoted by $\mathcal{A}^{nil}$, $\mathcal{A}^{qnil}$ and $\mathcal{A}^{\bullet}$, respectively. Also, $(\lambda 1 - a)^{-1}$ is an analytic function on $\mathbb{C} \setminus \sigma(a)$ and if $\lambda_0 \in \operatorname{iso}\sigma(a)$ then by

$$p = \frac{1}{2\pi i}\int_C (\lambda 1 - a)^{-1} d\lambda$$

is given the spectral idempotent of a corresponding to λ_0, where C is again a circle centered at λ_0 which separates λ_0 from the set $\sigma(a) \setminus \{\lambda_0\}$.

The concept of Drazin invertibility was originally introduced by M. P. Drazin in [27] for elements of an associative ring. We recall his definition, but for our purpose it is sufficient to consider the Banach algebra case.

Definition 1.5.1. An element $a \in \mathcal{A}$ is said to be Drazin invertible if there exists an element $b \in \mathcal{A}$ and some $k \in \mathbb{N}$ such that

$$ab = ba, \quad bab = b, \quad a^k ba = a^k.$$

The least $k \in \mathbb{N}$ such that the above equations hold is the index of a. If $a \in \mathcal{A}$ is invertible, then the index of a is 0. The element b is a Drazin inverse of a, and it is denoted by a^D.

If $a \in \mathcal{A}$ is Drazin invertible then its Drazin inverse is unique, and explicit expression for a^D is given in the following proposition.

Proposition 1.5.2. ([64, Proposition 1]) *Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ is Drazin invertible of degree k if and only if there exists an idempotent $p \in \mathcal{A}$ such that:*

$$ap = pa, \qquad a + p \text{ is invertible}, \qquad a^k p = 0. \tag{1.16}$$

If (1.16) *is satisfied, then $a^D = (a+p)^{-1}(1-p)$.*

If $\mathcal{A} = L(\mathcal{X})$ it is possible to give another sufficient and necessary conditions such that $T \in L(\mathcal{X})$ is Drazin invertible, and in the following theorem we collect some of them.

Theorem 1.5.3. *Let $T \in L(\mathcal{X})$ and $0 \in \sigma(T)$. The following statements are equivalent:*
(i) *T is Drazin invertible;*
(ii) $\mathrm{asc}(T) < \infty$ *and* $\mathrm{dsc}(T) < \infty$*;*
(iii) $0 \in \Pi(T)$*;*
(iv) *There exists $(M, N) \in Red(T)$ such that T_M is invertible and T_N is nilpotent.*

Proof. (i) $\iff$ (ii) is proved in [47, Theorem 4]. For (iii) $\implies$ (ii) see [67, Theorem 5.8-A], while (ii) $\implies$ (iii) is shown in [53, Thereom 2.1].
(ii) $\implies$ (iv). Let $\mathrm{asc}(T) < \infty$ and $\mathrm{dsc}(T) < \infty$. It is a classical result that $\mathrm{asc}(T) = \mathrm{dsc}(T) = p < \infty$, and $\mathcal{X} = R(T^p) \oplus N(T^p)$ with $R(T^p)$ and $N(T^p)$ closed; see for example [68, Theorem 3.7] and [30, Theorem IV.1.12]. Clearly, $R(T^p)$ and $N(T^p)$ are T-invariant, $T_{R(T^p)}$ is invertible and $T_{N(T^p)}$ is nilpotent. We set $M = R(T^p)$ and $N = N(T^p)$, and the implication follows; see also [64, Proposition 6].
(iv) $\implies$ (ii). Let $T = T_M \oplus T_N$ with T_M invertible and T_N nilpotent of degree d. Then, $\mathrm{asc}(T_M) = \mathrm{dsc}(T_M) = 0$ and $\mathrm{asc}(T_N) = \mathrm{dsc}(T_N) = d$. According to (1.14) and (1.15), $\mathrm{asc}(T) < \infty$ and $\mathrm{dsc}(T) < \infty$. □

Definition 1.5.4. Let $T \in L(\mathcal{X})$.
(i) T is left Drazin invertible if $\mathrm{asc}(T) < \infty$ and $R(T^{\mathrm{asc}(T)+1})$ is closed;
(ii) T is right Drazin invertible if $\mathrm{dsc}(T) < \infty$ and $R(T^{\mathrm{dsc}(T)})$ is closed.

Left and right Drazin invertible operators acting on a Hilbert space are characterized by M. Berkani [8, Theorem 3.12].

Theorem 1.5.5. ([8, Theorem 3.12]) *Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H})$. Then T is left (right) Drazin invertible if and only if there exists $(M, N) \in Red(T)$ such that T_M is bounded below (surjective) and T_N is nilpotent.*

The concept of Drazin invertibility were generalized by J. Koliha [48].

Definition 1.5.6. ([48, Definition 4.1]) An element $a \in \mathcal{A}$ is said to be generalized Drazin invertible if there exists an element $b \in \mathcal{A}$ such that

$$ab = ba, \quad bab = b, \quad aba - a \in \mathcal{A}^{qnil}.$$

The element b is a generalized Drazin inverse of a, and it will be denoted by $b = a^d$.

The following theorem gives necessary and sufficient conditions for the existence of a generalized Drazin inverse in a Banach algebra.

Theorem 1.5.7. ([48, Theorem 4.2]) *Let $\mathcal{A}$ be a Banach algebra. The following conditions on an element $a \in \mathcal{A}$ are equivalent:*

(i) *a is generalized Drazin invertible;*

(ii) *There exists an idempotent $p \in \mathcal{A}$ commuting with a such that $a+p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$;*

(iii) $0 \notin \operatorname{acc} \sigma(a)$.

In this case the generalized Drazin inverse is unique, and is given by $a^d = (a+p)^{-1}(1-p)$, where p is the spectral idempotent of a corresponding to 0.

Again, the Banach algebra $L(\mathcal{X})$ deserves a special attention. Before we proceed, we need to recall definitions of two important subspaces of $\mathcal{X}$ corresponding to every $T \in L(\mathcal{X})$.

The quasinilpotent part $H_0(T)$ of an operator $T \in L(\mathcal{X})$ is defined by

$$H_0(T) = \{x \in \mathcal{X} : \lim_{n \to +\infty} \|T^n x\|^{1/n} = 0\}.$$

It is easy to verify that $H_0(T) = \{0\}$ if T is bounded below. An operator $T \in L(\mathcal{X})$ is quasinilpotent if and only if $H_0(T) = \mathcal{X}$ [1, Theorem 1.68].

The analytical core of T, denoted by $K(T)$, is the set of all $x \in \mathcal{X}$ for which there exist $c > 0$ and a sequence (x_n) in $\mathcal{X}$ satisfying

$$Tx_1 = x, \;\; Tx_{n+1} = x_n \;\text{ for all }\; n \in \mathbb{N}, \;\; \|x_n\| \leq c^n \|x\| \;\text{ for all }\; n \in \mathbb{N}.$$

If T is surjective, then $K(T) = \mathcal{X}$ [1, Theorem 1.22].

Theorem 1.5.8. ([1, Theorem 1.41, Corollary 1.69]) *Suppose that $T \in L(\mathcal{X})$ admits a GKD (M, N). Then:*

(i) *$H_0(T) = H_0(T_M) \oplus H_0(T_N)$;*

(ii) *$K(T) = K(T_M)$ and $K(T)$ is closed.*

Theorem 1.5.9. *Let $T \in L(\mathcal{X})$. The following conditions are equivalent:*
(i) *T is generalized Drazin invertible;*
(ii) *There is a bounded projection P on $\mathcal{X}$ such that $R(P) = H_0(T)$ and $N(T) = K(T)$;*
(iii) *There exists $(M, N) \in Red(T)$ such that T_M is invertible and T_N is quasinilpotent;*
(iv) *$\mathcal{X} = K(T) \oplus H_0(T)$ with at least one of the component spaces closed.*

If T is generalized Drazin invertible, then the subspaces M and N from Condition (iii) *are uniquely determined: $M = K(T)$ and $N = H_0(T)$.*

Proof. (i) $\Longleftrightarrow$ (ii) is proved in [55, Théorème 1.6]. For (i) $\Longleftrightarrow$ (iv) see [66, Theorem 4] and [23, Theorem 6.7], and for (i) $\Longrightarrow$ (iii) see [48, Theorem 7.1].
(iii) $\Longrightarrow$ (i). There exists $\epsilon > 0$ such that $T_M - \lambda I$ and $T_N - \lambda I$ are invertible for $0 < |\lambda| < \epsilon$. According to Lemma 1.4.7, $T - \lambda I$ is invertible for $0 < |\lambda| < \epsilon$. It means that $0 \notin \operatorname{acc}\sigma(T)$, and T is generalized Drazin invertible by Theorem 1.5.7.

If T is generalized Drazin invertible then there exists $(M, N) \in Red(T)$ such that T_M is invertible and T_N quasinilpotent. Since (M, N) is a GKD for T, Theorem 1.5.8 implies

$$H_0(T) = H_0(T_M) \oplus H_0(T_N) \quad \text{and} \quad K(T) = K(T_M).$$

We note that $H_0(T_M) = \{0\}$, $H_0(T_N) = N$ and $K(T_M) = M$. Consequently, $M = K(T)$ and $N = H_0(T)$. $\square$

The set of all Drazin invertible elements and generalized Drazin invertible elements of a Banach algebra $\mathcal{A}$ will be denoted by $\mathcal{A}^D$ and $\mathcal{A}^{gD}$, respectively. The Drazin and generalized Drazin spectrum of $a \in \mathcal{A}$ are respectively the sets

$$\sigma_D(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin \mathcal{A}^D\} \quad \text{and} \quad \sigma_{gD}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin \mathcal{A}^{gD}\}.$$

The Drazin spectrum $\sigma_D(a)$ is compact [12, Proposition 2.5]. From Theorem 1.5.7 it follows that $\sigma_{gD}(a) = \operatorname{acc}\sigma(a)$, hence $\sigma_{gD}(a)$ is also compact. Unlike the spectrum of a, $\sigma_D(a)$ and $\sigma_{gD}(a)$ may be empty sets. For instance, if a is nilpotent element of $\mathcal{A}$ or is an idempotent, then $\sigma_D(a) = \emptyset$; $\sigma_{gD}(a) = \emptyset$ if a is quasinilpotent. The Drazin and generalized Drazin resolvent set of $a \in \mathcal{A}$ are defined by $\rho_D(a) = \mathbb{C} \setminus \sigma_D(a)$ and $\rho_{gD}(a) = \mathbb{C} \setminus \sigma_{gD}(a)$, respectively.

We give a brief exposition of the axiomatic theory of spectrum; see [49, 57, 58].

Definition 1.5.10. Let $\mathcal{A}$ be a Banach algebra. A non-empty subset $\mathcal{R}$ of $\mathcal{A}$ is called a regularity if it satisfies the following conditions:
(i) If $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in \mathcal{R}$ if and only if $a^n \in \mathcal{R}$;
(ii) If $a, b, c, d \in \mathcal{A}$ are mutually commuting elements satisfying $ac + bd = 1$, then necessary and sufficient for $ab \in \mathcal{R}$ is that $a \in \mathcal{R}$ and $b \in \mathcal{R}$.

According to [12, Theorem 2.3] and [54, Theorem 1.2], the sets $\mathcal{A}^D$ and $\mathcal{A}^{gD}$ are regularities.

Proposition 1.5.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T}:\mathcal{A}\to\mathcal{B}$. If $\mathcal{R}$ is a regularity in $\mathcal{B}$, then $\mathcal{T}^{-1}(\mathcal{R})$ is a regularity in $\mathcal{A}$.*

Proof. Since $\mathcal{T}(1)=1\in\mathcal{R}$ [57, Proposition 6.2], we see that $1\in\mathcal{T}^{-1}(\mathcal{R})$, so $\mathcal{T}^{-1}(\mathcal{R})$ is a non-empty subset of $\mathcal{A}$. For $a\in\mathcal{A}$ and $n\in\mathbb{N}$ we have

$$a\in\mathcal{T}^{-1}(\mathcal{R})\Leftrightarrow\mathcal{T}(a)\in\mathcal{R}\Leftrightarrow\mathcal{T}(a)^n\in\mathcal{R}\Leftrightarrow\mathcal{T}(a^n)\in\mathcal{R}\Leftrightarrow a^n\in\mathcal{T}^{-1}(\mathcal{R}).$$

It is also very easy to verify that $\mathcal{T}^{-1}(\mathcal{R})$ satisfies condition (ii) of Definition 1.5.10, so $\mathcal{T}^{-1}(\mathcal{R})$ is a regularity in $\mathcal{A}$. $\square$

Given a regularity $\mathcal{R}\subset\mathcal{A}$, it is possible to define the spectrum of $a\in\mathcal{A}$ corresponding to $\mathcal{R}$ as

$$\sigma_{\mathcal{R}}(a)=\{\lambda\in\mathbb{C}\colon a-\lambda 1\notin\mathcal{R}\}.$$

Let $a\in\mathcal{A}$ and let f be an analytic function on a neighbourhood U of $\sigma(a)$. It is possible to define $f(a)$ by

$$f(a)=\frac{1}{2\pi i}\int_\Gamma f(z)(z1-a)^{-1}dz,$$

where Γ is a contour surrounding $\sigma(a)$ in U; for details see [57]. Every spectrum defined by a regularity satisfies the spectral mapping theorem [57, Theorem 6.7].

Theorem 1.5.12. (spectral mapping theorem) *Let $\mathcal{R}$ be a regularity in a Banach algebra $\mathcal{A}$ and let $\sigma_{\mathcal{R}}$ be the corresponding spectrum. Then*

$$\sigma_{\mathcal{R}}(f(a))=f(\sigma_{\mathcal{R}}(a))$$

for every $a\in\mathcal{A}$ and every function f analytic on a neighbourhood of $\sigma(a)$ which is non-constant on each component of its domain of definition.

Chapter 2

GKD and spectra originating from Fredholm theory

In this chapter we study an operator $T \in L(\mathcal{X})$ which can be decomposed by $T = T_M \oplus T_N$, where $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. Clearly, such an operator admits a generalized Kato decomposition. Lemma 1.4.7 and Proposition 1.4.9 enable us to approach the problem in a unified way. It means that we may study all decompositions mentioned above (for any $\mathbf{R}_i$) by using the same method.

2.1 The classes $\mathbf{gDR}_i$

We consider the following classes of bounded linear operators:

$$\mathbf{gDR}_i = \left\{ T \in L(\mathcal{X}) : \begin{array}{c} \text{there exists } (M, N) \in Red(T) \text{ such that} \\ T_M \in \mathbf{R}_i \text{ and } T_N \text{ is quasinilpotent} \end{array} \right\}, 1 \leq i \leq 12.$$

Proposition 2.1.1. *Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. If T belongs to the set $\mathbf{gDR}_i$, then $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$.*

Proof. Let $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. Since $\mathbf{R}_i$ is open, there exists $\epsilon > 0$ such that $(T - \lambda I)_M = T_M - \lambda I_M \in \mathbf{R}_i$ for $|\lambda| < \epsilon$. On the other hand, $(T - \lambda I)_N = T_N - \lambda I_N \in L(N)^{-1} \subset \mathbf{R}_i$ for every $\lambda \neq 0$. Now, by applying Lemma 1.4.7 we obtain that $T - \lambda I \in \mathbf{R}_i$ for $0 < |\lambda| < \epsilon$, and so $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$. $\square$

Theorem 2.1.2. *Let $T \in L(\mathcal{X})$ and $7 \leq i \leq 12$. The following conditions are equivalent:*

(i) *There exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent, that is $T \in \mathbf{gDR}_i$;*

(ii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$;*

(iii) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$;*

(iv) *There exists a projection $P \in L(\mathcal{X})$ that commutes with T such that $T + P \in \mathbf{R}_i$ and TP is quasinilpotent.*

Proof. (i) $\Longrightarrow$ (ii). Let $T = T_M \oplus T_N$, where $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. Then $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$ by Proposition 2.1.1. From [57, Theorem 16.21] it follows that there exist two closed T-invariant subspaces M_1 and M_2 such that $M = M_1 \oplus M_2$, M_2 is finite dimensional, T_{M_1} is Kato and T_{M_2} is nilpotent. We have $\mathcal{X} = M_1 \oplus (M_2 \oplus N)$, $M_2 \oplus N$ is closed, $T_{M_2 \oplus N} = T_{M_2} \oplus T_N$ is quasinilpotent and thus T admits the GKD $(M_1, M_2 \oplus N)$.

(ii) $\Longrightarrow$ (iii). Clear.

(iii) $\Longrightarrow$ (i). Let $i \in \{7, 8, 9\}$. Assume that T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$, that is $0 \in \operatorname{acc} \rho_{\mathbf{R}_i}(T)$. Then there exists $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent. Since $0 \in \operatorname{acc} \rho_{\mathbf{R}_i}(T)$, according to Lemma 1.4.7(i), it follows that $0 \in \operatorname{acc} \rho_{\mathbf{R}_i}(T_M)$. From Proposition 1.4.9(i) it follows that $T_M \in \mathbf{R}_i$, and hence $T \in \mathbf{gDR}_i$.

Suppose that T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathcal{W}_+}(T)$, i.e. $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$. Then there exists $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent. We will show that $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T_M)$. Let $\epsilon > 0$. From $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$ it follows that there exists $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$ and $T - \lambda I \in \mathcal{W}_+(X)$. As T_N is quasinilpotent, $T_N - \lambda I_N$ is invertible, and according to Lemma 1.4.7(iii), we conclude that $T_M - \lambda I_M \in \mathcal{W}_+(M)$, that is $\lambda \in \rho_{\mathcal{W}_+}(T_M)$. Therefore, $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T_M)$ and from Proposition 1.4.9(ii) it follows that T_M is upper semi-Weyl, and so $T \in \mathbf{gD}\mathcal{W}_+(\mathcal{X})$. The cases $i = 11$ and $i = 12$ can be proved similarly.

(i) $\Longrightarrow$ (iv). Suppose that there exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. Let $P \in L(\mathcal{X})$ be a projection such that $N(P) = M$ and $R(P) = N$. Then $TP = PT$ and every element $x \in \mathcal{X}$ may be represented as $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in N$. Also,

$$\|(TP)^n x\|^{\frac{1}{n}} = \|T^n P x\|^{\frac{1}{n}} = \|(T_N)^n x_2\|^{\frac{1}{n}} \to 0 \ (n \to \infty),$$

since T_N is quasinilpotent. We obtain $H_0(TP) = \mathcal{X}$, so TP is quasinilpotent. Since $(T + P)_M = T_M$ and $(T + P)_N = T_N + I_N \in L(N)^{-1}$, we have that $(T + P)_M \in \mathbf{R}_i$ and $(T + P)_N \in \mathbf{R}_i$, and hence $T + P \in \mathbf{R}_i$ by Lemma 1.4.7(i) and (ii).

(iv) $\Longrightarrow$ (i). Assume that there exists a projection $P \in L(\mathcal{X})$ that commutes with T such that $T + P \in \mathbf{R}_i$ and TP is quasinilpotent. Put $N(P) = M$ and $R(P) = N$. Then $\mathcal{X} = M \oplus N$, $T(M) \subset M$ and $T(N) \subset N$. For every $x \in N$ we have

$$\|(T_N)^n x\|^{\frac{1}{n}} = \|T^n P^n x\|^{\frac{1}{n}} = \|(TP)^n x\|^{\frac{1}{n}} \to 0 \ (n \to \infty),$$

since TP is quasinilpotent. It follows that $H_0(T_N) = N$, so T_N is quasinilpotent. It remains to prove that $T_M \in \mathbf{R}_i$. For $i \in \{7, 8, 9\}$, by

Lemma 1.4.7(i) we deduce that $T_M = (T+P)_M \in \mathbf{R}_i$. Set $i = 10$. Since T_N is quasinilpotent, it follows that $T_N + I_N$ is invertible. From $T + P \in \mathcal{W}_+(\mathcal{X})$ and the decomposition

$$T + P = (T + P)_M \oplus (T + P)_N = T_M \oplus (T_N + I_N),$$

according to Lemma 1.4.7(iii), we conclude that $T_M \in \mathcal{W}_+(M)$. For $i = 10$ and $i = 12$ we apply a similar consideration. $\square$

In the case that $T - \lambda_0 I \in L(\mathcal{X})$ admits a GKD, Q. Jiang and H. Zhong gave a characterization of the SVEP at λ_0 by using the approximate point spectrum of T [41].

Theorem 2.1.3. ([41, Theorem 3.5, Theorem 3.9]) *Suppose that $T - \lambda_0 I \in L(\mathcal{X})$ admits a GKD. Then the following statements are equivalent:*

(i) *T has the SVEP at λ_0 (T' has the SVEP at λ_0);*

(ii) *$\sigma_{ap}(T)$ does not cluster at λ_0 ($\sigma_{su}(T)$ does not cluster at λ_0);*

(iii) *λ_0 is not an interior point of $\sigma_{ap}(T)$ (λ_0 is not an interior point of $\sigma_{su}(T)$).*

We extend Theorem 2.1.3 in two directions. Firstly, in Theorems 2.1.4 and 2.1.5 we provide further conditions equivalent to Conditions (i)-(iii) of Theorem 2.1.3. Secondly, under the hypotheses of Theorem 2.1.3, we show that the equivalence (ii) $\Longleftrightarrow$ (iii) remains valid in the case of essential spectra ($4 \le i \le 12$), see Corollary 2.1.7 below.

Theorem 2.1.4. *Let $T \in L(\mathcal{X})$. The following conditions are equivalent:*

(i) *$H_0(T)$ is closed and there exists a closed subspace M of $\mathcal{X}$ such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed;*

(ii) *There exists $(M, N) \in Red(T)$ such that T_M is bounded below and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{M}(\mathcal{X})$;*

(iii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{ap}(T)$;*

(iv) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{ap}(T)$;*

(v) *There exists a bounded projection P on $\mathcal{X}$ which commutes with T such that $T + P$ is bounded below and TP is quasinilpotent;*

(vi) *There exists $(M, N) \in Red(T)$ such that T_M is upper semi-Browder and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{B}_+(\mathcal{X})$;*

(vii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_+}(T)$;*

(viii) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathcal{B}_+}(T)$;*

(ix) *There exists a bounded projection P on $\mathcal{X}$ which commutes with T such that $T + P$ is upper semi-Browder and TP is quasinilpotent.*

In particular, if T satisfies any of Conditions (i)-(ix)*, then the subspace N in* (ii) *is uniquely determined and $N = H_0(T)$.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that $H_0(T)$ is closed and that there exists a closed T-invariant subspace M of $\mathcal{X}$ such that $\mathcal{X} = H_0(T) \oplus M$ and $T(M)$ is closed. For $N = H_0(T)$ we have that $(M, N) \in \mathit{Red}(T)$ and $H_0(T_N) = N$, which implies that T_N is quasinilpotent. From $N(T_M) = N(T) \cap M \subset H_0(T) \cap M = \{0\}$ it follows that T_M is injective and since $R(T_M) = T(M)$ is a closed subspace in M, we conclude that T_M is bounded below.

(ii) $\Longrightarrow$ (i). Assume that there exists $(M, N) \in \mathit{Red}(T)$ such that T_M is bounded below and T_N is quasinilpotent. Then (M, N) is a GKD for T, and so from Theorem 1.5.8 it follows that $H_0(T) = H_0(T_M) \oplus H_0(T_N) = H_0(T_M) \oplus N$. Since T_M is bounded below, we get that $H_0(T_M) = \{0\}$ and hence $H_0(T) = N$. Therefore, $H_0(T)$ is closed and complemented with M, $(M, H_0(T)) \in \mathit{Red}(T)$, and $T(M)$ is closed because T_M is bounded below.

(ii) $\Longrightarrow$ (iii). Since T_M is bounded below then it is Kato, hence T admits a GKD. Applying Proposition 2.1.1 we obtain that $0 \notin \operatorname{acc} \sigma_{ap}(T)$.

(vi) $\Longrightarrow$ (vii) can be proved analogously to the proof of the implication (i) $\Longrightarrow$ (ii) in Theorem 2.1.2. The implications (iii) $\Longrightarrow$ (iv) and (vii) $\Longrightarrow$ (viii) are clear.

(viii) $\Longrightarrow$ (ii). Let T admit a GKD and let $0 \notin \operatorname{int} \sigma_{\mathcal{B}_+}(T)$, i.e. $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$. There exists $(M, N) \in \mathit{Red}(T)$ such that T_M is Kato and T_N is quasinilpotent. From $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$ it follows that $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T_M)$ according to Lemma 1.4.7(i). From Proposition 1.4.9(iv) it follows that T_M is bounded below, and hence $T \in \mathbf{gD}\mathcal{M}(\mathcal{X})$.

(iv) $\Longrightarrow$ (ii). This implication can be proved by using Proposition 1.4.9(iii), analogously to the proof of the implication (viii) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (vi). Follows from the fact that every bounded below operator is upper semi-Browder.

The equivalences (v) $\Longleftrightarrow$ (ii) and (vi) $\Longleftrightarrow$ (ix) can be proved analogously to the equivalence (i) $\Longleftrightarrow$ (iv) in Theorem 2.1.2. ☐

Theorem 2.1.5. *For $T \in L(\mathcal{X})$ the following conditions are equivalent:*

(i) *$K(T)$ is closed and there exists a closed subspace N of $\mathcal{X}$ such that $N \subset H_0(T)$ and $(K(T), N) \in \mathit{Red}(T)$;*

(ii) *There exists $(M, N) \in \mathit{Red}(T)$ such that T_M is surjective and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{Q}(\mathcal{X})$;*

(iii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{su}(T)$;*

(iv) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{su}(T)$;*

(v) *There exists a bounded projection P on $\mathcal{X}$ which commutes with T such that $T + P$ is surjective and TP is quasinilpotent;*

(vi) *There exists $(M, N) \in \mathit{Red}(T)$ such that T_M is lower semi-Browder and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{B}_-(\mathcal{X})$;*

(vii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_-}(T)$;*

(viii) *T admits a GKD and* $0 \notin \operatorname{int} \sigma_{\mathcal{B}_-}(T)$*;*

(ix) *There exists a bounded projection P on $\mathcal{X}$ which commutes with T such that $T + P$ is lower semi-Browder and TP is quasinilpotent.*

In particular, if T satisfies any of Conditions (i)-(ix)*, then the subspace M in* (ii) *is uniquely determined: $M = K(T)$.*

Proof. (i) $\Longrightarrow$ (ii). Assume that $K(T)$ is closed and that there exists a closed T-invariant subspace N, such that $N \subset H_0(T)$ and $\mathcal{X} = K(T) \oplus N$. For $M = K(T)$ we have that $(M, N) \in Red(T)$, $R(T_M) = R(T) \cap M = R(T) \cap K(T) = K(T) = M$, and so T_M is surjective. Since $H_0(T_N) = H_0(T) \cap N = N$, we conclude that T_N is quasinilpotent.

(ii) $\Longrightarrow$ (i). Suppose that there exists $(M, N) \in Red(T)$ such that T_M is surjective and T_N is quasinilpotent. Then (M, N) is a GKD for T and from Theorem 1.5.8 we see that $K(T) = K(T_M)$. Since T_M is surjective, it follows that $K(T_M) = M$, and so $K(T) = M$ and $K(T)$ is closed. Thus $(K(T), N) \in Red(T)$ and since T_N is quasinilpotent, we have that $N = H_0(T_N) \subset H_0(T)$.

The rest of the proof is similar to the proofs of Theorems 2.1.4 and 2.1.2. □

In the following theorem we characterize generalized Drazin invertible operators.

Theorem 2.1.6. *Let $T \in L(\mathcal{X})$. The following conditions are equivalent:*

(i) *T is generalized Drazin invertible;*

(ii) *T admits a GKD and* $0 \notin \operatorname{int} \sigma(T)$*;*

(iii) *T admits a GKD and* $0 \notin \operatorname{int} \sigma_{\mathcal{B}}(T)$*;*

(iv) *T admits a GKD and* $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}(T)$*;*

(v) *There exists $(M, N) \in Red(T)$ such that T_M is Browder and T_N is quasinilpotent;*

(vi) *There exists a bounded projection P on $\mathcal{X}$ which commutes with T such that $T + P$ is Browder and TP is quasinilpotent.*

Proof. Similar to the proof of Theorem 2.1.4. □

Corollary 2.1.7. *Let $T \in L(\mathcal{X})$ and $\lambda_0 \in \mathbb{C}$. If $T - \lambda_0 I$ admits a GKD and $4 \leq i \leq 12$, then the following statements are equivalent:*

(i) *$\sigma_{\mathbf{R}_i}(T)$ does not cluster at λ_0;*

(ii) *λ_0 is not an interior point of $\sigma_{\mathbf{R}_i}(T)$.*

Proof. Follows from the equivalence (ii)$\Longleftrightarrow$(iii) of Theorem 2.1.2, equivalences (vii) $\Longleftrightarrow$ (viii) of Theorems 2.1.4 and 2.1.5, and from the equivalence (iii) $\Longleftrightarrow$ (iv) of Theorem 2.1.6. □

Corollary 2.1.8. *Let $T \in L(\mathcal{X})$ and let $0 \in \partial\sigma_{\mathbf{R}_i}(T)$, $1 \leq i \leq 12$. Then T admits a generalized Kato decomposition if and only if T belongs to the class* $\mathbf{gDR}_i$.

Proof. Follows from the equivalence (i) $\Longleftrightarrow$ (iii) of Theorem 2.1.2, the equivalences (ii) $\Longleftrightarrow$ (iv) and (vi) $\Longleftrightarrow$ (viii) of Theorems 2.1.4 and 2.1.5, and from the equivalence (i) $\Longleftrightarrow$ (ii) of Theorem 2.1.6. $\square$

We observe that the statement of Corollary 2.1.8 for $i = 3$ has already been proved in [41, Theorem 3.8].

Proposition 2.1.9. *Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. If $T \in \mathbf{gDR}_i$, then $T^n \in \mathbf{gDR}_i$ for every $n \in \mathbb{N}$.*

Proof. Let $1 \leq i \leq 12$ and $n \in \mathbb{N}$. If $T \in \mathbf{gDR}_i$ then there exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. It implies $T^n = (T_M)^n \oplus (T_N)^n$, $(T_M)^n \in \mathbf{R}_i$, and $(T_N)^n$ is quasinilpotent. Consequently, $T^n \in \mathbf{gDR}_i$. $\square$

Remark 2.1.10. Let $T \in L(\mathcal{X})$ and suppose that p is a nontrivial complex polynomial. According to [34, Theorem 2] and [39, Lemma 2.3.2], $\operatorname{acc} \sigma(p(T)) = p(\operatorname{acc} \sigma(T))$. Analysis similar to that in the proof of [39, Lemma 2.3.2] shows that $\operatorname{acc} \sigma_{\mathbf{R}_i}(p(T)) = p(\operatorname{acc} \sigma_{\mathbf{R}_i}(T))$, $1 \leq i \leq 9$. Indeed, let $\lambda \in \operatorname{acc} \sigma_{\mathbf{R}_i}(p(T))$. By the spectral mapping theorem, $\sigma_{\mathbf{R}_i}(p(T)) = p(\sigma_{\mathbf{R}_i}(T))$, $1 \leq i \leq 9$. Consequently, $\lambda \in \operatorname{acc} p(\sigma_{\mathbf{R}_i}(T))$, and hence $\lambda \neq p(s_n) \to \lambda \, (n \to \infty)$ for some sequence (s_n) in $\sigma_{\mathbf{R}_i}(T)$. By compactness of $\sigma_{\mathbf{R}_i}(T)$, there is a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} \to s \, (k \to \infty)$. We obtain $s \in \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$ and $p(s_{n_k}) \to p(s) \, (k \to \infty)$ since p is continuous. It means $\lambda = p(s) \in p(\operatorname{acc} \sigma_{\mathbf{R}_i}(T))$.

Conversely, assume that $\lambda \in p(\operatorname{acc} \sigma_{\mathbf{R}_i}(T))$. Then there exists an element $s \in \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$ such that $\lambda = p(s)$ and there is a sequence (s_n) in $\sigma_{\mathbf{R}_i}(T)$ such that $s_n \neq s$ for each $n \in \mathbb{N}$, and $s_n \to s \, (n \to \infty)$. Because $\sigma_{\mathbf{R}_i}(T)$ is closed, $s \in \sigma_{\mathbf{R}_i}(T)$. Also, the sequence $(p(s_n))$ converges to $p(s)$ since p is continuous. By the spectral mapping theorem, $p(s) \in \sigma_{\mathbf{R}_i}(p(T))$ and $p(s_n) \in \sigma_{\mathbf{R}_i}(p(T))$ for all $n \in \mathbb{N}$. Further, consider the polynomial $q(z) = p(z) - p(s)$ and suppose $p(s_n) = p(s)$ for an infinite number of elements s_n. It means that the set of zeros of q has an accumulation point and the standard argument of complex analysis implies $g \equiv 0$, what is a contradiction. We can only have $p(s_n) = p(s)$ for finitely many elements s_n, and thus there is a subsequence of $(p(s_n))$ not containing $p(s)$, but converging to $p(s)$, so $\lambda = p(s) \in \operatorname{acc} \sigma_{\mathbf{R}_i}(p(T))$.

Proposition 2.1.11. *Let $T \in L(\mathcal{X})$ admit a GKD. If $T^n \in \mathbf{gDR}_i$ for some $n \in \mathbb{N}$, then $T \in \mathbf{gDR}_i$, where $1 \leq i \leq 9$.*

Proof. Let $T \in L(\mathcal{X})$ admit a GKD and suppose that $T^n \in \mathbf{gDR}_i$ for some $n \in \mathbb{N}$. It follows that $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T^n)$. By Remark 2.1.10 ($p(z) = z^n$),

$$0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T) \Longleftrightarrow 0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T^n),$$

and so $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$. We apply Theorems 2.1.2, 2.1.4, 2.1.5 or 2.1.6, and obtain $T \in \mathbf{gDR}_i$. $\square$

2.2 Examples and comments

The inclusions $L(\mathcal{X})^{gD} \subset \mathbf{gD}\mathcal{M}(\mathcal{X})$ and $L(\mathcal{X})^{gD} \subset \mathbf{gD}\mathcal{Q}(\mathcal{X})$ may be strict.

Example 2.2.1. Each element in the space $\ell^2(\mathbb{N})$ is a sequence $x = (x_i) = (x_1, x_2, \cdots)$ of complex numbers such that

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty.$$

The space $\ell^2(\mathbb{N})$ is a Hilbert space with scalar product defined by

$$(x, y) = \sum_{i=1}^{\infty} x_i \bar{y}_i \quad \text{for every} \quad x, y \in \ell^2(\mathbb{N}).$$

The forward and backward unilateral shifts operators are defined respectively by

$$U(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots) \quad \text{and} \quad V(x_1, x_2, \cdots) = (x_2, x_3, \cdots),$$

where $x = (x_i) \in \ell^2(\mathbb{N})$. The operators U and V belong to $L(\ell^2(\mathbb{N}))$, and

$$\sigma(U) = \sigma(V) = \mathbb{D}, \quad \sigma_{ap}(U) = \sigma_{su}(V) = \mathbb{S}, \quad \sigma_{su}(U) = \sigma_{ap}(V) = \mathbb{D},$$

where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\mathbb{S} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. We conclude that U is bounded below (so U is generalized Drazin bounded below), but U is not generalized Drazin invertible since $0 \in \operatorname{acc} \sigma(U)$. Also, V is surjective (so V is generalized Drazin surjective), but V is not generalized Drazin invertible.

We also show that the inclusions $\mathbf{gD}\mathcal{M}(\mathcal{X}) \subset \mathbf{gD}\mathcal{W}_{+}(\mathcal{X})$ and $\mathbf{gD}\mathcal{Q}(\mathcal{X}) \subset \mathbf{gD}\mathcal{W}_{-}(\mathcal{X})$ can be proper. We note that in the next example we will use notions and facts presented in section 3.1.

Example 2.2.2. Let U and V be as in Example 2.2.1, and let $T = U \oplus V$. It is easy to see that $\alpha(U) = \beta(V) = 0$, $\beta(U) = \alpha(V) = 1$. Consequently, U and V are Fredholm operators, $\operatorname{ind}(U) = -1$, and $\operatorname{ind}(V) = 1$. According to Lemma 1.4.7, (3.2), (3.4) and (3.8), T is Fredholm and $\operatorname{ind}(T) = \operatorname{ind}(U) + \operatorname{ind}(V) = 0$. Accordingly, T is Weyl, and so T is generalized Drazin Weyl. By Example 2.2.1 and (3.12), $\sigma_{ap}(T) = \sigma_{ap}(U) \cup \sigma_{ap}(V) = \mathbb{D}$ and $\sigma_{su}(T) = \sigma_{su}(U) \cup \sigma_{su}(V) = \mathbb{D}$. Therefore, $0 \in \operatorname{acc} \sigma_{ap}(T)$ and $0 \in \operatorname{acc} \sigma_{su}(T)$, and from Theorems 2.1.4 and 2.1.5 it follows that T is neither generalized Drazin bounded below nor generalized Drazin surjective.

Remark 2.2.3. The following inclusions are true:

$$\begin{aligned}
\Phi_{+}(\mathcal{X}) \setminus \mathcal{W}_{+}(\mathcal{X}) &\subset \mathbf{gD\Phi}_{+}(\mathcal{X}) \setminus \mathbf{gD}\mathcal{W}_{+}(\mathcal{X}), \\
\Phi_{-}(\mathcal{X}) \setminus \mathcal{W}_{-}(\mathcal{X}) &\subset \mathbf{gD\Phi}_{-}(\mathcal{X}) \setminus \mathbf{gD}\mathcal{W}_{-}(\mathcal{X}), \\
\Phi(\mathcal{X}) \setminus \mathcal{W}(\mathcal{X}) &\subset \mathbf{gD\Phi}(\mathcal{X}) \setminus \mathbf{gD}\mathcal{W}(\mathcal{X}).
\end{aligned}$$

Let $T \in \Phi_+(\mathcal{X}) \setminus \mathcal{W}_+(\mathcal{X})$. Clearly, $\Phi_+(\mathcal{X}) \setminus \mathcal{W}_+(\mathcal{X}) \subset \Phi_+(\mathcal{X}) \subset \mathbf{gD\Phi}_+(\mathcal{X})$. In addition, there exists $\epsilon > 0$ such that $\mathrm{ind}(T-\lambda) = \mathrm{ind}(T) > 0$ for $|\lambda| < \epsilon$ [57, Theorem 18.4]. It follows that $T - \lambda \notin \mathcal{W}_+(\mathcal{X})$ for $|\lambda| < \epsilon$. Consequently, $0 \in \mathrm{acc}\,\sigma_{\mathcal{W}_+}(T)$, and applying Theorem 2.1.2 we obtain that $T \notin \mathbf{gD}\mathcal{W}_+(\mathcal{X})$.

The same argument can be used for the remaining inclusions.

The next example shows that the inclusions $\mathbf{gD}\mathcal{W}_+(\mathcal{X}) \subset \mathbf{gD\Phi}_+(\mathcal{X})$, $\mathbf{gD}\mathcal{W}_-(\mathcal{X}) \subset \mathbf{gD\Phi}_-(\mathcal{X})$ and $\mathbf{gD}\mathcal{W}(\mathcal{X}) \subset \mathbf{gD\Phi}(\mathcal{X})$ can be proper.

Example 2.2.4. Let $\mathcal{X} = \ell^2(\mathbb{N})$ and let U and V be as in Example 2.2.1. By Example 2.2.2, $U \in \Phi_-(\mathcal{X}) \setminus \mathcal{W}_-(\mathcal{X})$, $V \in \Phi_+(\mathcal{X}) \setminus \mathcal{W}_+(\mathcal{X})$, and $U, V \in \Phi(\mathcal{X}) \setminus \mathcal{W}(\mathcal{X})$. According to Remark 2.2.3, $U \in \mathbf{gD\Phi}_-(\mathcal{X}) \setminus \mathbf{gD}\mathcal{W}_-(\mathcal{X})$, $V \in \mathbf{gD\Phi}_+(\mathcal{X}) \setminus \mathbf{gD}\mathcal{W}_+(\mathcal{X})$, and $U, V \in \mathbf{gD\Phi}(\mathcal{X}) \setminus \mathbf{gD}\mathcal{W}(\mathcal{X})$.

Example 2.2.5. Let $T \in L(\mathcal{X})$ be a Riesz operator with infinite spectrum. The spectrum of T is a sequence converging to 0, $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T)$ and $\sigma_{\mathbf{R}_i}(T) = \{0\}$, $4 \leq i \leq 12$, (see [1, Section 3.9]). It follows that $0 \notin \mathrm{int}\,\sigma_{\mathbf{R}_i}(T) = \emptyset$, $1 \leq i \leq 12$, and $0 \notin \mathrm{acc}\,\sigma_{\mathbf{R}_i}(T) = \emptyset$, $4 \leq i \leq 12$. On the other hand, it was shown in [41] that T does not admit a GKD. It means that "T admits a GKD" can not be deleted from statements (iv), (vii) and (viii) of Theorems 2.1.4 and 2.1.5, as well as from statements (ii), (iii) and (iv) of Theorem 2.1.6, and also from statements (ii) and (iii) of Theorem 2.1.2.

Remark 2.2.6. We recall that if $T \in L(\mathcal{X})$ is generalized Drazin bounded below then a pair $(M, N) \in Red(T)$ such that T_M is bounded below and T_N is quasinilpotent has the property: $N = H_0(T)$; we are not sure whether M is uniquely determined (Theorem 2.1.4). Now, suppose that $T \in L(\mathcal{X})$ is generalized Drazin invertible. According to Theorem 1.5.9, then there exists a unique pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is quasinilpotent: $M = K(T)$ and $N = H_0(T)$. Since T_M is also bounded below then T is generalized Drazin bounded below by Theorem 2.1.4. Is there a pair $(M, H_0(T)) \in Red(T)$, $M \neq K(T)$, such that T_M is bounded below and $T_{H_0(T)}$ quasinilpotent? The answer is negative! Indeed, if such a pair exists then $0 \notin \mathrm{acc}\,\sigma(T_M)$ since $0 \notin \mathrm{acc}\,\sigma(T)$. Consequently, T_M is invertible by Corollary 1.4.10(iii) and hence $M = K(T)$. Similarly, if T is generalized Drazin invertible then there is a unique decomposition (M, N) of $\mathcal{X}$ which completely reduced T and such that T_M is surjective and T_N is quasinilpotent: $(M, N) = (K(T), H_0(T))$.

2.3 The classes $\mathbf{DR}_i$

Applying the same method as in the proof of Theorem 2.1.2 we can prove the following result.

Theorem 2.3.1. *Let $T \in L(\mathcal{X})$ and $1 \le i \le 12$. The following conditions are equivalent:*

(i) *There exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is nilpotent, that is $T \in \mathbf{DR}_i$;*

(ii) *T is of Kato type and $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$;*

(iii) *T is of Kato type and $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$;*

(iv) *There exists a projection $P \in L(\mathcal{X})$ that commutes with T such that $T + P \in \mathbf{R}_i$ and TP is nilpotent.*

Using Theorems 1.4.14, 1.4.15 and 1.5.3 we see that if $i = 9$ ($i = 12$) then Conditions (i)-(iv) of Theorem 2.3.1 are equivalent to the assertion that T is B-Fredholm (T is B-Weyl), while if $i = 3$ these conditions are equivalent to the fact that T is Drazin invertible.

Corollary 2.3.2. *Let $T \in L(\mathcal{X})$ and suppose that $0 \in \partial \sigma_{\mathbf{R}_i}(T)$, $1 \le i \le 12$. Then T is of Kato type if and only if T belongs to the class $\mathbf{DR}_i$.*

Proof. Follows from the equivalence (i) $\Longleftrightarrow$ (iii) of Theorem 2.3.1. □

Remark 2.3.3. Corollary 2.3.2($i = 3$) and Theorem 1.5.3 give the following result:

Let $T \in L(\mathcal{X})$ and suppose that $0 \in \partial \sigma(T)$. Then T is of Kato type if and only if 0 is a pole of the resolvent of T.

This result was proved by P. Aiena and E. Rosas [3, Theorem 2.9].

We recall that for every linear operator T acting on a Banach space $\mathcal{X}$ and every $n \in \mathbb{N}_0$ the operator $T_n : R(T^n) \to R(T^n)$ is defined as $T_n x = Tx$ for $x \in R(T^n)$. Clearly, T_n is linear operator and $T_0 = T$. Further, let $c'_n(T) = \dim N(T^{n+1})/N(T^n)$ and $c_n(T) = \dim R(T^n)/R(T^{n+1})$. According to [43, Lemma 1 and Lemma 2], $c'_n(T) = \dim(N(T) \cap R(T^n))$ and $c_n(T) = \operatorname{codim}(R(T)+N(T^n))$, so the sequences $(c'_n(T))_n$ and $(c_n(T))_n$ are non-increasing. In particular, $c'_0(T) = \alpha(T)$ and $c_0(T) = \beta(T)$. The sequence $((k_n(T))_n$ is given by

$$k_n(T) = \dim(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T)),$$

and equivalently

$$k_n(T) = \dim(R(T) + N(T^{n+1}))/(R(T) + N(T^n)).$$

From this it is easily seen that

$$c'_n(T) = k_n(T) + c'_{n+1}(T) \quad \text{and} \quad c_n(T) = k_n(T) + c_{n+1}(T), \tag{2.1}$$

and that an operator $T \in L(\mathcal{X})$ is Kato if and only if $R(T)$ is closed and $k_i(T) = 0$ for all $i \ge 0$.

Remark 2.3.4. (i) Suppose that $\mathcal{X}$ is a Banach space and let $T \in L(\mathcal{X})$. If $(M, N) \in Red(T)$ and if T_N is nilpotent, then the following statements are equivalent:
(a) $\operatorname{asc}(T_n) < \infty$ for every $n \in \mathbb{N}_0$;
(b) $\operatorname{asc}(T_n) < \infty$ for some $n \in \mathbb{N}_0$;
(c) $\operatorname{asc}(T_M) < \infty$.

The implication (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c). Let $\operatorname{asc}(T_n) < \infty$ for some $n \in \mathbb{N}_0$. It is evident that $c'_p(T_n) = 0$ for some p. From [8, Lemma 3.1] it follows that $c'_{n+p}(T) = c'_p(T_n) = 0$ and therefore $\operatorname{asc}(T) < \infty$. According to the proof of Lemma 1.4.7, we get $\operatorname{asc}(T_M) < \infty$.

(c) $\Longrightarrow$ (a). Suppose that $\operatorname{asc}(T_M) < \infty$ and let $n \in \mathbb{N}_0$. Since T_N is nilpotent then $\operatorname{asc}(T_N)$ is finite, and thus $\operatorname{asc}(T) < \infty$ by the proof of Lemma 1.4.7. There exists $p \geq n$ such that $c'_p(T) = 0$. From [8, Lemma 3.1] it follows $c'_{p-n}(T_n) = c'_p(T) = 0$, and thus $\operatorname{asc}(T_n) < \infty$.

Similarly, if $(M, N) \in Red(T)$ and if T_N is nilpotent, then the following statements are equivalent:
(a) $\operatorname{dsc}(T_n) < \infty$ for every $n \in \mathbb{N}_0$;
(b) $\operatorname{dsc}(T_n) < \infty$ for some $n \in \mathbb{N}_0$;
(c) $\operatorname{dsc}(T_M) < \infty$.

(ii) If T_n is upper (resp. lower) semi-Fredholm for some $n \geq 0$ then $R(T^m)$ is closed, T_m is upper (resp. lower) semi-Fredholm and $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for every $m \geq n$ [13].

Similar to the definitions of the classes $\mathbf{B\Phi}(\mathcal{X})$ and $\mathbf{B}\mathcal{W}(\mathcal{X})$, the classes $\mathbf{BR}_i$ are introduced and studied [8]. In what follows we establish a relationship between classes $\mathbf{BR}_i$ and $\mathbf{DR}_i$ in the case of Banach spaces. For the case of a Hilbert space see [8, Theorem 3.12].

Proposition 2.3.5. *Let $\mathcal{X}$ be a Banach space and $T \in L(\mathcal{X})$. If $i \in \{1, 2, 4, 5, 7, 8, 10, 11\}$ then the following statements are equivalent:*

(i) *T is of Kato type and $T \in \mathbf{BR}_i$;*

(ii) *$T \in \mathbf{DR}_i$.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that T is of Kato type and that $T \in \mathbf{B\Phi}_+(\mathcal{X})$. There exist two closed T-invariant subspaces M and N such that $\mathcal{X} = M \oplus N$, T_M is Kato and T_N is nilpotent of degree d. Also, there exists $n \geq 0$ such that $R(T^n)$ is closed and T_n is upper semi-Fredholm. From $c'_n(T) = \dim(N(T) \cap R(T^n)) = \alpha(T_n) < \infty$ and from the fact that $(c'_k(T))_k$ is a non-increasing sequence, there exists $p \geq \max\{d, n\}$ such that $c'_p(T) = c'_{p+1}(T) = \cdots < \infty$. Since $(T_N)^p = 0$, $c'_p(T_N) = 0$ and thus $c'_p(T_M) = c'_p(T_M) + c'_p(T_N) = c'_p(T) < \infty$. Since $k_j(T_M) = 0$ for each $j \geq 0$ then (2.1) gives $\alpha(T_M) = c'_0(T_M) = c'_p(T_M) < \infty$. Since T_M has closed range, it follows that T_M is upper semi-Fredholm.

In addition, if $T \in \mathbf{B}\mathcal{M}(\mathcal{X})$, then $c_n'(T) = \alpha(T_n) = 0$, so $\alpha(T_M) = c_p'(T_M) = c_p'(T) = 0$, and hence T_M is bounded below. Further, if $T \in \mathbf{B}\mathcal{B}_+(\mathcal{X})$, then T_M is upper semi-Browder by Remark 2.3.4.

Let $T \in \mathbf{B}\mathcal{W}_+(\mathcal{X})$. It follows that $R(T^p) = R((T_M)^p) \subset M$, $R(T^p)$ is closed and $\mathrm{ind}(T_p) = \mathrm{ind}(T_n) \leq 0$. Since T_M is upper semi-Fredholm, then $\mathrm{ind}(T_M) = \mathrm{ind}((T_M)_p)$, where $(T_M)_p : R((T_M)^p) \to R((T_M)^p)$. It is evident that $T_p = (T_M)_p$, hence $\mathrm{ind}(T_M) = \mathrm{ind}((T_M)_p) = \mathrm{ind}(T_p) = \mathrm{ind}(T_n) \leq 0$, i.e. $T_M \in \mathcal{W}_+(\mathcal{X})$, so $T \in \mathbf{D}\mathcal{W}_+(\mathcal{X})$.

The remaining part can be proved similarly.

(ii) $\Longrightarrow$ (i). Let $T \in \mathbf{D}\mathcal{W}_+(\mathcal{X})$. There exists $(M, N) \in \mathit{Red}(T)$ such that T_M is upper semi-Weyl and T_N is nilpotent. Then $R(T^p)$ is closed and $R(T^p) = R((T_M)^p) \subset M$ for sufficiently large p. From $T_p = (T_M)_p$ we conclude that T_p is upper semi-Fredholm and $\mathrm{ind}(T_p) = \mathrm{ind}((T_M)_p) = \mathrm{ind}(T_M) \leq 0$. It means that T_p is upper semi-Weyl, so $T \in \mathbf{B}\mathcal{W}_+(X)$. Using the similar technique we can prove the remaining part. $\square$

Recall that for a Riesz operator $T \in L(\mathcal{X})$ with infinite spectrum we have $0 \in \sigma_{gK}(T) \subset \sigma_{Kt}(T)$, so T is not of Kato type. It means that the condition that T is of Kato type can not be omitted from statement (iii) of Theorem 2.3.1 if $1 \leq i \leq 12$, as well as from statement (ii) of Theorem 2.3.1 if $4 \leq i \leq 12$. The following example ensures that the condition that T is of Kato type in statement (ii) of Theorem 2.3.1 can not be omitted if $i \in \{1, 2, 3\}$.

Example 2.3.6. The space $\ell^1(\mathbb{N})$ consists of all complex sequences $x = (x_i) = (x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} |x_i|$ converges. It is a Banach space with norm given by

$$\|x\| = \sum_{i=1}^{\infty} |x_i|.$$

Let $B : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ be defined by

$$B(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots), \quad (x_1, x_2, x_3, \cdots) \in \ell^1(\mathbb{N}).$$

Then B is a quasinilpotent (but not nilpotent) operator of the space $L(\ell^1(\mathbb{N}))$ [69, p. 280]. It means that 0 is not an accumulation point of $\sigma(B)$, $\sigma_{ap}(B)$ and $\sigma_{su}(B)$. What is more, B is not Drazin invertible [48, Example 8.1]. According to Corollary 2.3.2($i = 3$), B is not of Kato type.

2.4 Applications

For $T \in L(\mathcal{X})$ we define the spectra with respect to the sets $\mathbf{gDR}_i$, $1 \leq i \leq 12$, in a classical way:

$$\sigma_{\mathbf{gDR}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{gDR}_i\}, \ \ 1 \leq i \leq 12.$$

From Theorems 2.1.2, 2.1.4 and 2.1.5 it follows that

$$\left.\begin{aligned} \sigma_{\mathbf{gDR}_i}(T) &= \sigma_{gK}(T) \cup \operatorname{acc} \sigma_{\mathbf{R}_i}(T) \\ &= \sigma_{gK}(T) \cup \operatorname{int} \sigma_{\mathbf{R}_i}(T),\ 1 \leq i \leq 12. \end{aligned}\right\} \tag{2.2}$$

The theorems mentioned above also imply that $\sigma_{\mathbf{gDL}(\mathcal{X})^{-1}}(T) = \sigma_{\mathbf{gD}\mathcal{B}}(T)$ is the generalized Drazin spectrum of T, and that $\sigma_{\mathbf{gD}\mathcal{M}}(T) = \sigma_{\mathbf{gD}\mathcal{B}_+}(T)$ and $\sigma_{\mathbf{gD}\mathcal{Q}}(T) = \sigma_{\mathbf{gD}\mathcal{B}_-}(T)$. The following scheme is clear:

$$\begin{array}{ccccccccc}
 & & \sigma_{\mathbf{gD\Phi}_+}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{W}_+}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{M}}(T) & \subset & \sigma_{ap}(T) \\
 & \rotatebox{45}{$\subset$} & & & & & & \rotatebox{-45}{$\subset$} & \\
\sigma_{gK}(T) & \subset & \sigma_{\mathbf{gD\Phi}}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{W}}(T) & & & \subset & \sigma_{gD}(T) \\
 & \rotatebox{-45}{$\subset$} & & & & & & \rotatebox{45}{$\subset$} & \\
 & & \sigma_{\mathbf{gD\Phi}_-}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{W}_-}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{Q}}(T) & \subset & \sigma_{su}(T)
\end{array}$$

Proposition 2.4.1. *Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. If $\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$, then $\sigma_{gK}(T) = \sigma_{\mathbf{gDR}_i}(T)$. In particular, if $\sigma(T)$ is at most countable or contained in a line, then $\sigma_{gK}(T) = \sigma_{gD}(T)$.*

Proof. Since $\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$, then from Proposition 1.1.1(iii) we conclude that $\operatorname{int} \sigma_{\mathbf{R}_i}(T) = \emptyset$. The desired result follows from (2.2). □

As examples of operators with the spectrum contained in a line we mention self-adjoint and unitary operators on a Hilbert space. The spectrum of a Riesz operator is at most countable.

Proposition 2.4.2. *Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. The following statements hold:*

(i) $\sigma_{\mathbf{gDR}_i}(T) \subset \sigma_{\mathbf{DR}_i}(T) \subset \sigma_{\mathbf{R}_i}(T) \subset \sigma(T)$*;*

(ii) $\sigma_{\mathbf{gDR}_i}(T)$ *is a compact subset of* $\mathbb{C}$*;*

(iii) $\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{\mathbf{gDR}_i}(T)$ *consists of at most countably many isolated points.*

Proof. (i). It is obvious.

(ii). It suffices to prove that $\sigma_{\mathbf{gDR}_i}(T)$ is closed since it is bounded by (i). If $\lambda_0 \notin \sigma_{\mathbf{gDR}_i}(T)$, then $T - \lambda_0 I \in \mathbf{gDR}_i$ and by Proposition 2.1.1 there exists $\epsilon > 0$ such that $T - \lambda_0 I - \lambda I \in \mathbf{R}_i \subset \mathbf{gDR}_i$ for $0 < |\lambda| < \epsilon$. It means that $D(\lambda_0, \epsilon) \subset \mathbb{C} \setminus \sigma_{\mathbf{gDR}_i}(T)$, where $D(\lambda_0, \epsilon)$ is an open disc centered at λ_0 with radius ϵ. Consequently, $\sigma_{\mathbf{gDR}_i}(T)$ is closed.

(iii). If $\lambda \in \sigma_{\mathbf{R}_i}(T) \setminus \sigma_{\mathbf{gDR}_i}(T)$, then $\lambda \in \sigma_{\mathbf{R}_i}(T)$ and $T - \lambda I \in \mathbf{gDR}_i$. Applying Proposition 2.1.1 we obtain that $\lambda \in \operatorname{iso} \sigma_{\mathbf{R}_i}(T)$, and hence $\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{\mathbf{gDR}_i}(T)$ consists of at most countably many isolated points. □

Corollary 2.4.3. *Let $T \in L(\mathcal{X})$. Then the following inclusions hold:*

(i) $\operatorname{acc} \sigma_{ap}(T) \setminus \operatorname{acc} \sigma_{\mathcal{B}_+}(T) \subset \sigma_{gK}(T)$*;*

(ii) $\operatorname{acc} \sigma_{su}(T) \setminus \operatorname{acc} \sigma_{\mathcal{B}_-}(T) \subset \sigma_{gK}(T)$*;*

(iii) $\operatorname{acc}\sigma(T) \setminus \operatorname{acc}\sigma_{\mathcal{B}}(T) \subset \sigma_{gK}(T)$;

(iv) $\operatorname{int}\sigma_{ap}(T) \setminus \operatorname{int}\sigma_{\mathcal{B}_+}(T) \subset \sigma_{gK}(T)$;

(v) $\operatorname{int}\sigma_{su}(T) \setminus \operatorname{int}\sigma_{\mathcal{B}_-}(T) \subset \sigma_{gK}(T)$;

(vi) $\operatorname{int}\sigma(T) \setminus \operatorname{int}\sigma_{\mathcal{B}}(T) \subset \sigma_{gK}(T)$.

Proof. Follows from the equivalences (iii) $\Longleftrightarrow$ (vii) and (iv) $\Longleftrightarrow$ (viii) in Theorems 2.1.4 and 2.1.5. □

Remark 2.4.4. Let $T \in L(\mathcal{X})$ be a Riesz operator with infinite spectrum. As we mentioned earlier, T does not admit a GKD (more precisely, $\sigma_{gK}(T) = \{0\}$, see [42, Example 1]). It is interesting to note that the same follows from Corollary 2.4.3. Namely, $\sigma_{\mathcal{B}}(T) = \{0\}$ and so $0 \notin \operatorname{acc}\sigma_{\mathcal{B}}(T)$, while $0 \in \operatorname{acc}\sigma(T)$. Therefore, $0 \in \operatorname{acc}\sigma(T) \setminus \operatorname{acc}\sigma_{\mathcal{B}}(T)$ and hence $0 \in \sigma_{gK}(T)$ by Corollary 2.4.3. On the other hand, if $0 \neq \lambda \in \mathbb{C}$ then $T - \lambda I$ is Browder. Consequently, $T - \lambda I$ admits a GKD for $0 \neq \lambda \in \mathbb{C}$, and hence $\sigma_{gK}(T) = \{0\}$.

Theorem 2.4.5. *Let $T \in L(\mathcal{X})$. Then the following inclusions hold:*

$$\partial\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T), \quad 1 \leq i \leq 12. \tag{2.3}$$

In addition,

$$\begin{aligned} \partial\sigma_{\mathcal{B}_+}(T) \cap \operatorname{acc}\sigma_{ap}(T) &\subset \sigma_{gK}(T); \\ \partial\sigma_{\mathcal{B}_-}(T) \cap \operatorname{acc}\sigma_{su}(T) &\subset \sigma_{gK}(T); \\ \partial\sigma_{\mathcal{B}}(T) \cap \operatorname{acc}\sigma(T) &\subset \sigma_{gK}(T). \end{aligned}$$

Proof. According to Theorem 2.1.3 and Corollary 2.1.7,

$$\partial\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\sigma_{\mathbf{R}_i}(T) = \operatorname{acc}\sigma_{\mathbf{R}_i}(T) \setminus \operatorname{int}\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T), \ 1 \leq i \leq 12.$$

Moreover, suppose that $\lambda \in \partial\sigma_{\mathcal{B}_+}(T) \cap \operatorname{acc}\sigma_{ap}(T)$ and $T-\lambda$ admits a GKD. Then $\lambda \notin \operatorname{int}\sigma_{\mathcal{B}_+}(T)$ and from the equivalence (viii)$\Longleftrightarrow$(iii) in Theorem 2.1.4 we get that $\lambda \notin \operatorname{acc}\sigma_{ap}(T)$, a contradiction.

The remaining inclusions can be proved analogously. □

Corollary 2.4.6. *Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. Then the set $\partial\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{gK}(T)$ consists of at most countably many points.*

Proof. From (2.3) it follows that

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T) \cup \operatorname{iso}\sigma_{\mathbf{R}_i}(T), \quad 1 \leq i \leq 12,$$

which implies that

$$\partial\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{gK}(T) \subset \operatorname{iso}\sigma_{\mathbf{R}_i}(T).$$

Consequently, $\partial\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{gK}(T)$ is at most countable. □

Theorem 2.4.7. *Let $T \in L(\mathcal{X})$. Then*

$$\begin{array}{ccccccccc}
 & & \partial\sigma_{\mathbf{gD}\mathcal{M}}(T) & \subset & \partial\sigma_{\mathbf{gD}\mathcal{W}_+}(T) & \subset & \partial\sigma_{\mathbf{gD\Phi}_+}(T) & & \\
 & \subset & & & & & & \subset & \\
\partial\sigma_{gD}(T) & \subset & & \partial\sigma_{\mathbf{gD}\mathcal{W}}(T) & \subset & \partial\sigma_{\mathbf{gD\Phi}}(T) & & \subset & \partial\sigma_{gK}(T) \\
 & \subset & & & & & & \subset & \\
 & & \partial\sigma_{\mathbf{gD}\mathcal{Q}}(T) & \subset & \partial\sigma_{\mathbf{gD}\mathcal{W}_-}(T) & \subset & \partial\sigma_{\mathbf{gD\Phi}_-}(T) & &
\end{array}$$

In addition,

$$\begin{aligned}
\partial\sigma_{\mathbf{gD\Phi}}(T) \subset \partial\sigma_{\mathbf{gD\Phi}_+}(T),\ \partial\sigma_{\mathbf{gD\Phi}}(T) \subset \partial\sigma_{\mathbf{gD\Phi}_-}(T),\\
\partial\sigma_{\mathbf{gD}\mathcal{W}}(T) \subset \partial\sigma_{\mathbf{gD}\mathcal{W}_+}(T),\ \ \partial\sigma_{\mathbf{gD}\mathcal{W}}(T) \subset \partial\sigma_{\mathbf{gD}\mathcal{W}_-}(T),
\end{aligned}$$

and

$$\left.\begin{aligned}
\eta\sigma_{gK}(T) = \eta\sigma_{\mathbf{gD\Phi}_+}(T) = \eta\sigma_{\mathbf{gD}\mathcal{W}_+}(T) = \eta\sigma_{\mathbf{gD}\mathcal{M}}(T)\\
= \eta\sigma_{\mathbf{gD\Phi}_-}(T) = \eta\sigma_{\mathbf{gD}\mathcal{W}_-}(T) = \eta\sigma_{\mathbf{gD}\mathcal{Q}}(T)\\
= \eta\sigma_{\mathbf{gD\Phi}}(T) = \eta\sigma_{\mathbf{gD}\mathcal{W}}(T) = \eta\sigma_{gD}(T).
\end{aligned}\right\} \tag{2.4}$$

Proof. According to Proposition 1.1.3 it is sufficient to prove the inclusions:

$$\begin{array}{lll}
\partial\sigma_{gD}(T) \subset \sigma_{gK}(T); & \partial\sigma_{\mathbf{gD}\mathcal{M}}(T) \subset \sigma_{gK}(T); & \partial\sigma_{\mathbf{gD}\mathcal{W}_+}(T) \subset \sigma_{gK}(T);\\
\partial\sigma_{\mathbf{gD\Phi}_+}(T) \subset \sigma_{gK}(T); & \partial\sigma_{\mathbf{gD}\mathcal{Q}}(T) \subset \sigma_{gK}(T); & \partial\sigma_{\mathbf{gD}\mathcal{W}_-}(T) \subset \sigma_{gK}(T);\\
\partial\sigma_{\mathbf{gD\Phi}_-}(T) \subset \sigma_{gK}(T); & \partial\sigma_{\mathbf{gD}\mathcal{W}}(T) \subset \sigma_{gK}(T); & \partial\sigma_{\mathbf{gD\Phi}}(T) \subset \sigma_{gK}(T).
\end{array}$$

We will only prove $\partial\sigma_{gD}(T) \subset \sigma_{gK}(T)$ since the remaining inclusions can be proved analogously.

Suppose that $\lambda_0 \in \partial\sigma_{gD}(T)$. From (2.2) and from the fact that $\sigma_{gD}(T)$ is closed, it follows that

$$\lambda_0 \in \sigma_{gD}(T) = \sigma_{gK}(T) \cup \operatorname{int}\sigma(T). \tag{2.5}$$

We prove that

$$\lambda_0 \notin \operatorname{int}\sigma(T). \tag{2.6}$$

Suppose on the contrary that $\lambda_0 \in \operatorname{int}\sigma(T)$. Since $\operatorname{int}\sigma(T)$ is an open set, then there exists an $\epsilon > 0$ such that $D(\lambda_0, \epsilon) \subset \operatorname{int}\sigma(T)$. It follows that $D(\lambda_0, \epsilon) \subset \sigma_{gD}(T)$, which contradicts the fact that $\lambda_0 \in \partial\sigma_{gD}(T)$. Now, (2.5) and (2.6) imply that $\lambda_0 \in \sigma_{gK}(T)$. $\square$

Proposition 2.4.8. *Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. Then the following statements are equivalent:*

(i) $\sigma(T)$ *is at most countable;*

(ii) $\sigma_{gK}(T)$ *is at most countable;*

(iii) $\sigma_{\mathbf{gDR}_i}(T)$ *is at most countable.*

In that case $\sigma_{gK}(T) = \sigma_{\mathbf{gDR}_i}(T)$. In particular, $\sigma(T)$ is a finite set if and only if $\sigma_{gK}(T) = \emptyset$ if and only if $\sigma_{\mathbf{gDR}_i}(T) = \emptyset$.

Proof. The equivalence (ii) $\iff$ (iii) and identity $\sigma_{gK}(T) = \sigma_{\mathbf{gDR}_i}(T)$ are consequences of (2.4) and Lemma 1.1.4. It remains to prove (i) $\iff$ (ii).

(i) $\Longrightarrow$ (ii). Follows from $\sigma_{gK}(T) \subset \sigma(T)$;

(ii) $\Longrightarrow$ (i). From (2.4) and Lemma 1.1.4 we conclude that $\sigma_{gD}(T) = \operatorname{acc}\sigma(T)$ is at most countable. According to Proposition 1.1.1(vi), $\sigma(T)$ is at most countable.

We note that $\sigma(T)$ is finite if and only if $\sigma_{gD}(T)$ is empty (apply Proposition 1.1.1(v)). The remaining part follows from (2.4) and Lemma 1.1.4. $\square$

The fact that $\sigma_{gK}(T)$ is empty if and only if $\sigma(T)$ is finite has already been proved in [42, Theorem 5].

Corollary 2.4.9. ([42, Theorem 3]) *Let $T \in L(\mathcal{X})$ and let $\rho_{gK}(T)$ has only one component. Then*

$$\sigma_{gK}(T) = \sigma_{gD}(T).$$

Proof. Since $\rho_{gK}(T)$ has only one component, it follows that $\sigma_{gK}(T)$ has no holes, and so $\sigma_{gK}(T) = \eta\sigma_{gK}(T)$. From (2.4) it follows that $\sigma_{gD}(T) \supset \sigma_{gK}(T) = \eta\sigma_{gK}(T) = \eta\sigma_{gD}(T) \supset \sigma_{gD}(T)$, and hence $\sigma_{gD}(T) = \sigma_{gK}(T)$. $\square$

We now consider some special situations.

Theorem 2.4.10. *Let $T \in L(\mathcal{X})$ and $1 \le i \le 12$. If*

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \operatorname{acc}\sigma_{\mathbf{R}_i}(T), \tag{2.7}$$

then

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_{\mathbf{R}_i}(T) \tag{2.8}$$

and

$$\eta\sigma_{\mathbf{R}_i}(T) = \eta\sigma_{gK}(T) = \eta\sigma_{Kt}(T) = \eta\sigma_{eK}(T). \tag{2.9}$$

Proof. From (2.7) it follows that $\partial\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$. Now (2.3) implies that $\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T)$. (2.9) follows from (2.8) and Proposition 1.1.3. $\square$

Theorem 2.4.11. *Let $T \in L(\mathcal{X})$ and $1 \le i \le 12$. If*

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T) = \operatorname{acc}\sigma_{\mathbf{R}_i}(T), \tag{2.10}$$

then

$$\sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_{\mathbf{gDR}_i}(T) = \sigma_{\mathbf{DR}_i}(T) = \sigma_{\mathbf{R}_i}(T). \tag{2.11}$$

Proof. From (2.10) and Theorem 2.4.10 it follows that

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_{\mathbf{R}_i}(T),$$

and so $\sigma_{\mathbf{R}_i}(T) = \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T)$. Since $\sigma_{\mathbf{R}_i}(T) = \sigma_{gK}(T) \subset \sigma_{\mathbf{gDR}_i}(T) \subset \sigma_{\mathbf{DR}_i}(T) \subset \sigma_{\mathbf{R}_i}(T)$, then (2.11) is proved. $\square$

Corollary 2.4.12. *Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. If $\sigma_{\mathbf{R}_i}(T) = \partial\sigma(T)$, and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$ then*

$$\sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_{\mathbf{gDR}_i}(T) = \sigma_{\mathbf{DR}_i}(T) = \sigma_{\mathbf{R}_i}(T).$$

Proof. Firstly, we will prove $\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$. The inclusion $\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{\mathbf{R}_i}(T)$ is evident. Let $\lambda \in \sigma_{\mathbf{R}_i}(T)$ and let $D(\lambda, r)$ be an open disc centered at λ with radius $r > 0$. It is clear that $\sigma_{\mathbf{R}_i}(T) \cap D(\lambda, r) \neq \emptyset$. By assumption, $\lambda \in \partial\sigma(T)$, so $\emptyset \neq \rho(T) \cap D(\lambda, r) \subset \rho_{\mathbf{R}_i}(T) \cap D(\lambda, r)$. Consequently, $\lambda \in \partial\sigma_{\mathbf{R}_i}(T)$. On the other hand, using Proposition 1.1.2 we obtain

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma(T) = \partial\sigma(T) \cap \operatorname{acc}\sigma(T) = \operatorname{acc}\partial\sigma(T) = \operatorname{acc}\sigma_{\mathbf{R}_i}(T).$$

Now we have $\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T) = \operatorname{acc}\sigma_{\mathbf{R}_i}(T)$, and the result follows from Theorem 2.4.11. ☐

Example 2.4.13. Let U and V be as in Example 2.2.1. We have already mentioned that $\sigma(U) = \sigma(V) = \mathbb{D}$ and $\sigma_{ap}(U) = \sigma_{su}(V) = \mathbb{S}$. Also, $\sigma_\Phi(U) = \sigma_\Phi(V) = \mathbb{S}$ (see [76, Theorem 4.2]). According to [57, Proposition 19.1], $\mathbb{S} = \partial\sigma_\Phi(U) \subset \sigma_{\Phi_+}(U) \cap \sigma_{\Phi_-}(U)$. On the other hand, $\mathbb{S} = \sigma_\Phi(U) = \sigma_{\Phi_+}(U) \cup \sigma_{\Phi_-}(U)$, so $\sigma_{\Phi_+}(U) \subset \mathbb{S}$ and $\sigma_{\Phi_-}(U) \subset \mathbb{S}$. It follows that $\sigma_\Phi(U) = \sigma_{\Phi_+}(U) = \sigma_{\Phi_-}(U) = \mathbb{S}$. Now, the operator U satisfies the conditions of Corollary 2.4.12 ($i = 1, 7, 8, 9$), so we have

$$\begin{aligned}
\mathbb{S} &= \sigma_{gK}(U) = \sigma_{Kt}(U) = \sigma_{eK}(U) \\
&= \sigma_{\mathbf{gD}\mathcal{M}}(U) = \sigma_{\mathbf{gD}\mathcal{W}_+}(U) = \sigma_{\mathbf{gD\Phi}_+}(U) = \sigma_{\mathbf{gD\Phi}_-}(U) = \sigma_{\mathbf{gD\Phi}}(U) \\
&= \sigma_{\mathbf{D}\mathcal{M}}(U) = \sigma_{\mathbf{D}\mathcal{W}_+}(U) = \sigma_{\mathbf{D\Phi}_+}(U) = \sigma_{\mathbf{D\Phi}_-}(U) = \sigma_{B\Phi}(U).
\end{aligned}$$

What is more, from $\sigma_{eK}(U) = \sigma_{ap}(U) = \mathbb{S}$ and from (1.5) we obtain $\sigma_K(U) = \mathbb{S}$.

In the same manner as above we can see that $\sigma_\Phi(V) = \sigma_{\Phi_+}(V) = \sigma_{\Phi_-}(V) = \mathbb{S}$ and

$$\begin{aligned}
\mathbb{S} &= \sigma_{gK}(V) = \sigma_{Kt}(V) = \sigma_{eK}(V) = \sigma_K(V) \\
&= \sigma_{\mathbf{gD}\mathcal{Q}}(V) = \sigma_{\mathbf{gD}\mathcal{W}_-}(V) = \sigma_{\mathbf{gD\Phi}_-}(V) = \sigma_{\mathbf{gD\Phi}_+}(V) = \sigma_{\mathbf{gD\Phi}}(V) \\
&= \sigma_{\mathbf{D}\mathcal{Q}}(V) = \sigma_{\mathbf{D}\mathcal{W}_-}(V) = \sigma_{\mathbf{D\Phi}_-}(V) = \sigma_{\mathbf{D\Phi}_+}(V) = \sigma_{B\Phi}(V).
\end{aligned}$$

If $T \in L(\mathcal{X})$ then $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ denotes the spectral radius of T. A classical result indicates that $r(T) = \lim_{n\to\infty} ||T^n||^{1/n}$.

Example 2.4.14. A weighted right shift T on $\ell^2(\mathbb{N})$ is defined by

$$T(x_1, x_2, \cdots) = (0, w_1x_1, w_2x_2, \cdots) \quad \text{for all} \quad (x_1, x_2, \cdots) \in \ell^2(\mathbb{N}),$$

where (w_n) is a given weight sequence. We always assume that $0 < w_n \leq 1$ for all $n \in \mathbb{N}$. A routine calculation shows that T is a bounded linear operator on $\ell^2(\mathbb{N})$ and that $\|T^n\| = \sup\limits_{k\in\mathbb{N}} w_k \cdots w_{k+n-1}$ for every $n \in$

$\mathbb{N}$. It is immediate that $r(T) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}}(w_k \cdot \cdots \cdot w_{k+n-1})^{1/n}$. If we suppose that $\lim_{n\to\infty} \inf_{k\in\mathbb{N}}(w_k \cdots w_{k+n-1})^{1/n} = r(T)$, then [52, Proposition 1.6.15] implies $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$. Thus, $\sigma_{ap}(T) = \partial\sigma_{ap}(T) = \operatorname{acc}\sigma_{ap}(T)$. Now, from Theorem 2.4.11 it follows that

$$\begin{aligned}
\sigma_{gK}(T) &= \sigma_{Kt}(T) = \sigma_K(T) = \sigma_{ap}(T) \\
&= \sigma_{\mathbf{gD}\mathcal{M}}(T) = \sigma_{\mathbf{gD}\mathcal{W}_+}(T) = \sigma_{\mathbf{gD\Phi}_+}(T) \\
&= \sigma_{\mathbf{D}\mathcal{M}}(T) = \sigma_{\mathbf{D}\mathcal{W}_+}(T) = \sigma_{\mathbf{D\Phi}_+}(T) \\
&= \{\lambda \in \mathbb{C} : |\lambda| = r(T)\}.
\end{aligned}$$

Chapter 3

Generalized inverses of operator matrices

Let $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ be an upper triangular operator matrix acting on the Banach space $\mathcal{X} \oplus \mathcal{Y}$ or separable Hilbert space $\mathcal{H} \oplus \mathcal{K}$. The sets $\bigcap_C \sigma_*(M_C)$, where $C \in L(\mathcal{Y}, \mathcal{X})$ (or $C \in L(\mathcal{K}, \mathcal{H})$) and $\sigma_* = \{\sigma, \sigma_l, \sigma_r, \sigma_K, \sigma_\Phi, \sigma_\mathcal{W}, \sigma_\mathcal{B}, \cdots\}$, have been widely studied; for example see [6, 26, 32, 73]. In this chapter we investigate the set $\bigcap_C \sigma_{gD}(M_C)$, where $C \in L(\mathcal{Y}, \mathcal{X})$ or $C \in L(\mathcal{K}, \mathcal{H})$. What is more, the case of Drazin invertibility is also considered.

3.1 Upper triangular operator matrices

Let $\mathcal{X} = M_1 \oplus M_2$ and $\mathcal{Y} = N_1 \oplus N_2$, where M_1, M_2 are closed subspaces of $\mathcal{X}$, and N_1, N_2 are closed subspaces of $\mathcal{Y}$. It is a classical fact that there exist projections $P_1, P_2 \in L(\mathcal{X})$ such that $R(P_1) = M_1$, $N(P_1) = M_2$, $R(P_2) = M_2$, $N(P_2) = M_1$ (see [51, Korolar 8.4.4]). Similarly, there exist projections $Q_1, Q_2 \in L(\mathcal{Y})$ such that $R(Q_1) = N_1$, $N(Q_1) = N_2$, $R(Q_2) = N_2$, $N(Q_2) = N_1$. For given bounded linear operators $U : M_1 \to N_1$, $V : M_2 \to N_1$, $S : M_1 \to N_2$ and $W : M_2 \to N_2$ we may define $A : \mathcal{X} \to \mathcal{Y}$ by

$$A = \begin{pmatrix} U & V \\ S & W \end{pmatrix} : \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \to \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

In other words, for $x \in \mathcal{X}$ we define $Ax = Ux_1 + Vx_2 + Sx_1 + Wx_2$, where $x_1 \in M_1$ and $x_2 \in M_2$ are unique vectors such that $x = x_1 + x_2$. It is clear that A is linear. Since

$$\begin{aligned} \|Ax\| &\le \|U\|\|x_1\| + \|V\|\|x_2\| + \|S\|\|x_1\| + \|W\|\|x_2\| \\ &\le 2K(\|x_1\| + \|x_2\|) \le 2K(\|P_1\| + \|P_2\|)\|x\|, \end{aligned}$$

where $K = \max\{\|U\|, \|V\|, \|S\|, \|W\|\}$, it follows that $A \in L(\mathcal{X}, \mathcal{Y})$. On the other hand, for every $A \in L(\mathcal{X}, \mathcal{Y})$ it is possible to find bounded linear operators $U : M_1 \to N_1$, $V : M_2 \to N_1$, $S : M_1 \to N_2$ and $W : M_2 \to N_2$ such that $Ax = Ux_1 + Vx_2 + Sx_1 + Wx_2$, where $x_1 \in M_1$, $x_2 \in M_2$ and $x = x_1 + x_2$. Indeed, let $Uw = Q_1Aw$, $Sw = Q_2Aw$ for every $w \in M_1$, and let $Vz = Q_1Az$, $Wz = Q_2Az$ for every $z \in M_2$. Clearly, U, V, S, W are bounded linear operators, and

$$\begin{aligned} Ax = (Q_1 + Q_2)A(x_1 + x_2) &= Q_1Ax_1 + Q_1Ax_2 + Q_2Ax_1 + Q_2Ax_2 \\ &= Ux_1 + Vx_2 + Sx_1 + Wx_2. \end{aligned}$$

For a deeper discussion on this topic we refer the reader to [59, Glava 2].

Now we consider a particular situation. Let $\tilde{\mathcal{X}}$ be the set

$$\tilde{\mathcal{X}} = \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

$\tilde{\mathcal{X}}$ is a vector space with standard addition and multiplication by scalars. The space $\tilde{\mathcal{X}}$ endowed with the norm $||(x, y)|| = (||x||^2 + ||y||^2)^{\frac{1}{2}}$ becomes a Banach space. The sets $\mathcal{M}_1 = \{(x, 0) : x \in \mathcal{X}\}$ and $\mathcal{M}_2 = \{(0, y) : y \in \mathcal{Y}\}$ are closed subspaces of $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Let consider the operator $M : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ defined by

$$M(x, y) = (Ax + Cy, By) \text{ for every } (x, y) \in \tilde{\mathcal{X}},$$

where $A \in L(\mathcal{X})$, $B \in L(\mathcal{Y})$ and $C \in L(\mathcal{Y}, \mathcal{X})$ are given operators. It is very common to represent M as

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

since $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + Cy \\ By \end{pmatrix}$. Obviously, M is linear. Let define operators $U : \mathcal{M}_1 \to \mathcal{M}_1$, $V : \mathcal{M}_2 \to \mathcal{M}_1$, $S : \mathcal{M}_1 \to \mathcal{M}_2$ and $W : \mathcal{M}_2 \to \mathcal{M}_2$ by

$$\begin{aligned} U(x, 0) = (Ax, 0), &\quad V(0, y) = (Cy, 0), \\ S(x, 0) = (0, 0), &\quad W(0, y) = (0, By). \end{aligned}$$

Then, S is the zero operator and $\|U(x, 0)\| = \|Ax\| \le \|A\|\|(x, 0)\|$, so U is bounded. In the same manner we can see that V and W are also bounded. Now, $M(x, y) = U(x, 0) + V(0, y) + W(0, y)$, i.e.

$$M = \begin{pmatrix} U & V \\ 0 & W \end{pmatrix} : \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{pmatrix}. \tag{3.1}$$

By the preceding paragraph, M is bounded, and it is said that M is an upper triangular operator matrix. In addition, if A and B are fix, and C is arbitrary, we write $M_C(A,B) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. To shorten notation, we use M_C for $M_C(A,B)$ when no confusion can arise. In particular, if $C=0$ then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is denoted by $A\oplus B$. Clearly, $\mathcal{M}_1$ and $\mathcal{M}_2$ are $(A\oplus B)-$invariant, $(A\oplus B)_{\mathcal{M}_1} = U$, and $(A\oplus B)_{\mathcal{M}_2} = W$. For $\lambda\in\mathbb{C}$, let consider the mapping $J: N(A-\lambda I)\to N(U-\lambda I)$ defined by $Jx=(x,0)$. This mapping is well-defined since for every $x\in N(A-\lambda I)$ we have $(U-\lambda I)(x,0) = ((A-\lambda I)x, 0) = (0,0)$, so $Jx\in N(U-\lambda I)$. It is easy to check that J is an isomorphism between $N(A-\lambda I)$ and $N(U-\lambda I)$. Consequently,

$$\alpha(A-\lambda I) = \alpha(U-\lambda I). \tag{3.2}$$

In particular,

$$A-\lambda I \text{ is one-one if and only if } U-\lambda I \text{ is one-one.} \tag{3.3}$$

It is a matter of routine to show that:

$$\beta(A-\lambda I) = \beta(U-\lambda I), \tag{3.4}$$
$$A-\lambda I \text{ is onto if and only if } U-\lambda I \text{ is onto,} \tag{3.5}$$
$$A-\lambda I \text{ has closed range if and only if } U-\lambda I \text{ has closed range.} \tag{3.6}$$

(3.3), (3.5) and (3.6) imply

$$\sigma_{ap}(A) = \sigma_{ap}(U), \quad \sigma_{su}(A) = \sigma_{su}(U), \quad \sigma(A) = \sigma(U). \tag{3.7}$$

Similarly,

$$\alpha(B-\lambda I) = \alpha(W-\lambda I), \quad \beta(B-\lambda I) = \beta(W-\lambda I) \tag{3.8}$$
$$\sigma_{ap}(B) = \sigma_{ap}(W), \quad \sigma_{su}(B) = \sigma_{su}(W), \quad \sigma(B) = \sigma(W). \tag{3.9}$$

According to Lemma 1.4.7 and (3.2)-(3.9), we obtain:

$$\alpha((A\oplus B)-\lambda I) = \alpha(U-\lambda I) + \alpha(W-\lambda I) = \alpha(A-\lambda I) + \alpha(B-\lambda I), \tag{3.10}$$
$$\beta((A\oplus B)-\lambda I) = \beta(U-\lambda I) + \beta(W-\lambda) = \beta(A-\lambda I) + \beta(B-\lambda I), \tag{3.11}$$
$$\sigma_*(A\oplus B) = \sigma_*(U)\cup\sigma_*(W) = \sigma_*(A)\cup\sigma_*(B), \tag{3.12}$$

where $\sigma_* = \{\sigma_{ap}, \sigma_{su}, \sigma\}$.

Finally, suppose that some $M\in L(\tilde{\mathcal{X}})$ possesses the decomposition (3.1) (it means that M has an upper triangular form with respect to the decomposition $\tilde{\mathcal{X}} = \mathcal{M}_1\oplus\mathcal{M}_2$). Then there exist $A\in L(\mathcal{X})$, $B\in L(\mathcal{Y})$ and $C\in L(\mathcal{Y},\mathcal{X})$

such that $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, and this fact will be used later (see Proposition 3.2.4). Indeed, let

$$\begin{aligned} Ax &= T_1U(x,0) \quad \text{for every} \quad x \in \mathcal{X}, \\ By &= T_2W(0,y) \quad \text{for every} \quad y \in \mathcal{Y}, \\ Cy &= T_1V(0,y) \quad \text{for every} \quad y \in \mathcal{Y}, \end{aligned}$$

where bounded linear operators $T_1 : \mathcal{M}_1 \to \mathcal{X}$ and $T_2 : \mathcal{M}_2 \to \mathcal{Y}$ are defined by $T_1(x,0) = x$ and $T_2(0,y) = y$ for every $(x,0) \in \mathcal{M}_1$ and $(0,y) \in \mathcal{M}_2$. Clearly, $(x,0) = (T_1(x,0),0)$ and $(0,y) = (0,T_2(0,y))$ for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Consequently, A, B and C are bounded linear operators, $U(x,0) = (Ax,0)$, $W(0,y) = (0,By)$ and $V(0,y) = (Cy,0)$. Now we have $M(x,y) = U(x,0) + V(0,y) + W(0,y) = (Ax + Cy, By)$, so $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

3.2 Generalized Drazin invertibility of M_C

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces then $\mathcal{H} \times \mathcal{K} = \{(h,k) : h \in \mathcal{H}, k \in \mathcal{K}\}$ is a Hilbert space with the inner product defined by

$$((h_1,k_1),(h_2,k_2)) = (h_1,h_2)_1 + (k_1,k_2)_2,$$

where $(\cdot\,,\cdot)_1$ is the inner product in $\mathcal{H}$ and $(\cdot\,,\cdot)_2$ is the inner product in $\mathcal{K}$ (for example, see problem 124 in [18]). The Hilbert space $\mathcal{H} \times \mathcal{K}$ is usually denoted by $\mathcal{H} \oplus \mathcal{K}$. In addition, if $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces then $\mathcal{H} \oplus \mathcal{K}$ is also separable since the Cartesian product of two countable sets is countable. In what follows, $\mathcal{H}$ and $\mathcal{K}$ will be always separable Hilbert spaces.

H. K. Du and J. Pan [28] have considered the invertible completions of upper triangular operator matrices acting on the separable Hilbert space $\mathcal{H} \oplus \mathcal{K}$.

Theorem 3.2.1. ([28, Theorem 2]) *For given $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$, we have*

$$\bigcap_{C \in L(\mathcal{K},\mathcal{H})} \sigma(M_C) = \sigma_{ap}(A) \cup \sigma_{su}(B) \cup \{\lambda \in \mathbb{C} : \alpha(B - \lambda I) \neq \beta(A - \lambda I)\}.$$

J. K. Han, H. Y. Lee and W. Y. Lee [32] have extended the above result to Banach spaces.

Theorem 3.2.2. ([32, Theorem 2]) *A 2×2 operator matrices M_C is invertible for some $C \in L(\mathcal{Y},\mathcal{X})$ if and only if $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ satisfy the following conditions:*
(i) *A is left invertible;*
(ii) *B is right invertible;*
(iii) *$\mathcal{X}/R(A) \cong N(B)$.*

The generalized Drazin invertibility of upper triangular operator matrices are studied in [25, 70, 72]. Necessary conditions for the existence of $C \in L(\mathcal{Y}, \mathcal{X})$ such that M_C is generalized Drazin invertible are presented in [70].

Theorem 3.2.3. ([70, Theorem 3.16]) *Let $A \in L(\mathcal{X})$, $B \in L(\mathcal{Y})$ and $C \in L(\mathcal{Y}, \mathcal{X})$. If M_C is generalized Drazin invertible, then the following statements hold:*
(i) $\sigma_l(A)$ *does not cluster at* 0;
(ii) $\sigma_r(B)$ *does not cluster at* 0;
(iii) *There exists $\delta > 0$ such that $\beta(A - \lambda I) = \alpha(B - \lambda I)$ for $0 < |\lambda| < \delta$.*

We recall that D. Djordjević and P. Stanimirović showed that if $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ are generalized Drazin invertible, then $(M_C)^d$ exists and has an upper triangular form for every $C \in L(\mathcal{Y}, \mathcal{X})$ [25, Theorem 5.1]. In the following proposition we prove the converse.

Proposition 3.2.4. *Let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. If $(M_C)^d$ exists for some $C \in L(\mathcal{Y}, \mathcal{X})$ and has an upper triangular form, then A and B are generalized Drazin invertible.*

Proof. Suppose that there exists some $C \in L(\mathcal{Y}, \mathcal{X})$ such that M_C is generalized Drazin invertible, and let $(M_C)^d = \begin{pmatrix} U & V \\ 0 & W \end{pmatrix}$, where $U \in L(\mathcal{X})$, $V \in L(\mathcal{Y}, \mathcal{X})$ and $W \in L(\mathcal{Y})$. It is easy to check that the equations $M_C(M_C)^d = (M_C)^d M_C$ and $(M_C)^d M_C (M_C)^d = (M_C)^d$ imply

$$AU = UA, \quad BW = WB, \quad UAU = U, \quad WBW = W. \tag{3.13}$$

A routine calculation shows that $\begin{pmatrix} A - AUA & S \\ 0 & B - BWB \end{pmatrix}$ is quasinilpotent, where $S = C - AUC - AVB - CWB$, since $M_C - M_C(M_C)^d M_C$ is quasinilpotent. Consequently, $\begin{pmatrix} A - AUA - \lambda I & S \\ 0 & B - BWB - \lambda I \end{pmatrix}$ is invertible for every $0 \neq \lambda \in \mathbb{C}$. Using Theorem 3.2.2 we obtain $\sigma_l(A - AUA) = \{0\}$ and $\sigma_r(B - BWB) = \{0\}$. From Proposition 1.4.11 it follows that

$$\sigma(A - AUA) = \{0\} \quad \text{and} \quad \sigma(B - BWB) = \{0\}. \tag{3.14}$$

The equations (3.13) and (3.14) ensure that A and B are generalized Drazin invertible. $\square$

Remark 3.2.5. In this remark we will use standard notions related to Hilbert spaces and we refer to [60, Glava 3] for their definitions and properties. Let $\mathcal{H}_1$ be a closed subspace of a separable Hilbert space $\mathcal{H}$. With the inner product defined by restriction, $\mathcal{H}_1$ is a Hilbert space in its own right. Let M be a countable dense set in $\mathcal{H}$ and let $S = \{P(x) : x \in M\}$, where $P \in L(\mathcal{H})$ is the orthogonal projection on $\mathcal{H}_1$. It is clear that S is countable. We claim that $\overline{S} = \mathcal{H}_1$. Let $x \in \mathcal{H}_1$ and $\epsilon > 0$. There exists $z \in M$ such that $\|z - x\| < \epsilon$. The vectors $Pz - x$ and $z - Pz$ are mutually orthogonal, so

$$\|Pz - x\|^2 \leq \|Pz - x\|^2 + \|z - Pz\|^2 = \|z - x\|^2 < \epsilon^2.$$

Consequently, $\|Pz - x\| < \epsilon$, which proves that $\mathcal{H}_1$ is separable.

In the following theorem we give sufficient conditions under which $M_C \in L(\mathcal{H} \oplus \mathcal{K})$ is generalized Drazin invertible for some $C \in L(\mathcal{K}, \mathcal{H})$.

Theorem 3.2.6. *Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$ be such that the following statements are satisfied:*

(i) *A and B each admits a GKD;*
(ii) *$\sigma_{ap}(A)$ does not cluster at 0;*
(iii) *$\sigma_{su}(B)$ does not cluster at 0;*
(iv) *There exists $\delta > 0$ such that $\beta(A - \lambda I) = \alpha(B - \lambda I)$ for $0 < |\lambda| < \delta$.*

Then there exists $C \in L(\mathcal{K}, \mathcal{H})$ such that M_C is generalized Drazin invertible.

Proof. By assumptions and Theorems 2.1.4 and 2.1.5, there exist closed A-invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$, and there exist closed B-invariant subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{K}$, such that $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$, $\mathcal{K}_1 \oplus \mathcal{K}_2 = \mathcal{K}$, $A_{\mathcal{H}_1} = A_1$ is bounded below, $A_{\mathcal{H}_2} = A_2$ is quasinilpotent, $B_{\mathcal{K}_1} = B_1$ is surjective and $B_{\mathcal{K}_2} = B_2$ is quasinilpotent.

By Lemma 1.4.7 (see (1.10)), $\beta(A - \lambda I) = \beta(A_1 - \lambda I) + \beta(A_2 - \lambda I)$ and $\alpha(B - \lambda I) = \alpha(B_1 - \lambda I) + \alpha(B_2 - \lambda I)$ for every $\lambda \in \mathbb{C}$. Since A_2 and B_2 are quasinilpotent,

$$\beta(A - \lambda I) = \beta(A_1 - \lambda I), \tag{3.15}$$
$$\alpha(B - \lambda I) = \alpha(B_1 - \lambda I), \tag{3.16}$$

for every $\lambda \in \mathbb{C} \setminus \{0\}$. Further, according to Lemma 1.3.5 there exists $\epsilon > 0$ such that

$$\beta(A_1) = \beta(A_1 - \lambda I) \text{ and } \alpha(B_1) = \alpha(B_1 - \lambda I) \text{ for } |\lambda| < \epsilon. \tag{3.17}$$

Consider $\lambda_0 \in \mathbb{C}$ such that $0 < |\lambda_0| < \min\{\epsilon, \delta\}$, where δ is as in (iv). Using (3.15), (3.16), (3.17) and (iv) we obtain

$$\beta(A_1) = \beta(A_1 - \lambda_0 I) = \beta(A - \lambda_0 I) = \alpha(B - \lambda_0 I) = \alpha(B_1 - \lambda_0 I) = \alpha(B_1).$$

On the other hand, $\mathcal{H}_1$, $\mathcal{K}_1$, $\mathcal{H}_2$ and $\mathcal{K}_2$ are separable Hilbert spaces (see Remark 3.2.5), $\mathcal{H}_1 \oplus \mathcal{K}_1$ and $\mathcal{H}_2 \oplus \mathcal{K}_2$ are closed subspaces of $\mathcal{H} \oplus \mathcal{K}$, and $(\mathcal{H}_1 \oplus \mathcal{K}_1) \oplus (\mathcal{H}_2 \oplus \mathcal{K}_2) = \mathcal{H} \oplus \mathcal{K}$. Applying Theorem 3.2.1 we conclude that there exists an operator $C_1 \in L(\mathcal{K}_1, \mathcal{H}_1)$ such that the operator $\begin{pmatrix} A_1 & C_1 \\ 0 & B_1 \end{pmatrix}$: $\mathcal{H}_1 \oplus \mathcal{K}_1 \to \mathcal{H}_1 \oplus \mathcal{K}_1$ is invertible. Let define an operator $C \in L(\mathcal{K}, \mathcal{H})$ by

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}.$$

An easy computation shows that $\mathcal{H}_1 \oplus \mathcal{K}_1$ and $\mathcal{H}_2 \oplus \mathcal{K}_2$ are invariant for $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, and also

$$(M_C)_{\mathcal{H}_1 \oplus \mathcal{K}_1} = \begin{pmatrix} A_1 & C_1 \\ 0 & B_1 \end{pmatrix},$$
$$(M_C)_{\mathcal{H}_2 \oplus \mathcal{K}_2} = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}.$$

Since A_2 and B_2 are quasinilpotent, then $\sigma((M_C)_{\mathcal{H}_2 \oplus \mathcal{K}_2}) = \sigma(A_2) \cup \sigma(B_2) = \{0\}$. Consequently, $(M_C)_{\mathcal{H}_2 \oplus \mathcal{K}_2}$ is quasinilpotent. By Theorem 1.5.9, M_C is generalized Drazin invertible. $\square$

Corollary 3.2.7. (i) *Let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. Then:*

$$\operatorname{acc} \sigma_l(A) \cup \operatorname{acc} \sigma_r(B) \cup \mathcal{G} \subset \bigcap_{C \in L(\mathcal{Y}, \mathcal{X})} \sigma_{gD}(M_C),$$

where $\mathcal{G} = \{\lambda \in \mathbb{C} : \nexists \delta > 0$ such that $\beta(A - \lambda I - \lambda' I) = \alpha(B - \lambda I - \lambda' I)$ for $0 < |\lambda'| < \delta\}$.
(ii) *Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$. Then:*

$$\operatorname{acc} \sigma_{ap}(A) \cup \operatorname{acc} \sigma_{su}(B) \cup \mathcal{G} \subset \bigcap_{C \in L(\mathcal{K}, \mathcal{H})} \sigma_{gD}(M_C) \subset$$
$$\subset \operatorname{acc} \sigma_{ap}(A) \cup \operatorname{acc} \sigma_{su}(B) \cup \mathcal{G} \cup \sigma_{gK}(A) \cup \sigma_{gK}(B).$$

In particular, if $\sigma_{gK}(A) \subset \operatorname{acc} \sigma_{ap}(A)$ and $\sigma_{gK}(B) \subset \operatorname{acc} \sigma_{su}(B)$, then:

$$\bigcap_{C \in L(\mathcal{K}, \mathcal{H})} \sigma_{gD}(M_C) = \operatorname{acc} \sigma_{ap}(A) \cup \operatorname{acc} \sigma_{su}(B) \cup \mathcal{G}.$$

Proof. (i). Follows from Theorem 3.2.3.
(ii). The result follows from (i) and Theorem 3.2.6 if we notice that $\sigma_l(\cdot) = \sigma_{ap}(\cdot)$ and $\sigma_r(\cdot) = \sigma_{su}(\cdot)$ for operators acting on a Hilbert space. $\square$

Remark 3.2.8. We recall that $\sigma_{gK}(T) \subset \sigma_{ap}(T) \cap \sigma_{su}(T)$ for every $T \in L(\mathcal{X})$ (see (1.5)). It means that the above conditions $\sigma_{gK}(A) \subset \operatorname{acc} \sigma_{ap}(A)$ and $\sigma_{gK}(B) \subset \operatorname{acc} \sigma_{su}(B)$ are satisfied whenever $\sigma_{ap}(A) = \operatorname{acc} \sigma_{ap}(A)$ and $\sigma_{su}(B) =$

$\operatorname{acc}\sigma_{su}(B)$. In particular, if U and V are respectively forward and backward unilateral shift operators on the space $\ell^2(\mathbb{N})$ then $\sigma_{ap}(U) = \operatorname{acc}\sigma_{ap}(U) = \mathbb{S}$ and $\sigma_{su}(V) = \operatorname{acc}\sigma_{su}(V) = \mathbb{S}$ (see Example 2.2.1). Further, if T is a Riesz operator, then by Proposition 2.4.8 or by [42, Theorem 5], $\sigma_{gK}(T) = \operatorname{acc}\sigma_{ap}(T) = \operatorname{acc}\sigma_{su}(T) = \emptyset$ if T has finite spectrum, and from [42, Example 1] and Example 2.2.5 we have $\sigma_{gK}(T) = \operatorname{acc}\sigma_{ap}(T) = \operatorname{acc}\sigma_{su}(T) = \{0\}$ if the spectrum of T is infinite.

The following example is from [71], and here we use it to demonstrate that there exists a nontrivial situation such that the conditions of Theorem 3.2.6 are satisfied.

Example 3.2.9. Let U and V be as in Example 2.2.1. Let define an operator W as

$$W = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} : \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}).$$

U admits a GKD and $0 \notin \operatorname{acc}\sigma_{ap}(U)$. Further, it is evident that W admits a GKD. By (3.12), $\sigma_{su}(W) = \sigma_{su}(V) \cup \sigma_{su}(0) = \mathbb{S} \cup \{0\}$, so $0 \notin \operatorname{acc}\sigma_{su}(W)$. We recall that $\beta(U) = \alpha(V) = 1$. Also, from Lemma 1.3.5 we obtain $\beta(U) = \beta(U - \lambda I)$ and $\alpha(V) = \alpha(V - \lambda I)$ for $0 < |\lambda| < \delta$, where $\delta > 0$ is a constant. By virtue of (3.10), for $0 < |\lambda| < \delta$ we have

$$\beta(U-\lambda I) = \beta(U) = \alpha(V) = \alpha(V-\lambda I) = \alpha(V-\lambda I)+\alpha(0-\lambda I) = \alpha(W-\lambda I).$$

From Theorem 3.2.6 it follows that there exists $C \in L(\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}), \ell^2(\mathbb{N}))$ such that

$$M_C = \begin{pmatrix} U & C \\ 0 & W \end{pmatrix} : \ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})) \to \ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}))$$

is generalized Drazin invertible, i.e. $0 \notin \bigcap_{C\in L(\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N}),\ell^2(\mathbb{N}))} \sigma_{gD}(M_C)$.

For given $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$, the set $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma(M_C)$ is completely described [32, Corollary 3]. It follows that $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma(M_C)$ is non-empty since $\sigma_l(A)$ and $\sigma_r(B)$ are non-empty sets. On the other hand, $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma_{gD}(M_C)$ may be empty, and in the following result we give sufficient and necessary conditions under which $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma_{gD}(M_C) = \emptyset$.

Theorem 3.2.10. *Let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. The following statements are equivalent:*

(i) $\bigcap\limits_{C\in L(\mathcal{Y},\mathcal{X})} \sigma_{gD}(M_C) = \emptyset$;

(ii) $\sigma(A)$ *and* $\sigma(B)$ *are finite;*

(iii) $\sigma_{gD}(M_C) = \emptyset$ *for every* $C \in L(\mathcal{Y}, \mathcal{X})$;

(iv) $\sigma_{gD}(M_C) = \emptyset$ *for some* $C \in L(\mathcal{Y}, \mathcal{X})$;

(v) $\sigma_{gK}(M_C) = \emptyset$ *for some* $C \in L(\mathcal{Y}, \mathcal{X})$;

(vi) $\sigma_{gK}(M_C) = \emptyset$ *for every* $C \in L(\mathcal{Y}, \mathcal{X})$.

Proof. (i) $\Longrightarrow$ (ii). Suppose that $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma_{gD}(M_C) = \emptyset$. By Corollary 3.2.7, $\operatorname{acc}\sigma_l(A) = \emptyset$ and $\operatorname{acc}\sigma_r(B) = \emptyset$, so $\sigma_l(A)$ and $\sigma_r(B)$ are finite sets. That $\sigma(A)$ and $\sigma(B)$ are finite follows from Proposition 1.4.11.

(ii) $\Longrightarrow$ (iii). We have $\sigma_{gD}(A) = \operatorname{acc}\,\sigma(A) = \emptyset$ and $\sigma_{gD}(B) = \operatorname{acc}\,\sigma(B) = \emptyset$. From [25, Theoem 5.1] it follows that $\sigma_{gD}(M_C) \subset \sigma_{gD}(A) \cup \sigma_{gD}(B)$ for every $C \in L(\mathcal{Y},\mathcal{X})$. Consequently, $\sigma_{gD}(M_C) = \emptyset$ for every $C \in L(\mathcal{Y},\mathcal{X})$.

The implications (iii) $\Longrightarrow$ (iv), (iv) $\Longrightarrow$ (i) and (vi) $\Longrightarrow$ (v) are clear.

(iv) $\Longrightarrow$ (v) and (iii) $\Longrightarrow$ (vi). Follows from $\sigma_{gK}(M_C) \subset \sigma_{gD}(M_C)$.

(v) $\Longrightarrow$ (iv). Suppose that $\sigma_{gK}(M_C) = \emptyset$ for some $C \in L(\mathcal{Y},\mathcal{X})$. From [42, Theorem 5] it follows that $\sigma(M_C)$ is finite, i.e. $\sigma_{gD}(M_C) = \emptyset$. $\square$

In the Hilbert space setting it is possible to provide another condition which is equivalent to those in Theorem 3.2.10.

Remark 3.2.11. Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$. For arbitrary $C \in L(\mathcal{K},\mathcal{H})$, M. Barraa and M. Boumazgour showed in [6, Theorem 2.5] the inclusion

$$(\sigma_K(A) \setminus \sigma_p(B)) \cup (\sigma_K(B) \setminus \overline{\sigma_p(A^*)}) \subset \sigma_K(M_C),$$

where $A^* \in L(\mathcal{H})$ is the Hilbert-adjoint operator of A, and the bar stands for complex conjugation. It is well known that $\sigma_d(A) = \overline{\sigma_p(A^*)}$, and we will have in mind this observation when we apply the aforementioned result.

Theorem 3.2.12. *Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$.*

(a) *Then:* $(\operatorname{acc}\sigma_K(A)\setminus\operatorname{acc}\sigma_p(B))\cup(\operatorname{acc}\sigma_K(B)\setminus\operatorname{acc}\sigma_d(A)) \subset \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gK}(M_C)$.

(b) *In addition, the following assertions are equivalent:*

(i) $\bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gD}(M_C) = \emptyset$;

(ii) $\bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gK}(M_C)$, $\operatorname{acc}\sigma_p(B)$ *and* $\operatorname{acc}\sigma_d(A)$ *are all empty.*

Proof. (a) We will prove $\operatorname{acc}\sigma_K(A) \setminus \operatorname{acc}\sigma_p(B) \subset \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gK}(M_C)$. The inclusion $\operatorname{acc}\sigma_K(B) \setminus \operatorname{acc}\sigma_d(A) \subset \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gK}(M_C)$ can be proved similarly. To obtain a contradiction, let $\lambda \notin \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gK}(M_C)$ and $\lambda \in \operatorname{acc}\sigma_K(A) \setminus \operatorname{acc}\sigma_p(B)$. Then, there exists $C \in L(\mathcal{K},\mathcal{H})$ such that $M_C - \lambda I$ admits a GKD. According to [41, Theorem 2.2] it follows that there exists $\epsilon > 0$ such that $M_C - \lambda I - \lambda' I$ is Kato for $0 < |\lambda'| < \epsilon$. Since $\lambda \notin \operatorname{acc}\sigma_p(B)$, then there exists $\epsilon_1 > 0$ such that $B - \lambda I - \lambda' I$ is injective for $0 < |\lambda'| < \epsilon_1$. Without loss of generality we may assume $\epsilon = \epsilon_1$. Now, by [6, Theorem 2.5], $A - \lambda I - \lambda' I$ is Kato, i.e., $\lambda \notin \operatorname{acc}\sigma_K(A)$ what is not possible.

(b) (i) $\Longrightarrow$ (ii). $\sigma(A)$ and $\sigma(B)$ are finite by Theorem 3.2.10. Consequently, $\operatorname{acc}\sigma_p(B)$ and $\operatorname{acc}\sigma_d(A)$ are empty sets. That $\bigcap\limits_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gK}(M_C) = \emptyset$ follows from $\bigcap\limits_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gK}(M_C) \subset \bigcap\limits_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gD}(M_C)$.
(ii) $\Longrightarrow$ (i). Using (a) we obtain $\operatorname{acc}\sigma_K(A) = \operatorname{acc}\sigma_K(B) = \emptyset$, so $\sigma_K(A)$ and $\sigma_K(B)$ are finite. By Proposition 1.4.11, we have that $\sigma(A)$ and $\sigma(B)$ are also finite. We apply Theorem 3.2.10 to obtain $\bigcap\limits_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gD}(M_C) = \emptyset$. $\square$

In some particular situations generalized Drazin invertibility of M_C for some $C \in L(\mathcal{K},\mathcal{H})$ implies that there exists $C_1 \in L(\mathcal{K},\mathcal{H})$ such that M_{C_1} is invertible.

Proposition 3.2.13. *Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$ be Kato operators. If M_C is generalized Drazin invertible for some $C \in L(\mathcal{K},\mathcal{H})$, then A is bounded below, B is surjective, and $\beta(A) = \alpha(B)$, i.e., M_{C_1} is invertible for some $C_1 \in L(\mathcal{K},\mathcal{H})$.*

Proof. Theorem 3.2.3 implies $0 \notin \operatorname{acc}\sigma_{ap}(A) \cup \operatorname{acc}\sigma_{su}(B)$ and $\beta(A-\lambda I) = \alpha(B-\lambda I)$ for $0 < |\lambda| < \delta$, where $\delta > 0$ is some constant. A is bounded below and B is surjective by Corollary 1.4.10. The equality $\beta(A) = \alpha(B)$ follows from Lemma 1.3.5. The existence of the operator C_1 follows from Theorem 3.2.1. $\square$

The following result is an immediate consequence of Proposition 3.2.13 and Theorem 3.2.1.

Corollary 3.2.14. *Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$. Then:*

$$\sigma_K(A) \cup \sigma_K(B) \cup \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gD}(M_C) = \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma(M_C).$$

Corollary 3.2.15. *Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$. The following statements are equivalent:*
(i) $\bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gD}(M_C) = \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma(M_C)$;
(ii) $\sigma_K(A) \cup \sigma_K(B) \subset \bigcap_{C\in L(\mathcal{K},\mathcal{H})} \sigma_{gD}(M_C)$.

Proof. Apply Corollary 3.2.14. $\square$

The following proposition gives necessary and sufficient condition under which $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma_{gD}(M_C) = \bigcap_{C\in L(\mathcal{Y},\mathcal{X})} \sigma(M_C)$ holds in case of Banach spaces.

Proposition 3.2.16. *Let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. Suppose that*

$$\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B) \ \textit{ for every } \ C \in L(\mathcal{Y},\mathcal{X}). \tag{3.18}$$

Then:

(i) $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)=(\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C))\setminus(\rho_{gD}(A)\cap\rho_{gD}(B))$;

(ii) $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)=\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C)$ *if and only if* $\rho_{gD}(A)\cap\rho_{gD}(B)\subset(\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C))^c$.

Proof. (i). Applying [25, Theorem 5.1] we obtain

$$\left.\begin{array}{r}\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)\subset(\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C))\setminus(\rho_{gD}(A)\cap\rho_{gD}(B))\\ \subset\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C),\end{array}\right\}\tag{3.19}$$

and it is worth pointing out that these inclusions are true for every $A\in L(\mathcal{X})$ and $B\in L(\mathcal{Y})$. On the other hand, let $\lambda\notin\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)$. There exists some $C\in L(\mathcal{Y},\mathcal{X})$ such that $\lambda\notin\sigma_{gD}(M_C)$. By assumption, $\lambda\notin\sigma_{gD}(M_C)=\sigma_{gD}(A)\cup\sigma_{gD}(B)$. Consequently, $\lambda\in\rho_{gD}(A)\cap\rho_{gD}(B)$, and hence $\lambda\notin\left(\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C)\right)\setminus(\rho_{gD}(A)\cap\rho_{gD}(B))$.

(ii). The implication $\Longrightarrow$ follows from (3.19). Suppose that the equality (3.18) is satisfied and that $\rho_{gD}(A)\cap\rho_{gD}(B)\subset(\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C))^c$. Using (i) we deduce $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)=(\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C))\setminus(\rho_{gD}(A)\cap\rho_{gD}(B))=\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C)$, which is the desired conclusion. $\square$

Some sufficient conditions for the equality (3.18) can be found in [70].

The following example shows that if (3.18) is satisfied, then the equality $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)=\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C)$ is not always true.

Example 3.2.17. Let both $A\in L(\mathcal{X})$ and $B\in L(\mathcal{Y})$ be quasinilpotent. Clearly, $\sigma_{gD}(A)=\sigma_{gD}(B)=\emptyset$. From $\sigma_{gD}(M_C)\subset\sigma_{gD}(A)\cup\sigma_{gD}(B)$, we have $\sigma_{gD}(M_C)=\emptyset$ for every $C\in L(\mathcal{Y},\mathcal{X})$, so (3.18) holds. Further, we apply Theorem 3.2.10 to conclude $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)=\emptyset$. On the other hand, $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C)\subset\sigma(A)\cup\sigma(B)=\{0\}$. It follows that $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C)=\{0\}$ since it is a non-empty set.

In general, the condition $\rho_{gD}(A)\cap\rho_{gD}(B)\subset(\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C))^c$ does not imply $\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma_{gD}(M_C)=\bigcap_{C\in L(\mathcal{Y},\mathcal{X})}\sigma(M_C)$.

Example 3.2.18. Let U, V, W and M_C be as in Example 3.2.9. Since $\sigma(U)=\sigma(V)=\sigma(W)=\mathbb{D}$, it follows that $\sigma_{gD}(U)=\sigma_{gD}(W)=\mathbb{D}$, hence $\rho_{gD}(U)\cap\rho_{gD}(W)=\mathbb{C}\setminus\mathbb{D}$. Moreover, $\bigcap_{C\in L(\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N}),\ell^2(\mathbb{N}))}\sigma(M_C)\subset\sigma(U)\cup\sigma(W)=\mathbb{D}$. We see that

$$\rho_{gD}(U)\cap\rho_{gD}(W)=\mathbb{C}\setminus\mathbb{D}\subset\left(\bigcap_{C\in L(\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N}),\ell^2(\mathbb{N}))}\sigma(M_C)\right)^c.$$

From Example 3.2.9 we know that $0 \notin \bigcap_{C\in L(\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N}),\ell^2(\mathbb{N}))} \sigma_{gD}(M_C)$. On the other hand, we recall that the operator W is not surjective, hence $0 \in \bigcap_{C\in L(\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N}),\ell^2(\mathbb{N}))} \sigma(M_C)$ by Theorem 3.2.1. Consequently,

$$\bigcap_{C\in L(\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N}),\ell^2(\mathbb{N}))} \sigma_{gD}(M_C) \neq \bigcap_{C\in L(\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N}),\ell^2(\mathbb{N}))} \sigma(M_C).$$

3.3 Drazin invertibility of M_C

Using Theorem 3.2.6 and Theorem 1.5.5 we obtain [16, Theorem 2.1] in a simpler way.

Theorem 3.3.1 ([16])**.** *Let $A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$ be given operators on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, such that:*
(i) *A is left Drazin invertible;*
(ii) *B is right Drazin invertible;*
(iii) *There exists a constant $\delta > 0$ such that $\beta(A - \lambda I) = \alpha(B - \lambda I)$ for every $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \delta$.*
Then there exists an operator $C \in L(\mathcal{K}, \mathcal{H})$ such that M_C is Drazin invertible.

Proof. By Theorem 1.5.5 it follows that there exist pairs $(\mathcal{H}_1, \mathcal{H}_2) \in Red(A)$ and $(\mathcal{K}_1, \mathcal{K}_2) \in Red(B)$ such that $A_{\mathcal{H}_1} = A_1$ is bounded below, $B_{\mathcal{K}_1} = B_1$ is surjective, $A_{\mathcal{H}_2} = A_2$ and $B_{\mathcal{K}_2} = B_2$ are nilpotent. As in the proof of Theorem 3.2.6 we conclude that there exists $C \in L(\mathcal{K}, \mathcal{H})$ such that

$$M_C = (M_C)_{\mathcal{H}_1\oplus\mathcal{K}_1} \oplus (M_C)_{\mathcal{H}_2\oplus\mathcal{K}_2},$$
$$(M_C)_{\mathcal{H}_1\oplus\mathcal{K}_1} \text{ is invertible},$$
$$(M_C)_{\mathcal{H}_2\oplus\mathcal{K}_2} = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}.$$

For sufficiently large $n \in \mathbb{N}$ we have

$$\begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}^n = \begin{pmatrix} (A_2)^n & 0 \\ 0 & (B_2)^n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so $(M_C)_{\mathcal{H}_2\oplus\mathcal{K}_2}$ is nilpotent. According to Theorem 1.5.3, M_C is Drazin invertible. □

Under additional assumptions the converse implication in Theorem 3.3.1 is also true even in the context of Banach spaces.

Theorem 3.3.2. *Let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ be of Kato type. If there exists some $C \in L(\mathcal{Y}, \mathcal{X})$ such that M_C is Drazin invertible, then the following holds:*
(i) *A is left Drazin invertible;*
(ii) *B is right Drazin invertible;*
(iii) *There exists a constant $\delta > 0$ such that $\beta(A - \lambda I) = \alpha(B - \lambda I)$ for every $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \delta$.*

Proof. (iii) is satisfied and $0 \notin \operatorname{acc}\sigma_{ap}(A) \cup \operatorname{acc}\sigma_{su}(B)$ by Theorem 3.2.3. Now Theorem 2.3.1 implies that there exist $(\mathcal{X}_1, \mathcal{X}_2) \in Red(A)$ and $(\mathcal{Y}_1, \mathcal{Y}_2) \in Red(B)$ such that A_1 is bounded below, A_2 is nilpotent, B_1 is surjective and B_2 is nilpotent. Let $n \geq d$ where $d \in \mathbb{N}$ is such that $(A_2)^d = 0$ and $(A_2)^{d-1} \neq 0$. We have

$$N(A^n) = N((A_1)^n) \oplus N((A_2)^n) = \mathcal{X}_2,$$
$$N(A^{d-1}) = N((A_1)^{d-1}) \oplus N((A_2)^{d-1}) = N((A_2)^{d-1}) \subsetneq \mathcal{X}_2.$$

It follows that $\operatorname{asc}(A) = d < \infty$. From $R(A^n) = R((A_1)^n) \oplus R((A_2)^n) = R((A_1)^n)$ we conclude that $R(A^n)$ is closed, and therefore A is left Drazin invertible. In a similar way we prove that B is right Drazin invertible. □

M. Boumazgour proved that if both A and B are semi-Fredholm, and if M_C is Drazin invertible for some C, then conditions (i)-(iii) of Theorem 3.3.2 are satisfied [16, Corollary 2.3]. We recall that the class of semi-Fredholm operators belongs to the class of Kato type operators [57, Theorem 16.21]. According to this observation, it seems that Theorem 3.3.2 is an extension of [16, Corollary 2.3].

Chapter 4

Generalized Kato-Riesz decomposition and generalized Drazin-Riesz invertible operators

In this chapter we give necessary and sufficient conditions for an operator $T \in L(\mathcal{X})$ to admit a decomposition $T = T_M \oplus T_N$ with $T_M \in \mathbf{R}_i$ and T_N Riesz. The case $i = 3$ is of particular importance. It leads to the introduction of generalized Drazin-Riesz invertible operators (Definition 4.2.1), a class which is larger than the class of generalized Drazin invertible operators.

4.1 Generalized Kato-Riesz decomposition

Definition 4.1.1. An operator $T \in L(\mathcal{X})$ is said to admit a generalized Kato-Riesz decomposition, abbreviated as GKRD, if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is Riesz.

Proposition 4.1.2. *Let $T \in L(\mathcal{X})$. If T admits a GKRD (M, N), then $(N^\perp, M^\perp)$ is a GKRD for T'.*

Proof. Suppose that T admits a GKRD (M, N). It is easily seen that both $N^\perp$ and $M^\perp$ are invariant under T'. Let P_M denote the projection onto M along N. Clearly, $P_M \in L(\mathcal{X})$ and $(P_M)'$ is also a projection. Since $R(P_M)$ is closed, we have

$$N((P_M)') = R(P_M)^\perp = M^\perp \text{ and } R((P_M)') = N(P_M)^\perp = N^\perp.$$

Accordingly, $\mathcal{X}' = R((P_M)') \oplus N((P_M)') = N^\perp \oplus M^\perp$, and so $(N^\perp, M^\perp) \in Red(T')$.

If $P_N = I - P_M$ then $R(P_N) = N$, $N(P_N) = M$, $TP_N = P_N T$ and $(M, N) \in Red(TP_N)$. By $TP_N = (TP_N)_M \oplus (TP_N)_N = 0 \oplus T_N$ and Lemma 1.4.13,

TP_N is Riesz. Consequently, $T'(P_N)' = (P_N)'T'$ is Riesz, and $(N^\perp, M^\perp) \in Red(T'(P_N)')$. Since $R((P_N)') = N(P_N)^\perp = M^\perp$, we conclude that $(T'(P_N)')_{M^\perp} = (T')_{M^\perp}$. From Lemma 1.4.13 it follows that $(T')_{M^\perp}$ is Riesz. As in the proof of [1, Theorem 1.43], we obtain that $(T')_{N^\perp}$ is Kato, and the proposition follows. □

4.2 Generalized Drazin-Riesz invertible operators

Definition 4.2.1. An operator $T \in L(\mathcal{X})$ is generalized Drazin-Riesz invertible if there exists $S \in L(\mathcal{X})$ such that:

$$TS = ST, \quad STS = S, \quad T - TST \text{ is Riesz.}$$

Definition 4.2.2. An operator $T \in L(\mathcal{X})$ is said to be Riesz quasi-polar if there exists a projection $Q \in L(\mathcal{X})$ satisfying

$$TQ = QT, \quad T(I-Q) \text{ is Riesz}, \quad Q \in (L(\mathcal{X})T) \cap (TL(\mathcal{X})). \tag{4.1}$$

Theorem 4.2.3. *Let $T \in L(\mathcal{X})$. The following conditions are equivalent:*
(i) *There exists $(M, N) \in Red(T)$ such that T_M is invertible and T_N is Riesz;*
(ii) *T admits a GKRD and $0 \notin \operatorname{int}\sigma(T)$;*
(iii) *T admits a GKRD, and both T and T' have the SVEP at 0;*
(iv) *T is generalized Drazin-Riesz invertible;*
(v) *T is Riesz quasi-polar;*
(vi) *There exists a bounded projection $P \in L(\mathcal{X})$ which commutes with T such that $T + P$ is Browder and TP is Riesz;*
(vii) *There exists $(M, N) \in Red(T)$ such that T_M is Browder and T_N is Riesz;*
(viii) *T admits a GKRD and $0 \notin \operatorname{acc}\sigma_{\mathcal{B}}(T)$;*
(ix) *T admits a GKRD and $0 \notin \operatorname{int}\sigma_{\mathcal{B}}(T)$.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that there exists $(M, N) \in Red(T)$ such that T_M is invertible and T_N is Riesz. By the fact that T_M is Kato, we conclude that T admits a GKRD (M, N). Since T_M is invertible, $0 \in \rho(T_M)$, and there exists $\epsilon > 0$ such that $D(0, \epsilon) \subset \rho(T_M)$. As T_N is Riesz, it follows that $0 \in \operatorname{acc}\rho(T_N)$. Consequently, $0 \in \operatorname{acc}(\rho(T_M) \cap \rho(T_N)) = \operatorname{acc}\rho(T)$, so $0 \notin \operatorname{int}\sigma(T)$.
(ii) $\Longrightarrow$ (i). Suppose that T admits a GKRD (M, N) and $0 \notin \operatorname{int}\sigma(T)$. Then T_M is Kato and $0 \in \operatorname{acc}\rho(T)$. According to Lema 1.4.7(i), it follows that $0 \in \operatorname{acc}\rho(T_M)$. From Proposition 1.4.9 we deduce that T_M is invertible.
(ii) $\Longrightarrow$ (iii). Apply (1.9).
(iii) $\Longrightarrow$ (ii). Suppose that T admits a GKRD, and that both T and T' have the SVEP at 0. Then there exists $(M, N) \in Red(T)$ such that T_M is Kato and T_N

is Riesz. By Proposition 1.3.7, T_M has the SVEP at 0. According to Theorem 1.3.8, T_M is bounded below and so there exists $\epsilon_1 > 0$ such that $T_M - \lambda$ is bounded below for every $|\lambda| < \epsilon_1$, that is $D(0,\epsilon_1) \subset \rho_{ap}(T_M)$. Further, T_N is Riesz, then $D(0,\epsilon_1) \setminus C_1 \subset \rho_{ap}(T_N)$, where C_1 is at most countable set of Riesz points of T_N. Consequently, $D(0,\epsilon_1) \setminus C_1 \subset \rho_{ap}(T_M) \cap \rho_{ap}(T_N) = \rho_{ap}(T)$. From Proposition 4.1.2 it follows that T' admits the GKRD $(N^\perp, M^\perp)$. From what has already been proved, it may be concluded that there exists $\epsilon_2 > 0$ such that $D(0,\epsilon_2) \setminus C_2 \subset \rho_{ap}(T') = \rho_{su}(T)$, where C_2 is at most countable set of Riesz points of $(T')_{M^\perp}$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $C = C_1 \cup C_2$. Then C is at most countable and $D(0,\epsilon) \setminus C \subset \rho_{ap}(T) \cap \rho_{su}(T) = \rho(T)$. Consequently, $0 \notin \operatorname{int}\sigma(T)$.

(i) $\Longrightarrow$ (iv). Suppose that there exists $(M,N) \in \mathit{Red}(T)$ such that T_M is invertible and T_N is Riesz. Let $S = (T_M)^{-1} \oplus 0$, i.e.

$$S = \begin{pmatrix} (T_M)^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} M \\ N \end{pmatrix} \to \begin{pmatrix} M \\ N \end{pmatrix},$$

and let $x \in \mathcal{X}$. Then $x = u+v$, where $u \in M$ and $v \in N$, and $TSx = TS(u+v) = T(T_M)^{-1}u = u$ and $STx = ST(u+v) = S(T_Mu + T_Nv) = (T_M)^{-1}T_Mu = u$. Thus $TS = ST$. Further, $STSx = STS(u+v) = Su = S(u+v) = Sx$, hence $STS = S$. In addition,

$$\begin{aligned} T - T^2S &= \begin{pmatrix} T_M & 0 \\ 0 & T_N \end{pmatrix} - \begin{pmatrix} (T_M)^2 & 0 \\ 0 & (T_N)^2 \end{pmatrix} \cdot \begin{pmatrix} (T_M)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_M & 0 \\ 0 & T_N \end{pmatrix} - \begin{pmatrix} T_M & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & T_N \end{pmatrix}, \end{aligned}$$

and so $T - T^2S$ is Riesz by Lemma 1.4.13.

(iv) $\Longrightarrow$ (v). Suppose that T is generalized Drazin-Riesz invertible. Then there exists $S \in L(\mathcal{X})$ such that $ST = TS$, $STS = S$ and $T - T^2S$ is Riesz. Let $Q = TS$. Then Q is a bounded projection which commutes with T, $Q = TS = ST \in (L(\mathcal{X})T) \cap (TL(\mathcal{X}))$, and $T(I-Q) = T(I-TS) = T - T^2S$ is Riesz.

(v) $\Longrightarrow$ (vi). Suppose that T is Riesz quasi-polar. Then there exists a bounded projection $Q \in L(\mathcal{X})$ satisfying (4.1). Let $P = I - Q$. Then $P^2 = P \in L(\mathcal{X})$, $TP = PT$, and TP is Riesz. From $I - P = Q \in (L(\mathcal{X})T) \cap (TL(\mathcal{X}))$, it follows that there exist $U, V \in L(\mathcal{X})$ such that $I - P = UT = TV$. Then

$$(T+P)(UTV+P) = (UTV+P)(T+P) = I + TP. \tag{4.2}$$

Since TP is Riesz, [1, Theorem 3.111] implies that $I + TP$ is Browder. Now, from (4.2) and [33, Theorem 7.9.2], we deduce that $T + P$ is Browder.

(vi) $\Longrightarrow$ (vii). Suppose that there exists a projection $P \in L(\mathcal{X})$ such that $TP = PT$, $T+P$ is Browder, and TP is Riesz. For $M = N(P)$ and $N = R(P)$ we have that $(M, N) \in Red(T)$. As $T_N = (TP)_N$ and $T_M = (T+P)_M$, from Lemma 1.4.13 and Lemma 1.4.7(i) it follows that T_N is Riesz and T_M is Browder.

(vii) $\Longrightarrow$ (viii). Let $(M, N) \in Red(T)$ and $T = T_M \oplus T_N$, where T_M is Browder and T_N is Riesz. Then $0 \in \rho_{\mathcal{B}}(T_M)$ and there exists $\epsilon > 0$ such that $D(0, \epsilon) \subset \rho_{\mathcal{B}}(T_M)$. Since T_N is Riesz, $\sigma_{\mathcal{B}}(T_N) \subset \{0\}$ by [1, Theorem 3.111]. Consequently, $D(0, \epsilon) \setminus \{0\} \subset \rho_{\mathcal{B}}(T_M) \cap \rho_{\mathcal{B}}(T_N)$. According to Lemma 1.4.7(i), $\rho_{\mathcal{B}}(T_M) \cap \rho_{\mathcal{B}}(T_N) = \rho_{\mathcal{B}}(T)$, and so $D(0, \epsilon) \setminus \{0\} \subset \rho_{\mathcal{B}}(T)$. Therefore, $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}(T)$.

From [57, Theorem 16.21] it follows that there exist two closed T-invariant subspaces M_1 and M_2 such that $M = M_1 \oplus M_2$, M_2 is finite dimensional, T_{M_1} is Kato and T_{M_2} is nilpotent. Hence $\mathcal{X} = M_1 \oplus (M_2 \oplus N)$ and $M_2 \oplus N$ is closed. From Lemma 1.4.13 it follows that $T_{M_2 \oplus N} = T_{M_2} \oplus T_N$ is Riesz, and thus T admits the GKRD $(M_1, M_2 \oplus N)$.

(viii) $\Longrightarrow$ (ix). Obvious.

(ix) $\Longrightarrow$ (i). Suppose that T admits a GKRD and $0 \notin \operatorname{int} \sigma_{\mathcal{B}}(T)$. Then there exists $(M, N) \in Red(T)$ such that T_M is Kato and T_N is Riesz. Since $0 \in \operatorname{acc} \rho_{\mathcal{B}}(T)$, from Lemma 1.4.7(i) we see that $0 \in \operatorname{acc} \rho_{\mathcal{B}}(T_M)$. By Proposition 1.4.9(iv), T_M is invertible. ☐

Proposition 4.2.4. *Let $T \in L(\mathcal{X})$. The following statements are equivalent:*

(i) *$T = T_M \oplus T_N$, where T_M is invertible and T_N is Riesz with infinite spectrum;*

(ii) *T admits a GKRD and there exists a sequence of nonzero Riesz points of T which converges to 0.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that $T = T_M \oplus T_N$, where T_M is invertible and T_N is Riesz with infinite spectrum. Then T admits a GKRD(M, N) and $\sigma(T_N) = \{0, \mu_1, \mu_2, \dots\}$, where μ_n, $n \in \mathbb{N}$, are nonzero Riesz points of T_N, and

$$\lim_{n \to \infty} \mu_n = 0. \tag{4.3}$$

According to Theorem 4.2.3, $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}(T)$, i.e. there exists $\epsilon > 0$ such that $\mu \notin \sigma_{\mathcal{B}}(T)$ for $0 < |\mu| < \epsilon$. From (4.3) it follows that there exists $n_0 \in \mathbb{N}$ such that $0 < |\mu_n| < \epsilon$ for $n \geq n_0$. Hence $\mu_n \in \sigma(T) \setminus \sigma_{\mathcal{B}}(T)$ for all $n \geq n_0$. Since the set $\sigma(T) \setminus \sigma_{\mathcal{B}}(T)$ is exactly the set of all Riesz points of T, we see that $(\mu_n)_{n=n_0}^{\infty}$ is the sequence of nonzero Riesz points of T which converges to 0.

(ii) $\Longrightarrow$ (i). Suppose that $T = T_M \oplus T_N$, where T_M is Kato, T_N is Riesz, and let (λ_n) be the sequence of nonzero Riesz points of T such that $0 = \lim_{n \to \infty} \lambda_n$. Since $\lambda_n \in \rho_{\mathcal{B}}(T)$ for all $n \in \mathbb{N}$, it follows that $0 \in \operatorname{acc} \rho_{\mathcal{B}}(T)$. As in the proof of Theorem 4.2.3 we conclude that T_M is invertible. Thus there exists an $\epsilon > 0$ such that $D(0, \epsilon) \subset \rho(T_M)$, and there exists $n_0 \in \mathbb{N}$ such that $\lambda_n \in D(0, \epsilon)$ for all $n \geq n_0$. Consequently, $\lambda_n \notin \sigma(T_M)$ for all $n \geq n_0$, and since $\lambda_n \in \sigma(T) = \sigma(T_M) \cup \sigma(T_N)$, it follows that $\lambda_n \in \sigma(T_N)$ for all $n \geq n_0$. Therefore, the spectrum of T_N is infinite. $\square$

Corollary 4.2.5. *Let $T \in L(\mathcal{X})$ be generalized Drazin-Riesz invertible and let $0 \in \operatorname{acc} \sigma(T)$. Then there exists a sequence of nonzero Riesz points of T which converges to 0.*

Proof. According to Theorem 4.2.3, $T = T_M \oplus T_N$ with T_M invertible and T_N Riesz. Since $0 \in \operatorname{acc} \sigma(T)$, it follows that $0 \in \operatorname{acc} \sigma_N(T)$, so $\sigma_N(T)$ is infinite. The corollary follows by applying Proposition 4.2.4. $\square$

4.3 Generalized Drazin-Riesz semi-Fredholm operators

Definition 4.3.1. Let $T \in L(\mathcal{X})$ and $1 \leq i \leq 12$. We say that T belongs to the class $\mathbf{GDRR}_i$ if there exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is Riesz. If $T \in \mathbf{GDRR}_i$ for some i, then it is said that T is generalized Drazin-Riesz semi-Fredholm operator. In particular, the class $\mathbf{GDR\Phi}(\mathcal{X})$ consists of generalized Drazin-Riesz Fredholm operators.

In what follows we characterize the classes $\mathbf{GDRR}_i$. Theorems 4.3.2 and 4.3.3 can be proved by an analysis similar to that in the proof of Theorem 4.2.3.

Theorem 4.3.2. *Let $T \in L(\mathcal{X})$. The following conditions are equivalent:*

(i) *There exists $(M, N) \in Red(T)$ such that T_M is bounded below and T_N is Riesz, that is $T \in \mathbf{gDR}\mathcal{M}(\mathcal{X})$;*

(ii) *T admits a GKRD and $0 \notin \operatorname{int} \sigma_{ap}(T)$;*

(iii) *T admits a GKRD and T has the SVEP at 0;*

(iv) *There exists $(M, N) \in Red(T)$ such that T_M is upper semi-Browder and T_N is Riesz, that is $T \in \mathbf{gDR}\mathcal{B}_+(\mathcal{X})$;*

(v) *T admits a GKRD and $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_+}(T)$;*

(vi) *T admits a GKRD and $0 \notin \operatorname{int} \sigma_{\mathcal{B}_+}(T)$;*

(vii) *There exists a bounded projection $P \in L(\mathcal{X})$ which commutes with T such that $T + P$ is upper semi-Browder and TP is Riesz.*

Theorem 4.3.3. *Let $T \in L(\mathcal{X})$. The following conditions are equivalent:*

(i) *There exists $(M, N) \in Red(T)$ such that T_M is surjective and T_N is Riesz, that is $T \in \mathbf{gDR}\mathcal{Q}(\mathcal{X})$;*

(ii) *T admits a GKRD and $0 \notin \operatorname{int} \sigma_{su}(T)$;*

(iii) *T admits a GKRD and T' has the SVEP at 0;*

(iv) *There exists* $(M,N) \in Red(T)$ *such that* T_M *is lower semi-Browder and* T_N *is Riesz, that is* $T \in \mathbf{gDR}\mathcal{B}_-(\mathcal{X})$*;*

(v) T *admits a GKRD and* $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_-}(T)$*;*

(vi) T *admits a GKRD and* $0 \notin \operatorname{int} \sigma_{\mathcal{B}_-}(T)$*;*

(vii) *There exists a bounded projection* $P \in L(\mathcal{X})$ *which commutes with* T *such that* $T+P$ *is lower semi-Browder and* TP *is Riesz.*

Corollary 4.3.4. *Let* $T \in L(\mathcal{X})$ *and* $\lambda_0 \in \mathbb{C}$*. If* $T - \lambda_0 I$ *admits a GKRD, then the following statements are equivalent:*

(i) T *has the SVEP at* λ_0 (T' *has the SVEP at* λ_0)*;*

(ii) λ_0 *is not an interior point of* $\sigma_{ap}(T)$ (λ_0 *is not an interior point of* $\sigma_{su}(T)$)*;*

(iii) $\sigma_{\mathcal{B}_+}(T)$ *does not cluster at* λ_0 ($\sigma_{\mathcal{B}_-}(T)$ *does not cluster at* λ_0)*;*

(iv) λ_0 *is not an interior point of* $\sigma_{\mathcal{B}_+}(T)$ (λ_0 *is not an interior point of* $\sigma_{\mathcal{B}_-}(T)$)*.*

Proof. Follows from the equivalences (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi) of Theorems 4.3.2 and 4.3.3. $\square$

Remark 4.3.5. Let $T \in L(\mathcal{X})$ be a Riesz operator with infinite spectrum. By (1.7) and (1.8), both T and T' have the SVEP at 0. On the other hand, 0 is an accumulation point of $\sigma_{ap}(T)$ and $\sigma_{su}(T)$. Consequently, if $T - \lambda_0$ admits a GKRD decomposition, then the statement that T (T') has the SVEP at λ_0 is not in general equivalent to the statement that $\sigma_{ap}(T)$ ($\sigma_{su}(T)$) does not cluster at λ_0.

Theorem 4.3.6. *Let* $T \in L(\mathcal{X})$ *and* $7 \leq i \leq 12$*. The following conditions are equivalent:*

(i) *There exists* $(M,N) \in Red(T)$ *such that* $T_M \in \mathbf{R}_i$ *and* T_N *is Riesz, that is* $T \in \mathbf{gDRR}_i$*;*

(ii) T *admits a GKRD and* $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$*;*

(iii) T *admits a GKRD and* $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$*;*

(iv) *There exists a bounded projection* P *on* $\mathcal{X}$ *which commutes with* T *such that* $T+P \in \mathbf{R}_i$ *and* TP *is Riesz.*

Proof. (i)$\Longrightarrow$(ii). Suppose that there exists $(M,N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is Riesz. As in the proof of Theorem 4.2.3 we obtain that T admits a GKRD (see (vii)$\Longrightarrow$(viii)).

Since $\mathbf{R}_i$ is open, from $T_M \in \mathbf{R}_i$ it follows that there exists $\epsilon > 0$ such that $D(0,\epsilon) \subset \rho_{\mathbf{R}_i}(T_M)$. According to [1, Theorem 3.111], $\sigma_{\mathbf{R}_i}(T_N) \subset \{0\}$, and so $D(0,\epsilon) \setminus \{0\} \subset \rho_{\mathbf{R}_i}(T_M) \cap \rho_{\mathbf{R}_i}(T_N)$. By Lemma 1.4.7(i) and (ii), $\rho_{\mathbf{R}_i}(T_M) \cap \rho_{\mathbf{R}_i}(T_N) \subset \rho_{\mathbf{R}_i}(T)$, and hence $D(0,\epsilon) \setminus \{0\} \subset \rho_{\mathbf{R}_i}(T)$. Therefore, $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$.

(ii)$\Longrightarrow$(iii) Obvious.

(iii)$\Longrightarrow$(i). Suppose that T admits a GKRD and $0 \notin \operatorname{int} \sigma_{\Phi_+}(T)$. Then there exists $(M, N) \in \mathit{Red}(T)$ such that T_M is Kato and T_N is Riesz, and $0 \in \operatorname{acc} \rho_{\Phi_+}(T)$. According to Lemma 1.4.7(i), $0 \in \operatorname{acc} \rho_{\Phi_+}(T_M)$. From Proposition 1.4.9(i) it follows that T_M is upper semi-Fredholm. The cases $i = 8$ and $i = 9$ can be proved similarly.

Suppose that T admits a GKRD and $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$. Then there exists $(M, N) \in \mathit{Red}(T)$ such that T_M is Kato and T_N is Riesz. We show that $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T_M)$. Let $\epsilon > 0$. From $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$ it follows that there exists $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$ and $T - \lambda \in \mathcal{W}_+(\mathcal{X})$. As T_N is Riesz, $T_N - \lambda$ is Fredholm of index zero, and according to Lemma 1.4.7(iii), $T_M - \lambda \in \mathcal{W}_+(M)$, that is $\lambda \in \rho_{\mathcal{W}_+}(T_M)$. Therefore, $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T_M)$ and from Proposition 1.4.9 (ii) it follows that T_M is upper semi-Weyl, and so $T \in \mathbf{gDR}\mathcal{W}_+(\mathcal{X})$. The cases $i = 11$ and $i = 12$ can be proved similarly.

(i) $\Longrightarrow$ (iv). Suppose that there exists $(M, N) \in \mathit{Red}(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is Riesz. Let $P \in L(\mathcal{X})$ be a projection such that $N(P) = M$ and $R(P) = N$. Then $TP = PT$, and since $TP = (TP)_M \oplus (TP)_N = 0 \oplus T_N$, from Lemma 1.4.13 it follows that TP is Riesz. Also, $\sigma_{\mathbf{R}_i}(T_N) \subset \{0\}$, and so $(T + P)_N = T_N + I_N \in \mathbf{R}_i$, where I_N is identity on N. Since $(T + P)_M = T_M \in \mathbf{R}_i$, we see that $T + P \in \mathbf{R}_i$ by Lemma 1.4.7(i) and (ii).

(iv) $\Longrightarrow$ (i). Suppose that there exists a projection $P \in L(\mathcal{X})$ that commutes with T such that $T + P \in \mathbf{R}_i$ and TP is Riesz. For $M = N(P)$ and $N = R(P)$ we have that $(M, N) \in \mathit{Red}(T)$ and $T_N = (TP)_N$ is Riesz. For $i \in \{7, 8, 9\}$, from Lemma 1.4.7(i) it follows that $T_M = (T + P)_M \in \mathbf{R}_i$. Suppose that $i \in \{10, 11, 12\}$. Since T_N is Riesz, it follows that $T_N + I_N$ is Weyl. Now, from $T + P = (T + P)_M \oplus (T + P)_N = T_M \oplus (T_N + I_N)$ and Lemma 1.4.7(iii), it follows that $T_M \in \mathbf{R}_i$. $\square$

The following two corollaries follow at once from Theorems 4.2.3, 4.3.2, 4.3.3 and 4.3.6.

Corollary 4.3.7. *Let $T \in L(\mathcal{X})$ and $7 \leq i \leq 12$. If $T - \lambda_0$ admits a GKRD, then the following statements are equivalent:*

(i) *λ_0 is not an interior point of $\sigma_{\mathbf{R}_i}(T)$;*

(ii) *$\sigma_{\mathbf{R}_i}(T)$ does not cluster at λ_0.*

Corollary 4.3.8. *Let $T \in L(\mathcal{X})$ and let $1 \leq i \leq 12$. If $0 \in \partial\sigma_{\mathbf{R}_i}(T)$, then T admits a generalized Kato-Riesz decomposition if and only if T belongs to $\mathbf{gDRR}_i$.*

Theorem 4.3.9. *Let $T \in L(\mathcal{X})$ and let f be a complex analytic function in a neighborhood of $\sigma(T)$. If $T \in \mathbf{gDRR}_i$ and $f^{-1}(0) \cap \sigma_{\mathbf{R}_i}(T) = \{0\}$, then $f(T) \in \mathbf{gDRR}_i$, $1 \leq i \leq 12$.*

Proof. We give the proof only for the cases $i = 4$ and $i = 10$ since other cases can be proved similarly. Suppose that $T \in \mathbf{gDR}\mathcal{B}_+(\mathcal{X})$. Then, there exists

$(M, N) \in Red(T)$ such that T_M is upper semi-Browder and T_N is Riesz. The pair (M, N) completely reduces $(\lambda I - T)^{-1}$ for every $\lambda \in \rho(T)$. It follows that $f(T) = \frac{1}{2\pi i} \int\limits_{\gamma} f(\lambda)(\lambda I - T)^{-1} d\lambda$, where γ is a contour surrounding $\sigma(T)$ and which lies in the domain of f, is also reduced by the pair (M, N). It is routine to verify that $f(T)_M = f(T_M)$ and $f(T)_N = f(T_N)$. Consequently, $f(T) = f(T_M) \oplus f(T_N)$.

In addition, suppose that $f^{-1}(0) \cap \sigma_{\mathcal{B}_+}(T) = \{0\}$. Using the fact that $0 \notin \sigma_{\mathcal{B}_+}(T_M) \subset \sigma_{\mathcal{B}_+}(T)$, we obtain $0 \notin f(\sigma_{\mathcal{B}_+}(T_M))$. According to the spectral mapping theorem, $0 \notin \sigma_{\mathcal{B}_+}(f(T_M))$ [62, Theorem 3.4], so $f(T_M)$ is upper semi-Browder. Since $f(0) = 0$, it follows that $f(T_N)$ is Riesz by [1, Theorem 3.113 (i)]. Consequently, $f(T) \in \mathbf{gDR}\mathcal{B}_+(\mathcal{X})$.

Suppose that $T \in \mathbf{gDR}\mathcal{W}_+(\mathcal{X})$ and $f^{-1}(0) \cap \sigma_{\mathcal{W}_+}(T) = \{0\}$. Then there exists $(M, N) \in Red(T)$ such that T_M is upper semi-Weyl and T_N is Riesz. As above we conclude that $f(T) = f(T_M) \oplus f(T_N)$ and that $f(T_N)$ is Riesz. From $0 \notin \sigma_{\mathcal{W}_+}(T_M) \subset \sigma_{\mathcal{W}_+}(T)$, we obtain $0 \notin f(\sigma_{\mathcal{W}_+}(T_M))$. Since $\sigma_{\mathcal{W}_+}(f(T_M)) \subset f(\sigma_{\mathcal{W}_+}(T_M))$ [62, Theorem 3.3], it follows that $0 \notin \sigma_{\mathcal{W}_+}(f(T_M))$, and so $f(T_M)$ is upper semi-Weyl. Consequently, $f(T) \in \mathbf{gDR}\mathcal{W}_+(\mathcal{X})$. □

Proposition 4.3.10. *Let $T \in L(\mathcal{X})$ and let f be a complex analytic function in a neighborhood of $\sigma(T)$ such that $f^{-1}(0) \cap \operatorname{acc} \sigma(T) = \emptyset$. Then $f(T) = A + K$, where $A \in L(\mathcal{X})$ is generalized Dazin-Riesz Fredholm and $K \in L(\mathcal{X})$ is compact.*

Proof. Since $\sigma(\pi(T)) \subset \sigma(T)$, f is analytic in a neighborhood of $\sigma(\pi(T))$ and $f(\pi(T)) = \pi(f(T))$, where $\pi : L(\mathcal{X}) \to L(\mathcal{X})/K(\mathcal{X})$ is the natural homomorphism. According to [34, Theorem 2],

$$\operatorname{acc} \sigma(\pi(f(T)) = \operatorname{acc} \sigma(f(\pi(T)) \subset f(\operatorname{acc} \sigma(\pi(T)) \subset f(\operatorname{acc} \sigma(T)).$$

By the assumption it follows that $0 \notin f(\operatorname{acc} \sigma(T))$. Consequently, $0 \notin \operatorname{acc} \sigma(\pi(f(T))$, i.e. $\pi(f(T))$ is generalized Drazin invertible. Now, we apply [15, Theorem 3.11], which completes the proof. □

Corollary 4.3.11. *Let $T \in L(\mathcal{X})$ have finite spectrum and let f be a complex analytic function in a neighborhood of $\sigma(T)$. Then $f(T) = A + K$, where $A \in L(\mathcal{X})$ is generalized Drazin-Riesz Fredholm and $K \in L(\mathcal{X})$ is compact.*

Proof. Since $\operatorname{acc} \sigma(T) = \emptyset$, the condition $f^{-1}(0) \cap \operatorname{acc} \sigma(T) = \emptyset$ is automatically satisfied. The result follows by Proposition 4.3.10. □

An operator $T \in L(\mathcal{X})$ is polynomially Riesz if there exists a nonzero complex polynomial p such that $p(T)$ is Riesz. According to [79], there will be a unique polynomial π_T of minimal degree with leading coefficient 1 such that $\pi_T(T)$ is Riesz. The polynomial π_T is called the minimal polynomial of T.

Corollary 4.3.12. *Let $T \in L(\mathcal{X})$ be polynomially Riesz and let f be a complex analytic function in a neighborhood of $\sigma(T)$ such that $f^{-1}(0) \cap \pi_T^{-1}(0) = \emptyset$. Then $f(T) = A + K$, where $A \in L(\mathcal{X})$ is generalized Drazin-Riesz Fredholm and $K \in L(\mathcal{X})$ is compact.*

Proof. Notice that if $T \in L(\mathcal{X})$ is polynomially Riesz, then $\operatorname{acc}\sigma(T) \subset \sigma_{\mathcal{B}}(T) = \pi_T^{-1}(0)$, so $f^{-1}(0) \cap \operatorname{acc}\sigma(T) \subset f^{-1}(0) \cap \pi_T^{-1}(0) = \emptyset$. The assertion follows by Proposition 4.3.10. $\square$

Chapter 5

B-Fredholm Banach algebra elements

5.1 Motivation

As we mentioned earlier, the Atkinson theorem states that necessary and sufficient for a Banach space operator to be Fredholm is that its coset in the Calkin algebra is invertible, i.e. $\Phi(\mathcal{X}) = \pi^{-1}((L(\mathcal{X})/K(\mathcal{X}))^{-1})$. The introduction of a Fredholm theory relative to a Banach algebra homomorphism was motivated by this well-known result and Theorems 1.4.4 and 1.4.6. This generalization is due to R. Harte [34].

Definition 5.1.1. [34] Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. An element $a \in \mathcal{A}$ will be said to be

(i) Fredholm, if $\mathcal{T}(a)$ is invertible in $\mathcal{B}$;

(ii) Weyl, if there exist $b, c \in \mathcal{A}$, $b \in \mathcal{A}^{-1}$ and $c \in \mathcal{T}^{-1}(0)$, such that $a = b + c$;

(iii) Browder, if there exist $b, c \in \mathcal{A}$, $b \in \mathcal{A}^{-1}$, $c \in \mathcal{T}^{-1}(0)$ and $bc = cb$, such that $a = b + c$.

($\mathcal{T}^{-1}(0)$ denotes the kernel of the homomorphism $\mathcal{T}$.)

The sets of Fredholm, Weyl and Browder elements relative to the homomorphism $\mathcal{T} \colon \mathcal{A} \to \mathcal{B}$ will be denoted by $\mathcal{F}_{\mathcal{T}}(\mathcal{A})$, $\mathcal{W}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{B}_{\mathcal{T}}(\mathcal{A})$, respectively. Naturally, these sets lead to the introduction of the corresponding spectra.

Definition 5.1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. Given $a \in \mathcal{A}$, the Fredholm spectrum, the Weyl spectrum and the Browder spectrum of a relative to the homomorphism $\mathcal{T} \colon \mathcal{A} \to \mathcal{B}$ are respectively the following sets:

(i) $\sigma_{\mathcal{F}_{\mathcal{T}}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{F}_{\mathcal{T}}(\mathcal{A})\} = \sigma(\mathcal{T}(a))$;

(ii) $\sigma_{\mathcal{W}_{\mathcal{T}}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{W}_{\mathcal{T}}(\mathcal{A})\}$;

(iii) $\sigma_{\mathcal{B}_{\mathcal{T}}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{B}_{\mathcal{T}}(\mathcal{A})\}$.

It is clear that $\mathcal{B}_{\mathcal{T}}(\mathcal{A}) \subset \mathcal{W}_{\mathcal{T}}(\mathcal{A}) \subset \mathcal{F}_{\mathcal{T}}(\mathcal{A})$ and that $\sigma_{\mathcal{F}_{\mathcal{T}}}(a) \subset \sigma_{\mathcal{W}_{\mathcal{T}}}(a) \subset \sigma_{\mathcal{B}_{\mathcal{T}}}(a) \subset \sigma(a)$. Also it is known that the sets $\sigma_{\mathcal{F}_{\mathcal{T}}}(a), \sigma_{\mathcal{W}_{\mathcal{T}}}(a)$ and $\sigma_{\mathcal{B}_{\mathcal{T}}}(a)$ are non-empty and compact. This theory has been developed by many authors, see for example [5, 24, 34, 35, 36, 37, 56, 75, 76, 78].

According to [12, Theorem 3.4], $T \in \mathbf{B\Phi}(\mathcal{X})$ if and only if $\tilde{\pi}(T) \in (L(\mathcal{X})/F(\mathcal{X}))^D$, where $\tilde{\pi} : L(\mathcal{X}) \to L(\mathcal{X})/F(\mathcal{X})$ is the quotient homomorphism. Moreover, according to [9, Corollary 4.4], $T \in L(\mathcal{X})$ is a B-Weyl operator if and only if $T = S + F$, where $S \in L(\mathcal{X})^D$ and $F \in F(\mathcal{X})$. The following definition is motivated by these observations.

Definition 5.1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} \colon \mathcal{A} \to \mathcal{B}$. An element $a \in \mathcal{A}$ is said to be

(i) B-Fredholm, if $\mathcal{T}(a) \in \mathcal{B}^D$;

(ii) B-Weyl, if there exist $b, c \in \mathcal{A}$, $b \in \mathcal{A}^D$ and $c \in \mathcal{T}^{-1}(0)$, such that $a = b+c$;

(iii) generalized B-Fredholm, if $\mathcal{T}(a) \in \mathcal{B}^{gD}$;

The set of B-Fredholm (respectively B-Weyl, generalized B-Fredholm) elements of the unital Banach algebra $\mathcal{A}$ relative to the homomorphism $\mathcal{T} \colon \mathcal{A} \to \mathcal{B}$ will be denoted by $\mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ (respectively $\mathcal{BW}_{\mathcal{T}}(\mathcal{A})$, $\mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$).

The algebra $L(\mathcal{X})/F(\mathcal{X})$ is not a Banach algebra, so it seems that Definition 5.1.3 does not generalize the class of B-Fredholm operators properly. This fact was observed by M. Berkani and he has redefined the notion of B-Fredholm elements [10, 11]. According to [10, Definition 1.2], an element $a \in \mathcal{A}$ is B-Fredholm if $\pi(a)$ is Drazin invertible in $\mathcal{A}/J$, where $J \subset \mathcal{A}$ is an ideal and $\pi : \mathcal{A} \to \mathcal{A}/J$ is the natural homomorphism. Whatever, in this chapter we study the objects introduced in Definition 5.1.3.

5.2 B-Fredholm and generalized B-Fredholm elements

Definition 5.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. Given $a \in \mathcal{A}$, the B-Fredholm spectrum, the B-Weyl spectrum and the generalized B-Fredholm spectrum of a relative to the homomorphism $\mathcal{T} \colon \mathcal{A} \to \mathcal{B}$ are respectively the following sets:

(i) $\sigma_{\mathcal{BF}_{\mathcal{T}}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{BF}_{\mathcal{T}}(\mathcal{A})\} = \sigma_D(\mathcal{T}(a))$;

(ii) $\sigma_{\mathcal{BW}_{\mathcal{T}}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{BW}_{\mathcal{T}}(\mathcal{A})\}$;

(iii) $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})\} = \sigma_{gD}(\mathcal{T}(a))$.

Remark 5.2.2. It is not difficult to prove the following statements:

(i) $\mathcal{F}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathcal{BF}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$;

(ii) $\mathcal{BF}_{\mathcal{T}}(\mathcal{A}) + \mathcal{T}^{-1}(0) = \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$;
(iii) $\mathcal{W}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathcal{BW}_{\mathcal{T}}(\mathcal{A}) = \mathcal{A}^D + \mathcal{T}^{-1}(0)$;
(iv) $\mathcal{A}^D \subseteq \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{A}^{gD} \subseteq \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$;
(v) $\mathcal{GBF}_{\mathcal{T}}(\mathcal{A}) + \mathcal{T}^{-1}(0) = \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{A}^D \subseteq \mathcal{A}^{gD} \subseteq \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$;
(vi) $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a) \subseteq \sigma_{\mathcal{BF}_{\mathcal{T}}}(a) \subseteq \sigma_{\mathcal{F}_{\mathcal{T}}}(a)$, $\sigma_{\mathcal{BF}_{\mathcal{T}}}(a) \subseteq \sigma_D(a)$ and $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a) \subseteq \sigma_{gD}(a)$.

If $S \subset \mathcal{A}$ is an arbitrary set we will say that $a \in \mathrm{Poly}^{-1}(S)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in S$. In particular, $\mathrm{Poly}^{-1}(\{0\})$ is the set of algebraic elements of $\mathcal{A}$. According to [78], if a is algebraic then there is a unique polynomial p of minimal degree with leading coefficient 1 such that $p(a) = 0$; p is called the minimal polynomial of a.

In the following theorem the main properties of the (generalized) B-Fredholm spectrum will be studied.

Theorem 5.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$. If $a \in \mathcal{A}$, then the following statements hold.*
(i) *$\mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$ are regularities.*
(ii) *If $f\colon U \to \mathbb{C}$ is an analytic function defined on a neighbourhood of $\sigma(a)$ which is non-constant on each component of its domain of definition, then*

$$\sigma_{\mathcal{BF}_{\mathcal{T}}}(f(a)) = f(\sigma_{\mathcal{BF}_{\mathcal{T}}}(a)), \quad \text{and} \quad \sigma_{\mathcal{GBF}_{\mathcal{T}}}(f(a)) = f(\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a)).$$

(iii) *$\sigma_{\mathcal{BF}_{\mathcal{T}}}(a)$ and $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a)$ are closed.*
(iv) *$\sigma_{\mathcal{BF}_{\mathcal{T}}}(a) = \emptyset$ if and only if $a \in \mathrm{Poly}^{-1}(\mathcal{T}^{-1}(0))$, equivalently, $\mathcal{T}(a) \in \mathrm{Poly}^{-1}(\{0\})$.*
(v) *$\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a) = \emptyset$ if and only if $\operatorname{acc}\sigma_{\mathcal{F}_{\mathcal{T}}}(a) = \emptyset$.*
(vi) *$\sigma_{\mathcal{BF}_{\mathcal{T}}}(a)$ is countable if and only if $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a)$ is countable if and only if $\sigma_{\mathcal{F}_{\mathcal{T}}}(a)$ is countable.*

Proof. (i). Recalling that both $\mathcal{A}^D$ and $\mathcal{A}^{gD}$ are regularities, and applying Proposition 1.5.11, we obtain that $\mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$ are regularities.
(ii). Apply Theorem 1.5.12 to $\mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$.
(iii). Recall that $\sigma_{\mathcal{BF}_{\mathcal{T}}}(a) = \sigma_D(\mathcal{T}(a))$ and $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a) = \sigma_{gD}(\mathcal{T}(a))$. Then, use the fact that both $\sigma_D(\mathcal{T}(a))$ and $\sigma_{gD}(\mathcal{T}(a))$ are closed; see [12, Proposition 2.5] and [54, Proposition 1.5(ii)].
(iv). Since $\sigma_{\mathcal{BF}_{\mathcal{T}}}(a) = \sigma_D(\mathcal{T}(a))$, this statement can be deduced from [14, Theorem 2.1].
(v). Use $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a) = \sigma_{gD}(\mathcal{T}(a)) = \operatorname{acc}\sigma(\mathcal{T}(a)) = \operatorname{acc}\sigma_{\mathcal{F}_{\mathcal{T}}}(a)$.
(vi). Clearly, $\sigma_{\mathcal{BF}_{\mathcal{T}}}(a) = \sigma_D(\mathcal{T}(a))$, $\sigma_{\mathcal{GBF}_{\mathcal{T}}}(a) = \sigma_{gD}(\mathcal{T}(a))$, and $\sigma_{\mathcal{F}_{\mathcal{T}}}(a) = \sigma(\mathcal{T}(a))$. According to [14, Theorem 2.2], necessary and sufficient for $\sigma_D(\mathcal{T}(a))$ to be countable is that $\sigma(\mathcal{T}(a))$ is countable. Also, $\sigma(\mathcal{T}(a)) = \operatorname{acc}\sigma(\mathcal{T}(a)) \cup$

$\operatorname{iso}\sigma(\mathcal{T}(a)) = \sigma_{gD}(\mathcal{T}(a)) \cup \operatorname{iso}\sigma(\mathcal{T}(a))$, and we recall that the set $\operatorname{iso}\sigma(\mathcal{T}(a))$ is countable. As a result, $\sigma_{gD}(\mathcal{T}(a))$ is countable if and only if $\sigma_D(\mathcal{T}(a))$ is countable if and only if $\sigma(\mathcal{T}(a))$ is countable. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$. The range of the homomorphism $\mathcal{T}$ will be denoted by $R(\mathcal{T})$. Let $\mathcal{R}_{\mathcal{T}}(\mathcal{A}) = \{a \in \mathcal{A}\colon \mathcal{T}(a) \in \mathcal{B}^{qnil}\}$ be the set of Riesz elements of $\mathcal{A}$ relative to the homomorphism $\mathcal{T}$ and $\mathcal{N}_{\mathcal{T}}(\mathcal{A}) = \{a \in \mathcal{A}\colon \text{there exists}\, k \in \mathbb{N}\, \text{such that}\, a^k \in \mathcal{T}^{-1}(0)\} = \{a \in \mathcal{A}\colon \mathcal{T}(a) \in \mathcal{B}^{nil}\}$ be the set of $\mathcal{T}$-nilpotent elements of $\mathcal{A}$; see [15, 78]. Clearly, $\mathcal{N}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathcal{R}_{\mathcal{T}}(\mathcal{A})$.

On the other hand, the homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ will be said to have the lifting property, if given $q \in \mathcal{B}^\bullet$, there is $p \in \mathcal{A}^\bullet$ such that $\mathcal{T}(p) = q$, i.e., $\mathcal{T}(\mathcal{A}^\bullet) = \mathcal{B}^\bullet$, which is equivalent to the conjunction of the following two conditions: $\mathcal{T}^{-1}(\mathcal{B}^\bullet) = \mathcal{A}^\bullet + \mathcal{T}^{-1}(0)$ and $\mathcal{B}^\bullet \subset R(\mathcal{T})$. This property does not hold in general. In particular, if $\mathcal{B}^\bullet \subset R(\mathcal{T})$ and $\mathcal{T}$ has the Riesz property, i.e., if for every $z \in \mathcal{T}^{-1}(0)$, $\sigma(z)$ is either finite or is a sequence converging to 0, then $\mathcal{T}$ has the lifting property, see [24, Lemma 2]. Consequently, if $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ is surjective and has the Riesz property, then $\mathcal{T}$ has the lifting property. Next, (generalized) B-Fredholm elements will be characterized.

Theorem 5.2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$. Suppose that $\mathcal{T}$ has the lifting property. Then, the following statements hold.*

(i) *Necessary and sufficient for $a \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$ is that there exists $p \in \mathcal{A}^\bullet$ such that $a + p \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$, $pa(1-p)$ and $(1-p)ap \in \mathcal{T}^{-1}(0)$ and $pap \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$.*

(ii) *Necessary and sufficient for $a \in \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ is that there exists $p \in \mathcal{A}^\bullet$ such that $a + p \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$, $pa(1-p)$ and $(1-p)ap \in \mathcal{T}^{-1}(0)$ and $pap \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$.*

Proof. (i). If $a \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$, then $\mathcal{T}(a) \in \mathcal{B}^{gD}$. In particular, according to Theorem 1.5.7, there is $q \in \mathcal{B}^\bullet$ such that $q\mathcal{T}(a) = \mathcal{T}(a)q$, $\mathcal{T}(a) + q \in \mathcal{B}^{-1}$ and $\mathcal{T}(a)q = q\mathcal{T}(a)q \in \mathcal{B}^{qnil}$. Since $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ has the lifting property, there is $p \in \mathcal{A}^\bullet$ such that $\mathcal{T}(p) = q$.

Now, the identity $q\mathcal{T}(a) = \mathcal{T}(a)q$ implies that $pa - ap \in \mathcal{T}^{-1}(0)$. However, multiplying by $1-p$, it is easy to prove that $pa(1-p)$ and $(1-p)ap \in \mathcal{T}^{-1}(0)$. In addition, since $\mathcal{T}(a+p) \in \mathcal{B}^{-1}$, $a+p \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$. Finally, since $\mathcal{T}(pap) \in \mathcal{B}^{qnil}$, $pap \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$.

Suppose that there exists $p \in \mathcal{A}^\bullet$ such that $a + p \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$, $pa(1-p)$ and $(1-p)ap \in \mathcal{T}^{-1}(0)$ and $pap \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$. Consequently, $q = \mathcal{T}(p) \in \mathcal{B}^\bullet$ and $q\mathcal{T}(a) = \mathcal{T}(a)q$, $\mathcal{T}(a) + q \in \mathcal{B}^{-1}$ and $\mathcal{T}(a)q = q\mathcal{T}(a)q \in \mathcal{B}^{qnil}$. Thus, according to Theorem 1.5.7, $\mathcal{T}(a) \in \mathcal{B}^{gD}$, equivalently, $a \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$.

(ii). Apply the same argument used in the proof of statement (i), using in particular Proposition 1.5.2 instead of Theorem 1.5.7. □

Next some basic properties of the objects introduced in Definition 5.1.3 will be considered.

Theorem 5.2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$. Then, the following statements hold.*
(i) $\mathcal{T}^{-1}(0) \subseteq \mathcal{T}^{-1}(\mathcal{B}^\bullet) \subseteq \mathcal{BF}_\mathcal{T}(\mathcal{A})$.
(ii) $\mathcal{A}^\bullet \subseteq \mathcal{T}^{-1}(\mathcal{B}^\bullet) \subseteq \mathcal{BF}_\mathcal{T}(\mathcal{A})$.
(iii) $\mathcal{F}_\mathcal{T}(\mathcal{A})$ *is a proper subset of* $\mathcal{BF}_\mathcal{T}(\mathcal{A})$.
(iv) $\mathcal{W}_\mathcal{T}(\mathcal{A})$ *is a proper subset of* $\mathcal{BW}_\mathcal{T}(\mathcal{A})$.
(v) $\mathcal{A}^\bullet \setminus \mathcal{T}^{-1}(1) \subseteq \mathcal{BF}_\mathcal{T}(\mathcal{A}) \setminus \mathcal{F}_\mathcal{T}(\mathcal{A})$.
(vi) *If* $a,\ b \in \mathcal{BF}_\mathcal{T}(\mathcal{A})$ *are such that* $ab - ba \in \mathcal{T}^{-1}(0)$, *then* $ab \in \mathcal{BF}_\mathcal{T}(\mathcal{A})$.
(vii) *If* $a \in \mathcal{BW}_\mathcal{T}(\mathcal{A})$, *then* $a^n \in \mathcal{BW}_\mathcal{T}(\mathcal{A})$ *for every* $n \in \mathbb{N}$.
(viii) $\mathcal{BW}_\mathcal{T}(\mathcal{A}) \setminus \mathcal{W}_\mathcal{T}(\mathcal{A}) \subseteq \mathcal{BF}_\mathcal{T}(\mathcal{A}) \setminus \mathcal{F}_\mathcal{T}(\mathcal{A})$.
(ix) $\sigma_{\mathcal{BW}_\mathcal{T}}(a) = \bigcap_{c \in \mathcal{T}^{-1}(0)} \sigma_D(a+c)$ $(a \in \mathcal{A})$.
(x) *The set* $\sigma_{\mathcal{BW}_\mathcal{T}}(a)$ *is closed* $(a \in \mathcal{A})$.

Proof. (i). This statement can be easily derived from the inclusions

$$\{0\} \subseteq \mathcal{B}^\bullet \subseteq \mathcal{B}^D.$$

(ii). Clearly, $\mathcal{T}(\mathcal{A}^\bullet) \subseteq \mathcal{B}^\bullet$ and $\mathcal{B}^\bullet \subseteq \mathcal{B}^D$.
(iii). Since $\{0\} \cap \mathcal{B}^{-1} = \emptyset$, then $\mathcal{T}^{-1}(0) \cap \mathcal{F}_\mathcal{T}(\mathcal{A}) = \mathcal{T}^{-1}(0) \cap \mathcal{T}^{-1}(\mathcal{B}^{-1}) = \emptyset$. Consequently, $\mathcal{T}^{-1}(0) \subseteq \mathcal{BF}_\mathcal{T}(\mathcal{A}) \setminus \mathcal{F}_\mathcal{T}(\mathcal{A})$.
(iv). Clearly, $\mathcal{T}^{-1}(0) \subseteq \mathcal{BW}_\mathcal{T}(\mathcal{A})$. In addition, according to the proof of statement (iii), $\mathcal{T}^{-1}(0) \cap \mathcal{W}_\mathcal{T}(\mathcal{A}) \subseteq \mathcal{T}^{-1}(0) \cap \mathcal{F}_\mathcal{T}(\mathcal{A}) = \emptyset$. Therefore, $\mathcal{T}^{-1}(0) \subseteq \mathcal{BW}_\mathcal{T}(\mathcal{A}) \setminus \mathcal{W}_\mathcal{T}(\mathcal{A})$.
(v). Note that $\mathcal{A}^\bullet \setminus \mathcal{T}^{-1}(1) \subseteq \mathcal{A}^\bullet \subseteq \mathcal{BF}_\mathcal{T}(\mathcal{A})$. In addition, if $a \in \mathcal{A}^\bullet \setminus \mathcal{T}^{-1}(1)$, then $\mathcal{T}(a) \in \mathcal{B}^\bullet \setminus \mathcal{B}^{-1}$. In particular, $a \notin \mathcal{F}_\mathcal{T}(\mathcal{A})$.
(vi). Apply [12, Proposition 2.6].
(vii). Let $a \in \mathcal{BW}_\mathcal{T}(\mathcal{A})$. Then $a = b + c$, where $b \in \mathcal{A}^D$ and $c \in \mathcal{T}^{-1}(0)$. It will be proved that $a^n = b^n + x_n$, where $x_n \in \mathcal{T}^{-1}(0)$, for every $n \in \mathbb{N}$. In fact, for $n = 1$ it is obvious. Suppose that this statement is true for $k \in \mathbb{N}$. Then,

$$a^{k+1} = a^k a = (b^k + x_k)(b + c) = b^{k+1} + (b^k c + x_k b + x_k c).$$

Clearly, $b^k c + x_k b + x_k c \in \mathcal{T}^{-1}(0)$. As a result, since $b^{k+1} \in \mathcal{A}^D$, $a^{k+1} \in \mathcal{BW}_\mathcal{T}(\mathcal{A})$.
(viii). Clearly, $\mathcal{BW}_\mathcal{T}(\mathcal{A}) \setminus \mathcal{W}_\mathcal{T}(\mathcal{A}) \subseteq \mathcal{BW}_\mathcal{T}(\mathcal{A}) \subseteq \mathcal{BF}_\mathcal{T}(\mathcal{A})$. If $a \in \mathcal{BW}_\mathcal{T}(\mathcal{A}) \setminus \mathcal{W}_\mathcal{T}(\mathcal{A})$, then there exist $c \in \mathcal{A}^D$ and $d \in \mathcal{T}^{-1}(0)$ such that $a = c + d$. In addition, according to Proposition 1.5.2, there is $p \in \mathcal{A}^\bullet$ such that

$$cp = pc, \qquad c + p \in \mathcal{A}^{-1}, \qquad cp \text{ is nilpotent.}$$

Note that since $a = (c + p) + (d - p)$ and $a \notin \mathcal{W}_\mathcal{T}(\mathcal{A}) = \mathcal{A}^{-1} + \mathcal{T}^{-1}(0)$, $p \notin \mathcal{T}^{-1}(0)$. Let $0 \neq q = \mathcal{T}(p) \in \mathcal{B}^\bullet$. Then, $q\mathcal{T}(c) = \mathcal{T}(c)q$, $\mathcal{T}(c)+q \in \mathcal{B}^{-1}$ and

$\mathcal{T}(c)q$ is nilpotent. Thus, $\mathcal{T}(c)$ is Drazin invertible but not invertible ($q \neq 0$), which implies that $c \notin \mathcal{F}_{\mathcal{T}}(\mathcal{A})$. However, since $d \in \mathcal{T}^{-1}(0)$, $a \notin \mathcal{F}_{\mathcal{T}}(\mathcal{A})$.

(ix)-(x). These statements can be easily deduced. □

5.3 Perturbations of B-Fredholm elements

Given a nonempty $S \subseteq \mathcal{A}$, the commuting perturbation class of S is the set

$$P_{comm}(S) = \{a \in \mathcal{A} : S +_{comm} \{a\} \subset S\},$$

where, if $H, K \subseteq \mathcal{A}$

$$H +_{comm} K = \{c + d : (c, d) \in H \times K, cd = dc\}.$$

Evidently, $0 \in P_{comm}(S)$, and so $P_{comm}(S)$ is always a nonempty set.

Remark 5.3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider the homomorphism $\mathcal{T} \colon \mathcal{A} \to \mathcal{B}$.

(i) Let $K \subseteq \mathcal{B}$, and $1 \in K$ or $0 \in K$. Since $\mathcal{T}(1) = 1$ ($\mathcal{T}(0) = 0$), $\mathcal{T}^{-1}(K) \neq \emptyset$. Also, $\mathcal{T}(0) = 0 \in P_{comm}(K)$, and so $0 \in \mathcal{T}^{-1}(P_{comm}(K))$. Consequently, $\mathcal{T}^{-1}(P_{comm}(K))$ is nonempty. Let $a \in \mathcal{T}^{-1}(P_{comm}(K))$, $d \in \mathcal{T}^{-1}(K)$ and $ad = da$ (there is at least one candidate for d: $d = 1$ or $d = 0$). We have $\mathcal{T}(a) \in P_{comm}(K)$, $\mathcal{T}(d) \in K$ and $\mathcal{T}(a)\mathcal{T}(d) = \mathcal{T}(d)\mathcal{T}(a)$. It follows that $a + d \in \mathcal{T}^{-1}(K)$. We have just established the following inclusion:

$$\mathcal{T}^{-1}(P_{comm}(K)) \subseteq P_{comm}(\mathcal{T}^{-1}(K)). \tag{5.1}$$

In particular, $\mathcal{T}^{-1}(P_{comm}(\mathcal{B}^D)) \subseteq P_{comm}(\mathcal{T}^{-1}(\mathcal{B}^D)) = P_{comm}(\mathcal{BF}_{\mathcal{T}}(\mathcal{A}))$.

(ii) Clearly, $a \in \mathcal{T}^{-1}(\mathrm{Poly}^{-1}(\{0\}))$ if and only if $\mathcal{T}(a) \in \mathrm{Poly}^{-1}(\{0\})$ if and only if $p(\mathcal{T}(a)) = \mathcal{T}(p(a)) = 0$ for some nontrivial polynomial p if and only if $a \in \mathrm{Poly}^{-1}(\mathcal{T}^{-1}(0))$. Accordingly, $\mathcal{T}^{-1}(\mathrm{Poly}^{-1}(\{0\})) = \mathrm{Poly}^{-1}(\mathcal{T}^{-1}(0))$.

(iii) If K_1, $K_2 \subseteq \mathcal{A}$ are such that $0 \in K_1 \cap K_2$ or $1 \in K_1 \cap K_2$, then

$$P_{comm}(K_1) \cap P_{comm}(K_2) \subseteq P_{comm}(K_1 \cap K_2). \tag{5.2}$$

Indeed, let $a \in P_{comm}(K_1) \cap P_{comm}(K_2)$, $d \in K_1 \cap K_2$ and $ad = da$ (the above condition ensures that for every $a \in P_{comm}(K_1) \cap P_{comm}(K_2)$ there is at least one $d \in K_1 \cap K_2$ such that $ad = da$, for example $d = 0$ or $d = 1$). Then, $a + d \in K_1 \cap K_2$, and hence $a \in P_{comm}(K_1 \cap K_2)$.

On the other hand, it is well known that $b \in \mathcal{A}^{qnil}$ if and only if for every $a \in \mathcal{A}$ which commutes with b there is the equivalence:

$$a \in \mathcal{A}^{-1} \Longleftrightarrow a + b \in \mathcal{A}^{-1}, \tag{5.3}$$

that is,

$$a \notin \mathcal{A}^{-1} \Longleftrightarrow a + b \notin \mathcal{A}^{-1}. \tag{5.4}$$

The equivalences (5.3) and (5.4) hold also if $\mathcal{A}^{-1}$ is replaced by $\mathcal{A}^{-1}_{left}$ or $\mathcal{A}^{-1}_{right}$. Consequently,

$$\mathcal{A}^{qnil} = P_{comm}(\mathcal{A}^{-1}) = P_{comm}(\mathcal{A}^{-1}_{left}) = P_{comm}(\mathcal{A}^{-1}_{right}),$$

and also,

$$\mathcal{A}^{qnil} = P_{comm}(\mathcal{A} \setminus \mathcal{A}^{-1}) = P_{comm}(\mathcal{A} \setminus \mathcal{A}^{-1}_{left}) = P_{comm}(\mathcal{A} \setminus \mathcal{A}^{-1}_{right}). \tag{5.5}$$

Proposition 5.3.2. *Let $\mathcal{A}$ be a unital Banach algebra and consider an algebraic element $a \in \mathcal{A}$. Then, $a \in \mathcal{A}^{qnil}$ if and only if a is nilpotent.*

Proof. Every nilpotent element is quasinilpotent. On the other hand, if $a \in \mathcal{A}^{qnil}$, then let p be the minimal polynomial such that $p(a) = 0$. It is well known that $\sigma(a) = p^{-1}(\{0\})$. Since $\sigma(a) = \{0\}$, there must exist $k \in \mathbb{N}$ such that $p(x) = x^k$. Consequently, a is nilpotent. □

In the following theorem the commuting perturbation class of $\mathcal{A}^D$ and $\mathcal{A}^{gD}$ will be considered.

Theorem 5.3.3. *Let $\mathcal{A}$ be a unital Banach algebra. Then:*
(i) $\mathcal{A}^{nil} \subseteq P_{comm}(\mathcal{A}^D) \subseteq \mathrm{Poly}^{-1}(\{0\})$;
(ii) $\mathcal{A}^{qnil} \subseteq P_{comm}(\mathcal{A}^{gD})$;
(iii) $\mathcal{A}^{nil} \subseteq P_{comm}(\mathcal{A}^D \setminus \mathcal{A}^{-1})$;
(iv) $\mathcal{A}^{qnil} \subseteq P_{comm}(\mathcal{A}^{gD} \setminus \mathcal{A}^{-1})$.

Proof. (i). Let $b \in \mathcal{A}^{nil}$ and $a \in \mathcal{A}^D$ such that $ab = ba$. Since $b \in \mathcal{A}^D$ and $b^D = 0$, according to [74, Theorem 3], $a + b \in \mathcal{A}^D$. In order to prove the remaining inclusion, suppose that $b \in P_{comm}(\mathcal{A}^D)$. The elements $\lambda 1 (= \lambda)$ are Drazin invertible and commute with b for every $\lambda \in \mathbb{C}$. Therefore, $b + \lambda \in \mathcal{A}^D$ for every $\lambda \in \mathbb{C}$. Consequently, $\sigma_D(b) = \emptyset$. According to [14, Theorem 2.1], b is algebraic.
(ii). It follows from [74, Theorem 8] and from the fact that $b^d = 0$ if $b \in \mathcal{A}^{qnil}$.
(iii). Since $\mathcal{A}^{nil} \subseteq P_{comm}(\mathcal{A}^D)$ and $\mathcal{A}^{nil} \subseteq \mathcal{A}^{qnil} = P_{comm}(\mathcal{A} \setminus \mathcal{A}^{-1})$ (identity (5.5)), apply (5.2) to obtain $\mathcal{A}^{nil} \subseteq P_{comm}(\mathcal{A}^D \cap (\mathcal{A} \setminus \mathcal{A}^{-1})) = P_{comm}(\mathcal{A}^D \setminus \mathcal{A}^{-1})$.
(iv). It follows from (ii), (5.5) and (5.2). □

Corollary 5.3.4. *Let $\mathcal{A}$ be a unital Banach algebra. Then*

$$\mathrm{Poly}^{-1}(\{0\}) \cap \mathcal{A}^{qnil} \subseteq P_{comm}(\mathcal{A}^D). \tag{5.6}$$

Proof. Apply Proposition 5.3.2 and Theorem 5.3.3(i). □

Next algebraic (nilpotent) elements will be characterized using the Drazin spectrum.

Theorem 5.3.5. *Let $\mathcal{A}$ be a unital Banach algebra and consider $d \in \mathcal{A}^{qnil}$. Then the following statements are equivalent:*

(i) *The element d is algebraic.*

(ii) *Given $a \in \mathcal{A}$, $ad = da$ implies that $\sigma_D(a+d) = \sigma_D(a)$.*

Proof. If d is algebraic, then according to Proposition 5.3.2, $d \in \mathcal{A}^{nil}$, which is equivalent to $-d \in \mathcal{A}^{nil}$. According to Corollary 5.3.4, $d, -d \in P_{comm}(\mathcal{A}^D)$. Let $a \in \mathcal{A}$ such that $ad = da$. If $\lambda \in \mathbb{C}$ is such that $\lambda \notin \sigma_D(a)$, then $a-\lambda \in \mathcal{A}^D$, and since $d \in P_{comm}(\mathcal{A}^D)$, $a+d-\lambda \in \mathcal{A}^D$. In particular, $\lambda \notin \sigma_D(a+d)$. To prove the reverse, apply the same argument to $-d \in P_{comm}(\mathcal{A}^D)$, $a+d$ and $\lambda \notin \sigma_D(a+d)$.

Conversely, if $a = 0$, then $\sigma_D(d) = \sigma_D(0) = \emptyset$. However, according to [14, Theorem 2.1], d is algebraic. □

In the following theorem the commuting perturbation class of (generalized) B-Fredholm elements will be considered.

Theorem 5.3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. Then:*

(i) $\mathcal{N}_{\mathcal{T}}(\mathcal{A}) \subseteq P_{comm}(\mathcal{BF}_{\mathcal{T}}(\mathcal{A})) \subseteq \mathcal{T}^{-1}(\mathrm{Poly}^{-1}(\{0\})$.

(ii) $\mathcal{R}_{\mathcal{T}}(\mathcal{A}) \subseteq P_{comm}(\mathcal{GBF}_{\mathcal{T}}(\mathcal{A}))$.

(iii) $\mathcal{N}_{\mathcal{T}}(\mathcal{A}) \subseteq P_{comm}(\mathcal{BF}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{F}_{\mathcal{T}}(\mathcal{A}))$.

(iv) $\mathcal{R}_{\mathcal{T}}(\mathcal{A}) \subseteq P_{comm}(\mathcal{GBF}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{F}_{\mathcal{T}}(\mathcal{A}))$.

Proof. (i). According to Theorem 5.3.3(i) and (5.1),

$$\mathcal{N}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathcal{T}^{-1}(P_{comm}(\mathcal{B}^D)) \subseteq P_{comm}(\mathcal{T}^{-1}(\mathcal{B}^D)) = P_{comm}(\mathcal{BF}_{\mathcal{T}}(\mathcal{A})).$$

Let $a \in P_{comm}(\mathcal{BF}_{\mathcal{T}}(\mathcal{A}))$. Then for every $\lambda \in \mathbb{C}$, $a+\lambda \in \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$, equivalently, $\mathcal{T}(a)+\lambda \in \mathcal{B}^D$. However, according to [14, Theorem 2.1], $\mathcal{T}(a) \in \mathcal{B}$ is algebraic.

(ii)-(iv). Apply Theorem 5.3.3(ii)-(iv) and use an argument similar to the one in the proof of statement (i). □

Corollary 5.3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. Let $a \in \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ and $b \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$ such that $ab - ba \in \mathcal{T}^{-1}(0)$. Then, $a + b \in \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$.*

Proof. Apply Theorem 5.3.3(i). □

Corollary 5.3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. If $a \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{T}(a) \in \mathcal{B}$ is algebraic, then $a \in P_{comm}(\mathcal{BF}_{\mathcal{T}}(\mathcal{A}))$.*

Proof. According to Corollary 5.3.4, $\mathcal{T}(a) \in P_{comm}(\mathcal{B}^D)$. Therefore, $a \in \mathcal{T}^{-1}(P_{comm}(\mathcal{B}^D)) \subseteq P_{comm}(\mathcal{T}^{-1}(\mathcal{B}^D)) = P_{comm}(\mathcal{B}\mathcal{F}_{\mathcal{T}}(\mathcal{A}))$. □

Under the same assumptions as in Corollary 5.3.8, note that if $a \in \mathcal{A}$ is algebraic and $a \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$, then $\mathcal{T}(a) \in P_{comm}(\mathcal{B}^D)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. Recall that according to [75, Theorem 10.1], the following statements are equivalent:

(i) The element $d \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$;

(ii) If $a \in \mathcal{A}$ is such that $ad - da \in \mathcal{T}^{-1}(0)$, then $\sigma_{\mathcal{F}_{\mathcal{T}}}(a) = \sigma_{\mathcal{F}_{\mathcal{T}}}(a + d)$;

(iii) If $a \in \mathcal{A}$ is such that $ad = ad$, then $\sigma_{\mathcal{F}_{\mathcal{T}}}(a) = \sigma_{\mathcal{F}_{\mathcal{T}}}(a + d)$;

(iv) $\sigma_{\mathcal{F}_{\mathcal{T}}}(d) = \{0\}$.

In the following theorem, a similar result for the B-Fredholm spectrum will be considered.

Theorem 5.3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (not necessarily continuous) homomorphism $\mathcal{T} : \mathcal{A} \to \mathcal{B}$. Let $d \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$. Then, the following conditions are equivalent:*

(i) *$\mathcal{T}(d)$ is algebraic;*

(ii) *If $a \in \mathcal{A}$ is such that $ad - da \in \mathcal{T}^{-1}(0)$, then $\sigma_{\mathcal{B}\mathcal{F}_{\mathcal{T}}}(a + d) = \sigma_{\mathcal{B}\mathcal{F}_{\mathcal{T}}}(a)$;*

(iii) *If $a \in \mathcal{A}$ is such that $ad = da$, then $\sigma_{\mathcal{B}\mathcal{F}_{\mathcal{T}}}(a + d) = \sigma_{\mathcal{B}\mathcal{F}_{\mathcal{T}}}(a)$;*

(iv) *$\sigma_{\mathcal{B}\mathcal{F}_{\mathcal{T}}}(d) = \emptyset$.*

Proof. (i)$\Longrightarrow$(ii). Apply Theorem 5.3.5.

(ii)$\Longrightarrow$(iii). It is obvious.

(iii)$\Longrightarrow$(iv). Consider $a = 0$. Then, $\sigma_{\mathcal{B}\mathcal{F}}(d) = \sigma_{\mathcal{B}\mathcal{F}}(0) = \emptyset$.

(iv)$\Longrightarrow$(i). Apply Theorem 5.2.3(iv). □

5.4 Perturbations of (generalized) B-Fredholm elements with equal spectral idempotents

Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}^{gD} \setminus \mathcal{A}^{-1}$. Then, $0 \neq p = 1 - a^d a$ is the spectral idempotent corresponding to 0, and in this section it will be denoted by $p = a^{\pi}$. Note that $(1 - p)\mathcal{A}(1 - p)$ is a Banach algebra with the unity $1 - p$, $(1 - p)a$, $a^d \in ((1 - p)\mathcal{A}(1 - p))^{-1}$, and a^d is the inverse of $(1 - p)a$ in the algebra $(1 - p)\mathcal{A}(1 - p)$.

Remark 5.4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (non necessarily continuous) homomorphism $\mathcal{T} \colon \mathcal{A} \to \mathcal{B}$.

(a) Let $a \in \mathcal{A}$ such that $\mathcal{T}(a) \in \mathcal{B}^{gD}$. Let $\mathcal{T}(a)^{\pi} = q$ and suppose that there exist $p \in \mathcal{A}^{\bullet}$ such that $\mathcal{T}(p) = q$ and $w \in (1 - p)\mathcal{A}(1 - p)$ such that

$\mathcal{T}(w) = \mathcal{T}(a)^d = ((1-q)\mathcal{T}(a)(1-q))^{-1} \in ((1-q)\mathcal{B}(1-q))^{-1}$. Then, it is not difficult to prove the following statements.
(i) $(1-p)aw = 1-p+c_1$ and $wa(1-p) = 1-p+c_2$, where $c_i \in \mathcal{T}^{-1}(0) \cap (1-p)\mathcal{A}(1-p)$, $i = 1, 2$.
(ii) If $w' \in (1-p)\mathcal{A}(1-p)$ is such that $\mathcal{T}(w') = \mathcal{T}(a)^d$, then $w' - w \in \mathcal{T}^{-1}(0) \cap (1-p)\mathcal{A}(1-p)$.
(b) Suppose in addition that $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ is surjective and has the lifting property, and consider $a \in \mathcal{A}$ as before, i.e., $\mathcal{T}(a) \in \mathcal{B}^{gD}$ and $\mathcal{T}(a)^\pi = q$. In particular, there exist $p \in \mathcal{A}^\bullet$ such that $\mathcal{T}(p) = q$ and $z \in \mathcal{A}$ such that $\mathcal{T}(a)^d = \mathcal{T}(z)$. However, since $\mathcal{T}(a)^d \in (1-q)\mathcal{B}(1-q)$, it is possible to choose $z \in (1-p)\mathcal{A}(1-p)$.

The results of Remark 5.4.1 will be used in what follows.

Proposition 5.4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (non necessarily continuous) homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$. Let $a_1 \in \mathcal{A}$ such that $\mathcal{T}(a_1) \in \mathcal{B}^{gD}$ and $\mathcal{T}(a_1)^\pi = q$. Suppose that there exist $p \in \mathcal{A}^\bullet$ and $w_1 \in (1-p)\mathcal{A}(1-p)$ such that $\mathcal{T}(p) = q$ and $\mathcal{T}(w_1) = \mathcal{T}(a_1)^d$. Let $a_2 \in \mathcal{A}$ and define $z = 1 + \mathcal{T}(a_1)^d\mathcal{T}(a_2 - a_1)$. Then, the following statements hold.*
(i) *The element $z \in \mathcal{B}^{-1}$ if and only if $p + w_1a_2 \in \mathcal{F}_\mathcal{T}(\mathcal{A})$.*
(ii) *Suppose that $\mathcal{T}(a_2)\mathcal{T}(a_1)^\pi = \mathcal{T}(a_1)^\pi\mathcal{T}(a_2)$. Then, $z \in \mathcal{B}^{-1}$ if and only $p + w_1a_2(1-p) \in \mathcal{F}_\mathcal{T}(\mathcal{A})$.*

Proof. (i). Note that $z \in \mathcal{B}^{-1}$ if and only if $1 + \mathcal{T}(w_1(a_2 - a_1)) \in \mathcal{B}^{-1}$. Since $\mathcal{T}(w_1a_1) = \mathcal{T}(w_1a_1(1-p)) = 1-q$, necessary and sufficient for $z \in \mathcal{B}^{-1}$ is that $q + \mathcal{T}(w_1a_2) \in \mathcal{B}^{-1}$, which in turn is equivalent to $p + w_1a_2 \in \mathcal{F}_\mathcal{T}(\mathcal{A})$.
(ii). Since $\mathcal{T}(a_1)$ and $\mathcal{T}(a_2)$ commute with q, it follows that $zq = qz$. From $qzq = q \in (q\mathcal{B}q)^{-1}$ and [60, Teorema 5.7.7] we conclude that $z \in \mathcal{B}^{-1}$ if and only if $(1-q)z(1-q) = 1-q+\mathcal{T}(w_1(a_2-a_1)(1-p)) \in ((1-q)\mathcal{B}(1-q))^{-1}$. A routine calculation shows that $1-q+\mathcal{T}(w_1(a_2-a_1)(1-p)) = (1-q)\mathcal{T}(p + w_1a_2(1-p))(1-q)$ and $q\mathcal{T}(p + w_1a_2(1-p))q = q \in (q\mathcal{B}q)^{-1}$. According to [60, Teorema 5.7.7], $z \in \mathcal{B}^{-1}$ if and only if $\mathcal{T}(p + w_1a_2(1-p)) \in \mathcal{B}^{-1}$, and it is exactly when $p + w_1a_2(1-p) \in \mathcal{F}_\mathcal{T}(\mathcal{A})$. □

In the following two theorems, (generalized) B-Fredholm elements that have the same spectral idempotents relative to the homomorphism $\mathcal{T}$ will be characterized.

Theorem 5.4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (non necessarily continuous) homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$. Suppose in addition that $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ is surjective and has the lifting property. Let $a_1 \in \mathcal{GBF}_\mathcal{T}(\mathcal{A})$ and consider $p \in \mathcal{A}^\bullet$ such that $\mathcal{T}(p) = \mathcal{T}(a_1)^\pi$. Then, the following statements are equivalent.*
(i) *$a_2 \in \mathcal{GBF}_\mathcal{T}(\mathcal{A})$ and $\mathcal{T}(a_1)^\pi = \mathcal{T}(a_2)^\pi$.*

(ii) $pa_2(1-p)$ *and* $(1-p)a_2p \in \mathcal{T}^{-1}(0)$, $pa_2p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ *and* $p + a_2 \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$.

(iii) $pa_2(1-p)$ *and* $(1-p)a_2p \in \mathcal{T}^{-1}(0)$, $pa_2p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ *and* $p + w_1a_2(1-p) \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$, *where* $w_1 \in (1-p)\mathcal{A}(1-p)$ *is such that* $\mathcal{T}(w_1) = \mathcal{T}(a_1)^d$.

(iv) $a_2 \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$, $p + w_1a_2 \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$ *and* $w_1 = (p + w_1a_2)w_2 + c$, *where* w_1 *is as in statement* (iii), $w_2 \in \mathcal{A}$ *is such that* $\mathcal{T}(w_2) = \mathcal{T}(a_2)^d$ *and* $c \in \mathcal{T}^{-1}(0)$.

Proof. (i) $\Longrightarrow$ (ii). Follows from Theorem 5.2.4.
(ii) $\Longrightarrow$ (iii). $a_2 \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$ by Theorem 5.2.4. Since $\mathcal{T}(a_1)^\pi = \mathcal{T}(a_2)^\pi$, [61, Theorem 2.2] implies that $1 + \mathcal{T}(a_1)^d\mathcal{T}(a_2 - a_1) \in \mathcal{B}^{-1}$. According to Proposition 5.4.2(ii), $p + w_1a_2(1-p) \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$.
(iii) $\Longrightarrow$ (iv). It is easily seen that $\mathcal{T}(a_2)$ and $\mathcal{T}(p) = \mathcal{T}(a_1)^\pi$ commute, and that $\mathcal{T}(a_2)\mathcal{T}(a_1)^\pi \in \mathcal{B}^{qnil}$. According to Proposition 5.4.2(ii), $1+\mathcal{T}(a_1)^d\mathcal{T}(a_2 - a_1) \in \mathcal{B}^{-1}$. Now, by Proposition 5.4.2(i) and [61, Theorem 2.2], $p + w_1a_2 \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$, $a_2 \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$ and

$$\mathcal{T}(a_2)^d = (1 + \mathcal{T}(a_1)^d\mathcal{T}(a_2 - a_1))^{-1}\mathcal{T}(a_1)^d.$$

Since $\mathcal{T}(w_1a_1) = 1 - \mathcal{T}(p)$, the last identity is equivalent to

$$\mathcal{T}(p + w_1a_2)\mathcal{T}(w_2) = \mathcal{T}(w_1),$$

which in turn is equivalent to $w_1 = (p + w_1a_2)w_2 + c$, $c \in \mathcal{T}^{-1}(0)$.
(iv) $\Longrightarrow$ (i). $1 + \mathcal{T}(a_1)^d\mathcal{T}(a_2 - a_1) \in \mathcal{B}^{-1}$ by Proposition 5.4.2(i). Further, the identity $\mathcal{T}(a_2)^d = (1 + \mathcal{T}(a_1)^d\mathcal{T}(a_2 - a_1))^{-1}\mathcal{T}(a_1)^d$ holds (see (iii) $\Longrightarrow$ (iv)). Now, [61, Theorem 2.2] implies that $\mathcal{T}(a_1)^\pi = \mathcal{T}(a_2)^\pi$. □

Theorem 5.4.4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two unital Banach algebras and consider a (non necessarily continuous) homomorphism* $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$. *Suppose in addition that* $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ *is surjective and has the lifting property. Let* $a_1 \in \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ *and consider* $p \in \mathcal{A}^\bullet$ *such that* $\mathcal{T}(p) = \mathcal{T}(a_1)^\pi$. *Then, the following statements are equivalent.*

(i) $a_2 \in \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$ *and* $\mathcal{T}(a_1)^\pi = \mathcal{T}(a_2)^\pi$.

(ii) $pa_2(1-p)$ *and* $(1-p)a_2p \in \mathcal{T}^{-1}(0)$, $pa_2p \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$ *and* $p + a_2 \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$.

(iii) $pa_2(1-p)$ *and* $(1-p)a_2p \in \mathcal{T}^{-1}(0)$, $pa_2p \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$ *and* $p + w_1a_2(1-p) \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$, *where* $w_1 \in (1-p)\mathcal{A}(1-p)$ *is such that* $\mathcal{T}(w_1) = \mathcal{T}(a_1)^D$.

(iv) $a_2 \in \mathcal{BF}_{\mathcal{T}}(\mathcal{A})$, $p + w_1a_2 \in \mathcal{F}_{\mathcal{T}}(\mathcal{A})$ *and* $w_1 = (p + w_1a_2)w_2 + c$, *where* w_1 *is as in statement* (iii), $w_2 \in \mathcal{A}$ *is such that* $\mathcal{T}(w_2) = \mathcal{T}(a_2)^D$ *and* $c \in \mathcal{T}^{-1}(0)$.

Proof. The arguments from the preceding theorem apply to the case of B-Fredholm elements using nilpotent elements instead of quasi-nilpotent elements. What is more, when considering Drazin invertible Banach algebra elements, statements similar to the ones in [61, Theorem 2.2] hold, if nilpotent elements instead of quasi-nilpotent elements are used. □

In the following theorem we consider the product of two generalized B-Fredholm elements with equal spectral idempotents.

Theorem 5.4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras and consider a (non necessarily continuous) homomorphism $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ such that $\mathcal{T}$ is surjective and has the lifting property. Let $a_i \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$, $i = 1, 2$, such that $\mathcal{T}(a_1)^\pi = \mathcal{T}(a_2)^\pi = q$ and $a_1a_2 - a_2a_1 \in \mathcal{T}^{-1}(0)$. Let $p \in \mathcal{A}^\bullet$ such that $\mathcal{T}(p) = q$. Then, $a_1a_2 \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$, $\mathcal{T}(a_1a_2)^\pi = q$ and if w_1, w_2, $w_{12} \in (1-p)\mathcal{A}(1-p)$ are such that $\mathcal{T}(w_1) = \mathcal{T}(a_1)^d$, $\mathcal{T}(w_2) = \mathcal{T}(a_2)^d$ and $\mathcal{T}(w_{12}) = \mathcal{T}(a_1a_2)^d$, then $w_{12} = w_2w_1 + c$, $c \in \mathcal{T}^{-1}(0)$.*

Proof. Since $\mathcal{T}(a_1), \mathcal{T}(a_2) \in \mathcal{B}^{gD}$ and $\mathcal{T}(a_1)\mathcal{T}(a_2) = \mathcal{T}(a_2)\mathcal{T}(a_1)$, according to [48, Theorem 5.5], $\mathcal{T}(a_1a_2) \in \mathcal{B}^{gD}$ and $\mathcal{T}(a_1a_2)^d = \mathcal{T}(a_1)^d\mathcal{T}(a_2)^d = \mathcal{T}(a_2)^d\mathcal{T}(a_1)^d$. Consequently, $a_1a_2 \in \mathcal{GBF}_{\mathcal{T}}(\mathcal{A})$. Further, since $\mathcal{T}(a_1)^\pi = \mathcal{T}(a_2)^\pi = q$,

$$\begin{aligned}
\mathcal{T}(a_1a_2)^\pi &= 1 - \mathcal{T}(a_1a_2)\mathcal{T}(a_1a_2)^d = 1 - \mathcal{T}(a_1)\mathcal{T}(a_2)\mathcal{T}(a_2)^d\mathcal{T}(a_1)^d \\
&= 1 - \mathcal{T}(a_1)(1-q)\mathcal{T}(a_1)^d = 1 - (1-q)\mathcal{T}(a_1)\mathcal{T}(a_1)^d \\
&= 1 - (1-q)(1-q) = 1 - (1-q) \\
&= q.
\end{aligned}$$

Since $\mathcal{T}(a_1a_2)^d = \mathcal{T}(a_2)^d\mathcal{T}(a_1)^d$, $\mathcal{T}(w_{12}) = \mathcal{T}(w_2)\mathcal{T}(w_1)$. Consequently, $w_{12} = w_2w_1 + c$, $c \in \mathcal{T}^{-1}(0)$. $\square$

Chapter 6

Closed upper and lower semi-Browder operators

The necessary and sufficient conditions under which a bounded linear operator defined everywhere is upper (lower) semi-Browder are well-known [2, Theorems 2.62 and 2.63], [77, Theorems 3 and 4]. Moreover, the upper (lower) semi-Browder spectrum of such an operator is characterized; see [62], [57, Corollary 20.20, Theorem 20.21], [1, Corollaries 3.45 and 3.47], [2, Theorems 4.4 and 4.5]. Our main goal is to extend the aforementioned results to the class of closed operators. It is done by generalizing Theorems 3 and 4 of [77]; see Theorems 6.2.4 and 6.3.2, and their consequences. On the other hand, the present chapter is also motivated by [4].

6.1 Closed operators

Until now we have worked with linear operators $T : \mathcal{X} \to \mathcal{Y}$ such that $\mathcal{D}(T) = \mathcal{X}$, where $\mathcal{D}(T)$ is the domain of definition of T. In this chapter we consider operators not necessarily defined for all vectors of the domain space.

Definition 6.1.1. A linear operator T from $\mathcal{X}$ to $\mathcal{Y}$ is an operator such that:

(i) The domain $\mathcal{D}(T)$ of T is a vector subspace of $\mathcal{X}$;

(ii) For $x, y \in \mathcal{D}(T)$ and scalars α,

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha Tx.$$

If T is a linear operator from $\mathcal{X}$ to $\mathcal{Y}$, then $\mathcal{X}$ and $\mathcal{Y}$ are respectively called the domain and range spaces. At first glance, we complicate the matter by introducing operators not defined everywhere in the domain space. It seems that T could be regarded as an operator on $\mathcal{D}(T)$ to $\mathcal{Y}$. However, $\mathcal{D}(T)$ is in general not closed in $\mathcal{X}$ and hence is not a Banach space (with the norm of $\mathcal{X}$), so we do not adopt this point of view. If $\mathcal{D}(T)$ is dense in $\mathcal{X}$, T is said to be densely defined. The kernel $N(T)$ of T is the set of all $x \in \mathcal{D}(T)$ such that $Tx = 0$. The range $R(T)$ of T is defined as the set of all vectors of the form Tx with $x \in \mathcal{D}(T)$. The nullity and deficiency of T, denoted respectively by $\alpha(T)$ and $\beta(T)$, are defined as before: $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim \mathcal{Y}/R(T)$.

To define ascent and descent we consider the case in which $\mathcal{D}(T)$ and $R(T)$ are in the same space $\mathcal{X}$. We can then define the iterates $T^2, T^3, \dots$ of T. If $n > 1$, $\mathcal{D}(T^n)$ is the set $\{x \in \mathcal{X} : x, Tx, \dots, T^{n-1}x \in \mathcal{D}(T)\}$ and $T^n x = T(T^{n-1}x)$. We can then consider $N(T^n)$ and $R(T^n)$. It is well known that $N(T^n) \subset N(T^{n+1})$ and $R(T^{n+1}) \subset R(T^n)$ if $n \in \mathbb{N}_0$. We follow the convention that $T^0 = I$ (the identity operator on $\mathcal{X}$, with $\mathcal{D}(I) = \mathcal{X}$). Thus $N(T^0) = \{0\}$ and $R(T^0) = \mathcal{X}$. It is also well known that if $N(T^k) = N(T^{k+1})$, then $N(T^n) = N(T^k)$ when $n \geq k$. In this case the smallest nonnegative integer k such that $N(T^k) = N(T^{k+1})$ is called the ascent of T and it is denoted by $\mathrm{asc}(T)$. If no such k exists we define $\mathrm{asc}(T) = \infty$. Similarly, if $R(T^{k+1}) = R(T^k)$, then $R(T^n) = R(T^k)$ when $n \geq k$. The smallest k (in the case when it exists) such that $R(T^{k+1}) = R(T^k)$ holds, is called the descent of T and denoted by $\mathrm{dsc}(T)$. We write $\mathrm{dsc}(T) = \infty$ if $R(T^{n+1})$ is always a proper subset of $R(T^n)$.

The generalized kernel and the generalized range of a linear operator T from $\mathcal{X}$ to $\mathcal{X}$ are respectively the sets $N^\infty(T) = \cup_{n=1}^\infty N(T^n)$ and $R^\infty(T) = \bigcap_{n=1}^\infty R(T^n)$. The following lemma will be used later; see [68, Lemma 3.4] and [4, Lemma 2.1].

Lemma 6.1.2. *Let $T : \mathcal{D}(T) \to \mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, be a linear operator.*
(i) *If $\mathrm{asc}(T) < \infty$, then $N^\infty(T) \cap R^\infty(T) = \{0\}$.*
(ii) *If $\alpha(T) < \infty$ and $N^\infty(T) \cap R^\infty(T) = \{0\}$, then $\mathrm{asc}(T) < \infty$.*

Consider the space $\mathcal{X} \times \mathcal{Y}$ consisting of all ordered pairs (x, y) of elements $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We recall that $\mathcal{X} \times \mathcal{Y}$ is a vector space with standard linear operations and it becomes a Banach space if the norm is defined by

$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}.$$

Definition 6.1.3. Let T be a linear operator from $\mathcal{X}$ to $\mathcal{Y}$. The graph $G(T)$ of T is the set $\{(x, Tx) : x \in \mathcal{D}(T)\}$. Since T is linear, $G(T)$ is a subspace of $\mathcal{X} \times \mathcal{Y}$.

If the graph of T is closed in $\mathcal{X} \times \mathcal{Y}$, then T is said to be closed operator.

It is straightforward to show that T is closed if and only if for any sequence $(x_n) \subset \mathcal{D}(T)$ such that $\lim x_n = x$ and $\lim Tx_n = y$, x belongs to $\mathcal{D}(T)$ and $Tx = y$. The set of all closed operators from $\mathcal{X}$ to $\mathcal{Y}$ will be denoted by $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. In particular, $\mathcal{C}(\mathcal{X}, \mathcal{X}) = \mathcal{C}(\mathcal{X})$. Clearly, every $T \in L(\mathcal{X}, \mathcal{Y})$ is closed: $L(\mathcal{X}, \mathcal{Y}) \subset \mathcal{C}(\mathcal{X}, \mathcal{Y})$. On the other hand, the well-known closed graph theorem shows that $T \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{D}(T) = \mathcal{X}$ imply $T \in L(\mathcal{X}, \mathcal{Y})$.

The following theorem is also well-known and it enables the introduction of the conjugate of a linear operator.

Theorem 6.1.4. *Let M be a subspace dense in $\mathcal{X}$. If T is a bounded linear map from M into $\mathcal{Y}$, then there exists a unique continuous linear extension $\overline{T}$ of T to all of $\mathcal{X}$ and $\|T\| = \|\overline{T}\|$.*

Definition 6.1.5. Let T be a linear operator (not necessarily closed) with domain $\mathcal{D}(T)$ dense in $\mathcal{X}$ and range $R(T) \subset \mathcal{Y}$. The conjugate operator T' is defined as follows: its domain $\mathcal{D}(T')$ consists of all $y' \in Y'$ for which $y'T$ is continuous on $\mathcal{D}(T)$; for such a y' we define $T'y' = x'$, where $x' = \overline{y'T}$ is the bounded linear extension of $y'T$ to $\mathcal{X}$.

Theorem 6.1.4 assures the existence of such an x' which is unique, so T' is well defined. It is easy to see that $\mathcal{D}(T')$ is a subspace of Y' and that T' is a closed linear operator.

Lemma 6.1.6. *Let $T \in \mathcal{C}(\mathcal{X})$ be a densely defined operator and $S \in L(\mathcal{X})$. Then, $T - S \in \mathcal{C}(\mathcal{X})$ and $(T - S)' = T' - S'$.*

Proof. Since $\mathcal{D}(T - S) = \mathcal{D}(T)$, $T - S$ is densely defined and thus $(T - S)'$ exists. By [46, Problem 5.6, p. 164], $T - S$ is closed. For $y' \in \mathcal{X}'$, $y'(T - S)$ is bounded on $\mathcal{D}(T)$ if and only if $y'T$ is bounded on $\mathcal{D}(T)$, and so $\mathcal{D}((T - S)') = \mathcal{D}(T') = \mathcal{D}(T' - S')$. For $y' \in \mathcal{D}((T - S)') = \mathcal{D}(T' - S')$ it follows that

$$\begin{aligned} (T - S)'y' &= \overline{y'(T - S)} = \overline{y'T - y'S}, \\ (T' - S')y' &= T'y' - S'y' = \overline{y'T} - y'S. \end{aligned}$$

Since the functionals $\overline{y'T - y'S}$ and $\overline{y'T} - y'S$ coincide on $\mathcal{D}(T)$, they coincide on $\mathcal{X}$. Therefore, $(T - S)' = T' - S'$. $\square$

The Fredholm theory can be extended to closed operators. An operator $T \in \mathcal{C}(\mathcal{X})$ is bounded below if there exists $c > 0$ such that

$$c\|x\| \leq \|Tx\| \quad \text{for every} \quad x \in \mathcal{D}(T).$$

Recall that $T \in \mathcal{C}(\mathcal{X})$ is bounded below if and only if T is injective with closed range [65, Theorem 5.1, p. 70]. Further, we also consider the following subsets of $\mathcal{C}(\mathcal{X})$:

$$\begin{aligned} \Phi_+(\mathcal{X}) &= \{T \in \mathcal{C}(\mathcal{X}) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed}\}; \\ \Phi_-(\mathcal{X}) &= \{T \in \mathcal{C}(\mathcal{X}) : \beta(T) < \infty\}, \\ \Phi_\pm(\mathcal{X}) &= \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X}); \\ \Phi(\mathcal{X}) &= \Phi_+(\mathcal{X}) \cap \Phi_-(\mathcal{X}); \\ \mathcal{B}_+(\mathcal{X}) &= \{T \in \mathcal{C}(\mathcal{X}) : T \in \Phi_+(\mathcal{X}) \text{ and } \operatorname{asc}(T) < \infty\}; \\ \mathcal{B}_-(\mathcal{X}) &= \{T \in \mathcal{C}(\mathcal{X}) : T \in \Phi_-(\mathcal{X}) \text{ and } \operatorname{dsc}(T) < \infty\}; \\ \mathcal{B}(\mathcal{X}) &= \mathcal{B}_+(\mathcal{X}) \cap \mathcal{B}_-(\mathcal{X}). \end{aligned}$$

The classes $\Phi_+(\mathcal{X}), \Phi_-(\mathcal{X}), \Phi_\pm(\mathcal{X}), \Phi(\mathcal{X}), \mathcal{B}_+(\mathcal{X}), \mathcal{B}_-(\mathcal{X})$ and $\mathcal{B}(\mathcal{X})$ consist of all upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm, Fredholm, upper semi-Browder, lower semi-Browder and Browder operators, respectively. As we see, in this chapter "bounded below operator" means "closed bounded below operator"; moreover, $\Phi_+(\mathcal{X})$ will denote the set of all closed upper semi-Fredholm operators, $\mathcal{B}_+(\mathcal{X})$ will denote the set of all closed upper semi-Browder operators, etc. For closed upper and lower semi-Fredholm operators the index is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If $T \in \Phi_+(\mathcal{X})\backslash\Phi_-(\mathcal{X})$, then $\text{ind}(T) = -\infty$, and if $T \in \Phi_-(\mathcal{X})\backslash\Phi_+(\mathcal{X})$, then $\text{ind}(T) = +\infty$. The corresponding spectra of $T \in \mathcal{C}(\mathcal{X})$ are defined in a usual way.

A linear operator T, $T : \mathcal{D}(T) \to \mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, is Kato if $R(T)$ is closed and $N(T) \subset R(T^m)$ for each $m \in \mathbb{N}$. A subspace M of $\mathcal{X}$ is called invariant under T if $T(\mathcal{D}(T) \cap M) \subset M$. By the restriction of T to M we then mean the operator T_M from M to M defined as follows: $\mathcal{D}(T_M) = M \cap \mathcal{D}(T)$, $T_M x = Tx$ if $x \in \mathcal{D}(T_M)$. If M is invariant under T and if $T(\mathcal{D}(T) \cap M) = M$, we say that M is exactly invariant under T. The following result [45] is of crucial importance.

Theorem 6.1.7. (Kato decomposition) *Let $\mathcal{X}$ be a Banach space and $T \in \Phi_\pm(\mathcal{X})$. Then there exists $d \in \mathbb{N}$ such that T has a Kato decomposition of degree d, i.e. there exists a pair (M, N) of two closed subspaces of $\mathcal{X}$ such that:*

(i) *$\mathcal{X} = M \oplus N$;*

(ii) *$T(M \cap \mathcal{D}(T)) \subset M$, $T_M : M \cap \mathcal{D}(T) \to M$, is a closed and Kato operator;*

(iii) *$N \subset \mathcal{D}(T)$, $\dim N < \infty$, $T(N) \subset N$ and $T_N : N \to N$ is a bounded and nilpotent operator of degree d.*

6.2 Closed upper semi-Browder operators

Theorem 6.2.1. *Let $T : \mathcal{D}(T) \to \mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, be a linear operator and let $S \in L(\mathcal{X})$ be such that S is bijective, $S(\mathcal{D}(T)) = \mathcal{D}(T)$, and S commutes with T. Then*

$$N(T - S) \subset R^\infty(T).$$

Proof. Since S is bijective, S^{-1} exists. Let $x \in \mathcal{D}(T) = S(\mathcal{D}(T))$. There exists $u \in \mathcal{D}(T)$ such that $Su = x$. Consequently, $S^{-1}x = u \in \mathcal{D}(T)$. From $TSu = STu$ we conclude that $Tx = STS^{-1}x$, and hence that $S^{-1}Tx = TS^{-1}x$.

Let $x \in N(T - S) \subset \mathcal{D}(T - S) = \mathcal{D}(T)$. Then $Tx = Sx \in \mathcal{D}(T)$ and $T^2x = T(Tx) = T(Sx) = S(Tx) = S^2x \in \mathcal{D}(T)$. By induction we conclude

$$T^n x = S^n x \in \mathcal{D}(T) \quad \text{for every} \quad n \in \mathbb{N}_0. \tag{6.1}$$

Observe that from $Tx = Sx$ it follows that

$$TS^{-1}x = S^{-1}Tx = x = SS^{-1}x,$$

and hence $S^{-1}x \in N(T-S)$. Consequently,

$$(S^{-1})^m x \in N(T-S) \text{ for every } m \in \mathbb{N}_0. \tag{6.2}$$

Using (6.1) and (6.2) we obtain

$$T^n(S^{-1})^m x = S^n(S^{-1})^m x \in \mathcal{D}(T) \quad \text{for every} \quad n,\ m \in \mathbb{N}_0. \tag{6.3}$$

Fix $n_0 \in \mathbb{N}$. From $T^{n_0}x = S^{n_0}x = SS^{n_0-1}x$ it follows that $S^{-1}T^{n_0}x = S^{n_0-1}x$. Applying (6.1) and the fact that S^{-1} commutes with T we get $T^{n_0}S^{-1}x = S^{n_0-1}x$. Continuing this method and using (6.3) we obtain $T^{n_0}(S^{-1})^{n_0}x = x$, so $x \in R(T^{n_0})$. Since n_0 is arbitrary, $x \in R^\infty(T)$. □

The following result indicates that the space $R^\infty(T)$ is exactly invariant under T; see [38, Lemma 38.1] and the proof of [68, Theorem 4.1].

Lemma 6.2.2. *Let $T : \mathcal{D}(T) \to \mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, be a linear operator with $\alpha(T) < \infty$. Then $T(\mathcal{D}(T) \cap R^\infty(T)) = R^\infty(T)$.*

Let $P \in L(\mathcal{X})$ be a projector which commutes with $T \in \mathcal{C}(\mathcal{X})$. Put $\mathcal{X}_0 = R(P)$ and $\mathcal{X}_1 = N(P)$. Clearly,

$$T\left(\mathcal{X}_j \cap \mathcal{D}(T)\right) \subset \mathcal{X}_j \quad \text{for} \quad j = 0, 1.$$

It is easy to check that the restrictions T_j of T to $\mathcal{X}_j$, $j = 0, 1$, are closed operators. In addition, for a linear operator T from $\mathcal{X}$ to $\mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, and $\epsilon > 0$ we define

$$\mathrm{comm}_\epsilon^{-1}(T) = \{S \in L(\mathcal{X})^{-1} : S \text{ commutes with } T,\ \|S\| < \epsilon\}.$$

The following definition is due to M. A. Goldman and S. N. Kračkovskii [31], and it will be used in what follows.

Definition 6.2.3. Let $T : \mathcal{D}(T) \to \mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, be a linear operator and $S \in L(\mathcal{X})$. We say that S commutes with T if:

(i) $Sx \in \mathcal{D}(T)$ for every $x \in \mathcal{D}(T)$;

(ii) $STx = TSx$ for every $x \in \mathcal{D}(T)$.

In the following theorem we give several necessary and sufficient conditions for a closed operator to be upper semi-Browder.

Theorem 6.2.4. *Let $T \in \mathcal{C}(\mathcal{X})$. Then the following conditions are equivalent:*

(i) T is upper semi-Browder;

(ii) T is upper semi-Fredholm, and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ with $S(\mathcal{D}(T)) = \mathcal{D}(T)$, it follows that $T - S$ is bounded below;

(iii) *T is upper semi-Fredholm and* $0 \notin \operatorname{acc}\sigma_{ap}(T)$*;*

(iv) *There exists a projector $P \in F(\mathcal{X})$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, T_0 is nilpotent bounded operator, and T_1 is bounded below;*

(v) *There exists a projector $P \in F(\mathcal{X})$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, TP is nilpotent bounded operator, and $T + P$ is bounded below;*

(vi) *There exists $B \in F(\mathcal{X})$ which commutes with T such that $T - B$ is bounded below;*

(vii) *There exists $B \in K(\mathcal{X})$ which commutes with T such that $T - B$ is bounded below.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that $T \in \mathcal{C}(\mathcal{X})$ is upper semi-Browder. Since $T \in \Phi_+(\mathcal{X})$, $R(T^n)$ is closed for every $n \in \mathbb{N}$ by [45, Lemma 543]. According to [45, Theorem 1], there exists some $\epsilon_1 > 0$ such that if $B \in L(\mathcal{X})$ and $\|B\| < \epsilon_1$, then $T - B \in \Phi_+(\mathcal{X})$. Let $\mathcal{X}_1 = R^\infty(T)$. $\mathcal{X}_1$ is a Banach space and $T(\mathcal{D}(T) \cap \mathcal{X}_1) = \mathcal{X}_1$ by Lemma 6.2.2. The operator T_1 from $\mathcal{X}_1$ to $\mathcal{X}_1$ induced by T is closed with $\alpha(T_1) < \infty$ and $\beta(T_1) = 0$. From $T_1 \in \Phi(\mathcal{X}_1)$, again by [45, Theorem 1], it follows that there exists some $\epsilon_2 > 0$ such that for $B \in L(\mathcal{X}_1)$, $\|B\| < \epsilon_2$ implies $T_1 - B \in \Phi(\mathcal{X}_1)$, $\alpha(T_1 - B) \leq \alpha(T_1)$, $\beta(T_1 - B) \leq \beta(T_1)$, $\operatorname{ind}(T_1 - B) = \operatorname{ind}(T_1)$. Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, and let $S \in \operatorname{comm}_\epsilon^{-1}(T)$ be such that $S(\mathcal{D}(T)) = \mathcal{D}(T)$. Since S commutes with T, it follows that $S(R(T^n)) \subset R(T^n)$ for every $n \in \mathbb{N}$, and so $S(\mathcal{X}_1) = S(\bigcap_{n=1}^\infty R(T^n)) = \bigcap_{n=1}^\infty S(R(T^n)) \subset \bigcap_{n=1}^\infty R(T^n) = \mathcal{X}_1$. Let $S_1 : \mathcal{X}_1 \to \mathcal{X}_1$ be the operator induced by S. The operator S_1 is bounded, $\|S_1\| < \epsilon$ and $\beta(T_1) = 0$, so $\beta(T_1 - S_1) = 0$. From Theorem 6.2.1 we have

$$\alpha(T - S) = \alpha(T_1 - S_1) = \operatorname{ind}(T_1 - S_1) = \operatorname{ind}(T_1) = \alpha(T_1).$$

From [4, Lemma 2.1(iii)] it follows that $N(T) \cap R^\infty(T) = \{0\}$, and hence $\alpha(T_1) = 0$. Consequently, $\alpha(T - S) = 0$. Since $T - S$ has closed range, $T - S$ is bounded below.

(ii) $\Longrightarrow$ (iii). Put $S = \lambda I$ with $0 < |\lambda| < \epsilon$.

(iii) $\Longrightarrow$ (iv). Suppose that T is upper semi-Fredholm and there exists $\epsilon > 0$ such that $T - \lambda I$ is injective with closed range for $0 < |\lambda| < \epsilon$. From Theorem 6.1.7 it follows that there exist two closed subspaces M and N such that $\mathcal{X} = M \oplus N$, $T(M \cap \mathcal{D}(T)) \subset M$, the restriction T_M of T to M is a closed and Kato operator; $N \subset \mathcal{D}(T)$, $\dim N < \infty$, $T(N) \subset N$ and $T_N : N \to N$ is a bounded and nilpotent operator. Let P be a projector such that $R(P) = N$ and $N(P) = M$. Clearly, $P \in F(\mathcal{X})$, $R(P) \subset \mathcal{D}(T)$ and $P(\mathcal{D}(T)) \subset \mathcal{D}(T)$. For $x \in \mathcal{D}(T)$ there exist $u \in N(P) \cap \mathcal{D}(T)$ and $v \in R(P)$ such that $x = u + v$. Since $TPx = Tv$ and $PTx = P(Tu + Tv) = Tv$, we conclude that P commutes with T. For $T_0 = T_N$ and $T_1 = T_M$, T_0 is a nilpotent bounded operator and $T_1 - \lambda I$ is injective for $0 < |\lambda| < \epsilon$. Since T_1 is Kato, from [45, Theorem 3, p. 297] we conclude that T_1 is injective. Thus, T_1 is bounded below.

(iv) $\Longrightarrow$ (v). Suppose that there exists a projector $P \in F(\mathcal{X})$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, T_0 is a nilpotent bounded operator of degree

p, and T_1 is bounded below. Since $\|TPx\| = \|T_0Px\| \le \|T\|\|P\|\|x\|$ for every $x \in \mathcal{X}$, TP is bounded. In addition, for $x \in \mathcal{X}$ there exist $u \in N(P)$ and $v \in R(P)$ such that $x = u + v$. Then

$$\begin{aligned}(TP)^p x &= \underbrace{(TP)(TP)\dots(TP)(TP)}_{p}x = \underbrace{(TP)(TP)\dots(TP)}_{p-1}Tv\\ &= \underbrace{(TP)(TP)\dots(TP)}_{p-2}TTv = \dots = T^p v = (T_0)^p v = 0,\end{aligned}$$

and so TP is nilpotent. From $T \in \mathcal{C}(\mathcal{X})$ and $P \in L(\mathcal{X})$ it follows that $T + P \in \mathcal{C}(\mathcal{X})$. Since T_0 is a nilpotent bounded operator, $T_0 + I_0$ is invertible, where I_0 is the identity operator on $\mathcal{X}_0$. Consequently, $N(T + P) = N(T_1) \oplus N(T_0 + I_0) = \{0\}$ and $R(T+P) = R(T_1) \oplus R(T_0 + I_0) = R(T_1) \oplus R(P)$. Since $R(T_1)$ is closed and $\dim R(P) < \infty$, $R(T + P)$ is closed. Therefore, $T + P$ is bounded below.

(v) $\Longrightarrow$ (vi). Put $B = -P$.

(vi) $\Longrightarrow$ (vii). Obvious.

(vii) $\Longrightarrow$ (i). Let there exists $B \in K(\mathcal{X})$ which commutes with T such that $T - B$ is bounded below. Put $A = T - B$. Then $\mathrm{asc}(A) < \infty$ and $A + \lambda B \in \Phi_+(\mathcal{X})$ for $\lambda \in [0, 1]$ according to [46, Chapter 4, Theorem 5.26]. Since B commutes with A, from [31, Theorem 3] it follows that the function $\lambda \to \overline{N^\infty(A+\lambda B)} \cap R^\infty(A+\lambda B)$ is locally constant on the set $[0, 1]$, and hence this function is constant on $[0, 1]$. As $\mathrm{asc}(A) < \infty$, from Lemma 6.1.2(i) it follows that $\overline{N^\infty(A)} \cap R^\infty(A) = N^\infty(A) \cap R^\infty(A) = \{0\}$, and so $\overline{N^\infty(A + B)} \cap R^\infty(A+B) = \{0\}$. It implies $N^\infty(A+B) \cap R^\infty(A+B) = \{0\}$, and by Lemma 6.1.2(ii), we get $\mathrm{asc}(A + B) < \infty$. Therefore, $T = A + B \in \mathcal{B}_+(\mathcal{X})$. □

Corollary 6.2.5. *Let $T \in \mathcal{C}(\mathcal{X})$. Then:*

$$\sigma_{\mathcal{B}_+}(T) = \sigma_{\Phi_+}(T) \cup \operatorname{acc} \sigma_{ap}(T).$$

Proof. Follows from the equivalence (i)$\Longleftrightarrow$(iii) of Theorem 6.2.4. □

Corollary 6.2.6. *Let $T \in \mathcal{C}(\mathcal{X})$. Then, $\sigma_{\mathcal{B}_+}(T)$ is a closed set.*

Proof. From [45, Theorem 1], it follows that $\sigma_{\Phi_+}(T)$ is closed. Now, according to Corollary 6.2.5, $\sigma_{\mathcal{B}_+}(T)$ is the union of two closed sets, so $\sigma_{\mathcal{B}_+}(T)$ is closed. □

For $T \in \mathcal{C}(\mathcal{X})$ set

$$\mathcal{F}_T(\mathcal{X}) = \{F \in F(\mathcal{X}) : F \text{ commutes with } T\}$$

and

$$\mathcal{K}_T(\mathcal{X}) = \{K \in K(\mathcal{X}) : K \text{ commutes with } T\}.$$

Corollary 6.2.7. *Let $T \in \mathcal{C}(\mathcal{X})$. Then:*

$$\sigma_{\mathcal{B}_+}(T) = \bigcap_{F \in \mathcal{F}_T(\mathcal{X})} \sigma_{ap}(T+F) = \bigcap_{K \in \mathcal{K}_T(\mathcal{X})} \sigma_{ap}(T+K).$$

Proof. Suppose that $\lambda \notin \bigcap\limits_{K \in \mathcal{K}_T(\mathcal{X})} \sigma_{ap}(T+K)$. Then there exists $K \in \mathcal{K}_T(\mathcal{X})$ such that $\lambda \notin \sigma_{ap}(T+K)$, that is $T+K-\lambda$ is bounded below. Since $-K$ commutes with $T-\lambda$, from the equivalence (i)$\Longleftrightarrow$(vii) of Theorem 6.2.4, it follows that $T-\lambda \in \mathcal{B}_+(\mathcal{X})$, i.e. $\lambda \notin \sigma_{\mathcal{B}_+}(T)$. Therefore, $\sigma_{\mathcal{B}_+}(T) \subset \bigcap\limits_{K \in \mathcal{K}_T(\mathcal{X})} \sigma_{ap}(T+K) \subset \bigcap\limits_{F \in \mathcal{F}_T(\mathcal{X})} \sigma_{ap}(T+F)$.

To prove the converse, suppose that $\lambda \notin \sigma_{\mathcal{B}_+}(T)$. Then $T-\lambda \in \mathcal{B}_+(\mathcal{X})$, and from from the equivalence (i)$\Longleftrightarrow$(vi) of Theorem 6.2.4, it follows that there exists $F \in F(\mathcal{X})$ which commutes with $T-\lambda$ such that $T-\lambda-F$ is bounded below. Then $F_1 = -F \in F(\mathcal{X})$ commutes with T, and hence $F_1 \in \mathcal{F}_T(\mathcal{X})$. Moreover, $T+F_1-\lambda$ is bounded below, and so $\lambda \notin \sigma_{ap}(T+F_1)$. Consequently, $\bigcap\limits_{F \in \mathcal{F}_T(\mathcal{X})} \sigma_{ap}(T+F) \subset \sigma_{\mathcal{B}_+}(T)$. $\square$

In order to compare our results and the results proved in [4], we need the following definition [44].

Definition 6.2.8. Let $T : \mathcal{D}(T) \to \mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, and $S : \mathcal{D}(S) \to \mathcal{X}$, $\mathcal{D}(S) \subset \mathcal{X}$ be linear operators. We say that S commutes with T if:

(i) $\mathcal{D}(T) \subset \mathcal{D}(S)$;

(ii) $Sx \in \mathcal{D}(T)$ whenever $x \in \mathcal{D}(T)$;

(iii) $STx = TSx$ for $x \in \mathcal{D}(T^2)$.

Remark 6.2.9. Let $T : \mathcal{D}(T) \to \mathcal{X}$, $\mathcal{D}(T) \subset \mathcal{X}$, be a linear operator and $S \in L(\mathcal{X})$. The following assertions hold:

(i) If S commutes with T in the sense of Definition 6.2.3, then S also commutes with T in the sense of Definition 6.2.8;

(ii) S commutes with T in the sense of Definition 6.2.8 and $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ if and only if S commutes with T in the sense of Definition 6.2.3 and $\mathcal{D}(T^2) = \mathcal{D}(T)$.

If we observe that $\mathcal{D}(S) = \mathcal{X}$ and that $\mathcal{D}(T^2) \subset \mathcal{D}(T)$, then (i) follows immediately. Further, it is easily seen that $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ is equivalent to $\mathcal{D}(T^2) = \mathcal{D}(T)$. Now, (ii) is a consequence of this fact and (i).

[4, Theorem 3.2] states that if $T \in \mathcal{C}(\mathcal{X})$ is upper semi-Browder then there exists $A \in \mathcal{C}(\mathcal{X})$ and $B \in F(\mathcal{X})$ such that $T = A + B$, $\mathcal{D}(A) = \mathcal{D}(T)$, A is bounded below and B commutes with T in the sense of Definition 6.2.8; the converse assertion holds if $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$. According to Remark 6.2.9, the equivalence (i) $\Longleftrightarrow$ (vi) of Theorem 6.2.4 is an extension of [4, Theorem 3.2].

6.3 Closed lower semi-Browder operators

Lemma 6.3.1. *Let $T \in \mathcal{C}(\mathcal{X})$ be a densely defined operator, and let $S \in L(\mathcal{X})$. If S commutes with T, then S' commutes with T'.*

Proof. For $y' \in \mathcal{D}(T')$ it follows that

$$\|S'y'(Tx)\| = \|(y'S)(Tx)\| = \|(y'T)(Sx)\| \leq \|y'T\|\|S\|\|x\|$$

for every $x \in \mathcal{D}(T)$, and hence $S'y' \in \mathcal{D}(T')$. Therefore, $S'(\mathcal{D}(T')) \subset \mathcal{D}(T')$. It remains to prove the commutativity relation. For $y' \in \mathcal{D}(T')$ we have

$$(T'S')y' = \overline{(S'y')T},$$

$$(S'T')y' = \overline{y'T}S.$$

Since $(S'y')Tx = (y'S)(Tx) = (y'T)(Sx)$ for $x \in \mathcal{D}(T)$, $\overline{(S'y')T} = \overline{y'T}S$ by Theorem 6.1.4. Consequently, $(T'S')y' = (S'T')y'$. □

In the following theorem we characterize closed lower semi-Browder operators.

Theorem 6.3.2. *If $T \in \mathcal{C}(\mathcal{X})$, $\overline{\mathcal{D}(T)} = \mathcal{X}$ and $\rho_\Phi(T) \neq \emptyset$, then the following conditions are equivalent:*

(i) *T is lower semi-Browder;*

(ii) *T is lower semi-Fredholm, and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ with $S(\mathcal{D}(T)) = \mathcal{D}(T)$, it follows that $T - S$ is onto;*

(iii) *T is lower semi-Fredholm and $0 \notin \mathrm{acc}\,\sigma_{su}(T)$;*

(iv) *There exists a projector $P \in F(\mathcal{X})$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, T_0 is a nilpotent bounded operator and T_1 is surjective;*

(v) *There exists a projector $P \in F(\mathcal{X})$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, TP is a nilpotent bounded operator and $T + P$ is surjective;*

(vi) *There exists $B \in F(\mathcal{X})$ which commutes with T such that $T - B$ is surjective;*

(vii) *There exists $B \in K(\mathcal{X})$ which commutes with T such that $T - B$ is surjective.*

Proof. (i) $\Longrightarrow$ (ii). Let $T \in \mathcal{B}_-(\mathcal{X})$. Then T' is a closed operator and from [4, Proposition 3.1(iii)] it follows that $\mathrm{asc}(T') < \infty$. Further, since $R(T)$ is closed, $R(T')$ is also closed by [30, Theorem IV.1.2], and from [30, Theorem IV.2.3, i.] it follows that $\alpha(T') = \beta(T) < \infty$. Therefore, $T' \in \mathcal{B}_+(\mathcal{X}')$.

Let $S \in L(\mathcal{X})$ be an arbitrary bijection with $S(\mathcal{D}(T)) = \mathcal{D}(T)$ and let S commutes with T. Then $S' \in L(\mathcal{X}')$, $\|S'\| = \|S\|$, and S' is bijective. In addition, by Lemma 6.3.1, $S'(\mathcal{D}(T')) \subset \mathcal{D}(T')$ and S' commutes with T'. We shall show that $S'(\mathcal{D}(T')) = \mathcal{D}(T')$. It is sufficient to prove $\mathcal{D}(T') \subset S'(\mathcal{D}(T'))$.

Suppose that $y' \in \mathcal{D}(T')$. Then there exists the unique functional $z' \in \mathcal{X}'$ such that $y' = S'z' = z'S$. It follows that $z' = y'S^{-1}$. By Theorem 6.2.1, S^{-1} commutes with T, and so for $x \in \mathcal{D}(T)$ the following holds

$$\begin{aligned} \|(z'T)x\| &= \|y'\left(S^{-1}(Tx)\right)\| = \|y'\left(T(S^{-1}x)\right)\| = \\ &= \|(y'T)(S^{-1}x)\| \leq \|y'T\|\|S^{-1}\|\|x\|, \end{aligned}$$

which proves that $z' \in \mathcal{D}(T')$. Therefore, $\mathcal{D}(T') \subset S'(\mathcal{D}(T'))$.

By Theorem 6.2.4, there exists some $\epsilon > 0$ such that $T' - A$ is bounded below for every operator $A \in L(\mathcal{X}')$ such that $A \in \mathrm{comm}_\epsilon^{-1}(T')$ and $A(\mathcal{D}(T')) = \mathcal{D}(T')$. According to the preceding paragraph, for $S \in \mathrm{comm}_\epsilon^{-1}(T)$ with $S(\mathcal{D}(T)) = \mathcal{D}(T)$, it follows that $S' \in \mathrm{comm}_\epsilon^{-1}(T')$ and $S'(\mathcal{D}(T')) = \mathcal{D}(T')$. Consequently, $T' - S'$ is bounded below. By Lemma 6.1.6 and [30, Theorem IV.1.2], $R(T - S)$ is closed, and by [30, Theorem IV. 2.3, i.] we conclude that

$$\beta(T - S) = \alpha((T - S)') = \alpha(T' - S') = 0.$$

Therefore, $T - S$ is onto.

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (iv). Suppose that $T \in \Phi_-(\mathcal{X})$ and $0 \notin \mathrm{acc}\,\sigma_{su}(T)$. From Theorem 6.1.7 it follows that there exist two closed subspaces M and N such that $\mathcal{X} = M \oplus N$, $T(M \cap \mathcal{D}(T)) \subset M$, the restriction T_M of T to M is a closed and Kato operator; $N \subset \mathcal{D}(T)$, $\dim N < \infty$, $T(N) \subset N$ and $T_N : N \to N$ is a bounded and nilpotent operator of degree p. Let P be the projector on N parallel to M. Then $P \in F(\mathcal{X})$, $R(P) \subset \mathcal{D}(T)$ and $P(\mathcal{D}(T)) \subset \mathcal{D}(T)$. In the same way as in the proof of Theorem 6.2.4 (see (iii) $\Longrightarrow$ (iv)) we conclude that P commutes with T. Clearly, $T_0 = T_N$ is a bounded and nilpotent operator. Since there exists $\epsilon > 0$ such that $T - \lambda I$ is surjective for $0 < |\lambda| < \epsilon$, $T_M - \lambda I$ is surjective for $0 < |\lambda| < \epsilon$. As T_M is Kato, from [45, Theorem 3, p. 297], we conclude that $T_1 = T_M$ is surjective.

(iv) $\Longrightarrow$ (v). Suppose that there exists a projector $P \in F(\mathcal{X})$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, T_0 is nilpotent bounded operator of degree p and T_1 is surjective. We can now proceed analogously to the proof of the implication (iv) $\Longrightarrow$ (v) of Theorem 6.2.4. Consequently, we obtain that TP is a nilpotent bounded operator. From $R(T + P) = R(T_1) \oplus R(T_0 + I) = N(P) \oplus R(P) = \mathcal{X}$, we see that $T + P$ is a surjection.

(v) $\Longrightarrow$ (vi). Put $B = -P$.

(vi) $\Longrightarrow$ (vii). Obvious.

(vii) $\Longrightarrow$ (i). Let there exists $B \in K(\mathcal{X})$ which commutes with T such that $T - B$ is surjective. Put $A = T - B$. The operator B' is compact and commutes with T'. The operator $T - B$ is surjective, so it has closed range. From [30, Theorem IV.1.2] and Lemma 6.1.6 we see that $R(T' - B')$ is also closed and by [30, Theorem IV.2.3], $\alpha(T' - B') = \beta(T - B) = 0$. It follows that the operator

$T' - B'$ is bounded below and from Theorem 6.2.4 it follows that $T' \in \mathcal{B}_+(\mathcal{X}')$. Using again [30, Theorem IV.1.2] and [30, Theorem IV.2.3, i.] we deduce that $R(T)$ is closed and $\beta(T) = \alpha(T') < \infty$, so $T \in \Phi_-(\mathcal{X})$. According to [4, Proposition 3.1], $\operatorname{dsc}(T) = \operatorname{asc}(T') < \infty$, and hence $T \in \mathcal{B}_-(\mathcal{X})$. $\square$

Corollary 6.3.3. *Let $T \in \mathcal{C}(\mathcal{X})$, $\overline{\mathcal{D}(T)} = \mathcal{X}$ and $\rho_\Phi(T) \neq \emptyset$. Then*

$$\sigma_{\mathcal{B}_-}(T) = \sigma_{\Phi_-}(T) \cup \operatorname{acc} \sigma_{su}(T).$$

Proof. Note that $\rho_\Phi(T) \neq \emptyset$ implies $\rho_\Phi(T - \lambda) \neq \emptyset$ for all $\lambda \in \mathbb{C}$. What is more, $T - \lambda$ is closed and densely defined for every $\lambda \in \mathbb{C}$. Now, the result follows from the equivalence (i) $\Longleftrightarrow$ (iii) of Theorem 6.3.2. $\square$

Corollary 6.3.4. *Let $T \in \mathcal{C}(\mathcal{X})$, $\overline{\mathcal{D}(T)} = \mathcal{X}$ and $\rho_\Phi(T) \neq \emptyset$. Then, $\sigma_{\mathcal{B}_-}(T)$ is closed.*

Proof. Notice that $\sigma_{\Phi_-}(T)$ is closed [45, Theorem 1] and apply Corollary 6.3.3. $\square$

Corollary 6.3.5. *Let $T \in \mathcal{C}(\mathcal{X})$, $\overline{\mathcal{D}(T)} = \mathcal{X}$ and $\rho_\Phi(T) \neq \emptyset$. Then*

$$\sigma_{\mathcal{B}_-}(T) = \bigcap_{F \in \mathcal{F}_T(\mathcal{X})} \sigma_{su}(T + F) = \bigcap_{K \in \mathcal{K}_T(\mathcal{X})} \sigma_{su}(T + K).$$

Proof. Suppose that $\lambda \notin \bigcap_{K \in \mathcal{K}_T(\mathcal{X})} \sigma_{su}(T + K)$. Then there exists $K \in \mathcal{K}_T(\mathcal{X})$ such that $\lambda \notin \sigma_{su}(T + K)$, that is $T + K - \lambda$ is surjective. Since $-K$ commutes with $T - \lambda$, $\overline{\mathcal{D}(T - \lambda)} = \overline{\mathcal{D}(T)} = \mathcal{X}$, $\rho_\Phi(T - \lambda) \neq \emptyset$, from Theorem 6.3.2 ((i) $\Longleftrightarrow$ (vii)) it follows that $T - \lambda \in \mathcal{B}_-(\mathcal{X})$, i.e. $\lambda \notin \sigma_{\mathcal{B}_-}(T)$. Therefore, $\sigma_{\mathcal{B}_-}(T) \subset \bigcap_{K \in \mathcal{K}_T(\mathcal{X})} \sigma_{su}(T + K) \subset \bigcap_{F \in \mathcal{F}_T(\mathcal{X})} \sigma_{su}(T + F)$.

To prove the opposite inclusion, suppose that $\lambda \notin \sigma_{\mathcal{B}_-}(T)$. Then $T - \lambda \in \mathcal{B}_-(\mathcal{X})$, and since $\overline{\mathcal{D}(T - \lambda)} = \mathcal{X}$ and $\rho_\Phi(T - \lambda) \neq \emptyset$, from Theorem 6.3.2 ((i) $\Longleftrightarrow$ (vi)) it follows that there exists $F \in F(\mathcal{X})$ which commutes with $T - \lambda$ such that $T - \lambda - F$ is surjective. Then $F_1 = -F \in F(\mathcal{X})$ commutes with T and hence $F_1 \in \mathcal{F}_T(\mathcal{X})$. Moreover, $T + F_1 - \lambda$ is surjective, and so $\lambda \notin \sigma_{su}(T + F_1)$. Consequently, $\bigcap_{F \in \mathcal{F}_T(\mathcal{X})} \sigma_{su}(T + F) \subset \sigma_{\mathcal{B}_+}(T)$. $\square$

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BIOGRAPHY

Miloš Cvetković was born on April 11, 1983 in Vranje. He finished elementary school "Branko Radičević" and general-education high school "Jovan Skerlić" in Vladičin Han. In the academic year 2002/2003 he enrolled at the Faculty of Sciences and Mathematics in Niš, Department of Physics. He graduated from this faculty in 2008 with the Grade Point Average of 9.20/10. After graduation he worked at the general-education high school in Ivanjica and elementary school "1 October" in Bašaid (Kikinda) for a short period of time.

He enrolled for his Doctoral Academic Studies in the academic year 2010/2011 at the Department of Mathematics, the Faculty of Sciences and Mathematics in Niš. He has passed all the foreseen exams and has authored or co-authored 5 scientific papers published in international journals. He presented his findings at several conferences: "13th Serbian Mathematical Congress" (Vrnjačka Banja, 2014), "Analysis, Topology and Applications 2014" (Vrnjačka Banja, 2014) and "International Symposium on Geometric Function Theory and Applications" (Ohrid, 2015).

Since the beginning of 2011 he has been working as a researcher on the project "Functional Analysis, Stochastic Analysis and Applications" (No. 174007) funded by the Ministry of Education, Science and Technological Development of the Republic of Serbia. During the academic year 2016/2017 he held practical exercises in Functional Analysis at the Department of Mathematics at the Faculty of Sciences and Mathematics in Niš and he also worked as a Teaching Assistant at the High Technology Arts Vocational School in Leskovac.

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Изјављујем да је докторска дисертација, под насловом

ДЕКОМПОЗИЦИЈЕ КАТООВОГ ТИПА И УОПШТЕЊА ДРАЗИНОВЕ ИНВЕРТИБИЛНОСТИ

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У Нишу, 14.6.2017. године

Потпис аутора дисертације:

Милош Д. Цвегковић

(Име, средње слово и презиме)

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ИЗЈАВА О ИСТОВЕТНОСТИ ЕЛЕКТРОНСКОГ И ШТАМПАНОГ ОБЛИКА ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

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ДЕКОМПОЗИЦИЈЕ КАТООВОГ ТИПА И УОПШТЕЊА ДРАЗИНОВЕ ИНВЕРТИБИЛНОСТИ

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У Нишу, 14.6.2017. године

Потпис аутора дисертације:

Милош Д. Цветковић

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ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

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Mentor, **MN**:	**Snežana Č. Živković-Zlatanović**
Title, **TI**:	Kato type decompositions and generalizations of Drazin invertibility
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Scientific field, **SF**:	**mathematics**
Scientific discipline, **SD**:	**mathematical analysis**
Subject/Key words, **S/KW**:	**functional analysis/Kato decomposition, Kato operator, semi-Fredholm operators, generalized Drazin spectrum, operator matrices, Banach algebra, closed operator**
UC	**517.983.23:517.984.3**
Holding data, **HD**:	**library**
Note, **N**:	
Abstract, **AB**:	The main objective of this dissertation is to give necessary and sufficient conditions under which a bounded linear operator T can be represented as the direct sum of a nilpotent (quasinilpotent, Riesz) operator T_N and an operator T_M which belongs to any of the following classes: upper (lower) semi-Fredholm operators, Fredholm operators, upper (lower) semi-Weyl operators, Weyl operators, upper (lower) semi-Browder operators, Browder operators, bounded below operators, surjective operators and invertible operators. These results are applied to different types of spectra. In addition, we introduce the notions of the generalized Kato-Riesz decomposition and generalized Drazin-Riesz invertible operators. Moreover, we study the generalized Drazin spectrum of an upper triangular operator matrix acting on the product of Banach or separable Hilbert spaces. Further, we introduce the notion of a B-Fredholm Banach algebra element. These objects are characterized and their main properties are studied. We also extend some results from the Fredholm theory to unbounded closed operators..

Accepted by the Scientific Board on, **ASB**:		4.7.2016.
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Defended Board, **DB**:	President:	
	Member:	
	Member	
	Member	
	Member, Mentor	

Образац Q4.09.13 - Издање 1

Прилог 4/1

ПРИРОДНО - МАТЕМАТИЧКИ ФАКУЛТЕТ
НИШ

КЉУЧНА ДОКУМЕНТАЦИЈСКА ИНФОРМАЦИЈА

Редни број, **РБР**:	
Идентификациони број, **ИБР**:	
Тип документације, **ТД**:	монографска
Тип записа, **ТЗ**:	текстуални
Врста рада, **ВР**:	докторска дисертација
Аутор, **АУ**:	Милош Д. Цветковић
Ментор, **МН**:	Снежана Ч. Живковић-Златановић
Наслов рада, **НР**:	Декомпозиције Катоовог типа и уопштења Дразинове инвертибилности
Језик публикације, **ЈП**:	енглески
Језик извода, **ЈИ**:	српски
Земља публиковања, **ЗП**:	Србија
Уже географско подручје, **УГП**:	Србија
Година, **ГО**:	2017.
Издавач, **ИЗ**:	ауторски репринт
Место и адреса, **МА**:	Ниш, Вишеградска 33.
Физички опис рада, **ФО**: (поглавља/страна/ цитата/табела/слика/графика/прилога)	90 стр.
Научна област, **НО**:	математика
Научна дисциплина, **НД**:	математичка анализа
Предметна одредница/Кључне речи, **ПО**:	функционална анализа/Катоова декомпозиција, Катоов оператор, семи-Фредхолмови оператори, уопштени Дразинов спектар, операторске матрице, Банахова алгебра, затворен оператор
УДК	517.983.23:517.984.3
Чува се, **ЧУ**:	библиотека
Важна напомена, **ВН**:	
Извод, **ИЗ**:	Централна тема дисертације је успостављање потребних и довољних услова под којима ограничен линеаран оператор T може бити представљен као директна сума једног нилпотентног (квазинилпотентног, Рисовог) оператора T_N и једног оператора T_M који припада било којој од следећих класа: горњи (доњи) семи-Фредхолмови оператори, Фредхолмови оператори, горњи (доњи) семи-Вејлови оператори, Вејлови оператори, горњи (доњи) семи-Браудерови оператори, Браудерови оператори, оператори ограничени одоздо, сурјективни оператори и инвертибилни оператори. Добијени резултати се примењују на изучавање различитих типова спектара. Такође, уводе се појмови уопштене Като-Рисове декомпозиције и уопштених Дразин-Рис инвертибилних оператора. Проучава се уопштени Дразинов спектар горње троугаоних операторских матрица које делују на производ Банахових или сепарабилних Хилбертових простора. Уводи се појам Б-Фредхолмових елемената Банахове алгебре. Ови објекти се карактеришу и њихове главне особине се изучавају. Проширујемо неке резултате из Фредхолмове теорије на неограничене затворене операторе.

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Чланови комисије, **КО**:	Председник:	
	Члан:	
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Образац Q4.09.13 - Издање 1