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A new type of regularity with applications to the wave front sets

doctoral dissertation

Nova vrsta regularnosti sa primenama na talasni front

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Contents

Sažetak

Klase Ževrea ([14]) su uvedene u cilju preciznijeg objašnjenja regularnosti rešenja jednačine provođenja toplote, i na taj način su našle svoju primenu u teoriji linearnih parcijalnih jednačina, posebno u ispitivanju hipoeliptičnosti, lokane rešivosti i u analizi prostiranja singulariteta rešenja.

U ovoj tezi definišemo klasu glatkih funkcija koje imaju "slabiju regularnost" nego Ževre funkcije, i izučavamo njihove osnovne osobine. Pokazujemo da naše klase imaju svojsto algebre kao i da su zatvorene u odnosu na delovanje operatora izvoda konačnog reda. Šta više, konstruišemo diferencijalne operatore beskonačnog reda i to nas dovodi do definicije ultradiferencijabilnih klasa funkcija. Takođe dokazujemo osobinu zatvorenosti u odnosu na inverze, i taj rezultat je najvažniji deo u dokazu glavne teoreme koja je formulisana u poslednjoj glavi.

Koristeći tehnike mikrolokalne analize, uvodimo i izučavamo odgovarajuće talasne frontove. Naš glavni rezultat pokazuje kako se prostiru singulariteti rešenja linearnih parcijalnih diferencijalnih jednačina u okviru naše regularnosti.

Neki rezultati iz ove teze su objavljeni u [29], [30], [43], kao i u radu [31] koji je u pripremi.

Abstract

Since their introduction in the context of regularity properties of fundamental solution of the heat operator in [14], Gevrey classes were used in many questions related to the general theory of linear partial differential operators, such as hypoellipticity, local solvability and propagation of singularities.

We introduce a family of smooth functions which are "less regular" than the Gevrey functions, and study its basic properties. In particular we prove the standard results concerning algebra property and stability under finite order derivation. Moreover, we construct infinite order operators which leads us to the definition of class with ultradifferentiable property. We also prove that our classes are inverse-closed, and this result is the essential part in the proof of our main result presented in the final Chapter.

Moreover, using the techniques of microlocal analysis, we introduce and investigate the corresponding wave front sets. Our main results shows how the singularities of solutions to partial differential equations (PDE's in short) propagate in the framework of our regularity.

Some results of thesis are published in [29], [30] and [43], see also [31].

Preface

In this thesis we propose a new type of local regularity and analyze corresponding classes of smooth functions. In particular, we introduce two parameter depending *defining sequences* $M_p^{\tau,\sigma} := p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$, $p \in \mathbf{N}$, which control the derivatives of functions, we analyze their properties, and construct classes of smooth function which posses *ultradifferentiable property*. Such classes are *classes of ultradifferentiable functions* since they are closed under action of certain infinite order differential operators. Moreover, we study nature of singularities related to their duals using the techniques of *microlocal analysis*.

H.Komtasu in [21] developed methods of *local analysis* for studying classes of ultradifferentiable functions and their spaces of *ultradistributions* as their strong duals. Classes of test functions are inductive and projective limits of (countable) families of Banach spaces and therefore they have Frechét structure. In particular, the *regularity condition* related to Komatsu's classes is given by

$$|\partial^\alpha \phi(x)| \leq A h^{|\alpha|} M_{|\alpha|}, \quad x \in K, \alpha \in \mathbf{N}^d,$$

and we refer to Chapter 1, Section 1.2, for details.

However, we propose the regularity of the form

$$|\partial^\alpha \phi(x)| \leq A h^{|\alpha|^\sigma} M_{|\alpha|}^{\tau,\sigma}, \quad x \in K, \alpha \in \mathbf{N}^d,$$

wherefrom it follows that corresponding classes of functions are not equal to Komatsu's classes for any choice of parameters $\tau > 0$ and $\sigma > 1$. Moreover, sequence $M_p^{\tau,\sigma}$ fails to satisfy usual Komatsu's conditions (see Chapter 1, Section 2.1) and in that sense results of Chapter 2 generalizes standard results given in [20].

In [38] Siddiqi studied the *inverse closedness* property of *Carleman classes* by imposing an additional condition to the M_p, $p \in \mathbf{N}$. The basic question is: if ϕ belongs to some subclass of smooth functions and $\phi \neq 0$ (locally) does the ϕ^{-1} belongs to the same subclass? It turns out that definition Carleman classes corresponds to Komatsu's definition of ultradiferentiable functions of *Roumieu type*, and therefore in Chapter 2, Section 2.7, we extend the corresponding result for our classes using different techniques then the one from [38]. Some of presented techniques will be used in the proof of main results in the final Chapter 4.

For the particular choice $M_p = p!^t$, $t > 1$, Roumieu type Komatsu classes correspond to *Gevrey classes*. They were introduced by M. Gevrey to describe regularity properties of fundamental solution of the heat operator in [14], and thereafter used in the study of different aspects of general theory of linear partial differential operators such as hypoellipticity, local solvability and propagation of singularities. In particular, the well-posedness of the Cauchy problem for weakly hyperbolic linear partial differential equations (PDE's) can be characterized by the Gevrey index t, while the same problem is ill-posed in the class of analytic functions, (see [3, 35] and the references given there).

Since there is a gap between Gevrey classes and smooth functions, it is of interest to study the intermediate spaces of smooth functions which are contained in those gaps by introducing appropriate regularity conditions. On one hand, this may serve to describe hypoellipticity properties between smooth hypoellipticity and Gevrey hypoellipticity, which is one motivation for work presented in this thesis, cf. [24].

Another motivation comes from microlocal analysis, where the notion of wave front set plays a crucial role. Different authors studied different types of wave front sets. Roughly speaking, complements of wave front sets are conical sets of points $(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \backslash \{0\}$ for which the distribution u is regular in the neighborhood of x in the directions of derivatives determined by ξ.

We refer to [12], [16], [42] for *classical wave front sets*, $\mathrm{WF}(u)$, whose complement describes the C^∞ regularity. In particular, classical wave front set satisfy

$$\pi_1(\mathrm{WF}(u)) = \mathrm{singsupp}(u), \quad u \in \mathcal{D}'(U),$$

where π_1 is *standard projection* and $\mathrm{singsupp}\, u$ denotes *singular support* of distribution u. However, the fundamental property is *microlocal hypoellipticity* which explains how the singularities of solutions to the partial differential equations *propagate*. It is expressed by

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u) \subseteq \mathrm{WF}(Pu) \cup \mathrm{Char}(P),$$

where P is partial differential operator with smooth coefficients and $\mathrm{Char}(P)$ is set of its *characteristics*. We refer to Chapter 1, Section 1.4 for details.

Wave front sets with respect to C^L, denoted by WF_L, are introduced and investigated in [16]. For $L_p = p^t$, $t > 1$, they are *Gevrey wave front sets* while putting $t = 1$ the definition of *analytic wave front set* arises, which is the largest wave front set in the existing literature. They both have *microlocal hypoellipticity* property and for the analytic case this property immediately implies famous *Holmgren's uniqueness theorem* from theory of PDE's, as it is stated in [16].

Moreover, these wave front sets are related to classes of ultradifferentiable functions (Gevrey classes and classes of analytic functions) and therefore they are significant for our investigation. Wave front sets introduced in Chapters 3 and Chapter 4 are different from WF_L for any choice of sequence L_p, and hence the modification of standard arguments in the proofs of main results is needed. Roughly speking, we introduce wave front sets which detect singularities that are "stronger" then the classical C^∞ singularities and at the same time "weaker" than any Gevrey type singularities.

Further properties of Gevrey wave front sets are studied in [35]. Moreover, the classical results are extended to the spaces of *Gevrey ultradistributions* and partial differential operators with coefficients in Gevrey classes.

Different types of wave front sets that modify the classical wave front set are introduced in the literature. Although their definition goes beyond the scope of this thesis, we briefly mention the *Gabor wave front set* originally defined in [17] and further developed in [36], which is based on microlocal analysis on cones taken with respect to the whole of the phase space variables. Such approach is recently successfully applied to the study of Schrödinger equations in [4, 5, 33, 44], see also the references therein. Since versions of Gabor wave front set can be adapted to analytic and Gevrey regularity (cf. [1, 40, 41]) it is natural to assume that the same holds in the framework of regularity proposed in thesis, and this will be the subject of future investigation.

0.1 Outline and Acknowledgements

Chapter "Introduction" contains some of the basic notions and notations from the theory of ultradifferentiable functions, partial differential equations and microlocal analysis that will be used in the thesis.

Chapter "Classes of ultradifferentiable functions" introduces a new classes of functions, $\mathcal{E}_{\tau,\sigma}$, $\tau > 0$, $\sigma > 1$, which describes a new type of local regularity investigated in this thesis. It contains original results concerning basic properties of the defining sequence $M_p^{\tau,\sigma}$, $p \in \mathbf{N}$ (sequence that controls the derivatives) and basic topological properties of $\mathcal{E}_{\tau,\sigma}$. Main result is given in Theorem 2.6.2, Section 2.6 where it is proven that classes $\mathcal{E}_{\infty,1}$ are closed under action of certain ultradifferentiable operators. In the final section, we discuss the inverse-closedness property.

A new type of wave front sets, $\mathrm{WF}_{\tau,\sigma}$, $\tau > 0$, $\sigma > 1$, are defined and analyzed in Chapter "Wave front sets related to $\mathcal{E}_{\tau,\sigma}$". In section 3.2 is proven that the local regularity described by the complement of $\mathrm{WF}_{\tau,\sigma}$ is regularity investigated in Chapter 2. For the analysis we choose *admissible* sequences of cut-off test functions, similar to one used in [16] to analyze local analyticity. Moreover, in order to describe asymptotic behaviour in microlocalization we intodroduce a procedure called *enumeration*. The notion of singular support related to classes $\mathcal{E}_{\tau,\sigma}$ is defined in Section 3.3 and the main result of the Chapter is presented in Theorem 3.3.1. In the final section, we discuss unions and intersections of $\mathrm{WF}_{\tau,\sigma}$ with respect to parameters τ, σ which lead us to the definition of $\mathrm{WF}_{0+,\infty}$, wave front set with *pseudo-local property*.

The main result of the final Chapter "Microlocal analysis of solutions to PDE's" is Theorem 4.2.1. It contains the proof of microlocal hypoellipticity of PDO's with the coefficients in classes $\mathcal{E}_{\infty,1+}$. In the Section 4.1, the PDO's with constant coefficients are considered, and the non-trivial modifications for the case of the non-constant coefficients is presented in Section 4.2. In particular, we prove that $\mathrm{WF}_{0+,\infty}$ have the pseudo-property.

The results presented in this thesis are obtained in collaboration with Professors Nenad Teofanov and Stevan Pilipović. I wish to express my sincere thanks to my mentor Professor Nenad Teofanov for understanding, endless support and numerous discussions in the process of writing the thesis. Also I express my gratitude to Professor Stevan Pilipović, who initiated the research presented in this thesis, for the opportunity to work within his group and for focusing this research in the right direction. I am forever in debt to my parents and my girls Katarina, Jovana i Biljana, for their endless love and support during all these years. This research is carried out under the Project no. 174024.

Chapter 1

Introduction

This chapter contains familiar results from theory of distributions, ultradifferentiable functions and partial differential equations that will be used in thesis. We begin by fixing the notation.

1.1 Notation

Nonnegative integers, integers, positive integers, real numbers, positive real numbers and complex numbers are denoted by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Z}_+$, $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{C}$, respectively. The integer part (the floor function) of $x \in \mathbf{R}_+$ is denoted by $\lfloor x \rfloor := \max\{m \in \mathbf{N} : m \leq x\}$. For a multiindices $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ we write $\partial^\alpha = \partial^{\alpha_1} \ldots \partial^{\alpha_d}$, $|\alpha| = |\alpha_1| + \ldots |\alpha_d|$, $\alpha! = \alpha_1! ... \alpha_d!$ and $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for $1 \leq i \leq d$. Moreover, $D^{\alpha_i} = \left(\frac{1}{i}\right)^{|\alpha|} \partial^\alpha$. We will also use the Stirling formula: $N! = N^N e^{-N} \sqrt{2\pi N} e^{\frac{\theta_N}{12N}}$, for some $0 < \theta_N < 1$, $N \in \mathbf{Z}_+$. If $U \subseteq \mathbf{R}^d$ is open, the we use the notation $K \subset\subset U$ if K is compact set with smooth boundary contained in U. Closure of set U is denoted by $\overline{U}$.

By $C^m(K)$, $m \in \mathbf{N}$, we denote the Banach space of m-times continuously differentiable functions on a compact set with smooth boundary $K \subset\subset U$, where $U \subseteq \mathbf{R}^d$ is an open set, and $C^\infty(K)$ is the corresponding set of smooth functions on K, see [20]. Moreover, by $C_0^\infty(K)$ we denote the space of smooth functions supported in K. With $\operatorname{supp} u$ we denote the support of function (distribution) u. Convolution is denoted with $f * g(x) = \int_{\mathbf{R}^d} f(x-y) g(y) dy$, whenever the integral make sense. Open ball of radius $r > 0$ centered at $x_0 \in \mathbf{R}^d$ is denoted by $B_r(x_0)$, and $\operatorname{card} A$ denotes the cardinal number of A. We use the standard notation $\langle x \rangle = (1 + |x|^2)^{1/2}$ for $x \in \mathbf{R}^d$. Fourier transform is denoted by

$$\mathcal{F}_{x\to\xi}u(x) = \widehat{u}(\xi) = \int u(x)e^{-2\pi i x\xi}\,dx.$$

Recall some of the properties for multinomial coefficients

$$\begin{aligned}\binom{|a|}{a_1, a_2, \dots a_m} &:= \binom{|a|}{a_1}\binom{|a|-a_1}{a_2}\dots\binom{|a|-a_1-\cdots-a_{m-2}}{a_{m-1}}\\ &= \frac{|a|!}{a_1!a_2!\dots a_m!},\end{aligned} \tag{1.1.1}$$

where $|a| = a_1 + a_2 + \cdots + a_m$, $a_k \in \mathbf{N}$, $k \le m$. Following formula generalizes Pascal triangle equality for binomial formula

$$\binom{|a|}{a_1, \dots, a_m} = \sum_{k=1}^{m}\binom{|a|-1}{a_1, \dots, a_k - 1, \dots a_m}, \quad |a| \ge 1 \tag{1.1.2}$$

Moreover, since $\binom{n}{k} \le 2^n$, $k \le n$, $n \in \mathbf{N}$ we note that

$$\binom{|a|}{a_1, a_2, \dots a_m} \le 2^{|a|}2^{|a|-a_1}\dots 2^{|a|-a_1-\cdots-a_{m-2}} \le 2^{a_1+2a_2+\cdots+ma_m}. \tag{1.1.3}$$

We will also use the *genarilezed Newton's formula* expressed by

$$(t_1 + t_2 + \cdots + t_d)^n = \sum_{a_1+a_2+\cdots+a_d=n}\binom{n}{a_1, \dots, a_d}t_1^{a_1}\dots t_d^{a_d}, \quad n \in \mathbf{N},$$

wherefrom we conclude that for multiindex $\alpha \in \mathbf{N}^d$ this formula implies that $|\alpha|! \le d^{|\alpha|}\alpha!$ (by setting $t_1 = t_2 = \cdots = t_d$). Converse inequality $\alpha! \le |\alpha|!$ is trivial.

For locally convex topological spaces X and Y, $X \hookrightarrow Y$ means that $X \subseteq Y$ and that the identity mapping from X to Y is continuous, and we use $\varprojlim$ and $\varinjlim$ to denote the projective and inductive limit topologies respectively. By X' we denote the dual of X and by $\langle\cdot,\cdot\rangle_X$ the dual pairing between X and X'. Set of continuous linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$.

Recall, a linear map $B \in \mathcal{L}(X, Y)$, X, Y are Banach spaces, is *quasi-nuclear* if there exists a sequence $\{x'_j\}$ in X' such that $\sum_{j=1}^{\infty}\|x'_j\|_{X'} < \infty$ and

$\|Bx\|_Y \leq \sum_{j=1}^{\infty} |\langle x, x'_j\rangle_X|$. In particular, a quasi-nuclear map $A \in \mathcal{L}(X,Y)$ is *nuclear* if there exists bounded sequences $x'_j \in X'$ and $y_j \in Y$, $j \in \mathbf{Z}_+$, and a sequence $\lambda_j \in \mathbf{C}$, $j \in \mathbf{Z}_+$, such that $\sum_{j=1}^{\infty} |\lambda_j| < \infty$ and $Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, x'_j\rangle_X y_j$. We refer to [39, Section III.7] and [26] for an extension of nuclear and quasi-nuclear mappings to arbitrary locally convex topological spaces.

With $\mathcal{D}'(U), \mathcal{E}'(U)$ we denote the spaces of *Schwartz distributions* and distributions with compact support, respectively. They are duals of spaces of smooth compactly supported functions and smooth functions on U, denoted by $\mathcal{D}(U) = C_0^\infty(U)$ and $\mathcal{D}(U) = C^\infty(U)$, respectively. Recall that $u \in \mathcal{D}'(U)$ if for every $K \subset\subset U$ there exists constants $C > 0$ and $M > 0$ such that

$$|\langle u, \varphi\rangle| \leq C \sum_{|\alpha|\leq M} \sup_{x\in K} |\partial^\alpha \varphi(x)|, \quad \varphi \in C_0^\infty(K),$$

and if M can be chosen independently of K the we say that u is *distribution of order M*.

We will also use the simplified version of Paley-Wiener theorem: If $\varphi \in \mathcal{D}(U)$ then for every $M > 0$ there exists $C > 0$ such $|\widehat{\varphi}(\xi)| \leq C\langle\xi\rangle^{-M}$. Moreover, for $u \in \mathcal{E}'(U)$, there exists $M, C > 0$ such that $|\widehat{u}(\xi)| \leq C\langle\xi\rangle^{M}$, where M is order of u. For more general versions Paley-Wiener theorem we refer to [20], [21].

To end these section we recall (see [42]) that the sequence $\{\chi_N\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{D}(U)$ if there exists compact set $K \subset\subset U$ such that $\operatorname{supp}\chi_N \subseteq K$ for every $N \in \mathbf{N}$ and for every $m \in \mathbf{N}$ there exists constants $C_m > 0$ such that $\sup_{|\alpha|\leq m} \sup_{x\in K} |D^\alpha \chi_N(x)| \leq C_m$ where C_m does not depend on $\mathbf{N}$.

Moreover the sequence $\{u_N\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{E}'(U)$ if there exists $K \subset\subset U$ such that $\operatorname{supp} u_N \subseteq K$ for every $N \in \mathbf{N}$, and for every $\varphi \in \mathcal{E}(U)$ there exists constant $C > 0$ so that $|\langle u_N, \phi\rangle| \leq C$, where C depends only on choice of test function φ.

Therefore, we conclude that if $u \in \mathcal{D}'(U)$ and $\{\chi_N\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{D}(U)$ then $\{\chi_N u\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{E}'(U)$. In particular, simple application of Leibniz rule implies that

$$|\langle \chi_N u, \varphi\rangle| = |\langle u, \chi_N\varphi\rangle| \leq C \sum_{|\alpha|\leq M} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \sup_{x\in K} |\partial^\beta \chi_N(x)||\partial^{\alpha-\beta}\varphi(x)|,$$

wherefrom the conclusion follows immediately.

1.2 Classical spaces of ultradifferentiable functions

In this section we recall Komatsu's approach to the theory of ultradifferentiable functions, see [20], and the notion of wave front set in the context of the Gevrey regularity.

By $M_p = (M_p)_{p\in\mathbf{N}}$ we denote a sequence of positive numbers such that the following conditions hold:

$$\begin{array}{llll} (M.0) & & M_0 = 1; & \\ (M.1) & & M_p^2 \le M_{p-1}M_{p+1}, & p\in\mathbf{Z}_+; \\ (M.2) & (\exists C>0) & M_{p+q} \le C^{p+1}M_pM_q, & p,q\in\mathbf{N}; \\ (M.3)' & & \sum\limits_{p=1}^{\infty}\frac{M_{p-1}}{M_p} < \infty. & \end{array}$$

Then M_p also satisfies weaker conditions: $(M.1)'$ $M_pM_q \le M_{p+q}$ and $(M.2)'$ $(\exists C>0)$ $M_{p+q} \le C_q^{p+1}M_p$, $p,q\in\mathbf{N}$.

Let the sequence M_p satisfy the conditions $(M.0)-(M.3)'$ and let $U\subseteq\mathbf{R}^d$ be an open set. A function $\phi\in C^\infty(U)$ is an *ultradifferentiable function of class* (M_p) (resp. *of class* $\{M_p\}$) if for each compact subset $K\subset\subset U$ and each $h>0$, there exists $C>0$ (resp. for each compact subset $K\subset\subset U$ there exists $h>0$ and $C>0$) such that

$$\sup_{x\in K}|\partial^\alpha\phi(x)| \le Ch^{|\alpha|}M_{|\alpha|}, \quad \alpha\in\mathbf{N}^d. \tag{1.2.1}$$

For a fixed compact set $K\subset\mathbf{R}^d$ and $h>0$, $\phi\in\mathcal{E}^{\{M_p\},h}(K)$ if $\phi\in C^\infty(K)$ and if (1.2.1) holds for some $C>0$. If $\phi\in C^\infty(\mathbf{R}^d)$ and all the derivatives vanishes on the boundary of K, then $\phi\in D_K^{\{M_p\},h}$. These spaces are Banach spaces under the norm

$$\|\phi\|_{\mathcal{E}^{\{M_p\},h}(K)} = \sup_{\alpha\in\mathbf{N}^d, x\in K}\frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|}M_{|\alpha|}}.$$

Locally convex spaces (in the sequel *l*.c.s.) of ultradifferentiable functions of class $\{M_p\}$ and of class (M_p) are respectively given by

$$\mathcal{E}^{\{M_p\}}(K) = \varinjlim_{h\to\infty}\mathcal{E}^{\{M_p\},h}(K) = \bigcup_{h\to\infty}\mathcal{E}^{\{M_p\},h}(K)$$

$$\mathcal{E}^{\{M_p\}}(U) = \varprojlim_{K\subset\subset U}\mathcal{E}^{\{M_p\}}(K) = \bigcap_{K\subset\subset U}\mathcal{E}^{\{M_p\}}(K),$$

$$\mathcal{E}^{(M_p)}(K) = \varprojlim_{h\to 0} \mathcal{E}^{\{M_p\},h}(K) = \bigcap_{h\to 0} \mathcal{E}^{\{M_p\},h}(K)$$

$$\mathcal{E}^{(M_p)}(U) = \varprojlim_{K\subset\subset U} \mathcal{E}^{(M_p)}(K) = \bigcap_{K\subset\subset U} \mathcal{E}^{(M_p)}(K),$$

and their strong duals are respectively called the space of ultradistributions of Roumieu type of class M_p and the space of ultradistributions of Beurling type of class M_p.

Spaces of ultradifferentiable functions of class $\{M_p\}$ (resp. of class (M_p)) with support in K is given by

$$\mathcal{D}_K^{\{M_p\}} = \varinjlim_{h\to\infty} \mathcal{D}^{\{M_p\},h}(K) = \bigcup_{h\to\infty} \mathcal{D}_K^{\{M_p\},h}$$

$$\mathcal{D}^{\{M_p\}}(U) = \varprojlim_{K\subset\subset U} \mathcal{D}_K^{\{M_p\}} = \bigcap_{K\subset\subset U} \mathcal{D}_K^{\{M_p\}},$$

$$\mathcal{D}_K^{(M_p)} = \varprojlim_{h\to 0} \mathcal{D}_K^{\{M_p\},h} = \bigcap_{h\to 0} \mathcal{D}_K^{\{M_p\},h}$$

$$\mathcal{D}^{(M_p)}(U) = \varprojlim_{K\subset\subset U} \mathcal{D}_K^{(M_p)} = \bigcap_{K\subset\subset U} \mathcal{D}_K^{(M_p)},$$

and its strong dual is the space of compactly supported ultradistributions of Roumieu type of class M_p (resp. of Beurling type of class M_p).

In what follows, $\mathcal{E}^*(U)$ and $\mathcal{D}^*(U)$ stand for $\mathcal{E}^{\{M_p\}}(U)$ or $\mathcal{E}^{(M_p)}(U)$, and for $\mathcal{D}^{\{M_p\}}(U)$ or $\mathcal{D}^{(M_p)}(U)$, respectively.

Remark 1.2.1. If M_p is the *Gevrey sequence*, $M_p = p!^t$, $t > 1$, then $\mathcal{E}^{\{p!^t\}}(U)$ is the *Gevrey class* of ultradifferentiable functions which we denote by $\mathcal{E}_t(U)$. Note that $p!^t$, $t > 1$, satisfies $(M.0) - (M.3)'$, while for $0 < t \leq 1$ sequence $M_p = p!^t$ fails to satisfy $(M.3)'$. For $t = 1$, the corresponding spaces consists of analytic function on U, while for $0 < t < 1$ spaces $\mathcal{E}^{\{p!^t\}}(U)$ consists of entire functions. In particular, it is well known that $\mathcal{D}^{\{p!^t\}}(U) = \{0\}$ when $0 < t \leq 1$. (see Theorem 1.3.8. in [16]). With $\mathcal{D}'_t(U)$ and $\mathcal{E}'_t(U)$, $t > 1$ we will denote spaces of *Gevrey ultradistributions* and its subspace of ultradistributions with compact support.

Recall, ([20], [21]), operators of the form $P(x, D) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x) D^\alpha$ are called ultradifferentiable operators of the class $*$, if $a_\alpha \in \mathcal{E}^*(U)$ and for the case $* = \{M_p\}$ (resp. $* = (M_p)$) for every $K \subset\subset U$, there exists $h > 0$ such that for any $L > 0$ there exists $A > 0$ so that (resp. for every $K \subset\subset U$ there exists constant $L > 0$ such that for every $h > 0$ there exists $A > 0$ so that)

$$\sup_{x\in K}|D^\beta a_\alpha(x)| \leq Ah^{|\beta|}M_{|\beta|}\frac{L^{|\alpha|}}{M_{|\alpha|}}, \quad \alpha,\beta\in \mathbf{N}^d.$$

Following Theorem captures the basic properties of the ultradifferentiable classes.

Theorem 1.2.1. *([20]) Let M_p, $p\in\mathbf{N}$, satisfy properties $(M.0)-(M.2)$, and $\phi,\psi\in\mathcal{E}^*(U)$ where U is open in $\mathbf{R}^d$. Then pointwise product $\phi\psi\in\mathcal{E}^*(U)$, and $\mathcal{E}^*(U)$ is closed under finite order derivation.*

*Further, if $P(x,D)=\sum_{|\alpha|=0}^{\infty}a_\alpha(x)D^\alpha$ is ultradifferentiable operator of the class *, the the mapping*

$$P(x,D)\,:\,\mathcal{E}^*(U)\to\mathcal{E}^*(U),$$

is continuous with respect to the topology of $\mathcal{E}^*(U)$.

Moreover, $\mathcal{E}^*(U)$ are nuclear.

To conclude this section, we recall that *associated function* to the sequence M_p, $p\in\mathbf{N}$, $M_0=1$, is given by

$$T_{M_p}(r)=\sup_{p\in\mathbf{N}} = \frac{r^p}{M_p}.$$

For more details and properties of $T_{M_p}(r)$ we refer to [20], see also Remark 2.1.4.

Remark 1.2.2. In the following chapters we will also use several spaces whose definition is equivalent to the definition of Komatsu's classes, with different conditions on the sequences. In fact, in [16] author introduce classes $C^L(U)$ in the following way: $\phi\in C^L(U)$ if and only if for every compact set $K\subset\subset U$ there exists constant C_K such that

$$\sup_{x\in K}|D^\alpha\phi(x)|\leq C_K(C_K L_\alpha)^{|\alpha|}, \quad \alpha\in\mathbf{N}^d, \tag{1.2.2}$$

where L_p, $p\in\mathbf{N}$, is the increasing sequence of positive numbers satisfying conditions

$$L_0=1, \quad p\leq L_p, \quad L_{p+1}\leq CL_p, \quad C>0,\, p\in\mathbf{N}. \tag{1.2.3}$$

In particular, $C^L(U)$ are of Roumieu type of class $M_p:=L_p^p$, $p\in\mathbf{N}$.

Moreover, the definition of *Carleman classes* $C_M(U)$ (see [38], [19]) is given by: $\phi\in C_M(U)$, where M_p, $p\in\mathbf{N}$, satisfies property $(M.1)$, if for every compact set $K\subset\subset U$

$$\sup_{p\in\mathbf{N}}(\sup_{x\in K}|f(x)|/M_p)^{1/p} < \infty.$$

Similarly, $C_M(U)$ are of Roumieu type of class M_p, $p \in \mathbf{N}$.

1.3 Approximate solutions to PDE's

In this section we analyze the solutions of certain partial differential equations. We recall some of the basic notions from the theory of PDE's.

Let $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$, $a_\alpha \in C^\infty(U)$, be the differential operator of order m on U, and let $P_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$, $(x,\xi) \in U \times \mathbf{R}^d\backslash\{0\}$ denotes its principal symbol.

Set $\tilde{\Gamma}$ in $U \times \mathbf{R}^d\backslash\{0\}$ is *conical* if $(x,\xi) \in \tilde{\Gamma}$ implies $(x,t\xi) \in \tilde{\Gamma}$ for every $t > 0$. Moreover, the function $\varphi(x,\xi)$ defined on $\tilde{\Gamma}$ is said to be *homogeneous* of order $k \in \mathbf{Z}$ if it satisfies $\varphi(x,t\xi) = t^k\varphi(x,\xi)$ for every $t > 0$. We immediately note that principal symbol P_m is homogeneous of order m.

Recall ([34]), the characteristic variety of operator $P(x,D)$ at point $\overline{x} \in U$ is given by

$$\mathrm{Char}_{\overline{x}}(P) = \{(\overline{x},\xi) \in U \times \mathbf{R}^d\backslash\{0\} \,|\, P_m(\overline{x},\xi) = 0\},$$

Characteristic set of operator $P(x,D)$ on the open set U and is given by

$$\mathrm{Char}(P) = \bigcup_{\overline{x}\in U} \mathrm{Char}_{\overline{x}}(P)$$

By the homogeneity of the principal symbol it follows that $\mathrm{Char}(P)$ is closed conical subset of $U \times \mathbf{R}^d\backslash\{0\}$.

Let $(x_0,\xi_0) \notin \mathrm{Char}(P)$. Since $P_m \in C^\infty(U \times \mathbf{R}^d\backslash\{0\})$, $P_m(x_0,\xi_0) \neq 0$ implies the existence of the conical neighborhood $\tilde{\Gamma}$ of (x_0,ξ_0) such that $P_m(x,\xi) \neq 0$, $(x,\xi) \in \tilde{\Gamma}$. Moreover, by the homogeneity of the principal symbol we obtain

$$P_m(x,\frac{\xi}{|\xi|}) = \frac{1}{|\xi|^m}|P_m(x,\xi)| \geq C, \quad (x,\xi) \in \tilde{\Gamma}. \tag{1.3.1}$$

For $x \in K \subset\subset \pi_1(\tilde{\Gamma})$, where π_1 denotes the standard projection, we obtain the fundamental inequality

$$C_1|\xi|^m \leq P_m(x,\xi) \leq C_2|\xi|^m, \quad C_1, C_2 > 0, (x,\xi) \in \tilde{\Gamma} \cap (K \times \mathbf{R}^d \backslash \{0\}), \tag{1.3.2}$$

where second inequality is trivial.

Note that for $u, f \in \mathcal{D}'(U)$, $P(x,D)u = f$, $v \in C_0^\infty(U)$, partial integration implies that $\langle u, P^T v\rangle = \langle f, v\rangle$, where $P^T v = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha(a_\alpha v)$, is transpose operator for P. It is clear that if $P_m(x,\xi)$ is principal symbol of P then $P_m(x,-\xi)$ is principal symbol of P^T.

Following theorem gives the asymptotical connection between principal symbol and the operator.

Theorem 1.3.1. *([34]) Let $P(x,D)$ be the differential operator with smooth coefficients of order m on U. If $\phi(x)$ is smooth real valued function on an open set $U \subseteq \mathbf{R}^d$, then asymptotically holds*

$$e^{-it\phi(x)}P(x,D)(e^{it\phi(x)}) \sim t^m P_m(x, \nabla_x\phi), \quad t \to \infty \tag{1.3.3}$$

for every $x \in U$, where ∇_x denotes gradient of the function.

Remark 1.3.1. For the future references, we discuss formula (1.3.3) in more detail. Let $\xi \in \mathbf{R}^d\backslash\{0\}$ be arbitrary but fixed. For $(x,\xi) \in U \times \mathbf{R}^d\backslash\{0\}$ set $\varphi(x,\xi) = x \cdot \xi$. Clearly, $\nabla_x\varphi = \xi$. Since the principal symbol P_m and the chosen function φ are homogeneous of order m and 1, respectively, Theorem 1.3.1 implies

$$\lim_{|\xi|\to\infty} \frac{e^{-ix\cdot\xi}P(x,D)(e^{ix\cdot\xi})}{P_m(x,\xi)} = 1, \tag{1.3.4}$$

for every $x \in U$.

Moreover, if we choose $\chi \in C_0^\infty(U)$, $\operatorname{supp}\chi \subseteq K$, then simple calculation gives

$$\begin{aligned} e^{-ix\cdot\xi}P(x,D)(e^{ix\cdot\xi}\chi(x)) &= e^{-ix\cdot\xi}\sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha(e^{ix\cdot\xi}\chi(x)) \\ &= e^{-ix\cdot\xi}\sum_{|\alpha|\leq m} a_\alpha(x)\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}(e^{ix\cdot\xi})D^\beta\chi(x) \\ &= e^{-ix\cdot\xi}P(x,D)(e^{ix\cdot\xi})\chi(x) \\ &+ \sum_{|\alpha|\leq m} a_\alpha(x)\sum_{\beta\leq\alpha,|\beta|\geq 1}\binom{\alpha}{\beta}\xi^{\alpha-\beta}D^\beta\chi(x), \end{aligned} \tag{1.3.5}$$

for $x \in K$ and $\xi \in \mathbf{R}^d \backslash \{0\}$. Since a_α, $|\alpha| \leq m$ are smooth functions we obtain

$$\Big| \sum_{|\alpha|\leq m} a_\alpha(x) \sum_{\beta\leq\alpha, |\beta|\geq 1} \binom{\alpha}{\beta} \xi^{\alpha-\beta} D^\beta \chi(x) \Big| \leq 2^m \sup_{x\in K} \sup_{\alpha\in\mathbf{N}^d} |D^\alpha \chi(x)||\xi|^{m-1}, \tag{1.3.6}$$

(1.3.2), (1.3.4) and (1.3.5) implies

$$\lim_{|\xi|\to\infty} \frac{e^{-ix\cdot\xi} P(x,D)(e^{ix\cdot\xi}\chi(x))}{P_m(x,\xi)} = \chi(x) \tag{1.3.7}$$

uniformly for $x \in K$. In that sense, if we observe the equation of the form

$$P(x,D)v(x,\xi) = \chi(x)e^{-ix\cdot\xi} \quad x \in K, \xi \in \mathbf{R}^d \backslash \{0\},$$

the approximative solution (for large ξ) is given by $v(x,\xi) = \dfrac{e^{ix\cdot\xi}\chi(x)}{P_m(\xi)}$.

1.4 Localization of distributions and wave front sets

In this section we recall some of the known wave fronts sets and their basic properties. We begin with *standard wave front set*, $WF(u)$. The regularity proposed by the complement of $WF(u)$ is related to $C^\infty(U)$ classes. Recall that Γ is conical if for every $\xi \in \Gamma$, $t\xi \in \Gamma$ for every $t > 0$.

Definition 1.4.1. Let $u \in \mathcal{D}'(U)$ and $(x_0,\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0,\xi_0) \notin WF(u)$ if and only if there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a smooth compactly supported function ϕ equal to 1 on Ω and

$$|\widehat{\phi u}(\xi)| \leq \frac{C_N}{|\xi|^N}, \quad N \in \mathbf{N}, \, \xi \in \Gamma, C_N > 0.$$

It can be shown ([42], [12]) that the Definition 1.4.1 does not depend on choice of *cutoff function* ϕ. In that sense, the WF(u) is equivalently defined in the following way:

$(x_0,\xi_0) \notin WF(u)$ if and only if there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a compactly supported distribution $v \in \mathcal{E}'(U)$ such that $v = u$ on Ω and

$$|\widehat{v}(\xi)| \leq \frac{C_N}{|\xi|)^N}, \quad N \in \mathbf{N}, \, \xi \in \Gamma, C_N > 0.$$

Next we recall the definition of *wave front sets with respect to* C^L (cf. [16]). Let L_p, $p \in \mathbf{N}$ be the increasing sequence of positive numbers satisfying (1.2.3). Then WF_L is defined in the following way

Definition 1.4.2. Let $u \in \mathcal{D}'(U)$ and $(x_0, \xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin WF_L(u)$ if and only if there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and

$$|\widehat{u_N}(\xi)| \leq \frac{C^{N+1} L_N^N}{|\xi|^N}, \quad N \in \mathbf{N}, \, \xi \in \Gamma, C > 0.$$

For the choice $L_p = p$, $p \in \mathbf{N}$, we obtain *analytic wave front set* $\mathrm{WF}_A(u)$ while for $L_p = p^t$, $t > 1$, *Gevrey wave front set* $\mathrm{WF}_t(u)$. The regularity of their complement is related to classes of analytic functions, $\mathcal{E}^{\{p!\}}(U)$ and Gevrey classes $\mathcal{E}_t$, $t > 1$. Therefore, following inclusion is immediate

$$\mathrm{WF}(u) \subseteq \mathrm{WF}_t(u) \subseteq \mathrm{WF}_A(u), \quad u \in \mathcal{D}'(U).$$

Note that in the Definition 1.4.2 we demand the existence of sequence of compactly supported distributions satisfying the desired decay estimates on Fourier side. This is due to the specific choice of cutoff functions. In particular, in [16] author used bounded sequence $\{\chi_N\}_{N\in\mathbf{N}}$ in $C_0^\infty(K)$, such that $\chi_N = 1$ on open set $\Omega \subset K$ for every $N \in \mathbf{N}$ and satisfies estimates of the following form

$$|D^{\alpha+\beta}\chi_N| \leq C_\beta^{|\alpha|+1} L_N^{|\alpha|}, \quad |\alpha| \leq N, N \in \mathbf{N}, \beta \in \mathbf{N}^d,$$

Since we cannot construct compactly supported analytic functions ($\mathcal{D}^{\{p!\}} = \{0\}$), this type of sequences turns out to be best possible choice to analyze microlocal analyticity. In particular, function χ_N, $N \in \mathbf{N}$ satisfies analytic estimates up to order N.

In our approach we will impose additional *admissibility conditions* to the sequences $\{\chi_N\}_{N\in\mathbf{N}}$ by introducing two parameters $\tau > 0$ and $\sigma > 1$, for which the estimates for derivatives are small up to certain order (see Chapter3, Definition 3.2.1).

Let us introduce the notion of *singular support* of distribution.

Definition 1.4.3. Let $u \in \mathcal{D}'(U)$. Then $x_0 \notin \mathrm{singsupp}(u)$ ($x_0 \notin \mathrm{singsupp}_L(u)$) if and only if there exists open neighborhood Ω of x_0 such that $u \in C^\infty(\Omega)$ ($u \in C^L(\Omega)$).

Remark 1.4.1. For $L_p = p!^t$, $t > 1$, the corresponding singular support is denoted by $\mathrm{singsupp}_t(u)$.

Following theorem relates the regularity of classes of smooth functions and regularity proposed by the complement of the wave front sets.

Theorem 1.4.1. *([16]) Let* $\mathrm{WF}(u)$*, and* $\mathrm{WF}_L(u)$*,* $u \in \mathcal{D}'(U)$*, be the standard and* C^L *wave front sets, respectively. Then*

$$\pi_1(\mathrm{WF}(u)) = \mathrm{singsupp}(u),$$

and

$$\pi_1(\mathrm{WF}_L(u)) = \mathrm{singsupp}_L(u),$$

where $\pi_1(x,\xi) = x$ *denotes the standard projection on* $\mathbf{R}^d \times \mathbf{R}^d \backslash \{0\}$.

Moreover, one of the main properties of wave front sets is *microlocal hypoellipticity* stated in the following theorem.

Theorem 1.4.2. *([16]) Let* $\mathrm{WF}(u)$*, and* $\mathrm{WF}_A(u)$*,* $u \in \mathcal{D}'(U)$*, be the standard and analytic wave front sets, and* $P_\infty(x,D)$*,* $P_A(x,D)$ *be the partial differential operators of order* m *with smooth and analytic coefficients on* U*, respectively. Then*

$$\mathrm{WF}(P_\infty u) \subseteq \mathrm{WF}(u) \subseteq \mathrm{WF}(P_\infty u) \cup \mathrm{Char} P_\infty(x,D),$$

and

$$\mathrm{WF}_A(P_A u) \subseteq \mathrm{WF}_A(u) \subseteq \mathrm{WF}_A(P_A u) \cup \mathrm{Char} P_A(x,D),$$

where Char *denotes characteristic set of an operator.*

Since the Gevrey classes are non-quasianalytic, by using compactly supported functions in $\mathcal{D}_t(U)$, $t > 1$, the definition of Gevrey wave front sets can be extended to the spaces of Gevrey ultradistributions. Although, this goes beyond the scope of this thesis, for the completeness we recall the definitions and main results presented in [35].

Recall, $\mathcal{D}'_t(U)$ and $\mathcal{E}'_t(U)$, $t > 1$, denotes the spaces of Gevrey ultradistributions and Gevrey ultradistributions with compact support, respectively.

Definition 1.4.4. Let $u \in \mathcal{D}'_t(U)$ and $(x_0,\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0,\xi_0) \notin WF_t(u)$ if and only if there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and $v \in \mathcal{E}'_t(U)$ such that $v = u$ on Ω and

$$|\widehat{v}(\xi)| \leq \frac{C^{N+1} N^N}{|\xi|^{N/t}}, \quad N \in \mathbf{N},\, \xi \in \Gamma, C > 0.$$

Remark 1.4.2. Note that if we interchange N with tN, $t>1$, in (1.4.4) we obtain

$$|\widehat{u}_N(\xi)| \leq \frac{C^{tN+1}(tN)^{(tN)}}{|\xi|^N} \leq \frac{C'^{N+1}N!^t}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\},\ t>1, \tag{1.4.1}$$

for some $C'>0$, where for the second inequality follows by Stirling's formula. This procedure we will call *enumeration* and it will be the essential tool in the proof of the results in Chapter 3 and Chapter 4.

We finish this section with following Theorem that generalizes properties of Gevrey wave front set in the context of Gevrey ultradistributions. For the proof of microlocal hypoellipticity author uses arguments of the theory of *pseudo-differential operators*, rather than standard, used for WF_A in [16].

Theorem 1.4.3. *([35]) Let* $\mathrm{WF}_t(u)$, $u \in \mathcal{D}_t'(U)$, *be the Gevrey wave front set. Then*

$$\pi_1(\mathrm{WF}_t(u)) = \mathrm{singsupp}_t(u).$$

Moreover, if $P_A(x,D)$ *is partial differential operator of order* m *with analytic coefficients, then*

$$\mathrm{WF}_t(P_A u) \subseteq \mathrm{WF}_t(u) \subseteq \mathrm{WF}_t(P_A u) \cup \mathrm{Char} P_A(x,D). \tag{1.4.2}$$

Chapter 2

Classes of ultradifferentiable functions

In this section we introduce subclasses of smooth function whose definition propose a new type of local regularity. They are different from any of Komatsu's spaces (see Chapter 1, Section 1.2), and therefore we need to adapt standard arguments developed in [20] to our case.

In the first section we establish the basic properties of the sequences $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, $\tau > 0$, $\sigma > 1$, and since they control the derivatives of functions, we are able to prove some standard results in our case. Although it turns out that our sequence does not satisfy Komatsu's condition $(M.2)'$, we are able to prove that our classes are algebras and that they are closed under the finite order derivation. Moreover, they are nuclear.

Our classes contains union of Gevrey classes $\cup_{t>1}\mathcal{E}_t$, and therefore thay are non-quasianalytic. Moreover, we are able to construct a compactly supported function in our classes, which does not belong to $\cup_{t>1}\mathcal{D}_t$. The main result of this Chapter concerns the construction of certain (ultra)differentiable operators, and construction of classes that are classes that are closed under their action. In particular, we construct classes of ultradifferentiable functions.

We also prove the inverse-closedness property. The proof is based on standard argument based on Faá di Bruno formula (see [38], [37]). However, it turns out that in our case we do not need additional properties of sequences $M_p^{\tau,\sigma}$ other than logarithmic convexity. We use this property to construct a function on our classes that does not belong to $\cup_{t>1}\mathcal{E}_t$. In the final section we give the definition of dual spaces, although the singularities of these spaces will be analyzed in the following chapter using the methods of microlocal analysis.

Some results from this Chapter are published in [29].

2.1 Properties of sequences $M_p^{\tau,\sigma}$

The following two lemmas captures the basic properties of $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, $\tau > 0$, $\sigma > 1$, which we will use in the next sections.

Lemma 2.1.1. *Let $\tau > 0$, $\sigma > 1$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{Z}_+$, $M_0^{\tau,\sigma} = 1$. Then there exist $A, B, C > 0$ such that*

$$M_p^{\tau,\sigma} \leq AC^{p^\sigma} \lfloor p^\sigma \rfloor !^{\tau/\sigma} \quad \text{and} \quad \lfloor p^\sigma \rfloor !^{\tau/\sigma} \leq BM_p^{\tau,\sigma}. \tag{2.1.1}$$

Proof. By $p^\sigma \leq \lfloor p^\sigma \rfloor + 1$ and $p^\sigma \leq 2\lfloor p^\sigma \rfloor$, $p \in \mathbf{Z}_+$, we have

$$p^{\tau p^\sigma} \leq p^{\tau(\lfloor p^\sigma \rfloor + 1)} \leq p^\tau \Big(2\lfloor p^\sigma \rfloor\Big)^{\tau \lfloor p^\sigma \rfloor/\sigma} \leq e^{\tau p^\sigma} 2^{\tau\lfloor p^\sigma \rfloor/\sigma} \lfloor p^\sigma \rfloor^{\tau\lfloor p^\sigma \rfloor/\sigma},$$

and the left hand side inequality in (2.1.2) follows from the Stirling formula.

The right hand side inequality in (2.1.2) follows directly from the Stirling formula:

$$\lfloor p^\sigma \rfloor !^{\tau/\sigma} \leq \Big(e^{-\lfloor p^\sigma \rfloor} \sqrt{2\pi \lfloor p^\sigma \rfloor} \lfloor p^\sigma \rfloor^{\lfloor p^\sigma \rfloor}\Big)^{\tau/\sigma} \leq B \lfloor p^\sigma \rfloor^{\tau \lfloor p^\sigma \rfloor/\sigma} \leq Bp^{\tau p^\sigma},$$

for some $B > 0$. □

Remark 2.1.1. Moreover we note that if $\tau > 0$, $\sigma > 1$, $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ and $\widetilde{M_p^{\tau,\sigma}} = \lfloor p^\sigma \rfloor !^{\tau/\sigma}$, $p \in \mathbf{N}$, Then

$$\widetilde{M_p^{\tau,\sigma}} \sim (2\pi)^{\tau/(2\sigma)} p^{\tau/2} e^{-(\tau/\sigma)p^\sigma} M_p^{\tau,\sigma}, \quad p \to \infty, \tag{2.1.2}$$

This is obtain as consequence of Stirling's formula and the fact that $\lfloor p^\sigma \rfloor \sim p^\sigma$ as $p \to \infty$, which follows easily by noting that $p^\sigma = \lfloor p^\sigma \rfloor + \varepsilon_p$, where $0 \leq \varepsilon_p < 1$, $p \in \mathbf{N}$.

Lemma 2.1.2. *Let there be given $\tau > 0$ and $\sigma > 1$. Then there exists a positive increasing sequence $C_q \geq 1$, $q \in \mathbf{N}$, and a constant $C > 1$ so that $M_p^{\tau,\sigma}$ satisfies:*

$(M.1)$ $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}$, $p \in \mathbf{Z}_+$

$\widetilde{(M.2)}'$ $M_{p+q}^{\tau,\sigma} \leq C_q^{p^\sigma} M_p^{\tau,\sigma}$, $p, q \in \mathbf{N}$,

$\widetilde{(M.2)}$ $M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma + q^\sigma} M_p^{\tau 2^{\sigma-1},\sigma} M_q^{\tau 2^{\sigma-1},\sigma}$, $p, q \in \mathbf{N}$.

Proof. For the proof, we put $\tau = 1$ without loss of generality.

Note that for $p = 1$, $(M.1)$ is satisfied. Further note that the second derivative of the function $f(t) = t^\sigma \ln t$, $t > 0$, is positive for $t > e^{\frac{1-2\sigma}{\sigma(\sigma-1)}}$ and we can conclude that sequence $\ln M_p$ is convex for $p-1 \in \mathbf{Z}_+$. This implies $(M.1)$.

Note that if $p=0$ (or $q=0$) conditions $\widetilde{(M.2)}'$ and $\widetilde{(M.2)}$ are trivially satisfied. In the sequel we assume that $p,q \in \mathbf{Z}_+$.

For $\widetilde{(M.2)}'$ write $\sigma = n+\delta$ where $n \in \mathbf{Z}_+$ and $0<\delta\leq 1$ with $n = \lfloor \sigma \rfloor$, $0<\delta<1$, when $\sigma \notin \mathbf{Z}_+$, and $n=\sigma-1$, $\delta=1$, when $\sigma \in \mathbf{Z}_+$. Then by binomial formula we have,

$$
\begin{aligned}
(p+q)^\sigma &\leq (p+q)^n(p^\delta+q^\delta) = p^\sigma + \sum_{k=1}^{n}\binom{n}{k}p^{\sigma-k}q^k \\
&+ \sum_{k=0}^{n}\binom{n}{k}p^{n-k}q^{k+\delta} \leq p^\sigma + 2^n(p^{\sigma-1}q^n + p^n q^\sigma) \\
&\leq p^\sigma + 2^{n+1}q^\sigma p^{\sigma-\delta}\,,
\end{aligned}
$$

where for the last inequality we have used that $n=\sigma-\delta$ and $0<\delta\leq 1$. In particular,

$$
(p+q)^\sigma \ln(p+q) \leq p^\sigma \ln(p+q) + 2^{n+1}q^\sigma p^{\sigma-\delta}\ln(p+q)\,. \tag{2.1.3}
$$

We estimate the first summand on the righthand side of (2.1.3) as

$$
\begin{aligned}
p^\sigma \ln(p+q) &= p^\sigma\Big(\ln p + \ln\Big(1+\frac{q}{p}\Big)\Big) \leq p^\sigma \ln p + p^{\sigma-1}q \\
&\leq p^\sigma \ln p + qp^\sigma\,.
\end{aligned} \tag{2.1.4}
$$

For the second summand of (2.1.3) we note that since $\ln p \leq Ap^\delta$, $0<\delta\leq 1$, we have $p \leq C^{p^\delta}$, for some $C>1$. Thus we obtain,

$$
\begin{aligned}
2^{n+1}q^\sigma p^{\sigma-\delta}\ln(p+q) &= 2^{n+1}q^\sigma p^{\sigma-\delta}\Big(\ln p + \ln\Big(1+\frac{q}{p}\Big)\Big) \\
&\leq 2^{n+1}q^\sigma p^\sigma \ln C + 2^{n+1}q^\sigma p^\sigma \ln(1+q)\,.
\end{aligned} \tag{2.1.5}
$$

Applying the estimates (2.1.4) and (2.1.5) in (2.1.3) and taking the exponentials, $\widetilde{(M.2)}'$ follows.

For the proof of $\widetilde{(M.2)}$ we use well know inequality $(p+q)^\sigma \leq 2^{\sigma-1}(p^\sigma+q^\sigma)$ to conclude

$$
(p+q)^{(p+q)^\sigma} \leq (p+q)^{2^{\sigma-1}p^\sigma}(p+q)^{2^{\sigma-1}q^\sigma}. \tag{2.1.6}
$$

First term on the righthand side of (2.1.6) we estimate as follows

$$\begin{aligned} 2^{\sigma-1}p^{\sigma}\ln(p+q) &= 2^{\sigma-1}p^{\sigma}\Big(\ln p+\ln\Big(1+\frac{q}{p}\Big)\Big) \leq 2^{\sigma-1}p^{\sigma}\ln p \\ &+ 2^{\sigma-1}qp^{\sigma-1} \leq 2^{\sigma-1}p^{\sigma}\ln p + 2^{\sigma-1}(p+q)^{\sigma}\,. \end{aligned}$$

Taking the exponentials we obtain $(p+q)^{2^{\sigma-1}p^{\sigma}} \leq p^{2^{\sigma-1}p^{\sigma}} e^{2^{\sigma-1}(p+q)^{\sigma}}$. Second term in (2.1.6) we estimate in the similar way. □

Remark 2.1.2. Note that for fixed τ and σ condition $(M.1)$ is standard Komatsu's conditions (and hence the notation). From the proof of $\widetilde{(M.2)}'$ it is clear that $M_p^{\tau,\sigma}$ does not satisfies condition $(M.2)'$. Natural extension of $(M.2)$ would be

$$M_{p+q}^{\tau,\sigma} \leq C^{p^{\sigma}+q^{\sigma}} M_p^{\tau,\sigma} M_q^{\tau,\sigma}, \quad C>0, p,q \in \mathbf{N}. \tag{2.1.7}$$

Our sequence fails to satisfy (2.1.7). To see that, suppose the opposite. Without loss of generality we may assume that $\tau = 1$. Then, if we put $p = q \neq 0$ in (2.1.7) we obtain

$$p^{(2p)^{\sigma}} \leq (C_1 p)^{2p^{\sigma}}, \quad p \in \mathbf{Z}_+\,, \tag{2.1.8}$$

where $C_1 = \frac{C}{2^{2^{\sigma-1}}}$. Taking the logarithm, (2.1.8) implies $2^{\sigma-1}\ln p \leq \ln C_1 p$ which is satisfied only for finitely many $p \in \mathbf{Z}_+$.

Remark 2.1.3. We would like to point out that $\widetilde{(M.2)}$ does not imply $\widetilde{(M.2)}'$, for fixed $\tau > 0$ and $\sigma > 1$. To see that assume that $M_p^{\tau,\sigma}$ satisfies $\widetilde{(M.2)}$. Then we write $2^{\sigma-1} = 1+\varepsilon$, for some $\varepsilon > 0$ and note that

$$M_{p+q}^{\tau,\sigma} \leq C^{p^{\sigma}+q^{\sigma}} M_p^{2^{\sigma-1}\tau,\sigma} M_q^{2^{\sigma-1}\tau,\sigma} = C^{p^{\sigma}+q^{\sigma}} M_q^{2^{\sigma-1}\tau,\sigma} M_p^{\varepsilon\tau,\sigma} M_p^{\tau,\sigma}$$

for every $p, q \in \mathbf{Z}_+$ and some $C > 1$. Thus, in order to $\widetilde{(M.2)}'$ be fulfilled, we note that for fixed $q \in \mathbf{Z}_+$ and some constants $C', C'' > 0$ it must hold

$$C^{p^{\sigma}+q^{\sigma}} M_q^{2^{\sigma-1}\tau,\sigma} M_p^{\varepsilon\tau,\sigma} \leq C'^{p^{\sigma}},$$

and hence it follows that $M_p^{\varepsilon\tau,\sigma} \leq C''^{p^{\sigma}}$ which is not true for p sufficiently large.

However, we adopt the Komatsu's notation since the corresponding properties of the sequence $M_p^{\tau,\sigma}$ would imply the desired properties of the spaces from Chapter 2, similarly as in [20].

We will also use following technical lemma in our computations.

Lemma 2.1.3. *For $\tau > 0$ and $\sigma > 1$ it holds*

$$\tilde{T}_{\tau,\sigma}(h) := \sup_{r>0} \frac{h^{r^\sigma}}{r^{\tau r^\sigma}} = e^{\frac{\tau}{\sigma e} h^{\frac{\sigma}{\tau}}} \quad , h > 0. \tag{2.1.9}$$

Proof. Set $f(r) = \frac{h^{r^\sigma}}{r^{\tau r^\sigma}}$, $r > 0$. Clearly, f reaches it's supremum and, by the methods explained in [13], it is sufficient to find the $\max_{r>0} \ln f(r)$. By the standard differential calculus, note that $(\ln f(r))' = r^{\sigma-1}(\sigma \ln h - \tau\sigma \ln r - \tau)$, $r > 0$, and hence the only stationary point of $\ln f(r)$ is $r_0 := h^{\frac{1}{\tau}} e^{-\sigma}$. Thus, $\max_{r>0} \ln f(r) = f(r_0) = h^{\frac{\sigma}{\tau}} \frac{\tau}{e\sigma}$. This implies that $\max_{r>0} f(r) = e^{\frac{\tau}{\sigma e} h^{\frac{\sigma}{\tau}}}$. □

Remark 2.1.4. In the theory of ultradifferentiable functions, for a given sequence M_p, $p \in \mathbf{N}$, the function T given by $T(h) = \sup_{p>0} \frac{h^p M_0}{M_p}$, $h > 0$, is called *the associated function* of the sequence M_p, $p \in \mathbf{N}$ (in [20] the function $\sup_{p>0} \ln \frac{h^p M_0}{M_p}$ is considered instead of $T(h)$). It plays an important role in the study of the spaces of ultradifferentiable functions and their dual spaces. Notice that $\tilde{T}_{\tau,\sigma}$ given by (2.1.9) is not the associated function of the sequence $M_p^{\tau,\sigma}$. It is known that the associated function $T_\tau(h)$ of the sequence $p!^\tau$, $\tau > 0$, satisfies the estimate of the form $C_1 e^{\frac{\tau}{e} h^{\frac{1}{\tau}}} \leq T_\tau(h) \leq C_2 e^{\frac{\tau}{e} h^{\frac{1}{\tau}}}$, for some $C_1, C_2 > 0$, and for every $h > 0$, cf. [13, Chapter IV.2]. This implies that

$$C'(T_\tau(h^\sigma))^{1/\sigma} \leq \tilde{T}_{\tau,\sigma}(h) \leq C''(T_\tau(h^\sigma))^{1/\sigma}$$

for some $C', C'' > 0$, $h > 0$ and for any given $\tau > 0$, $\sigma > 1$.

2.2 Test spaces

Let us introduce our basic spaces. Fix $K \subset\subset U$ and $h > 0$. Smooth function ϕ on U belongs to Banach space $\mathcal{E}_{\tau,\sigma,h}(K)$ if there exists $A > 0$ such that

$$|\partial^\alpha \phi(x)| \leq A h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}, \quad \alpha \in \mathbf{N}^d. \tag{2.2.1}$$

Norm is given by

$$||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha \in \mathbf{N}^d} \sup_{x \in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}}\,, \tag{2.2.2}$$

and we immediately obtain following embeddings

$$\mathcal{E}_{\tau_1,\sigma_1,h_1}(K) \hookrightarrow \mathcal{E}_{\tau_2,\sigma_2,h_2}(K)\,, \quad 0 < h_1 \leq h_2,\ 0 < \tau_1 \leq \tau_2,\ 1 < \sigma_1 \leq \sigma_2\,. \tag{2.2.3}$$

Remark 2.2.1. When $\sigma = 1$, we have $\mathcal{E}_{\tau,1,h}(K) = \mathcal{E}^{\{p!^\tau\},h}(K)$. For more information about classes $\mathcal{E}^{\{p!^\tau\},h}(K)$ see [20].

Let $\mathcal{D}^K_{\tau,\sigma,h}$ be the set of functions in $\mathcal{E}_{\tau,\sigma,h}(K)$ with the supports in K. Then, in the topological sense, we set

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}_{\tau,\sigma,h}(K), \tag{2.2.4}$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h\to 0} \mathcal{E}_{\tau,\sigma,h}(K), \tag{2.2.5}$$

$$\mathcal{D}^K_{\{\tau,\sigma\}} = \varinjlim_{h\to\infty} \mathcal{D}^K_{\tau,\sigma,h}\,, \tag{2.2.6}$$

$$\mathcal{D}^K_{(\tau,\sigma)} = \varprojlim_{h\to 0} \mathcal{D}^K_{\tau,\sigma,h} \tag{2.2.7}$$

$$\mathcal{D}_{\{\tau,\sigma\}}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}^K_{\{\tau,\sigma\}}, \tag{2.2.8}$$

$$\mathcal{D}_{(\tau,\sigma)}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}^K_{(\tau,\sigma)}. \tag{2.2.9}$$

These spaces are l.c.s., and their topology will be discussed in the next section. We will use the notation $\mathcal{E}_{\tau,\sigma}(U)$ for $\mathcal{E}_{\{\tau,\sigma\}}(U)$ or $\mathcal{E}_{(\tau,\sigma)}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$ for $\mathcal{D}_{\{\tau,\sigma\}}(U)$ or $\mathcal{D}_{(\tau,\sigma)}(U)$ (resp. $\mathcal{D}^K_{\tau,\sigma}(U)$ for $\mathcal{D}^K_{\{\tau,\sigma\}}(U)$ or $\mathcal{D}^K_{(\tau,\sigma)}(U)$).

Remark 2.2.2. From Lemma 2.1.1 it follows that the norms (2.2.2) and

$$\|\phi\|_{\widetilde{\mathcal{E}}_{\tau,\sigma,h}(K)} = \sup_{\alpha\in\mathbf{N}^d} \sup_{x\in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} \lfloor |\alpha|^\sigma \rfloor !^{\tau/\sigma}} < \infty, \quad h > 0. \tag{2.2.10}$$

also can be used in the definition of $\mathcal{E}_{\tau,\sigma}(K)$. Moreover, for $\sigma = 1$ classes given with (2.2.4) (resp. (2.2.8)) and (2.2.5) (resp. (2.2.9)) coincides with Komatsu's classes $\mathcal{E}^{\{p!^\tau\}}(U)$ ($\mathcal{D}^{\{p!^\tau\}}(U)$) and $\mathcal{E}^{(p!^\tau)}(U)$ ($\mathcal{D}^{(p!^\tau)}(U)$), $\tau > 0$.

Remark 2.2.3. Let us show that in the definition of the classes $\mathcal{E}_{\tau,\sigma}(U)$

$$\|\partial^\alpha \phi\|_{\infty,K} := \sup_{x\in K} |\partial^\alpha \phi(x)|$$

that appears in (2.2.2) can be replaced by

$$\|\partial^\alpha \phi\|_{L^p(K)} := \int_K |\partial^\alpha \phi(x)|\, dx, \quad p \geq 1.$$

Let $K \subset\subset U$ be arbitrary but fixed. Note that for every $h > 0$

$$\sup_{\alpha\in\mathbf{N}^d}\frac{\|\partial^\alpha\phi\|_{L^p(K)}}{h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\le C_1||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}, \tag{2.2.11}$$

for some $C_1>0$ which depends only on K. This follows directly from the simple inequality $\|\partial^\alpha\phi\|_{L^p(K)}\le |K|^{1/p}\|\partial^\alpha\phi\|_{\infty,K}$, where $|K|:=\int_K dx$.

For the converse inequality we prove that for every $h>0$ there exists positive constants c_h and C_2 such that

$$||\phi||_{\mathcal{E}_{\tau,\sigma,c_h h}(K)}\le C_2\sup_{\alpha\in\mathbf{N}^d}\frac{\|\partial^\alpha\phi\|_{L^p(K)}}{h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}, \tag{2.2.12}$$

We use the Sobolev embedding theorem (see [42]). For any $\alpha\in\mathbf{N}^d$ choose $\beta\in\mathbf{N}^d$ such that $|\beta|=|\alpha|+d/p+1$ for $p>1$ or $|\beta|=|\alpha|+d$ for $p=1$. Then the Sobolev theorem implies that

$$\|\partial^\alpha\phi\|_{\infty,K}\le C\|\partial^\beta\phi\|_{L^p(K)},\quad C>0, p\ge 1.$$

Moreover, for $\alpha\in\mathbf{N}^d$

$$\begin{aligned}\frac{\|\partial^\alpha\phi(x)\|_{\infty,K}}{(c_h h)^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}} &\le \frac{\|\partial^\beta\phi(x)\|_{L^p(K)}}{h^{|\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}}\frac{h^{|\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}}{(c_h h)^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\\ &\le C\frac{\|\partial^\beta\phi(x)\|_{L^p(K)}}{h^{|\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}}\frac{h^{|\beta|^\sigma}C'^{|\alpha|^\sigma}}{(c_h h)^{|\alpha|^\sigma}}, \end{aligned}\tag{2.2.13}$$

where for the second inequality we have use the $(\widetilde{M.2})'$ property of the sequence $M_p^{\tau,\sigma}$. Now if we choose $c_h=\max\{C',C'h^{2^{\sigma-1}}\}$, noting that and take supremum with respect to $\alpha\in\mathbf{N}^d$, noting that β depends on α, (2.2.13) implies (2.2.12) by observing simple inequalities

$$h^{|\beta|^\sigma}\le C_h h^{2^{\sigma-1}|\alpha|^\sigma}, h\ge 1,\quad h^{|\beta|^\sigma}\le C'_h h^{|\alpha|^\sigma}, 0<h<1,$$

for suitable constants $C_h, C'_h>0$.

Remark 2.2.4. By standard arguments (see [20]), spaces in (2.2.4) and (2.2.5) can be represented as inductive (resp. projective) limits of countable family of spaces as follows: Let $\{K_i\}_{i\in\mathbf{N}}$ be the sequence of compact sets with smooth boundary such that $K_i\subset K_{i+1}$, $i\in\mathbf{N}$ and $\cup_{i\in\mathbf{N}}K_i=U$. Then, for $j\in\mathbf{N}$,

$$\mathcal{E}_{\{\tau,\sigma\}}(U)=\varprojlim_{i\to\infty}\varinjlim_{j\to\infty}\mathcal{E}_{\tau,\sigma,j}(K_i),\quad \mathcal{D}_{\{\tau,\sigma\}}(U)=\varinjlim_{i\to\infty}\varinjlim_{j\to\infty}\mathcal{D}^{K_i}_{\tau,\sigma,j},$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{i\to\infty} \varprojlim_{j\to\infty} \mathcal{E}_{\tau,\sigma,1/j}(K_i), \quad \mathcal{D}_{(\tau,\sigma)}(U) = \varinjlim_{i\to\infty} \varprojlim_{j\to\infty} \mathcal{D}^{K_i}_{\tau,\sigma,1/j}.$$

Following proposition describes the basic relations between our classes.

Proposition 2.2.1. *Let $\sigma_1 \geq 1$. Then for every $\sigma_2 > \sigma_1$ and $\tau > 0$*

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \varprojlim_{\tau\to 0^+} \mathcal{E}_{\tau,\sigma_2}(U). \tag{2.2.14}$$

Moreover, if $0 < \tau_1 < \tau_2$, then for every $\sigma > 1$ it holds

$$\mathcal{E}_{\{\tau_1,\sigma\}}(U) \hookrightarrow \mathcal{E}_{(\tau_2,\sigma)}(U) \hookrightarrow \mathcal{E}_{\{\tau_2,\sigma\}}(U). \tag{2.2.15}$$

In particular, for $\sigma > 1$

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varinjlim_{\tau\to\infty} \mathcal{E}_{(\tau,\sigma)}(U). \tag{2.2.16}$$

$$\varprojlim_{\tau\to 0^+} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{\tau\to 0^+} \mathcal{E}_{(\tau,\sigma)}(U). \tag{2.2.17}$$

Proof. Since in (2.2.16) and (2.2.17) we observe unions and intersections of $\mathcal{E}_{\{\tau,\sigma\}}(U)$ (resp. $\mathcal{E}_{(\tau,\sigma)}(U)$) with respect to τ, the equalities follow immediately from (2.2.15).

To prove (2.2.14), take arbitrary $h > 0$, $\tau > 0$ and $\varepsilon > 0$ and set $\sigma_2 = \sigma_1 + \varepsilon$. Let $\phi \in \mathcal{E}_{\tau_0,\sigma_1,h}(K)$ for some $\tau_0 > 0$ and $K \subset\subset U$.

We begin by noting that

$$||\phi||_{\mathcal{E}_{\tau,\sigma_2,h}(K)} \leq \sup_{\alpha\in\mathbf{N}^d} \frac{h^{|\alpha|^{\sigma_1}}|\alpha|^{\tau_0|\alpha|^{\sigma_1}}}{h^{|\alpha|^{\sigma_2}}|\alpha|^{\tau|\alpha|^{\sigma_2}}} ||\phi||_{\mathcal{E}_{\tau_0,\sigma_1,h}(K)}. \tag{2.2.18}$$

Further observe that there exists $A_\varepsilon > 0$ such that $\tau_0 p^{\sigma_1} \ln p \leq A_\varepsilon \tau_0 p^{\sigma_1+\varepsilon} = A_\varepsilon \tau_0 p^{\sigma_2}$, and thus $p^{\tau_0 p^{\sigma_1}} \leq e^{A_\varepsilon \tau_0 p^{\sigma_2}}$ (note that A_ε blows up as $\varepsilon \to 0^+$). Now, for $C := e^{A_\varepsilon \tau_0}$ and $c_h := \max\{1/h, 1\}$, we obtain

$$\sup_{\alpha\in\mathbf{N}^d} \frac{h^{|\alpha|^{\sigma_1}}|\alpha|^{\tau_0|\alpha|^{\sigma_1}}}{h^{|\alpha|^{\sigma_2}}|\alpha|^{\tau|\alpha|^{\sigma_2}}} \leq \sup_{\alpha\in\mathbf{N}^d} \frac{(C\,c_h)^{|\alpha|^{\sigma_2}}}{|\alpha|^{\tau|\alpha|^{\sigma_2}}} \leq \tilde{T}_{\tau,\sigma_2}(Cc_h) = e^{\frac{\tau}{e\sigma_2}(C\,c_h)^{\frac{\sigma_2}{\tau}}}, \tag{2.2.19}$$

where $\tilde{T}_{\tau,\sigma}$ is function from Lemma 2.1.3. Now (2.2.14) follows from (2.2.18) and (2.2.19).

Note that second embedding in (2.2.15) is trivial. For the proof of the first embedding, let $\phi \in \mathcal{E}_{\tau_1,\sigma,k}(K)$ for some $k > 0$. We note that

$$||\phi||_{\mathcal{E}_{\tau_2,\sigma,h}(K)} \leq \sup_{\alpha\in\mathbf{N}^d} \frac{k^{|\alpha|^{\sigma}}|\alpha|^{\tau_1|\alpha|^{\sigma}}}{h^{|\alpha|^{\sigma}}|\alpha|^{\tau_2|\alpha|^{\sigma}}} ||\phi||_{\mathcal{E}_{\tau_1,\sigma,k}(K)}, \quad h, k > 0\,.$$

Moreover,

$$\sup_{\alpha\in\mathbf{N}^d}\frac{k^{|\alpha|^\sigma}|\alpha|^{\tau_1|\alpha|^\sigma}}{h^{|\alpha|^\sigma}|\alpha|^{\tau_2|\alpha|^\sigma}}\le\tilde{T}_{\tau_2-\tau_1,\sigma}(k/h)=e^{\frac{\tau_2-\tau_1}{e\sigma}(k/h)^{\frac{\sigma}{\tau_2-\tau_1}}},$$

wherefrom for any given $h>0$ we obtain

$$||\phi||_{\mathcal{E}_{\tau_2,\sigma,h}(K)}\le C||\phi||_{\mathcal{E}_{\tau_1,\sigma,k}(K)},$$

for some $C>0$. This proves the proposition. □

In the topological sense, let us introduce the following notation

$$\mathcal{E}_{0^+,\sigma}(U):=\varprojlim_{\tau\to 0^+}\mathcal{E}_{\tau,\sigma}(U), \tag{2.2.20}$$

$$\mathcal{E}_{\infty,\sigma}(U):=\varinjlim_{\tau\to\infty}\mathcal{E}_{\tau,\sigma}(U), \tag{2.2.21}$$

$$\mathcal{E}_{\tau,1^+}(U):=\varprojlim_{\sigma\to 1^+}\mathcal{E}_{\tau,\sigma}(U), \tag{2.2.22}$$

$$\mathcal{E}_{\tau,\infty}(U):=\varinjlim_{\sigma\to\infty}\mathcal{E}_{\tau,\sigma}(U), \tag{2.2.23}$$

$$\mathcal{E}_{0^+,1^+}(U):=\varprojlim_{\sigma\to 1^+}\mathcal{E}_{0^+,\sigma}(U), \tag{2.2.24}$$

$$\mathcal{E}_{\infty^+,1^+}(U):=\varprojlim_{\sigma\to 1^+}\mathcal{E}_{\infty,\sigma}(U), \tag{2.2.25}$$

$$\mathcal{E}_{0^+,\infty}(U):=\varinjlim_{\sigma\to\infty}\mathcal{E}_{0^+,\sigma}(U), \tag{2.2.26}$$

$$\mathcal{E}_{\infty,\infty}(U):=\varinjlim_{\sigma\to\infty}\mathcal{E}_{\infty,\sigma}(U), \tag{2.2.27}$$

As immediate consequence of Proposition 2.2.1 we obtain the following corollary.

Corollary 2.2.1. *We have the following strict embeddings*

$$\varinjlim_{t\to\infty}\mathcal{E}_t(U)\hookrightarrow\mathcal{E}_{0^+,1^+}(U)\hookrightarrow\mathcal{E}_{\infty,1^+}(U)\hookrightarrow\mathcal{E}_{0^+,\infty}(U)\hookrightarrow\mathcal{E}_{\infty,\infty}(U). \tag{2.2.28}$$

where $\mathcal{E}_t(U)$, $t>1$, denotes the Gevrey class of Roumieu or Beurling type.

Proof. The embedding $\varinjlim_{t\to\infty} \mathcal{E}_t(U) \hookrightarrow \mathcal{E}_{0^+,1^+}(U)$ in (2.2.28) follows directly from Proposition 2.2.1 when $\sigma_2 > \sigma_1 = 1$. Note that embedding $\mathcal{E}_{0^+,1^+}(U) \hookrightarrow \mathcal{E}_{\infty,1^+}(U)$ is trivial. Further note that $u \in \mathcal{E}_{\infty,1^+}(U)$ implies that for every $\sigma > 1$ and some $\tau_0 > 0$ $u \in \mathcal{E}_{\tau_0,\sigma}(U)$. Fix $\sigma_1 > 1$, and let $\sigma_2 > \sigma_1 > 1$. Then following embeddings holds

$$\mathcal{E}_{\tau_0,\sigma_1}(U) \hookrightarrow \mathcal{E}_{0^+,\sigma_2}(U) \hookrightarrow \mathcal{E}_{0^+,\infty}(U),$$

where first embedding follows from (2.2.14). Since the embedding $\mathcal{E}_{0^+,\infty}(U) \hookrightarrow \mathcal{E}_{\infty,\infty}(U)$ is trivial, the assertion follows. □

Note that classes $\mathcal{E}_{\tau,\sigma}$, for every $\tau > 0$, $\sigma > 1$ are larger then Gevrey classes but their inductive limits with respect to τ and σ are continuously injected in $C^\infty(U)$. In particular, they are non-quasianalytic. The precise result is given in the next section.

2.3 Compactly supported test functions

In this section we construct a compactly supported function in $\mathcal{D}_{\{\tau,\sigma\}}(U)$, U is an open set in $\mathbf{R}^d$, following the ideas presented in [16], Sections 1.3 and 1.4. Clearly, compactly supported Gevrey functions are in $\mathcal{D}_{\{\tau,\sigma\}}(U)$. The purpose of this section is to construct a compactly supported function in $\mathcal{D}_{\{\tau,\sigma\}}(U)$ which is not in $\mathcal{D}_t(U)$, for any $t > 1$.

We note that it is enough to show the statement for some neighborhood of origin. See Theorems 1.3.5. and 1.4.2. in [16] for details.

Let us start with the following lemma.

Lemma 2.3.1. *Let $\tau > 0$, $\sigma > 1$. The sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$ satisfies Komatsu's condition $(M.3)'$. In particular,*

$$\sum_{p=1}^{\infty} \frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} < \infty. \tag{2.3.1}$$

Proof. From $2 \le (1+\frac{1}{p})^p \le e$ and $p^\sigma \ln(1+\frac{1}{p}) = p^{\sigma-1}\ln(1+\frac{1}{p})^p$, it follows that

$$\tau \ln 2\, p^{\sigma-1} \le \tau p^\sigma \ln\left(1+\frac{1}{p}\right) \le \tau p^{\sigma-1}, \quad p \in \mathbf{Z}_+. \tag{2.3.2}$$

This implies

$$2^{\tau p^{\sigma-1}} \le \left(1+\frac{1}{p}\right)^{\tau p^\sigma} \le e^{\tau p^{\sigma-1}}, \quad p \in \mathbf{Z}_+. \tag{2.3.3}$$

Since $p^\sigma \geq (p-1)^{\sigma-1}p = (p-1)^\sigma + (p-1)^{\sigma-1}$, $p \in \mathbf{Z}_+$, the left hand side of (2.3.3) we gives,

$$\begin{aligned}
\sum_{p=1}^{\infty} \frac{(p-1)^{\tau(p-1)^\sigma}}{p^{\tau p^\sigma}} &\leq \sum_{p=1}^{\infty} \frac{(p-1)^{\tau(p-1)^\sigma}}{p^{\tau((p-1)^\sigma+(p-1)^{\sigma-1})}} \\
&= \sum_{p=1}^{\infty} \left((1-\frac{1}{p})^{\tau(p-1)^\sigma}\right) \frac{1}{p^{\tau(p-1)^{\sigma-1}}} \qquad (2.3.4) \\
&\leq \sum_{p=1}^{\infty} \frac{1}{(2p)^{\tau(p-1)^{\sigma-1}}} < \infty,
\end{aligned}$$

which proves the lemma.

□

We immediately obtain the following corollary.

Corollary 2.3.1. *Let there be given $\tau > 0$ and $\sigma > 1$. Then there exists a compactly supported function $\phi \in \mathcal{E}_{\{\tau,\sigma\}}(U)$ such that $0 \leq \phi \leq 1$ and $\int_{\mathbf{R}^d} \phi\, dx = 1$.*

Proof. By the Denjoy-Carleman theorem (see Theorem 1.3.8. in [16]), conditions $(M.1)$ and $(M.3)'$ implies the existence of compactly supported function in Komatsu's space $\mathcal{E}^{\{p^{\tau p^\sigma}\}}(U)$, for the appropriate set U. Since the constant h in (2.2.1) is taken to the power $|\alpha|^\sigma$, $\alpha \in \mathbf{N}^d$, classes $\mathcal{E}_{\{\tau,\sigma\}}(U)$ are larger then $\mathcal{E}^{\{p^{\tau p^\sigma}\}}(U)$, and therefore classes $\mathcal{D}_{\{\tau,\sigma\}}(U)$ are nontrivial. However, our goal is to construct a function in $\mathcal{D}_{\{\tau,\sigma\}}(U)$ which is not in $\mathcal{D}_t(U)$, for any $t > 1$. We apply methods explained in Theorem 1.3.5 in [16], and to make the conclusion clear, we repeat some of the steps for our case.

We start with one dimensional case. Let χ be the characteristic function of interval $(0,1)$, and for $c > 0$ let $H_c(x) = \frac{1}{c}\chi(\frac{x}{c})$. Clearly $\int_{\mathbf{R}} H_c dx = 1$ and we recall that

$$(H_c * f)'(x) = \frac{1}{c}\Big(\int_{x-c}^{x} f(t)dt\Big)' = \frac{f(x)-f(x-c)}{c}, \qquad (2.3.5)$$

for any continuous function f on $\mathbf{R}$.

Further we set

$$a_p := \frac{1}{(2(p+1))^{\tau p^{\sigma-1}}}, \quad p \in \mathbf{N},$$

and note that (2.3.4) imply

$$\frac{M_p^{\tau,\sigma}}{M_{p+1}^{\tau,\sigma}} \le 2^{-\tau p^{\sigma-1}} \frac{1}{(p+1)^{\tau p^{\sigma-1}}} = a_p, \quad p \in \mathbf{N}\,. \tag{2.3.6}$$

Put

$$u_m(x) = H_{a_0} * H_{a_1} \cdots * H_{a_m}, \quad m \in \mathbf{N}.$$

Then, by [16, Theorem 1.3.5] it follows that the sequence $\{u_m\}_{m\in\mathbf{N}}$ has a uniform limit $u \in C^\infty(\mathbf{R})$ supported in $[0,a]$ where $a = \sum\limits_{p=0}^{\infty} a_p < \infty$, and $\int_{\mathbf{R}} u\,dx = 1$.

Next we estimate the derivatives $u_m^{(p)}$, $p \le m-1$. After applying p iterations of (2.3.5) and by using (2.3.6) we obtain

$$\begin{aligned}
|u_m^{(p)}(x)| &= \prod_{k=0}^{p-1} \frac{|H_{a_p} * \cdots * H_{a_m}(x) - H_{a_p} * \cdots * H_{a_m}(x-a_k)|}{a_k} \\
&\le 2^p \prod_{k=0}^{p-1} \frac{1}{a_k} \sup_{x\in\mathbf{R}} |H_{a_p} * H_{a_{p+1}} \cdots * H_{a_m}(x)| \\
&\le 2^p \Big(\prod_{k=0}^{p-1} \frac{1}{a_k}\Big) \sup_{x\in\mathbf{R}} |H_{a_p}(x)| \Big(\prod_{k=p+1}^{m} \int_{\mathbf{R}} H_{a_k}(x)dx\Big) \\
&= 2^p \prod_{k=0}^{p} \frac{1}{a_k} \le 2^p \prod_{k=0}^{p} \frac{M_{k+1}^{\tau,\sigma}}{M_k^{\tau,\sigma}} = 2^p \frac{M_{p+1}^{\tau,\sigma}}{M_0^{\tau,\sigma}} \\
&= 2^p (p+1)^{\tau(p+1)^\sigma} \le C^{p^\sigma} p^{\tau p^\sigma},
\end{aligned} \tag{2.3.7}$$

where we used $\widetilde{(M.2)}'$ for the last inequality.

From the uniform convergence it follows that the derivatives of u also satisfy (2.3.7), so that $u \in \mathcal{D}_{\{\tau,\sigma\}}^{[0,a]}$.

Next, we extend this to higher dimensions by putting $\psi(x) = u(x+a/2)$ and $\phi(x) = \prod_{k=1}^{d} \psi(x_k)$ for $x = (x_1, x_2, \dots, x_d)$. Since the sequence $M_p^{\tau,\sigma}$ fulfills the $(M.1)$ property, we obtain

$$|\partial^\alpha \phi(x)| = \prod_{k=1}^{d} |\partial^{\alpha_k} \psi(x_k)| \le \prod_{k=1}^{d} C^{\alpha_k{}^\sigma} \alpha_k{}^{\tau \alpha_k{}^\sigma} \le C^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma},$$
$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d),$$

wherefrom $\phi \in \mathcal{D}_{\{\tau,\sigma\}}^{K}$ with $K = [-a/2, a/2]^d$.

Although K does not have a smooth boundary, by [16, Lemma 1.4.3] one can find appropriate open set U and conclude that $\phi \in \mathcal{D}_{\{\tau,\sigma\}}(U)$.

At the same time $\phi \notin \mathcal{D}_t(U)$, for any $t > 1$. In order to use the same construction and obtain a function in $\bigcup_{t>1} \mathcal{D}_t(U)$ one should choose sequences $\{\widetilde{a}_p\}_{p\in\mathbf{N}}$ of the form $\widetilde{a}_p := \dfrac{p!^t}{(p+1)!^t} = \dfrac{1}{(p+1)^t}$, for some $t > 1$, $p \in \mathbf{N}$ (see Theorem 1.3.5. in [16]).

Then the corresponding function $\widetilde{u}$ belongs to $\bigcup_{t>1} \mathcal{D}^{\{p!^t\}}_{[0,\widetilde{a}]}$, $u(x) \neq 0$, $x \in (0, \widetilde{a})$, where $\widetilde{a} = \sum_{k=0}^{\infty} \widetilde{a}_k$. The key observation is that we would use $(M.2)'$ property of the sequence $p!^t$, $t > 1$, instead of $\widetilde{(M.2)}'$ in (2.3.7).

Let us explain that $a < \widetilde{a}$, that is for every $\sigma, t > 1$ and $\tau > 0$ there exists $p_0 \in \mathbf{N}$ such that $\widetilde{a}_p > a_p$ for every $p > p_0$. In fact, we need to show simple inequality

$$(2(p+1))^{\tau p^{\sigma-1}} > (p+1)^t, \quad p > p_0 \,. \tag{2.3.8}$$

Taking the logarithm we note that

$$\tau p^{\sigma-1}(\ln 2 + \ln(p+1)) > \tau p^{\sigma-1} \ln(p+1) > t \ln(p+1),$$

for $p > \lceil (t/\tau)^{1/(\sigma-1)} \rceil := p_0$, and (2.3.8) follows. If we set $\tau = t > 1$, then inequality (2.3.8) holds for any $p \in \mathbf{N}$ and hence we conclude that $a < \widetilde{a}$. In particular, $u \neq \widetilde{u}$ for two reasons: u vanishes outside $[0, a]$ while $\widetilde{u} \neq 0$ on $(a, \widetilde{a})$, and the estimates on derivatives of u and $\widetilde{u}$ are different (see the calculation in (2.3.7)).

We refer to [16, Lemma 1.3.6] for a discussion about the precision of the presented construction. □

Remark 2.3.1. Since the construction presented in Lemma 2.3.1 and Corollary 2.3.1 holds for every $\tau > 0$ and $\sigma > 1$, by (2.2.15), it follows that ϕ from Corollary 2.3.1 also belongs to $\mathcal{D}_{(\tau,\sigma)}(U)$ for some choice of $\tau > 0$ and $\sigma > 1$.

Remark 2.3.2. Let us show that the classes $\mathcal{D}_{\tau,\sigma}(\mathbf{R}^d)$ are closed under translation and dilatation. In fact, without loss of generality we may assume that $\phi \in \mathcal{D}^{K_{r_0}}_{\tau,\sigma,C}$ where $K_{r_0} = \overline{B_r(0)}$ is closed ball of radius r centered at origin.

The conclusion that $T_{x_0}\phi(x) := \phi(x - x_0) \in \phi \in \mathcal{D}^{K_{r x_0}}_{\tau,\sigma,C}$, $x_0 \in \mathbf{R}^d$, is straightforward. For $\lambda \in \mathbf{R}^d \backslash \{0\}$ set $D_\lambda \phi(x) := \phi(\lambda x)$. To prove that $D_\lambda \phi(x) \in \mathcal{D}^{K_{\frac{r_0}{\lambda}}}_{\tau,\sigma,\widetilde{\lambda}C}$, for $\widetilde{\lambda} = \max\{1, \lambda\}$ observe

$$\sup_{x\in K_{\frac{r_0}{\lambda}}} |\partial^\alpha D_\lambda \phi(x)| = \sup_{x\in K_{\frac{r_0}{\lambda}}} |\lambda^{|\alpha|}\phi(\lambda x)| = \sup_{y\in K_{r_0}} |\lambda^{|\alpha|}\phi(y)| \leq A(\widetilde{\lambda}C)^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma},$$

for $\alpha \in \mathbf{N}^d$, and the conclusion follows. In particular, if ϕ from Corollary 2.3.1 is constructed on $B_r(0)$, $r>0$ then the function $\tilde{\phi}(x) := \phi(\frac{x-x_0}{\lambda}) \in \mathcal{E}_{\{\tau,\sigma\}}(B_{\lambda r}(x_0))$.

2.4 Topological properties

Although our sequence does not satisfy Komatsu's condition $(M.2)'$, in this section we show that the spaces in (2.2.4)-(2.2.9) are nuclear. We are following the ideas in the proof of [20, Theorem 2.6], and consider only the Roumieu case and the Beurling case follows by the similar arguments (see Remark 2.4.1).

Let us show that for every $h>0$ there exists $k>h$ such that identity mapping $X\to Y$ is quasi-nuclear, where $X=\mathcal{E}_{\tau,\sigma,h}(K)$ and $Y=\mathcal{E}_{\tau,\sigma,k}(K)$ (resp. $X=\mathcal{D}^K_{\tau,\sigma,h}$ and $Y=\mathcal{D}^K_{\tau,\sigma,k}$). This means that seminorms on $\mathcal{E}_{\tau,\sigma}(K) := \varinjlim_{h\to\infty} \mathcal{E}_{\tau,\sigma,h}(K)$ (resp. $\mathcal{D}^K_{\tau,\sigma} := \varinjlim_{h\to\infty} \mathcal{D}^K_{\tau,\sigma,h}$) are prenuclear, cf. [39, page 177]. By [39, Theorem IV 10.2] this implies that $\mathcal{E}_{\tau,\sigma}(K)$ (resp. $\mathcal{D}^K_{\tau,\sigma}$) is a nuclear space.

The classes under consideration can be represented as projective and inductive limits of countable family of spaces, cf. Remark 2.2.4. The nuclearity of $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$ then follows from [39, Theorem III 7.4].

Theorem 2.4.1. *Let there be given $\tau>0$ and $\sigma>1$, and let K be compact in $\mathbf{R}^d$. Then the spaces $\mathcal{E}_{\tau,\sigma}(K)$, $\mathcal{D}^K_{\tau,\sigma}$ are nuclear.*

Proof. We follow the idea presented in [20]. Let $\phi \in \mathcal{E}_{\tau,\sigma,h}(K)$ and let $u_{\alpha,j}$, $\alpha\in\mathbf{N}^d$, $j\in\mathbf{Z}^d$, be the sequence linear functionals on $\mathcal{E}_{\tau,\sigma,h}(K)$ given by

$$\langle \phi, u_{\alpha,j}\rangle = \frac{\langle \partial^\alpha \phi, v_j\rangle_{C^{d+1}(K)}}{k^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\,, \tag{2.4.1}$$

where $v_j \in (C^{d+1}(K))'$ is defined by the following procedure:

Choose $l>0$ such that K is contained in the interior of $L=[-l,l]^d$ and let $C^{d+1}_L(\pi L)$ be the space of all $d+1$ time differentiable functions on πL with supported in L. Let $B\in\mathcal{L}(C^{d+1}(K), C^{d+1}_L(\pi L))$ be the (Whitney's) extension operator such that $Bf|_K = f$ and let $t_j \in (C^{d+1}_L(\pi L))'$ be given by

$$\langle f, t_j\rangle := \int_{\pi L} f(y) e^{-iyj/l}\,dy, \quad j\in\mathbf{Z}^d.$$

From [20, Lemma 2.3] it follows that the identity operator from $C^{d+1}(K)$ to $C(K)$, given by

$$f(x) = \frac{1}{(2\pi l)^d} \sum_{j\in\mathbf{Z}^d} e^{-ixj/l} \langle Bf, t_j\rangle_{C_L^{d+1}(\pi L)}, \quad x \in K,$$

is quasi-nuclear. In particular, if we put $v_j = t_j \circ B$, $j \in \mathbf{Z}^d$, it follows that

$$\sum_{j\in\mathbf{Z}^d}^{\infty} ||v_j||_{(C^{d+1}(K))'} < \infty, \quad \text{and} \quad ||f||_{C(K)} \leq \sum_{j\in\mathbf{Z}^d}^{\infty} |\langle f, v_j\rangle_{C^{d+1}(K)}| \,. \qquad (2.4.2)$$

By (2.2.2), (2.4.1) and the righthand side of (2.4.2) we obtain

$$\|\phi\|_Y = \sup_{\alpha\in\mathbf{N}^d} \sup_{x\in K} \frac{|\partial^\alpha \phi(x)|}{k^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}} \leq \sum_{\alpha\in\mathbf{N}^d} \sum_{j\in\mathbf{Z}^d} |\langle \phi, u_{\alpha,j}\rangle| \,.$$

It remains to show that $\sum_{\alpha,j} ||u_{\alpha,j}||_{X'} < \infty$.

Note that for $|\alpha| \geq 1$, $\alpha \in \mathbf{N}^d$ and $h \geq 1$, by $\widetilde{(M.2)}'$ we obtain

$$\begin{aligned} |\langle \phi, u_{\alpha,j}\rangle| &\leq \sup_{|\beta|\leq d+1} \frac{h^{|\alpha+\beta|^\sigma} |\alpha+\beta|^{\tau|\alpha+\beta|^\sigma}}{k^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}} ||\phi||_X ||v_j||_{(C^{d+1}(K))'} \\ &= \sup_{|\beta|\leq d+1} \frac{h^{(1+\frac{|\beta|}{|\alpha|})^\sigma |\alpha|^\sigma} |\alpha+\beta|^{\tau|\alpha+\beta|^\sigma}}{k^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}} ||\phi||_X ||v_j||_{(C^{d+1}(K))'} \\ &\leq \Big(\frac{h^{(d+2)^\sigma}}{k}\Big)^{|\alpha|^\sigma} C_d^{|\alpha|^\sigma} ||\phi||_X ||v_j||_{(C^{d+1}(K))'}, \end{aligned} \qquad (2.4.3)$$

for some $C_d > 1$ which includes constant from $\widetilde{(M.2)}'$.

For $0 < h < 1$, note that $h^{|\alpha+\beta|^\sigma} \leq h^{|\alpha|^\sigma}$ and thus by $\widetilde{(M.2)}'$ it follows that

$$\sup_{|\beta|\leq d+1} \frac{h^{|\alpha+\beta|^\sigma} |\alpha+\beta|^{\tau|\alpha+\beta|^\sigma}}{k^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}} \leq \Big(\frac{h}{k}\Big)^{|\alpha|^\sigma} C_d^{|\alpha|^\sigma}. \qquad (2.4.4)$$

Now we choose $k > 0$ such that $k > \max\{2C_d h, 2C_d h^{(d+2)^\sigma}\}$, so the estimates (2.4.3), and (2.4.4) imply

$$\sum_{\alpha\in\mathbf{N}^d} \sum_{j\in\mathbf{Z}^d} ||u_{\alpha,j}||_{X'} < \sum_{\alpha\in\mathbf{N}^d} \sum_{j\in\mathbf{Z}^d} \Big(\frac{1}{2}\Big)^{|\alpha|^\sigma} ||v_j||_{(C^{d+1}(K))'} < \infty.$$

We conclude that for every $h > 0$ there exists $k > h$ such that the identity mapping $\mathcal{E}_{\tau,\sigma,h}(K) \to \mathcal{E}_{\tau,\sigma,k}(K)$ (resp. $\mathcal{D}^K_{\tau,\sigma,h} \to \mathcal{D}^K_{\tau,\sigma,k}$) is quasi-nuclear, and the theorem is proved. □

Remark 2.4.1. Note that from the proof of Theorem 2.4.1 it is clear that form every given $k>0$ there exists $h<k$ such that the identity mapping $\mathcal{E}_{\tau,\sigma,h}(K)\to\mathcal{E}_{\tau,\sigma,k}(K)$ (resp. $\mathcal{D}^K_{\tau,\sigma,h}\to\mathcal{D}^K_{\tau,\sigma,k}$) is quasi-nuclear. This implies the nuclearity of the spaces $\varprojlim_{h\to 0^+}\mathcal{E}_{\tau,\sigma,h}(K)$ (resp. $\varprojlim_{h\to 0^+}\mathcal{D}^K_{\tau,\sigma,h}$), and hence the results for the Beurling case follows directly.

2.5 Algebra property

Since $M_p^{\tau,\sigma}=p^{\tau p^\sigma}$, $\tau>0$, $\sigma>1$, satisfies properties $(M.1)$ and $\widetilde{(M.2)}'$, the following proposition implies the algebra property as well as stability under finite order derivation.

Proposition 2.5.1. *Let U be open in $\mathbf{R}^d$, $\tau>0$ and $\sigma>1$. Then spaces $\mathcal{E}_{\tau,\sigma}(U)$ are closed under the pointwise multiplication of functions. Moreover, they are closed under the finite order differentiation.*

Proof. Let $K\subset\subset\mathbf{R}^d$ and for $h>0$ set $c_h=\min\{h,h^{2^{\sigma-1}}\}$. Then for $\phi,\psi\in\mathcal{E}_{\tau,\sigma,c_h}(K)$, by Leibniz formula we obtain

$$\begin{aligned}||\phi\psi||_{\mathcal{E}_{\tau,\sigma,2h}(K)} &\le \sup_{\alpha\in\mathbf{N}^d}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\frac{c_h^{|\alpha-\beta|^\sigma}c_h^{|\beta|^\sigma}|\alpha-\beta|^{\tau|\alpha-\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}}{(2h)^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\\ &\cdot\ ||\phi||_{\mathcal{E}_{\tau,\sigma,c_h}(K)}||\psi||_{\mathcal{E}_{\tau,\sigma,c_h}(K)}. \qquad (2.5.1)\end{aligned}$$

Further note that for $h\ge 1$, $c_h=h$ and for $\beta\le\alpha$ it holds $|\alpha-\beta|^\sigma+|\beta|^\sigma\le|\alpha|^\sigma$. Then $(M.1)$ property of the sequence $M_p^{\tau,\sigma}$ implies

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\frac{c_h^{|\alpha-\beta|^\sigma}c_h^{|\beta|^\sigma}|\alpha-\beta|^{\tau|\alpha-\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}}{(2h)^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\le\frac{2^{|\alpha|}h^{|\alpha|^\sigma}}{(2h)^{|\alpha|^\sigma}}\le 1,\quad\alpha\in\mathbf{N}^d.$$

For $0<h<1$, $c_h=h^{2^{\sigma-1}}$, and for $\beta\le\alpha$, $(1/h)^{|\alpha|^\sigma}\le(1/h)^{2^{\sigma-1}|\alpha-\beta|^\sigma}(1/h)^{2^{\sigma-1}|\beta|^\sigma}$. Combining this with $(M.1)$ we obtain

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\frac{c_h^{|\alpha-\beta|^\sigma}c_h^{|\beta|^\sigma}|\alpha-\beta|^{\tau|\alpha-\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}}{(2h)^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\le\frac{2^{|\alpha|}}{2^{|\alpha|^\sigma}}\le 1,\quad\alpha\in\mathbf{N}^d.$$

For the proof of closedness under differentiation fix $\beta\in\mathbf{N}^d$, and for $h>0$ set $c'_h=\max\{h,h^{2^{\sigma-1}}\}$. Then for $x\in K$, $\widetilde{(M.2)}'$ property of the sequence $M_p^{\tau,\sigma}$ implies

$$|(\partial^{\alpha+\beta}\phi(x))| \leq ||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} h^{|\alpha+\beta|^\sigma} |\alpha+\beta|^{\tau|\alpha+\beta|^\sigma} \quad (2.5.2)$$

$$\leq ||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} C_h'^{|\beta|^\sigma} (C_{|\beta|} c_h')^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}, \quad (2.5.3)$$

where $C_h' = \max\{1, h^{2^{\sigma-1}}\}$ and $C_{|\beta|}$ is constant that appears in $\widetilde{(M.2)}'$ (see Lemma 2.1.2 for $q = |\beta|$). This implies that for every $h > 0$ there exists $C_{h,\beta} > 0$ such that $||\partial^\beta \phi||_{\mathcal{E}_{\tau,\sigma,C_{|\beta|}c_h'}(K)} \leq C_{h,\beta} ||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}$. Hence, the statement follows. □

Remark 2.5.1. Let $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$ be the partial differential operator of order m with $a_\alpha \in \mathcal{E}_{\tau,\sigma}(U)$. Then, by the proof of Proposition 2.5.1, it follows that $P : \mathcal{E}_{\tau,\sigma}(U) \to \mathcal{E}_{\tau,\sigma}(U)$ is continuous linear map with respect to topology of spaces $\mathcal{E}_{\tau,\sigma}(U)$ (see (2.2.4) and (2.2.5)). Moreover, if one of the function in the proof of Proposition 2.5.1 is chosen from $\mathcal{D}^K_{\tau,\sigma}$, then clearly $\phi\psi$ vanishes outside of K, and hence $\phi\psi \in \mathcal{D}^K_{\tau,\sigma}$. In particular, if $a_\alpha \in \mathcal{D}^K_{\tau,\sigma}$, $|\alpha| \leq m$, then $P : \mathcal{E}_{\tau,\sigma}(U) \to \mathcal{D}^K_{\tau,\sigma}$ is continuous linear operator.

2.6 Ultradifferentiable property

In this section we study the continuity properties of certain ultradifferentiable operators $P(x,\partial)$ acting on $\mathcal{E}_{\tau,\sigma}(U)$. Recall, if the defining sequence M_p fulfills the condition $(M.2)'$ then the corresponding test function space is closed under the action of ultradifferentiable operators. Since the sequence $M_p^{\tau,\sigma}$ does not satisfy (2.1.7) the space $\mathcal{E}_{\tau,\sigma}(U)$ can not be closed under the action of $P(x,\partial)$. However, if we consider

$$\mathcal{E}_{\infty,\sigma}(U) := \varinjlim_{\tau\to\infty} \mathcal{E}_{(\tau,\sigma)}(U) = \varinjlim_{\tau\to\infty} \mathcal{E}_{\{\tau,\sigma\}}(U),$$

then the following results hold true.

Theorem 2.6.1. *Let U be open in $\mathbf{R}^d$, $\tau > 0$ and $\sigma > 1$. If $P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha$ is a constant coefficient differential operator of infinite order such that for some $L > 0$ and $A > 0$ (resp. every $L > 0$ there exists $A > 0$) such that*

$$|a_\alpha| \leq A \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}, \quad (2.6.1)$$

then $\mathcal{E}_{\infty,\sigma}(U)$ is closed under action of $P(\partial)$. In particular,

$$P(\partial): \quad \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)\,, \tag{2.6.2}$$

is continuous linear mapping, where $\mathcal{E}_{\tau,\sigma}(U)$ *denotes* $\mathcal{E}_{\{\tau,\sigma\}}(U)$ *(resp.* $\mathcal{E}_{\{\tau,\sigma\}}(U)$*).*

Proof. Take $\phi \in \mathcal{E}_{\tau,\sigma,h}(K)$ for $h > 0$ arbitrary but fixed. Then, for $x \in K$, using $\widetilde{(M.2)}$ we obtain

$$\begin{aligned} |\partial^\beta(a_\alpha \partial^\alpha \phi(x))| &\le A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}} h^{|\alpha+\beta|^\sigma} (|\alpha+\beta|)^{\tau|\alpha+\beta|^\sigma} \\ &\le A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}} h^{|\alpha+\beta|^\sigma} C^{|\alpha|^\sigma} C^{|\beta|^\sigma} |\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma} |\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma} \\ &\le A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} (LCc_h)^{|\alpha|^\sigma} (Cc_h)^{|\beta|^\sigma} |\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma}, \end{aligned} \tag{2.6.3}$$

where for the last inequality we have used that for $\sigma > 1$

$$|\alpha|^\sigma + |\beta|^\sigma \le |\alpha+\beta|^\sigma \le 2^{\sigma-1}(|\alpha|^\sigma + |\beta|^\sigma)$$

with $c_h = \max\{h, h^{2^{\sigma-1}}\}$ and $C > 1$ is the constant that appears in $\widetilde{(M.2)}$ (see Lemma 2.1.2).

Note that $c_h = h$ when $0 < h \le 1$ and $c_h = h^{2^{\sigma-1}}$ when $h > 1$. Hence, for the case $\mathcal{E}_{(\infty,\sigma)}(U)$ (resp. $\mathcal{E}_{\{\infty,\sigma\}}(U)$) we can choose $h > 0$ (resp. $L > 0$) such that

$$LCc_h < 1/2. \tag{2.6.4}$$

Since the series $\sum_{|\alpha|=0}^{\infty} (1/2)^{|\alpha|^\sigma}$ is convergent, taking the sum with respect to α and the supremum with respect to β and $x \in K$ from (2.6.3) it follows that

$$||P(\partial)\phi||_{\mathcal{E}_{\tau 2^{\sigma-1},\sigma,Cc_h}(K)} \le C'||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}, \tag{2.6.5}$$

for some $C' > 0$ and the Theorem is proved. □

Since $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$ satisfies $(M.1)$ we can prove the more general statement than one in theorem 2.6.1. Let us introduce the following definition.

Definition 2.6.1. A differential operator of infinite order

$$P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x) \partial^\alpha \tag{2.6.6}$$

is said to be an ultradifferential operator of class (τ,σ) (resp. $\{\tau,\sigma\}$) on an open set $U \subseteq \mathbf{R}^d$ if coefficients $a_\alpha(x)$ belongs to $\mathcal{E}_{(\tau,\sigma)}(U)$ (resp. $\mathcal{E}_{\{\tau,\sigma\}}(U)$) and satisfies the following condition: for every $K \subset\subset U$ there exists constant $L>0$ such that for any $h>0$ there exists $A>0$ (resp. for every $K \subset\subset U$ there exists $h>0$ such that for any $L>0$ there exists $A>0$) such that,

$$\sup_{x\in K} |\partial^\beta a_\alpha(x)| \leq A h^{|\beta|^\sigma} |\beta|^{\tau|\beta|^\sigma} \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}, \quad \alpha,\beta \in \mathbf{N}^d. \tag{2.6.7}$$

We say that $P(x,\partial)$ is of the class τ,σ if it is of the class (τ,σ) or $\{\tau,\sigma\}$.

Remark 2.6.1. We note that $(\tau,1)$ (resp. $\{\tau,1\}$) are Komatsu's ultradifferentiable operators of class $(p!^\tau)$ (resp. $\{p!^\tau\}$).

Theorem 2.6.2. *Let $\tau>0$, $\sigma>1$, and $P(x,\partial)$ be a differential operator of class (τ,σ) (resp. $\{\tau,\sigma\}$). Then $\mathcal{E}_{\infty,\sigma}(U)$ is closed under action of $P(x,\partial)$. In particular,*

$$P(x,\partial): \quad \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)\,, \tag{2.6.8}$$

is continuous linear mapping, where $\mathcal{E}_{\tau,\sigma}(U)$ denotes $\mathcal{E}_{\{\tau,\sigma\}}(U)$ (resp. $\mathcal{E}_{\{\tau,\sigma\}}(U)$).

Proof. Let $a_\alpha, \phi \in \mathcal{E}_{\tau,\sigma,h}(K)$, $\alpha \in \mathbf{N}^d$, for $h>0$ arbitrary but fixed. Then, by (2.6.7), for $x \in K$ we obtain

$$\begin{aligned}
|\partial^\beta(a_\alpha(x)\partial^\alpha\phi(x))| &\leq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}|\partial^{\beta-\gamma}a_\alpha(x)||\partial^{\alpha+\gamma}\phi(x)| \\
&\leq A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma}(|\beta-\gamma|)^{\tau|\beta-\gamma|^\sigma} \\
&\cdot \quad \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}h^{|\alpha+\gamma|^\sigma}(|\alpha+\gamma|)^{\tau|\alpha+\gamma|^\sigma} \\
&\leq A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}\frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}(|\alpha+\beta|)^{\tau|\alpha+\beta|^\sigma} \\
&\cdot \quad \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma+|\alpha+\gamma|^\sigma} \\
&\leq A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}(CL)^{|\alpha|^\sigma}C^{|\beta|^\sigma}|\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma}C_{h,\beta}, \quad (2.6.9)
\end{aligned}$$

where we have used $(M.1)'$, $\widetilde{(M.2)}$ properties of the sequence $M_p^{\tau,\sigma}$ and $C_{h,\beta} = \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma+|\alpha+\gamma|^\sigma}$.

To estimate $C_{h,\beta}$, note that for $\gamma \leq \beta$,

$$|\beta-\gamma|^\sigma + |\alpha+\gamma|^\sigma \geq \frac{1}{2^{\sigma-1}}(|\alpha|^\sigma + |\beta|^\sigma),$$

and hence

$$C_{h,\beta} \leq 2^{|\beta|} h^{\frac{1}{2^{\sigma-1}}|\alpha|^\sigma} h^{\frac{1}{2^{\sigma-1}}|\beta|^\sigma}, \quad 0<h<1.$$

For $h \geq 1$ and $\gamma \leq \beta$ we note that

$$|\beta-\gamma|^\sigma + |\alpha+\gamma|^\sigma \leq |\alpha+\beta|^\sigma \leq 2^{\sigma-1}(|\alpha|^\sigma + |\beta|^\sigma).$$

This implies that

$$C_{h,\beta} \leq 2^{|\beta|} h^{2^{\sigma-1}|\alpha|} h^{2^{\sigma-1}|\beta|}, \quad h \geq 1.$$

Now if we put $c_h = \max\{h^{\frac{1}{2^{\sigma-1}}}, h^{2^{\sigma-1}}\}$, from (2.6.9) it follows that

$$|\partial^\beta(a_\alpha(x)\partial^\alpha\phi(x))| \leq B||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}(c_h CL)^{|\alpha|^\sigma}(2c_h C)^{|\beta|^\sigma}|\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma}. \quad (2.6.10)$$

Since $c_h = h^{\frac{1}{2^{\sigma-1}}}$, $0<h\leq 1$ and $c_h = h^{2^{\sigma-1}}$ for $h>1$, we can choose $h>0$ (resp. $L>0$) such that $LCc_h < 1/2$. Taking the sum with respect to $\alpha \in \mathbf{N}^d$ and supremums with respect to $\beta \in \mathbf{N}^d$ and $x \in K$, by (2.6.10) it follows

$$||P(x,\partial)\phi||_{\mathcal{E}_{\tau 2^{\sigma-1},\sigma,2Cc_h}(K)} \leq C'||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}$$

for some $C'>0$ and this completes the proof. □

2.7 Inverse closedness property of classes $\mathcal{E}_{\{\tau,\sigma\}}$

In this section we prove the ivnverse-closedness property of our classes $\mathcal{E}_{\{\tau,\sigma\}}(U)$. We use this result to construct a function that does not belong to any of the Gevrey classes $\mathcal{E}_\tau$, $\tau>1$, but it is in our class $\mathcal{E}_{\{\tau,\sigma\}}$ for some $\tau>0$ and $\sigma>1$.

We say that a *l.c.s.* of smooth functions on U, $\mathcal{A} \subseteq C^\infty(U)$, is algebra if pointwise multiplication of functions is continuous operation from $\mathcal{A}\times\mathcal{A}$ to $\mathcal{A}$. Moreover we assume the $\mathcal{A}$ is *unital*, i.e., it contains neutral element (in particular, it contains function $\phi(x)=1$, $x\in U$). We recall the definition of inverse-closedness property of algebra $\mathcal{A}$.

Definition 2.7.1. Algebra $\mathcal{A}$ is *inverse-closed* in $C^\infty(U)$ if for any $\varphi \in \mathcal{A}$ for which $\varphi(x) \neq 0$ on U follows that $\dfrac{1}{\varphi} \in \mathcal{A}$.

Remark 2.7.1. Note that Proposition 2.5.1 implies that our classes $\mathcal{E}_{\{\tau,\sigma\}}(U)$, $\tau > 0$, $\sigma > 1$, are algebras. It is consequence of the property $(M.1)$ of the defining sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$. It is well known (see [21], [20]), that spaces $\mathcal{E}^{\{p!^\tau\}}$, $\tau > 0$ are also algebras of smooth functions.

In the sequel we introduce the notion of *almost increasing sequences.*

Definition 2.7.2. A sequence M_p, $p \in \mathbf{N}$, of positive numbers is almost increasing if for some $C > 0$, $M_p < CM_q$ when $p < q$.

Following result that concerns almost increasing sequences will be used in the following Chapters.

Lemma 2.7.1. *[19] Let M_p, $p \in \mathbf{N}$, be the sequence of positive numbers that satisfies property $(M.1)$. Then the sequence $\left(\frac{M_p}{p^p}\right)^{1/p}$ is almost increasing if and only if there exists $C > 0$ such that for all $j \in \mathbf{N}$ and all $k_i \in \mathbf{N}$ it holds*

$$\prod_{i=1}^{j} \frac{M_{k_i}}{k_i!} \le C^k \frac{M_k}{k!},$$

where $k = \sum_{i=1}^{j} k_i$.

Remark 2.7.2. Observe that $\left(\dfrac{M_p^{\tau,\sigma}}{p^p}\right)^{1/p} = p^{\tau p^{\sigma-1}-1}$, where $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$. Hence we conclude that $\left(\dfrac{M_p}{p^p}\right)^{1/p}$ is almost increasing, since

$$p^{\tau p^{\sigma-1}-1} < q^{\tau q^{\sigma-1}-1}, \quad q > p > \lceil (1/\tau)^{1/(\sigma-1)} \rceil.$$

Moreover, by Lemma 2.7.1 it follows

$$\prod_{i=1}^{j} k_i^{\tau k_i^\sigma} \le C^k \frac{k_1! \cdots k_j!}{k!} k^{\tau k^\sigma}, \tag{2.7.1}$$

for $k = \sum_{i=1}^{j} k_i$.

We recall ([38]) following result concerning inverse-closedness of Carleman classes, whose definition coincides with Komatsu's definition of $\mathcal{E}^{\{M_p\}}(U)$ (classes of ultradifferentiable functions of Roumieu type) (see [20]).

Theorem 2.7.1. *Let $\mathcal{E}^{\{M_p\}}(U)$ be Komatsu's class of smooth functions for which defining sequence M_p, $p \in \mathbf{N}$, satisfies property $(M.1)$. Then the following conditions are equivalent:*

a) $\lim_{p\to\infty} M_p^{1/p} = \infty$ *and* $\left(\frac{M_p}{p^p}\right)^{1/p}$ *is almost increasing.*

b) Algebra $\mathcal{E}^{\{M_p\}}(U)$ *is inverse-closed in* $C^\infty(U)$.

Remark 2.7.3. We immediately note that Gevrey classes, including classes of analytic functions on U, $\mathcal{E}^{\{p!^\tau\}}(U)$, $\tau \geq 1$, $(M_p = p!^\tau)$ are inverse-closed in $C^\infty(U)$. Also note that Theorem 2.7.1 does not hold for $\mathcal{E}^{\{p!^\tau\}}(U)$, $0 < \tau < 1$, which consists of entire functions, since defining sequence fails to satisfy 3. condition.

However, since our classes are larger then Komatsu's classes $\mathcal{E}^{\{p^{\tau p^\sigma}\}}(U)$ it turns out that we do not need to impose any additional conditions to sequence $M_p^{\tau,\sigma}$.

In the proof of the main result in this section, we will use the generalized version of Faá di Bruno formula presented in [22] so we fix the notation. A multiindex $\alpha \in \mathbf{N}^d$ is said to be *decomposed* into *parts* $p_1, \dots, p_s \in \mathbf{N}^d$ with *multiplicities* $m_1, \dots, m_s \in \mathbf{N}$, respectively, if it holds

$$\alpha = m_1 p_1 + m_2 p_2 + \cdots + m_s p_s, \tag{2.7.2}$$

where $m_i \in \{0, 1, \dots, |\alpha|\}$, $|p_i| \in \{1, \dots, |\alpha|\}$ for $i \in \{1, \dots, s\}$. Note that $s \leq |\alpha|$ and the *total multiplicity* m, given by $m = m_1 + \cdots + m_s$, also satisfies $m \leq |\alpha|$.

Moreover if $p_i = (p_{i_1}, \dots, p_{i_d})$, $i \in \{1, \dots, s\}$, we order the parts in the following way: $p_i << p_j$ when $i < j$ if and only that there exists $k \in \{1, \dots, d\}$ such that $p_{i_1} = p_{j_1}, \dots, p_{i_{k-1}} = p_{j_{k-1}}$ and $p_{i_k} < p_{j_k}$.

The list (s, p, m) is called the decomposition of α and the set of all decompositions of the form (2.7.2) is denoted by π.

For smooth functions $f : U \to \mathbf{C}$ and $g : V \to U$, where U, V are open in $\mathbf{R}$ and $\mathbf{R}^d$, respectively, the generalized Faa di Bruno formula is given by

$$\partial^\alpha(f(g)) = \alpha! \sum_{(s,p,m)\in\pi} f^{(m)}(g) \prod_{k=1}^{s} \frac{1}{m_k!}\Big(\frac{1}{p_k!}\partial^{p_k} g\Big)^{m_k}. \tag{2.7.3}$$

We illustrate the notation with the following example.

Example 2.7.1. Let $d = 2$ and consider the partial differential operator $\frac{\partial^3}{\partial x_1 \partial x_2^2}$ In particular, $\alpha = (1, 2)$. Then the simple calculation gives

$$\frac{\partial^3}{\partial x_1 \partial x_2^2} f(g) = f'(g)\frac{\partial^3 g}{\partial x_1 \partial x_2^2} + f''(g)\Big(\frac{\partial g}{\partial x_1}\frac{\partial^2 g}{\partial x_2^2} + 2\frac{\partial g}{\partial x_2}\frac{\partial^2 g}{\partial x_1 \partial x_2}\Big)$$

$$+f'''(g)\frac{\partial g}{\partial x_1}\Big(\frac{\partial g}{\partial x_2}\Big)^2. \tag{2.7.4}$$

Observe that there are four summands in (2.7.4). They correspond to the following elements of π in (2.7.3) when $\alpha = (1,2)$ (respectively):

$$\begin{aligned}(1,2) &= 1\cdot(1,2), \quad (m=1)\\ (1,2) &= 1\cdot(0,2)+1\cdot(0,1), \quad (m=2)\\ (1,2) &= 1\cdot(0,1)+1\cdot(1,1), \quad (m=2)\\ (1,2) &= 2\cdot(0,1)+1\cdot(1,0), \quad (m=3).\end{aligned}$$

In particular, we read $1\cdot(0,1)+1\cdot(1,1)$ as $\Big(\frac{\partial g}{\partial x_2}\Big)^1\Big(\frac{\partial^2 g}{\partial x_1\partial x_2}\Big)^1$, and the corresponding coefficient is equal to

$$1!2!\frac{1}{1!1!}\Big(\frac{1}{0!1!}\Big)^1\Big(\frac{1}{1!1!}\Big)^1 = 2,$$

and this is in agreement with (2.7.4).

For the complete proof of (2.7.3) and more examples we refer to [22].

Remark 2.7.4. Observe that $\operatorname{card}\pi \le (1+|\alpha|)^{d+2}$. In particular, each m_i, $1\le i\le s$, may take values form 0 to $|\alpha|$ which gives $|\alpha|+1$ possibilities. Moreover, if $p_i = (p_{i_1},\dots,p_{i_d})$, $1\le i\le s$, for each p_{i_k}, $1\le k\le d$, we also have$|\alpha|+1$ possibilities which is in total $(|\alpha|+1)^d$. Since $s\le|\alpha|$ we conclude that

$$\operatorname{card}\pi \le |\alpha|(|\alpha|+1)^{d+1} \le (1+|\alpha|)^{d+2}.$$

Now we can prove the following theorem.

Theorem 2.7.2. *Let* $U\subseteq \mathbf{R}^d$ *be open. Classes* $\mathcal{E}_{\{\tau,\sigma\}}(U)$, $\tau>0$, $\sigma>1$, *are inverse-closed in* $C^\infty(U)$.

Proof. For the proof we use the generalized Faá di Bruno formula given by (2.7.3). Let $K\subset\subset U$ be arbitrary but fixed. Further let $\phi\in\mathcal{E}_{\{\tau,\sigma\}}(U)$, and moreover $\phi(x)\neq 0$ for $x\in U$. Since K is chosen arbitrary, it is sufficient to prove that

$$\sup_{x\in K}|\partial^\alpha\Big(\frac{1}{\phi(x)}\Big)| \le A'h'^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}, \quad \alpha\in\mathbf{N}^d \tag{2.7.5}$$

for some $A',h'>0$.

In (2.7.3) set $f(x) = \frac{1}{x}$, $x\neq 0$. Since $f^{(m)}(x) = \frac{(-1)^m m!}{x^{m+1}}$, $m\in\mathbf{N}$, observe that for $x\in K$

$$\partial^\alpha\Big(\frac{1}{\phi(x)}\Big) = \alpha! \sum_{(s,p,m)\in\pi} \frac{(-1)^m m!}{(\phi(x))^{m+1}} \prod_{k=1}^{s} \frac{1}{m_k!}\Big(\frac{1}{p_k!}\partial^{p_k}\phi(x)\Big)^{m_k}, \qquad (2.7.6)$$

and therefore

$$|\partial^\alpha\Big(\frac{1}{\phi(x)}\Big)| \leq |\alpha|! \sum_{(s,p,m)\in\pi} \frac{m!}{m_1!\dots m_s!|(\phi(x))|^{m+1}} \prod_{k=1}^{s} \Big(\frac{1}{p_k!}|\partial^{p_k}\phi(x)|\Big)^{m_k} \qquad (2.7.7)$$

Since $|\phi(x)| \geq C$ for $x \in K$ and $m = \sum_{k=1}^{s} m_k \leq |\alpha|$ observe that

$$\frac{m!}{m_1!\dots m_s!|(\phi(x))|^{m+1}} \leq C^{m+1} \leq C^{|\alpha|^\sigma+1}, \qquad (2.7.8)$$

for suitable constant $C > 0$.

Moreover, for $\phi \in \mathcal{E}_{\{\tau,\sigma\}}(U)$ it follows that

$$|\partial^{p_k}\phi(x)| \leq Ah^{|p_k|^\sigma}|p_k|^{\tau|p_k|^\sigma}, \quad 1 \leq k \leq s, \qquad (2.7.9)$$

and since $\alpha_j = \sum_{k=1}^{s} m_k p_{k_j}$ we obtain

$$\sum_{k=1}^{s} m_k|p_k| = \sum_{j=1}^{d}\sum_{k=1}^{s} m_k p_{k_j} = \sum_{j=1}^{d} \alpha_j = |\alpha|. \qquad (2.7.10)$$

Now by (2.7.9) and (2.7.10) we have

$$\begin{aligned}\prod_{k=1}^{s} \Big(\frac{1}{p_k!}|\partial^{p_k}\phi(x)|\Big)^{m_k} &\leq \prod_{k=1}^{s}\Big(Ah^{|p_k|^\sigma}|p_k|^{\tau|p_k|^\sigma}\Big)^{m_k} \\ &\leq A^m \prod_{k=1}^{s}\Big(h^{|\alpha|^{\sigma-1}}|\alpha|^{\tau|\alpha|^{\sigma-1}}\Big)^{m_k|p_k|} \\ &= A^{|\alpha|}\Big(h^{|\alpha|^{\sigma-1}}|\alpha|^{\tau|\alpha|^{\sigma-1}}\Big)^{\sum_{k=1}^{s} m_k|p_k|} \\ &= (Ah)^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}. \qquad (2.7.11)\end{aligned}$$

Finally, using (2.7.12), (2.7.8) and (2.7.11) we obtain

$$\begin{aligned}|\partial^\alpha\Big(\frac{1}{\phi(x)}\Big)| &\leq |\alpha|! \sum_{(s,p,m)\in\pi} \frac{m!}{m_1!\dots m_s!|(\phi(x))|^{m+1}} \prod_{k=1}^{s} \Big(\frac{1}{p_k!}|\partial^{p_k}\phi(x)|\Big)^{m_k} \\ &\leq Ah^{|\alpha|^\sigma+1}|\alpha|^{\tau|\alpha|^\sigma}\alpha!(|\alpha|+1)^{d+2} \\ &\leq A'h'^{|\alpha|^\sigma+1}|\alpha|^{\tau|\alpha|^\sigma},\end{aligned}$$

for suitable $A' > 0$ and $h' > 0$, where we have used the bound for number of terms in (2.7.3) (see Remark 2.7.4), and the last inequality follows from $\alpha!(|\alpha|+1)^{d+2} \leq |\alpha|^{d+|\alpha|+1} \leq C''^{|\alpha|^\sigma}$ for some $C'' > 0$. This completes the proof. □

Example 2.7.2. To end this section we use inverse-closedness arguments to construct a function $\phi \in \mathcal{E}_{\{\tau,\sigma\}}(U)$, $\tau > 0$, $\sigma > 1$ that does not belong to $\bigcup_{\tau>1} \mathcal{E}_\tau(U)$, on some open set U, where $\mathcal{E}_\tau$, $\tau > 1$ are Gevrey classes.

In Section 2.3 we have constructed a compactly supported function $\psi \in \mathcal{D}^{[-a/2,a/2]}_{\{\tau,\sigma\}}$ where

$$a = \sum_{p=0}^{\infty} a_p, \quad a_p = 2^{-\tau p^{\sigma-1}} \frac{1}{(p+1)^{\tau p^{\sigma-1}}},$$

whose all derivatives vanishes at the end points and $\psi(x) \neq 0$, $x \in (-a/2, a/2)$. We also concluded that

$$\psi \notin \bigcup_{\tau>1} \mathcal{D}^{\{p!^\tau\}}_{[-a/2,a/2]}, \tag{2.7.12}$$

where $\mathcal{D}^{\{p!^\tau\}}_{[-a/2,a/2]}$, $\tau > 1$ is space of Gevrey functions with support $[-a/2, a/2]$ (see the proof of Corollary 2.3.1).

Let $\phi(x) = \dfrac{1}{\psi(x)}$, $x \in (-a/2, a/2)$. Then by the calculation done in the proof Theorem 2.7.2 (for $d = 1$) we conclude that for every $K \subset\subset (-a/2, a/2)$ it holds

$$\sup_{x\in K} |\phi^{(k)}(x)| \leq A h^{k^\sigma} k^{\tau k^\sigma}, \quad k \in \mathbf{N}. \tag{2.7.13}$$

for some $A, h > 0$. In, particular $\phi \in \mathcal{E}_{\{\tau,\sigma\}}(-a/2, a/2)$. To conclude that $\phi \notin \bigcup_{\tau>1} \mathcal{E}_\tau(-a/2, a/2)$ we suppose the opposite, $\phi \in \mathcal{E}_\tau(-a/2, a/2)$, for some $\tau > 0$. Since Gevrey classes are inverse-closed in C^∞ (see Theorem 2.7.1) it follows that $\psi \in \mathcal{E}_\tau(-a/2, a/2)$, and since all the derivatives of ψ vanishes at $\pm a/2$, it follows that $\psi \in \mathcal{D}^{\{p!^\tau\}}_{[-a/2,a/2]}$. This is in contradiction with (2.7.12).

Note that ψ is compactly supported, while $\phi \notin \mathcal{D}_{\tau,\sigma}$, so that we obtained an example of element from $\mathcal{E}_{\{\tau,\sigma\}} \backslash \mathcal{D}_{\{\tau,\sigma\}}$, which is not in $\bigcup_{\tau>0} \mathcal{E}_\tau$.

To pass to higher dimensions, using the $(M.1)$ property of sequence $M_p^{\tau,\sigma}$, $p \in \mathbf{N}$, we conclude that $f(x) = \prod_{k=1}^{d} \phi(x_k)$, $x = (x_1, x_2, \dots, x_d)$, belongs to $\mathcal{E}_{\{\tau,\sigma\}}(U)$ for $U = (-a/2, a/2)^d$. To avoid the fact that the boundary of U is not smooth, by [16, Lemma 1.4.3] it is possible to find appropriate open set with smooth boundary.

2.8 Dual spaces

To complete this Chapter, we define the dual spaces for $\mathcal{D}_{\tau,\sigma}(U)$ and $\mathcal{E}_{\tau,\sigma}(U)$.

Definition 2.8.1. The space of ultradistributions of type τ,σ, $\mathcal{D}'_{\tau,\sigma}(U)$, $\tau>0$, $\sigma>1$ is the set of of all linear functionals u on $\mathcal{D}_{\tau,\sigma}(U)$ satisfying the following estimates: in the case (τ,s) (resp. $\{\tau,\sigma\}$) for every $K\subset\subset U$ there exists constants $A,h>0$ (resp. for every $h>0$ there exists $A>0$) such that

$$|\langle u,\phi\rangle|\leq A\sup_{\alpha\in\mathbf{N}^d}h^{|\alpha|^\sigma}(|\alpha|^{\tau|\alpha|^\sigma})^{-1}\sup_{x\in K}|D^\alpha\phi(x)|,\quad \phi\in\mathcal{D}^K_{\tau,\sigma}. \tag{2.8.1}$$

$\mathcal{E}'_{\tau,\sigma}(U)$ is subspace of $\mathcal{D}'_{\tau,\sigma}(U)$ of ultardistributions of type τ,σ with compact support.

Remark 2.8.1. From the Proposition 2.2.1 we immediately obtain following embeddings

$$\mathcal{D}'(U)\hookrightarrow\mathcal{D}'_{\tau,\sigma}(U)\hookrightarrow\mathcal{D}'_t(U),$$

and

$$\mathcal{E}'(U)\hookrightarrow\mathcal{E}'_{\tau,\sigma}(U)\hookrightarrow\mathcal{E}'_t(U),$$

for every $\tau>0$ and $t,\sigma>1$ where $\mathcal{D}'_t(U)$ is space of Gevrey ultradistributions and $\mathcal{E}'_t(U)$ is space of Gevrey ultradistribution with compact support.

Chapter 3

Wave front sets related to $\mathcal{E}_{\tau,\sigma}$

In the previous Chapter we have established new type of local regularity and proved the basic properties of the related classes. The goal of this Chapter is to define a new type of wave front sets and to show that local regularity of their complement is regularity proposed by $\mathcal{E}_{\tau,\sigma}$, $\tau > 0$, $\sigma > 1$.

Following the ideas presented in [16] we introduce the notion of singular supports related to our classes and prove that they are equal to the standard projection of introduced wave front sets. This is done by careful analysis of sequences of cut-off test functions which lead to specific *admissibility condition*. The elements of such sequences have small estimates on derivatives up to the finite order, and it turns out that they are convenient for our analysis.

One of the main ingredients of the following proofs is the procedure which we call *enumeration* (see also proof of the Lemma 3.1.1). We say that two conditions of the form (3.2.1) are equivalent if one is obtained from another after replacing N with positive, increasing sequence a_N such that $a_N \to \infty$, $N \to \infty$. This procedure we call enumeration, and write $N \to a_N$ and u_N instead of u_{a_N}.

Note that this procedure involves a change of variables (with respect to $N \in \mathbf{N}$) which "speeds up" or "slows down" the decay estimates of single members of the corresponding sequences, while retaining the asymptotic behavior when $N \to \infty$. In other words, although the estimates of the terms of a sequence before and after enumeration are different for each $N \in \mathbf{N}$, its asymptotic behavior remains the same.

We also prove the *microlocal propery* of finite order partial differential operators (PDO's in short) with the coefficients in classes $\mathcal{E}_{\{\tau,\sigma\}}$. In last section, we define intersections and unions of our wave front sets, which leads us to the definition of wave front set with *microlocal hypoellipticity* property.

Main results of this Chapter are published in [43].

3.1 Different types of local regularity

As a part of motivation for our work, we start with the following lemma which describes decay properties on the Fourier transform side. Basic ingredient of the proof is procedure that we call *enumeration* which will be frequently used in the forthcoming proofs.

Lemma 3.1.1. *Let U be the open set in $\mathbf{R}^d$, $\Omega \subseteq K \subset\subset U$ and $u \in C_0^\infty(U)$. Further, let $\{u_N\}_{N\in\mathbf{N}}$, be the sequence of compactly supported smooth functions such that,* $\operatorname{supp} u_N \subseteq K$*, $u_N = u$ on Ω, and which satisfies one of the following regularity conditions:*

$$|\widehat{u}_N(\xi)| \leq A\, \frac{h^{N^t} \lfloor N^t \rfloor !}{|\xi|^{\lfloor N^t \rfloor}}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\},\ t > 0 \tag{3.1.1}$$

$$|\widehat{u}_N(\xi)| \leq A\, \frac{h^N N!^t}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\},\ t > 1 \tag{3.1.2}$$

$$|\widehat{u}_N(\xi)| \leq A\, \frac{h^N N!^t}{|\xi|^{\lfloor N^t \rfloor}}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\},\ 0 < t < 1. \tag{3.1.3}$$

for some (different) constants $A, h > 0$. Then

$$(3.1.1) \Rightarrow (3.1.2) \Rightarrow (3.1.3)\,.$$

Proof. Note that after enumeration $N \to N^{1/t}$ (3.1.1) is equivalent to

$$|\widehat{u}_N(\xi)| \leq A\, \frac{h^N N!}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\}. \tag{3.1.4}$$

and hence, putting u_N instead of $u_{N^{1/t}}$ condition (3.1.1) is equivalent to the condition related to local analyticity (see [16]). Now from (3.1.4) is clear that (3.1.1) $\Rightarrow$ (3.1.2).

If we apply the same enumeration as for (3.1.1) to the condition (3.1.3) note that we obtain

$$|\widehat{u}_N(\xi)| \leq A\, \frac{h^{N^{1/t}} \lfloor N^{1/t} \rfloor !^t}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\}, \tag{3.1.5}$$

that is, after enumeration and letting $t = 1/\sigma$, by Proposition 2.2.1 and Remark 2.2.2, (3.1.3) is equivalent to

$$|\widehat{u}_N(\xi)| \leq A\, \frac{h^{N^\sigma} N^{N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\}. \tag{3.1.6}$$

If (3.1.5) we put u_N instead of u_{N^σ} ($\sigma = 1/t$), by the simple inequality $N!^t \le CN^{N^\sigma}$, $t, \sigma > 1$, $N \in \mathbf{N}$ it follows that (3.1.2) $\Rightarrow$ (3.1.3). This proves the lemma. □

Remark 3.1.1. Moreover, for $\tau > 0$ and $\sigma > 1$, we introduce new regularity condition

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^N N!^{1/\sigma}}{|\xi|^{\lfloor (N/\tau)^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\}. \tag{3.1.7}$$

By the similar arguments as in Lemma 3.1.1, we note that by applying Stirling's formula and enumeration $N \to \tau N^\sigma$, (3.1.7) is equivalent to

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\}, \tag{3.1.8}$$

and hence the simple inequality of the form $N!^t \le CN^{\tau N^\sigma}$, $\tau > 0$, $t, \sigma > 1$, $N \in \mathbf{N}$ implies that (3.1.1) $\Rightarrow$ (3.1.2) $\Rightarrow$ (3.1.7). Note that for $\sigma = 1/t$, (3.1.7) $\Leftrightarrow$ (3.1.3) when $\tau = 1$, while (3.1.7) $\Rightarrow$ (3.1.3) when $\tau \in (0, 1)$.

3.2 New type of local regularity of distributions

In this section, by using (3.1.7), we define a new type of wave front set of an distribution. We will use the technique from the proof of [16, Proposition 8.4.2], developed for the analytic wave front set, and construct a (bounded) sequence of cutoff functions in a similar way.

Let $\tau > 0$, $\sigma > 1$, $\Omega \subseteq K \subset\subset U \subseteq \mathbf{R}^d$, where Ω and U are open in $\mathbf{R}^d$, and the closure of Ω is contained in K. We use standard notation $\mathcal{D}'(U)$ for Schwartz distributions and $\mathcal{E}'(U)$ for their subspace of distributions with compact support.

Let $u \in \mathcal{D}'(U)$. Following the idea presented in [16], we analyze the nature of regularity related to the condition

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\}. \tag{3.2.1}$$

where $\{u_N\}_{N \in \mathbf{N}}$ is bounded sequence in $\mathcal{E}'(U)$ such that $u_N = u$ in Ω and A, h are some positive constants.

An essential part of the upcoming definition of the wave front set concerns the sequences of the following form.

Definition 3.2.1. Let $\tau > 0$, $\sigma > 1$, and $\Omega \subseteq K \subset\subset U$ such that the closure of Ω is contained in K. A sequence $\{\chi_N\}_{N\in\mathbf{N}}$ of functions in $C_0^\infty(K)$ is said to be *τ,σ-admissible with respect to K* if

a) $\chi_N = 1$ in a neighborhood of Ω, for every $N \in \mathbf{N}$,

b) there exists a positive sequence A_β such that

$$\sup_{x\in K} |D^{\alpha+\beta}\chi_N(x)| \le A_\beta^{|\alpha|+1} \lfloor N^{1/\sigma}\rfloor^{|\alpha|}, \quad |\alpha| \le \lfloor (N/\tau)^{1/\sigma}\rfloor, \tag{3.2.2}$$

for every $N \in \mathbf{N}$ and $\beta \in \mathbf{N}^d$.

Remark 3.2.1. A similar approach based on the sequence of functions $\{\varphi_N\}_{N\in\mathbf{N}}$ "analytic up to the order N" is used to extended results on Schwartz distributions from [16] to Gevrey type ultradistributions, cf. [35, Proposition 1.4.10, Corollary 1.4.11]. When $\tau > 0$, $\sigma > 1$ and $\beta = 0$ in (3.2.2) we obtain

$$\sup_{x\in K} |\partial^\alpha \chi_N(x)| \le A^{|\alpha|+1} \frac{\lfloor N^{1/\sigma}\rfloor^{|\alpha|}}{|\alpha|^{\frac{1}{\sigma}|\alpha|}} |\alpha|^{\frac{1}{\sigma}|\alpha|}$$

$$\le A^{|\alpha|+1} \sup_{r>0} \frac{N^{r/\sigma}}{r^{r/\sigma}} |\alpha|^{\frac{1}{\sigma}|\alpha|} = A^{|\alpha|+1} e^{\frac{1}{e\sigma}N} |\alpha|^{\frac{1}{\sigma}|\alpha|}, \quad |\alpha| \le \lfloor (N/\tau)^{1/\sigma}\rfloor,$$

and χ_N is therefore "quasi-analytic up to the order $\lfloor (N/\tau)^{1/\sigma}\rfloor$". Note that "the order of quasi-analyticity" of χ_N tends to infinity as $\tau \to 0^+$ for fixed $N \in \mathbf{N}$ and $\sigma > 1$.

Following Lemma is direct consequence of [16, Theorem 1.3.5 and Theorem 1.4.2].

Lemma 3.2.1. *Let there be given $r > 0$, $\tau > 0$, $\sigma > 1$ and $x_0 \in \mathbf{R}^d$. There exists τ,σ-admissible sequence $\{\chi_N\}_{N\in\mathbf{N}}$ with respect to $\overline{B_{2r}(x_0)}$ such that $\chi_N = 1$ on $B_r(x_0)$, for every $N \in \mathbf{N}$.*

Proof. Fix $r > 0$. Following the notation of the quoted Theorems, we set $d_k = \dfrac{r}{4\lfloor (N/\tau)^{1/\sigma}\rfloor}$, $k \le \lfloor (N/\tau)^{1/\sigma}\rfloor$, $N \in \mathbf{N}$. Note that

$$\sum_{k=1}^{\lfloor (N/\tau)^{1/\sigma}\rfloor} d_k = \frac{r}{4} < \frac{r}{2},$$

for every $N \in \mathbf{N}$.

Since the infimum of distances between points in $\overline{B_{5r/4}(x_0)}$ and $\mathbf{R}^d \backslash B_{7r/4}(x_0)$ is $r/2$, Theorem 1.4.2 from [16] implies that for every $N \in \mathbf{N}$ there exists a smooth function $\widetilde{\chi_N}$ such that $\operatorname{supp}\widetilde{\chi_N} \subseteq B_{7r/4}(x_0)$, $\widetilde{\chi}_N = 1$ on $B_{5r/4}(x_0)$, and

$$\sup_{x\in K}|D^\alpha\widetilde{\chi_N}(x)| \leq A^{|\alpha|}\prod_{k=1}^{|\alpha|} d_k = A^{|\alpha|}\lfloor (N/\tau)^{1/\sigma}\rfloor^{|\alpha|} \leq C^{|\alpha|}\lfloor N^{1/\sigma}\rfloor^{|\alpha|}, \quad (3.2.3)$$

for $|\alpha| \leq \lfloor (N/\tau)^{1/\sigma}\rfloor$, $N \in \mathbf{N}$, where the constant $C > 0$ depends on τ and σ.

Next we choose a non-negative function $\theta \in C_0^\infty(B_{r/4}(x_0))$, $\int \theta(x)dx = 1$ and note that $\chi_N = \theta * \widetilde{\chi_N}$ clearly satisfies (3.2.2) for every $N \in \mathbf{N}$, if we let β derivatives act on θ and α derivatives act on $\widetilde{\chi_N}$. Hence $\{\chi_N\}_{N\in\mathbf{N}}$ is a τ,σ-admissible sequence with respect to $\overline{B_{2r}(x_0)}$ and the lemma is proved. □

Remark 3.2.2. We would like to emphasize some of the properties of τ,σ-admissible sequences. Note that if we put $\alpha = 0$ in (3.2.2), we obtain that the sequence $\{\chi_N\}_{N\in\mathbf{N}}$ is bounded in $C^\infty(U)$. Moreover, by applying the Fourier transform, by standard calculations it follows

$$|\widehat{\chi}_N(\xi)| \leq A_\beta^{|\alpha|+1}\lfloor N^{1/\sigma}\rfloor^{|\alpha|}\langle\xi\rangle^{-|\alpha|-|\beta|}, \quad |\alpha| \leq \lfloor (N/\tau)^{1/\sigma}\rfloor, \quad (3.2.4)$$

for every $N \in \mathbf{N}, \xi \in \mathbf{R}^d$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. Moreover, since $\{\chi_N\}_{N\in\mathbf{N}}$ is bounded in $C^\infty(U)$, for $u \in \mathcal{D}'(U)$, $\{\chi_N u\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{E}'(U)$.

Let $\{u_N\}_{N\in\mathbf{N}}$ be bounded sequence in $\mathcal{E}'(U)$. Recall that Paley-Wiener type theorems, and the fact that $e^{-ix\cdot\xi} \in C^\infty(\mathbf{R}^d_x)$, for every $\xi \in \mathbf{R}^d$, implies that

$$|\widehat{u}_N(\xi)| = |\langle u_N, e^{-i\cdot\xi}\rangle| \leq C\langle\xi\rangle^M, \quad (3.2.5)$$

for some $C > 0$ independent of N, where M is order of distribution u.

We first show how the condition (3.2.1) can be related to the regularity of elements from $\mathcal{E}_{\{\tau,\sigma\}}(U)$.

Proposition 3.2.1. *Let $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, $\Omega \subseteq U$ with the closure contained in U and let $\{u_N\}_{N\in\mathbf{N}}$ be a bounded sequence in $\mathcal{E}'(U)$, $u_N = u$ on Ω and such that (3.2.1) holds. Then $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$.*

Proof. After the enumeration $N \to N^\sigma$ and by Lemma 2.1.1, condition (3.2.1) is equivalent to

$$|\widehat{u}_N(\xi)| \leq A\,\frac{k^{N^\sigma}N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d\backslash\{0\}. \quad (3.2.6)$$

for some $A, k > 0$.

By the Fourier inversion formula and the fact that $u_N = u$ in Ω we obtain

$$(h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma})^{-1}|D^\alpha u(x)|$$
$$= (h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma})^{-1}\Big|\Big(\int_{|\xi|\le 1} + \int_{|\xi|>1}\Big)\xi^\alpha \widehat{u}_N(\xi)e^{2\pi i x\xi}d\xi\Big|$$
$$\le I_1 + I_2, \quad N\in\mathbf{N}, \alpha\in\mathbf{N}^d, x\in\Omega, \quad (3.2.7)$$

where $h > 0$ will be chosen later on. Using (3.2.5) we estimate I_1 by

$$I_1 = (h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma})^{-1}\Big|\int_{|\xi|\le 1}\xi^\alpha \widehat{u}_N(\xi)e^{2\pi i x\xi}d\xi\Big|$$
$$\le C(h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma})^{-1}\int_{|\xi|\le 1}\langle\xi\rangle^M d\xi. \quad (3.2.8)$$

If $h \ge 1$ we conclude that $I_1 \le C_1$ where C_1 does not depend on α. To estimate I_2, note that by (3.2.6) we have

$$\begin{aligned} I_2 &= (h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma})^{-1}\Big|\int_{|\xi|>1}\xi^\alpha \widehat{u}_N(\xi)e^{2\pi i x\xi}d\xi\Big| \\ &\le A(h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma})^{-1}k^{N^\sigma}N^{\tau N^\sigma}\int_{|\xi|>1}|\xi|^{|\alpha|-N}d\xi \le C(k^{2^{\sigma-1}}/h)^{|\alpha|^\sigma}, \end{aligned}$$

where for the last inequality we chose $N = |\alpha|+d+1$, and use $\widetilde{(M.2)}'$ property of $M_p^{\tau,\sigma}$, $p\in\mathbf{N}$. Now, for $h > k^{2^{\sigma-1}}$ we conclude that $I_2 \le C_2$, and C_2 does not depend on α. Hence, if we take $h > \max\{1, k^{2^{\sigma-1}}\}$, we conclude that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$, and the statement is proved. □

Therefore the condition (3.2.1) implies local regularity related to the classes $\mathcal{E}_{\{\tau,\sigma\}}(U)$ from Chapter 2. For the opposite direction, if $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$ we need to observe $\tilde\tau,\sigma$-admissible sequences, where $\tilde\tau = \tau^{\sigma/(\sigma-1)}$. The precise statement is the following.

Proposition 3.2.2. *Let $\Omega \subseteq K \subset\subset U$, $\overline{\Omega} \subset K$, $u \in \mathcal{D}'(U)$, and let $\{\chi_N\}_{N\in\mathbf{N}}$ be the $\tilde\tau,\sigma$-admissible sequence with respect to K, where $\tilde\tau = \tau^{\sigma/(\sigma-1)}$, $\tau > 0$, $\sigma > 1$. If $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$, then $\{\chi_N u\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{E}'(U)$, $\chi_N u = u$ on Ω, and*

$$|\widehat{\chi_N u}(\xi)| \le A\frac{h^N N!^{\tilde\tau^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor}}, \quad N\in\mathbf{N},\ \xi\in\mathbf{R}^d\backslash\{0\}. \quad (3.2.9)$$

That is, after enumeration $N \to \tilde\tau N$, $\{\chi_N u\}_{N\in\mathbf{N}}$ satisfies (3.2.1)*. for some $A, h > 0$.*

Proof. Put $u_N = \chi_N u$, $N \in \mathbf{N}$. By the Remark 3.2.2, $\{u_N\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{E}'(U)$. Note also that $u_N = u$ on Ω and $\operatorname{supp} u_N \subseteq K$.

Since $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$, from (3.2.2) for $|\alpha| \leq \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor$, $x \in \Omega$, and for some $k > 1$ we obtain

$$\begin{aligned}
|D^\alpha u_N(x)| &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta}\chi_N(x)||D^\beta u(x)| \\
&\leq ||u||_{\mathcal{E}_{\tau,\sigma,k}(\Omega)} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} A^{|\alpha-\beta|+1} \lfloor N^{1/\sigma} \rfloor^{|\alpha-\beta|} k^{|\beta|^\sigma} |\beta|^{\tau|\beta|^\sigma} \\
&\leq A||u||_{\mathcal{E}_{\tau,\sigma,k}(\Omega)} (2A)^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor} \lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor} k^{N/\tilde{\tau}} \lfloor N^{1/\sigma} \rfloor^{\frac{\tau N}{\tilde{\tau}}} \\
&\leq A||u||_{\mathcal{E}_{\tau,\sigma,k}(\Omega)} B^N N^{\frac{1}{\sigma}(\frac{1}{\tau})^{1/(\sigma-1)} N^{1/\sigma}} N^{\frac{1}{\sigma}(\frac{1}{\tau})^{1/(\sigma-1)} N} \qquad (3.2.10)
\end{aligned}$$

for some $B > 0$, where for the last inequality we have used that $\tilde{\tau} = \tau^{\sigma/(\sigma-1)}$.

Next we note that there exists $c > 0$, such that

$$N^{1/\sigma} \ln N \leq c N^{1/\sigma} N^{1-1/\sigma} = cN,$$

wherefrom $N^{\frac{1}{\sigma}(\frac{1}{\tau})^{1/(\sigma-1)} N^{1/\sigma}} \leq C^N$ for some $C > 1$ (which depends on τ and σ). Hence (3.2.10) can be estimated by

$$|D^\alpha u_N(x)| \leq A||u||_{\mathcal{E}_{\tau,\sigma,k}(\Omega)} h^N N^{\frac{1}{\sigma}(\frac{1}{\tau})^{1/(\sigma-1)} N}, \qquad (3.2.11)$$

for some $h > 0$. Applying the Fourier transform to (3.2.11) for $|\alpha| = \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor$ we obtain

$$|\widehat{u}_N(\xi)| \leq A||u||_{\mathcal{E}_{\tau,\sigma,k}(\Omega)} \frac{h^N N^{\frac{1}{\sigma}(\frac{1}{\tau})^{1/(\sigma-1)} N}}{|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\}. \qquad (3.2.12)$$

Finally, after the enumeration $N \to \tilde{\tau} N$, we note that (3.2.12) and Stirling's formula imply (3.2.9), and the proposition is proved. □

Remark 3.2.3. When $\sigma = 1$ and $\tau \neq 1$ the proof of Proposition 3.2.2 does not hold. In particular, when $0 < \tau < 1$ the order of quasi-analyticity of χ_N given by $\lfloor N^{1/\sigma} / \tau^{\frac{1}{\sigma-1}} \rfloor$ tends to infinity when $\sigma \to 1^+$ for fixed $N \in \mathbf{N}$, while for $\tau > 1$ it tends to zero. (see Remark 3.2.1). This suggests that in the study of the "critical" behavior when $\sigma \to 1^+$ the dependence of the parameter τ on σ becomes inevitable. For $\tau = \sigma = 1$ Proposition 3.2.2 coincides with necessity part of [16, Proposition 8.4.2.].

Now we define the wave front set in $\mathcal{D}'(U)$ with respect to the condition (3.2.1). Propositions 3.2.1 and 3.2.2 imply that the decay estimates from the condition (3.2.1) are related to the regularity defined by $\mathcal{E}_{\{\tau,\sigma\}}(U)$ from Chapter 1.

Definition 3.2.2. Let $\tau > 0$ and $\sigma > 1$, $\mathcal{D}'(U)$, $t > 1$, and $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $\mathrm{WF}_{(\tau,\sigma)}(u)$) if there exists open conic neighborhood $\Omega \times \Gamma$ of (x_0, ξ_0) and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and (3.2.1) holds for some constants $A, h > 0$ (resp. for every $h > 0$ there exists $A > 0$).

Remark 3.2.4. It follows immediately from the definition that $\mathrm{WF}_{\{\tau,\sigma\}}(u)$, $u \in \mathcal{D}'(U)$, is closed subset of $U \times \mathbf{R}^d \backslash \{0\}$. Note that for $\tau > 0$ and $\sigma > 1$

$$\mathrm{WF}_{\{\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{1,1\}}(u) = \mathrm{WF}_A(u).$$

Moreover, when $0 < \tau < 1$ and $\sigma = 1$ we have $\mathrm{WF}_A(u) \subseteq \mathrm{WF}_{\{\tau,1\}}(u)$. However, since Proposition 3.2.2 does not hold when $0 < \tau < 1$ and $\sigma = 1$ (see Remark 3.2.3), we are not able to prove the usual relation between $\mathrm{WF}_{\{\tau,1\}}(u)$ and the singular support of u, see Theorem 3.3.1. This suggests that the singularities related to $WF_{\{\tau,1\}}$ should be studied by a different approach (see [32]).

Remark 3.2.5. We would like to point out that our wave front sets $\mathrm{WF}_{\{\tau,\sigma\}}$ are not equal to WF_L, studied in [16], for any choice of $\tau > 0$, $\sigma > 1$ and the sequence L_p, $p \in \mathbf{N}$. Recall that Hörmander started the construction presented in Section 8.4. in [16] by imposing two conditions on the sequence L_p: (1) $L_p \geq p$ and (2) $L_{p+1} \leq CL_p$, for some $C > 0$ which does not depend on p. We also note that $L_p = M_p^{1/p}$, $p \in N$, where M_p are sequences used by Komatsu in [20], for the definition of classes of ultradifferentiable functions.

It would be sufficient to prove that sequence $L_p^{\tau,\sigma} := (M_p^{\tau,\sigma})^{1/p}$ does not satisfy Hörmander's condition (2), for any $\tau > 0$ and $\sigma > 1$. Since $L_p^{\tau,\sigma} = p^{\tau p^{\sigma-1}}$, it is clear that is satisfies (1) for every $\tau > 0$ and $\sigma > 1$. Assume that $\sigma > 2$. By the computations presented in Lemma 2.3.1 it follows that

$$\frac{(p+1)^{\tau(p+1)^{\sigma-1}}}{p^{\tau p^{\sigma-1}}} \geq (2p)^{\tau p^{\sigma-2}}, \quad p \in \mathbf{N},$$

which clearly implies we cannot choose $C > 0$ such that $(M_p^{\tau,\sigma})^{1/p}$ satisfy condition (2) for $\sigma > 2$.

For the case $1 < \sigma < 2$, we note that precise estimate gives

$$(p+1)^{\tau(p+1)^{\sigma-1}} \leq (p+1)^{\tau p^{\sigma-1}}(p+1)^{\tau} = p^{\tau p^{\sigma-1}}(1+\frac{1}{p})^{\tau p^{\sigma-1}}(p+1)^{\tau},$$

and if we assume that (2) holds for some large $C > 0$, the calculation above would imply that $(1+\frac{1}{p})^{\tau p^{\sigma-1}}(p+1)^{\tau} \leq C$ and this is true only for finitely many $p \in \mathbf{N}$.

3.3 Singular support related to the classes $\mathcal{E}_{\tau,\sigma}$

In this section define the notion of singular support related to $\mathcal{E}_{\tau,\sigma}$, $\tau > 0$, $\sigma > 1$ and prove that it is equal to the standard projection of $\mathrm{WF}_{\tau,\sigma}$. The following inequality, which holds for some $C > 0$, will be frequently used in the sequel:

$$\lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tau)^{1/\sigma} \rfloor} \leq N^{1/\sigma (1/\tau)^{1/\sigma} N} \leq C^N N!^{\tau^{-1/\sigma}/\sigma} \tag{3.3.1}$$

(the second inequality follows from Stirling's formula).

We start with the following Lemma.

Lemma 3.3.1. *Let* $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$ *and set* $\tau = \tau^{\sigma/(\sigma-1)}$. *Let* $K \subset\subset U$, F *be a closed cone, and* $\{\chi_N\}_{N\in\mathbf{N}}$ *be a* $\tilde{\tau}, \sigma$*-admissible sequence with respect to* K. *Then* $\{\chi_N u\}_{N\in\mathbf{N}}$ *is a bounded sequence in* $\mathcal{E}'(U)$, *and if* $\mathrm{WF}_{\{\tau,\sigma\}}(u) \cap (K \times F) = \emptyset$, *then for some* $A, h > 0$ *we have*

$$|\widehat{\chi_N u}(\xi)| \leq A \frac{h^N N!^{\tilde{\tau}^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N}, \xi \in F. \tag{3.3.2}$$

Proof. Let $(x_0, \xi_0) \in K \times F$, and set $r_0 := r_{x_0,\xi_0} > 0$. Furthermore, let $\{\chi_N\}_{N\in\mathbf{N}}$ be the $\tilde{\tau}, \sigma$-admissible sequence with respect to $\overline{B_{r_0}(x_0)}$, $\overline{B_{r_0}(x_0)} \subseteq \Omega \subseteq K$. Boundedness of $\{\chi_N u\}_{N\in\mathbf{N}}$ follows by Remark 3.2.2.

Since $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ we choose u_N, Ω and Γ as in Definition 3.2.2 so that

$$|\widehat{u}_N(\xi)| \leq A \frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\ \xi \in \Gamma, \tag{3.3.3}$$

for some $A, h > 0$. The condition (3.3.3) is equivalent to

$$|\widehat{u}_N(\xi)| \leq A \frac{h^N N!^{\tilde{\tau}^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\ \xi \in \Gamma, \tag{3.3.4}$$

after applying Stirling's formula and the enumeration $N \to N/\tilde{\tau}$.

Let Γ_0 be an open conical neighborhood of ξ_0 with the closure contained in Γ and choose $\varepsilon > 0$ such that $\xi - \eta \in \Gamma$ when $\xi \in \Gamma_0$ and $|\eta| < \varepsilon|\xi|$. Then, since $\chi_N u = \chi_N u_N$, we write

$$\widehat{\chi_N u}(\xi) = \Big(\int_{|\eta|<\varepsilon|\xi|} + \int_{|\eta|\geq\varepsilon|\xi|} \Big) \widehat{\chi}_N(\eta) \widehat{u}_N(\xi-\eta)\, d\eta = I_1 + I_2, \quad \xi \in \Gamma_0, N \in \mathbf{N}.$$

To estimate I_1, note that for $|\eta| < \varepsilon|\xi|$ we have

$$|\xi - \eta| \geq |\xi| - |\eta| > (1-\varepsilon)|\xi|.$$

Thus, by using (3.2.4) for $\alpha = 0$ and $|\beta| = d+1$ and (3.3.4), we obtain

$$\begin{aligned}|I_1| = \Big| \int_{|\eta|<\varepsilon|\xi|} \widehat{\chi}_N(\eta)\widehat{u}_N(\xi-\eta)\,d\eta \Big| \\ \leq \int_{|\eta|<\varepsilon|\xi|} |\widehat{\chi}_N(\eta)| A \frac{h^N N!^{\tilde{\tau}^{-1/\sigma}/\sigma}}{|\xi-\eta|^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}} d\eta \\ \leq A \frac{h^N N!^{\tilde{\tau}^{-1/\sigma}/\sigma}}{((1-\varepsilon)|\xi|)^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}} \int_{\mathbf{R}^d} \langle \eta \rangle^{-d-1} d\eta \\ \leq A_1 \frac{h_1^N N!^{\tilde{\tau}^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}}, \quad \xi \in \Gamma_0, N \in \mathbf{N}. \quad (3.3.5)\end{aligned}$$

To estimate I_2, note that for $|\eta| \geq \varepsilon|\xi|$ we have

$$|\xi - \eta| \leq |\xi| + |\eta| \leq (1 + 1/\varepsilon)|\eta|,$$

and thus, using (3.2.4) for $|\alpha| = \lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor$, together with (3.2.5) and (4.1.2), for every $\beta \in \mathbf{N}^d$ and some $M > 0$ we obtain

$$\begin{aligned}|I_2| &= \Big| \int_{|\eta|\geq\varepsilon|\xi|} \widehat{\chi}_N(\eta)\widehat{u}_N(\xi-\eta)\,d\eta \Big| \\ &\leq \frac{A_\beta^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor+1} \lfloor N^{1/\sigma}\rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}}{(\varepsilon|\xi|)^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}} \int_{|\eta|\geq\varepsilon|\xi|} \langle\eta\rangle^{-|\beta|} C \langle \xi-\eta\rangle^M \, d\eta \\ &\leq \frac{A_\beta^{N+1} \lfloor N^{1/\sigma}\rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}}{(\varepsilon|\xi|)^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}} \int_{\mathbf{R}^d} \langle\eta\rangle^{-|\beta|} \langle (1+1/\varepsilon)\eta\rangle^M , d\eta \\ &\leq \frac{A'^{N+1} N!^{\tilde{\tau}^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}} \quad \xi \in \Gamma_0,\end{aligned}$$

for some $A' > 0$, where for the last inequality we have chosen $|\beta| = M+d+1$.

Thus, the statement follows for $(x,\xi) \in B_{r_0}(x_0) \times \Gamma_0$.

In order to extend the result to $K \times F$ we use the same idea as in the proof of [16, Lemma 8.4.4]. Since the intersection of F with the unit sphere is a compact set, there exists a finite number of balls $B_{r_{x_0,\xi_j}}(x_0)$, and cones

Γ_j that covers F, $j \leq n$, $n \in \mathbf{Z}_+$, and note that (3.3.2) remains valid if $\{\chi_N\}_{N\in\mathbf{N}}$ is chosen so that $\operatorname{supp} \chi_N \subseteq B_{r_{x_0}} := \bigcap_{j=1}^{n} B_{r_{x_0,\xi_j}}(x_0)$, $\xi_j \in \Gamma_j$.

Moreover, since K is compact set, it is covered by a finite number of balls $B_{r_{x_k}}$, $k \leq n$, $n \in \mathbf{Z}_+$. By [20, Lemma 5.1.] there exist non-negative functions $\chi_k \in C_0^\infty(B_{r_{x_k}/2})$, $k \leq n$, such that $\sum_{k=1}^{n} \chi_k = 1$ on a neighborhood of K. Next, for every $N \in \mathbf{N}$ we choose a non-negative function $\phi_N \in C_0^\infty(B_{r_{x_k}/2})$ such that $\int \phi_N(x) = 1$ and

$$\sup_{x\in K} |D^\alpha \phi_N(x)| \leq C^{|\alpha|} \lfloor N^{1/\sigma} \rfloor^{|\alpha|},$$

for $|\alpha| \leq \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor$, where the constant $C > 0$ depends on τ and σ, cf. [16, Theorem 1.4.2.]. Now, for $\chi_{N,k} = \phi_N * \chi_k$, we have $\sum_{k=1}^{n} \chi_{N,k} = 1$ in a neighborhood of K, and each $\chi_{N,k}$, $1 \leq k \leq n$, satisfies (3.2.2).

To conclude the proof we note that if $\{\chi_N\}_{N\in\mathbf{N}}$ is a $\tilde{\tau}, \sigma$-admissible sequence with respect to K, then $\chi_N \chi_{N,k}$ also satisfies estimate of type (3.2.2), for $1 \leq k \leq n$. This follows by simple application of Leibniz rule. Thus, (3.3.2) holds if we replace χ_N by $\chi_N \chi_{N,k}$. Since $\sum_{k=1}^{n} \chi_N \chi_{N,k} = \chi_N$, the result follows. □

Now we can define singular support of distributions with respect to classes $\mathcal{E}_{\{\tau,\sigma\}}$.

Definition 3.3.1. Let $\tau > 0$ and $\sigma > 1$, $u \in \mathcal{D}'(U)$ and $x_0 \in U$. Then $x_0 \notin \operatorname{singsupp}_{\{\tau,\sigma\}}(u)$ if and only if there exists neighborhood Ω of x_0 such that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$.

Following theorem is consequence of Propositions 3.2.1, 3.2.2, and Lemma 3.3.1.

Theorem 3.3.1. *Let $\tau > 0$ and $\sigma > 1$, $u \in \mathcal{D}'(U)$. Let $\pi_1 : U \times \mathbf{R}^d \backslash \{0\} \to U$ be the standard projection given with $\pi_1(x,\xi) = x$. Then*

$$\operatorname{singsupp}_{\{\tau,\sigma\}}(u) = \pi_1(\mathrm{WF}_{\{\tau,\sigma\}}(u)) \,. \tag{3.3.6}$$

Proof. Choose $x_0 \notin \pi_1(\mathrm{WF}_{\{\tau,\sigma\}}(u))$. Then we can choose compact neighborhood K of x_0 such that $\mathrm{WF}_{\{\tau,\sigma\}}(u) \cap (K \times \mathbf{R}^d \backslash \{0\}) = \emptyset$. By Lemma 3.3.1, there exists a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on some open set Ω, and

$$|\widehat{u}_N(\xi)| \leq A \, \frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N}, \, \xi \in \mathbf{R}^d \backslash \{0\}. \tag{3.3.7}$$

holds for some $A, h > 0$. Then Proposition 3.2.1 implies that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$, that is, $x_0 \notin \text{singsupp}_{\{\tau,\sigma\}}(u)$.

Conversely, if $x_0 \notin \text{singsupp}_{\{\tau,\sigma\}}(u)$, then there exist neighborhood Ω of x_0 such that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$. Then by the Proposition 3.2.2, there exists a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and (3.3.7) holds. This completes the proof. □

To conclude this section we make a short comment about $\text{WF}_{(\tau,\sigma)}(u)$, $u \in \mathcal{D}'(U)$. Recall that for $0 < \tau < \rho$ and $\sigma > 1$ it holds

$$\mathcal{E}_{\{\tau,\sigma\}}(U) \hookrightarrow \mathcal{E}_{(\rho,\sigma)}(U) \hookrightarrow \mathcal{E}_{\{\rho,\sigma\}}(U). \tag{3.3.8}$$

(see Proposition 2.2.15). By what we have proved so far, the regularity related to the complement of $\text{WF}_{\{\tau,\sigma\}}$ is regularity of the classes $\mathcal{E}_{\{\tau,\sigma\}}$. Thus, combining result of the Proposition 2.2.15 with the results of this section we obtain the following Corollary.

Corollary 3.3.1. *Let* $u \in \mathcal{D}'(U)$, $t > 1$. *Then for* $0 < \tau < \rho$ *and* $\sigma > 1$ *it holds*

$$\text{WF}(u) \subseteq \text{WF}_{\{\rho,\sigma\}}(u) \subseteq \text{WF}_{(\rho,\sigma)}(u) \subseteq \text{WF}_{\{\tau,\sigma\}}(u) \subseteq \bigcap_{t>1} \text{WF}_t(u) \subseteq \text{WF}_A(u)\,,$$

where WF_t *and* WF_A *are Gevrey and analytic wave front sets, respectively.*

3.4 Microlocal property of PDO's with respect to $\text{WF}_{\{\tau,\sigma\}}$

In this section we prove the microlocal property of finite order PDO's with coefficients in $\mathcal{E}_{\{\tau,\sigma\}}(U)$ with respect to $\text{WF}_{\{\tau,\sigma\}}(u)$, $u \in \mathcal{D}'(U)$. In particular, following theorem holds.

Theorem 3.4.1. *Let*

$$P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$$

be a differential operator of order m *on* U *with* $a_\alpha \in \mathcal{E}_{\{\tau,\sigma\}}(U)$, $|\alpha| \leq m$, *and let* $u \in \mathcal{D}'(U)$, $\tau > 0, \sigma > 1$. *Then*

$$\text{WF}_{\{\tau,\sigma\}}(P(x,D)u) \subseteq \text{WF}_{\{\tau,\sigma\}}(u),$$

The statement directly follows from the next lemma.

Lemma 3.4.1. *Let* $u \in \mathcal{D}'(U)$, $\tau > 0, \sigma > 1$. *Then*

$$\mathrm{WF}_{\{\tau,\sigma\}}(\partial_j u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(u), \quad 1 \le j \le d.$$

If, in addition $\phi \in \mathcal{E}_{\{\tau,\sigma\}}(U)$, *then*

$$\mathrm{WF}_{\{\tau,\sigma\}}(\phi u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(u). \tag{3.4.1}$$

Proof. To prove the first part of the Lemma, fix $1 \le j \le d$. If $(x_0, \xi_0) \notin \mathrm{WF}_{\tau,\sigma}(u)$, then by the definition there exists a conical neighborhood $\Omega \times \Gamma$ of (x_0, ξ_0), and a bounded sequence $\{u_N\}$ in $u \in \mathcal{E}'(U)$ such that $u_N = u$ on Ω such that after the enumeration $N \to N^\sigma$ we obtain

$$|\widehat{u}_N(\xi)| \le A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\, \xi \in \Gamma, \tag{3.4.2}$$

for some $A, h > 0$ (resp. for every $h > 0$ there exists $A > 0$.)

For $x_0 \in \Omega$, we note that

$$|\widehat{\partial_j u}_{N+1}(\xi)| \le A|\xi| \frac{h^N (N+1)^{\tau(N+1)^\sigma}}{|\xi|^{N+1}} \le A_1 \frac{h_1^N N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\, \xi \in \Gamma, \tag{3.4.3}$$

where for the second inequality we used the $\widetilde{(M.2)}'$ property of $M_p^{\tau,\sigma}$,

For the second part, set $\tilde{\tau} = \tau^{\frac{\sigma}{\sigma-1}}$ and fix $(x_0, \xi_0) \notin WF_{\{\tau,\sigma\}}(u)$. Then by the definition, there exists open conic neighborhood $\Omega \times \Gamma$ of (x_0, ξ_0) and a bounded sequence $\{u_N\}_{N \in \mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and

$$|\widehat{u}_N(\xi)| \le A \frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\, \xi \in \Gamma. \tag{3.4.4}$$

Choose a compact neighborhood $K_{x_0} \subset\subset \Omega$ of x_0, and let $\{\chi_N\}_{N \in N}$ be $\tilde{\tau}, \sigma$-admissible sequence with respect K_{x_0}. Set $\widetilde{\chi}_N = \phi \chi_N$, $N \in \mathbf{N}$, and note that $\widetilde{\chi}_N u = \widetilde{\chi}_N u_N$. Since $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ satisfies $\widetilde{(M.2)}'$ (see Lemma 2.1.2) for some positive increasing sequence C_q, $q \in \mathbf{N}$, and $h > 1$ we obtain

$$\begin{aligned}
|D^{\alpha+\beta} \widetilde{\chi}_N(x)| &\le \sum_{\delta \le \alpha} \sum_{\gamma \le \beta} \binom{\alpha}{\delta} \binom{\beta}{\gamma} |D^{\alpha-\delta+\beta-\gamma} \widetilde{\chi}_N(x)| |D^{\gamma+\delta} \phi(x)| \\
&\le \sum_{\delta \le \alpha} \sum_{\gamma \le \beta} \binom{\alpha}{\delta} \binom{\beta}{\gamma} A_\beta^{|\alpha-\delta|+1} \lfloor N^{1/\sigma} \rfloor^{|\alpha-\delta|} h^{|\gamma+\delta|^\sigma+1} |\gamma+\delta|^{\tau|\gamma+\delta|^\sigma} \\
&\le (2h')^{|\beta|^\sigma+1} \sum_{\delta \le \alpha} \binom{\alpha}{\delta} A_\beta^{|\alpha-\delta|+1} \lfloor N^{1/\sigma} \rfloor^{|\alpha-\delta|} (C_\beta h')^{|\delta|^\sigma} |\delta|^{\tau|\delta|^\sigma},
\end{aligned} \tag{3.4.5}$$

for $x \in K_{x_0}$, $|\alpha| \le \lfloor (N/\tilde\tau)^{1/\sigma} \rfloor$, $\beta \in \mathbf{N}^d$, where $h' = h^{2^{\sigma-1}}$. Now it is clear that by putting $|\alpha| = 0$ in (3.4.5) we obtain,

$$|D^\beta \widetilde\chi_N(x)| \le C'_\beta, \quad x \in K_{x_0},$$

and hence by applying Fourier transform it follows

$$|\widehat{\widetilde\chi}_N(\xi)| \le C'_\beta \langle \xi \rangle^{-|\beta|}, \quad \beta \in \mathbf{N}^d, \xi \in \Gamma,$$

for suitable $C'_\beta > 0$. In particular, since $\mathcal{E}_{\{\tau,\sigma\}}(U) \hookrightarrow C^\infty(U)$ it follows that $\widetilde\chi_N = \phi\chi_N$, $N \in \mathbf{N}$, is bounded in $C^\infty(U)$ and hence $\widetilde\chi_N u$, $N \in \mathbf{N}$, is bounded in $\mathcal{E}'(U)$.

Moreover, note that by the same type of estimates as in (3.2.10) for $|\alpha| = \lfloor (N/\tilde\tau)^{1/\sigma} \rfloor$ and $\beta \in \mathbf{N}^d$, by (3.4.5) we obtain that

$$|D^{\alpha+\beta} \widetilde\chi_N(x)| \le C''^{N+1}_\beta N^{\frac{1}{\sigma}(\frac{1}{\tilde\tau})^{1/\sigma} N}, \beta \in \mathbf{N}^d, x \in K_{x_0}$$

and hence after applying Fourier transform it follows

$$|\widehat{\widetilde\chi}_N(\xi)| \le C''^{N+1}_\beta N^{\frac{1}{\sigma}(\frac{1}{\tilde\tau})^{1/\sigma} N} \langle \xi \rangle^{-|\alpha|-|\beta|}, \quad \beta \in \mathbf{N}^d, \xi \in \Gamma, \tag{3.4.6}$$

for some constants $C''_\beta > 0$.

Now using (3.4.4) and (3.4.6) and arguing in the same way as in the proof of Lemma 3.3.1, one can find open cone $\Gamma_0 \subseteq \Gamma$ such that

$$|\widehat{\widetilde\chi_N u}(\xi)| \le A \, \frac{h^N N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma} N}}{|\xi|^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\ \xi \in \Gamma_0,$$

for suitable $A, h > 0$. After enumeration $N \to \tilde\tau N$ the statement follows. □

3.5 Intersections and unions of $\mathrm{WF}_{\tau,\sigma}$

In this section we consider the intersections and unions of the wave front sets $\mathrm{WF}_{\tau,\sigma}$, $\tau > 0$, $\sigma > 1$. It turns out that, from the microlocal point of view, the regularity related to the complement of these unions and intersections coincides with the regularity proposed by the classes (2.2.24)-(2.2.27) from Chapter 2.

In particular, for $u \in \mathcal{D}'(U)$, we consider

$$\mathrm{WF}_{0^+,1^+}(u) = \bigcap_{\sigma>1} \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \tag{3.5.1}$$

$$\mathrm{WF}_{\infty,1^+}(u) = \bigcap_{\sigma>1} \bigcup_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \tag{3.5.2}$$

$$\mathrm{WF}_{0^+,\infty}(u) = \bigcup_{\sigma>1} \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \tag{3.5.3}$$

$$\mathrm{WF}_{\infty,\infty}(u) = \bigcup_{\sigma>1} \bigcup_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u). \tag{3.5.4}$$

where we recall that $\mathrm{WF}_{\tau,\sigma}(u)$ denotes $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ or $\mathrm{WF}_{(\tau,\sigma)}(u)$ for $\tau \in [0,\infty]$ and $\sigma \in [1,\infty]$.

Remark 3.5.1. Recall (cf. Corollary 3.3.1),

$$\mathrm{WF}_{\{\rho,\sigma\}}(u) \subseteq \mathrm{WF}_{(\rho,\sigma)}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(u)\,, u \in \mathcal{D}'(U). \tag{3.5.5}$$

Moreover, we conclude that $\bigcap\limits_{\tau>0} \mathrm{WF}_{\{\tau,\sigma\}}(u) = \bigcap\limits_{\tau>0} \mathrm{WF}_{(\tau,\sigma)}(u)$ and $\bigcup\limits_{\tau>0} \mathrm{WF}_{\{\tau,\sigma\}}(u) = \bigcup\limits_{\tau>0} \mathrm{WF}_{(\tau,\sigma)}(u)$, and hence in (3.5.1)-(3.5.4) it is sufficient to observe the intersections and unions of $\mathrm{WF}_{\{\tau,\sigma\}}(u)$.

We start with the following technical result.

Lemma 3.5.1. *Let* $u \in \mathcal{D}'(U)$*, and* $\sigma_2 > \sigma_1 \geq 1$*. Then*

$$\bigcup_{\tau>0} \mathrm{WF}_{\tau,\sigma_2}(u) \subseteq \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma_1}(u)\,.$$

Proof. Let $(x_0,\xi_0) \notin \bigcap_{\tau>0} \mathrm{WF}_{\{\tau,\sigma_1\}}(u)$. Then there exists $\tau_0 > 0$ such that $(x_0,\xi_0) \notin \mathrm{WF}_{\{\tau_0,\sigma_1\}}(u)$. Hence there exists open conic neighborhood $\Omega \times \Gamma$ of (x_0,ξ_0) and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω such that, after enumeration $N \to N^{\sigma_1}$ (see also Lemma 2.1.2),

$$|\widehat{u}_N(\xi)| \leq A \frac{h^{N^{\sigma_1}} N^{\tau_0 N^{\sigma_1}}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \Gamma, \tag{3.5.6}$$

for some constants $A, h > 0$.

We need to prove that for every $\tau > 0$, $(x_0,\xi_0) \notin \mathrm{WF}_{\{\tau,\sigma_2\}}(u)$. This follows easily from (3.5.6), noting that (see the proof of the [29, Proposition 2.1.]) for every $\tau > 0$ and $h > 0$ there exists $A_1 > 0$ such that

$$h^{N^{\sigma_1}} N^{\tau_0 N^{\sigma_1}} \leq A_1 h^{N^{\sigma_2}} N^{\tau N^{\sigma_2}}, \quad N \in \mathbf{N},$$

and the Lemma is proved.

□

As an immediate consequence of the Lemma 3.5.1 we obtain the following Corollary.

Corollary 3.5.1. *For $u \in \mathcal{D}'(U)$, it holds*

$$\begin{aligned} \mathrm{WF}(u) &\subseteq \mathrm{WF}_{0^+,1^+}(u) \subseteq \mathrm{WF}_{\infty,1^+}(u) \\ &\subseteq \mathrm{WF}_{0^+,\infty}(u) \subseteq \mathrm{WF}_{\infty,\infty}(u) \subseteq \bigcap_{\tau>1} \mathrm{WF}_\tau(u)\,, \end{aligned} \tag{3.5.7}$$

where WF *and* WF_τ *are standard and Gevrey wave front sets, respectively.*

Proof. Note that the last inclusion follow directly from Lemma 3.5.1 for $\sigma_2 > \sigma_1 = 1$ by taking the unions and intersections with respect to $\tau > 1$. The only nontrivial inclusion left is the third one. Assume that $(x_0,\xi_0) \notin \mathrm{WF}_{0^+,\infty}(u)$, that is, for every $\sigma > 1$ $(x_0,\xi_0) \notin \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u)$. Fix some $\sigma = \sigma_1 > 1$ and let $\sigma_2 > \sigma_1$. By Lemma 3.5.1 it follows that $(x_0,\xi_0) \notin \bigcup_{\tau>0} \mathrm{WF}_{\tau,\sigma_2}(u)$. Hence it follows that that there exists $\sigma > 1$ such that for every $\tau > 0$ $(x_0,\xi_0) \notin \mathrm{WF}_{\tau,\sigma}(u)$ and therefore $(x_0,\xi_0) \notin \mathrm{WF}_{\infty,1^+}(u)$. □

Let us extend the definition of singular support (see Chapter3) related to classes $\mathcal{E}_{\{\tau,\sigma\}}(U)$, $\tau \in (0,\infty)$ and $\sigma \in (1,\infty)$ to the borderline cases.

Definition 3.5.1. Let $\tau \in [0,\infty]$ and $\sigma \in [1,\infty]$, $u \in \mathcal{D}'(U)$ and $x_0 \in U$. Then $x_0 \notin \mathrm{singsupp}_{\{\tau,\sigma\}}(u)$ if and only if there exists neighborhood Ω of x_0 such that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$.

Let $\pi_1 : U \times \mathbf{R}^d \backslash \{0\} \to U$ denotes the standard projection given by $\pi_1(x,\xi) = x$. From Propositions 3.2.1, 3.2.2, and Lemma 3.3.1 it follows that for a given $u \in \mathcal{D}'(U)$, $\tau > 0$ and $\sigma > 1$, we have $\mathrm{singsupp}_{\{\tau,\sigma\}}(u) = \pi_1(\mathrm{WF}_{\{\tau,\sigma\}}(u))$.

For the borderline cases $\tau \in \{0,\infty\}$ and $\sigma \in \{1,\infty\}$ we have the following.

Theorem 3.5.1. *Let $\pi_1 : U \times \mathbf{R}^d \backslash \{0\} \to U$ be the standard projection given with $\pi_1(x,\xi) = x$. Then*

$$\pi_1(\mathrm{WF}_{\infty,\infty}(u)) = \mathrm{singsupp}_{0^+,0^+}(u)\,, \tag{3.5.8}$$

$$\pi_1(\mathrm{WF}_{0^+,0^+}(u)) = \mathrm{singsupp}_{\infty,\infty}(u)\,, \tag{3.5.9}$$

$$\pi_1(\mathrm{WF}_{\infty,1^+}(u)) = \mathrm{singsupp}_{0^+,\infty}(u)\,, \tag{3.5.10}$$

$$\pi_1(\mathrm{WF}_{0^+,\infty}(u)) = \mathrm{singsupp}_{\infty,1^+}(u). \tag{3.5.11}$$

Proof. Recall that classes $\mathcal{E}_{\infty,1^+}$ have the ultradifferentiable property, that is, for every $\sigma > 1$ classes $\mathcal{E}_{\infty,\sigma}$ are closed under the action of ultradifferentiable operators of the class τ,σ (see Definition 2.6.1 and Theorem 2.6.2 from Chapter 2).

We prove here only $\pi_1(\mathrm{WF}_{0^+,\infty}(u)) = \mathrm{singsupp}_{\infty,1^+}(u)$ and leave the other equalities to the reader.

Assume that $x_0 \notin \pi_1(\mathrm{WF}_{0^+,\infty}(u))$, so that there is a compact neighborhood $K \subset\subset U$ of x_0 such that

$$K \times \mathbf{R}^d \backslash \{0\} \subseteq (\mathrm{WF}_{0^+,\infty}(u))^c = \bigcap_{\sigma>1} \bigcup_{\tau>0} (\mathrm{WF}_{\{\tau,\sigma\}}(u))^c, \tag{3.5.12}$$

where $(\mathrm{WF}_{\{\tau,\sigma\}}(u))^c$ denotes the complement of the set $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ in $U \times \mathbf{R}^d \backslash \{0\}$. Therefore, if $(x,\xi) \in K \times \mathbf{R}^d \backslash \{0\}$ then for every $\sigma > 1$ there exist $\tau_0 > 0$ such that $(x,\xi) \notin \mathrm{WF}_{\{\tau_0,\sigma\}}(u)$.

Let $\sigma > 1$ be arbitrary but fixed, and set $\tilde{\tau}_0 = \tau_0^{\sigma/(\sigma-1)}$. From Lemma 3.3.1, it follows that there is a $\tilde{\tau}_0, \sigma$-admissible sequence $\{\chi_N\}_{N \in \mathbf{N}}$ such that $u_N = \chi_N u$, $N \in \mathbf{N}$ is a bounded sequence in $\mathcal{E}'(U)$, $u_N = u$ on some $\Omega \subseteq K$, and

$$|\widehat{\chi_N u}(\xi)| \leq A \frac{h^N N!^{\tilde{\tau}_0^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde{\tau}_0)^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N}\,, \xi \in \mathbf{R}^d \backslash \{0\}\,,$$

which after enumeration $N \to \tilde{\tau}_0 N$ becomes

$$|\widehat{\chi_N u}(\xi)| \leq A \frac{h^N N!^{\tau_0/\sigma}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N}\,, \xi \in \mathbf{R}^d \backslash \{0\}\,. \tag{3.5.13}$$

By Proposition 3.2.1 it follows that $u \in \mathcal{E}_{\{\tau_0,\sigma\}}(U)$, and since σ can be chosen arbitrary, we conclude that $u \in \mathcal{E}_{\infty,1^+}(U)$ (see Proposition 2.2.1). Therefore $\mathrm{singsupp}_{\infty,1^+}(u) \subset \pi_1(\mathrm{WF}_{0^+,\infty}(u))$.

For the opposite inclusion, assume that $x_0 \notin \mathrm{singsupp}_{\infty,1^+}(u)$. Then $u \in \mathcal{E}_{\infty,1^+}(\Omega)$, for some Ω which is a neighborhood of x_0. In particular, for every $\sigma > 1$ there exists $\tau_0 > 0$ such that $u \in \mathcal{E}_{\tau_0,\sigma}(\Omega)$. Fix $\sigma > 1$ and put $\tilde{\tau} = \tau_0^{\sigma/(\sigma-1)}$. Now we use a $\tilde{\tau}_0, \sigma$-admissible sequence $\{\chi_N\}_{N \in \mathbf{N}}$ and Proposition 3.2.2 implies (3.5.13). It follows that $(x_0,\xi) \in (\mathrm{WF}_{\{\tau_0,\sigma\}}(u))^c$ for every $\sigma > 1$ and for some $\tau_0 > 0$. Hence, by the equality in (3.5.12) it follows that $(x_0,\xi) \notin \mathrm{WF}_{0^+,\infty}(u)$ for every $\xi \in \mathbf{R}^d \backslash \{0\}$ and therefore $x_0 \notin \pi_1(\mathrm{WF}_{0^+,\infty}(u))$, wherefrom $\pi_1(\mathrm{WF}_{0^+,\infty}(u)) \subset \mathrm{singsupp}_{\infty,1^+}(u)$, which finishes the proof. $\square$

Chapter 4

Microlocal analysis of solutions to the PDE's

In this Chapter we analyze propagation of singularities of solutions to linear PDE's within our framework. Following the ideas presented in Theorem 8.6.1. in [16] for WF_A we use τ, σ- admissible sequences in our construction. In the first section we consider constant coefficients PDO's. In that case Hörmanders operators R_j commutes, and by that we are able to modify the standard construction and to be more rigorous in the calculation. However, it is not possible to prove the microlocal hypoellipticity of PDO's with respect to $\mathrm{WF}_{\tau,\sigma}$, and by considering $\mathrm{WF}_{0^+,\infty}$ we obtain the desired result.

We also prove the more general result when PDO's are with coefficients in $\mathcal{E}_{\{\tau,\sigma\}}$. In this case the proof is non-trivially different since it involves some of the techniques used in Chapter 2, Section 2.7.

Main results of this Chapter are published in [30]. See also [31].

4.1 Microlocal hypoellipticity of constant coefficients PDO's

In the sequel we prove the following result.

Theorem 4.1.1. *Let $\tau > 0$, $\sigma > 1$. Let $u \in \mathcal{D}'(U)$ and $P(D)$ the constant coefficient differential operator of order m. Then if $P(D)u = f$ in $\mathcal{D}'(U)$, it holds*

$$\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(f) \subseteq \mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(f) \cup \mathrm{Char}(P) \tag{4.1.1}$$

Proof. The first embedding in (4.2.49) follows form Corollary 3.4.1. We will prove the second embedding for the operators with constant coefficients, and note that the proof is more technical for the non-constant coefficient since we can not use commutation relations of some operators in that case. However, the idea of the proof is the same, so this more general case will be discussed later.

Therefore it remains to prove that

$$\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(P(D)u) \cup \mathrm{Char}(P(D)).$$

The following inequality, which holds for $\tau > 0$, $\sigma > 1$ and for some $C > 0$, will be frequently used:

$$\lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tau)^{1/\sigma} \rfloor} \leq N^{N\tau^{-1/\sigma}/\sigma} \leq C^N N!^{\tau^{-1/\sigma}/\sigma}. \tag{4.1.2}$$

Assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(P(D)u) \cup \mathrm{Char}(P(D))$. Then there exists a compact set K containing x_0 and a closed cone Γ containing ξ_0 such that $P_m(x, \xi) \neq 0$ when $(x, \xi) \in K \times \Gamma$ and $(K \times \Gamma) \cap \mathrm{WF}_{\{\tau,\sigma\}}(P(D)u) = \emptyset$.

Let $\tilde{\tau} = \tau^{\frac{\sigma}{\sigma-1}}$ and let $\{\chi_N\}_{N \in \mathbf{N}}$, be a $\tilde{\tau}, \sigma$-admissible sequence with respect to K.

Put $u_N = \chi_{2^\sigma N} u$, $N \in \mathbf{N}$, so that

$$\widehat{u}_N(\xi) = \int u(x) \chi_{2^\sigma N}(x) e^{-ix\xi} dx, \quad \xi \in \mathbf{R}^d, \; N \in \mathbf{N}.$$

The easy part of the proof is the estimate of $|\widehat{u}_N(\xi)|$, $N \in \mathbf{N}$, for "small" values of $\xi \in \Gamma$, that is when $|\xi| \leq \lfloor N^{1/\sigma} \rfloor$. In fact, since $\{u_N\}_{N \in \mathbf{N}}$ is bounded in $\mathcal{E}'(U)$, Paley-Wiener theorems (see [21]), and the fact that $e^{-ix\cdot\xi} \in C^\infty(\mathbf{R}^d_x)$, for every $\xi \in \mathbf{R}^d$, implies that $|\widehat{u}_N(\xi)| = |\langle u_N, e^{-i\cdot\xi} \rangle| \leq C \langle \xi \rangle^M$, for some $C, M > 0$ independent of N. Hence, from (4.1.2) we have

$$|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} |\widehat{u_N}(\xi)| \leq \lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} |\widehat{u_N}(\xi)| \leq A C^N N^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma} N},$$

where $A, C > 0$ do not depend on N. After enumeration $N \to \tilde{\tau} N$ we obtain

$$|\widehat{u_N}(\xi)| \leq A \frac{C^N N^{\frac{\tilde{\tau}^{1-1/\sigma}}{\sigma} N}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}} \leq A \frac{h^N N!^{\frac{\tau}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}},$$

which estimates $|\widehat{u_N}(\xi)|$ when $\xi \in \Gamma$, $|\xi| \leq \lfloor N^{1/\sigma} \rfloor$, $N \in \mathbf{N}$.

It remains to estimate $|\widehat{u}_N(\xi)|$, when $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$ and for $N \in \mathbf{N}$ large enough (so that $N \to \infty$ implies $|\xi| \to \infty$).

As in the proof of [16, Theorem 8.6.1], in Subsection 4.1.1 we use the technique of approximate solution (see also [34, Theorem 1, Section 1.6]) to obtain

$$\chi_{2^\sigma N}(x) e^{-ix\cdot\xi} = P^T(D) \left(\frac{e^{-ix\cdot\xi}}{P_m(\xi)} w_N(x, \xi) \right) + e_N(x, \xi) e^{-ix\xi} \tag{4.1.3}$$

$x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, that is, the following representation holds:

$$\begin{aligned}\widehat{u}_N(\xi) &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int u(x) P^T(D) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(\xi)} \right) dx \\ &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int P(D) u(x) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(\xi)} \right) dx, \quad (4.1.4)\end{aligned}$$

where

$$w_N(x,\xi) = \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \ldots a_m} (R_1^{a_1} R_2^{a_2} \ldots R_m^{a_m} \chi_{2^\sigma N})(x,\xi), \qquad (4.1.5)$$

$$e_N(x,\xi) = \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde\tau})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tilde\tau})^{\frac{1}{\sigma}} \rfloor - m+k} \binom{|a|}{a_1, ..., a_m} (R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})(x,\xi), \qquad (4.1.6)$$

$x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, and we put $\mathfrak{S} = a_1 + 2a_2 + \cdots + m a_m$.

The derivation of (4.1.4) and the calculation of $w_N(x,\xi)$ and $e_N(x,\xi)$ is done in Subsections 4.1.1 and 4.1.2, so we continue with the estimation of the first term in (4.1.4).

Estimated number of terms in $e_N(x,\xi)$ given in Subsection 4.1.1, and the estimates of $D^\beta(R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})$ given by (4.1.30) (Subsection 4.1.3) imply

$$\begin{aligned}|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| &\leq A \sum_{|\alpha| \leq M} |D_x^\alpha (e_N(x,\xi) e^{-ix\xi})| \\ &\leq A \sum_{|\alpha| \leq M} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D_x^{\alpha-\beta} e^{-ix\xi}| |D_x^\beta e_N(x,\xi)| \\ &\leq A |\xi|^M |\xi|^{-\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde\tau)^{1/\sigma} \rfloor - M} C^N N!^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}} \\ &= A \frac{C^N N!^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde\tau)^{1/\sigma} \rfloor}}, x \in K, \xi \in \Gamma, \qquad (4.1.7)\end{aligned}$$

for suitable constants $A, C > 0$ and $|\xi|$ large enough. After enumeration $N \to \tilde\tau 2^{\sigma-1} N$, (4.1.7) is equivalent to

$$|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| \leq A \frac{C^N N!^{\frac{\tau 2^{\sigma-1}}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad x \in K, \xi \in \Gamma,$$

which estimates the first term on the righthand side of (4.1.4). In fact, we will use a slightly weaker estimate which is obtained from (4.1.7) after enumeration

$$N \to N + \lceil \tilde\tau 2^{\sigma-1} (M+d+1)^\sigma \rceil. \qquad (4.1.8)$$

It remains to estimate the second term on the righthand side of (4.1.4) for $|\xi| > \lfloor N^{1/\sigma} \rfloor$. This is the hardest part of the proof. By the Lemma 3.3.1 there exists a bounded sequence $\{f_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $f_N = f = P(D)u$ in a neighborhood of K and there exists a cone V such that $\overline{\Gamma} \subset V$ and

$$|\mathcal{F}(f_N)(\eta)| \leq A \frac{h^N N!^{\frac{\tilde{\tau}-1/\sigma}{\sigma}}}{|\eta|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}}, \quad \eta \in V. \tag{4.1.9}$$

Since $\{\chi_{2^\sigma N}(x)\}_{N\in\mathbf{N}}$ is bounded in $C_0^\infty(U)$, by the Paley-Wiener theorem (see also Remark 3.2.2) it follows that for every $\tilde{M} > 0$ there exists $C > 0$ which does not depend on N so that $|\widehat{\chi}_{2^\sigma N}(\eta)| \leq C\langle\eta\rangle^{-\tilde{M}}$, $N \in \mathbf{N}$. From $\operatorname{supp}\chi_N \subseteq K$, $N \in \mathbf{N}$, it follows that

$$\pi_1(\operatorname{supp} w_N(x,\xi)) \subseteq K, \quad N \in \mathbf{N},$$

and since $f_N = f$ in a neighborhood of K, we have $w_N f = w_N f_{N'}$ in $\mathcal{D}'(U)$, where we put $N' = N - \lceil 2^{\sigma-1}\tilde{\tau}(M+d+1)^\sigma \rceil$. Therefore (and since $\mathcal{F}(g_1 \cdot g_2)(\xi) = (\mathcal{F}(g_1) * \mathcal{F}(g_2))(\xi)))$

$$\begin{aligned}\langle f(\cdot)e^{-i\xi\cdot}, w_N(\cdot,\xi)/P_m(\xi)\rangle &= \frac{1}{P_m(\xi)}\mathcal{F}_{x\to\xi}(f_{N'}(x)w_N(x,\xi))(\xi) \\ &= \frac{1}{P_m(\xi)}\int_{\mathbf{R}^d}\mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta)\,d\eta = I_1 + I_2,\end{aligned}$$

where

$$I_1 = \frac{1}{P_m(\xi)}\int_{|\eta|<\varepsilon|\xi|}\mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta,\xi)\,d\eta, \tag{4.1.10}$$

$$I_2 = \frac{1}{P_m(\xi)}\int_{|\eta|\geq\varepsilon|\xi|}\mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta,\xi)\,d\eta, \tag{4.1.11}$$

and $0 < \varepsilon < 1$ is chosen so that $\xi - \eta \in V$ when $\xi \in \Gamma$, $\xi > \lfloor N^{1/\sigma} \rfloor$, and $|\eta| < \varepsilon|\xi|$.

Since $|\eta| < \varepsilon|\xi|$ implies $|\xi-\eta| \geq (1-\varepsilon)|\xi|$, by using the computation of

$\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)$ from Subsection 4.1.4, we estimate I_1 as follows:

$$\begin{aligned}|I_1| &\leq \frac{1}{|P_m(\xi)|}\int_{|\eta|<\varepsilon|\xi|}|\mathcal{F}(f_{N'})(\xi-\eta)||\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta\\ &\leq \int_{|\eta|<\varepsilon|\xi|}A\frac{h^{N'}N'!^{\tau/\sigma}}{|\xi-\eta|^{\lfloor N'^{1/\sigma}\rfloor}}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta\\ &\leq A\frac{h^{N'}N'!^{\tau/\sigma}}{((1-\varepsilon)|\xi|)^{\lfloor N'^{1/\sigma}\rfloor}}\int_{|\eta|<\varepsilon|\xi|}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta\\ &\leq A_1\frac{h_1^{N'}N'!^{\tau/\sigma}}{|\xi|^{\lfloor N'^{1/\sigma}\rfloor}}C^{\lfloor (N/\tau)^{1/\sigma}\rfloor}\int_{\mathbf{R}^d}|\widehat{\chi}_{2^\sigma N}(\eta)|\,d\eta\\ &\leq A_2\frac{h_2^{N'}N'!^{\tau/\sigma}}{|\xi|^{\lfloor N'^{1/\sigma}\rfloor}},\quad \xi\in\Gamma,|\xi|>\lfloor N^{1/\sigma}\rfloor.\end{aligned}\tag{4.1.12}$$

We used the Paley-Wiener theorem for $\{\widehat{\chi}_{2^\sigma N}\}$ and trivial inequality $|P_m(\xi)|\geq 1$ when $|\xi|>\lfloor N^{1/\sigma}\rfloor$.

It remains to estimate I_2. Note that $|\eta|\geq\varepsilon|\xi|$ implies $|\xi-\eta|\leq(1+1/\varepsilon)|\eta|$, and by Paley-Wiener type estimates we have $|\mathcal{F}(f_{N'})(\eta)|\leq C\langle\eta\rangle^M$, where $C>0$ does not depend on N'. Therefore

$$\begin{aligned}|I_2| &\leq \frac{1}{|P_m(\xi)|}\int_{|\eta|\geq\varepsilon|\xi|}|\mathcal{F}f_{N'}(\xi-\eta)||\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta\\ &\leq A\int_{|\eta|\geq\varepsilon|\xi|}\langle\xi-\eta\rangle^M\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+d+1}\frac{|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|}{\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+d+1}}\,d\eta\\ &\leq C^{N+1}\frac{\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+M+d+1}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor}}|\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta,\xi)|,\end{aligned}$$

when $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$.

To finish the proof, we show that if $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$ then there exists $h>0$ such that

$$\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+M+d+1}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\leq h^{N+1}N^{\frac{\tilde\tau-1/\sigma}{\sigma}N}.\tag{4.1.13}$$

Since $N'=N-\lceil 2^{\sigma-1}\tilde\tau(M+d+1)^\sigma\rceil$, it follows that

$$(N/\tilde\tau)^{1/\sigma}=\Big(\frac{N'+\lceil 2^{\sigma-1}\tilde\tau(M+d+1)^\sigma\rceil}{\tau}\Big)^{1/\sigma}\geq 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}+M+d+1.\tag{4.1.14}$$

If $\mathfrak{S}\leq\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor-m$, $|\beta|=\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor$ then

$$\mathfrak{S}+|\beta|<2\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor\leq\lfloor 2(N/\tilde\tau)^{1/\sigma}\rfloor,\tag{4.1.15}$$

From (4.1.15), when $x \in K$ and $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$ it follows that

$$|D^\beta w_N(x,\xi)| \le \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \dots a_m} \sup_{x\in K} |(D^\beta R_1^{a_1} R_2^{a_2} \dots R_m^{a_m} \chi_{2^\sigma N})(x,\xi)|$$

$$\le \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \dots a_m} |\xi|^{-\mathfrak{S}} C^{\mathfrak{S}+|\beta|+1} \lfloor N^{1/\sigma} \rfloor^{\mathfrak{S}+|\beta|}$$

$$\le \lfloor N^{1/\sigma} \rfloor^{|\beta|} \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \dots a_m} C^{\mathfrak{S}+|\beta|+1} \le C'^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor + 1} \lfloor N^{1/\sigma} \rfloor^{|\beta|}.$$

Since $\pi_1(\operatorname{supp} w_N(x,\xi)) \subseteq K$ and $|\beta| = \lfloor (N/\tilde\tau)^{1/\sigma} \rfloor$, we obtain

$$|\eta|^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor} |\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)| \le C'^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor + 1} \lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor} \le C''^{N+1} N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma} N}, \tag{4.1.16}$$

where we used the first part of (4.1.2). Now (4.1.14) and (4.1.16) gives

$$\sup_{\eta \in \mathbf{R}^d} \langle \eta \rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}} (N'/\tilde\tau)^{1/\sigma} \rfloor + M + d + 1} |\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|$$

$$\le \sup_{\eta \in \mathbf{R}^d} \langle \eta \rangle^{\lfloor (N/\tilde\tau)^{1/\sigma} \rfloor} |\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)| \le C''^{N+1} N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma} N}, \tag{4.1.17}$$

and (4.1.13) follows. Therefore

$$|I_2| \le A \frac{h^N N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma} N}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}} (N'/\tilde\tau)^{1/\sigma} \rfloor}}, \tag{4.1.18}$$

for suitable constants $A, h > 0$. After enumeration given by (4.1.8), and using $(M.2)'$ property of the sequence $N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma} N}$, we conclude that (4.1.18) is equivalent to

$$|I_2| \le A \frac{h^N N!^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde\tau)^{1/\sigma} \rfloor}}, \tag{4.1.19}$$

for some $A, h > 0$. After enumeration $N \to \tilde\tau 2^{\sigma-1} N$ we finally obtain

$$|\widehat{u}_N(\xi)| \le A \frac{h^N N!^{\frac{\tau 2^{\sigma-1}}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}},$$

for some $A, h > 0$, and the proof is finished.

4.1.1 Derivation of the representation of $\widehat{u}_N(\xi)$

Formally, we are searching for $v(x,\xi)$ so that

$$\widehat{u}_N(\xi) = \int u(x)\chi_{2^\sigma N}(x)e^{-ix\xi}dx = \int u(x)P^T(D)v(x,\xi)dx,$$

$\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma}\rfloor$, where $P^T(D) = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}a_\alpha D^\alpha$ is the transpose operator of $P(D)$, and $v(x,\xi)$ is the solution of the equation

$$P^T(D)v(x,\xi) = \chi_{2^\sigma N}(x)e^{-ix\xi}, \quad x \in K, \xi \in \Gamma, |\xi| > \lfloor N^{1/\sigma}\rfloor. \qquad (4.1.20)$$

Note that $\dfrac{e^{-ix\xi}\chi_{2^\sigma N}}{P_m(\xi)}$ solves equation (4.1.20) approximately for large ξ. Led by the calculation done in Chapter 1, Section 1.3, we choose $v(x,\xi)$ of the form $v(x,\xi) = \dfrac{e^{-ix\xi}w(x,\xi)}{P_m(\xi)}$, for some $w(\cdot,\xi) \in C^\infty(K)$, where $x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma}\rfloor$.

Then (4.1.20) becomes

$$(I - R(\xi))w(x,\xi) = \chi_{2^\sigma N}(x) \quad x \in K, \xi \in \Gamma, |\xi| > \lfloor N^{1/\sigma}\rfloor, \qquad (4.1.21)$$

where $R(\xi) = \sum_{j=1}^m R_j(\xi)$, $R_j(\xi) = p_j(\xi)\sum_{|\alpha|\leq j} a_\alpha D^\alpha$, and $p_j(\xi)$ are homogeneous functions of order $-j$. In fact, formal calculation gives

$$\begin{aligned}
&e^{ix\xi}P^T(D)\Big(\frac{w(x,\xi)e^{-ix\xi}}{P_m(\xi)}\Big) \\
&\quad = e^{ix\xi}\frac{1}{P_m(\xi)}\sum_{|\alpha|\leq m}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-1)^{|\alpha|}a_\alpha D^{\alpha-\beta}(e^{-ix\xi})D^\beta w(x,\xi) \\
&\quad\quad = \sum_{|\alpha|\leq m}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-1)^{|\alpha|}a_\alpha\Big(\frac{(-\xi)^{\alpha-\beta}}{P_m(\xi)}\Big)D^\beta w(x,\xi),
\end{aligned}$$

for $x \in K$ and $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma}\rfloor$. Since $\dfrac{(-\xi)^{\alpha-\beta}}{P_m(\xi)}$ is homogeneous of order $|\alpha| - |\beta| - m$ with respect to ξ, it follows that (4.1.20) would imply (4.1.21).

Now, successive applications of the operator R in (4.1.21) give

$$R^{k-1}(\xi)w(x,\xi) - R^k(\xi)w(x,\xi) = R^{k-1}(\xi)\chi_{2^\sigma N}(x), \quad x \in K, \xi \in \Gamma, |\xi| > \lfloor N^{1/\sigma}\rfloor,$$

for every $k \in \{1, \dots, N\}$, so that after summing up those N equalities we obtain

$$w(x,\xi) - R^N(\xi) w(x,\xi) = \sum_{k=0}^{N-1} R^k(\xi) \chi_{2^\sigma N}(x),$$

which gives formal approximate solution

$$\begin{aligned} w(x,\xi) &= \sum_{k=0}^{\infty} R^k \chi_{2^\sigma N}(x,\xi) \\ &= \sum_{|a|=0}^{\infty} \binom{|a|}{a_1, a_2, \dots, a_m} R_1^{a_1} R_2^{a_2} \dots R_m^{a_m} \chi_{2^\sigma N}(x,\xi). \quad (4.1.22) \end{aligned}$$

The operators $R_k^{a_k}(\xi)$, $1 \le k \le m$, are of order less then or equal to ka_k and homogeneous of order $-ka_k$ with respect to ξ. Since $P(D)$ have constant coefficients, the operators R_j commute, and we used the generalized Newton formula, cf. [35].

We proceed with the following approximation procedure. We consider partial sums

$$w_N(x,\xi) = \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \dots a_m} (R_1^{a_1} R_2^{a_2} \dots R_m^{a_m} \chi_{2^\sigma N})(x,\xi),$$

$\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, and $N \in \mathbf{N}$ is large enough, so that (4.1.21) takes the form (4.1.3) and the error term e_N is given by:

$$e_N(x,\xi) = \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+k} \binom{|a|}{a_1, ..., a_m} (R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})(x,\xi).$$

The precise calculation which leads to (4.1.3) is given in Subsection 4.1.2. Note that the number of terms in (4.1.6) is bounded by $4 \cdot 2^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor}$, since from $\binom{n}{k} \le 2^n$, $k \le n$, $n \in \mathbf{N}$, we obtain

$$\binom{|a|}{a_1, a_2, \dots a_m} \le 2^{|a|} 2^{|a|-a_1} \dots 2^{|a|-a_1-\dots-a_{m-2}} \le 2^{a_1+2a_2+\dots+ma_m},$$

and therefore

$$\sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+k} \binom{|a|}{a_1, \dots, a_m} \le \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+k} 2^{a_1+2a_2 \dots + ma_m}$$

$$\leq 2^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m + 1} \sum_{k=1}^{m} 2^k \leq 4 \cdot 2^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor}, \tag{4.1.23}$$

where we put $\mathfrak{S} = a_1 + 2a_2 + \cdots + ma_m$.

4.1.2 The calculation of the error term

For multinomial coefficients

$$\binom{|a|}{a_1, a_2, \ldots a_m} := \binom{|a|}{a_1}\binom{|a| - a_1}{a_2} \ldots \binom{|a| - a_1 - \cdots - a_{m-2}}{a_{m-1}}$$
$$= \frac{|a|!}{a_1! a_2! \ldots a_m!}, \quad |a| = a_1 + a_2 + \cdots + a_m,\ a_k \in \mathbf{N},\ k \leq m, \tag{4.1.24}$$

a generalization of Pascal's triangle equality for the binomial formula gives

$$\binom{|a|}{a_1, ..., a_m} = \sum_{k=1}^{m} \binom{|a| - 1}{a_1, ..., a_k - 1, ...a_m}, \quad |a| \geq 1, \tag{4.1.25}$$

wherefrom for $|a| \geq 1$, and putting $\mathfrak{S} = a_1 + 2a_2 + \cdots + ma_m$ we obtain

$$\sum_{\mathfrak{S}=0}^{(\lfloor \frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$= \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m} \Big(\sum_{k=1}^{m} \binom{|a| - 1}{a_1, ..., a_k - 1, ...a_m} \Big) R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$= \sum_{k=1}^{m} \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m - k} \binom{|a|}{a_1, ..., a_k, ...a_m} R_1^{a_1} ... R_k^{a_k + 1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$= \sum_{k=1}^{m} R_k \Big(\sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m - k} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N} \Big), \tag{4.1.26}$$

where for the second equality we interchange the summation and substitute a_k with $a_k + 1$.

Hence, for $|a| \geq 0$ we have

$$
\begin{aligned}
(I-R)w_N = & \sum_{\mathfrak{S}=0}^{(\lfloor \frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m} \binom{|a|}{a_1,...,a_m} R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N} \\
& - \sum_{k=1}^{m} R_k \Big(\sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m-k} \binom{|a|}{a_1,...,a_m} R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N} \\
& + \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m-k+1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m} \binom{|a|}{a_1,...,a_m} R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N} \Big) \\
= \chi_{2^\sigma N} & - \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m-k+1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m} \binom{|a|}{a_1,...,a_m} R_1^{a_1}...R_k^{a_k+1}...R_m^{a_m}\chi_{2^\sigma N} \\
= \chi_{2^\sigma N} & - \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m+1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m+k} \binom{|a|}{a_1,...,a_m} R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N}, \quad (4.1.27)
\end{aligned}
$$

where for the second equality we used (4.1.26) and for the last one we substitute a_k with $a_k - 1$.

Therefore, if we set

$$
e_N(x,\xi) = \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor - m+1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor - m+k} \binom{|a|}{a_1,...,a_m} (R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N})(x,\xi),
$$

then the computation of this subsection gives the equality (4.1.3), which in turn implies the fundamental representation (4.1.4).

4.1.3 Estimates for $D^\beta(R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N})$

Note that for N large enough we have

$$
(\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M)^\sigma \leq 2^{\sigma-1}(N/\tilde{\tau} + M^\sigma) < 2^\sigma N/\tilde{\tau}
$$

so that for $|\beta| \leq M$ the following estimate holds:

$$
\mathfrak{S} + |\beta| \leq \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M = \lfloor (N/\tilde{\tau})^{1/\sigma} + M \rfloor < \lfloor 2(N/\tilde{\tau})^{1/\sigma} \rfloor .
$$

Thus, for $x \in K$, $\xi \in \Gamma$, and $\mathfrak{S} \geq \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor - m$, by using (4.1.2) we obtain

$$\begin{aligned} |D^\beta (R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})(x,\xi)| &\leq |\xi|^{-\mathfrak{S}} A^{\mathfrak{S}+|\beta|+1} \lfloor N^{1/\sigma} \rfloor^{\mathfrak{S}+|\beta|} \\ &\leq |\xi|^{m-\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} A^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M + 1} \lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M} \\ &\leq |\xi|^{m-\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} C^{N+1} N^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma} N}, \qquad (4.1.28) \end{aligned}$$

for some $C > 0$, which is, after enumeration $N \to N + 2^{\sigma-1}\tilde{\tau}(m+M)^\sigma$ bounded by

$$|\xi|^{m-\lfloor ((N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma)/\tilde{\tau})^{1/\sigma} \rfloor} A^{N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma+1} \times (N + 2^{\sigma-1}\tilde{\tau}(m+M)^\sigma)^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma}(N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma)},$$

for some $A > 0$. Moreover,

$$\begin{aligned} \Big(\frac{N + 2^{\sigma-1}\tilde{\tau}(m+M)^\sigma}{\tilde{\tau}} \Big)^{1/\sigma} &\geq 2^{\frac{1-\sigma}{\sigma}} ((N/\tilde{\tau})^{1/\sigma} + 2^{\frac{\sigma-1}{\sigma}}(m+M)) \\ &= 2^{\frac{1-\sigma}{\sigma}} (N/\tilde{\tau})^{1/\sigma} + m + M. \qquad (4.1.29) \end{aligned}$$

Finally, (4.1.29), $(M.2)'$ property of $N^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma} N}$ and Stirling's formula give the estimate

$$|D^\beta R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}(x)| \leq |\xi|^{-\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde{\tau})^{1/\sigma} \rfloor - M} C^{N+1} N!^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma}} \qquad (4.1.30)$$

for some $C > 0$.

4.1.4 The computation of $\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)$

From

$$(R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})(x,\xi) = \prod_{j=1}^m p_j^{a_j}(\xi) \sum_{|\alpha| \leq \mathfrak{S}} c_\alpha D^\alpha \chi_{2^\sigma N}(x)$$

for suitable constants c_α, it follows that

$$\mathcal{F}_{x\to\eta}(R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})(\eta,\xi) = \prod_{j=1}^m p_j^{a_j}(\xi) \sum_{|\alpha| \leq \mathfrak{S}} c''_\alpha \eta^\alpha \widehat{\chi}_{2^\sigma N}(\eta),$$

so that

$$\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi) = \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \dots a_m} \Big(\prod_{j=1}^m p_j^{a_j}(\xi) \Big) \sum_{|\alpha| \leq \mathfrak{S}} c''_\alpha \eta^\alpha \widehat{\chi}_{2^\sigma N}(\eta).$$

Note that the number of terms in $\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)$ is bounded by $C2^{\lfloor (N/\tau)^{1/\sigma}\rfloor}$ for some $C>0$ which does not depend on N.

When $|\eta|\le\varepsilon|\xi|$, $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$, and N sufficiently large we have

$$|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\le \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor-m}\binom{|a|}{a_1,a_2\dots a_m}\Big(\prod_{j=1}^{m}(|p_j(\xi)||\varepsilon\xi|^j)^{a_j}\Big)\sum_{|\alpha|\le\mathfrak{S}}c''_\alpha|\widehat{\chi}_{2^\sigma N}(\eta)|$$
$$\le AC^{\lfloor (N/\tau)^{1/\sigma}\rfloor}|\widehat{\chi}_{2^\sigma N}(\eta)|,$$

for some $A,C>0$, and we used

$$\prod_{j=1}^{m}(|p_j(\xi)||\varepsilon\xi|^j)^{a_j}\le A\varepsilon^{\mathfrak{S}}\le A,\quad \xi\in\Gamma,|\xi|>\lfloor N^{1/\sigma}\rfloor,$$

which follows from $\varepsilon<1$ and the fact that $\prod_{j=1}^{m}(|p_j(\xi)||\xi|^j)^{a_j}$ is homogeneous of order zero. □

4.2 Microlocal hypoellipticity of non-constant coefficients PDO's

In the finial section we extend the result of the Theorem 4.1.1 to the case of the partial differential operators with non-constant coefficients. In particular, we prove the following result.

Theorem 4.2.1. *Let* $u\in\mathcal{D}'(U)$ *and* $P(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha$ *be partial differential operator of order* m *with the coefficients* $a_\alpha(x)\in\mathcal{E}_{\{\tau,\sigma\}}(U)$. *Then if* $P(x,D)u=f$ *in* $\mathcal{D}'(U)$, *it holds*

$$\mathrm{WF}_{\{2^{2\sigma-1}\tau,\sigma\}}(f)\subseteq\mathrm{WF}_{\{2^{2\sigma-1}\tau,\sigma\}}(u)\subseteq\mathrm{WF}_{\{\tau,\sigma\}}(f)\cup\mathrm{Char}(P(x,D)).\quad(4.2.1)$$

Proof. The first embedding in (4.2.1) is given by Theorem 3.4.1, so it remains to prove that

$$\mathrm{WF}_{\{2^{2\sigma-1}\tau,\sigma\}}(u)\subseteq\mathrm{WF}_{\{\tau,\sigma\}}(f)\cup\mathrm{Char}(P(x,D)).$$

Assume that $(x_0,\xi_0)\notin\mathrm{WF}_{\{\tau,\sigma\}}(P(x,D)u)\cup\mathrm{Char}(P(x,D))$. Then there exists a compact set K containing x_0 and a closed cone Γ containing ξ_0 such that $P_m(x,\xi)\ne0$ when $(x,\xi)\in K\times\Gamma$ and

$$(K\times\Gamma)\cap\Big(\mathrm{WF}_{\{\tau,\sigma\}}(P(x,D)u)\cup\mathrm{Char}(P(x,D))\Big)=\emptyset.$$

Since K is fixed, the distributions involved in the proof are of finite order.

Let $\tilde{\tau} = \tau^{\frac{\sigma}{\sigma-1}}$ and let $\{\chi_N\}_{N\in\mathbf{N}}$, be a $\tilde{\tau}, \sigma$-admissible sequence with respect to K.

Put $u_N = \chi_{2^\sigma N} u$, $N \in \mathbf{N}$, so that

$$\widehat{u}_N(\xi) = \int u(x)\chi_{2^\sigma N}(x) e^{-ix\xi} dx, \quad \xi \in \mathbf{R}^d,\ N \in \mathbf{N}.$$

The estimate of $|\widehat{u}_N(\xi)|$ when $|\xi| \le \lfloor N^{1/\sigma} \rfloor$ is the easiest part of the proof. In fact, the estimate (3.2.5) together with

$$\lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tau)^{1/\sigma} \rfloor} \le N^{N\tau^{-1/\sigma}/\sigma} \le C^N N!^{\tau^{-1/\sigma}/\sigma}.$$

gives

$$|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} |\widehat{u_N}(\xi)| \le \lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} |\widehat{u_N}(\xi)| \le A C^N N^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma} N},$$

where $A, C > 0$ do not depend on N. After enumeration $N \to \tilde{\tau} N$ we obtain

$$|\widehat{u_N}(\xi)| \le A \frac{C^N N^{\frac{\tilde{\tau}^{1-1/\sigma}}{\sigma} N}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}} \le A \frac{h^N N!^{\frac{\tau}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}},$$

which estimates $|\widehat{u_N}(\xi)|$ when $\xi \in \Gamma$, $|\xi| \le \lfloor N^{1/\sigma} \rfloor$, $N \in \mathbf{N}$.

It remains to estimate $|\widehat{u}_N(\xi)|$ when $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$ and for $N \in \mathbf{N}$ large enough. As in the proof of [16, Theorem 8.6.1], we search for appropriate functions $w_N(x,\xi)$ and $e_N(x,\xi)$, $x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, so that

$$\chi_{2^\sigma N}(x) = e^{ix\cdot\xi} P^T(x,D) \left(\frac{e^{-ix\cdot\xi}}{P_m(x,\xi)} w_N(x,\xi) \right) + e_N(x,\xi), \tag{4.2.2}$$

$x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$.

The identity (4.2.2) implies that

$$\begin{aligned} \widehat{u}_N(\xi) &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int u(x) P^T(x,D) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(x,\xi)} \right) dx \\ &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int P(x,D) u(x) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(x,\xi)} \right) dx, \end{aligned} \tag{4.2.3}$$

$x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$.

Put

$$\mathcal{K}_1 = \{k \in \mathbf{N} \,|\, 0 \le mk \le \lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m\}, \tag{4.2.4}$$

and

$$\mathcal{K}_2 = \{k \in \mathbf{N} \,|\, \lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m < mk \le \lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor \}. \tag{4.2.5}$$

We refer to Subsection 4.2.1 for calculations which lead to

$$w_N(x,\xi) = \sum_{k\in\mathcal{K}_1} R^k == \sum_{k\in\mathcal{K}_1} \sum_{\mathfrak{S}_k=0}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} (R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N})(x,\xi), \tag{4.2.6}$$

$$e_N(x,\xi) = \sum_{k\in\mathcal{K}_2} R^k = \sum_{k\in\mathcal{K}_2} \sum_{\mathfrak{S}_k=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor} (R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N})(x,\xi), \tag{4.2.7}$$

$x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, $\mathfrak{S}_k = j_1+j_2+\dots+j_k$, $j_i \in \{1,\dots,m\}$, $1 \le i \le k$.

Moreover

$$R_j(x,\xi) = \sum_{|\alpha|\le j} c_{\alpha,j}(x,\xi) D^\alpha,$$

for suitable functions $c_{\alpha,j}(x,\xi)$ which are homogeneous of order $-j$ and

$$|D^\beta c_{\alpha,j}(x,\xi)| \le |\xi|^{-j} A h^{|\beta|^\sigma} |\beta|^{\tau|\beta|^\sigma}, \quad \beta \in \mathbf{N}^d, x \in K, \xi \in \Gamma \tag{4.2.8}$$

for some $A, h > 0$ and $|\alpha| \le j$. Again, we refer to Subsection 4.2.1 for details concerning the representation (4.2.3) and proceed with estimating terms on the right hand side of (4.2.3).

The number of summands in $w_N(x,\xi)$ and $e_N(x,\xi)$ is bounded by $A \cdot C^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}$ for some constants $A, C > 0$, which is the same number as in the case of operators with constant coefficients (when operators R_j commute), see Remark 4.2.1.

Since the operators R_j, $1 \le j \le m$ do not commute, in the sequel we use different arguments than in [29]. If M denotes the order of distribution u, then the estimates of $D^\beta(R_{j_1}...R_{j_k}\chi_{2^\sigma N})$ from Subsection 4.2.3 (cf. (4.2.48)) imply

$$|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| \le A \sum_{|\alpha|\le M} |D_x^\alpha (e_N(x,\xi) e^{-ix\xi})| \tag{4.2.9}$$

$$\le A|\xi|^M |\xi|^{-\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde{\tau})^{1/\sigma} \rfloor - M} C^N N!^{\frac{\tilde{\tau}-1/\sigma}{\sigma}} = A \frac{C^N N!^{\frac{\tilde{\tau}-1/\sigma}{\sigma}}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde{\tau})^{1/\sigma} \rfloor}},$$

$x \in K$, $\xi \in \Gamma$, for suitable constants $A, C > 0$ and $|\xi|$ large enough. After enumeration $N \to \tilde{\tau} 2^{\sigma-1} N$ we conclude that (4.2.9) is equivalent to

$$|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| \le A \frac{C^N N!^{\frac{\tau 2^{\sigma-1}}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad x \in K, \xi \in \Gamma,$$

which is the estimate for the first term on the righthand side of (4.2.3).

To estimate the second term on the righthand side of (4.2.3) for $|\xi| > \lfloor N^{1/\sigma} \rfloor$, we choose $\{f_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ (see Lemma 3.3.1) such that $f_N = f = P(x,D)u$ in a neighborhood of K and there exists a cone V such that $\overline{\Gamma} \subset V$ and

$$|\mathcal{F}(f_N)(\eta)| \le A \frac{h^N N!^{\frac{\tilde{\tau}-1/\sigma}{\sigma}}}{|\eta|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}}, \quad \eta \in V. \tag{4.2.10}$$

Note that $w_N f = w_N f_{N'}$ in $\mathcal{D}'(U)$, where we put

$$N' = N - \lceil 2^{\sigma-1}\tilde{\tau}(M+d+1)^\sigma \rceil,$$

and M denotes the order of distribution f. Therefore

$$\begin{aligned} \langle f(\cdot)e^{-i\xi\cdot}, w_N(\cdot,\xi)/P_m(\cdot,\xi)\rangle &= \mathcal{F}_{x\to\xi}(f_{N'}(x)\frac{w_N(x,\xi)}{P_m(x,\xi)})(\xi) \\ &= \int_{\mathbf{R}^d} \mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)\, d\eta = I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{|\eta|<\varepsilon|\xi|} \mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)\, d\eta, \tag{4.2.11}$$

$$I_2 = \int_{|\eta|\ge\varepsilon|\xi|} \mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)\, d\eta, \tag{4.2.12}$$

and $0 < \varepsilon < 1$ is chosen so that $\xi - \eta \in V$ when $\xi \in \Gamma$, $\xi > \lfloor N^{1/\sigma} \rfloor$, and $|\eta| < \varepsilon|\xi|$.

Let $j_1, \dots, j_k \in \{1, \dots, m\}$ be fixed. Since $\chi_{2^\sigma N}(\cdot)$, $N \in \mathbf{N}$, is bounded in $C_0^\infty(K)$, note that $R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N}(\cdot,\xi)$, $N \in \mathbf{N}$, is also bounded in $C_0^\infty(K)$ for every $\xi \in \Gamma$. Moreover, since the coefficients of $P_m(\cdot,\xi)$ are in $C^\infty(U)$ and $P_m(x,\xi) \neq 0$ when $x \in K$ and $\xi \in \Gamma$, it follows that $\dfrac{R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N}(\cdot,\xi)}{P_m(\cdot,\xi)}$, $N \in \mathbf{N}$, is bounded in $C_0^\infty(K)$ when $\xi \in \Gamma$, and moreover it is homogeneous of order $-m - \mathfrak{S}_k$. Hence, by Paley-Wiener type estimates it follows that for every $\tilde{M} > 0$ there exists $C > 0$, independent of $N \in \mathbf{N}$, such that

$$\begin{aligned} |\mathcal{F}_{x\to\eta}\Big(\frac{R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N}(x,\xi)}{P_m(x,\xi)}\Big)(\eta,\xi)| &\le C|\xi|^{-m-\mathfrak{S}_k}\langle\eta\rangle^{-\tilde{M}} \\ &\le C\langle\eta\rangle^{-\tilde{M}}, \quad \eta \in \mathbf{R}^d, \end{aligned}$$

when $\xi \in \Gamma$ and $|\xi| > \lfloor N^{1/\sigma} \rfloor \ge 1$.

When $\tilde{M} = d+1$ this estimate, together with (4.2.6), implies that there exists $C > 0$ such that

$$\begin{aligned} \Big|\mathcal{F}_{x\to\eta}\Big(\frac{w_N(x,\xi)}{P_m(x,\xi)}\Big)\Big| &\leq \sum_{k\in\mathcal{K}_1} \sum_{\mathfrak{S}_k=0}^{\lfloor(\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m} \Big|\mathcal{F}_{x\to\eta}\Big(\frac{R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N}(x,\xi)}{P_m(x,\xi)}\Big)(\eta,\xi)\Big| \\ &\leq C^{\lfloor(N/\tilde{\tau})^{1/\sigma}\rfloor+1}\langle\eta\rangle^{-d-1}. \end{aligned} \tag{4.2.13}$$

Since $|\eta| < \varepsilon|\xi|$ implies $|\xi-\eta| \geq (1-\varepsilon)|\xi|$, by using (4.2.13), we estimate I_1 as follows:

$$\begin{aligned} |I_1| &\leq \int_{|\eta|<\varepsilon|\xi|} |\mathcal{F}(f_{N'})(\xi-\eta)||\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)}(\eta,\xi)|\,d\eta \\ &\leq \int_{|\eta|<\varepsilon|\xi|} A\frac{h^{N'}N'!^{\tau/\sigma}}{|\xi-\eta|^{\lfloor N'^{1/\sigma}\rfloor}}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)}(\eta,\xi)|\,d\eta \\ &\leq A\frac{h^{N'}N'!^{\tau/\sigma}}{((1-\varepsilon)|\xi|)^{\lfloor N'^{1/\sigma}\rfloor}}\int_{\mathbf{R}^d} C^{\lfloor(N/\tilde{\tau})^{1/\sigma}\rfloor+1}\langle\eta\rangle^{-d-1}\,d\eta \\ &\leq A_2\frac{h_2^{N'}N'!^{\tau/\sigma}}{|\xi|^{\lfloor N'^{1/\sigma}\rfloor}}, \quad \xi\in\Gamma, |\xi| > \lfloor N^{1/\sigma}\rfloor. \end{aligned} \tag{4.2.14}$$

It remains to estimate I_2. Note that $|\eta| \geq \varepsilon|\xi|$ implies $|\xi-\eta| \leq (1+1/\varepsilon)|\eta|$. Moreover, since f is distribution of order M by Paley-Wiener type estimates we have $|\mathcal{F}(f_{N'})(\eta)| \leq C\langle\eta\rangle^M$, where $C > 0$ does not depend on N'. Therefore

$$\begin{aligned} |I_2| &\leq \int_{|\eta|\geq\varepsilon|\xi|} |\mathcal{F}f_{N'}(\xi-\eta)||\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)|\,d\eta \\ &\leq \int_{|\eta|\geq\varepsilon|\xi|} \langle\xi-\eta\rangle^M\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde{\tau})^{1/\sigma}\rfloor+d+1}\frac{|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)|}{\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde{\tau})^{1/\sigma}\rfloor+d+1}}\,d\eta \\ &\leq C^{N+1}\frac{\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde{\tau})^{1/\sigma}\rfloor+M+d+1}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)|}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde{\tau})^{1/\sigma}\rfloor}}, \end{aligned}$$

when $\xi\in\Gamma$, $|\xi| > \lfloor N^{1/\sigma}\rfloor$.

To finish the proof, we show that if $\xi\in\Gamma$, $|\xi| > \lfloor N^{1/\sigma}\rfloor$ then there exists $h > 0$ such that

$$\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde{\tau})^{1/\sigma}\rfloor+M+d+1}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)| \leq h^{N+1}N^{\frac{2^\sigma\tilde{\tau}-1/\sigma}{\sigma}N}. \tag{4.2.15}$$

Arguing in the similar way as in the proof of [30, Theorem 1.1], it is sufficient to prove

$$\sup_{x\in K}|D^\beta \frac{w_N(x,\xi)}{P_m(x,\xi)}| \le C^{N+1} N^{\frac{2^\sigma \tilde{\tau}-1/\sigma}{\sigma}N}, \quad \beta \le \lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor \tag{4.2.16}$$

for some $C>0$, when $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$. Recall (see Subsection 4.2.1),

$$\sup_{x\in K}\Big|D^\gamma \frac{1}{P_m(x,\xi)}\Big| \le |\xi|^{-m} C^{|\gamma|^\sigma+1}|\gamma|^{\tau|\gamma|^\sigma}, \quad \gamma\in\mathbf{N}^d, \xi\in\Gamma,$$

for some $C>0$. Moreover, from (4.2.48) (see Subsection 4.2.3) it follows that

$$\sup_{x\in K}|D^\gamma w_N(x,\xi)| \le C'^{N+1}\sum_{k\in\mathcal{K}_1}\sum_{\mathfrak{S}_k=0}^{\lfloor(\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor-m}|\xi|^{-\mathfrak{S}_k} \sum_{a_k\le\mathfrak{S}_k+|\gamma|}\binom{\mathfrak{S}_k+|\gamma|}{a_k} C^{\mathfrak{S}_k+|\gamma|-a_k}(\mathfrak{S}_k+|\gamma|-a_k)^{\tau(\mathfrak{S}_k+|\gamma|-a_k)^\sigma}\lfloor N^{1/\sigma}\rfloor^{a_k}, \tag{4.2.17}$$

for some $C'>0$, when $\xi\in\Gamma$.

Hence, for $x\in K$ and $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$ we obtain

$$\begin{aligned}|D^\beta\frac{w_N(x,\xi)}{P_m(x,\xi)}| &\le \sum_{k\in\mathcal{K}_1}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}|D^{\beta-\gamma}\frac{1}{P_m(x,\xi)}||D^\gamma w_N(x,\xi)|\\ &\le C'^{N+1}\sum_{k\in\mathcal{K}_1}\sum_{\gamma\le\beta}\sum_{\mathfrak{S}_k=0}^{\lfloor(\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor-m}\sum_{a_k\le\mathfrak{S}_k+|\gamma|}\binom{\beta}{\gamma}\binom{\mathfrak{S}_k+|\beta|}{a_k}\\ &\quad C^{|\beta-\gamma|^\sigma+1}|\beta-\gamma|^{\tau|\beta-\gamma|^\sigma}C^{\mathfrak{S}_k+|\gamma|-a_k}(\mathfrak{S}_k+|\gamma|-a_k)^{\tau(\mathfrak{S}_k+|\gamma|-a_k)^\sigma}\lfloor N^{1/\sigma}\rfloor^{a_k-\mathfrak{S}_k-m}\\ &\le C''^{N+1}\sum_{k\in\mathcal{K}_1}\lfloor N^{1/\sigma}\rfloor^{|\beta|-m}\sum_{\gamma\le\beta}\sum_{\mathfrak{S}_k=0}^{\lfloor(\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor-m}\sum_{a_k\le\mathfrak{S}_k+|\gamma|}\\ &\quad\binom{\beta}{\gamma}\binom{\mathfrak{S}_k+|\beta|}{a_k}(\mathfrak{S}_k+|\beta|-a_k)^{\tau(\mathfrak{S}_k+|\beta|-a_k)^\sigma},\end{aligned} \tag{4.2.18}$$

for $\beta\le\lfloor(N/\tilde{\tau})^{1/\sigma}\rfloor$, where we used $(M.1)$ property of the sequence $M_p^{\tau,\sigma}=p^{\tau p^\sigma}$.

Further observe that

$$\begin{aligned}(\mathfrak{S}_k+|\beta|-a_k)^{\tau(\mathfrak{S}_k+|\beta|-a_k)^\sigma} &\leq (\lfloor 2(N/\tilde{\tau})^{1/\sigma}\rfloor)^{\tau(\lfloor 2(N/\tilde{\tau})^{1/\sigma}\rfloor)^\sigma}\\ &\leq C'^N N^{\frac{2^\sigma\tilde{\tau}^{-1/\sigma}}{\sigma}N}, \end{aligned} \qquad (4.2.19)$$

and

$$\lfloor N^{1/\sigma}\rfloor^{|\beta|-m} \leq \lfloor N^{1/\sigma}\rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor} \leq C''^N, \qquad (4.2.20)$$

for some $C', C'' > 0$. Using the estimate for number of terms in w_N, by (4.2.18), (4.2.19) and (4.2.20), the estimate (4.2.16) follows.

Now by the similar arguments as in the proof of [30, Theorem 1.1], we conclude that

$$\begin{aligned}&\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde{\tau})^{1/\sigma}\rfloor+M+d+1}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)|\\ &\quad\leq \sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor (N/\tilde{\tau})^{1/\sigma}\rfloor}|\mathcal{F}_{x\to\eta}(\frac{w_N(x,\xi)}{P_m(x,\xi)})(\eta,\xi)| \leq C''^{N+1}N^{\frac{2^\sigma\tilde{\tau}^{-1/\sigma}}{\sigma}N}, \end{aligned} \qquad (4.2.21)$$

where we used (4.1.2) and

$$\pi_1(\operatorname{supp}\frac{w_N(x,\xi)}{P_m(x,\xi)}) \subseteq K.$$

Hence (4.2.15) follows.

Therefore

$$|I_2| \leq A\frac{h^N N^{\frac{2^\sigma\tilde{\tau}^{-1/\sigma}}{\sigma}N}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde{\tau})^{1/\sigma}\rfloor}}, \qquad (4.2.22)$$

for suitable constants $A, h > 0$. After enumeration

$$N \to N + \lceil\tilde{\tau}2^{\sigma-1}(M+d+1)^\sigma\rceil, \qquad (4.2.23)$$

(so that $N' \to N$) and using $(M.2)'$ property of the sequence $N^{\frac{2^\sigma\tilde{\tau}^{-1/\sigma}}{\sigma}N}$, we conclude that (4.2.22) is equivalent to

$$|I_2| \leq A\frac{h^N N!^{\frac{2^\sigma\tilde{\tau}^{-1/\sigma}}{\sigma}}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N/\tilde{\tau})^{1/\sigma}\rfloor}}, \qquad (4.2.24)$$

for some $A, h > 0$. After enumeration $N \to \tilde{\tau}2^{\sigma-1}N$ we finally obtain

$$|\widehat{u}_N(\xi)| \leq A\frac{h^N N!^{\frac{\tau 2^{2\sigma-1}}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}},$$

for some $A, h > 0$, and the proof is finished.

□

Remark 4.2.1. We would like to point out that the number of terms in sums (4.2.6) and (4.2.7) is bounded by same expression as for the case when operators R_j commute. In particular, note that in that case by using Newton's generalized formula we obtain

$$\begin{aligned} e_N(x,\xi) &= \sum_{k\in\mathcal{K}_2} R^k = \sum_{k\in\mathcal{K}_2} \sum_{a_1+\cdots+a_m=k} \binom{|a|}{a_1,\ldots,a_m} R_1^{a_1}\ldots R_m^{a_m} \\ &= \sum_{\lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor - m < m(a_1+\cdots+a_m) \le \lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor} \binom{|a|}{a_1,\ldots,a_m} R_1^{a_1}\ldots R_m^{a_m}. \quad (4.2.25) \end{aligned}$$

Since $m > 1$, it follows that $a_1 + 2a_2 + \ldots ma_m \le m(a_1 + a_2 + \ldots a_m)$. Hence we conclude that

$$\begin{aligned} \sum_{\lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor - m < m(a_1+\cdots+a_m) \le \lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor} \binom{|a|}{a_1,\ldots,a_m} &\le \sum_{m(a_1+\cdots+a_m) \le \lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor} \binom{|a|}{a_1,\ldots,a_m} \\ &\le \sum_{a_1+\cdots+ma_m \le \lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor} \binom{|a|}{a_1,\ldots,a_m} \\ &\le AC^{\lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor}, \end{aligned}$$

for some $A, C > 0$ where the last inequality follows by (4.1.23).

4.2.1 Representing $\widehat{u}_N(\xi)$ by approximate solution

If

$$\widehat{u}_N(\xi) = \int u(x)\chi_{2^\sigma N}(x)e^{-ix\xi}dx = \int u(x)P^T(x,D)v(x,\xi)dx,$$

$\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma}\rfloor$, where $P^T(x,D) = \sum_{|\alpha|\le m} b_\alpha(x)D^\alpha$ is the transpose operator of $P(x,D)$ and $b_\alpha(x) \in \mathcal{E}_{\{\tau,\sigma\}}(U)$ (since $\mathcal{E}_{\{\tau,\sigma\}}(U)$ is closed under finite order differentiation), then $v(x,\xi)$ is the solution of the equation

$$e^{ix\xi}P^T(x,D)v(x,\xi) = \chi_{2^\sigma N}(x), \quad x \in K, \xi \in \Gamma, |\xi| > \lfloor N^{1/\sigma}\rfloor. \qquad (4.2.26)$$

Similarly as in [16] and [29] we assume that $v(x,\xi) = \dfrac{e^{-ix\xi}w(x,\xi)}{P_m(x,\xi)}$, for some $w(\cdot,\xi) \in C^\infty(K)$, so that we may rewrite the left hand side of (4.2.26) as

$$\begin{aligned} e^{ix\xi}P^T(x,D)(\frac{w(x)e^{-ix\xi}}{P_m(x,\xi)}) &= e^{ix\xi}\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}b_\alpha(x)D^{\alpha-\beta}(e^{-ix\xi})D^\beta\Big(\frac{w(x)}{P_m(x,\xi)}\Big) \\ &= \sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\sum_{\gamma\le\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}b_\alpha(x)(-\xi)^{\alpha-\beta} \\ &\quad D^\gamma\Big(\frac{1}{P_m(x,\xi)}\Big)D^{\beta-\gamma}w(x), \end{aligned} \tag{4.2.27}$$

for $(x,\xi) \in K\times\Gamma$, the conical neighborhood of (x_0,ξ_0) such that $K\times\Gamma\cap(\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(f)\cup \mathrm{Char}(P)) = \emptyset$.

Next we use the powerful technique of approximate solution (cf. [16, 34]) to rewrite (4.2.27) in the following convenient form:

$$\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\sum_{\gamma\le\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}b_\alpha(x)(-\xi)^{\alpha-\beta}D^\gamma\Big(\frac{1}{P_m(x,\xi)}\Big)D^{\beta-\gamma} = I - R(x,\xi), \tag{4.2.28}$$

where

$$R(x,\xi) = \sum_{j=1}^{m} R_j(x,\xi), \quad R_j(x,\xi) = \sum_{|\alpha|\le j} c_{\alpha,j}(x,\xi)D^\alpha, \tag{4.2.29}$$

for suitable functions $c_{\alpha,j}(x,\xi)$ which are homogeneous of order $-j$ and

$$|D^\beta c_{\alpha,j}(x,\xi)| \le |\xi|^{-j}Ah^{|\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}, \quad \beta\in\mathbf{N}^d, x\in K, \xi\in\Gamma \tag{4.2.30}$$

for some $A,h>0$ and $|\alpha|\le j$, cf. [29].

This representation, together with (4.2.26) and (4.2.27), gives

$$(I-R(x,\xi))w(x,\xi) = \chi_{2^\sigma N}(x) \quad x\in K, \xi\in\Gamma, |\xi| > \lfloor N^{1/\sigma}\rfloor, \tag{4.2.31}$$

which can be solved as follows.

Successive applications of the operator R to both sides of (4.2.31) gives

$$R^{k-1}(x,\xi)w(x,\xi) - R^k(x,\xi)w(x,\xi) = R^{k-1}(x,\xi)\chi_{2^\sigma N}(x),$$

$x\in K$, $\xi\in\Gamma$, $|\xi| > \lfloor N^{1/\sigma}\rfloor$ for every $k\in\{1,\dots,N\}$, so that after summing up those N equalities we obtain

$$w(x,\xi) - R^N(x,\xi)w(x,\xi) = \sum_{k=0}^{N-1} R^k(x,\xi)\chi_{2^\sigma N}(x),$$

which gives formal approximate solution

$$w(x,\xi) = \sum_{k=0}^{\infty} R^k \chi_{2^\sigma N}(x,\xi) \tag{4.2.32}$$

If we consider the partial sums of the form

$$w_N(x,\xi) = \sum_{k\in\mathcal{K}_1} \sum_{\mathfrak{S}_k=0}^{\lfloor(\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}}\rfloor - m} (R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N})(x,\xi), \tag{4.2.33}$$

where $\mathcal{K}_1$ is given by (4.2.4), we obtain

$$(I-R)w_N(x,\xi) = \chi_{2^\sigma N}(x) - e_N(x,\xi), \quad N\in\mathbf{N}, x\in K, \xi\in\Gamma. \tag{4.2.34}$$

It remains to calculate the error term $e_N(x,\xi)$, which is done in Subsection 4.2.2.

We finish this subsection by showing (4.2.28) implies the estimate (4.2.30). An essential argument in this part of the proof is the inverse-closedness property presented in Theorem 2.7.2.

Recall,

$$D^\alpha\Big(\frac{1}{P_m(x,\xi)}\Big) = \alpha! \sum_{(s,p,j)\in\pi} \frac{(-1)^j j!}{(P_m(x,\xi))^{j+1}} \prod_{k=1}^{s} \frac{1}{j_k!}\Big(\frac{1}{p_k!}D^{p_k}P_m(x,\xi)\Big)^{j_k}, \tag{4.2.35}$$

for $\alpha\in\mathbf{N}^d$, where sum is taken over all decompositions (s,p,j) of the form

$$\alpha = j_1p_1 + j_2p_2 + \dots + j_sp_s, \tag{4.2.36}$$

with $j = \sum_{i=1}^{s} j_i \in \{0,1,\dots,|\alpha|\}$, $p_i\in\mathbf{N}^d$, $|p_i|\in\{1,\dots,|\alpha|\}$ for $i\in\{1,\dots,s\}$, $s\le|\alpha|$. (see Section 2.7)

Since the coefficients of $P_m(x,\xi)$ are in $\mathcal{E}_{\{\tau,\sigma\}}(U)$ it follows that

$$\sup_{x\in K} |D^{p_k}P_m(x,\xi)| \le Ah^{|p_k|^\sigma}|p_k|^{\tau|p_k|^\sigma}|\xi|^m, \tag{4.2.37}$$

for some $A,h>0$. Moreover, from $(K\times\Gamma)\cap\mathrm{Char}(P)=\emptyset$ it follows that

$$\sup_{x\in K}|P_m(x,\xi)| \ge C'|\xi|^m. \tag{4.2.38}$$

Hence, by using (4.2.35), (4.2.37) and (4.2.38) we obtain

$$
\begin{aligned}
|D^\alpha\Big(\frac{1}{P_m(x,\xi)}\Big)| &\leq |\alpha|! \sum_{(s,p,j)\in\pi} \frac{j!}{j_1!\dots j_s!|P_m(x,\xi)|^{j+1}} \prod_{k=1}^{s}\Big(\frac{1}{p_k!}|D^{p_k}P_m(x,\xi)|\Big)^{j_k} \\
&\leq |\alpha|! \sum_{(s,p,j)\in\pi} \frac{|\xi|^{mj} j!}{|\xi|^{m(j+1)} j_1!\dots j_s!|P_m(x,\xi)|^{j+1}} \\
&\cdot \prod_{k=1}^{s}\Big(\frac{1}{p_k!}Ah^{|p_k|^\sigma}|p_k|^{\tau|p_k|^\sigma}\Big)^{j_k} \\
&\leq |\xi|^{-m}A'h'^{|\alpha|^\sigma+1}|\alpha|^{\tau|\alpha|^\sigma},
\end{aligned}
$$

for some A, A', $h, h' > 0$, where the last inequality follows by calculation from the proof of Theorem 2.7.2.

In particular, we have proved that $\dfrac{1}{P_m(x,\xi)} \in \mathcal{E}_{\{\tau,\sigma,h\}}(K)$ for some $h>0$ and for every $\xi\in\Gamma$. From the algebra property of extended Gevrey classes it follows that $b_\alpha(x)\partial^\gamma \dfrac{1}{P_m(x,\xi)} \in \mathcal{E}_{\{\tau,\sigma,h'\}}(K)$ for some $h'>0$, where $|\gamma| \leq |\alpha| \leq m$ and $b_\alpha(x)$ are the coefficients of $P^T(x,D)$.

4.2.2 The calculation of the error term

Here we show that (4.2.33) and (4.2.34) imply

$$
e_N(x,\xi) = \sum_{k\in\mathcal{K}_2} \sum_{\mathfrak{S}_k=\lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor-m+1}^{\lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor} (R_{j_1}R_{j_2}\dots R_{j_k}\chi_{2^\sigma N})(x,\xi), \tag{4.2.39}
$$

for $N\in\mathbf{N}$,$x\in K$, $\xi\in\Gamma$, where $\mathfrak{S}_k = j_1+j_2+\dots+j_k$, $j_i\in\{1,\dots,m\}$, $1\leq i\leq k$, and $\mathcal{K}_2$ is given by (4.2.5).

Notice that the order of operator R^k, $k\in\mathbf{N}$, is mk. Hence we compute

$$
\begin{aligned}
&\sum_{k\in\mathcal{K}_1} R^k - \sum_{k\in\mathcal{K}_2} R^{k+1} \\
&= \sum_{k\in\mathcal{K}_2} R^k - \sum_{\{k\in\mathbf{N}\,|\,m\leq mk\leq\lfloor(\frac{N}{\tau})^{\frac{1}{\sigma}}\rfloor\}} R^k \\
&= I - \sum_{k\in\mathcal{K}_2} R^k
\end{aligned}
\tag{4.2.40}
$$

where $\mathcal{K}_1$ is given by (4.2.4) and in the last equality we used

$$\mathcal{K}_1 \cap \{k \in \mathbf{N} \,|\, m \le mk \le \lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor\}$$

$$= \{k \in \mathbf{N} \,|\, m \le mk \le \lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m\}.$$

Moreover, since the operators R_j, $1 \le j \le m$, do not commute we can write

$$\sum_{k \in \mathcal{K}_1} R^k = \sum_{k \in \mathcal{K}_1} \sum_{\mathfrak{S}_k = 0}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} R_{j_1} R_{j_2} \dots R_{j_k},$$

and

$$\sum_{k \in \mathcal{K}_2} R^k = \sum_{k \in \mathcal{K}_2} \sum_{\mathfrak{S}_k = \lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor} R_{j_1} R_{j_2} \dots R_{j_k}$$

where $\mathfrak{S}_k = j_1 + j_2 + \cdots + j_k$, $j_i \in \{1, \dots, m\}$, $1 \le i \le k$.

In particular, we conclude that if w_N and e_N are given by (4.2.6) and (4.2.7), (4.2.40) implies

$$(I - R) w_N(x, \xi) = \chi_{2^\sigma N}(x) - e_N(x, \xi), \quad N \in \mathbf{N}, x \in K, \xi \in \Gamma.$$

4.2.3 Estimates for $D^\beta (R_{j_1} ... R_{j_k} \chi_{2^\sigma N})$

Put

$$\mathfrak{S}_k = j_1 + \cdots + j_k, \quad \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor - m \le \mathfrak{S}_k \le \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor,$$

for $k \in \mathbf{N}$ such that $mk \le \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor$ (see Subsection 4.2.2), and let $|\beta| \le M$ where M is order of distribution u.

In the sequel we follow the idea presented in [16, Lemmas 8.6.2 and 8.6.3]. Recall, $R_j(x, \xi) = \sum_{|\alpha| \le j} c_{\alpha, j}(x, \xi) D^\alpha$, and note that successive applications of the Leibniz rule implies that $D^\beta (R_{j_1} ... R_{j_k} \chi_{2^\sigma N})$ can be written as a sum of terms of the form

$$(D^{\gamma_0} c_{\alpha_{j_1}, j_1}(x, \xi))(D^{\gamma_1} c_{\alpha_{j_2}, j_2}(x, \xi)) \dots (D^{\gamma_{k-1}} c_{\alpha_{j_k}, j_k}(x, \xi))(D^{\gamma_k} \chi_{2^\sigma N}).$$

Put $a_i = |\gamma_i|$ so that

$$a_0 + \cdots + a_k = \mathfrak{S}_k + |\beta|, \tag{4.2.41}$$

$$a_0 \le |\beta|, \tag{4.2.42}$$

and

$$a_i \leq \sum_{t=1}^{i} j_t + |\beta|, \quad 1 \leq i \leq k. \tag{4.2.43}$$

From (4.2.30) it follows that

$$|D^{\gamma_{i-1}} c_{\alpha_{j_i}, j_i}(x,\xi)| \leq |\xi|^{-j_i} A h^{a_{i-1}^\sigma} a_{i-1}^{\tau a_{i-1}^\sigma}, \quad \gamma_{i-1} \in \mathbf{N}^d, x \in K, \xi \in \Gamma, \tag{4.2.44}$$

for some $A, h > 0$ and $|\alpha_{j_i}| \leq j_i$, $i = 1, \dots, k$. Moreover,

$$|D^{\gamma_k} \chi_{2^\sigma N}| \leq C^{a_k+1} \lfloor N^{1/\sigma} \rfloor^{a_k}, \quad x \in K,$$

since $a_k \leq \mathfrak{S}_k + |\beta| < \lfloor 2(N/\tilde{\tau})^{1/\sigma} \rfloor$.

Observe that the number of multiindices $\gamma_0, \dots, \gamma_k$ with the property (4.2.41) is $\binom{\mathfrak{S}_k + |\beta|}{a_0, \dots, a_k}$. In the sequel we write $\sum$ when the sum is taken over all multiindices $\gamma_0, \dots, \gamma_k$ which satisfies (4.2.41)-(4.2.43).

Hence, for $x \in K$ and $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$ we estimate

$$\begin{aligned}
&|D^\beta R_{j_1} \dots R_{j_k} \chi_{2^\sigma N}(x,\xi)| \leq \\
&\qquad \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \dots, a_k} \Big(\prod_{i=1}^{k} |D^{\gamma_{i-1}} c_{\alpha_{j_i}, j_i}(x,\xi)| \Big) \cdot |D^{\gamma_k} \chi_{2^\sigma N}| \\
&\quad \leq |\xi|^{-\mathfrak{S}_k} \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \dots, a_k} \Big(\prod_{i=1}^{k} A h^{a_{i-1}^\sigma} a_{i-1}^{\tau a_{i-1}^\sigma} \Big) \cdot \Big(C^{a_k+1} \lfloor N^{1/\sigma} \rfloor^{a_k} \Big) \\
&\quad \leq |\xi|^{m - \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} A^{\frac{1}{m} \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} h^{\lfloor 2(N/\tilde{\tau})^{1/\sigma} \rfloor^\sigma} C^{\lfloor 2(N/\tilde{\tau})^{1/\sigma} \rfloor + 1} \\
&\qquad \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \dots, a_k} \Big(\prod_{i=1}^{k} a_{i-1}^{\tau a_{i-1}^\sigma} \Big) \cdot \lfloor N^{1/\sigma} \rfloor^{a_k} \\
&\quad \leq |\xi|^{m - \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} h'^{N+1} \sum \binom{\mathfrak{S}_k + |\beta|}{a_0, \dots, a_k} \Big(\prod_{i=1}^{k} a_{i-1}^{\tau a_{i-1}^\sigma} \Big) \cdot \lfloor N^{1/\sigma} \rfloor^{a_k},
\end{aligned}$$

for some $h' > 0$. By the almost increasing property of $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ it follows that

$$\begin{aligned}
\prod_{i=1}^{k} a_{i-1}^{\tau a_{i-1}^\sigma} &\leq C^{a_0 + \dots + a_{k-1}} \frac{a_0! \cdots a_{k-1}!}{(a_0 + \dots + a_{k-1})!} (a_0 + \dots + a_{k-1})^{\tau (a_0 + \dots + a_{k-1})^\sigma} \\
&= C^{\mathfrak{S}_k + |\beta| - a_k} \frac{a_0! \cdots a_{k-1}!}{(\mathfrak{S}_k + |\beta| - a_k)!} (\mathfrak{S}_k + |\beta| - a_k)^{\tau (\mathfrak{S}_k + |\beta| - a_k)^\sigma},
\end{aligned}$$

wherefrom

$$\sum \binom{\mathfrak{S}_k+|\beta|}{a_0,\dots,a_k}\Big(\prod_{i=1}^{k} a_{i-1}^{\tau a_{i-1}^{\sigma}}\Big)\cdot \lfloor N^{1/\sigma}\rfloor^{a_k} \le \sum \frac{a_0!\cdots a_{k-1}!}{(\mathfrak{S}_k+|\beta|-a_k)!}$$
$$\cdot\, C^{\mathfrak{S}_k+|\beta|-a_k}\frac{(\mathfrak{S}_k+|\beta|)!}{a_0!\dots a_{k-1}!a_k!}(\mathfrak{S}_k+|\beta|-a_k)^{\tau(\mathfrak{S}_k+|\beta|-a_k)^{\sigma}}\cdot \lfloor N^{1/\sigma}\rfloor^{a_k}$$
$$= \sum_{a_k\le \mathfrak{S}_k+|\beta|}\binom{\mathfrak{S}_k+|\beta|}{a_k} C^{\mathfrak{S}_k+|\beta|-a_k}(\mathfrak{S}_k+|\beta|-a_k)^{\tau(\mathfrak{S}_k+|\beta|-a_k)^{\sigma}}\lfloor N^{1/\sigma}\rfloor^{a_k}. \tag{4.2.45}$$

Further, for N large enough we have

$$(\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor + M)^{\sigma} \le 2^{\sigma-1}(N/\tilde\tau + M^{\sigma}) < 2^{\sigma}N/\tilde\tau$$

so that for $|\beta|\le M$ the following estimate holds:

$$\mathfrak{S}_k+|\beta| \le \lfloor (N/\tilde\tau)^{1/\sigma}\rfloor + M < \lfloor 2(N/\tilde\tau)^{1/\sigma}\rfloor.$$

This, together with $(M.2)'$ property of $M_p^{\tau,\sigma}=p^{\tau p^{\sigma}}$ gives

$$\begin{aligned}(\mathfrak{S}_k+|\beta|-a_k)^{\tau(\mathfrak{S}_k+|\beta|-a_k)^{\sigma}} &\le (\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor + M)^{\tau(\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor+M)^{\sigma}}\\ &\le C'^{\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor^{\sigma}}\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor^{\tau\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor^{\sigma}}\\ &\le C''^{N} N^{\frac{\tilde\tau-1/\sigma}{\sigma}N}.\end{aligned} \tag{4.2.46}$$

Moreover since

$$\lfloor N^{1/\sigma}\rfloor^{a_k} \le \lfloor N^{1/\sigma}\rfloor^{\lfloor 2(N/\tilde\tau)^{1/\sigma}\rfloor} \le C^{N}, \tag{4.2.47}$$

for sufficiently large $C>0$, by (4.2.46) and (4.2.47) we obtain

$$\sum_{a_k\le \mathfrak{S}_k+|\beta|}\binom{\mathfrak{S}_k+|\beta|}{a_k} C^{\mathfrak{S}_k+|\beta|-a_k}(\mathfrak{S}_k+|\beta|-a_k)^{\tau(\mathfrak{S}_k+|\beta|-a_k)^{\sigma}}\lfloor N^{1/\sigma}\rfloor^{a_k}$$
$$\le h^{N+1}N^{\frac{\tilde\tau-1/\sigma}{\sigma}N},$$

for some $h>0$, which implies that for $x\in K$, $\xi\in\Gamma$

$$|D^{\beta}R_{j_1}...R_{j_k}\chi_{2^{\sigma}N}(x,\xi)| \le |\xi|^{m-\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor}h^{N+1}N^{\frac{\tilde\tau-1/\sigma}{\sigma}N}.$$

Choosing the appropriate enumerations and similar estimates as in [29, Subsection 4.3] we conclude that

$$|D^{\beta}R_{j_1}...R_{j_k}\chi_{2^{\sigma}N}(x,\xi)| \le |\xi|^{-\lfloor 2^{\frac{1-\sigma}{\sigma}}(N/\tilde\tau)^{1/\sigma}\rfloor - M}h^{N+1}N!^{\frac{\tilde\tau-1/\sigma}{\sigma}} \tag{4.2.48}$$

for some $h>0$, which gives the desired estimate.

Remark 4.2.2. Note that Lemma 2.7.1 is used twice in the proof of Theorem 4.2.1. That lead us to the conclusion that the Theorem does not hold for wave fronts sets $\mathrm{WF}_{\{\tau,1\}}$, when $0 < \tau < 1$, if we choose partial differential operators with quasi-analytic coefficients. In particular, $M_p = p!^\tau$, $0 < \tau < 1$ fails to satisfy Komatsu's condition $(M.4)''$ since Stirling's formula implies that

$$\left(\frac{M_p}{p^p}\right)^{1/p} \sim Cp^{\tau-1}, \quad p \to \infty,$$

for some $C > 0$. Hence the representation (4.2.29), with $c_{\alpha,j}$, $|\alpha| \leq j$, $1 \leq j \leq m$ satisfying the desired estimate of the form (4.2.30), is not possible.

Moreover, for our analysis we have chosen $\tilde{\tau}, \sigma-$ admissible sequences where $\tilde{\tau} = \tau^{\sigma/(\sigma-1)}$, and hence we obtain "critical behaviour" for the case $0 < \tau < 1$ and $\sigma = 1$. (see Remark 3.2.1, Chapter 3).

This leads us to the lead us to the wave front set, $\mathrm{WF}_{0^+,\infty}$, with the *pseudo-local property*. In particular, following Corollary is an immediate consequence of the Remark 3.5.1 and Theorem 4.2.1.

Theorem 4.2.2. *Let $u \in \mathcal{D}'(U)$ and $P(x,D)$ the differential operator of order m with coefficients in $\mathcal{E}_{\infty,1^+}(U)$. Then if $P(x,D)u = f$ in $\mathcal{D}'(U)$, it holds*

$$\mathrm{WF}_{0^+,\infty}(f) \subseteq \mathrm{WF}_{0^+,\infty}(u) \subseteq \mathrm{WF}_{0^+,\infty}(f) \cup \mathrm{Char}(P(x,D)). \tag{4.2.49}$$

Proof. Let $\sigma > 1$ be arbitrary but fixed, $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ partial deifferential operator such that $a_\alpha \in \mathcal{E}_{\infty,\sigma}(U)$, $|\alpha| \leq m$. In particular, $a_\alpha \in \mathcal{E}_{\{\tau_\alpha,\sigma\}}(U)$ for some τ_α. Set $\tau = \max_{|\alpha|\leq m} \tau_\alpha$ to conclude that $a_\alpha \in \mathcal{E}_{\{\tau,\sigma\}}(U)$ for all $|\alpha| \leq m$. Now the statement follows directly from Theorem 4.2.1. □

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Ispite sa doktorskih studija položio je u predviđenom roku sa prosečnom ocenom 10,00, koautor je tri naučna rada čime je stekao uslov za odbranu doktorske disertacije.

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Izvod: U ovoj tezi definišemo novu klasu glatkih funkcija i izučavamo njihove osnovne osobine. Pokazujemo da naše klase imaju svojsto algebre kao i da su zatvorene u odnosu na delovanje operatora izvoda konačnog reda. Šta više, konstruišemo diferencijalne operatore beskonačnog reda i to nas dovodi do definicije ultradiferencijabilnih klasa funkcija. Takođe dokazujemo osobinu zatvorenosti u odnosu na inverze, i taj rezultat je najvažniji deo u dokazu glavne teoreme koja je formulisana u poslednjoj glavi. Koristeći tehnike mikrolokalne analize, uvodimo i izučavamo odgovarajuće talasne frontove, i

pokazujemo odgovarajuća tvrđenja vezana za singularni nosač distribucije. Naš glavni rezultat pokazuje kako se prostiru singulariteti rešenja linearnih parcijalnih diferencijalnih jednačina u okviru naše regularnosti.
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Abstract: We introduce a family of smooth functions which are "less regular" than the Gevrey functions, and study its basic properties. In particular we prove the standard results concerning algebra property and stability under finite order derivation. Moreover, we construct infinite order operators which leads us to the definition of class with ultradifferentiable property. We also prove that our classes are inverse-closed, and this result is the essential part in the proof of our main result presented in the final Chapter. Moreover, using the techniques of microlocal analysis, we introduce and investigate the

corresponding wave front sets, and the prove the results related to singular support of a distribution. Our main results shows how the singularities of solutions to partial differential equations (PDE's in short) propagate in the framework of our regularity.
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