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Maja Nedović

THE SCHUR COMPLEMENT AND H-MATRIX THEORY

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To my family

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Apstrakt

U teoriji matrica, od devetnaestog veka pa do danas, pitanje regularnosti date kvadratne kompleksne matrice ima posebnu ulogu. Prepoznata su brojna svojstva matrica i strukture koje garantuju da je determinanta posmatrane matrice različita od nule. Tokom dvadesetog veka, lista uslova dovoljnih za regularnost matrica postala je veoma bogata. Neki od ovih uslova poseduju i dodatne kvalitete, koji ih čine pogodnim za praktičnu primenu. Naime, u poslednjih nekoliko decenija, paralelno sa razvojem računara, raste i potreba za takvim matematičkim rezultatima i takvim uslovima koji su elegantni i lako proverljivi (u smislu da njihova provera nije računski zahtevna). Ovo naročito dolazi do izražaja u matematičkim modelima velikih dimenzija, koji se pojavljuju sve češće u inženjerstvu, ekonomiji, molekularnoj biologiji. Jednostavnost i elegancija u formulacijama matematičkih rezultata, iako oduvek na ceni, danas imaju i dodatnu vrednost, jer omogućavaju smislenu praktičnu primenu.

Ideja sa kojom započinjemo ima sve navedene kvalitete. To je ideja (stroge) dijagonalne dominacije, formulisana za kvadratne (isprva realne, a zatim i kompleksne) matrice. Lévy i Desplanques (1881), dokazali su da stroga dominacija dijagonalnog elementa po modulu u odnosu na sumu modula vandijagonalnih elemenata u svakoj vrsti garantuje regularnost matrice. Iako jednostavna, oduvek prisutna u teoriji matrica i isprva formulisana sa, verovatno, drugačijim motivima, ova klasična ideja je i danas prisutna u aktuelnim istraživanjima. Teorije koje su nastale njenim uopštavanjem, (prvenstveno teorija M- i H-matrica, vidi [4]), pokazale su se fundamentalnim u numeričkoj linearnoj algebri. Osim teorijskog značaja, ove ideje imaju i veliku praktičnu vrednost i primenu u raznim oblastima istraživanja, u inženjerstvu, robotici, ekonomiji, molekularnoj i populacionoj biologiji.

Kažemo da je data matrica, $A = [a_{ij}] \in \mathbb{C}^{n,n}$, strogo dijagonalno dominantna (SDD) ako

$$|a_{ii}| > r_i(A), \text{ za sve } i \in N = \{1, 2, \dots, n\},$$

gde je

$$r_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|.$$

Poslednju veličinu, $r_i(A)$, nazivamo (brisanom) sumom $i-$te vrste.

Ispostavilo se da su mnoge značajne teme u primenjenoj linearnoj algebri na neki način povezane sa pričom o SDD svojstvu. Najpoznatiji takav primer je, svakako, veza SDD svojstva sa problemima lokalizacije karakterističnih korena

kvadratnih kompleksnih matrica. Danas smo svesni da se priča o regularnosti SDD matrica, kao i nekih drugih tpova matrica koji predstavljaju uopštenja SDD matrica, može ekvivalentno ispričati u terminima lokalizacije karakterističnih korena. Iako implicitno prisutna u radovima s početka dvadesetog veka, ova veza je precizno formulisana i u potpunosti iskorišćena tek u knjizi „Geršgorin i njegovi krugovi”, čiji je autor Richard Varga, vidi [82].

Rezultat Geršgorina (1931) objavljen u radu [35] omogućio je jednostavnu lokalizaciju karakterističnih korena date kvadratne kompleksne matrice. Svakoj vrsti matrice pridružen je jedan krug u kompleksnoj ravni sa centrom u dijagonalnom elementu i poluprečnika koji je jednak brisanoj sumi vrste. Unija ovih n krugova u kompleksnoj ravni sadrži sve karakteristične korene date matrice. Drugim rečima, ako je $i-$ti Geršgorinov krug definisan na sledeći način,

$$\Gamma_i(A) = \{z \in \mathbb{C} | |z - a_{ii}| \leq r_i(A)\},$$

tada je *Geršgorinov skup* dat sa

$$\Gamma(A) = \bigcup_{i \in N} \Gamma_i(A).$$

Ako sa $\sigma(A)$ označimo spektar matrice A, koji predstavlja skup svih karakterističnih korena matrice A,

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid det(\lambda I - A) = 0\},$$

gde I označava jediničnu matricu reda n, tada, Geršgorinova teorema tvrdi

$$\sigma(A) \subseteq \Gamma(A).$$

Olga Tausski Tod, u radovima [78, 79, 80] razvila je i dalje promovisala ovu ideju, kao i Ostrowski [65, 66], Brauer [5] i Brualdi [10]. Ljiljana Cvetković i Vladimir Kostić, [16, 17, 18, 49], definisali su pojam matrične klase DD-tipa, koji objedinjuje različite klase matrica definisane uslovima zasnovanim na dijagonalnoj dominaciji, kao i odgovarajuće rezultate u oblasti lokalizacije karakterističnih korena.

Teorija M- i H-matrica, koja je poslužila kao osnova za brojna istraživanja u raznim oblastima numeričke i linearne algebre, naročito u ispitivanju konvergencije iterativnih postupaka i stabilnosti dinamičkih sistema, takođe je u tesnoj vezi sa strogom dijagonalnom dominacijom.

Kažemo da je data matrica, $A = [a_{ij}] \in \mathbb{R}^{n,n}$, M-matrica, ako je A Z-matrica, što znači da su joj svi vandijagonalni elementi nepozitivni, ako je A regularna i $A^{-1} \geq 0$. H-matrice predstavljaju kompleksnu generalizaciju M-matrica. Za datu matricu, $A = [a_{ij}] \in \mathbb{C}^{n,n}$, njena pridružena matrica, $\langle A \rangle = [m_{ij}]$, definisana je na sledeći način,

$$m_{ii} = |a_{ii}|\ ,\ \ m_{ij} = -|a_{ij}|\ ,\ \ i, j = 1, 2, \ldots, n\ ,\ \ i \neq j.$$

Kažemo da je A H-matrica, ako je njena pridružena matrica, $\langle A \rangle$, M-matrica.

Ostrowski (1937) prvi uvodi oznake koje danas koristimo, M i H, (Minkowski, Hadamard) za navedene tipove matrica. Minkowski-matrice, ili M-matrice, su svojom strukturom i zanimljivim svojstvima privukle pažnju dve grupe istraživača. S jedne strane, proučavali su ih matematičari koji su se bavili problemima primenjene linearne algebre, a sa druge strane, ekonomisti koji su se bavili pitanjima ravnoteže tržišta. Na ovaj način, teorija M-matrica razvijala se dvojako, kroz različita tumačenja i različite terminologije. Mnogi rezultati u linearnoj algebri, ali i u primenama u ekonomiji, ekologiji, inženjerstvu, formulisani su, implicitno ili eksplicitno, u terminima teorije M-matrica. Brojne ekvivalentne karakterizacije M-matrica nalazimo u knjizi [4].

Imajući u vidu sve navedeno, nije teško razumeti zbog čega je teorija H-matrica, koja predstavlja uopštenje prethodne priče, i danas veoma živa oblast istraživanja. Za sve one koji se bave proučavanjem svojstava matrica i uslova regularnosti, lokalizacijama karakterističnih korena, analizom konvergencije iterativnih postupaka za rešavanje retkih sistema linearnih jednačina velikih dimenzija, teorija H-matrica predstavlja neprocenjiv alat.

U okvirima ove teze, najznačajnija karakterizacija H-matrica biće upravo ona koja definiše vezu H-matrica sa pojmom stroge dijagonalne dominacije. Fiedler i Pták (1962) pokazali su da klasa H-matrica zapravo predstavlja klasu generalizovano dijagonalno dominantnih matrica. Preciznije, za svaku H-matricu A, postoji dijagonalna regularna matrica W, sa osobinom da je AW SDD. Možemo, takođe, pretpostaviti da W ima pozitivne dijagonalne elemente.

Kao što je formulisano u [84], klasa H-matrica je *dijagonalno izvedena* iz klase SDD matrica.

Na sličan način, možemo formulisati karakterizacije za neke konkretne potklase unutar klase H-matrica. Takve ekvivalentne definicije potklasa H-matrica nazivamo skalirajućim karakterizacijama i koristimo ih u narednim poglavljima. Ova tehnika skaliranja pokazaće se veoma korisnom u proučavanju karakterističnih korena, maksimum-norme inverzne matrice (što je motivisano potrebom da se oceni uslovni broj), kao i pri ispitivanju svojstava Šurovog komplementa.

Šurov komplement predstavlja matricu koja se javlja u procesu Gausove eliminacije u blok-varijanti. James Joseph Sylvester (1851) proučavao je svojstva ove matrice, iako ona tada još uvek nije bila poznata pod imenom koje danas koristimo.

Neka je $M \in \mathbb{C}^{n,n}$ podeljena na blokove na sledeći način,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

gde je $A \in \mathbb{C}^{k,k}$, $1 \leq k \leq n$, regularna (vodeća) glavna podmatrica matrice M. Šurov komplement (SC) od A u M označavamo M/A i definišemo na sledeći način

$$M/A = D - CA^{-1}B.$$

Ako je podmatrica A određena skupom indeksa α, koristićemo i oznaku M/α.

Brualdi i Schneider u radu [11] navode niz implicitnih ranih pojavljivanja SC matrice u radovima raznih autora. U knjizi [84], takođe su navedeni neki rezultati tog tipa (Banachiewicz (1937), Aitken (1939), Guttman (1946)).

Naziv koji danas koristimo za ovu matricu, (Schur complement), prvi put se javlja u radovima Emilie Haynsworth (1968), u izučavanju inercije hermitske matrice. Haynsworth je pokazala da je inercija aditivna u odnosu na SC - rezultat danas poznat pod nazivom „Haynsworth inertia additivity formula”. Inercija hermitske matrice $A \in \mathbb{C}^{n,n}$ je uređena trojka $(p(A), q(A), z(A))$, u kojoj nenegativni celi brojevi $p(A)$, $q(A)$ i $z(A)$, predstavljaju, redom, broj pozitivnih, negativnih i nula karakterističnih korena date matrice A, uključujući višestrukost. Emilie Haynsworth je dokazala da za hermitsku matricu $A \in \mathbb{C}^{n,n}$ i njenu regularnu glavnu podmatricu A_{11}, važi sledeća formula

$$In(A) = In(A_{11}) + In(A/A_{11}).$$

Razlog za njen izbor imena za matricu A/A_{11} je Issai Schur i njegova čuvena lema (1917) koja daje vezu determinante matrice i determinante njene podmatrice, vidi [72]. Naime, za $M \in \mathbb{C}^{n,n}$ podeljenu na blokove na već opisan način, pri čemu je A regularna, važi

$$det(M/A) = detM/detA.$$

Odatle i sama oznaka, M/A . Kako je SC matrica manjeg formata od početne matrice sa kojom je u specifičnoj vezi, svojstva SC matrice postaju zanimljiva tema za istraživanje, a naročito njihova veza sa svojstvima početne matrice. Razlog za to je jasan - SC omogućava redukovanje dimenzije mnogih problema u praksi.

Imajući u vidu sva prethodna razmatranja, jasno je da motivacija za proučavanje SDD matrica i, opštije, H-matrica, dolazi iz nekoliko različitih oblasti.

Prvo, pokazalo se da je SDD svojstvo odlična polazna tačka za definisanje novih uslova za regularnost matrice. Na ovaj način mogu se definisati uslovi slabiji od SDD koji, ne samo što su dovoljni za regularnost, već u isto vreme predstavljaju dovoljne uslove da posmatrana matrica pripada klasi H-matrica. Tako dobijamo nove, šire potklase klase H-matrica koje su opisane lako proverljivim (računski nezahtevnim) uslovima. Definisanje potklasa H-matrica, međutim, nije jedini cilj ovakvih razmatranja, već tek početak brojnih primena.

Već smo naveli da uslovi dovoljni za regularnost matrica, kao što je SDD svojstvo, proizvode ekvivalentne rezultate u oblasti lokalizacije karakterističnih korena. Tvrđenje da je svaka SDD matrica regularna (Lévy, Desplanques, 1881) ekvivalentno je teoremi Geršgorina (1931) o lokalizaciji karakterističnih korena pomoću unije n krugova u kompleksnoj ravni. Na isti način i druga tvrđenja o regularnosti nekih klasa matrica proizvode odgovarajuće lokalizacione skupove. Ukoliko nam je poznata veza između matričnih klasa, poznat nam je i odnos odgovarajućih lokalizacionih skupova - široj klasi matrica odgovara uža lokalizaciona oblast. Iako ponikle iz teorije H-matrica, ove lokalizacije važe za proizvoljne kompleksne matrice.

U teoriji matrica, a i u primenama (u inženjerstvu) nisu nam uvek potrebne konkretne vrednosti karakterističnih korena, niti njihove aproksimacije u formi tačaka u kompleksnoj ravni. Često se javlja potreba da se formulišu (što jednostavniji) dovoljni uslovi da spektar matrice bude unutar neke posebne oblasti u ravni. Ovo je naročito važna tema u proučavanju stabilnosti dinamičkih sistema. Poznato je da, u ovakvim problemima, činjenica da svi karakteristični koreni pripadaju levoj poluravni, ili, u drugom kontekstu, centralnom jediničnom krugu, garantuje stabilnost. U novije vreme, međutim, javljaju se i takvi problemi gde i neke druge, vrlo specifične oblasti u ravni igraju jednako važnu ulogu i potrebno je ispitati pod kojim uslovima će spektar biti unutar takvih oblasti. Možemo, stoga, slobodno reći da je analiza stabilnosti dinamičkih sistema jedna od najvažnijih oblasti u kojoj teorija H-matrica i iz nje nastali rezultati o lokalizaciji igraju suštinski značajnu ulogu.

Mnogi aktuelni problemi u linearnoj algebri dolaze iz praktičnih primena. U slučajevima kada je matematički model opisan matricom koja je dobijena kao rezultat raznih eksperimenata i merenja, moramo biti svesni grešaka koje su prisutne u takvim podacima od samog početka. Moguće je da veoma male greške i veoma male promene u početnim podacima drastično utiču na rešenje. Ovde nastupaju teorija perturbacija i analiza loše uslovljenih matrica. Veličina koju nazivamo uslovnim brojem pokazuje koliko je matrica dobro ili loše uslovljena. Uslovni broj najčešće definišemo kao proizvod neke norme date matrice i norme njene inverzne matrice. Iz tog razloga, korisno je definisati gornju ocenu norme inverzne matrice (naravno, bez izračunavanja inverzne matrice).

Varah (1975) daje jednostavan način za ocenu norme beskonačno (maksimum-norme) inverzne matrice za datu SDD matricu. U radu [81], pokazano je da za SDD matricu $A = [a_{ij}] \in \mathbb{C}^{n,n}$ važi sledeća ocena

$$||A^{-1}||_\infty \leq \frac{1}{\min\limits_{i\in N}(|a_{ii}| - r_i(A))}.$$

Iako klasičan, ovaj rezultat je veoma usamljen primer. Naime, za matrice koje nisu SDD, sličnih rezultata skoro da i nema. Pokazaćemo da se korišćenjem teorije H-matrica i tehnike skaliranja mogu dobiti ocene maksimum-norme inverzne matrice i za matrice koje nisu SDD. Drugim rečima, da je moguće uopštiti rezultat Varaha na šire klase matrica. Ovo uopštenje pruža korist i ukoliko ga primenimo u suprotnom smeru. Naime, nove ocene maksimum norme inverzne matrice ne samo da pokrivaju šire klase matrica, već mogu dati vrednost bližu tačnoj vrednosti norme čak i kada ih primenimo na SDD matrice.

Teorija H-matrica od značaja je za ispitivanje konvergencije iterativnih postupaka za rešavanje (retkih) sistema linearnih jednačina velikih dimenzija, vidi [19], kao i za izučavanje subdirektnih suma matrica, [7]. Dijagonalna dominacija je i početna tačka za definisanje raznih ocena determinante date matrice, kao što je T. Szulc pokazao u radu [74].

Osim određivanja karakterističnih korena i singularnih vrednosti date matrice, jedan od najčešćih ciljeva u linearnoj algebri i matričnoj analizi jeste pronalaženje odgovarajuće dekompozicije matrice. Potrebno je, često, načiniti korak od matrice velikih dimenzija ka matricama manjeg formata. Jedan od načina za redukciju dimenzije problema je i prelazak na SC matricu, ili na drugi način dobijene blok-matrice. Postavlja se pitanje koja svojstva matrica se prenose na SC i pod kojim uslovima. Ukoliko početna, velika, matrica poseduje neka korisna svojstva, da li će ta svojstva biti prisutna u SC matrici uvek, ili samo za neke, pogodne izbore particije skupa indeksa? Koje matrične klase su zatvorene na SC transformaciju? U knjizi [84], u četvrtom poglavlju čiji su autori Johnson i Smith, brojna poznata svojstva matrica su analizirana sa ciljem da se utvrdi da li su invarijantna na SC. Za mnoge klase matrica pokazano je da jesu zatvorene na SC, u slučajevima kada je on definisan, a za klase koje nisu zatvorene prikazani su kontraprimeri. Interesantni su i slučajevi takvih klasa koje, za neke izbore skupa indeksa, jesu zatvorene na SC, dok za drugačije particije zatvorenost ne važi. Izvedeni su i neki opšti rezultati o klasama matrica koje jesu zatvorene na SC ili na glavne podmatrice i o klasama njima inverznih matrica. U ovoj tezi, još neka svojstva, značajna u teoriji H-matrica, ispitana su na ovaj način.

Još jedno zanimljivo pitanje se nameće u razmatranju svojstava SC matrice. Da li možemo pronaći vezu između (lokalizacije) karakterističnih korena početne matrice i (lokalizacije) karakterističnih korena SC matrice? Drugim rečima, šta možemo reći o korenima SC matrice samo na osnovu analize elemenata početne matrice?

Kada je u pitanju inercija H-matrica i SDD matrica, znamo da odgovor leži u dijagonalnim elementima polazne matrice, vidi [50]. Liu i Huang u radu [55] dali su odgovor na slično pitanje za matricu Šurovog komplementa date H-matrice sa realnim dijagonalnim elementima. Broj karakterističnih korena sa pozitivnim realnim delom i broj karakterističnih korena sa negativnim realnim delom su određeni brojem pozitivnih (negativnih) dijagonalnih elemenata u polaznoj matrici i u podmatrici određenoj izborom skupa indeksa α. U radu [83], dato je uopštenje pomenutog rezultata na H-matrice sa kompleksnom dijagonalom. Uslovi za matricu A i skup indeksa α koji garantuju da SC, A/α, ima $|J_{R+}(A)|-|J^{\alpha}_{R+}(A)|$ karakterističnih korena sa pozitivnim realnim delom i $|J_{R-}(A)|-|J^{\alpha}_{R-}(A)|$ karakterističnih korena sa negativnim realnim delom, gde je

$$\begin{aligned} J_{R+}(A) &= \{i \mid Re(a_{ii}) > 0,\ i \in N\},\\ J_{R-}(A) &= \{i \mid Re(a_{ii}) < 0,\ i \in N\},\\ J^{\alpha}_{R+}(A) &= \{i \mid Re(a_{ii}) > 0,\ i \in \alpha\},\\ J^{\alpha}_{R-}(A) &= \{i \mid Re(a_{ii}) < 0,\ i \in \alpha\}, \end{aligned}$$

su dati u istom radu.

U radu [59], naglašeno je da klase regularnih matrica koje poseduju svojstvo SC-zatvorenosti predstavljaju važan aparat u numeričkoj analizi i matričnoj analizi. Ovo je posebno izraženo u ispitivanju konvergencije iterativnih postupaka i izvođenju raznih matričnih nejednakosti. U istom radu, proučavana je separacija krugova SC matrice. Naime, za datu SDD matricu A, iz regularnosti sledi da nijedan Geršgorinov disk formiran za matricu A ne sadrži koordinatni početak. Veličine $|a_{ii}|-r_i(A)$ mere separaciju diskova od nule. Kako je SC date SDD matrice ponovo SDD matrica, ima smisla uporediti separaciju diskova SC matrice sa separacijom diskova polazne matrice. U [59], Liu i Zhang su pokazali da je separacija diskova SC matrice date SDD matrice veća ili jednaka separaciji diskova polazne matrice. Primetimo, ovaj problem se suštinski razlikuje od problema utvrđivanja broja karakterističnih korena u levoj (desnoj) poluravni za SC matricu. Sada želimo da saznamo koliko su (najmanje) karakteristični koreni udaljeni od koordinatnog početka, drugim rečima, koliko je posmatrana matrica daleko od singularnosti. Navodimo kraću verziju glavnog rezultata iz [59].

Neka je $A=[a_{ij}]\in\mathbb{C}^{n,n}$ SDD matrica, $\alpha=\{i_1,i_2,...,i_k\}\subset N$, $\overline{\alpha}=N\backslash\alpha=\{j_1,j_2,...,j_l\}$, $k+l=n$. Označimo $A/\alpha=[a'_{ts}]$. Tada,

$$|a'_{tt}|-r_t(A/\alpha)\geq|a_{j_tj_t}|-r_{j_t}(A)>0.$$

Kao primena ovog rezultata, u istom radu su diskutovane ocene determinante i karakterističnih korena. Preciznije, data je veza između lokalizacija (separacija) korena matrice A/α i korena matrice $A(\overline{\alpha})$, za SDD matrice sa realnom dijagonalom. U ovoj tezi, pokazaćemo kako se pomoću tehnike skaliranja mogu dobiti novi rezultati ovog tipa.

J. M. Pena, u radu [68], takođe se bavi Geršgorinovim diskovima SC matrica. U ovom radu date su strategije pivotiranja pri Gausovoj eliminaciji, koje garantuju smanjivanje poluprečnika Geršgorinovih krugova SC matrica tokom procesa. S tim u vezi, neki rezultati o SC-zatvorenosti raznih klasa matrica, dobijeni u ovoj tezi, mogu se protumačiti i kao posebne strategije blok-pivotiranja, sa ciljem da matrice nastale tokom procesa Gausove eliminacije bivaju sve bliže strogoj dijagonalnoj dominaciji.

Kako ćemo u ovoj tezi razmatrati razne potklase klase H-matrica i njihove primene, navodimo ovde, ukratko, pet različitih kategorija potklasa H-matrica, dobijenih različitim idejama generalizovanja SDD svojstva. Razmotrićemo razne načine narušavanja stroge dijagonalne dominacije, tako da novodobijeni uslov i dalje definiše neku potklasu klase H-matrica.

Prva i najprirodnija takva ideja bila je zadržati sve SDD vrste, osim jedne. Možemo definisati uslov koji zahteva da, za svaki izbor dve različite vrste u matrici, proizvod (zbir) dijagonalnih elemenata dominira nad proizvodom (zbirom) brisanih suma odgovarajućih vrsta. Na ovaj način je definisana poznata klasa Ostrowski-matrica. Primetimo da takav uslov dozvoljava narušavanje stroge dijagonalne dominaciju u najviše jednoj vrsti.

Dalje, moguće je SDD uslov oslabiti kroz particiju početne matrice i definisanje uslova dominacije na nekim delovima matrice, odnosno, korišćenjem samo delova suma vrsta. Kao što ćemo videti u ovoj tezi, klase dobijene particijama skupa indeksa pokazale su se veoma korisnim u različitim primenama. One obuhvataju i matrice u kojima imamo više vrsta koje nisu SDD.

Jedna od ideja za generalizaciju SDD svojstva je i (konveksna) kombinacija suma vrsta i suma kolona u posmatranoj matrici.

Takođe, možemo posmatrati umesto običnih, brisanih suma vrsta, drugačije, rekurzivno definisane sume vrsta. Ova poslednja ideja rezultovala je definisanjem Nekrasov (Gudkov) matrica, koje su poslužile kao osnova za brojne generalizacije i primene.

Prirodno relaksiranje uslova stroge dijagonalne dominacije postiže se i relaksiranjem strogih nejednakosti. Kod uslova definisanih na ovaj način, važnu ulogu igraju i svojstva koja su grafovske prirode, poput osobine nerazloživosti matrica.

U svim ovim generalizacijama, neophodno je zadržati SDD svojstvo bar u jednoj vrsti, da bi nove klase matrica, definisane takvim uslovima i dalje predstavljale potklase klase H-matrica. Navedene osnovne ideje generalizacije SDD osobine prikazane su sledećim dijagramom.

$$\begin{array}{ccccc}
 & & \begin{bmatrix} Multiplikativni \\ i\ aditivni \\ uslovi \end{bmatrix} & & \\
 & & \uparrow & & \\
\begin{bmatrix} Uslovi \\ nestriktne \\ dominacije \end{bmatrix} & \leftarrow & \begin{bmatrix} SDD \end{bmatrix} & \rightarrow & \begin{bmatrix} Uslovi \\ definisani \\ particijom \end{bmatrix} \\
 & & \swarrow \quad \searrow & & \\
 & & \begin{bmatrix} Rekurzivno \\ definisane \\ sume\ vrsta \end{bmatrix} \begin{bmatrix} Kombinacija \\ reda\ i \\ kolone \end{bmatrix} & &
\end{array}$$

U sledećem dijagramu navodimo najznačajnije oblasti primene svih navedenih potklasa H-matrica.

$$\begin{array}{ccccc}
 & & \begin{bmatrix} Uslovi \\ za \\ regularnost \end{bmatrix} \begin{bmatrix} Lokalizacija \\ spektra \\ matrice \end{bmatrix} & & \\
 & & \nwarrow \quad \nearrow & & \\
\begin{bmatrix} Ocena \\ max\ norme \\ inverza \end{bmatrix} & \leftarrow & \begin{bmatrix} PRIMENE \end{bmatrix} & \rightarrow & \begin{bmatrix} Svojstva \\ SC \\ matrice \end{bmatrix} \\
 & & \swarrow \quad \downarrow \quad \searrow & & \\
 & & \begin{bmatrix} Konvergencija \\ iterativnih \\ metoda \end{bmatrix} \begin{bmatrix} Subdirektne \\ sume \\ matrica \end{bmatrix} \begin{bmatrix} Ocene \\ za \\ determinantu \end{bmatrix} & &
\end{array}$$

Originalni rezultati u ovoj tezi spadaju u gornje četiri oblasti primene potklasa H-matrica, navedene u drugom dijagramu. Doprinos teze je najvećim delom u definisanju novih uslova za regularnost, novih ocena maksimum norme inverzne matrice, novih rezultata koji se tiču lokalizacije spektra i novih rezultata o svojstvima SC matrice. Ova poslednja stavka obuhvata rezultate o zatvorenosti nekih klasa matrica na SC, kao i definisanje preliminarnih lokalizacija (separacija) karakterističnih korena SC matrice na osnovu elemenata polazne matrice.

Struktura teze

Teza se sastoji od šest poglavlja, uključujući prvo, uvodno poglavlje i šesto, zaključno poglavlje.

U drugom poglavlju, dat je kratak pregled poznatih rezultata iz teorije M- i H-matrica. Naglašen je odnos H-matrica i stroge dijagonalne dominacije kroz skalirajuću karakterizaciju H-matrica. Navedeni su, ukratko, i neki drugi pristupi teoriji H-matrica koji se mogu pronaći u literaturi.

U trećem poglavlju, razmatrane su brojne potklase klase H-matrica, grupisane na način prikazan prvim dijagramom. Posmatrane su Ostrowski i Pupkov matrice, kao predstavnici prve ideje generalizacije SDD osobine, a zatim klase zasnovane na particijama skupa indeksa. Ovde spadaju Dashnic-Zusmanovich matrice, Σ-SDD i $PH-$matrice. Za ove potklase navedene su i odgovarajuće skalirajuće karakterizacije. Prikazane su klase bazirane na uslovima koji uključuju i sume vrsta i sume kolona, $\alpha 1-$ i $\alpha 2-$ matrice. U petom odeljku, razmatrane su klase opisane uslovima koji uključuju rekurzivno definisane sume vrsta, kao što su Nekrasov i Gudkov matrice. U šestom odeljku, analizirane su klase definisane nestriktnim nejednakostima. U trećem poglavlju, pored poznatih tvrđenja, nalaze se i originalni rezultati. Originalan doprinos je konstrukcija skalirajuće matrice za već poznate klase matrica, (Napomene 1 i 2 i Teorema 20, koja je objavljena u radu [77]). Takođe, originalni rezultati su i tvrđenja koja daju nove uslove za regularnost kroz definicije klasa $\{P_1,P_2\}$-Nekrasov matrica i $\{P_1,P_2\}$-semi-Nekrasov matrica. Rezulati koji se tiču klase $\{P_1,P_2\}$-Nekrasov matrica, (Lema 5, Lema 6, Teorema 23, Teorema 24 i Teorema 25), objavljeni su u radu [22]. Klasa $\{P_1,P_2\}$-semi-Nekrasov matrica je prvi put uvedena u tezi, a originalan doprinos su Lema 7, Lema 8 i Teorema 33.

Četvrto poglavlje se velikim delom sastoji od novih rezultata. U njemu razmatramo primenu H-matrica u oceni maximum norme inverzne matrice i u lokalizaciji karakterističnih korena. Prvu vrstu primene, za ocenu norme, daje Teorema 35, kao i Teorema 36. U njima smo konstruisali dve gornje ocene maksimum norme inverzne matrice za polaznu matricu koja pripada $\{P_1,P_2\}$-Nekrasov klasi. Takođe smo numeričkim primerom ilustrovali činjenicu da naše ocene, ne samo što pokrivaju širu klasu matrica (poznata Varah-ova ocena važi samo za SDD matrice), već, ukoliko ih primenimo na SDD ili Nekrasov matrice, mogu dati precizniju ocenu od poznatih rezultata za SDD (Nekrasov) matrice. Rezultati o oceni norme inverzne matrice objavljeni su u radu [22]. Kada je u pitanju lokalizacija korena, ustanovljen je odnos nekih poznatih oblasti lokalizacije (objavljeno u [25]).

U petom poglavlju razmatrana je primena svih prethodno navedenih rezultata na probleme u vezi sa SC matricom. U prvom odeljku, dat je kratak pregled poznatih SC rezultata. Drugi odeljak posvećen je problemu zatvorenosti pojedinih klasa matrica na SC. Originalni rezultati su Teorema 54 i originalna tehnika dokazivanja (Teorema 55 i Teorema 56), sve objavljeno u radu [21]. Teorema 57, Posledica 1 i Posledica 2 kao i Teoreme 58, 59 i 60, prvi put su prikazane u tezi. Svi ovi rezultati odnose se na klase bazirane na particiji skupa indeksa. Teoreme 62 i 63, koje se odnose na Σ-Nekrasov matrice, publikovane su u radu [26].

U trećem odeljku petog poglavlja bavimo se problemom lokalizacije i separacije karakterističnih korena SC matrice, koje se mogu definisati pomoću elemenata polazne matrice. Predstavljena su dva različita tipa originalnih rezultata. Prvi tip je lokalizacija vertikalnim trakama, koja zapravo daje vezu separacije korena SC matrice i separacije odgovarajuće podmatrice u polaznoj matrici. Dalje, prikazana je lokalizacija korena pomoću krugova Geršgorinovog tipa za SC matricu. Originalan doprinos su Teoreme 80, 81, 83, 84, 85 i 87, publikovane u radovima [27, 77, 28].

Šesto poglavlje sastoji se od zaključnih napomena.

Dakle, u ovoj tezi predmet istraživanja su potklase H-matrica i njihove primene, naročito u proučavanju svojstava SC matrica. Originalne rezultate možemo svrstati u nekoliko tipova.

1. Formulisani su novi dovoljni uslovi za regularnost matrica, koji definišu neke potklase klase H-matrica.
2. Za poznate potklase klase H-matrica, predložena je konstrukcija skalirajuće matrice.
3. Konstruisane su gornje ocene maksimum norme inverzne matrice koje pokrivaju i matrice koje nisu bile obuhvaćene poznatim rezultatima ovog tipa, pri čemu za neke matrice za koje već postoje poznate ocene u literaturi, naše nove ocene daju bolje rezultate.
4. Dokazana je zatvorenost nekih potklasa klase H-matrica na SC. Drugim rečima, dokazano je da su neka matrična svojstva invarijantna u odnosu na SC.
5. Pokazano je kako se pomoću tehnike skaliranja mogu dobiti rezultati o separaciji karakterističnih korena SC matrice.
6. Pomoću tehnike skaliranja su dobijene (preliminarne) lokalizacije karakterističnih korena SC matrice, Geršgorinovog tipa, dakle, u formi unije krugova u kompleksnoj ravni, koji su definisani elementima polazne matrice.

Abstract

H-matrix theory is nowadays one of the basic tools in linear algebra and its applications, especially for researchers dealing with problems of convergence of iterative methods, eigenvalue localization and stability of dynamical systems. Subclasses of (nonsingular) H-matrices are an excellent starting point for deriving new nonsingularity conditions, matrix inequalities, bounds for eigenvalues, determinants and norms of inverse matrices, properties of Schur complements and many other applications. Therefore, the main goal of this research is to introduce new nonsingularity results and new applications of both well-known and new subclasses of H-matrices to eigenvalue localization problems, upper bounds for the maximum-norm of the inverse matrix and to the treatment of Schur complement properties.

The Schur complement is a matrix obtained as a result of block-Gaussian elimination and it is an important tool in reducing the order of large mathematical models that arise in economy and engineering. It is, therefore, interesting to know which matrix properties are invariant under such transformation. In other words, which classes of matrices are closed under taking Schur complements. In [84], an extensive list of matrix classes and their properties related to Schur complements is presented. In this thesis, we investigate more matrix properties based on diagonal dominance in relation to Schur complements. Also, we present new results on eigenvalue localization for the Schur complement matrix that can be obtained from the entries of the original matrix. A well-known result of Fiedler and Pták that gives a scaling characterization of H-matrices and explains their relation to diagonal dominance was a base for developing a scaling technique that proved to be very useful in dealing with both Schur complement closure properties and Schur complement eigenvalue problems.

The outline of the thesis is the following. The thesis consists of six chapters including the introductory chapter and concluding chapter in the end.

Definitions and characterizations of M- and H-matrices, followed by a brief overview of well-known results in this field, are given in the second chapter, together with relation between H-matrices and SDD matrices, a scaling characterization of H-matrices and some more general definitions and classifications of H-matrices that can be found in literature.

Sections 1–6 of Chapter 3 present many different subclasses of the class of H-matrices, grouped by five main ideas of how to break SDD condition in order to obtain useful generalizations of SDD class. For some of these subclasses given in Chapter 3, together with classical and well-known definitions, corresponding

scaling characterizations are presented. Original contribution is given through definitions of $\{P_1, P_2\}$-Nekrasov matrices and $\{P_1, P_2\}$-semi-Nekrasov matrices and results on construction of a scaling matrix for the given Nekrasov matrix.

Chapter 4 consists of two sections dealing with applications of H-matrix theory in the fields of, respectively, determination of upper bounds for the max-norm of the inverse (Section 1) and eigenvalue localization problems (Section 2). Original contributions in connection with determining an upper bound for the max-norm of the inverse are given, as well as numerical examples showing that, even for matrices from well-known classes for which there already exist some max-norm bounds in literature, our bounds can be closer to the exact value.

Chapter 5 discusses applications of H-matrix theory results from Chapter 3 to the Schur complement related problems. In Section 1, definition of the Schur complement and a brief overview of well-known results in this field are recalled.

Section 2 deals with the question which matrix classes mentioned in Chapter 3 are closed under taking Schur complement. In other words, which matrix properties are invariant under Schur complement transformation. Original contributions related to this topic are given, together with new proofs for some already known results, based on scaling technique. Section 3 of the fifth chapter deals with eigenvalue localization and separation for the Schur complement, based on entries of the original matrix. Two different types of localization are presented: vertical bands and Geršgorin type circles.

Contents

Chapter 1

Introduction

1.1 History

The question of nonsingularity of a given matrix was an interesting topic for many researchers of the nineteenth century. Many relations between different matrix properties and structures on one hand and nonzero determinant on the other hand, were recognized. This search continues in the twentieth century as well. The list of various matrix properties that guarantee nonsingularity of matrices has become very rich. Some of these conditions sufficient for nonsingularity of matrices are elegant and simple and relatively easy to check. This quality of simplicity has become very important, especially with the start of computer era and our growing interest in algorithms that do not involve too many calculations, as mathematical models and matrices we are dealing with are getting larger and larger. Just as an illustration, in studying the stability of the aircraft in 1940s, a certain matrix of order 6 was in the center of the problem. Nowadays, these models involve more details and matrices are usually of a much greater order than just 50 or 60 years ago.

All these factors led to an increased need for applications of modern mathematical results in engineering and computer science. Simplicity, beauty and elegance of some quite old results and ideas in mathematics are still qualities that attract the attention of mathematicians, but they also have another, very important role today - these qualities are even more valuable when it comes to practical applications. Simple, yet powerful results in matrix theory that were developed in the first place for very different reasons and different motives, nowadays are finding their way into ecology, molecular and population biology, engineering, robotics, food webs, neural networks, wireless sensor networks and all sorts of applications.

The idea that we start from has all these qualities - it is old, simple and elegant, yet powerful and present in modern research as it has a great theoretical and practical value. It is the concept of **strict diagonal dominance (SDD)**, in matrix theory.

We say that a matrix, $A = [a_{ij}] \in \mathbb{C}^{n,n}$, is **strictly diagonally dominant (SDD) matrix** if

$$|a_{ii}| > r_i(A), \text{ for all } i \in N = \{1, 2, \ldots, n\},$$

where

$$r_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|.$$

We usually denote the sum of moduli of off–diagonal entries in the *i*–th row of matrix A by $r_i(A)$, and call it the *i–th deleted row sum* of matrix A.
As the concept of matrix itself starts with James Joseph Sylvester (1814–1897), the idea to give a special role to the diagonal of a matrix and to compare it with the rest of the matrix in order to guarantee nonsingularity appeared in the work of Lévy in 1881, see [51]. It can also be found in the work of Desplanques, [31], Minkowski, [62] and Hadamard, [37]. First, the real case was covered, and later the complex case, as well.

However, it turned out that many important topics in linear algebra of the twentieth century were in some way connected to (strict) diagonal dominance. The most famous example is, certainly, the relation of SDD property to **eigenvalue localization problems**. Namely, the story of nonsingularity of SDD matrices (and some other classes of matrices that generalize SDD class) can be equivalently told in the language of eigenvalue localization results. Although this relation was implicitly present in some early papers of the twentieth century, it was not explicitly formulated and it was not fully exploited until the work of Varga in 2004, see [82].

The story started in 1931, when **Semjon Aranovič Geršgorin** (1901–1933), published the famous paper on how to simply localize the eigenvalues of a given matrix of order *n* using an area formed as the union of *n* disks, see [35].

The *i–th Geršgorin disk* for a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is formed in the following way,

$$\Gamma_i(A) = \{z \in \mathbb{C} | |z - a_{ii}| \leq r_i(A)\},$$

while *Geršgorin set* is

$$\Gamma(A) = \bigcup_{i \in N} \Gamma_i(A).$$

If we denote by $\sigma(A)$ the spectrum of A , i.e., the set of all eigenvalues of A,

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid det(\lambda I - A) = 0\},$$

where I denotes the identity matrix of order n, then Geršgorin's theorem states that

$$\sigma(A) \subseteq \Gamma(A).$$

Through work of Olga Taussky Tod (1906–1995), one decade after Geršgorin's paper, see [78, 79], (see also [80]) and then through works of Ostrowski [65, 66], Brauer [5] and Brualdi [10], this result was further developed and promoted. Through work of Richard Varga (see the book "Geršgorin and his circles", [82]) the relation between Geršgorin's eigenvalue localization result and nonsingularity of strictly diagonally dominant matrices has finally come to light.

In the work of Ljiljana Cvetković and Vladimir Kostić, (see [16, 17, 18, 49]), the relation between these two streams of research (nonsingularity on one hand and eigenvalue localization on the other) was further analyzed and a new concept of DD-type class of matrices was introduced, as a unifying framework for classes of matrices defined by conditions based on diagonal dominance, together with a new treatment of the corresponding eigenvalue localization results.

Another important topic that turned out to be strongly connected to diagonal dominance is the **theory of M- and H-matrices**. Let us recall the most often seen characterization of a (nonsingular) M-matrix.

A real matrix, $A = [a_{ij}] \in \mathbb{R}^{n,n}$, is an M-matrix, if A is a Z-matrix, (meaning that all the off–diagonal entries are nonpositive), A is nonsingular and $A^{-1} \geq 0$. In other words, it is a real square matrix with nonpositive off–diagonal entries and nonnegative inverse.

The H-matrix concept is a complex generalization of the (real) M-matrix concept. Letters M- and H- appeared in the works of Ostrowski (1937) and they come from the names of two great mathematicians, Minkowski and Hadamard.

Minkowski matrices, or M-matrices, with their special structure and interesting properties immediately captured the attention of two groups of researchers - mathematicians working in the field of linear algebra and its applications (especially in connection with eigenvalue localization problems and study of convergence of iterative methods for solving large sparse systems of linear equations) and economists who studied M-matrices in connection with stability of general equilibrium and analysis of economic systems. Both groups of researchers developed different aspects of M-matrix concept, as well as many different definitions, interpretations and applications (a rich list of equivalent characterizations of M-matrices can be found in [4]). The theory of M-matrices has become fundamental to linear algebra and it has contributed to many areas of mathematical research and applications. Many results in modern robotics, ecology and engineering rely on mathematical foundation that is (explicitly or implicitly) formulated in terms of M-matrices.

Having this in mind, it is not surprising that H-matrix theory, a complex extension of M-matrix theory, is a very active field of research nowadays. For those who deal with applied linear algebra problems, such as study of matrix properties that guarantee nonsingularity of matrices, or in the field of eigenvalue localization and analysis of convergence of iterative methods for solving large sparse systems of linear equations, H-matrix theory represents one of the most powerful tools. It is also an underlying mathematical theory in many mathematical models constructed for solving problems that arise in biology, engineering and other fields of applications.

In literature, there can be found different definitions of the term, as well as different approaches to the subject.

The most interesting characterization of H-matrices that will be often used in the following chapters starts, again, from the concept of strict diagonal dominance. Namely, in the work of Fiedler and Pták in 1962, see [33], it is shown that H-matrices are actually **generalized SDD matrices**. In other words, **for any given H-matrix A, there exists a diagonal nonsingular matrix W, such that AW is SDD.**

As said in [84], the wide class of (nonsingular) H-matrices is *diagonally derived* from the class of SDD matrices.

We will use this characterization of H-matrices together with similar characterizations of some subclasses of H-matrices when dealing with many different problems, such as eigenvalue localization or determining an upper bound for the maximum norm of the inverse matrix. This approach will be called the **scaling technique**, while corresponding characterizations of some matrix classes by the form of these diagonal matrices, (W), will be called **scaling characterizations**. We will use these scaling characterizations also when dealing with the Schur complement related questions.

The Schur complement (SC) story starts, again, with Sylvester (1851), as an idea to study in more detail entries of the matrix that appears in block Gaussian elimination process. The concept itself was also probably known to Gauss. This is how the Schur complement matrix is most often defined.

Let $M \in \mathbb{C}^{n,n}$ be partitioned in blocks in the following way,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{C}^{k,k}$, $1 \leq k \leq n$, is a nonsingular leading principal submatrix of M. The Schur complement of A in M is denoted by M/A and defined to be

$$M/A = D - CA^{-1}B.$$

If the Schur complement of M is formed with respect to a submatrix A determined by the index set α, it is often denoted by M/α.

There were some researchers of the nineteenth and twentieth century who, implicitly, dealt with the Schur complement matrix (see the paper of Brualdi and Schneider, [11], for the list of implicit early appearances of SC), especially those who were interested in finding the inverse of a nonsingular partitioned matrix (Banachiewicz inversion formula, (1937), Aitken block-diagonalization formula (1939), Guttman rank additivity formula (1946), all listed in [84]).

However, the name Schur complement appeared for the first time in 1968 in the work of Emilie Virginia Haynsworth (1916–1985), see [38, 39], who studied the inertia of (partitioned) Hermitian matrices and showed that the inertia is additive on the Schur complement - the result that we today refer to as the *Haynsworth inertia additivity formula*. Inertia of a Hermitian matrix $A \in \mathbb{C}^{n,n}$ is the ordered triple $(p(A), q(A), z(A))$, where nonnegative integers $p(A)$, $q(A)$ and $z(A)$, give, respectively, the numbers of positive, negative and zero eigenvalues of A, including multiplicities. Haynsworth proved that for a Hermitian matrix $A \in \mathbb{C}^{n,n}$ and its nonsingular principal submatrix A_{11}, the following formula holds,

$$In(A) = In(A_{11}) + In(A/A_{11}).$$

One may assume that (after a permutation similarity) the partitioned Hermitian matrix in question is of the following form,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^H & A_{22} \end{bmatrix},$$

where A_{11} is a nonsingular principal submatrix in A. Haynsworth was the one who chose the name of the great mathematician, Issai Schur (1875–1941), to denote this special matrix, A/A_{11}. The reason for her choice was the paper by Issai Schur published in 1917 in the Crelle's Journal, in which a lemma that we now call the *Schur determinant lemma* and *Schur determinant formula* were introduced for the first time, see [72]. Schur determinant lemma gives a relation between the determinant of a matrix and the determinant of its submatrix. The Schur's formula states that for $M \in \mathbb{C}^{n,n}$ partitioned in blocks as in (1.1), with A nonsingular, it holds

$$det(M/A) = detM/detA.$$

The study of Schur complement became interesting because it represents a matrix of a smaller format but in a specific way connected to the given, larger matrix, which is convenient for practical use, too.

Issai Schur was Alfred Brauer's Ph.D dissertation adviser, while Alfred Brauer was Emilie Haynsworth's dissertation adviser, see [84]. The topic of Haynsworth's dissertation was determinantal bounds for diagonally dominant matrices, which brings us back to where we first started - to the story of SDD.

1.2 Motivation

What is the main motivation to study SDD matrix class and related matrix classes? In other words, what are the benefits of developing the H-matrix theory?

First of all, it turned out that SDD property is an excellent starting point for constructing **new nonsingularity results**, i.e., **simple sufficient conditions for nonsingularity of matrices** based on different types of diagonal dominance. So, the first and most obvious motivation for us is to find weaker, but still simple enough, sufficient conditions for nonsingularity of matrices, that are at the same time **sufficient conditions for a given matrix to be an H-matrix**. By these conditions we define wider and wider subclasses of nonsingular matrices, and subclasses of H-matrices, by simple and easy to check conditions. But, obtaining a weaker sufficient condition for nonsingularity (i.e., obtaining a wider class of nonsingular matrices starting from the SDD class) is not the only, or even the most important benefit.

It is well-known that each nonsingularity result of this type produces an equivalent result in the field of **eigenvalue localization**. We know that the fact that SDD matrices are nonsingular is equivalent to the famous Geršgorin's theorem. Notice that Geršgorin's result applies not only to SDD matrices, but to all complex matrices. And the same reasoning holds for nonsingularity conditions weaker than SDD – each of them produces an eigenvalue localization area that applies to any complex matrix. The wider the matrix class is – the tighter is localization area obtained in this way. Again, all these localizations, although inspired by the study of H-matrices, apply to all complex square matrices.

In matrix theory and applications (especially in engineering) we don't always need the exact eigenvalues. We don't always need approximations of eigenvalues by dots in the complex plane. More often we need to determine some simple sufficient conditions for the eigenvalues to belong to some special area in the complex plane. For instance, when it comes to problems of stability of dynamical systems, it is important to know whether the eigenvalues belong to the complex left half-plane, or under what additional conditions the spectrum, the set of all the eigenvalues, will be a subset of the unit circle. In modern research, some other, special areas in the complex plane play the role of left half-plane or unit circle. Therefore, the motivation to deal with generalizations of SDD property and H-matrix theory comes from the eigenvalue localization problems and analysis of stability of dynamical systems.

There are more linear algebra problems that come from practical applications. Namely, if we deal with system matrix obtained practically, by a real–life measuring process, we have to be aware of the errors present in the matrix entries from the very start. It is possible that small changes in the starting data affect the solution drastically. This is where **the perturbation theory** steps in, together with the analysis of the so-called *ill–conditioned matrices*. A quantity called *the condition number* shows how "ill" the matrix could be. As the condition number is usually determined in the following way,

$$\kappa(A) = \|A\| \|A\|^{-1},$$

as the product of a matrix norm and a norm of the inverse matrix, it is very useful to find ways to determine **the upper bound for the norm of the inverse matrix** without calculating the inverse.

A result of Varah, see [81], gives a simple and elegant upper bound for the maximum norm of the inverse matrix of an SDD matrix. Namely, for an SDD matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, the following bound applies,

$$||A^{-1}||_\infty \leq \frac{1}{\min\limits_{i \in N} (|a_{ii}| - r_i(A))}.$$

Starting from this result, it is possible to construct upper bounds that can be applied to more matrices, even those that are not SDD. Moreover, these new upper bounds, obtained by further developing H-matrix theory, when applied to SDD matrices can give tighter estimates than Varah bound. Therefore, another benefit that comes out of H-matrix theory is the ability to determine upper bounds for the norm of the inverse for matrices that don't belong to SDD class.

Another important topic in linear algebra is **convergence of iterative procedures** for solving large systems of linear equations. H-matrix theory contributes to this area of research, see [19], as well as when dealing with **subdirect sums of matrices**, see [7]. For finding **determinant bounds**, diagonal dominance and H-matrices play a significant role, as one can see in the paper by Szulc, [74].

Besides determining eigenvalues or singular values of a given matrix, one of the most common tasks in applied linear algebra and matrix theory is to find an appropriate factorization, an appropriate decomposition of a matrix. When dealing with large matrices, it is very useful to find a way to **replace a problem of a large format with problems of smaller formats**. Or, in other words, it is useful to make a step from the large matrix to a matrix of a smaller dimension. One way to do this is to partition the matrix into blocks and deal with block-matrices.

Another way to achieve this transition to a smaller matrix is to perform on the partitioned (2x2) block-matrix one step of block-Gaussian elimination and produce a zero matrix under the block-diagonal. What is left to consider is the Schur complement, which is a smaller matrix than the original one. As it is connected to the starting, "parent" matrix, but smaller in dimension, it is an interesting question what is the relation between these matrices. More precisely, if the original

matrix has a ”good” property, (for instance, a property which guarantees convergence of some iterative procedures), will the property stay invariant when taking the Schur complement? Which of the well-known matrix properties are invariant under Schur complement transformation? Or, which of the well-known matrix classes are closed under taking Schur complements?
Another interesting question is the following – what can we say about the eigenvalues of the Schur complement matrix only by analyzing the entries in the original matrix? Also, is there a way to bound the norm of the inverse for the Schur complement matrix just by analyzing the entries in the original matrix? One part of the thesis deals with questions of this type.

1.3 Thesis outline

The thesis consists of six chapters including the introductory chapter and concluding chapter in the end.

As it is clear from the title, H-matrix theory is the basis for most of the work presented in this thesis. Therefore, definitions and characterizations of M- and H-matrices followed by a brief overview of well-known results in this field are given in the first section of the second chapter. The second section of Chapter 2 deals with relation between H-matrices and SDD matrices and scaling characterization of H-matrices, while in Section 3 of Chapter 2 some more general definitions and classifications of H-matrices that can be found in the literature are recalled.

Sections 1–6 of Chapter 3 present many different subclasses of the class of H-matrices, grouped by five main ideas of how to break the SDD condition in order to obtain useful generalizations of SDD class. These five main ideas, or, five main directions of generalization, are listed in Section 1 and then discussed in the remainder of the second chapter. First, multiplicative and additive conditions are presented (Section 2) together with related Ostrowski and Pupkov matrices. Section 3 deals with Dashnic-Zusmanovich matrices, Σ-SDD matrices and PH-matrices, all based on the idea of partitioning the index set of a matrix. In the fourth section of this chapter, conditions defined by column sums are recalled through classes of $\alpha 1$–matrices and $\alpha 2$–matrices. Then, in Section 5, classes of Nekrasov matrices, Gudkov matrices, $\{P_1, P_2\}$-Nekrasov matrices and Σ-Nekrasov matrices, all based on recursively defined row sums, are discussed, while in Section 6, strict inequalities in definitions of some subclasses are replaced by nonstrict inequalities which brings us to semi-SDD, semi-Nekrasov and $\{P_1, P_2\}$-semi Nekrasov matrices.

The last section of this chapter, Section 7, deals with relations between subclasses. With this we close the third chapter in which the main goal was to present all the different subclasses of H-matrices we are going to work with.

For some of these subclasses given in Chapter 3, together with classical and well-known definitions, corresponding scaling characterizations are presented.

Original contributions related to scaling characterizations of some well-known subclasses of H-matrices and introducing new subclasses of H-matrices, i.e., new nonsingularity conditions, can be found in Remarks 1 and 2, Theorem 20, published in paper [77] in 2015, then, Lemmas 5, 6 and Theorems 23, 24 and 25, published in 2015 in the paper [22]. The class of $\{P_1,P_2\}$-Nekrasov matrices was introduced in [22]. Also, the class of $\{P_1,P_2\}$-semi-Nekrasov matrices is introduced in the thesis for the first time, together with related results given in Lemma 7, Lemma 8 and Theorem 33.

As mentioned before, the motivation for finding scaling characterizations (or, if not full characterization, at least some of the diagonal scaling matrices) lies in the fact that this scaling approach is often more elegant and revealing when we deal with eigenvalues, norm bounds, convergence of iterative procedures or the Schur complement matrix of a given matrix from the observed subclass.

Chapter 4 consists of two sections dealing with applications of H-matrix theory in the fields of, respectively, determination of upper bounds for the max-norm of the inverse (Section 1) and eigenvalue localization problems (Section 2). Original contributions in connection with determining an upper bound for the max-norm of the inverse are given in Theorems 35 and 36 of the first section, as well as numerical examples showing that, even for matrices from well-known classes for which there already exist some max-norm bounds in the literature, our bounds can be closer to the exact value. These results are published in [22]. In the field of eigenvalue localization, original contribution is Theorem 42 of Section 2. Starting from Dashnic-Zusmanovich class of matrices, we considered the corresponding eigenvalue localization area applicable to all square complex matrices and showed that it is a subset of Brauer's ovals of Cassini. This result is published in [25].

Chapter 5 discusses applications of H-matrix theory results from Chapter 3 to the Schur complement related problems.

In Section 1, definition of the Schur complement and a brief overview of well-known results in this field are recalled.

Section 2 deals with the question which matrix classes mentioned in Chapter 3 are closed under taking Schur complement. Original contributions related to this topic are given in Theorem 54, in new proofs of Theorems 55 and 56, based on scaling technique, all published in [21], then, in Theorem 57 with Corollaries 1 and 2 and in Theorems 58, 59 and 60, all presented in the thesis for the first time. These results all apply to partition-based matrix classes (Dashnic-Zusmanovich matrices, Σ-SDD matrices and PH-matrices).

Also, Theorems 62 and 63 related to Σ-Nekrasov matrices are published in [26].

Section 3 of the fifth chapter deals with eigenvalue localizations for the Schur complement, based on entries of the original matrix. Two different types of localization are presented : vertical bands and Geršgorin-type disks. Original contributions are given in Theorems 80, 81, 83, 84 and 85, published in [27, 77, 28].

Chapter 6 consists of concluding remarks.

1.4 Notation

$\mathbb{R}$ - the set of real numbers

$\mathbb{C}$ - the set of complex numbers

$\mathbb{R}^n$ - the set of column vectors of n real components

$\mathbb{C}^n$ - the set of column vectors of n complex components

$\mathbb{R}^{m,n}$ - the set of all $m \times n$ matrices with real entries

$\mathbb{C}^{m,n}$ - the set of all $m \times n$ matrices with complex entries

$A = [a_{ij}]$ - a matrix A with entries a_{ij}

$x \geq 0$ - each entry of vector x is nonnegative

$x > 0$ - each entry of vector x is positive

$A \geq 0$ - each entry of matrix A is nonnegative

$A > 0$ - each entry of matrix A is positive

I - identity matrix

e - vector with all components equal to 1

$N = \{1, 2, \ldots, n\}$ - the set of indices

$N_r(A)$ - the set of indices denoting SDD rows in A.

$N_c(A)$ - the set of indices denoting SDD columns in A.

$J_{R+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in N\}$ - the set of indices corresponding to diagonal entries with positive real part

$J_{R-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in N\}$ - the set of indices corresponding to diagonal entries with negative real part

$J^{\alpha}_{R+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in \alpha\}$ - the set of indices in α corresponding to diagonal entries with positive real part

$J^{\alpha}_{R-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in \alpha\}$ - the set of indices in α corresponding to diagonal entries with negative real part

$S \subseteq N$ - S is a subset in N

$S \subset N$ - S is a proper subset in N

$\emptyset \neq S \subset N$ - S is a nonempty proper subset in N

$\overline{S} = N \setminus S$ - complement of S in the index set N

$\det A$ - determinant of matrix A

$|A|$ - matrix obtained from matrix A by taking moduli of all entries

$\langle A \rangle$ - comparison matrix for matrix A

A^{-1} - inverse of matrix A

$A^{-} \in \mathbb{C}^{n,m}$ - a generalized inverse of matrix A

$A^{\dagger} \in \mathbb{C}^{n,m}$ - Moore–Penrose generalized inverse of matrix A

A^T - transpose of matrix A

$\overline{A}$ - conjugate of matrix A

A^H - conjugate transpose of matrix A

$\sigma(A)$ - spectrum (the set of all eigenvalues) of matrix A

$\rho(A)$ - spectral radius (the maximum of moduli of eigenvalues of matrix A)

$In(A)$ - inertia of Hermitian matrix A

$rank(A)$ - rank of matrix A

$diag(d)$ - diagonal matrix with diagonal entries equal to components of vector d

$d(A)$ - column vector of diagonal entries of matrix $|A|$

$r_i(A)$ - the i–th deleted row sum of matrix A, the sum of moduli of off-diagonal entries in the i–th row of matrix A

$r(A)$ - column vector of deleted row sums for matrix A

$r_i^S(A)$ - the i–th S - deleted row sum of matrix A, the sum of moduli of off–diagonal entries in the i–th row of matrix A with column indices from S

$c_i(A)$ - the i–th deleted column sum of matrix A, the sum of moduli of off–diagonal entries in the i–th column of matrix A

$c(A)$ - column vector of deleted column sums for matrix A

$R_i(A)$ - the i–th row sum of matrix A, the sum of all entries in the i–th row of matrix A including the diagonal entry

$\pi = \{p_j\}_{j=0}^{\ell}$ - partition that divides the index set N into ℓ disjoint nonempty subsets $S_1, S_2, \ldots, S_\ell$, where $S_j = \{p_{j-1}+1, p_{j-1}+2, \ldots, p_j\}$, $j = 1, \ldots, \ell$

$A^{(i_1, i_2, \ldots i_l)}$ - aggregated matrix of order ℓ with respect to the corresponding partition of matrix A, where $i_k \in S_k$, $k = 1, \ldots, \ell$

$h_i(A)$ - the i–th deleted Nekrasov row sum for matrix A

$h(A)$ - column vector of deleted Nekrasov row sums for matrix A

$h_i^S(A)$ - the i–th S - deleted Nekrasov row sum for matrix A

δ_{st} - Kronecker delta function

$P = [p_{ij}] = [\delta_{i\pi(j)}] \in \mathbb{R}^{n,n}$ - permutation matrix of order n

$h^P(A) = Ph(P^TAP)$ - permuted column vector of deleted Nekrasov row sums for permuted matrix P^TAP

$h_i^P(A) = (Ph(P^TAP))_i$ - the i–th component of vector $h^P(A)$

$l_i(A) = \sum_{j=1}^{i-1} |a_{ij}|$, $i = 2,3,...,n$ - the lower i–th deleted row sum of matrix A

$u_i(A) = \sum_{j=i+1}^{n} |a_{ij}|$ - the upper i–th deleted row sum of matrix A

$q_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|}$, $i = 2,3,...,n$ - the lower i–th Nekrasov row sum of A

$q(A)$ - column vector with components $q_i(A)$

$q^P(A) = Pq(P^TAP)$ - permuted column vector of lower Nekrasov row sums for permuted matrix P^TAP

$q_i^P(A) = (Pq(P^TAP))_i$ - the i–th component of vector $q^P(A)$

$\Gamma_i(A)$ - the i–th Geršgorin disk for matrix A

$\Gamma(A)$ - the Geršgorin set for matrix A

$\Gamma_i^{r^x}(A)$ - the i–th Geršgorin disk for matrix $X^{-1}AX$ where $X = diag(x)$

$\Gamma^{r^x}(A)$ - the Geršgorin set for matrix $X^{-1}AX$ where $X = diag(x)$

$\Gamma_S^{r^x}(A)$ - the union of disks $\Gamma_i^{r^x}(A)$ over $i \in S$

$K(A)$ - the Brauer Cassini eigenvalue localization set for matrix A

$\psi(A)$ - the Dashnic Zusmanovich eigenvalue localization set for matrix A

$\Omega(A)$ - equimodular set for matrix A

$||A||$ - a norm of matrix A

$||A||_\infty$ - infinity (maximum) norm of matrix A

$A(\alpha,\beta)$ - submatrix in matrix A of rows indexed by α and columns indexed by β

$A(\alpha)$ - submatrix in matrix A of rows indexed by α and columns indexed by α

A/α - the Schur complement of matrix A with respect to index set α

$A/\alpha,\beta$ - the Schur complement of matrix A with respect to index sets α and β

M/A - the Schur complement of matrix M with respect to submatrix A

$A \circ B$ - Hadamard product of matrices A and B

$A/_{\circ}\alpha$ - the diagonal Schur complement of matrix A with respect to index set α

$P(A/\alpha)$ - the Perron complement of matrix A with respect to index set α

Chapter 2

The class of H–matrices

2.1 M-matrices and H-matrices : background

As said in the introductory chapter, H-matrices present a generalization of (real) M-matrices to the complex case. Therefore, we start with recalling definition and some interesting properties of M-matrices. In order to do that, we should first point out a special structure, a special **sign-pattern** present in all M-matrices.

In [4] it is pointed out that in practical applications in biology, physics, economy, sociology and statistics, because of the real–life conditions, some matrix structures occur more often than others. One of the most common structures that occurs in practice is the following - a square, real matrix with nonnegative diagonal entries and nonpositive off–diagonal entries. In linear algebra, matrices with such a sign-pattern also play an important role when dealing with eigenvalue problems or linear complementarity problems. Furthermore, these matrices are strongly related to the theory of nonnegative matrices.

In order to define M-matrices, we first recall the definition of Z-matrices, that are real matrices with a specific sign–pattern. Throughout this section, by $\mathbb{C}^n(\mathbb{R}^n)$ we denote complex (real) n-dimensional vector space, by $\mathbb{C}^{n,n}(\mathbb{R}^{n,n})$ the collection of all $n \times n$ matrices with complex (real) entries, and by $N := \{1, 2, \dots, n\}$ the set of indices.

Definition 1 *A matrix* $A = [a_{ij}] \in \mathbb{R}^{n,n}$ *is a* Z*-matrix if* $a_{ij} \leq 0$ *for all* $i, j \in N$, $i \neq j$.

In other words, it is a square real matrix with nonpositive off–diagonal entries.

In the following example, the matrix A_0 is a Z-matrix.

Example 1

$$A_0 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Now we are ready to define M-matrices.

Definition 2 *A matrix* $A = [a_{ij}] \in \mathbb{R}^{n,n}$ *is an* M*-matrix if* A *is a* Z*-matrix and* A *is inverse-nonnegative, that is,* A^{-1} *exists and* $A^{-1} \geq 0$.

In this definition of M-matrices, the notion $A^{-1} \geq 0$ stands for $(A^{-1})_{ij} \geq 0$, for all $i, j \in N$.

It is clear from the definition that M-matrices are real, *nonsingular* matrices. Here, if not specified differently, the term M-matrices will be used for nonsingular M-matrices. It is important at this point to emphasize nonsingularity because in the literature there are more general definitions of M-matrices that include some singular M-matrices. We will mention briefly these generalizations later, in the end of the second chapter.

When talking about inverse-positivity, it is not just the class of M-matrices itself that deserves our attention in relation to study of iterative methods, see [4]. It is also the class of **inverse** M**-matrices** that has been a very active field of research. For more details and interesting properties of this matrix class, see [45].

If we look again at the Z-matrix A_0, mentioned in Example 1, we notice that A_0 is nonsingular and its inverse is

$$A_0^{-1} = \frac{1}{7} \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 10 & 8 & 6 & 4 & 2 \\ 4 & 8 & 12 & 9 & 6 & 3 \\ 3 & 6 & 9 & 12 & 8 & 4 \\ 2 & 4 & 6 & 8 & 10 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \geq 0,$$

therefore, it is an M-matrix.

It is easy to prove one more well-known sign-structural property of M-matrices, in addition to the Z-form already mentioned in the Definition 2.

Lemma 1 ([4]) *If* $A = [a_{ij}] \in \mathbb{R}^{n,n}$ *is an* M*-matrix, then* $a_{ii} > 0$ *for all* $i \in N$.

In other words, all diagonal entries of an M-matrix are positive.

This can be confirmed in the following way. If $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is an M-matrix, then, A is a Z-matrix, meaning that all the off-diagonal entries of A are nonpositive. If by z we denote the vector obtained in the following way:

$$z := A^{-1}e,$$

where $e = [1, 1, \ldots, 1]^T$ is the column vector in $\mathbb{R}^n$ with all components equal to 1, then, clearly, since A is nonsingular and $A^{-1} \geq 0$,

$$z \geq 0,$$

$$Az = e > 0.$$

This implies

$$(Az)_i = \sum_{j=1}^{n} a_{ij}z_j > 0, \;\; i \in N,$$

and

$$a_{ii}z_i > -\sum_{j=1, j\neq i}^{n} a_{ij}z_j \geq 0, \;\; i \in N.$$

As for all $i \in N$ it holds that $a_{ii}z_i > 0$, this implies that for all $i \in N$, $a_{ii} > 0$. Therefore, a necessary condition for a square real matrix to be an M-matrix is to have positive diagonal entries.

Another important characterization of M-matrices is given in the following theorem.

Theorem 1 ([4]) *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a Z-matrix. Then, A is an M-matrix if and only if there exists a vector $z \in \mathbb{R}^n$, $z > 0$ such that $Az > 0$.*

Let us recall more well-known results on M-matrices:

Theorem 2 ([4]) *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a Z-matrix. Then, A is an M-matrix if and only if any of the following conditions holds:*
a) All of the principal minors of A are positive.
b) Every real eigenvalue of each principal sub-matrix of A is positive.
c) $A + D$ is nonsingular for each nonnegative diagonal matrix D.
d) A does not reverse the sign of any vector, that is, if $x \neq 0$ and $y = Ax$, then for some $i \in N$, $x_i y_i > 0$.
e) A is monotone, that is $Ax \geq 0 \Rightarrow x \geq 0$.

As already said before, M-matrices are *real*, nonsingular matrices. In order to make a complex generalization of this class of matrices we define for each square, complex matrix its *comparison* matrix, which is always a real matrix.

Definition 3 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$. The comparison matrix of the given matrix A is denoted with $\langle A \rangle$, and defined as $\langle A \rangle = [m_{ij}]$, where*

$$m_{ii} = |a_{ii}| \ , \ \ m_{ij} = -|a_{ij}| \ , \ \ i, j = 1, 2, \ldots, n \ , \ \ i \neq j.$$

Notice that

$$\langle A \rangle = \begin{bmatrix} |a_{11}| & -|a_{12}| & -|a_{13}| & \cdots & -|a_{1n}| \\ -|a_{21}| & |a_{22}| & -|a_{23}| & \cdots & -|a_{2n}| \\ -|a_{31}| & -|a_{32}| & |a_{33}| & \cdots & -|a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -|a_{n1}| & -|a_{n2}| & -|a_{n3}| & \cdots & |a_{nn}| \end{bmatrix} =$$

$$= 2 \begin{bmatrix} |a_{11}| & 0 & 0 & \cdots & 0 \\ 0 & |a_{22}| & 0 & \cdots & 0 \\ 0 & 0 & |a_{33}| & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & |a_{nn}| \end{bmatrix} - \begin{bmatrix} |a_{11}| & |a_{12}| & |a_{13}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & |a_{23}| & \cdots & |a_{2n}| \\ |a_{31}| & |a_{32}| & |a_{33}| & \cdots & |a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |a_{n1}| & |a_{n2}| & |a_{n3}| & \cdots & |a_{nn}| \end{bmatrix}$$

$$= 2|D(A)| - |A|.$$

Now we are ready to give the definition of the H-matrix, the one which represents the basis and the frame for all the results presented in the following chapters.

Definition 4 *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix if its comparison matrix $\langle A \rangle$ is an M-matrix, i.e., $\langle A \rangle^{-1}$ exists and $\langle A \rangle^{-1} \geq 0$.*

Obviously, from the definition and from Lemma 1, we see that an H-matrix has no zero diagonal elements.

Notice that there are many different matrices with the same comparison matrix. For instance, all the matrices in the equimodular set of the given matrix A have their comparison matrices equal to $\langle A \rangle$.

Definition 5 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$. Equimodular set for the matrix A is defined as follows*

$$\Omega(A) := \{B = [b_{ij}] \in \mathbb{C}^{n,n} : \ |b_{ij}| = |a_{ij}|, \ i, j \in N\}.$$

Also, in general, in the equimodular set of a given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ there can be both singular and nonsingular matrices, as the following simple example shows.

Example 2 *Consider matrices A and B,*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Obviously, A is singular while B is nonsingular. But, comparison matrix for both matrices A and B is

$$\langle A \rangle = \langle B \rangle = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Notice that the comparison matrix, $\langle A \rangle = \langle B \rangle$, is, in this case, a singular matrix.

The class of H-matrices is invariant to some basic matrix transformations. As stated in the next theorem, if the matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix, then the same holds for its conjugate, $\overline{A}$, transpose, A^T and permuted matrix, $P^T AP$, for any given permutation matrix P of order n.

Permutations of the form $P^T AP$, although similarity transformations, will more often be called simultaneous permutations of rows and columns in the following chapters. We use this term in order to emphasize the fact that the set of entries in the row corresponding to one particular diagonal entry does not change through such permutations. The same observation holds for the set of entries in the corresponding column.

Theorem 3 *The class of H-matrices is closed under conjugations, transpositions and similarity (simultaneous) permutations of rows and columns.*

It is easy to see that, for $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and $\overline{A}$ the conjugate of A, $\langle A \rangle = \langle \overline{A} \rangle$. Therefore, $\langle A \rangle^{-1}$ is nonnegative if and only if $\langle \overline{A} \rangle^{-1}$ is.
As $\langle A^T \rangle = \langle A \rangle^T$, and as $\langle A^T \rangle^{-1} = (\langle A \rangle^T)^{-1} = (\langle A \rangle^{-1})^T$, it follows that $\langle A^T \rangle^{-1} \geq 0$ if and only if $\langle A \rangle^{-1} \geq 0$.
As far as similarity permutations are considered, let $P = [p_{ij}] := [\delta_{i\pi(j)}] \in \mathbb{R}^{n,n}$ be a permutation matrix of order n, where π is an injective mapping from N to N, while δ_{st} is Kronecker delta function, i.e.,

$$\delta_{st} = 1, \; for \; s = t,$$

$$\delta_{st} = 0, \; for \; s \neq t.$$

Then, it is easy to see that $\langle P^T AP \rangle = P^T \langle A \rangle P$, which implies

$$\langle P^T AP \rangle^{-1} = P^T \langle A \rangle^{-1} P.$$

Therefore, the matrix $\langle P^TAP\rangle^{-1}$ is nonnegative if and only if $\langle A\rangle^{-1}$ is nonnegative. In other words, P^TAP is an H-matrix if and only if A is an H-matrix.

We close this section with one more well-known and interesting property of H-matrices that can be found in [65, 42].

Theorem 4 (Ostrowski) *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix, then*

$$|A^{-1}| \le \langle A\rangle^{-1}.$$

2.2 Relation to diagonal dominance

The classes that we are here particularly interested in are strongly related to the well-known class of strictly diagonally dominant matrices (SDD).

Definition 6 *Given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we say that A is strictly diagonally dominant (SDD) matrix if*

$$|a_{ii}| > r_i(A), \text{ for all } i \in N,$$

where

$$r_i(A) = \sum_{j\in N\setminus\{i\}} |a_{ij}|.$$

This means that in each row, the diagonal entry by modulus is strictly greater than the sum of moduli of off–diagonal entries. The quantity $r_i(A)$ is called the *i–th deleted row sum.*

In other words, the square complex matrix A is SDD if

$$d(A) > r(A),$$

where

$$r(A) := [r_1(A), ..., r_n(A)]^T$$

is the column vector of deleted row sums, and the column vector of moduli of diagonal entries is denoted by

$$d(A) := [|a_{11}|, ..., |a_{nn}|]^T.$$

The fact that SDD matrices are nonsingular is an old result that appeared in the works of Lévy (1881), [51], Desplanques (1887), Minkowski (1900) and Hadamard (1903).

Theorem 5 (Lévy-Desplanques) *If the matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an SDD matrix, then it is nonsingular.*

It is clear that SDD property does not depend on the order of rows. Also, only moduli of entries play a role in determining whether a given matrix is SDD. Therefore, the following statement clearly holds.

Theorem 6 *The class of SDD matrices is closed under conjugations and simultaneous permutations of rows and columns.*

In other words, if the matrix $A \in \mathbb{C}^{n,n}$ is SDD, then the conjugate matrix, $\overline{A}$ is also SDD.
If P is a permutation matrix of order n and the matrix $A \in \mathbb{C}^{n,n}$ is SDD, then the permuted matrix P^TAP is also SDD. This is because in the matrix P^TAP only the order of rows is changed, and the order of the entries inside the row is changed, but the i–th row in A and the corresponding row in P^TAP consist of the same set of entries, and diagonal entries remain on the diagonal.

We introduced the class of H-matrices in the way that is most often seen in books, starting from the definition of M-matrices. There are many similar definitions that emphasize the relation between these two concepts. But, among many different characterizations of the class of H-matrices, one of them is most revealing when it comes to clarifying the relation of this class to the concept of diagonal dominance. The following theorem gives the relation of SDD property and the class of H-matrices, see [33].

Theorem 7 (Fiedler-Pták) *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix if and only if there exists a diagonal nonsingular matrix W such that AW is an SDD matrix. Moreover, we can always assume that W has only positive diagonal entries.*

In other words, the class of H-matrices is exactly the class of generalized diagonally dominant (GDD) matrices, as they are often called in the literature.

The idea for constructing this diagonal scaling matrix W is the following. If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix, then the comparison matrix $\langle A \rangle$ is an M-matrix. From Theorem 1, there exists a positive vector z such that

$$\langle A \rangle z > 0.$$

It can be proved that for a diagonal matrix W constructed as

$$W = diag(z),$$

it holds that W is nonsingular and AW is SDD.

Reverse, if there exists nonsingular, diagonal matrix W such that AW is SDD, for vector z defined as

$$z = [|w_{11}|,\ |w_{22}|,\ \ldots,|w_{nn}|]^T,$$

it holds that

$$\langle A \rangle z > 0.$$

Therefore, in some considerations (for example, in scaling $PH-$matrices) the search for a diagonal scaling matrix is conducted as the search for a corresponding vector from Theorem 1.

As stated in [84], the class of H-matrices is diagonally derived from the class of SDD matrices. This means that the wider class is obtained by multiplying all the SDD matrices from the right with all the nonsingular diagonal matrices of the corresponding order.

It is clear from the form of AW that this transformation represents scaling of whole columns in the given matrix,

$$AW = \begin{bmatrix} a_{11}w_1 & a_{12}w_2 & \dots & a_{1n}w_n \\ a_{21}w_1 & a_{22}w_2 & \dots & a_{2n}w_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}w_1 & a_{n2}w_2 & \dots & a_{nn}w_n \end{bmatrix}.$$

For the given H-matrix, A, there exists a diagonal matrix W with positive diagonal entries that transforms A into an SDD matrix by multiplying it from the right, but for the fixed A, the matrix W with this property is not unique - one can transform a given H-matrix to SDD in many different ways.

Example 3 *Consider the matrix*

$$A = \begin{bmatrix} 25 & 1 & 1 & 3 \\ 0 & 5 & 2 & 6 \\ 5 & 1 & 5 & 3 \\ 5 & 1 & 2 & 15 \end{bmatrix}.$$

It is obvious that A is not an SDD matrix. But, it is an H-matrix, because it can be transformed to an SDD matrix by multiplying A from the right with diagonal nonsingular matrix. As one scaling matrix we can take the following matrix W,

$$W = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix},$$

because

$$AW = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 0 & 5 & 2 & 2 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 2 & 5 \end{bmatrix},$$

which is obviously SDD.

Having in mind the statement of Theorem 3, we recall a column–concept analogous to SDD.

Definition 7 *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is strictly diagonally dominant by columns if*

$$|a_{ii}| > c_i(A), \ \textit{for all} \ \ i \in N,$$

where

$$c_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ji}|$$

is the i-th deleted column sum.

In other words, $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is strictly diagonally dominant by columns if

$$d(A) > c(A),$$

where

$$c(A) := [c_1(A), ..., c_n(A)]^T$$

is the column vector of deleted column sums, and the column vector of moduli of diagonal entries is, again, denoted by

$$d(A) := [|a_{11}|, ..., |a_{nn}|]^T.$$

Matrices that are SDD by columns are, clearly, nonsingular and, moreover, they are H-matrices.

It is interesting to point out one well-known necessary condition for a given square complex matrix A to be an H-matrix. It is easy to see that the following is true.

Theorem 8 *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix, then A has at least one SDD row and at least one SDD column.*

Namely, if $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix, then, there exists a diagonal nonsingular matrix $W = diag(w_1, ..., w_n)$ with positive diagonal entries, such that AW is SDD. Let

$$w_k := \min_{i \in N} \{w_i\}.$$

As AW is SDD, it holds

$$|a_{kk}| w_k > \sum_{j \in N \setminus \{k\}} |a_{kj}| w_j,$$

which implies

$$|a_{kk}| > \sum_{j\in N\setminus\{k\}} |a_{kj}|\frac{w_j}{w_k} \geq \sum_{j\in N\setminus\{k\}} |a_{kj}| = r_k(A).$$

Therefore, the k-th row is SDD. The same reasoning works for columns, as well.

In Chapter 3, the main idea is to relax the SDD condition, i.e., to allow one or more non-SDD rows in a matrix, but in such a way that the new, weaker condition still defines a subclass of nonsingular H-matrices. In order to achieve this, according to Theorem 8, we have to preserve at least one SDD row and at least one SDD column.

2.3 General H–matrices

Classes of M- and H-matrices, as we defined and used them in this dissertation, consist only of **nonsingular** matrices. However, there are more general definitions of M- and H-matrices that include singular matrices. In [4, 82] one can find the following, more general definition of M-matrices.

For a given Z-matrix, $A = [a_{ij}] \in \mathbb{R}^{n,n}$, let us first denote by μ the maximum of diagonal entries in A, as done in [82], i.e.,

$$\mu = \max_{i\in N}\{a_{ii}\}.$$

Then,

$$A = \mu I - B,$$

where

$$B \geq 0.$$

In [82], for a Z-matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and $A = \mu I - B$ a splitting of A as described above, with $B \geq O$, it is said that A is a (general) M-matrix if $\mu \geq \rho(B)$.

More precisely, A is a nonsingular (general) M-matrix if $\mu > \rho(B)$, and a singular (general) M-matrix if $\mu = \rho(B)$.

Given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, then A is a (general) H-matrix if $\langle A\rangle$ is a (general) M-matrix.

Also in [82], the proof is given for the following statement. Given any $A = [a_{ij}] \in \mathbb{C}^{n,n}$ for which $\langle A\rangle$ is a nonsingular M-matrix, then A is a nonsingular H-matrix.

Note that the reverse statement is not true. If A is a general H-matrix, it can be invertible while its comparison matrix, $\langle A\rangle$, is singular!

Example 4 *If we recall matrices A and B of Example 2, we see that for the matrix B,*

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

it holds that $\langle B \rangle = I - C$, with $C \geq 0$, and $\rho(C) = 1$. Therefore, $\langle B \rangle$ is a singular general M-matrix, which implies that B is a general H-matrix. But, it is clear that B is nonsingular!

Therefore, when talking about general M- and general H-matrices, there exist nonsingular general H-matrices with singular comparison matrix.

Let us also point out that in general M- and general H-matrices (unlike before) there can be zero diagonal entries, as the following simple example shows.

Example 5 *Consider the following matrix,*

$$A = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

Clearly, $\langle A \rangle = A = I - B$ with $B \geq 0$ and $\rho(B) = 1$. Therefore, A is a singular general M-matrix and a singular general H-matrix, although there is a zero diagonal entry.

In [8] these facts are emphasized and a classification of the set of general H-matrices is obtained. Furthermore, in the paper [9], the Schur complement properties for general H-matrices are examined.

We point out here that the term "equimodular set" has a slightly different meaning in [8, 9] than the one that was used in [82] in connection with eigenvalue localization problems and Geršgorin type theorems.

In the paper [8] it is stated that there are three essentially different types of general H-matrices.

First, when the given general H-matrix A has nonsingular comparison matrix, $\langle A \rangle$, then all the matrices in the set $\Omega(A)$ are nonsingular. The class of H-matrices with nonsingular comparison matrix is called *invertible class*. This class coincides with H-matrices as defined at the beginning. In the following chapters we deal only with this first type of H-matrices.

Second, the class of general H-matrices with singular comparison matrix and the property that all the matrices in $\Omega(A)$ set are singular is called the *singular class*. The third, *mixed class* of general H-matrices, consists of those general H-matrices for which the comparison matrix is singular, but there exists a nonsingular matrix in the set $\Omega(A)$.

Invertible H-matrices are characterized also as such matrices for which the spectral radius of the corresponding Jacobi matrix is strictly less than 1. For matrices in mixed class, the spectral radius of the corresponding Jacobi matrix is equal to 1. Also, for a matrix in the mixed class, all diagonal entries are nonzero. The singular class is characterized by the existence of zero diagonal entries.

Now, if we consider again matrices B of Example 2 and A of Example 5, it is easy to see that matrix B is in the mixed class, while matrix A belongs to the singular class.

For further generalizations of SDD property and H-matrix theory, see the paper [1] on diagonally dominant infinite matrices as operators on l^p spaces.

From everything said in this chapter, it is clear that the theory of M- and H-matrices, that once started as homage to H. Minkowski and J. Hadamard paid by Ostrowski in 1937, has grown into a very interesting and applicable research area.

Today, there are more than 70 equivalent characterizations of M-matrices. A very extensive research on different conditions that define M-matrices and their relations, can be found in the book [4]. Also, in [82], it is once again stated that this theory has proved to be an incredibly useful tool in linear algebra.

Chapter 3

Subclasses of H–matrices

3.1 Breaking the SDD

Throughout this section, we deal with classes of nonsingular matrices ”between” the class of SDD matrices and the class of H-matrices. We could interpret definitions of subclasses of the class of H-matrices as sufficient conditions for a matrix to be an H-matrix. They are also sufficient conditions for nonsingularity of matrices. When it comes to practical applications, it is very useful to have simple conditions, easy to check, with not too many calculations. In that way, we are able to recognize some H-matrices without calculating the inverse matrix, which is crucial, especially when dealing with large matrices. But, this is not the only motive for introducing and analyzing these subclasses. It is well-known that SDD matrices have many nice properties, but some of them do not hold for H-matrices in general, while they do hold for matrices in some special subclasses of H-matrices. Also, bounds for eigenvalues and for the norm of the inverse matrix are developed for some special subclasses of H-matrices and these results at the same time can give tighter estimates when applied back to SDD matrices.

Therefore, the main goal in the following section is to ”break” the SDD property in such a way that the new condition, now weaker than SDD condition, is still simple enough, and defines a new subclass of nonsingular H-matrices.

There are many ways to relax the SDD property - one could allow one or more rows not to be SDD, or one could replace strict inequalities with non-strict inequalities. One way is, also, to replace deleted row sums in the definition of SDD with different type of row sums, or, to combine row sums and column sums in defining new dominance–based conditions.

We will group subclasses of H-matrices into five different categories, based on different ideas of how to "break" the SDD property. These ideas, as directions of generalizations of SDD, are presented in the following diagram.

$$
\begin{array}{ccccc}
 & & \begin{bmatrix} Multiplicative \\ and\ additive \\ conditions \end{bmatrix} & & \\
 & & \uparrow & & \\
\begin{bmatrix} Conditions \\ by\ nonstrict \\ inequalities \end{bmatrix} & \leftarrow & \begin{bmatrix} SDD \end{bmatrix} & \rightarrow & \begin{bmatrix} Partition- \\ based \\ conditions \end{bmatrix} \\
 & & \swarrow \quad \searrow & & \\
 & \begin{bmatrix} Recursively \\ defined \\ row\ sums \end{bmatrix} & & \begin{bmatrix} Columns \\ and\ rows \\ combined \end{bmatrix} &
\end{array}
$$

In the first chapter, motivation for introducing these conditions is explained. For each new subclass of H-matrices, (i.e., for each elegant new condition sufficient for a given matrix to be a nonsingular H-matrix), new possibilities for applications arise. The most important fields where application of new subclasses of H-matrices produces new results are briefly recalled in the following diagram.

$$
\begin{array}{ccccc}
 & \begin{bmatrix} Conditions \\ for \\ nonsingularity \end{bmatrix} & & \begin{bmatrix} Eigenvalue \\ localization \\ sets \end{bmatrix} & \\
 & & \nwarrow \quad \nearrow & & \\
\begin{bmatrix} Max-norm \\ bounds\ for \\ the\ inverse \end{bmatrix} & \leftarrow & \begin{bmatrix} APPLICATIONS \end{bmatrix} & \rightarrow & \begin{bmatrix} Schur \\ complement \\ properties \end{bmatrix} \\
 & \swarrow & \downarrow & \searrow & \\
\begin{bmatrix} Convergence\ of \\ iterative \\ methods \end{bmatrix} & & \begin{bmatrix} Subdirect \\ sums\ of \\ matrices \end{bmatrix} & & \begin{bmatrix} Bounds \\ for \\ determinants \end{bmatrix}
\end{array}
$$

In the remainder of Chapter 3, many subclasses of the class of H-matrices that arise from relaxing the SDD condition will be presented. In Chapter 4, applications of these subclasses to determining max-norm bounds of the inverse will be considered, together with applications to eigenvalue localization problems. In Chapter 5, applications of these subclasses to Schur complement related problems will be presented.

3.2 Multiplicative and additive conditions

In order to define a wider class of nonsingular matrices starting from the SDD class, one idea is to consider products (sums) of diagonal entries and compare them to products (sums) of the corresponding deleted row sums. We will call conditions involving products of deleted row sums *multiplicative conditions*, while conditions involving sums of deleted row sums are called *additive conditions*. We start this section with a well-known multiplicative condition, a famous result of Ostrowski, that was the starting point for many different generalizations and applications. We proceed with the result of Pupkov and Solov'ev that belongs to additive conditions.

3.2.1 Ostrowski matrices

When it comes to generalizations of the SDD property, one of the first ideas is to compare product of two diagonal entries and product of corresponding two deleted row sums, for every choice of two different rows in a given matrix. This is the idea of Ostrowski, see [65], also discussed in [82].

Definition 8 *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is an Ostrowski matrix if*

$$|a_{ii}||a_{jj}| > r_i(A)r_j(A)\ ,\ \textit{for all}\ \ i,j \in N,\ \ i \neq j.$$

Obviously, it is possible for at most one row in an Ostrowski matrix not to be SDD. The Ostrowski class consists of matrices with at most one non–SDD row.

Theorem 9 ([65]) *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is an Ostrowski matrix, then* A *is nonsingular.*

There is a straight forward proof of Ostrowski for this statement, but here, in order to emphasize once more the role of scaling matrices, we discuss a construction of a scaling matrix for a given Ostrowski matrix. By constructing a nonsingular diagonal matrix that (by multiplication from the right) transforms a given Ostrowski matrix to an SDD matrix, it can be verified that every Ostrowski matrix is an H-matrix and nonsigularity directly follows.

The form of the scaling matrix for the given Ostrowski matrix is itself revealing and useful for our further investigations of Ostrowski matrices and their Schur complements.

If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an Ostrowski matrix, then, all diagonal entries of A are nonzero and for at most one index k in the index set N it holds that $|a_{kk}| \le r_k(A)$. It is obvious that Ostrowski class is closed under simultaneous permutations of rows and columns, as these permutations do not affect the set of values of deleted row sums (only their order). Let us, therefore, assume, without loss of generality, that for $k = 1$,

$$|a_{11}| \le r_1(A), \ \ |a_{ii}| > r_i(A), \ i \in \{2,3,\dots,n\}$$

and,

$$|a_{11}||a_{ii}| > r_1(A)r_i(A), \ i \in \{2,3,\dots,n\}.$$

Now, we construct a diagonal matrix with positive diagonal entries as follows.

$$W = \begin{bmatrix} \gamma & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

The matrix AW is SDD if it holds that

$$\gamma|a_{11}| > r_1(A), \tag{3.1}$$

$$|a_{ii}| > r_i(A) + (\gamma - 1)|a_{i1}|, \ \ i \in \{2,3,\dots,n\}. \tag{3.2}$$

For each index $i \in \{2,3,\dots,n\}$, such that $a_{i1} = 0$, the condition (3.2) holds for any γ, so, conditions (3.1) and (3.2) are both fulfilled if we choose γ as follows:

$$\frac{r_1(A)}{|a_{11}|} < \gamma < \min_{\substack{i \ne 1 \\ a_{i1} \ne 0}} \frac{|a_{ii}| - r_i(A) + |a_{i1}|}{|a_{i1}|}. \tag{3.3}$$

In order to do that, this interval for γ has to be a nonempty interval. In other words, we need

$$\frac{r_1(A)}{|a_{11}|} < \frac{|a_{ii}| - r_i(A) + |a_{i1}|}{|a_{i1}|},$$

for each $i \in \{2,3,\dots,n\}$, such that $a_{i1} \ne 0$. This is true, because

$$r_1(A)|a_{i1}| < |a_{11}|\big(|a_{ii}| - r_i(A) + |a_{i1}|\big).$$

More precisely, for each $i \in \{2,3,\dots,n\}$, we have

$$|a_{11}|\big(|a_{ii}| - r_i(A) + |a_{i1}|\big) = |a_{11}||a_{ii}| - |a_{11}|\big(r_i(A) - |a_{i1}|\big) >$$

$$> r_1(A)r_i(A) - |a_{11}|\big(r_i(A) - |a_{i1}|\big) =$$

$$= r_1(A)\big(r_i(A) - |a_{i1}| + |a_{i1}|\big) - |a_{11}|\big(r_i(A) - |a_{i1}|\big) =$$

$$= \big(r_1(A) - |a_{11}|\big)\big(r_i(A) - |a_{i1}|\big) + r_1(A)|a_{i1}| \geq r_1(A)|a_{i1}|,$$

as both $\big(r_1(A) - |a_{11}|\big)$ and $\big(r_i(A) - |a_{i1}|\big)$ are nonnegative. Therefore, the interval (3.3) for the parameter γ is not empty and there exists a diagonal matrix W, with positive diagonal entries, such that AW is SDD, meaning that A is an H-matrix.

Notice that this consideration provides a "good" interval for choosing this (one) positive parameter, γ, for the given Ostrowski matrix.

3.2.2 Pupkov-Solov'ev matrices

In the same manner as it is done in Ostrowski condition, one could try the same idea with sums instead of products. It turns out that the condition obtained in this way does not guarantee nonsingularity, as the following example shows.

Example 6 *The matrix*

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

satisfies the condition $|a_{11}| + |a_{22}| > r_1(A) + r_2(A)$, *but it is, clearly, a singular matrix.*

But, in the papers of Pupkov, [69, 70], and Solov'ev, [73], one can find an additive condition of this type that does guarantee nonsingularity of a given matrix. We call matrices that satisfy this condition Pupkov-Solov'ev matrices.

Definition 9 *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a Pupkov-Solov'ev matrix if*

$$|a_{ii}| > \min(r_i(A), \max_{j \neq i}\{|a_{ji}|\}), \text{ for all } i \in N,$$

and

$$|a_{ii}| + |a_{jj}| > r_i(A) + r_j(A)\ , \text{ for all } j \in N,\ j \neq i.$$

Theorem 10 ([70, 73]) *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a Pupkov-Solov'ev matrix, then* A *is nonsingular.*

Notice that in a Pupkov-Solov'ev matrix there can be at most one non-SDD row, as well as it was the case with Ostrowski matrices. Also, Pupkov-Solov'ev class is closed under simultaneous permutations of rows and columns. It is easy to prove this statement, as deleted row sums do not change their values under simultaneous permutations (only their order is changed) and the set of entries in each column of the matrix does not change (only their order is changed).

This result was a starting point for further generalizations, (some of which involved the notion of irreducibility and applications to block-matrices), new nonsingularity conditions and new characterizations of M-matrices, see [40, 75]. Some generalizations of Pupkov-Solov'ev results and interpretations in eigenvalue localization problems were also considered in [82], where it is emphasized that this eigenvalue inclusion set is not easy to implement and that there are still many open questions on how this set compares to other well-known eigenvalue inclusion sets. However, although not as simple for application as Geršgorin's result, this condition is interesting when it comes to interpretations of Pupkov-Solov'ev property in different mathematical models.

3.3 Partition–based conditions

As we have seen in the previous section, it is possible to relax the SDD condition and allow one row to be non–SDD, yet still remain inside the class of nonsingular H-matrices. This can be done if we compare products (sums) of deleted row sums to products (sums) of corresponding diagonal entries. What happens if we want to relax the SDD condition even more and allow more than one row with broken strict diagonal dominance?

In the literature there are conditions of this type based on the idea of partitioning the index set of a matrix and defining dominance–based conditions on the matrix parts, i.e., using parts of deleted row sums.

Well–known classes of Dashnic–Zusmanovich (DZ) matrices, S-SDD and Σ-SDD matrices, as well as classes of $PM^{\pi}-$ and $PH^{\pi}-$matrices, are all based on different partitions of the index set of a matrix.

3.3.1 Dashnic–Zusmanovich (DZ) matrices

In papers [29, 30], an interesting nonsingularity result is presented. It led to some very useful and applicable generalizations that appeared in papers [34, 43, 24] related to S-SDD matrices, and, later, in [47, 48], related to $PH^{\pi}-$matrices.

Definition 10 *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a Dashnic–Zusmanovich (DZ) matrix if there exists an index* $i \in N$ *such that*

$$|a_{ii}|\,(|a_{jj}| - r_j(A) + |a_{ji}|\,) > r_i(A)|a_{ji}|\,,\ \textit{for all}\ \ j \neq i\,,\ \ j \in N.$$

DZ matrices are nonsingular matrices, which was proved in [29, 30].

Theorem 11 ([29, 30]) *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a DZ matrix, then it is nonsingular.*

They are also H-matrices, as directly follows from the scaling characterization of this matrix class. Although there is a straight forward proof for this statement, we discuss here scaling characterization in more detail, as the construction of scaling matrices will be useful in dealing with Schur complements of DZ matrices in the fifth chapter. Even more, the fact that the form of scaling matrices gives *a complete, equivalent characterization* of DZ matrix class will be crucial when investigating Schur complement closure properties.

3.3.2 Scaling characterization of DZ matrices

DZ class can be characterized as the subclass of H-matrices for which the corresponding scaling matrix W belongs to the set $\mathcal{F}$, defined as the set of diagonal matrices whose diagonal entries are equal to 1, all except one diagonal entry, which is a positive number.

$$\mathcal{F} = \{W = diag(w_1, ..., w_n) : w_i = \gamma > 0 \ \text{for one} \ i \in N,\ w_j = 1 \ \text{for} \ j \neq i\} \quad (3.4)$$

As one can see, each DZ matrix can be transformed to an SDD matrix only by multiplying all entries in one column of the matrix with a corresponding positive number, γ, that we will call a *scaling parameter*. For $i = 1$, the scaling matrix and the scaled matrix are formed as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \gamma & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} a_{11}\gamma & a_{12} & \dots & a_{1n} \\ a_{21}\gamma & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\gamma & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The following statement is implicitly present in papers of Dashnic and Zusmanovich, [29, 30], in terms of eigenvalue localization sets. In [49] it is presented in a different form, as a complete characterization of the matrix class considered.

Theorem 12 ([49]) *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a DZ matrix if and only if there exists a matrix* $W \in \mathcal{F}$ *such that* AW *is an SDD matrix.*

Let us discuss here in more detail a construction of the scaling matrix and the choice of the scaling parameter γ, as it will be useful in the following chapters.

For a DZ matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$, there exists an index $i\in N$, such that

$$|a_{ii}|\,(|a_{jj}|-r_j(A)+|a_{ji}|\,)>r_i(A)|a_{ji}|\,,\ \text{for all}\ j\neq i\,,\ j\in N.$$

Assume, without loss of generality, that $i=1$, i.e.,

$$|a_{11}|\,(|a_{jj}|-r_j(A)+|a_{j1}|\,)>r_1(A)|a_{j1}|\,,\ \text{for all}\ j\neq 1\,,\ j\in N.$$

Obviously, it follows that $|a_{11}|>0$, i.e., $a_{11}\neq 0$.

If, for some $j\in N\setminus\{1\}$, it holds that $a_{j1}=0$, then, from A being a DZ matrix, the j-th row in A is already SDD. In other words, if $a_{j1}=0$, then $|a_{jj}|>r_j(A)$.

Consider, now, for a diagonal matrix $W=diag(\gamma,1,...,1)$, with $\gamma>0$, the matrix AW, formed as above. AW is SDD if and only if

$$\gamma|a_{11}|>r_1(A),\ \text{and}$$

$$|a_{jj}|>|a_{j1}|\gamma+r_j(A)-|a_{j1}|,\ \text{for all}\ j\in N,\ j\neq 1.$$

This will be true if, for all $j\in N$, $j\neq 1$, for which $a_{j1}\neq 0$, it holds that

$$\frac{r_1(A)}{|a_{11}|}<\gamma<\frac{|a_{jj}|-r_j(A)+|a_{j1}|}{|a_{j1}|}.$$

As the given matrix is a DZ matrix, the interval for scaling parameter γ is not empty. Therefore, there exists $W\in\mathcal{F}$ such that AW is an SDD matrix.

Reverse, if we assume that there exists a diagonal nonsingular scaling matrix $W\in\mathcal{F}$, formed as described above, such that AW is SDD, then, the interval $(\gamma_1(A),\gamma_2(A))$ for the scaling parameter γ, with

$$\gamma_1(A)=\frac{r_1(A)}{|a_{11}|},\quad \gamma_2(A)=\min_{j\in N\setminus\{1\},\,a_{j1}\neq 0}\frac{|a_{jj}|-r_j(A)+|a_{j1}|}{|a_{j1}|},$$

is not empty, which implies that A is a DZ matrix.

As it is done in this consideration, one can always construct a corresponding diagonal scaling matrix, W, for the given, fixed, DZ matrix A. Notice that the previous theorem gives even more information than that. It precisely describes the class of scaling matrices for the class of DZ matrices, meaning that the class of scaling matrices, in this case, is itself an equivalent definition of the DZ class. In other words, different from the situation with Ostrowski matrices, this consideration does not just provide a construction of a scaling matrix. It contains a reverse statement, too - if a matrix can be scaled to SDD in this way, it is a DZ matrix.

This reverse direction is crucial in investigation of SC closure properties.

3.3.3 Σ-SDD matrices

The following class of matrices is made of SDD class by partitioning the index set into two disjoint subsets. It was defined in a slightly different way in [34] and discussed in [24].

Let us denote the part of the i–th deleted row sum that corresponds to the subset S by

$$r_i^S(A) := \sum_{k\in S, k\neq i} |a_{ik}|.$$

Obviously, for arbitrary subset S and each index $i \in N$,

$$r_i(A) = r_i^S(A) + r_i^{\overline{S}}(A).$$

Definition 11 *Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, and given any nonempty subset S of N, then A is an S-strictly diagonally dominant (S-SDD) matrix if*

$$|a_{ii}| > r_i^S(A) \text{ for all } i \in S \text{ and}$$

$$\left(|a_{ii}| - r_i^S(A)\right)\left(|a_{jj}| - r_j^{\overline{S}}(A)\right) > r_i^{\overline{S}}(A) r_j^S(A) \text{ for all } i \in S,\ j \in \overline{S}.$$

If there exists a nonempty subset S of N, such that $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is an S-SDD matrix, then we say that A belongs to the class of Σ-SDD matrices.

Notice from the previous definition that in S-SDD matrix A it also holds

$$|a_{jj}| > r_j^{\overline{S}}(A) \text{ for all } j \in \overline{S}.$$

Therefore, both submatrices $A(S)$ and $A(\overline{S})$ are SDD.

3.3.4 Scaling characterization of Σ-SDD matrices

Σ-SDD class can be characterized as the subclass of H-matrices for which the corresponding scaling matrix W belongs to the set $\mathcal{W}$, defined as the set of all diagonal matrices whose diagonal entries are either 1 or γ, where γ is an arbitrary positive number, i.e.

$$\mathcal{W} = \bigcup_{S\subset N} \mathcal{W}^S, \tag{3.5}$$

$$\mathcal{W}^S = \{W = diag(w_1, w_2, ..., w_n) : w_i = \gamma > 0 \text{ for } i \in S \text{ and } w_i = 1 \text{ otherwise}\}.$$

Theorem 13 ([24]) *Let S be a nonempty subset in N. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is an S-SDD matrix if and only if there exists a matrix $W \in \mathcal{W}^S$ such that AW is an SDD matrix.*

Again, we discuss a construction of a corresponding scaling matrix for the given S-SDD matrix in more detail and recall bounds for the scaling parameter γ. It will be useful for investigating eigenvalues of Schur complements of S-SDD matrices in Chapter 5. Also, we emphasize the equivalence, i.e., the reverse direction in the statement of Theorem 13, as it is important in SC closure problems.

For $S \subseteq N$, define diagonal matrix $W(S,\gamma) \in \mathcal{W}^S$ in the following way:

$$W(S,\gamma) = diag(w_1, w_2, ..., w_n) ,$$

where

$$w_i = \begin{cases} \gamma, & i \in S, \\ 1, & i \in \overline{S}. \end{cases}$$

One could show that, for the given subset S of the index set, the matrix A is an S-SDD matrix if and only if there exists a positive number, γ, such that $AW(S,\gamma)$ is an SDD matrix.

If $S = N$, it is clear that A is SDD if and only if $AW(S,\gamma) = \gamma A$ is SDD.

If assumed that

$$S = \{1,2,\dots,k\}, \ \overline{S} = \{k+1,k+2,\dots,n\},$$

matrices we are dealing with look like this:

$$\underbrace{\left[\begin{array}{ccc|ccc} a_{11} & \dots & a_{1k} & a_{1,k+1} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} & a_{k,k+1} & \dots & a_{kn} \\ \hline a_{k+1,1} & \dots & a_{k+1,k} & a_{k+1,k+1} & \dots & a_{k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nk} & a_{n,k+1} & \dots & a_{nn} \end{array}\right]}_{A} \underbrace{\left[\begin{array}{ccc|ccc} \gamma & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \gamma & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{array}\right]}_{W(S,\gamma)}$$

$$= \left[\begin{array}{ccc|ccc} a_{11}\gamma & \dots & a_{1k}\gamma & a_{1,k+1} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1}\gamma & \dots & a_{kk}\gamma & a_{k,k+1} & \dots & a_{kn} \\ \hline a_{k+1,1}\gamma & \dots & a_{k+1,k}\gamma & a_{k+1,k+1} & \dots & a_{k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}\gamma & \dots & a_{nk}\gamma & a_{n,k+1} & \dots & a_{nn} \end{array}\right].$$

Now, one could verify that the matrix A is an S-SDD matrix if and only if there exists a positive γ, such that $AW(S,\gamma)$ is an SDD matrix.

First, assume that A is an S-SDD matrix. Choose γ from the interval

$$(\gamma_1(A),\gamma_2(A)), \tag{3.6}$$

with

$$0 \le \gamma_1(A) := \max_{i\in S} \frac{r_i^{\overline{S}}(A)}{|a_{ii}| - r_i^S(A)}\ ,\ \ \gamma_2(A) := \min_{j\in\overline{S},\, r_j^S(A)\neq 0} \frac{|a_{jj}| - r_j^{\overline{S}}(A)}{r_j^S(A)},$$

where, if $r_j^S(A) = 0$ for all $j \in \overline{S}$, then, $\gamma_2(A)$ is defined to be $+\infty$. Note that, according to the definition of S-SDD matrices, the interval $(\gamma_1(A),\gamma_2(A))$ is not empty. Now, it is easy to check that $AW(S,\gamma)$ is an SDD matrix.

Reverse, if assumed that for some positive γ, $AW(S,\gamma)$ is an SDD matrix, then γ has to be chosen from the interval $(\gamma_1(A),\gamma_2(A))$, which means that this interval is not empty. But, this implies that matrix A is an S-SDD matrix.

This consideration gives us the way of constructing a diagonal scaling matrix W for the given S-SDD matrix, A. But, also, it gives one more characterization of the S-SDD class, using the form of scaling matrices. In many problems, this scaling characterization is more revealing and more elegant than the classical definition. For instance, it is now obvious, from scaling characterizations, that DZ class is a subclass of Σ-SDD class, and, that it is obtained by choosing S to be a singleton. Also, from the previous theorem directly follows the next one.

Theorem 14 ([24]) *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a Σ-SDD matrix, then it is an H-matrix and therefore nonsingular.*

3.3.5 $PM^{\pi}-$ and $PH^{\pi}-$matrices

These classes of matrices also arise from SDD via partitioning the index set, but unlike in forming the S-SDD class, here, partitioning into more than two disjoint subsets is allowed.

For the sum of all the entries in one row of the matrix the following notation is used,

$$R_i(A) = \sum_{j=1}^{n} a_{ij},\ \ i = 1,\dots,n.$$

Note that, unlike in a deleted row sum, $r_i(A)$, here, diagonal entry is included. Also, in $R_i(A)$, entries are taken as they are, a_{ij} instead of $|a_{ij}|$.

Let us now recall the definitions of $PM^{\pi}-$ and $PH^{\pi}-$matrices, as they were given in [47, 48]. In order to do that, a partition of the index set of a matrix is needed and, also, we need to recall the definition of aggregated matrices.

For the given partition $\pi = \{p_j\}_{j=0}^{\ell}$, that divides the index set N into ℓ disjoint nonempty subsets $S_1, S_2, \ldots, S_\ell$, where

$$S_j = \{p_{j-1}+1, p_{j-1}+2, \ldots, p_j\}, \;\; j = 1,2,\ldots,\ell \tag{3.7}$$

and for the matrix A in block form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1\ell} \\ A_{21} & A_{22} & \cdots & A_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\ell 1} & A_{\ell 2} & \cdots & A_{\ell\ell} \end{bmatrix} = [A_{ij}]_{\ell\times\ell}, \tag{3.8}$$

we define the collection of aggregated matrices of order ℓ, as follows,

$$A^{(i_1,i_2,\ldots,i_\ell)} = \begin{bmatrix} R_{i_1}(A_{11}) & R_{i_1}(A_{12}) & \cdots & R_{i_1}(A_{1\ell}) \\ R_{i_2}(A_{21}) & R_{i_2}(A_{22}) & \cdots & R_{i_2}(A_{2\ell}) \\ \vdots & \vdots & \ddots & \vdots \\ R_{i_\ell}(A_{\ell 1}) & R_{i_\ell}(A_{\ell 2}) & \cdots & R_{i_\ell}(A_{\ell\ell}) \end{bmatrix}, \tag{3.9}$$

where $i_k \in S_k$, $k = 1,\ldots,\ell$.

Definition 12 ([47]) *Given any $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ and given a partition of N, $\pi = \{p_j\}_{j=0}^{\ell}$, A is a $PM-$matrix with respect to the partition π, i.e., A is a $PM^{\pi}-$matrix, if A is a $Z-$matrix and all the aggregated matrices (3.9) are (nonsingular) $M-$matrices.*

In the same manner as done before, a complex generalization is defined.

Definition 13 ([47]) *Given any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ and given a partition of N, $\pi = \{p_j\}_{j=0}^{\ell}$, A is a $PH-$matrix with respect to the partition π, i.e., A is a $PH^{\pi}-$matrix, if $\langle A\rangle$ is a $PM-$matrix with respect to the same partition N, i.e., if $\langle A\rangle$ is a PM^{π}-matrix.*

Considering matrices in $\mathbb{C}^{n,n}$ and the finest possible partition of the index set N, $\pi = \{0,1,2,\ldots,n\}$, $PM^{\pi}-$ ($PH^{\pi}-$)matrices represent the class of (nonsingular) $M-$matrices ($H-$matrices) in $\mathbb{C}^{n,n}$. If $\ell = 1$, or, in other words, for $\pi = \{0,n\}$, the class of $PH^{\pi}-$matrices in $\mathbb{C}^{n,n}$ is the class of SDD matrices in $\mathbb{C}^{n,n}$.

For some purposes, it is more convenient to use a different notation. Namely, if for a given matrix A in $\mathbb{C}^{n,n}$ there exists any partition π of the index set N into k disjoint nonempty subsets, (not necessarily of the form (3.7)), such that A is a $PH^{\pi}-$matrix, we say that A belongs to $PH(k)$ class of matrices. Notice that, in this notation, the class of SDD matrices in $\mathbb{C}^{n,n}$ is actually $PH(1)$ class, while $PH(n)$ class is the class of H-matrices in $\mathbb{C}^{n,n}$.

If $\pi = \{0,m,n\}$, i.e., if the index set N is divided into two disjoint nonempty subsets:

$$S = \{1,2,\ldots,m\} \quad \text{and} \quad \overline{S} = \{m+1,m+2,\ldots,n\},$$

then aggregated matrices (3.9) have the following form:

$$A^{(i_1,i_2)} = \begin{bmatrix} R_{i_1}(A_{11}) & R_{i_1}(A_{12}) \\ R_{i_2}(A_{21}) & R_{i_2}(A_{22}) \end{bmatrix}, \; i_1 \in S, \; i_2 \in \overline{S}.$$

Following the notation, aggregated matrices for $\langle A \rangle$ look like this:

$$\langle A \rangle^{(i_1,i_2)} = \begin{bmatrix} |a_{i_1 i_1}| - r_{i_1}(A_{11}) & -R_{i_1}(|A_{12}|) \\ -R_{i_2}(|A_{21}|) & |a_{i_2 i_2}| - r_{i_2}(A_{22}) \end{bmatrix}, \; i_1 \in S, \; i_2 \in \overline{S}.$$

The matrix A is a $PH^{\pi}-$matrix if all the aggregated matrices $\langle A \rangle^{(i_1,i_2)}$ are $M-$matrices, i.e., if all their principal minors are positive,

$$|a_{i_1 i_1}| - r_{i_1}(A_{11}) > 0, \; i_1 \in S, \tag{3.10}$$

and, for all $i_1 \in S, \; i_2 \in \overline{S}$,

$$(|a_{i_1 i_1}| - r_{i_1}(A_{11}))(|a_{i_2 i_2}| - r_{i_2}(A_{22})) - R_{i_1}(|A_{12}|) R_{i_2}(|A_{21}|) > 0. \tag{3.11}$$

As for all $i_1 \in S, \; i_2 \in \overline{S}$,

$$r_{i_1}(A_{11}) = r^{S}_{i_1}(A), \quad R_{i_1}(|A_{12}|) = r^{\overline{S}}_{i_1}(A),$$

$$R_{i_2}(|A_{22}|) = r^{S}_{i_2}(A), \quad r_{i_2}(A_{22}) = r^{\overline{S}}_{i_2}(A),$$

we see that conditions (3.10) and (3.11) represent the definition of S-SDD matrices. In other words, the class $PH(2)$ is exactly the class of Σ-SDD matrices in $\mathbb{C}^{n,n}$.

The next result is given in [47].

Theorem 15 ([47]) *If there exists a partition π of the index set N such that $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ is a $PH^{\pi}-$matrix, then A is an $H-$matrix.*

In [47, 48], the following interesting properties for this class of matrices are proved. First, if $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ is a $PH^{\pi}-$matrix and if ν is a partition of the index set N finer than π, then, A is also a $PH^{\nu}-$matrix. A partition ν of the index set N into m disjoint nonempty subsets $M_1, M_2, \ldots, M_m$ is said to be finer than a partition π of the index set N into ℓ disjoint nonempty subsets $S_1, S_2, \ldots, S_\ell$, if $m > \ell$ and each of the sets $S_1, S_2, \ldots, S_\ell$ is a union of some sets $M_1, M_2, \ldots, M_m$. In that case, it is said also that partition π is coarser than partition ν.

Having this in mind, it is clear that for classes of matrices denoted by $PH(k)$ the following relation is true

$$PH(1) \subseteq PH(2) \subseteq ... \subseteq PH(n).$$

Second, if A in $\mathbb{C}^{n,n}$ is a $PH^{\pi}-$matrix where π divides the index set N into disjoint nonempty sets $S_1, S_2, \ldots, S_\ell$, then, there exists an index $k \in \{1,2,...,\ell\}$ such that all the rows in A indexed by indices from S_k are SDD. This property represents a generalization of a necessary condition for a given matrix to be an H-matrix that was presented in Theorem 8. Namely, Theorem 8 states that an H-matrix has at least one SDD row. Here, a necessary condition for a given matrix to be a $PH^{\pi}-$matrix is to have at least one block-row (with respect to a given partition π) consisting only of SDD rows.

Also, it is easy to note another interesting property of $PH^{\pi}-$matrices. Namely, if A in $\mathbb{C}^{n,n}$ is a $PH^{\pi}-$matrix where π divides the index set N into disjoint nonempty subsets $S_1, S_2, \ldots, S_\ell$, then all diagonal blocks of A in the block form with respect to π are SDD. In other words, $A(S_1),...,A(S_\ell)$ are all SDD submatrices in A. In the special case of S-SDD matrices, we already mentioned before that $A(S)$ and $A(\overline{S})$ are both SDD.

3.3.6 Scaling characterization of $PM^{\pi}-$ and $PH^{\pi}-$matrices

Classes SDD, DZ, Σ-SDD and H-matrices can all be treated as special cases of $PH-$classes, with different choices of partitions. It becomes obvious also from the scaling characterization of $PH^{\pi}-$matrices, that was given in [48].
First, let us recall the main equivalence result of [48] related to $PM^{\pi}-$matrices.

Theorem 16 ([48]) *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a Z-matrix and let $\pi = \{p_j\}_{j=0}^{\ell}$ be the partition of the index set N, that divides the index set N into ℓ disjoint nonempty subsets $S_1, S_2, \ldots, S_\ell$, where*

$$S_j = \{p_{j-1}+1, p_{j-1}+2, \ldots, p_j\}, \ \ j = 1,2,\ldots,\ell.$$

Then, A is a $PM^{\pi}-$matrix if and only if there exists a positive vector $x = [x_i] \in \mathbb{R}^n$ of the form

$$x_i = c_j \ \textit{for all} \ i \in S_j, \ \ j = 1,2,...,\ell,$$

such that the vector Ax is positive, i.e., such that AX is SDD where $X = diag(x)$.

In [48], the proof of Theorem 16 is given by a construction of the vector $x = [x_i] \in \mathbb{R}^n$. We recall here the idea of the proof and we use it later to construct a scaling matrix for the given $PM^{\pi}-$matrix ($PH^{\pi}-$matrix), A, in order to find a preliminary eigenvalue localization for the Schur complement of the given matrix A. The equivalence in Theorem 16 will be important for proving SC closure of $PM^{\pi}-$ ($PH^{\pi}-$) class in the fifth chapter.

For a $PM^{\pi}-$matrix A, in [48], it is first assumed that there are no zero off–diagonal entries. Also, since all the aggregated matrices are nonsingular M-matrices and therefore have positive diagonal entries, this implies

$$(A_{ii}e)_j > 0, \text{ for all } j \in S_i, \; i = 1, ..., \ell.$$

Because of this, we also may assume that

$$A_{ii}e = e, \; i = 1, ..., \ell.$$

In order to explain this, consider the diagonal matrix

$$D = diag(R_1(D_A), ..., R_n(D_A)),$$

where

$$D_A = diag(A_{11}, ..., A_{\ell\ell})$$

and consider $D^{-1}A$ instead of A. In this way, without loss of generality, it is assumed that

$$A_{ii}e = e, \; i = 1, ..., \ell.$$

Having this in mind, aggregated matrices can be represented as

$$A^{(i_1, i_2, ..., i_\ell)} = I_l - B^{(i_1, i_2, ..., i_\ell)}, \; i_j \in S_j, \; j = 1, ..., \ell,$$

where all the matrices $B^{(i_1, i_2, ..., i_\ell)}$ have positive off–diagonal entries, all diagonal entries equal to zero and, furthermore,

$$\rho(B^{(i_1, i_2, ..., i_\ell)}) < 1 \text{ for all } i_j \in S_j, \; j = 1, ..., \ell,$$

where $\rho(B^{(i_1, i_2, ..., i_\ell)})$ denotes the Perron root of the considered matrix, i.e., its nonnegative eigenvalue equal to the spectral radius. For

$$\rho_0 = \rho(B^{(i_1^{(0)}, i_2^{(0)}, ..., i_\ell^{(0)})}) = \max_{(i_1, i_2, ..., i_\ell)} \{\rho(B^{(i_1, i_2, ..., i_\ell)})\},$$

it holds that $\rho_0 < 1$. If $c = [c_j] \in \mathbb{R}^\ell$ denotes the Perron vector of the matrix $B^{(0)} = B^{(i_1^{(0)}, i_2^{(0)}, \ldots, i_\ell^{(0)})}$, i.e., $B^{(0)}c = \rho_0 c$, then it is proved in [48] that the scaling vector can be formed as

$$x_i = c_j \ \text{ for all } \ i \in S_j, \ \ j = 1, 2, ..., \ell. \tag{3.12}$$

Now we recall the scaling characterization of $PH^\pi-$matrices given in [48], only with different notation.

Theorem 17 ([48]) *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a $PH^\pi-$matrix, where π is the partition of the index set N into nonempty disjoint sets $S_1, ..., S_\ell$, if and only if there exists a matrix $W \in \mathcal{W}^\pi$ such that AW is an SDD matrix, where*

$$\mathcal{W}^\pi = \{W = diag(w_1, w_2, ..., w_n) : w_i = \gamma_j > 0 \textit{ for all } i \in S_j, \ \ j = 1, ..., \ell \ \}. \tag{3.13}$$

3.3.7 Construction of a scaling matrix for $PH^\pi-$matrices

We discuss here a modified construction of a diagonal scaling matrix for a given $PM^\pi-$matrix, in a less expensive way, in some cases.

As in [48], let A be a $PM^\pi-$matrix, where $\pi = \{p_j\}_{j=0}^{\ell}$ is the partition of the index set N, that divides N into ℓ disjoint nonempty subsets $S_1, S_2, \ldots, S_\ell$ and assume first that A has no zero off–diagonal entries. As all the aggregated matrices $A^{(i_1, i_2, \ldots, i_\ell)}$ are nonsingular M-matrices, their diagonal entries are positive. We may assume, without loss of generality, that all diagonal entries in aggregated matrices are equal to 1. Therefore, aggregated matrices can be represented in the following way

$$A^{(i_1, i_2, \ldots, i_\ell)} = I_l - B^{(i_1, i_2, \ldots, i_\ell)}, \ \ i_j \in S_j, \ \ j = 1, ..., \ell,$$

where all the matrices $B^{(i_1, i_2, \ldots, i_\ell)}$ have positive off–diagonal entries and zero diagonal entries.

In order to avoid calculation of spectral radius for all the matrices $B^{(i_1, i_2, \ldots, i_\ell)}$, let us construct the matrix $B^* = [b_{kt}]$ of order ℓ, as follows

$$b_{kk} = 0, \ \ b_{kt} = \max_{i_k \in S_k} \sum_{j \in S_t} |a_{i_k j}|. \tag{3.14}$$

Obviously,

$$B^* \geq B^{(i_1, i_2, \ldots, i_\ell)} \ \text{ for all } \ i_j \in S_j, \ j = 1, ..., \ell.$$

Also, B^* has all positive off–diagonal entries, so it is irreducible, and, by Perron-Frobenius theorem, if we denote by

$$\rho^* = \rho(B^*)$$

the Perron root of B^*, i.e., its nonnegative eigenvalue equal to the spectral radius, then, there exists a unique and positive vector c^* such that

$$B^* c^* = \rho^* c^*.$$

Having this in mind, for any aggregated matrix $A^{(i_1,i_2,\dots,i_\ell)}$, it holds that

$$A^{(i_1,i_2,\dots,i_\ell)} c^* = c^* - B^{(i_1,i_2,\dots,i_\ell)} c^* \geq c^* - B^* c^* = c^* - \rho^* c^* = (1-\rho^*) c^* > 0,$$

where the last inequality holds if

$$\rho^* < 1.$$

Under this condition, for the vector x defined as in (3.12), we have

$$Ax > 0.$$

Therefore, a scaling matrix for A can be constructed as

$$X = diag(x).$$

Now, if A has some zero off–diagonal entries, one may consider the matrix

$$A_\varepsilon \leq A$$

obtained from A by replacing its zero off–diagonal entries by $(-\varepsilon)$, where $\varepsilon > 0$. For ε sufficiently small, all the aggregated matrices $A_\varepsilon^{(i_1,i_2,\dots,i_\ell)}$ are nonsingular M-matrices. Therefore,

$$Ax \geq A_\varepsilon x > 0,$$

for the vector x obtained as before.

We summarize this discussion with the following remarks.

Remark 1 *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a $PM^\pi-$matrix, where $\pi = \{p_j\}_{j=0}^{\ell}$ is the partition of the index set N. If $\rho(B^*) < 1$ where B^* is defined as in (3.14), and if its corresponding Perron eigenvector is denoted by c^* and $X = diag(x)$ with x defined as in (3.12), then the matrix AX is SDD.*

Remark 2 *Let* $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ *be a* $PH^{\pi}-$*matrix, where* $\pi = \{p_j\}_{j=0}^{\ell}$ *is the partition of the index set* N. *If* $\rho(B^*) < 1$ *where* B^* *is defined as in (3.14), only for* $\langle A \rangle$ *instead of* A, *and if its corresponding Perron eigenvector is denoted by* c^* *and* $X = diag(x)$ *with* x *defined as in (3.12), then the matrix* AX *is SDD.*

Example 7 *Consider the given matrix, A, partitioned as follows*

$$A = \left[\begin{array}{ccc|ccc|ccc} 1800 & -150 & -150 & -1 & -1 & -1 & -10 & -20 & -10 \\ -150 & 1950 & -150 & -1 & -1 & -1 & -10 & -20 & -10 \\ -150 & -150 & 1650 & -1 & -1 & -1 & -20 & -10 & -10 \\ \hline -150 & -300 & -150 & 12 & -1 & -1 & -10 & -30 & -10 \\ -150 & -300 & -150 & -1 & 21 & -1 & -10 & -40 & -50 \\ -150 & -150 & -450 & -1 & -1 & 25 & -10 & -30 & -60 \\ \hline -150 & -150 & -150 & -1 & -2 & -1 & 100 & -10 & -10 \\ -150 & -150 & -150 & -1 & -2 & -1 & -10 & 130 & -20 \\ -150 & -150 & -150 & -1 & -2 & -1 & -100 & -10 & 300 \end{array}\right].$$

It can be shown that A *belongs to* $PM(3)$, *or, more precisely,* $A \in PM^{\pi}$ *for* $\pi = \{0,3,6,9\}$. *For*

$$D_A = diag(A_{11}, A_{22}, A_{33})$$

and

$$D = diag(R_1(D_A), ..., R_9(D_A)),$$

we consider the matrix $D^{-1}A$ *and form the matrix* B^* *as described in (3.14). The spectral radius of* B^* *is*

$$\rho^* = \rho(B^*) = 0.825769 < 1,$$

while the corresponding positive eigenvector is

$$c^* = [0.00526575,\ 0.995354,\ 0.0961377]^T.$$

For

$$x = [c_1^*,\ c_1^*,\ c_1^*,\ c_2^*,\ c_2^*,\ c_2^*,\ c_3^*,\ c_3^*,\ c_3^*]$$

and for

$$X = diag(x),$$

it holds that the matrix AX *is SDD.*

Notice that, in this example, for the given matrix A *and the given partition* π, *we have 27 aggregated matrices. Therefore, in order to construct a scaling matrix as described in the proof of Theorem 16, one should calculate spectral radius for 27 matrices and then take the maximum. Considering* B^* *instead, we calculated spectral radius only once.*

3.4 Columns and rows combined

In the previous section we considered generalizations of the SDD property via partitions of the index set of a matrix. The following two classes of matrices are based on a different direction of generalization. Namely, one could try to define new conditions that involve both row and column sums. If we consider matrices $A = [a_{ij}] \in \mathbb{C}^{n,n}$ that satisfy the following condition,

$$|a_{ii}| > \min\{r_i(A), c_i(A)\}, \text{ for all } i \in N,$$

it turns out that they can be singular, as the following example shows.

Example 8 *Consider the matrix*

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 1.5 \end{bmatrix}$$

Clearly, $|a_{ii}| > \min\{r_i(A), c_i(A)\}$, *for all* $i \in N$, *but* A *is singular.*

3.4.1 $\alpha 1$–matrices

Having in mind the Example 8, let us recall results of Ostrowski based on convex combinations of row sums and column sums.

Definition 14 *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *is an* $\alpha 1$*–matrix if there exists* $\alpha \in [0,1]$ *such that*

$$|a_{ii}| > \alpha r_i(A) + (1-\alpha) c_i(A)\ , \textit{ for all } i \in N.$$

Theorem 18 (Ostrowski) *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *is an* $\alpha 1$*–matrix, then* A *is nonsingular.*

The proof of nonsingularity follows from Theorem 19, for which the proof can be found in [82], and from the generalized arithmetic–geometric mean inequality.

Moreover, if $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is an $\alpha 1$–matrix, then A is an H-matrix.

For this statement, the proof can be found in [49]. This class of matrices has to be treated in a different manner, as we do not know a full scaling characterization. Therefore, the proof used in [49] illustrates a more general technique convenient for all DD-type matrix-classes. The main idea is to represent the comparison matrix as

$$\langle A \rangle = D - B,$$

where $D = diag(|a_{11}|, |a_{22}|, ..., |a_{nn}|)$ and to prove that $\rho(D^{-1}B) < 1$.

3.4.2 α2–matrices

We recall one more class of this type, introduced by Ostrowski and based on generalized geometric mean.

Definition 15 *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is an α2–matrix if there exists $\alpha \in [0,1]$ such that*

$$|a_{ii}| > (r_i(A))^{\alpha}(c_i(A))^{1-\alpha}\ , \ \textit{for all}\ i \in N.$$

In [82] one can find the proof for the following statement.

Theorem 19 ([82]) *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is an α2–matrix, then A is nonsingular.*

Moreover, every α2–matrix is an H-matrix.

3.5 Recursively defined row sums

3.5.1 Nekrasov matrices

In the paper by Gudkov, [36], there is a condition for the nonsingularity of a matrix in the form of a system of inequalities that are consequences of a result due to Mehmke and Nekrasov. As this condition defines a class of nonsingular matrices, we call these matrices Nekrasov matrices. As it is said in [41], the original setting was in terms of the convergence of the Gauss–Seidel iteration. This class of matrices was further discussed in many papers and it was used to obtain max-norm bounds of the inverse, bounds for determinants, and, also, this class was a starting point for many different generalizations, made in order to expand this nonsingularity result to wider classes of matrices, see [22], [23], [26], [44], [52], [76].

Definition 16 ([36, 52]) *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$. The values $h_i(A)$, $i \in N$ defined recursively by*

$$h_1(A) := r_1(A), \quad h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|,\ \ i = 2,3,\ldots n, \tag{3.15}$$

are called Nekrasov row sums.

Definition 17 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$. We say that A is a Nekrasov matrix if*

$$|a_{ii}| > h_i(A),\ \textit{for all}\ i \in N. \tag{3.16}$$

In other words, using vectors as before, we could say that the class of Nekrasov matrices is defined by

$$d(A) > h(A),$$

where

$$h(A) := [h_1(A), ..., h_n(A)]^T.$$

Comparing the deleted row sums used in definition of SDD property and recursively obtained, modified row sums used in definition of Nekrasov property, (Nekrasov row sums), one can easily see that the Nekrasov condition is weaker and the class of Nekrasov matrices is wider than the SDD class.

Namely, if $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an SDD matrix, it is enough to see that

$$h_i(A) \leq r_i(A) < |a_{ii}|, \; i \in N.$$

This can be verified using mathematical induction.

From definition of $h_i(A)$, we know that $h_1(A) = r_1(A)$. Let us assume that $h_i(A) \leq r_i(A)$, $i \in \{1, 2, ..., k-1\}$. Then,

$$h_k(A) = \sum_{j=1}^{k-1} |a_{kj}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=k+1}^{n} |a_{kj}| \leq$$

$$\leq \sum_{j=1}^{k-1} |a_{kj}| \frac{r_j(A)}{|a_{jj}|} + \sum_{j=k+1}^{n} |a_{kj}| \leq$$

$$\leq \sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}| = r_k(A).$$

Notice that Nekrasov row sums are obtained from deleted row sums by placing specific weights on entries in the lower triangular part of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ a_{21}\clubsuit & a_{22} & a_{23} & a_{24} & \dots & a_{2n} \\ a_{31}\clubsuit & a_{32}\bigstar & a_{33} & a_{34} & \dots & a_{3n} \\ a_{41}\clubsuit & a_{42}\bigstar & a_{43}\blacklozenge & a_{44} & \dots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}\clubsuit & a_{n2}\bigstar & a_{n3}\blacklozenge & a_{n3}\spadesuit & \dots & a_{nn} \end{bmatrix}.$$

This could be interpreted as follows. The dominance obtained in the first row is used to ”shrink” the entries in the first column and in that way ”help” the following rows to reach dominance. The process continues in the same manner: the

dominance obtained in the second row is used to shrink all the entries in the second column belonging to the lower triangle, and so on.
Having this in mind, given a matrix A, by $A = D - L - U$ we denote the standard splitting of A into its diagonal, (D), strictly lower, $(-L)$, and strictly upper, $(-U)$, triangular parts.

Let us now recall two well known lemmas.

Lemma 2 ([71]) *Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, with $a_{ii} \neq 0$ for all $i \in N$, then*

$$h_i(A) = |a_{ii}| \Big((|D| - |L|)^{-1} \, |U| e \Big)_i,$$

where $e \in \mathbb{C}^n$ is the vector with all components equal to 1.

Lemma 3 ([76]) *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is a Nekrasov matrix if and only if*

$$(|D| - |L|)^{-1} |U| e < e,$$

where $e \in \mathbb{C}^n$ is the vector with all components equal to 1.

Lemma 4 ([76]) *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is a Nekrasov matrix, then $I - (|D| - |L|)^{-1} |U|$ is an SDD matrix, where I is the identity matrix of order n.*

Although both SDD and Nekrasov class are related to the idea of diagonal dominance, there is an important difference between them. The class of SDD matrices is closed under simultaneous permutations of rows and columns, while Nekrasov class is not. Simultaneous permutations do not affect the set of values of deleted row sums, but they do change the set of values of Nekrasov row sums, for, in calculating recursively defined Nekrasov sums, the order is crucial.

3.5.2 A scaling matrix for Nekrasov matrices

Because of the way Nekrasov class is defined, involving recursively calculated row sums, finding the whole class of corresponding diagonal scaling matrices is not an easy task. But, in some applications we do not need a complete scaling characterization as an equivalence statement. For instance, when determining an Geršgorin-like eigenvalue localization area for the Schur complement of a Nekrasov matrix using only the entries of the original matrix, it is enough to know how to find at least some (if not all of them) diagonal scaling matrices for the given Nekrasov matrix.

Inspired by [36], we propose the following theorem which gives one way to construct a scaling matrix for the given Nekrasov matrix.

The following result is published in the paper [77], which is a joint work of T. Szulc, Lj. Cvetković and the author.

Theorem 20 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be a Nekrasov matrix with all nonzero Nekrasov row sums. Then, for a diagonal positive matrix*

$$D = diag(d_1, \dots, d_n),$$

where

$$d_i = \varepsilon_i \frac{h_i(A)}{|a_{ii}|},\ i = 1, \dots, n,$$

and $(\varepsilon_i)_{i=1}^n$ *is an increasing sequence of numbers with*

$$\varepsilon_1 = 1,$$

$$\varepsilon_i \in \left(1, \frac{|a_{ii}|}{h_i(A)}\right),\ i = 2, \dots, n,$$

the matrix AD *is an SDD matrix.*

Proof: Set

$$AD = A' = [a'_{ij}]$$

and observe that, by the definition of ε_i,

$$d_i = \varepsilon_i \frac{h_i(A)}{|a_{ii}|} < 1,\ i \in N. \tag{3.17}$$

Consider the first row of the matrix A. We have

$$|a'_{11}| = h_1(A) = \sum_{j=2}^{n} |a_{1j}|,$$

from which, by nonzero Nekrasov row sums of A and by (3.17) we get

$$|a'_{11}| > \sum_{j=2}^{n} \varepsilon_j \frac{h_j(A)}{|a_{jj}|} |a_{1j}| = \sum_{j=2}^{n} |a'_{1j}|.$$

Consider, now, any i-th row of A with $i > 1$. We have

$$|a'_{ii}| = \varepsilon_i h_i(A),$$

which, as the sequence $(\varepsilon_i)_{i=1}^n$ is increasing, becomes

$$|a'_{ii}| > \varepsilon_{i-1} h_i(A) = \sum_{j=1}^{i-1} \varepsilon_{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \varepsilon_{i-1} \sum_{j=i+1}^{n} |a_{ij}| \geq$$

$$\geq \sum_{j=1}^{i-1} \varepsilon_j |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \varepsilon_{i-1} \sum_{j=i+1}^{n} |a_{ij}| = \sum_{j=1}^{i-1} |a'_{ij}| + \varepsilon_{i-1} \sum_{j=i+1}^{n} |a_{ij}|.$$

Therefore,

$$|a'_{ii}| > \sum_{j=1}^{i-1} |a'_{ij}| + \varepsilon_{i-1} \sum_{j=i+1}^{n} |a_{ij}|. \tag{3.18}$$

As any $\varepsilon_i \geq 1,\ i = 1, \ldots, n,$ from (3.18) we obtain

$$|a'_{ii}| > \sum_{j=1}^{i-1} |a'_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|.$$

From this, by (3.17), we get

$$|a'_{ii}| > \sum_{j=1}^{i-1} |a'_{ij}| + \sum_{j=i+1}^{n} \varepsilon_j \frac{h_j(A)}{|a_{jj}|} |a_{ij}| = \sum_{j=1, j\neq i}^{n} |a'_{ij}|.$$

This completes the proof.□

Notice that the condition ”nonzero Nekrasov row sums” could be replaced by ”nonzero deleted row sums”, as $r_i(A) \neq 0$ for all $i \in N$ implies $h_i(A) \neq 0$ for all $i \in N$.

Theorem 21 ([36]) *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a Nekrasov matrix then it is nonsingular, moreover, it is an H-matrix.*

Although there is a straight forward proof for Theorem 21, the proof of Theorem 20 shows that, for matrices with nonzero Nekrasov row sums, the previous statement can also be proved through the construction of a scaling matrix.

3.5.3 P-Nekrasov and Gudkov matrices

Since SDD property is invariant under permutation of indices, while the condition (3.16) is not, one easily obtains a wider class.

Definition 18 *Given a permutation matrix P of order n, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a P-Nekrasov matrix if $P^T AP$ is a Nekrasov matrix, i.e., if*

$$|(P^T AP)_{ii}| > h_i(P^T AP), \text{ for all } i \in N, \tag{3.19}$$

or, in other words,

$$d(P^T AP) > h(P^T AP).$$

The union of all P-Nekrasov classes by permutation matrices P is known as Gudkov class, see [23].

Definition 19 *A matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$ is a Gudkov matrix if there exists a permutation matrix P such that A is a P-Nekrasov matrix.*

As Nekrasov matrices are nonsingular H-matrices, this statement holds for Gudkov matrices, too.

Theorem 22 ([36]) *If $A=[a_{ij}]\in\mathbb{C}^{n,n}$ is a Gudkov matrix, then A is an H-matrix and therefore it is nonsingular.*

3.5.4 $\{P_1,P_2\}$-Nekrasov matrices

$\{P_1,P_2\}$-Nekrasov matrices were introduced and studied in paper [22], a joint work of Lj. Cvetković, V. Kostić and the author. All the results in this subsection are original contribution, published in [22]. In the same paper we presented new upper bounds for the max-norm of the inverse matrix for a given $\{P_1,P_2\}$-Nekrasov matrix. These bounds can be found in Chapter 4.

The following, new nonsingularity result arises as a generalization of Nekrasov property by using two different permutations of the index set. The main motivation comes from the following observation: matrices that are Nekrasov matrices up to simultaneous permutations of rows and columns, are nonsingular. But, testing all the permutations of the index set for the given matrix is too expensive. In some cases, this nonsingularity criterion allows us to use quantities already calculated in order to conclude that the given matrix is nonsingular.

Given a matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$, $n\geq 2$, and given two permutation matrices, $P_1,P_2\in\mathbb{R}^{n,n}$, let us suppose that A is neither P_1-Nekrasov nor P_2-Nekrasov matrix. We want to define a new condition involving permuted sums, such that a matrix satisfying this condition is nonsingular.

Suppose that for the given matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$, $n\geq 2$ and two given permutation matrices P_1 and P_2,

$$d(A)>\min\left\{h^{P_1}(A),h^{P_2}(A)\right\}, \tag{3.20}$$

where

$$h^{P_k}(A)=P_kh(P_k^TAP_k),\ \ k=1,2,$$
$$h_i^{P_k}(A)=(P_kh(P_k^TAP_k))_i,\ \ k=1,2.$$

We call such a matrix $\{P_1,P_2\}$-Nekrasov matrix.

The following example shows that it can happen that a matrix is neither P_1 - Nekrasov nor P_2 - Nekrasov, but it is $\{P_1,P_2\}$ - Nekrasov.

Example 9 *Consider the matrix*

$$A = \begin{bmatrix} 7 & 1.5 & 1.5 & 1.5 \\ 0 & 7 & 1.5 & 6 \\ 7 & 1.5 & 7 & 0 \\ 1.5 & 1.5 & 1.5 & 7 \end{bmatrix}.$$

For identical and counter-identical permutation, i.e., for $P_1 = I$ *and*

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

it is easy to see that as

$$h^{P_1}(A) = [4.5,\ 7.5,\ 6.10714,\ 3.8801]^T$$

and

$$h^{P_2}(A) = [4.00255,\ 5.67857,\ 8.5,\ 4.5]^T,$$

A is neither P_1 *- Nekrasov nor* P_2 *- Nekrasov, but it is* $\{P_1, P_2\}$ *- Nekrasov.*

For matrices satisfying $\{P_1, P_2\}$ - Nekrasov property we prove the following results of the same type as Lemma 2 and Lemma 3.

Lemma 5 *Given any matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *with* $a_{ii} \neq 0$ *for all* $i \in N$, *and given a permutation matrix,* $P \in \mathbb{R}^{n,n}$, *then*

$$h_i^P(A) = |a_{ii}| \Big(P(|\tilde{D}| - |\tilde{L}|)^{-1} |\tilde{U}| e \Big)_i, \tag{3.21}$$

where $e \in \mathbb{C}^n$ *is the vector with all components equal to* 1 *and* $\tilde{D}$ *is diagonal,* $(-\tilde{L})$ *strictly lower and* $(-\tilde{U})$ *strictly upper triangular part of the matrix* $P^T AP$, *i.e.,* $P^T AP = \tilde{D} - \tilde{L} - \tilde{U}$ *is the standard splitting of the matrix* $P^T AP$.

Proof : By definition we have

$$h_i^P(A) = \Big(Ph(P^T AP) \Big)_i.$$

Notice that there exists the unique index $j \in N$ for which

$$h_i^P(A) = h_j(P^T AP).$$

It is the very index $j \in N$ for which $P_{ij} = 1$ holds.
From Lemma 2, we obtain

$$h_j(P^T AP) = |(P^T AP)_{jj}| \Big((|\tilde{D}| - |\tilde{L}|)^{-1} |\tilde{U}| e \Big)_j =$$

$$= |a_{ii}| \Big(P(|\tilde{D}| - |\tilde{L}|)^{-1} |\tilde{U}| e \Big)_i,$$

and, as $e = P^T e$,

$$h_i^P(A) = |a_{ii}| \Big(P(|\tilde{D}| - |\tilde{L}|)^{-1} |\tilde{U}| P^T e \Big)_i. \ \square$$

From Lemma 5, we see that, for two given permutation matrices $P_1, P_2 \in \mathbb{R}^{n,n}$, it holds

$$h_i^{P_k}(A) = |a_{ii}| \Big(P_k(|D_k| - |L_k|)^{-1} |U_k| e \Big)_i, \ k = 1,2, \tag{3.22}$$

where $P_k^T AP_k = D_k - L_k - U_k$ is the standard splitting of matrices $P_k^T AP_k$, $k = 1,2$.

Let us now construct a special matrix, $C \in \mathbb{C}^{n,n}$, for $\{P_1, P_2\}$-Nekrasov matrix, $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, as follows.

$$C = \begin{bmatrix} C(1) \\ \hline C(2) \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline C(n) \end{bmatrix} \in \mathbb{C}^{n,n} \tag{3.23}$$

with

$$C(i) = e_i^T P_{k_i} (|D_{k_i}| - |L_{k_i}|)^{-1} |U_{k_i}| P_{k_i}^T,$$

where e_i is the standard basis vector, whose components are equal to zero, all except the i–th component, which is equal to 1, and, for each index i, the corresponding index $k_i \in \{1,2\}$ is chosen in such way that

$$\min \Big\{ h_i^{P_1}(A), h_i^{P_2}(A) \Big\} = h_i^{P_{k_i}}(A). \tag{3.24}$$

In other words, we construct the matrix C in the following way. We choose each row to be the corresponding row from either

$$P_1(|D_1| - |L_1|)^{-1} |U_1| P_1^T, \text{ or}$$

$$P_2(|D_2| - |L_2|)^{-1} |U_2| P_2^T,$$

depending on comparison of $h_i^{P_1}(A)$, $h_i^{P_2}(A)$, i.e., we choose the row from the very matrix where minimum of these two sums is obtained.

What we do is the following - in permuted matrices, $A_1 = P_1^T A P_1$ and $A_2 = P_2^T A P_2$, we calculate new Nekrasov row sums in the usual way, from the first row to the last one. We track one row from the original matrix A and find it in permuted matrices A_1 and A_2. Then, we compare these two corresponding Nekrasov row sums and find the minimum of the two. Notice that the row that we track always consists of the same values in both permuted matrices, but what changes is the lower triangular part, and this is why in this way we obtain different Nekrasov row sums in permuted matrices, while permutations do not affect ordinary deleted row sums.

Lemma 6 *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is a $\{P_1, P_2\}$-Nekrasov matrix, then the matrix $I - C$ is an SDD matrix, where I is the identity matrix and C defined as in (3.23).*

Proof : Let us suppose that A is a $\{P_1, P_2\}$-Nekrasov matrix. Then,

$$d(A) > \min\left\{h^{P_1}(A), h^{P_2}(A)\right\}.$$

For the i-th component we have

$$|a_{ii}| > \min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\},$$

where, from Lemma 5,

$$h_i^{P_k}(A) = |a_{ii}|\left(P_k(|D_k| - |L_k|)^{-1}|U_k|P_k^T e\right)_i.$$

This implies

$$|a_{ii}| > \min\left\{|a_{ii}|\left(P_1(|D_1| - |L_1|)^{-1}|U_1|P_1^T e\right)_i, |a_{ii}|\left(P_2(|D_2| - |L_2|)^{-1}|U_2|P_2^T e\right)_i\right\}.$$

As it follows from (3.20) that $a_{ii} \neq 0$, therefore

$$1 > \min\left\{\left(P_1(|D_1| - |L_1|)^{-1}|U_1|P_1^T e\right)_i, \left(P_2(|D_2| - |L_2|)^{-1}|U_2|P_2^T e\right)_i\right\}.$$

This means that the matrix

$$B := I - C$$

has all row sums positive. Notice that $(|D_k| - |L_k|)$ is a nonsingular M-matrix, so, all the off–diagonal entries of the matrix B are nonpositive, and B is an SDD matrix. □

Theorem 23 *Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ and two permutation matrices $P_1, P_2 \in \mathbb{R}^{n,n}$, if A is a $\{P_1, P_2\}$-Nekrasov matrix, then A is nonsingular.*

Proof : Let us suppose the opposite, that A is $\{P_1, P_2\}$-Nekrasov matrix and singular. Then, there exists a nonzero vector x such that $Ax = 0$. Let $\{P_1, P_2\}$ be the given set of two permutation matrices. For $k = 1, 2$, we have

$$P_k^T A P_k P_k^T x = 0,$$

which can be expressed as

$$D_k P_k^T x = L_k P_k^T x + U_k P_k^T x, \tag{3.25}$$

where (D_k) is diagonal, $(-L_k)$ strictly lower and $(-U_k)$ strictly upper triangular part of the matrix $P_k^T A P_k$.

After using the triangular inequality and rearranging, (3.25) becomes

$$(|D_k| - |L_k|)|P_k^T x| \leq |U_k||P_k^T x|.$$

Since (3.20) implies that all diagonal entries of the matrix A are nonzero, then $|D_k| - |L_k|$ is a nonsingular M-matrix, and, therefore,

$$|P_k^T x| \leq (|D_k| - |L_k|)^{-1}|U_k||P_k^T x|.$$

Since $|P_k^T x| = |P_k^T||x| = P_k^T|x|$, multiplying the last inequality from the left hand side with P_k, we obtain

$$|x| \leq \Big(P_k(|D_k| - |L_k|)^{-1}|U_k|P_k^T\Big)|x|, \;\; k = 1, 2.$$

From the above, one derives the inequality

$$|x_i| \leq \Big(\Big(P_{k_i}(|D_{k_i}| - |L_{k_i}|)^{-1}|U_{k_i}|P_{k_i}^T\Big)|x|\Big)_i, \;\; i \in N,$$

which still holds if in each row i we choose the corresponding k_i as in (3.24). Then, the coefficient matrix in the right hand side turns to the matrix C defined in (3.23).

Therefore,

$$\big(I - C\big)|x| \leq 0. \tag{3.26}$$

As the matrix on the left hand side of inequality (3.26) has all row sums positive, which, together with the fact that all its off–diagonal entries are nonpositive, implies (see [4]) that it is a nonsingular M-matrix, then, from (3.26) it follows that $|x| \leq 0$ for nonzero vector x. This contradiction completes the proof. □

Theorem 24 *Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ and two permutation matrices $P_1, P_2 \in \mathbb{R}^{n,n}$, if A is a $\{P_1, P_2\}$-Nekrasov matrix, then A is an H-matrix.*

Proof : Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ be a $\{P_1, P_2\}$-Nekrasov matrix. Let $\langle A \rangle = D - B$ be the standard splitting of the matrix $\langle A \rangle$ into diagonal and off–diagonal part. The matrix D is nonsingular from the $\{P_1, P_2\}$-Nekrasov condition. Let us first prove that $\rho(D^{-1}B) < 1$. Suppose the opposite, that $\rho(D^{-1}B) \geq 1$, meaning that there is an eigenvalue $\lambda \in \sigma(D^{-1}B)$ such that $|\lambda| \geq 1$. This implies

$$\det(\lambda I - D^{-1}B) = 0,$$

and

$$\det(D^{-1})\det(\lambda D - B) = 0,$$

which, as matrix D is nonsingular, implies

$$\det(\lambda D - B) = 0.$$

In other words, the matrix $F := \lambda D - B$ is singular. But, then, for each $i \in N$,

$$|f_{ii}| = |\lambda||a_{ii}| \geq |a_{ii}| > \min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} \geq \min\left\{h_i^{P_1}(F), h_i^{P_2}(F)\right\}.$$

The last inequality holds from the following observations

$$h_i^{P_k}(A) = |a_{ii}|\Big(P_k(|D_k| - |L_k|)^{-1}|U_k|P_k^T e\Big)_i, \quad i \in N,\ k = 1, 2,$$

and, on the other hand,

$$h_i^{P_k}(F) = |\lambda||a_{ii}|\Big(P_k(|\lambda||D_k| - |L_k|)^{-1}|U_k|P_k^T e\Big)_i =$$

$$= |\lambda||a_{ii}|\left(P_k\left(|D_k| - \frac{1}{|\lambda|}|L_k|\right)^{-1}\frac{1}{|\lambda|}|U_k|P_k^T e\right)_i =$$

$$= |a_{ii}|\left(P_k\left(|D_k| - \frac{1}{|\lambda|}|L_k|\right)^{-1}|U_k|P_k^T e\right)_i, \quad i \in N,\ k = 1, 2.$$

Note that the matrices $M_A := |D_k| - |L_k|$ and $M_F := |D_k| - \frac{1}{|\lambda|}|L_k|$ are both non-singular M-matrices, with

$$M_F - M_A = \left(1 - \frac{1}{|\lambda|}\right)|L_k| \geq 0.$$

Therefore,

$$M_A^{-1} \geq M_F^{-1} (\geq 0).$$

Hence,

$$h_i^{P_k}(A) \geq h_i^{P_k}(F), \;\; i \in N, \; k = 1, 2.$$

This means that matrix F is also $\{P_1, P_2\}$-Nekrasov, and therefore nonsingular. This is a contradiction with $\det(\lambda D - B) = \det F = 0$.
As such eigenvalue $\lambda \in \sigma(D^{-1}B)$ does not exist, we conclude that $\rho(D^{-1}B) < 1$, and from

$$(D^{-1}\langle A \rangle)^{-1} = (I - D^{-1}B)^{-1} = \sum_{k \geq 0} (D^{-1}B)^k \geq 0,$$

we have

$$\langle A \rangle^{-1} \geq 0,$$

therefore, A is an H-matrix. □

Instead of a set of two permutation matrices in $\mathbb{R}^{n,n}$, we can observe a set of p arbitrary permutation matrices in $\mathbb{R}^{n,n}$, $\Pi_n = \{P_k\}_{k=1}^p$, and define the Π_n-Nekrasov property.

Namely, given a set of p permutation matrices in $\mathbb{R}^{n,n}$, $\Pi_n = \{P_k\}_{k=1}^p$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is called Π_n-Nekrasov, if

$$d(A) > \min_{k=1,\ldots,p} h^{P_k}(A). \tag{3.27}$$

Same as before, we can prove the following.

Theorem 25 *Given a set Π_n of permutation matrices in $\mathbb{R}^{n,n}$, every Π_n-Nekrasov matrix is nonsingular, moreover, it is an H-matrix.*

This can be stated also in the following form.

Theorem 26 *Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, if for each index $i \in N$ there exists a permutation matrix $P_i \in \mathbb{R}^{n,n}$, such that*

$$|a_{ii}| > h_i^{P_i}(A),$$

then, A is nonsingular, moreover, it is an H-matrix.

3.5.5 Σ-Nekrasov matrices

Recursively defined Nekrasov row sums could be combined with the partition-approach. In [23], a class of this type was introduced.

Definition 20 *Given any matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}, n\geq 2,$ and given any nonempty subset S of N, then A is an S-Nekrasov matrix if*

$$|a_{ii}|>h_i^S(A) \ \ \textit{for all} \ \ i\in S$$

and

$$\left(|a_{ii}|-h_i^S(A)\right)\left(|a_{jj}|-h_j^{\overline{S}}(A)\right)>h_i^{\overline{S}}(A)h_j^S(A) \ \ \textit{for all} \ \ i\in S,\ j\in\overline{S},$$

where

$$h_1^S(A)=r_1^S(A),$$

$$h_i^S(A)=\sum_{j=1}^{i-1}|a_{ij}|\frac{h_j^S(A)}{|a_{jj}|}+\sum_{j=i+1,\ j\in S}^{n}|a_{ij}|,\quad i=2,3,\dots n.$$

If there exists a nonempty subset S of N, such that $A=[a_{ij}]\in\mathbb{C}^{n,n}, n\geq 2,$ is an S-Nekrasov matrix, then we say that A belongs to the class of Σ-Nekrasov matrices.

3.5.6 A scaling matrix for Σ-Nekrasov matrices

Concerning the class of Σ-Nekrasov matrices, at this point we want to emphasize that it can be fully characterized (if and only if condition) as the subclass of H-matrices for which the corresponding class of scaling matrices, that scale Σ-Nekrasov matrices to Nekrasov matrices, is $\mathcal{W}$ as defined in (3.5).

Theorem 27 ([23]) *A matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}, n\geq 2,$ is a Σ-Nekrasov matrix if and only if there exists a matrix $W\in\mathcal{W}$ such that AW is a Nekrasov matrix.*

We discuss here a construction of a corresponding scaling matrix, as we will use it when dealing with Schur complements of Σ-Nekrasov matrices.

If A is an S-Nekrasov matrix, define the interval

$$J_A(S)=(\gamma_1^S(A),\gamma_2^S(A)), \tag{3.28}$$

where

$$\gamma_1^S(A):=\max_{i\in S}\frac{h_i^{\overline{S}}(A)}{|a_{ii}|-h_i^S(A)}\ ,$$

$$\gamma_2^S(A) := \min_{j\in\overline{S}, h_j^S(A)\neq 0} \frac{|a_{jj}| - h_j^{\overline{S}}(A)}{h_j^S(A)}.$$

If $h_j^S(A) = 0$ for all $j \in \overline{S}$, take $\gamma_2^S(A) = +\infty$.

As A is S-Nekrasov matrix, we know that

$$|a_{ii}| > h_i^S(A) \text{ for all } i \in S,$$

$$|a_{jj}| > h_j^{\overline{S}}(A) \text{ for all } j \in \overline{S} \quad \text{and}$$

$$\left(|a_{ii}| - h_i^S(A)\right)\left(|a_{jj}| - h_j^{\overline{S}}(A)\right) > h_i^{\overline{S}}(A) h_j^S(A) \text{ for all } i \in S,\ j \in \overline{S},$$

which implies

$$\frac{h_i^{\overline{S}}(A)}{|a_{ii}| - h_i^S(A)} < \frac{|a_{jj}| - h_j^{\overline{S}}(A)}{h_j^S(A)},$$

for all $i \in S$, $j \in \overline{S}$ such that $h_j^S(A) \neq 0$.

Then, obviously, interval $J_A(S)$ is not empty, so one could choose $\gamma \in J_A(S)$ and define a diagonal matrix W as follows

$$W = diag(w_1, w_2, ..., w_n),$$

where

$$w_i = \begin{cases} \gamma, & i \in S, \\ 1, & i \in \overline{S}. \end{cases}$$

Now, one could verify that AW is a Nekrasov matrix, i.e.,

$$|(AW)_{ii}| > h_i(AW) \text{ for all } i \in N.$$

By induction,

$$h_i(AW) = \gamma h_i^S(A) + h_i^{\overline{S}}(A) \quad \text{for all} \quad i \in N.$$

As $\gamma \in J_A(S)$, then

$$\frac{h_i^{\overline{S}}(A)}{|a_{ii}| - h_i^S(A)} < \gamma < \frac{|a_{jj}| - h_j^{\overline{S}}(A)}{h_j^S(A)},$$

for all $i \in S$, $j \in \overline{S}$ and $h_j^S(A) \neq 0$, which implies

$$\gamma |a_{ii}| > \gamma h_i^S(A) + h_i^{\overline{S}}(A),$$

for all $i \in S$, and

$$|a_{jj}| > \gamma h_j^S(A) + h_j^{\overline{S}}(A),$$

for all $j \in \overline{S}$.

Therefore,

$$|(AW)_{ii}| > h_i(AW) \text{ for all } i \in N.$$

The reverse can be proven similarly. This observation also can serve as a proof for the next statement.

Theorem 28 ([23]) *Σ-Nekrasov matrices are H-matrices and therefore nonsingular.*

Namely, as the matrix AW from the previous theorem is a Nekrasov matrix, and an H-matrix, it can be scaled to SDD. In other words, there exists a diagonal matrix X such that AWX is SDD. Therefore, WX is a scaling matrix for A, which implies that A is an H-matrix.

Obviously, from Theorem 20 and Theorem 27 together, we are now able to construct a scaling matrix for a given Σ-Nekrasov matrix. In other words, we know how to find a diagonal matrix that will transform the given Σ-Nekrasov matrix to an SDD matrix. For the S-Nekrasov matrix A, a corresponding diagonal scaling matrix can be formed as WD, where $W \in \mathcal{W}$ with $\gamma \in J_A(S)$, while D is constructed as in Theorem 20.

3.6 Conditions by nonstrict inequalities

3.6.1 Irreducibility and the existence of nonzero chains

In order to relax SDD condition, one idea is to replace strict inequalities with nonstrict inequalities. As we know that H-matrices have at least one SDD row, it is natural for us to consider the following class of matrices, called **diagonally dominant (DD) matrices**.

Definition 21 *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a diagonally dominant (DD) matrix if*

$$|a_{ii}| \geq r_i(A), \text{ for all } i \in N,$$

and for at least one index $k \in N$,

$$|a_{kk}| > r_k(A).$$

In Geršgorin's paper in 1931, it was assumed that DD matrices are nonsingular, which is not true in general, as the following simple example shows.

Example 10 *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Obviously, A is DD, but singular.

In 1948, Olga Taussky-Todd introduced the notion of irreducibility, a graph-theoretic property of matrices, which, together with DD, forms a sufficient condition for nonsingularity.

Definition 22 *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *is reducible if there exists a permutation matrix* $P \in \mathbb{R}^{n,n}$, *such that*

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{C}^{l,l}$, $A_{22} \in \mathbb{C}^{n-l,n-l}$, *for some* $1 \leq l < n$. *If such a permutation matrix does not exist, we say that A is irreducible. For* $A = [a_{ij}] \in \mathbb{C}^{1,1}$, *A is irreducible if its (only) entry is nonzero.*

It is a well-known result that a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is irreducible if and only if its graph is strongly connected.

With irreducibility added, we define **irreducibly diagonally dominant (IDD) matrices** as follows.

Definition 23 *An irreducible matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is called irreducibly diagonally dominant (IDD) matrix if*

$$|a_{ii}| \geq r_i(A), \text{ for all } i \in N,$$

and for at least one index $k \in N$,

$$|a_{kk}| > r_k(A).$$

Theorem 29 ([78]) *Given any* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, *if A is irreducibly diagonally dominant (IDD) matrix then A is nonsingular, moreover, it is an H-matrix.*

In the paper [74], it is said that, in order to form another condition sufficient for nonsingularity of a given matrix, the irreducibility condition could also be replaced with the existence of **nonzero chains**. In other words, if $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is DD and for each index $i \in N$ such that $|a_{ii}| = r_i(A)$, (non-SDD row) there is a sequence of nonzero entries of A of the form $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_{l-1} i_l}$, with $|a_{i_l i_l}| > r_{i_l}(A)$, (SDD row), then, we say that A is **chain diagonally dominant (CDD)** matrix. In [74] it is proved that every CDD matrix is nonsingular, moreover, it is a Gudkov matrix. Also, every IDD matrix is a CDD matrix as well. The chain condition itself is weaker than irreducibility condition.

Similarly, Σ-SDD condition can be modified by replacing strict inequalities with nonstrict inequalities (all but one) and by adding either irreducibility or the chain condition. In this way, new conditions that guarantee nonsingularity of matrices are obtained in [17].

When considering Nekrasov condition, one would expect that by replacing ordinary deleted row sums, $r_i(A)$, in the definition of IDD matrices with recursively defined sums, $h_i(A)$, this new condition would be sufficient for nonsingularity. However, this is not true, as it was shown and further discussed in [2], [76] and [52].

Example 11 *Consider the matrix*

$$A = \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}.$$

The matrix A is clearly irreducible, as all the entries are nonzero and

$$|a_{11}| = 4 > h_1(A) = r_1(A) = 2,$$

$$|a_{22}| = 2 = h_2(A) = 2,$$

but, obviously, A is singular.

When considering nonstrict conditions sufficient for nonsingularity of matrices, one interesting concept would be **semi-strict diagonal dominance (semi-SDD)** introduced by Beauwens in 1976 in the paper [3]. It turned out that this condition of Beauwens guarantees nonsingularity even if ordinary deleted row sums are replaced by Nekrasov row sums.

3.6.2 Semi-SDD matrices

Let us introduce a notation for the part of deleted row sum before the diagonal and for the part of deleted row sum after the diagonal, as follows.

$$l_1(A) := 0,$$

$$l_i(A) := \sum_{j=1}^{i-1} |a_{ij}|, \; i = 2, 3, ..., n,$$

$$u_i(A) := \sum_{j=i+1}^{n} |a_{ij}|, \; i = 1,2,...,n-1,$$
$$u_n(A) := 0.$$

Then, obviously,

$$r_i(A) = l_i(A) + u_i(A), \; i = 1,2,...,n,$$

and, also,

$$l_n(A) = r_n(A),$$
$$u_1(A) = r_1(A).$$

Definition 24 *Given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we say that A is a lower semi-strictly diagonally dominant matrix, (lower semi-SDD), if the following conditions hold:*

$$|a_{ii}| \geq r_i(A), \; i = 1,2,...,n,$$
$$|a_{ii}| > l_i(A), \; i = 1,2,...,n.$$

Definition 25 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and let P be a given n-by-n permutation matrix. If $P^T AP$ is a lower semi-SDD matrix, we say that A is a P-semi-SDD matrix.*

Definition 26 *Given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we say that A is a semi-strictly diagonally dominant matrix, (semi-SDD), if there exists a permutation matrix P such that A is a P-semi-SDD matrix.*

It is a well-known fact that lower semi-SDD matrices form a subclass of nonsingular H-matrices. Clearly, the same statement holds for semi-SDD matrices.

3.6.3 Semi-Nekrasov matrices

Let us introduce notation for the lower part of a Nekrasov row sum.

$$q_1(A) := 0,$$
$$q_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|}, \; i = 2,3,...,n.$$

Then, obviously,

$$h_i(A) = q_i(A) + u_i(A), \; i = 1,2,...,n,$$

and, also

$$h_1(A) = r_1(A) = u_1(A),$$
$$h_n(A) = q_n(A).$$

Definition 27 *Given* $A = [a_{ij}] \in \mathbb{C}^{n,n}$*, we say that* A *is a lower semi-Nekrasov matrix if the following conditions hold:*

$$|a_{ii}| \geq h_i(A),\ i = 1,2,...,n,$$

$$|a_{ii}| > q_i(A),\ i = 1,2,...,n.$$

It is proved that the following relation between classes of lower semi-SDD and lower semi-Nekrasov matrices holds, based on a comparison of deleted row sums and Nekrasov row sums.

Theorem 30 ([76]) *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a lower semi-SDD matrix, then* A *is also a lower semi-Nekrasov matrix.*

Definition 28 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$*, and let* P *be a given* n*-by-*n *permutation matrix. If* $P^T AP$ *is a lower semi-Nekrasov matrix, we say that* A *is a* P*-semi-Nekrasov matrix.*

Definition 29 *Given* $A = [a_{ij}] \in \mathbb{C}^{n,n}$*, we say that* A *is a semi-Nekrasov matrix if there exists a permutation matrix* P *such that* A *is a* P*-semi-Nekrasov matrix.*

In the paper [76], the following interesting result is proved.

Theorem 31 ([76]) *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a lower semi-Nekrasov matrix, then* A *is a Gudkov matrix and therefore nonsingular.*

The proof of this statement, given in [76], is based on a step-by-step construction of the permutation matrix that transforms the given lower semi-Nekrasov matrix, $A = [a_{ij}] \in \mathbb{C}^{n,n}$, to a Nekrasov matrix.

Also, a direct corollary of this is the next statement.

Theorem 32 ([76]) *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a semi-Nekrasov matrix, then* A *is a Gudkov matrix and hence a nonsingular* H*-matrix.*

Notice that, if a matrix A is a semi-SDD matrix, it cannot always be transformed to SDD only by means of simultaneous permutations of rows and columns, as the set of values of deleted row sums, $r_i(A),\ i = 1,2,...,n,$ is invariant under such permutations. We already stated before that the set of entries in each row of a given matrix does not change under simultaneous permutations (only their order is changed) and that diagonal entries remain on the diagonal. However, as lower semi-SDD matrices are also H-matrices, one can transform a lower semi-SDD matrix to an SDD matrix by diagonal scaling.

On the other hand, from Theorem 31, we see that if A is a semi-Nekrasov matrix it can be transformed to a Nekrasov matrix only by simultaneous permutations of rows and columns, because the set of values of Nekrasov row sums, $h_i(A),\ i = 1,2,...,n,$ does change when the order of rows is changed.

3.6.4 $\{P_1, P_2\}$-semi-Nekrasov matrices

In the same manner as done in the definition of $\{P_1, P_2\}$-Nekrasov class, let us now define another sufficient condition for nonsingularity of matrices. Results in this subsection are original contribution and they are presented here for the first time. Let P_1, P_2, be two permutation matrices in $\mathbb{R}^{n,n}$ and for the given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, consider corresponding simultaneous permutations of rows and columns, i.e., consider permuted matrices $P_1^T AP_1$ and $P_2^T AP_2$. If none of them is a lower semi-Nekrasov matrix, i.e., if A is neither P_1-semi-Nekrasov nor P_2-semi-Nekrasov, is it possible to use already calculated sums to confirm nonsingularity in a different way?

As before, we denote

$$h^{P_k}(A) = P_k h(P_k^T AP_k), \ \ k = 1,2,$$

$$h_i^{P_k}(A) = (P_k h(P_k^T AP_k))_i, \ \ k = 1,2.$$

Also, we introduce some new notations for the left parts of Nekrasov sums in permuted matrices:

$$q(A) = [q_1(A),...,q_n(A)]^T,$$

$$q^{P_k}(A) = P_k q(P_k^T AP_k), \ \ k = 1,2,$$

$$q_i^{P_k}(A) = (P_k q(P_k^T AP_k))_i, \ \ k = 1,2.$$

Definition 30 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and let P_1, P_2, be two permutation matrices in $\mathbb{R}^{n,n}$. We say that A is a $\{P_1, P_2\}$-semi Nekrasov matrix if for every $i = 1,2,....,n$, there exists $k_i \in \{1,2\}$, such that both*

$$|a_{ii}| \geq h_i^{P_{k_i}}(A) \tag{3.29}$$

and

$$|a_{ii}| > q_i^{P_{k_i}}(A) \tag{3.30}$$

hold.

We calculate Nekrasov row sums for permuted matrices, $P_1^T AP_1$ and $P_2^T AP_2$, in the usual way, from the first row to the last row. If, for each row in matrix A, the lower semi-Nekrasov condition holds for the corresponding row in at least one of the two permuted matrices, $P_1^T AP_1$ or $P_2^T AP_2$, then we say that A is $\{P_1, P_2\}$-semi-Nekrasov matrix.

Example 12 *Consider the matrix*

$$A = \begin{bmatrix} 7 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 7 & \frac{3}{2} & 6 \\ 7 & 1 & 7 & \frac{10}{7} \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 7 \end{bmatrix}.$$

For identical and counter-identical permutation, i.e., for $P_1 = I$ *and*

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

it is easy to see that as

$$h^{P_1}(A) = [4.5,\ 7.5,\ 7,\ 4.07143]^T$$

and

$$h^{P_2}(A) = [4.11141,\ 5.76822,\ 8.91837,\ 4.5]^T,$$

A *is neither* P_1*-semi-Nekrasov nor* P_2*-semi-Nekrasov, but it is* $\{P_1, P_2\}$*-semi Nekrasov.*

In a similar fashion as done in the case of $\{P_1, P_2\}$-Nekrasov matrices, the following can be proved.

Lemma 7 *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is a lower semi-Nekrasov matrix, then* $I - (|D| - |L|)^{-1}|U|$ *is a lower semi-SDD matrix.*

Proof: Assume that A is a lower semi-Nekrasov matrix. Denote by

$$T := (|D| - |L|)^{-1}|U|.$$

Then,

$$(Te)_i = \frac{h_i(A)}{|a_{ii}|} \leq 1.$$

In other words,

$$Te \leq e,$$

which implies nonstrict diagonal dominance in each row of the matrix $I - T$. So, the first condition for $I - T$ to be a lower semi-SDD matrix is fulfilled. It remains to prove that

$$|I - T|_{ii} > l_i(I - T),\ \ i = 1, 2, \ldots, n.$$

For one, fixed index $i \in N$, there are two possibilities.

First, if the $i-$th row in the matrix A is strictly Nekrasov dominant, i.e., if

$$|a_{ii}| > h_i(A),$$

then the strict diagonal dominance in the $i-$th row of $I-T$ holds, and therefore,

$$|I-T|_{ii} > l_i(I-T).$$

Second, if for some $i \in N$ it holds that

$$|a_{ii}| = h_i(A)$$

and

$$|a_{ii}| > q_i(A),$$

this implies $u_i(A) > 0$, and the existence of an index $k \in \{i+1, i+2, ..., n\}$, such that $a_{ik} \neq 0$, follows. By the construction of the matrix T, we see that in this case

$$(I-T)_{ik} \neq 0.$$

Let us explain this in more detail. As $(|D|-|L|)$ is an M-matrix, we see that

$$(|D|-|L|)^{-1} \geq 0.$$

Also, $|U| \geq 0$, which implies

$$T \geq 0.$$

From

$$(|D|-|L|)T = |U|,$$

it follows that

$$((|D|-|L|)T)_{ik} = |U|_{ik}$$

and

$$\sum_{j \leq i} (|D|-|L|)_{ij} T_{jk} = |a_{ik}|.$$

This implies

$$|a_{ii}| T_{ik} = \sum_{j<i} |a_{ij}| T_{jk} + |a_{ik}|.$$

We conclude that, if $|a_{ik}| > 0$, then $T_{ik} > 0$ and $(I-T)_{ik} \neq 0$. Therefore,

$$|I-T|_{ii} > l_i(I-T).$$

This completes the proof.□

Notice that the matrix $I-T$ has all positive diagonal entries and nonpositive off–diagonal entries, and it is, therefore, a nonsingular M-matrix.

Let us now construct a special matrix, $G \in \mathbb{C}^{n,n}$, for the given $\{P_1, P_2\}$-semi Nekrasov matrix, $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, as follows.

$$G = \begin{bmatrix} \underline{G(1)} \\ \underline{G(2)} \\ \cdot \\ \cdot \\ \underline{\cdot} \\ G(n) \end{bmatrix} \in \mathbb{C}^{n,n} \tag{3.31}$$

with

$$G(i) = e_i^T P_{k_i} (|D_{k_i}| - |L_{k_i}|)^{-1} |U_{k_i}| P_{k_i}^T,$$

where e_i is the standard basis vector, whose components are equal to zero, all except the $i-$th component, which is equal to 1, and, for each index i, the corresponding index $k_i \in \{1, 2\}$ is chosen in such way that conditions (3.29) and (3.30) of Definition 30 hold.
In other words, we construct the matrix G in the following way. We choose each row in G to be the corresponding row from either

$$P_1(|D_1| - |L_1|)^{-1} |U_1| P_1^T,$$

or

$$P_2(|D_2| - |L_2|)^{-1} |U_2| P_2^T.$$

We choose index 1, if in permuted matrix $P_1^T A P_1$, lower semi-Nekrasov condition holds for the row in consideration. Otherwise, we choose 2.

Lemma 8 *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a $\{P_1, P_2\}$-semi Nekrasov matrix, then $I-G$ is a lower semi-SDD matrix, where I is the n-by-n identity matrix and G is defined as in (3.31). Also, $I-G$ is a nonsingular M-matrix.*

Proof: Follows from the previous statement, Lemma 5 and the construction of the matrix G.□

Theorem 33 *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a $\{P_1, P_2\}$-semi Nekrasov matrix, then it is nonsingular.*

Proof : The proof is based on considerations similar to those used in the proof of Theorem 23.□

3.7 Relations between some subclasses

In previous sections, many different subclasses of H-matrices are presented. For some subclasses, we already discussed relations between them. Let us now summarize the most important conclusions of this type.

First of all, the following line of inclusions is almost in full explained before.

$$\text{SDD} = PH(1) \subset \text{Ostrowski} \subset \text{DZ} \subset \Sigma - \text{SDD} = PH(2) \subset H = PH(n).$$

Let us just briefly explain the following relation,

$$\text{Ostrowski} \subset \text{DZ}.$$

Let A be an Ostrowski matrix. Then,

$$|a_{ii}||a_{jj}| > r_i(A)r_j(A),$$

for all $i, j \in N$, $i \neq j$. Clearly, there can be at most one non-SDD row in A. Assume that

$$|a_{ii}| \leq r_i(A)$$

and

$$|a_{jj}| > r_j(A),$$

for all $j \in N$, $j \neq i$. Then, it holds

$$|a_{ii}|(|a_{jj}| - r_j(A) + |a_{ji}|) = |a_{ii}||a_{jj}| + |a_{ii}|(|a_{ji}| - r_j(A)) >$$

$$> r_i(A)r_j(A) + r_i(A)(|a_{ji}| - r_j(A)) = r_i(A)|a_{ji}|,$$

which means that A is a Dashnic-Zusmanovich matrix.

If there is no such index i, i.e., if A is SDD, then,

$$|a_{ii}|(|a_{jj}| - r_j(A) + |a_{ji}|) > |a_{ii}||a_{ji}| > r_i(A)|a_{ji}|,$$

therefore, A is a DZ-matrix.

Dashnic-Zusmanovich class is obviously a subclass of Σ-SDD. Namely, as said before, DZ class is a special case of Σ-SDD, with S chosen to be a singleton. Also, Σ-SDD is clearly a special case of $PH-$class, when partitions of the index set into two disjoint components are considered.

Furthermore, for classes involving column sums, from arithmetic-geometric mean relation, it follows

$$\text{SDD} \subset \alpha_1 \subset \alpha_2.$$

When it comes to classes based on recursively defined row sums, the following relations hold,

$$\text{Nekrasov} \subset \Sigma - \text{Nekrasov},$$

and

$$\Sigma - \text{SDD} \subset \Sigma - \text{Nekrasov}.$$

Let us explain this last relation. From Theorem 27, if A is a Σ-SDD matrix, then there exists a diagonal matrix $W \in \mathcal{W}$, such that AW is SDD. But, every SDD matrix is also a Nekrasov matrix. This implies that A is a Σ-Nekrasov matrix, as it can be scaled to Nekrasov matrix by $W \in \mathcal{W}$.

When comparing classes based on recursion to classes based on partition, it is interesting to point out that Nekrasov class and Σ-SDD class stand in general position. The intersection of these classes is nonempty (the SDD class is in that intersection), but neither of them is a subclass of the other, as the following example shows.

Example 13 *The matrix*

$$A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 4 & 8 & 4 & 2 \\ 4 & 8 & 1 & 4 \end{bmatrix}$$

is S-SDD with $S = \{1,2\}$*, but it is not a Nekrasov matrix, while the matrix*

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 2 & 0 \\ 1.8 & 2 & 2 \end{bmatrix}$$

is a Nekrasov matrix, but it is not Σ*-SDD.*

Also, for classes based on nonstrict conditions, the following relation is clear,

$$\text{SDD} \subset \text{lower semi-SDD} \subset \text{lower semi-Nekrasov}.$$

Regarding graph theoretic properties such as irreducibility and the existence of nonzero chains, obviously,

$$\text{IDD} \subset \text{CDD},$$

while classes of IDD matrices and lower semi-SDD matrices stand in a general position, as well as classes IDD and SDD, as the following simple example shows.

Example 14 *Consider matrices*

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}.$$

The matrix A *is clearly lower semi-SDD (it is, moreover, SDD), but it is not IDD, as it is reducible. On the other hand, matrix* B *is IDD, but it is neither SDD nor lower semi-SDD.*

Chapter 4

Max-norm bounds and eigenvalues

4.1 Max-norm bounds

4.1.1 Max-norm bounds for the inverse of $\{P_1, P_2\}$-Nekrasov matrices

In [81], a max-norm bound is given for the inverse of SDD matrices. This result of Varah served as a starting point for further generalizations.

Theorem 34 ([81]) *Given an SDD matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *the following bound applies,*

$$||A^{-1}||_\infty \leq \frac{1}{\min\limits_{i \in N} (|a_{ii}| - r_i(A))}.$$

This result was the basis for obtaining bounds for maximum norm of the inverse matrix for matrices belonging to classes Σ-SDD, PH- , Nekrasov and Σ-Nekrasov, see [61, 47, 20].

Now, we are going to use statements proved in Chapter 3 together with bound of Varah in order to obtain a max-norm bound for the inverse of a $\{P_1, P_2\}$-Nekrasov matrix.

The following two upper bounds for the maximum-norm of the inverse, given in Theorem 35 and Theorem 36 are original contributions and the result of joint work of Lj. Cvetković, V. Kostić and the author. These results are published in the paper [22].

Theorem 35 *Suppose that, for a given set of permutation matrices* $\{P_1,P_2\}$*, a matrix* $A=[a_{ij}]\in\mathbb{C}^{n,n}$*,* $n\geq 2$*, is a* $\{P_1,P_2\}$*-Nekrasov matrix. Then,*

$$||A^{-1}||_\infty \leq \frac{\max\limits_{i\in N}\left(\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|}\right)}{\min\limits_{i\in N}\left(1-\min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|},\frac{h_i^{P_2}(A)}{|a_{ii}|}\right\}\right)}, \tag{4.1}$$

where

$$z_1(A):=1,\ \ z_i(A):=\sum_{j=1}^{i-1}|a_{ij}|\frac{z_j(A)}{|a_{jj}|}+1,\ \ i=2,3,\dots n, \tag{4.2}$$

the corresponding vector is $z(A):=[z_1(A),...,z_n(A)]^T$*,* $z^P(A)=Pz(P^TAP)$*, and for the given* $i\in N$ *the corresponding index* $k_i\in\{1,2\}$ *is chosen in such way that*

$$\min\left\{h_i^{P_1}(A),h_i^{P_2}(A)\right\}=h_i^{P_{k_i}}(A).$$

Proof : Let A be a $\{P_1,P_2\}$-Nekrasov matrix. From Lemma 6, then

$$B:=I-C$$

is an SDD matrix. Therefore, for the inverse matrix of the matrix B the Varah bound holds:

$$||B^{-1}||_\infty \leq \frac{1}{\min\limits_{i\in N}(|b_{ii}|-r_i(B))}.$$

In the same manner as before, we obtain

$$|b_{ii}|-r_i(B)=$$

$$=1-\min\left\{\left(P_1\left(|D_1|-|L_1|\right)^{-1}|U_1|P_1^Te\right)_i,\left(P_2\left(|D_2|-|L_2|\right)^{-1}|U_2|P_2^Te\right)_i\right\}=$$

$$=1-\min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|},\frac{h_i^{P_2}(A)}{|a_{ii}|}\right\},\ i\in N.$$

It remains to find a link between matrices B^{-1} and A^{-1}. It is easy to see that, for a fixed $k\in\{1,2\}$,

$$I-P_k(|D_k|-|L_k|)^{-1}|U_k|P_k^T=P_k(|D_k|-|L_k|)^{-1}\left((|D_k|-|L_k|)P_k^T-|U_k|P_k^T\right)=$$

$$=P_k(|D_k|-|L_k|)^{-1}\left(|D_k|-|L_k|-|U_k|\right)P_k^T=P_k(|D_k|-|L_k|)^{-1}(P_k^T\langle A\rangle P_k)P_k^T=$$

$$=P_k(|D_k|-|L_k|)^{-1}P_k^T\langle A\rangle.$$

Therefore, we have obtained that for $k \in \{1,2\}$, it holds

$$I - P_k(|D_k| - |L_k|)^{-1}|U_k|P_k^T = P_k(|D_k| - |L_k|)^{-1}P_k^T \langle A \rangle.$$

Now, if we allow different values of k in different rows, keeping the same value of k in the same row on the left and the right hand side of this equality, we obtain in that way our "mixed - rows" matrix from (3.23) by choosing k_i in each row i as described in (3.24).

In other words, we have

$$B = I - C = \widetilde{C}\langle A \rangle, \tag{4.3}$$

where we denote by $\widetilde{C}$ the matrix defined as follows.

$$\widetilde{C} = \begin{bmatrix} \widetilde{C}(1) \\ \hline \widetilde{C}(2) \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \widetilde{C}(n) \end{bmatrix} \in \mathbb{C}^{n,n}$$

with

$$\widetilde{C}(i) = e_i^T P_{k_i}(|D_{k_i}| - |L_{k_i}|)^{-1} P_{k_i}^T,$$

where e_i is the standard basis vector, whose components are equal to zero, all except the $i-$th component, which is equal to 1, and, for each index i, the corresponding index $k_i \in \{1,2\}$ is chosen, as in (3.24), such that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

Therefore, from (4.3),

$$\langle A \rangle^{-1} = B^{-1}\widetilde{C}$$

and

$$||A^{-1}||_\infty \leq ||\langle A \rangle^{-1}||_\infty = ||B^{-1}\widetilde{C}||_\infty \leq ||B^{-1}||_\infty ||\widetilde{C}||_\infty \leq$$

$$\leq \frac{1}{\min\limits_{i \in N}\left(1 - \min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|}, \frac{h_i^{P_2}(A)}{|a_{ii}|}\right\}\right)} ||\widetilde{C}||_\infty.$$

From [20], we know that

$$(I - |L||D|^{-1})^{-1}e = z(A),$$

with recursively defined values

$$z_1(A) := 1,\ \ z_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1,\ \ i = 2,3,\dots n,$$

and the corresponding vector

$$z(A) := [z_1(A),...,z_n(A)]^T.$$

In the same fashion as done with Nekrasov sums, we defined permuted vector as

$$z^P(A) = Pz(P^T AP).$$

Having this in mind, it is easy to see that

$$||\widetilde{C}||_\infty = ||\widetilde{C}e||_\infty = \max_{i\in N} \left(\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|} \right),$$

where, for each index i, the corresponding index $k_i \in \{1,2\}$ is chosen in such way that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

This completes the proof. □

Theorem 36 *Suppose that, for a given set of permutation matrices $\{P_1,P_2\}$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is a $\{P_1,P_2\}$-Nekrasov matrix. Then,*

$$||A^{-1}||_\infty \leq \frac{\max_{i\in N} \left(z_i^{P_{k_i}}(A) \right)}{\min_{i\in N} \left(|a_{ii}| - \min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} \right)}, \tag{4.4}$$

where

$$z_1(A) := 1,\ \ z_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1,\ \ i = 2,3,\dots n,$$

the corresponding vector is $z(A) := [z_1(A),...,z_n(A)]^T$, $z^P(A) = Pz(P^T AP)$, and for the given $i \in N$ the corresponding index $k_i \in \{1,2\}$ is chosen such that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

Proof: Instead of the matrix B from (4.3), consider the matrix $B' = |D|B$. Then,

$$|b'_{ii}| - r_i(B') = |a_{ii}| - \min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\},$$

and

$$\langle A\rangle^{-1} = (B')^{-1}|D|\widetilde{C}.$$

Therefore,

$$||A^{-1}||_\infty \leq ||\langle A\rangle^{-1}||_\infty = ||(B')^{-1}|D|\widetilde{C}||_\infty \leq ||(B')^{-1}||_\infty|||D|\widetilde{C}||_\infty \leq$$

$$\leq \frac{1}{\min\limits_{i\in N}\left(|a_{ii}| - \min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\}\right)}|||D|\widetilde{C}||_\infty,$$

where

$$|||D|\widetilde{C}||_\infty = |||D|\widetilde{C}e||_\infty = \max_{i\in N}\left(z_i^{P_{k_i}}(A)\right),$$

where, for each index i, the corresponding index $k_i \in \{1,2\}$ is chosen in such a way that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

This completes the proof. □

Example 15 *Consider the following matrices :*

$$A_1 = \begin{bmatrix} 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 114 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 14 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 814 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 8 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 7 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 4 & 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 8 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 8 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1.5 & 0.1 & 0 & 0.1 & 0 & 0 \\ 0.1 & 2 & 0.1 & 1.9 & 0 & 0 \\ 0 & 0.1 & 23 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0.5 & 44 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 44 & 0.4 \\ 0 & 0 & 0.5 & 0 & 1 & 1 \end{bmatrix}$$

Matrix A_1 is an SDD matrix, while A_2 is a Nekrasov matrix. Matrix A_3 is neither SDD nor Nekrasov, but it does satisfy our new $\{P_1,P_2\}$-Nekrasov condition, where P_1 is the identical permutation of order 6 and P_2 is counter-identical permutation of order 6. In the following table, we compare the results for max-norm bounds of the inverse matrix obtained using Theorem 35 and Theorem 36, (with P_1 and P_2 being identical and counter-identical permutation matrix of the corresponding order), to those of Varah, for SDD matrices, and to the bounds for Nekrasov matrices presented in [20] (in the table we call them Nekrasov I and Nekrasov II).

Bound	*Varah*	*Nekrasov I*	*Nekrasov II*	$\{P_1,P_2\}$*-Nek I*	$\{P_1,P_2\}$*-Nek II*
A_1	*0.5*	*0.2443*	*0.3108*	***0.2132***	*0.2443*
A_2	-	*2.2282*	*2.8729*	*0.7726*	***0.5992***
A_3	-	-	-	***1.114***	*1.1255*

Exact values for the max-norm of the inverse matrix are as follows:
$\|A_1^{-1}\|_\infty = 0.1796$, $\|A_2^{-1}\|_\infty = 0.3445$, $\|A_3^{-1}\|_\infty = 1.0578$.

As one can see from this table, our bounds are better than Varah for some SDD matrices, and, in some cases, they are better than bounds for Nekrasov matrices presented in [20]. If the matrix is neither SDD nor Nekrasov, like, for example, A_3, the only bounds that can be applied are bounds (4.1) and (4.4).

4.2 Geršgorin theorems

4.2.1 Eigenvalue localization as another formulation of nonsingularity results

In [82] it is pointed out that the result on nonsingularity of SDD matrices and the well-known Geršgorin's theorem (1931) that gives the eigenvalue localization set for the given square complex matrix are actually equivalent.

Theorem 37 (Geršgorin Theorem) *For any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and any $\lambda \in \sigma(A)$, there is a positive integer k in N such that*

$$|\lambda - a_{kk}| \leq r_k(A).$$

The set

$$\Gamma_i(A) = \{z \in \mathbb{C} | |z - a_{ii}| \leq r_i(A)\}$$

is called the $i-$th Geršgorin disk. The union of these disks contains all the eigenvalues and it is called the Geršgorin set,

$$\sigma(A) \subseteq \Gamma(A) = \bigcup_{i \in N} \Gamma_i(A).$$

The next statement gives a clear relation of nonsingularity results on one hand and eigenvalue localization sets on the other hand. It was explicitly formulated in the book "Geršgorin and His Circles" by Richard Varga, [82].

Theorem 38 (Varga) *Geršgorin Theorem is equivalent to Lévy–Desplanques Theorem.*

Having this in mind, we see that the story of nonsingularity has an equivalent formulation in the language of eigenvalue inclusion sets. The idea of the proof is to consider the existence of a zero eigenvalue of the given matrix.

Theorem 39 (Geršgorin Weighted Theorem) *For any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and any $x > 0$ in $\mathbb{R}^n$, then,*

$$\sigma(A) \subseteq \Gamma^{r^x}(A) = \bigcup_{i \in N} \Gamma_i^{r^x}(A),$$

where

$$\Gamma_i^{r^x}(A) = \{z \in \mathbb{C} | |z - a_{ii}| \leq r_i(X^{-1}AX)\},$$

and $X = diag(x)$.

For any nonempty subset S of the index set N, let $\Gamma_S^{r^x}(A)$ denote the union of those weighted Geršgorin disks whose indices belong to S, i.e.,

$$\Gamma_S^{r^x}(A) = \bigcup_{i\in S} \Gamma_i^{r^x}(A).$$

Theorem 40 (Geršgorin Isolation Theorem) *For any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and any $x > 0$ in $\mathbb{R}^n$ for which the relation*

$$\Gamma_S^{r^x}(A) \bigcap \Gamma_{\overline{S}}^{r^x}(A) = \emptyset$$

holds for some proper subset S of N, then, $\Gamma_S^{r^x}(A)$ contains exactly $|S|$ eigenvalues of A.

In the same manner, nonsingularity results for some other subclasses of H-matrices can be the starting point for obtaining eigenvalue localization results.

Let

$$K_{ij}(A) = \{z \in \mathbb{C} : \ |z - a_{ii}||z - a_{jj}| \leq r_i(A) r_j(A)\},$$
$$K(A) = \bigcup_{i,j\in N,\, i\neq j} K_{ij}(A).$$

Theorem 41 (Brauer-Cassini Theorem) *For any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ and any $\lambda \in \sigma(A)$, there is a pair of distinct integers i and j in N such that*

$$|\lambda - a_{ii}||\lambda - a_{jj}| \leq r_i(A) r_j(A),$$

or, equivalently, $\lambda \in K_{ij}(A)$. As this holds for each $\lambda \in \sigma(A)$, then $\sigma(A) \subset K(A)$.

Let

$$\psi_{ij}(A) = \{z \in \mathbb{C} : \ |z - a_{ii}|(|z - a_{jj}| - r_j(A) + |a_{ji}|) \leq r_i(A)|a_{ji}|\},$$
$$\psi(A) = \bigcap_{i\in N} \bigcup_{j\in N,\, j\neq i} \psi_{ij}(A).$$

Theorem 42 (Dashnic-Zusmanovich) *For any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ and any $\lambda \in \sigma(A)$, for every $i \in N$ there exists an index $j \in N$, $j \neq i$, such that*

$$|\lambda - a_{ii}|(|\lambda - a_{jj}| - r_j(A) + |a_{ji}|) \leq r_i(A)|a_{ji}|,$$

or, equivalently, $\lambda \in \psi_{ij}(A)$. As this holds for each $\lambda \in \sigma(A)$, then $\sigma(A) \subset \psi(A)$.

In [25], we proved the following relation between these localization sets.

Theorem 43 *For any matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *it holds*

$$\psi(A) \subseteq K(A) \subseteq \Gamma(A).$$

Proof: Assume first that $z \in K(A)$. Then, there exist $i, j \in N$, $i \neq j$, such that $z \in K_{ij}(A)$, i.e., there exist $i, j \in N$, $i \neq j$, such that $|z - a_{ii}||z - a_{jj}| \leq r_i(A)r_j(A)$. But this implies that either $|z - a_{ii}| \leq r_i(A)$, or $|z - a_{jj}| \leq r_j(A)$. It follows that $z \in \Gamma_i(A)$, or $z \in \Gamma_j(A)$, i.e., $z \in \Gamma_i(A) \bigcup \Gamma_j(A) \subseteq \Gamma(A)$, therefore, $K(A) \subseteq \Gamma(A)$. Let us now prove that

$$\psi(A) \subseteq \Gamma(A).$$

Take $z \in \psi(A)$. Then, for every $i \in N$ there exists $j \in N$, $j \neq i$, such that

$$|z - a_{ii}|(|z - a_{jj}| - r_j(A) + |a_{ji}|) \leq r_i(A)|a_{ji}|.$$

There are two possibilities. First, if there exists an index $i \in N$ such that $|z - a_{ii}| \leq r_i(A)$, then, $z \in \Gamma_i(A) \subseteq \Gamma(A)$. Second, if for all $i \in N$, it holds $|z - a_{ii}| > r_i(A)$, then, $\frac{r_i(A)}{|z - a_{ii}|} < 1$ and there exists $j \in N$, $j \neq i$, such that

$$|z - a_{ii}|(|z - a_{jj}| - r_j(A) + |a_{ji}|) \leq r_i(A)|a_{ji}|,$$

which implies

$$|z - a_{jj}| - r_j(A) + |a_{ji}| \leq |a_{ji}|,$$

and

$$|z - a_{jj}| \leq r_j(A).$$

Therefore,

$$z \in \Gamma_j(A) \subseteq \Gamma(A).$$

Now, we are able to prove that $\psi(A) \subseteq K(A)$. Let $z \in \psi(A)$. Then, as proved above, $z \in \Gamma(A)$, i.e., there exists $k \in N$ such that $|z - a_{kk}| \leq r_k(A)$. For k there exists $j \in N$, $j \neq k$, such that

$$|z - a_{kk}|(|z - a_{jj}| - r_j(A) + |a_{jk}|) \leq r_k(A)|a_{jk}|.$$

Now, we have

$$|z - a_{kk}||z - a_{jj}| \leq r_k(A)|a_{jk}| + |z - a_{kk}|(r_j(A) - |a_{jk}|) \leq$$

$$\leq r_k(A)|a_{jk}| + r_k(A)(r_j(A) - |a_{jk}|) = r_k(A)r_j(A),$$

which implies that $z \in K(A)$.□

Geršgorin disks and Brauer's ovals of Cassini both depend on deleted row sums and diagonal entries of the given matrix A. The set $\psi_{ij}(A)$ that arises from

Dashnic-Zusmanovich result depends on some additional quantities, a_{ij}, but although this brings an increase in calculations, it also brings a tighter eigenvalue inclusion area.

As a generalization of $\psi(A)$ set, a result in eigenvalue localization is obtained, starting from the nonsingularity result for the class of Σ-SDD matrices, and even tighter eigenvalue inclusion area, called CKV set, is formed, see [24].

Chapter 5

The Schur complement and H–matrices

5.1 The Schur complement

5.1.1 Basic properties

As said in the first chapter, the term Schur complement was introduced by Emilie Haynsworth in 1968 and the reason for choosing the name of Issai Schur was the lemma in his paper [72] published in 1917.

Long before the name Schur complement appeared for the first time, the concept itself was already implicitly present in the work of many mathematicians. As said in [12], James Joseph Sylvester investigated some properties of the Schur complement matrix in the 19−th century. In [11], other early manifestations of the Schur complement are recalled.

Here is the definition of the Schur complement.

Definition 31 *Let* $M \in \mathbb{C}^{n,n}$ *be partitioned in blocks in the following way*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{5.1}$$

where $A \in \mathbb{C}^{k,k}$, $1 \le k \le n$, *is a nonsingular leading principal submatrix of* M. *The Schur complement of* A *in* M *is denoted by* M/A *and defined to be*

$$M/A = D - CA^{-1}B.$$

From this definition we see that there is a clear relation of the Schur complement concept to Gaussian elimination.

If $M \in \mathbb{C}^{n,n}$ is partitioned in blocks as in (5.1), where $A \in \mathbb{C}^{k,k}$, $1 \leq k \leq n$, is a nonsingular leading principal submatrix of M, then, applying block Gaussian elimination in order to transform block D to a zero matrix brings us to

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D-CA^{-1}B \end{bmatrix}. \tag{5.2}$$

Therefore, the Schur complement arises as a side–product of Gaussian elimination, and one of its most obvious applications is in solving systems of linear equations. By means of Schur complements a large system can be replaced by systems of smaller formats. Consider the system

$$Mx = b,$$

with M partitioned as in (5.1), x partitioned into vectors u and v conformally with M and b partitioned into $b1$ and $b2$ in the same manner. Then,

$$Au + Bv = b1$$

$$Cu + Dv = b2.$$

Therefore,

$$(D - CA^{-1}B)v = b2 - CA^{-1}b1.$$

We could first solve this system to find v and then find u from the first equation.

Besides the fact that the Schur complement appears as the intermediate step in Gaussian elimination, over the years mathematicians have found many different manifestations and applications of the Schur complement matrix. It turned out that the concept itself deserves a special name and a special treatment, especially when examining various matrices of the form $V - PQ^{-1}R$, their applications in statistics and mathematical programming and generalizations to singular or non-square blocks, see [6, 15, 67, 13].

Now, let us recall the well-known lemma, that was the reason for naming this special matrix after Issai Schur.

Lemma 9 (Schur determinantal lemma, [72]) *Let $A,B,C,D \in \mathbb{C}^{n,n}$. Suppose that matrices A and C commute. Then, for the matrix $M \in \mathbb{C}^{2n,2n}$,*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{5.3}$$

it holds

$$\det M = \det(AD - CB).$$

The proof is based on the following considerations. First, if I denotes the identity matrix of order n, consider the matrix equality

$$\begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ 0 & D-CA^{-1}B \end{bmatrix} \tag{5.4}$$

and take determinants in this equality, which implies

$$\det(A^{-1})\det(M) = \det(D-CA^{-1}B).$$

Therefore,

$$\det(M) = \det(A)\det(D-CA^{-1}B) = \det(AD-ACA^{-1}B) = \det(AD-CB).$$

It can be shown that the statement of this lemma holds for a singular matrix A, too. In literature, this result is often mentioned by the name **Schur determinant formula** and written in the form

$$\det(M) = \det(A)\det(D-CA^{-1}B) = \det(A)\det(M/A).$$

This is where the motivation for the notation of the Schur complement, M/A, comes from. Also, from this formula it is clear that, if A is nonsingular, then M is nonsingular if and only if M/A is nonsingular.

Let us recall some more early appearances of the Schur complement, as done in [84].

Theorem 44 (Banachiewicz inversion formula, 1937.) *Let $M \in \mathbb{C}^{n,n}$ be a nonsingular matrix partitioned as in (5.1), where $A \in \mathbb{C}^{k,k}$, $1 \le k \le n$, is nonsingular. Then, M/A is nonsingular and*

$$M^{-1} = \begin{bmatrix} A^{-1}+A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix}. \tag{5.5}$$

Notice that as a corollary of this statement we have that the lower right block in M^{-1} is exactly $(M/A)^{-1}$.

Also, for $M \in \mathbb{C}^{n,n}$ partitioned as in (5.1), where $A \in \mathbb{C}^{k,k}$, $1 \le k \le n$, is a nonsingular leading principal submatrix in M, the following formula of Aitken (1939) holds

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix}. \tag{5.6}$$

The Aitken block-diagonalization formula holds even if M is not a square matrix.

Directly from Aitken formula, the result of Gutman (1946) on rank additivity follows:

$$rank(M) = rank(A) + rank(M/A).$$

Notice that the submatrix A in the definition of the Schur complement need not be always set in the upper left corner in the partitioned matrix. If we consider again the partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

with $D \in \mathbb{C}^{n-k,n-k}$, $1 \leq k \leq n$, nonsingular, then the corresponding Schur complement of D in M would be

$$M/D = A - BD^{-1}C.$$

This brings us to the result of Duncan (1944), which states that, for $M \in \mathbb{C}^{n,n}$ a nonsingular matrix partitioned as in (5.1), with $A \in \mathbb{C}^{k,k}$, $D \in \mathbb{C}^{n-k,n-k}$, $1 \leq k \leq n$, being both nonsingular submatrices in M,

$$(M/D)^{-1} = A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1}.$$

The proof is obtained by applying Banachiewicz inversion formula both with respect to A and with respect to D.

Now we recall the well-known result of Haynsworth, published in 1968, on the inertia of Schur complement matrix. The inertia of the Hermitian matrix $M \in \mathbb{C}^{n,n}$ is the ordered triple of nonnegative integers

$$In(M) = (p(M), q(M), z(M)),$$

that gives, respectively, the number of positive, negative and zero eigenvalues of the matrix M.

Theorem 45 (Haynsworth inertia additivity formula, 1968.) *Let $M \in \mathbb{C}^{n,n}$ be a Hermitian matrix and $A \in \mathbb{C}^{k,k}$, $1 \leq k \leq n$, a nonsingular leading principal submatrix of M. Then,*

$$In(M) = In(A) + In(M/A).$$

There is another very important property of the Schur complement that comes from Crabtree and Haynsworth, often called *the quotient formula*.

Theorem 46 (Crabtree - Haynsworth, 1969.) *Let* $M \in \mathbb{C}^{n,n}$ *be a nonsingular matrix partitioned as follows*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

with $A \in \mathbb{C}^{k,k}$, $1 \leq k \leq n$, *being a nonsingular submatrix in* M, *also partitioned as follows*

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

with $P \in \mathbb{C}^{l,l}$, $1 \leq l \leq k$, *nonsingular submatrix in* A. *Then,* A/P *is a nonsingular leading principal submatrix of* M/P *and*

$$M/A = (M/P)/(A/P).$$

This result will be very useful when analyzing invariance of different matrix properties under Schur complement transformation. Therefore, we will recall it again later, in a slightly different form.

In definitions of the Schur complement the block A is usually allowed to be any principal nonsingular submatrix, determined by any proper nonempty subset α of the index set N.

Definition 32 *Given* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, *let* $A(\alpha, \beta)$ *denote the submatrix of* A *lying in the rows indexed by* α *and columns indexed by* β. *Let* $A(\alpha, \alpha)$ *be abbreviated to* $A(\alpha)$, *while* $A(\alpha)$ *is assumed to be a nonsingular matrix. The Schur complement of* A *with respect to a proper subset of* N, α, *is denoted by* A/α *and defined in the following way*

$$A/\alpha = A(\overline{\alpha}) - A(\overline{\alpha}, \alpha)(A(\alpha))^{-1} A(\alpha, \overline{\alpha}).$$

This can be made even more general if Schur complements are defined with respect to a nonsingular block $A(\alpha, \beta)$ where α and β are subsets of the index set N of the same cardinality. But, in this thesis, we will think of the Schur complement mostly in the sense of Definition 32. In many cases, even the simpler definition, Definition 31, will be good enough, because the $A(\alpha)$ block can be made an upper left corner in the matrix M by means of simultaneous permutations of rows and columns.

In [11], there are various determinantal identities involving Schur complements. The entries of the Schur complement matrix can be expressed using minors as in the following statement,

$$(A/\alpha)_{st} = \frac{\det A(\alpha \cup \{j_s\}, \alpha \cup \{j_t\})}{\det A(\alpha)}.$$

As eigenvalue problems are a very important area of research in applied linear algebra and matrix analysis, it is interesting to know whether some information on eigenvalues of the Schur complement matrix can be obtained from the entries of the original matrix. Results of this type related to Hermitian matrices can be found in [84]. As Cauchy interlacing theorem states, eigenvalues of a principal submatrix of a Hermitian matrix interlace eigenvalues of the parent matrix. More precisely, let eigenvalues of a Hermitian matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be arranged in a decreasing order and denoted by

$$\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_n(A).$$

Let $\alpha = \{1,2,...,k\} \subset N$, i.e., let $A(\alpha)$ be a principal submatrix in A. Then, for each $i = 1,2,...,k$,

$$\lambda_i(A) \geq \lambda_i(A(\alpha)) \geq \lambda_{i+n-k}(A).$$

If we consider a Hermitian matrix and its Schur complement, the same statement is not true in general, but a result of this type for positive semidefinite matrices holds, see [84].

Namely, if $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a positive semidefinite matrix, $\alpha = \{1,2,...,k\} \subseteq N$ and $A(\alpha)$ a nonsingular principal submatrix of A, then,

$$\lambda_i(A) \geq \lambda_i(A(\overline{\alpha})) \geq \lambda_i(A/\alpha) \geq \lambda_{i+k}(A), \ \ i = 1,2,...,n-k.$$

Notice that, in this case, eigenvalues are real and this result gives a relation of eigenvalues of the parent matrix to eigenvalues of the submatrix and eigenvalues of SC. In the following sections, we will investigate relations of eigenvalues of the parent matrix to eigenvalues of the submatrix and SC in cases when matrices in consideration are not positive semidefinite, but some special H-matrices with eigenvalues that are not necessarily real.

5.1.2 Generalized SC

Let us briefly recall generalizations of the original definition of the Schur complement that arise when block A in M is allowed to be square but singular, or, a rectangular submatrix in a rectangular matrix M.

For $A \in \mathbb{C}^{m,n}$, a generalized inverse for the given matrix A is $A^- \in \mathbb{C}^{n,m}$, such that

$$AA^-A = A.$$

Generalized inverse is not necessarily unique. If the matrix A is square and nonsingular, the generalized inverse is unique and it is exactly the ordinary inverse. If A is singular it has many generalized inverses.

The generalized inverse that is most often seen in the literature is the **Moore-Penrose** generalized inverse. It is the unique matrix that satisfies condition for generalized inverse and some additional conditions.

Namely, let $A \in \mathbb{C}^{m,n}$. The Moore-Penrose generalized inverse for the given matrix A is $A^{\dagger} \in \mathbb{C}^{n,m}$, such that

$$AA^{\dagger}A = A,$$

$$A^{\dagger}AA^{\dagger} = A^{\dagger},$$

$$(AA^{\dagger})^{H} = AA^{\dagger},$$

$$(A^{\dagger}A)^{H} = A^{\dagger}A.$$

The Moore-Penrose inverse is used in the definition of the generalized Schur complement.

Definition 33 *Let* $A \in \mathbb{C}^{m,n}$, $\alpha \subseteq \{1,2,...,m\}$, $\beta \subseteq \{1,2,...,n\}$. *The generalized Schur complement of* $A(\alpha,\beta)$ *in* A *is denoted by* $A/\alpha,\beta$, *and defined as follows*

$$A/\alpha,\beta = A(\overline{\alpha},\overline{\beta}) - A(\overline{\alpha},\beta)(A(\alpha,\beta))^{\dagger}A(\alpha,\overline{\beta}).$$

In literature, there are many results and applications related to generalized SC, but in this thesis we will investigate classical SC of a given square matrix, with respect to a nonsingular principal submatrix.

5.2 Matrix properties invariant under SC

It is well-known, see [14], that the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant. Also, in the same paper, there is a similar result on H-matrices - if a matrix is an H-matrix, then its Schur complement is an H-matrix, too. Recent research showed that the same type of statement holds for some other matrix classes and matrix properties.

Notice that, when considering closure properties of different matrix classes under taking Schur complements, there are some technical remarks that should be pointed out.

First, for a Schur complement of the given matrix A taken with respect to some index set α to be defined at all, we assume that submatrix $A(\alpha)$ is nonsingular. Therefore, in [84], it is assumed that all the classes considered in this context are principally nonsingular, meaning that all the principal submatrices are nonsingular. When considering H-matrices, this condition is fulfilled, as this class is already principally nonsingular.

Second, in order to talk about "closure" properties of matrix classes, these matrix classes cannot be limited to one fixed dimension, as Schur complement is a matrix of a smaller format than the original matrix. Also, if we say that certain matrix property is invariant under taking Schur complements, it means that the matrix property is defined in the same manner for different formats of matrices.

In [84], in the chapter by Johnson and Smith, a rich list of different matrix classes was analyzed and results on closure properties of these classes under taking Schur complements are presented. We recall here basic terminology used in [84].

For a given proper nonempty subset α of the index set N, a matrix class $\mathcal{C}$ is said to be α-*SC-closed* if for any $A \in \mathcal{C}$, $A/\alpha \in \mathcal{C}$. A matrix class $\mathcal{C}$ is said to be *SC-closed* if $\mathcal{C}$ is α-SC-closed for all α.

A matrix class $\mathcal{C}$ is said to be $\alpha - \mathit{hereditary}$ if for all $A \in \mathcal{C}$, $A(\alpha) \in \mathcal{C}$. A matrix class $\mathcal{C}$ is *hereditary* if $\mathcal{C}$ is α-hereditary for all α.

We recall the corollary of Banachiewichz inversion formula, that was mentioned earlier, as it is strongly related to considerations of SC-closure. Here, it is stated for an arbitrary nonempty proper subset α of the index set N, not necessarily corresponding to the leading principal submatrix. Namely, for $A \in \mathbb{C}^{n,n}$ nonsingular, α any nonempty proper subset in N such that $A(\alpha)$ is nonsingular, it holds that

$$A^{-1}(\overline{\alpha}) = (A/\alpha)^{-1}.$$

The Crabtree–Haynsworth property can be stated as follows, namely, for any nonempty proper subset α of the index set N, not necessarily corresponding to the leading principal submatrix.

In other words, for $A \in \mathbb{C}^{n,n}$ principally nonsingular, α a nonempty proper subset in N and β a nonempty proper subset in $\overline{\alpha}$, then,

$$(A/\alpha)/\beta = A/(\alpha \cup \beta).$$

Having this property of Schur complement in mind, we see that SC-closure follows from α-SC-closure for all singletons α.

The two following statements give further clarification of closure properties. The first one explains the relation between the question of SC-closure and the question of closure under taking principal submatrices. The second result shows that Schur complement transformation "runs through" multiplication with diagonal matrices, which is the most useful observation when considering H-matrices that can also be defined through diagonal scaling.

Theorem 47 ([84]) *A matrix class $\mathcal{C}$ is (α-)SC-closed if and only if the class of inverse matrices, $\mathcal{C}^{-1}$, is ($\overline{\alpha}$-)hereditary.*

Theorem 48 ([84]) *Let X and Y be nonsingular diagonal matrices and let A be principally nonsingular. Then,*

$$(XA)/\alpha = X(\overline{\alpha})(A/\alpha),$$

$$(AY)/\alpha = (A/\alpha)Y(\overline{\alpha})$$

and

$$(XAY)/\alpha = X(\overline{\alpha})(A/\alpha)Y(\overline{\alpha}).$$

5.2.1 Schur complements of SDD, M- and H-matrices

In [14], closure properties of classes of SDD, M- and H-matrices under taking Schur complements and under taking principal submatrices, are considered. Also, in the same paper, classes of inverse matrices of SDD, M- and H-matrices are investigated and it is proved that they are SC-closed and hereditary. Some results on inertia of H-matrices involving Schur complements are presented as well.

Theorem 49 ([14]) *Given SDD matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a nonempty proper subset α of the index set N, then, the Schur complement of A with respect to α, A/α, is also an SDD matrix.*

The proof is based on the following observations. For a given SDD matrix $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, it is enough to consider the matrix $A/\{1\}$. In other words, it is enough to prove that $A/\{1\}$ is SDD. As SDD class is closed under simultaneous permutations of rows and columns, from $\{1\}$-SC-closure it follows α-SC-closure for all singletons α. And this fact, together with the Crabtree-Haynsworth property of SC, implies α-SC-closure for any α.

In [14] it is proved by calculation that the first row in $A/\{1\}$ is SDD, and for the remaining rows it can be proved in the same manner.

Based on similar arguments, the SC-closure property for the class of matrices that are SDD by columns can be proved.

Theorem 50 ([84]) *Given any $A = [a_{ij}] \in \mathbb{C}^{n,n}$ such that A is SDD by columns and given any nonempty proper subset α of the index set N, then, the Schur complement of A with respect to α, A/α, is also SDD by columns.*

Now, using the scaling characterization of H-matrices and SC-closure of SDD class, together with Theorem 48, one obtains the following result of Carlson and Markham.

Theorem 51 ([14]) *Given any H-matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and given any proper nonempty subset α of the index set N, then, the Schur complement of A with respect to α, A/α, is also an H-matrix.*

Matrix A is an H-matrix if and only if there exists a diagonal nonsingular matrix, W, such that AW is SDD. But, from Theorem 49, the Schur complement of AW is SDD, and

$$(AW)/\alpha = (A/\alpha) \cdot (W/\alpha) = (A/\alpha) \cdot W(\overline{\alpha}).$$

This implies that the matrix A/α can be scaled to SDD by multiplying from the right by diagonal nonsingular matrix $W(\overline{\alpha})$. In other words, A/α is an H-matrix.

Notice that this can be explained in the following way - as the class of diagonal nonsingular matrices is hereditary, i.e., closed under taking principal submatrices, this implies SC-closure of the class of H-matrices. This argument will be crucial for investigating SC closure of those subclasses of H-matrices for which we have the scaling characterization.

Theorem 52 ([14]) *Given any M-matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and given any nonempty proper subset α of the index set N, then, the Schur complement of A with respect to α, A/α, is also an M-matrix.*

If $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is an M-matrix and α a nonempty proper subset in N, then, A is also an H-matrix, therefore, A/α is an H-matrix. It remains to see that A/α is a Z-matrix. Consider

$$A/\alpha = A(\overline{\alpha}) - A(\overline{\alpha},\alpha)(A(\alpha))^{-1}A(\alpha,\overline{\alpha}).$$

We see that $A(\overline{\alpha})$ is a Z-matrix, $A(\overline{\alpha},\alpha) \leq 0$ and $A(\alpha,\overline{\alpha}) \leq 0$. As $(A(\alpha))^{-1} \geq 0$, it follows that A/α is a Z-matrix and, therefore, an M-matrix.

We will use this argument when considering SC closure of $PM-$matrices later in this chapter.

In [14] it is said that Theorem 52 is due to Crabtree. Even a more revealing property of M-matrices related to Schur complements is discussed. Namely, if A is a Z-matrix, then, for $\emptyset \subset \alpha \subset N$, A is an M-matrix if and only if both $A(\alpha)$ and A/α are M-matrices.

The class of Z-matrices is not closed under Schur complement transformations, in general, as the following example shows.

Example 16 *Consider the matrix*

$$A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

Obviously, A is a Z-matrix, but

$$A/\{1\} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix},$$

which is not a Z-matrix.

However, it is easy to prove that if A is a Z-matrix and $A(\alpha)$ is an M-matrix, then A/α is also a Z-matrix.

Obviously, classes of SDD, SDD by columns, Z-, M- and H-matrices are all hereditary. Therefore, having in mind the statement of Theorem 47, corresponding classes of inverse matrices are all SC-closed.

5.2.2 SC of Ostrowski matrices

In paper by Li and Tsatsomeros, [53], SC-closure properties of Ostrowski class are considered. It is shown that this class is SC-closed and, even more, a condition that guarantees that the resulting Schur complement matrix of an Ostrowski matrix will be SDD, is given. We recall this result and we give an explanation for the second part of the statement, based on scaling.

Theorem 53 ([53]) *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be an Ostrowski matrix, α a nonempty proper subset of the index set and $N_r = \{i \in N \mid |a_{ii}| > r_i(A)\}$. Then, A/α is an Ostrowski matrix. Moreover, if $\overline{N}_r = N \backslash N_r \subset \alpha$, then, A/α is SDD .*

This can be stated also as follows. We know that there can be at most one non-SDD row in an Ostrowski matrix. If the Schur complement of a given Ostrowski matrix is taken with respect to an index set α that contains the index of the "bad" row, then, the resulting matrix is not just Ostrowski matrix, but also SDD.

Note that the second part of the result of Li and Tsatsomeros can be proved very easily, if we use the scaling matrix for a given Ostrowski matrix, constructed as in the discussion of Theorem 9. Namely, assume that $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ is an Ostrowski matrix with one non-SDD row indexed by $k \in N$ and let $W = diag(w_1, ..., w_n)$ be a nonsingular diagonal matrix with $w_i = 1$, for all $i \in N \backslash \{k\}$ and $w_k = \gamma > 0$, such that AW is SDD. Given any nonempty proper subset α of the index set N such that $k \in \alpha$, it holds that $W(\overline{\alpha})$ is the identity matrix of the corresponding order and therefore

$$(AW)/\alpha = (A/\alpha) \cdot W(\overline{\alpha}) = A/\alpha.$$

As the Schur complement of SDD matrix is SDD, $(AW)/\alpha$ is SDD, and so is A/α. This completes the proof of the second part of Theorem 53.

It is clear that Ostrowski class is both SC-closed and hereditary. Therefore, the corresponding inverse class, i.e., the class of inverse matrices for Ostrowski matrices, is also both SC-closed and hereditary.

5.2.3 SC-closure of partition-based classes

Results presented in this subsection are original contribution. Theorems 54, 55 and 56 are published in [21], together with original, scaling proofs. This paper is a joint work with Lj. Cvetković, V. Kostić and T. Szulc. Theorems 57, 58, 59 and 60 are presented here for the first time, as well as Corollaries 1 and 2.

First, we prove SC-closure property for the class of DZ matrices.

Theorem 54 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a DZ matrix. Then, for any nonempty proper subset α of N, A/α is also a DZ matrix. Moreover, if for the given matrix A, there exists a scaling matrix $W \in \mathcal{F}$, with $w_i = \gamma > 0$ for one $i \in N$ and $w_j = 1$ for $j \neq i$, where $\{i\} \subseteq \alpha$ or $N \setminus \{i\} = \alpha$, then, A/α is SDD.*

Proof: Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a Dashnic-Zusmanovich matrix. Then, from Theorem 12, there exists a matrix $W \in \mathcal{F}$ (defined by (3.4)), such that AW is an SDD matrix. As the Schur complement of an SDD matrix is SDD, AW/α is SDD, too. Since

$$(AW)/\alpha = (A/\alpha) \cdot W(\overline{\alpha}),$$

with $W(\overline{\alpha}) \in \mathcal{F}$, Theorem 12 provides that A/α is a Dashnic-Zusmanovich matrix. In order to complete the proof, it suffices to notice that, if $W \in \mathcal{F}$ is a scaling matrix, such that $w_i = \gamma > 0$, where $\{i\} \subseteq \alpha$, then,

$$W(\overline{\alpha}) = I,$$

with I denoting the identity matrix of the corresponding order. In case that

$$N \setminus \{i\} = \alpha,$$

we have

$$W(\overline{\alpha}) = \gamma \cdot I.$$

As neither of matrices I and $\gamma \cdot I$ affects SDD property of A/α, this implies that A/α is SDD. □

The following two statements are given in [57] and consider SC-closure of the class of Σ-SDD matrices. We gave new, original proofs for these results, based on scaling characterization of Σ-SDD matrices, in the same manner as done for DZ matrices. These proofs are published in [21].

Theorem 55 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a Σ-SDD matrix. Then for any nonempty proper subset α of N, A/α is also a Σ-SDD matrix. More precisely, if A is an S-SDD matrix, then A/α is an $(S \setminus \alpha)$-SDD matrix.*

Proof: Let A be a Σ-SDD matrix. Then, from Theorem 13, there exists a matrix $W \in \mathcal{W}$ (defined by (3.5)), such that AW is an SDD matrix. As the Schur complement of an SDD matrix is SDD, too, we conclude that $(AW)/\alpha$ is SDD. We have

$$(AW)/\alpha = (A/\alpha) \cdot W(\overline{\alpha}).$$

Since $W(\overline{\alpha}) \in \mathcal{W}$, i.e., the class $\mathcal{W}$ is closed under taking principal submatrices, from Theorem 13, we obtain that A/α is a Σ-SDD matrix.

To complete the proof, it is enough to see that the matrix $W(\overline{\alpha})$ is of the form

$$W(\overline{\alpha}) = diag(w_{i_1}, w_{i_2}, ..., w_{i_l})$$

with

$$w_{i_j} = \gamma > 0 \text{ for } i_j \in S \setminus \alpha \text{ and } w_{i_j} = 1 \text{ otherwise. } \square$$

Theorem 56 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be an S-SDD matrix. Then, for any nonempty proper subset α of N such that $S \subseteq \alpha$ or $\overline{S} \subseteq \alpha$, A/α is an SDD matrix.*

Proof: Let A be an S-SDD matrix. Then, from Theorem 13, there exists a matrix $W \in \mathcal{W}$ (defined by (3.5)), such that AW is an SDD matrix.

As the Schur complement of an SDD matrix is SDD, then, $(AW)/\alpha$ is SDD. Again,

$$(AW)/\alpha = (A/\alpha) \cdot W(\overline{\alpha}),$$

where $W(\overline{\alpha})$ is either the identity matrix of the corresponding format, I, in case that $S \subseteq \alpha$, or

$$W(\overline{\alpha}) = \gamma \cdot I,$$

in case that $\overline{S} \subseteq \alpha$. Therefore, $W(\overline{\alpha})$ cannot affect the strict diagonal dominance of A/α, implying that A/α is SDD.$\square$

A natural generalization of previous results to $PH-$matrices follows. It is presented here for the first time.

The next theorem states that the Schur complement of a $PH^{\pi}-$matrix is, again, a $PH-$matrix with respect to a different partition. Namely, as the Schur complement matrix has a different, smaller order than the original matrix A, it will be a $PH-$matrix with respect to the partition of the new index set which is actually the original partition restricted to $\overline{\alpha}$.

Theorem 57 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a $PH^{\pi}-$matrix where $\pi = \{p_j\}_{j=0}^{\ell}$ is the partition of the index set N. Given any nonempty proper subset α of N, then A/α is a $PH^{\pi|_{\overline{\alpha}}}-$matrix (where $\pi|_{\overline{\alpha}}$ denotes the partition π restricted to $\overline{\alpha}$).*

Proof: Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a $PH^{\pi}-$matrix. Then, from Theorem 17, there exists a matrix $W \in \mathcal{W}^{\pi}$ (defined by (3.13)), such that AW is an SDD matrix. As the Schur complement of an SDD matrix is SDD, we conclude that $(AW)/\alpha$ is SDD. Then, we have

$$(AW)/\alpha = (A/\alpha) \cdot W(\overline{\alpha}).$$

Since $W(\overline{\alpha}) \in \mathcal{W}^{\pi|_{\overline{\alpha}}}$, from Theorem 17, we obtain that A/α is a $PH^{\pi|_{\overline{\alpha}}}-$matrix.□

Moreover, if $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a $PH(k)-$matrix, then, for any nonempty proper subset α of N, A/α is also a $PH(k)-$matrix (if k does not exceed the format of A/α).

To complete this consideration, it is enough to point out two facts. First, the matrix $W(\overline{\alpha})$ is of the form

$$W(\overline{\alpha}) = diag(w_{i_1}, w_{i_2}, ..., w_{i_r}),$$

with

$$w_{i_k} = \gamma_j > 0 \text{ for all } i_k \in S_j, \quad j = 1, ..., \ell.$$

Second, if A is a $PH(m)-$matrix, then A is also a $PH(k)-$matrix for any $m \leq k \leq n$. This is because if A is a $PH^{\pi}-$matrix with respect to a given partition π, then, A is a $PH-$matrix also with respect to any partition finer than π.

Special cases of Theorem 57 are considered in the following corollaries.

Corollary 1 *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a $PH^{\pi}-$matrix with respect to the partition $\pi = \{p_j\}_{j=0}^{\ell}$ of the index set N, as in (3.7). Then, for any nonempty proper subset α of N such that $S_1 \cup S_2 \cup ... \cup S_{\ell-1} \subseteq \alpha$, A/α is an SDD matrix.*

Proof: Let A be a $PH^{\pi}-$matrix. Then, from Theorem 17, there exists a matrix $W \in \mathcal{W}^{\pi}$ (defined by (3.13)), such that AW is an SDD matrix. As the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, $(AW)/\alpha$ is strictly diagonally dominant. It is easy to see that

$$(AW)/\alpha = (A/\alpha) \cdot W(\overline{\alpha}),$$

where $W(\overline{\alpha})$ is of the form

$$W(\overline{\alpha}) = \gamma_{\ell} \cdot I$$

and it will not affect the strict diagonal dominance of A/α. Therefore, A/α is SDD.□

Corollary 2 *Let $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ be a $PH^{\pi}-$matrix with respect to the partition $\pi = \{p_j\}_{j=0}^{\ell}$ of the index set N, as in (3.7). Then, for any nonempty proper subset α of N such that $S_1 \cup S_2 \cup ... \cup S_{\ell-2} \subseteq \alpha$, A/α is a Σ-SDD matrix.*

Clearly, these results hold for any partition (not only of the ”leading” type), as simultaneous permutations can be applied. Also, if A is in $PH(k)$ and if α contains sets $S_{i_1},\ldots,S_{i_r}$, corresponding to r different blocks, then A/α is in $PH(k-r)$. As the number of blocks can decrease, we see that Schur complements can get ”closer” to SDD= $PH(1)$.

Notice that the class $PH(k)$ is hereditary, therefore, the class of inverse matrices is both hereditary and SC-closed.

Based on similar arguments used in Theorem 57 and in discussion of Theorem 52, we can prove SC-closure for the $PM(k)$ matrix class. As this class is also hereditary, it follows that the class of corresponding inverse matrices is both hereditary and SC-closed.

Theorem 58 *Let $A=[a_{ij}]\in\mathbb{R}^{n,n}$ be a $PM^{\pi}-$matrix where $\pi=\{p_j\}_{j=0}^{\ell}$ is the partition of the index set N. Given any nonempty proper subset α of N, then A/α is a $PM^{\pi|_{\overline{\alpha}}}-$matrix (where $\pi|_{\overline{\alpha}}$ denotes the partition π restricted to $\overline{\alpha}$).*

Proof: Let A be a $PM^{\pi}-$matrix. Then, A is a Z-matrix and a $PH^{\pi}-$matrix. From Theorem 57, we know that A/α is a $PH^{\pi|_{\overline{\alpha}}}-$matrix. It remains to prove that A/α is in Z-form, i.e., that A/α is an M-matrix, too. But, this is clear, because A is an M-matrix and from Theorem 52 we know that its Schur complement is an M-matrix.□

From everything discussed in this section, it is easy to see that more general statements hold.

Theorem 59 *Let $\mathcal{C}$ be a subclass of H-matrices derived from SDD class through scaling, with a scaling characterization given by the class $\mathcal{D}(\mathcal{C})$, a subclass in the class of nonsingular diagonal matrices, $\mathcal{D}$. Let α be a nonempty proper subset of the index set. If $\mathcal{D}(\mathcal{C})$ is $\overline{\alpha}$-hereditary, then, $\mathcal{C}$ is α-SC-closed.*

Proof: Let $A=[a_{ij}]\in\mathbb{C}^{n,n}$ be a given matrix from the class $\mathcal{C}$. Let W be a corresponding scaling matrix for A from $\mathcal{D}(\mathcal{C})$. Let α be a nonempty proper subset of the index set. As

$$(AW)/\alpha=(A/\alpha)\cdot W(\overline{\alpha}),$$

where $W(\overline{\alpha})$ is, again, from $\mathcal{D}(\mathcal{C})$, this implies that A/α is in $\mathcal{C}$. □

Theorem 60 *Let $\mathcal{C}$ be a subclass of H-matrices derived from SDD class through scaling, with a scaling characterization given by the class $\mathcal{D}(\mathcal{C})$, a subclass in the class of nonsingular diagonal matrices, $\mathcal{D}$. Then, if $\mathcal{D}(\mathcal{C})$ is hereditary, $\mathcal{C}$ is SC-closed.*

5.2.4 Schur complements of $\alpha 1$– and $\alpha 2$–matrices

The class of $\alpha 1$–matrices is not closed under Schur complements in general. The counter-example can be found in [54]. In the same paper, it is shown that the class of $\alpha 2$–matrices is not closed under Schur complement, either.

However, with $N_r = \{i \in N \mid |a_{ii}| > r_i(A)\}$ as before and $N_c = \{i \in N \mid |a_{ii}| > c_i(A)\}$, in the same paper it is proved that for a given $\alpha 1$– ($\alpha 2$–) matrix, $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, if $\alpha \subseteq N_r \cap N_c$, then, A/α is also $\alpha 1$– ($\alpha 2$–) matrix.

5.2.5 SC-closure of Nekrasov matrices

In [44], the following result based on results from [2] is presented.

Theorem 61 ([44]) *The Nekrasov property is hereditary for Gaussian elimination.*

In other words, if we consider a matrix obtained through Gaussian elimination from a Nekrasov matrix (and of the same format as the original matrix), it is again a Nekrasov matrix. The proof is based on following observations. For B being the matrix obtained from A through one step of Gaussian elimination, it is proved that B is a Nekrasov matrix.

For

$$\frac{h_i(A)}{|a_{ii}|}, \quad i = 1, ..., n,$$

the quantity called the *Nekrasov multiplier for the row i,* it is proved that for a Nekrasov matrix A, no Nekrasov multiplier for any row can increase in Gaussian elimination. In other words,

$$\frac{h_i(B)}{|b_{ii}|} \leq \frac{h_i(A)}{|a_{ii}|}, \quad i = 2, ..., n.$$

This inequality can be proved as done in [2].

An interpretation of this inequality would be the following. Relative Nekrasov dominant degree increases (or, more precisely, cannot decrease) through Schur complement transformation. In other words, Nekrasov dominance can only get stronger when taking SC.

It is easy to see that from the previous statement follows the next one.

Corollary 3 *The Nekrasov class is $\{1\}$-SC-closed. Moreover, it is α-SC-closed for all α of the form $\alpha = \{1, 2, ..., m\}$.*

In other words, if the Schur complement of a Nekrasov matrix is taken with respect to a leading principal submatrix, the resulting matrix is also a Nekrasov matrix.

Using the scaling characterization, as done in the paper [26], which is a joint work of Lj. Cvetković and the author, we obtain:

Theorem 62 *If* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is an* S*-Nekrasov matrix, then* $A/\{1\}$ *is* $S \setminus \{1\}$*-Nekrasov matrix.*

Proof: Let A be an S-Nekrasov matrix. Then, from Theorem 13, there exists a matrix $W \in \mathcal{W}$ (defined by (3.5)), such that AW is a Nekrasov matrix. As the $\{1\}$-Schur complement of a Nekrasov matrix is a Nekrasov matrix, too, we conclude that $AW/\{1\}$ is a Nekrasov matrix. Also,

$$(AW)/\{1\} = (A/\{1\}) \cdot W(\{\overline{1}\}).$$

Since $W(\{\overline{1}\}) \in \mathcal{W}$ is of the form

$$W(\{\overline{1}\}) = diag(w_{i_1}, w_{i_2}, ..., w_{i_l}),$$

with

$$w_{i_j} = \gamma > 0 \ \text{ for } \ i_j \in S \setminus \{1\} \ \text{ and } \ w_{i_j} = 1 \ \text{ otherwise,}$$

from Theorem 13, we obtain that $A/\{1\}$ is an $S \setminus \{1\}$-Nekrasov matrix. □

Direct corollary of this statement and the Crabtree-Haynsworth property of Schur complement is the following.

Theorem 63 *The* Σ*-Nekrasov class is* $\{1\}$*-SC-closed. Moreover, it is* α*-SC-closed for all* $\alpha = \{1, 2, ..., m\}$.

In other words, Σ-Nekrasov class is closed under taking Schur complements with respect to leading principal submatrices.

5.2.6 SC of IDD and lower semi-SDD matrices

The class of IDD matrices is neither hereditary nor SC-closed, as stated in [84]. The same holds for the class of reducible matrices.

When lower semi-SDD matrices are considered, in the paper of Ikramov there is a proof for the following statement.

Theorem 64 ([44]) *The lower semi-SDD property is hereditary for Gaussian elimination.*

As strict and nonstrict diagonal dominance are both hereditary for Gaussian elimination, it remains to prove that matrix B, obtained from A through one step of Gaussian elimination, satisfies the condition

$$|b_{ii}| > l_i(B), \ \ i = 1, \dots, n.$$

As in the case of Nekrasov property, a stronger statement is proved, namely,

$$\frac{l_i(B)}{|b_{ii}|} \le \frac{l_i(A)}{|a_{ii}|}, \ \ i = 3, \dots, n.$$

This inequality is obtained in [44] by induction.

It is easy to see that from the previous statement follows the next one.

Corollary 4 *The lower semi-SDD class is* $\{1\}$*-SC-closed. Moreover, it is* α*-SC-closed for all* $\alpha = \{1, 2, \dots, m\}$.

Again, the closure property holds when Schur complements are taken with respect to leading principal submatrices.

5.2.7 Diagonal Schur complement

In [56], Schur complements and related diagonal-Schur complements are said to be important tools in matrix theory, control theory, numerical analysis and statistics. Therefore, in the same paper, different matrix properties associated with diagonal dominance are considered and invariance of these properties under Schur and diagonal-Schur complements is discussed. In [26], starting from definitions and results from [56] on diagonal-Schur complements, we obtained some original results and new proofs based on scaling characterizations of matrix classes considered.

The diagonal-Schur complement of $A = [a_{ij}] \in \mathbb{C}^{n,n}$, with respect to a proper subset α of the index set N, is denoted by $A/_\circ\alpha$ and defined to be

$$A(\overline{\alpha}) - \{A(\overline{\alpha}, \alpha)(A(\alpha))^{-1} A(\alpha, \overline{\alpha})\} \circ I,$$

where, as before, $A(\alpha, \beta)$ stands for the submatrix of A lying in the rows indexed by α and the columns indexed by β, while $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. For $A = [a_{ij}] \in \mathbb{C}^{m,n}$ and $B = [b_{ij}] \in \mathbb{C}^{m,n}$, the Hadamard product of A and B is the matrix $[a_{ij} b_{ij}]$, which we denote by $A \circ B$. It is assumed that $A(\alpha)$ is a nonsingular matrix.

We say that a matrix class $\mathcal{C}$ is α-*diagonal-SC-closed* if for any $A \in \mathcal{C}$, $A/_\circ\alpha \in \mathcal{C}$. A matrix class $\mathcal{C}$ is said to be *diagonal-SC-closed* if $\mathcal{C}$ is α-diagonal-SC-closed for all α.

It is important to note that the Crabtree-Haynsworth property holds for the Schur complement, but does not hold in general for the diagonal-Schur complement.

However, it is easy to see that the following useful property considering multiplication with diagonal matrices holds.

Lemma 10 ([56]) *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *and let* α *be a nonempty proper subset of* N, *such that* $A(\alpha)$ *is nonsingular. Let* $W \in \mathbb{R}^{n,n}$ *be a diagonal matrix with positive diagonal entries. Then,*

$$(AW)/_{\circ}\alpha = (A/_{\circ}\alpha)W(\overline{\alpha}).$$

In [55, 56], the class of SDD matrices is proved to be diagonal-SC-closed, as well as the class of Ostrowski matrices and the class of H-matrices. Using scaling characterization of DZ matrices, we proved the following diagonal-SC-closure result, based on arguments similar to those used in the proof of Theorem 54. This result is published in [26].

Theorem 65 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be a Dashnic-Zusmanovich matrix. Then, for any nonempty proper subset* α *of N,* $A/_{\circ}\alpha$ *is also a Dashnic-Zusmanovich matrix.*

Moreover, if for the given matrix A, there exists a scaling matrix $W \in \mathcal{F}$, with $w_i = \gamma > 0$, where $\{i\} \subseteq \alpha$ or $N \setminus \{i\} = \alpha$, then, $A/_{\circ}\alpha$ is a strictly diagonally dominant matrix.

In the same paper, [26], using scaling characterization of S-SDD matrices, we presented simplified proofs for the next two results from [56].

Theorem 66 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be a* Σ*-SDD matrix. Then, for any nonempty proper subset* α *of* N, $A/_{\circ}\alpha$ *is also a* Σ*-SDD matrix. More precisely, if* A *is an S-SDD matrix, then* $A/_{\circ}\alpha$ *is an* $(S \setminus \alpha)$*-SDD matrix.*

Theorem 67 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be an S-SDD matrix. Then, for any nonempty proper subset* α *of* N, *such that* $S \subseteq \alpha$ *or* $\overline{S} \subseteq \alpha$, $A/_{\circ}\alpha$ *is an SDD matrix.*

Proofs are based on arguments similar to those used in proofs of Theorem 55 and Theorem 56.

It is easy to see that for $PH-$matrices, statements for diagonal-Schur complements analogous to Theorem 57 and Corollaries 1 and 2 are true.

Theorem 68 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be a* $PH^{\pi}-$*matrix, where* $\pi = \{p_j\}_{j=0}^{\ell}$ *is the partition of the index set* N. *Given any nonempty proper subset* α *of N, then,* $A/_{\circ}\alpha$ *is a* $PH^{\pi|_{\overline{\alpha}}}-$*matrix (where* $\pi|_{\overline{\alpha}}$ *denotes the partition* π *restricted to* $\overline{\alpha}$*).*

Corollary 5 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be a* $PH^{\pi}-$*matrix with respect to the partition* $\pi = \{p_j\}_{j=0}^{\ell}$ *of the index set* N*. Then, for any nonempty proper subset* α *of* N*, such that* $S_1 \cup S_2 \cup \ldots \cup S_{\ell-1} \subseteq \alpha$*,* $A/_{\circ}\alpha$ *is an SDD matrix.*

Corollary 6 *Let* $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ *be a* $PH^{\pi}-$*matrix with respect to the partition* $\pi = \{p_j\}_{j=0}^{\ell}$ *of the index set* N*. Then, for any nonempty proper subset* α *of* N*, such that* $S_1 \cup S_2 \cup \ldots \cup S_{\ell-2} \subseteq \alpha$*,* $A/_{\circ}\alpha$ *is a* Σ*-SDD matrix.*

A similar reasoning holds for $PM-$matrices, as well. Namely, $PH-$class is already proved to be diagonal-SC-closed, while Z-form is preserved because $A(\overline{\alpha})$ is a Z-matrix and the change affects only diagonal entries. Therefore, the following statement is true.

Theorem 69 *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be a* $PM^{\pi}-$*matrix, where* $\pi = \{p_j\}_{j=0}^{\ell}$ *is the partition of the index set* N*. Given any nonempty proper subset* α *of* N*, then,* $A/_{\circ}\alpha$ *is a* $PM^{\pi|_{\overline{\alpha}}}-$*matrix (where* $\pi|_{\overline{\alpha}}$ *denotes the partition* π *restricted to* $\overline{\alpha}$*).*

Again, more general results follow.

Theorem 70 *Let* $\mathcal{C}$ *be a subclass of* H*-matrices derived from SDD class through scaling, with a scaling characterization given by the class* $\mathcal{D}(\mathcal{C})$*, a subclass in the class of nonsingular diagonal matrices,* $\mathcal{D}$*. Let* α *be a nonempty proper subset of the index set. If* $\mathcal{D}(\mathcal{C})$ *is* $\overline{\alpha}$*-hereditary, then,* $\mathcal{C}$ *is diagonal-*α*-SC-closed.*

Theorem 71 *Let* $\mathcal{C}$ *be a subclass of* H*-matrices derived from SDD class through scaling, with a scaling characterization given by the class* $\mathcal{D}(\mathcal{C})$*, a subclass in the class of nonsingular diagonal matrices,* $\mathcal{D}$*. Then, if* $\mathcal{D}(\mathcal{C})$ *is hereditary,* $\mathcal{C}$ *is diagonal-SC-closed.*

For classes based on recursively defined row sums, the following statements are true, see [26]. Proofs are based on induction and scaling characterization of Σ-Nekrasov matrices.

Theorem 72 *The Nekrasov class is* $\{1\}$*-diagonal-SC-closed.*

Theorem 73 *If A is S-Nekrasov matrix, then* $A/_{\circ}\{1\}$ *is* $S\setminus\{1\}$*-Nekrasov matrix.*

Theorem 74 *The* Σ*-Nekrasov class is* $\{1\}$*-diagonal-SC-closed.*

5.2.8 Schur complements of general H-matrices

We already mentioned that there is a more general definition of H-matrices that includes some singular matrices, too. As said in [8], the class of general H-matrices can be divided into three subclasses - invertible, mixed and singular class. In the paper [9], Schur complements of general H-matrices were studied. It is shown that the Schur complement of a general H-matrix is, again, a general H-matrix (if the Schur complement exists). Even more, for matrices in the invertible class and for matrices in the singular class, it is established that Schur complements belong to the same subclass of general H-matrices as the original matrix. In other words, invertible class and singular class are both SC-closed. Also, for a given *singular* matrix belonging to the mixed class, the Schur complement is, again, in the mixed class. However, for the given *nonsingular* matrix belonging to the mixed class, in the same paper it is shown that one of the two cases can occur. The Schur complement either remains in the same, mixed, class, or, the Schur complement belongs to the invertible class. Some conditions on the graph of the given matrix are proposed in [9], for determining which of the two possible cases will occur.

5.2.9 The Perron complement of PH–matrices

In [63], Meyer introduced the notion of the Perron complement (PC), in connection with computing the stationary distribution vector for a Markov chain. In the same paper, Meyer noticed a connection of Perron complements to Schur complements.

The Perron complement of a nonnegative irreducible matrix, $A \in \mathbb{C}^{n,n}$, with respect to a proper subset of N, α, is denoted by $P(A/\alpha)$ and defined to be

$$P(A/\alpha) = A(\overline{\alpha}) + A(\overline{\alpha},\alpha)[\rho(A)I - A(\alpha)]^{-1}A(\alpha,\overline{\alpha}),$$

where $\rho(A)$ denotes the spectral radius of the matrix A, $A(\alpha,\beta)$ stands for the submatrix of $A \in \mathbb{C}^{n,n}$, lying in the rows indexed by α and the columns indexed by β, while $A(\alpha,\alpha)$ is abbreviated to $A(\alpha)$, as before.

Meyer also gave the following result on Perron complements.

Theorem 75 ([63]) *Let $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ be a nonnegative and irreducible matrix with spectral radius $\rho(A)$, and let α be a nonempty proper subset of N. Then, the Perron complement $P(A/\alpha)$ is also nonnegative and irreducible matrix, with spectral radius $\rho(A)$.*

Perron (and Schur) complements have been studied in [64], where it is stated that Perron complements of inverse M-matrices are inverse M-matrices. Fallat and

Neumann obtained conditions for PC-closure of totally-nonnegative (TN) matrices in [32], together with a quotient formula for PC analogous to the Crabtree-Haynsworth quotient property for SC. The authors also considered an ordering between PC and SC of TN matrices. Johnson and Xenophotos in [46] investigated primitivity of PC.

In [85], the following result on Perron complements of SDD matrices is obtained.

Theorem 76 ([85]) *Let $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ be an SDD, nonnegative and irreducible matrix with spectral radius $\rho(A)$, and let α be a nonempty proper subset of N. Then, if $\rho(A) \geq \max_{i\in\alpha}\sum_{j=1}^{n}|a_{ij}|$, $P(A/\alpha)$ is also SDD, nonnegative and irreducible matrix.*

Based on scaling characterizations, we obtain a result on PC of PH-matrices.

Theorem 77 *Let $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ be a nonnegative, irreducible $PH-$matrix. For a nonempty proper subset α of N, if $\rho(A) \geq \max_{i\in\alpha} 2|a_{ii}|$, then $P(A/\alpha)$ is also a nonnegative irreducible $PH-$matrix. More precisely, if A is a $PH-$matrix with respect to the partition $\pi = \{p_j\}_{j=0}^{\ell}$ of the index set N, then $P(A/\alpha)$ is a $PH-$matrix with respect to the partition $\pi|_{\overline{\alpha}}$, which is π restricted to $\overline{\alpha}$.*

Proof: Let A be a $PH-$matrix with respect to the partition $\pi = \{p_j\}_{j=0}^{\ell}$ of the index set N. Then, from Theorem 17, there exists a matrix $W \in \mathcal{W}^{\pi}$ (defined by (3.13)), such that AW is an SDD matrix. As AW is an SDD matrix, so is the matrix $B = W^{-1}AW$. It is easy to see that

$$P(B/\alpha) = (W(\overline{\alpha}))^{-1}P(A/\alpha)W(\overline{\alpha}).$$

As B is an SDD matrix similar to A, it holds

$$\rho(B) = \rho(A) \geq 2\max_{i\in\alpha}|a_{ii}| > \max_{i\in\alpha}|a_{ii}| + \max_{i\in\alpha}\sum_{j=1, j\neq i}^{n}|a_{ij}|\frac{w_j}{w_i} =$$

$$= \max_{i\in\alpha}|b_{ii}| + \max_{i\in\alpha}\sum_{j=1, j\neq i}^{n}|b_{ij}| \geq \max_{i\in\alpha}\sum_{j=1}^{n}|b_{ij}|.$$

This means that B satisfies condition of Theorem 76, therefore $P(B/\alpha)$ is an SDD matrix. We see that $P(A/\alpha)$ can be scaled to SDD matrix by multiplying from the right with diagonal nonsingular matrix $W(\overline{\alpha})$ that belongs to $\mathcal{W}^{\pi|_{\overline{\alpha}}}$. This implies that $P(A/\alpha)$ is a $PH-$matrix with respect to the partition $\pi|_{\overline{\alpha}}$, which is π restricted to $\overline{\alpha}$.$\square$

Corollary 7 *Let* $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ *be a nonnegative irreducible* $PH-$*matrix with respect to the partition* $\pi = \{p_j\}_{j=0}^{\ell}$ *of the index set* N*. For a nonempty proper subset* α *of* N, *such that* $S_1 \cup S_2 \cup ... \cup S_{\ell-1} \subseteq \alpha$*, if* $\rho(A) \geq \max_{i\in\alpha} 2|a_{ii}|$*, then* $P(A/\alpha)$ *is a nonegative irreducible SDD matrix.*

Corollary 8 *Let* $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ *be a nonnegative irreducible* $PH-$*matrix with respect to the partition* $\pi = \{p_j\}_{j=0}^{\ell}$ *of the index set* N*. For a nonempty proper subset* α *of* N, *such that* $S_1 \cup S_2 \cup ... \cup S_{\ell-2} \subseteq \alpha$*, if* $\rho(A) \geq \max_{i\in\alpha} 2|a_{ii}|$*, then* $P(A/\alpha)$ *is a nonegative irreducible* Σ*-SDD matrix.*

Notice that the same statement holds for any choice of a nonempty proper subset α of N such that $\overline{\alpha} \subseteq S_j \cup S_k$ for some $j,k \in \{1,2,...,l\}$.

5.3 Eigenvalue localization for SC

Eigenvalue localization and separation are problems that attract the attention of many researchers. There are many relations of eigenvalue problems to H-matrix theory and there are different formulations of and different approaches to such problems.

First, as we have seen already in Chapter 4, nonsingularity results for some matrix classes (subclasses in the class of H-matrices) served as a starting point for defining new eigenvalue inclusion sets. In fact, as emphasized in [82], these two streams of research are equivalent, although formulated in a different manner. Geršgorin's result is equivalent to Lévy-Desplanques Theorem, i.e., the fact that all the eigenvalues of a given square complex matrix are contained in the Geršgorin's set is equivalent to the statement that every SDD matrix is nonsingular. Starting from some other subclasses of (nonsingular) H-matrices, such as Ostrowski or S-SDD, corresponding eigenvalue inclusion results were obtained. Brauer's Ovals of Cassini or CKV set, see [82], are some examples for localizations of this type. Notice that these results, often called Geršgorin-type theorems, apply to all square complex matrices. Therefore, in that context, H-matrix theory inspired new results that are applicable even outside its frame. However, not all of the H-subclasses that we deal with produce localization sets as elegant as Geršgorin's set is. For a practical use, these sets should be defined in a computationally not too demanding way.

Second, what is there to be said about eigenvalues of those matrices that actually belong to the class of H-matrices? As we know, for some practical purposes, especially related to questions of stability, it is of a great use to know for a given matrix in which half-plane the eigenvalues are located. If eigenvalues are all in the left half-plane, then (one type of) stability is guaranteed. It is an important question also to find the number of the eigenvalues that belong in the left (right) half-plane. In other words, due to its relation to questions of stability of dynamical systems, inertia, or inertia-triple, that is defined as the ordered triple of integers that represent the number of eigenvalues with positive, zero and negative real part, respectively, for a given square complex matrix (in general), is often a subject to investigation. Luckily, for real SDD and H-matrices, an inertia-related question can be answered by observing the location of the diagonal entries. In [50], one can find Inertia Principle for both real and complex case.

The same inertia question can be posed for the Schur complement matrix, as well. In recent years, many researchers dealt with eigenvalue distribution problems for Schur complements of some special matrices. Liu and Huang in [55] gave the result on the number of eigenvalues with positive real part and the number of eigenvalues with negative real part for the Schur complement, A/α, of an H-matrix, A, with real diagonal entries. In the paper [83], there is a generalization of this result to H-matrices with complex diagonal entries. Different conditions on the matrix A and the index set α are given to ensure that the Schur complement A/α has $|J_{R+}(A)| - |J_{R+}^{\alpha}(A)|$ eigenvalues with positive real part and $|J_{R-}(A)| - |J_{R-}^{\alpha}(A)|$ eigenvalues with negative real part, where

$$J_{R+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in N\},$$
$$J_{R-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in N\},$$
$$J_{R+}^{\alpha}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in \alpha\}$$

and

$$J_{R-}^{\alpha}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in \alpha\}.$$

In [59], it is stated that matrix classes with properties of nonsingularity and SC-closure represent important tools in numerical analysis and in matrix analysis. This is especially true when dealing with convergence of iterative methods and deriving matrix inequalities. In the same paper, a disk separation of the Schur complement is studied. Namely, for a given SDD matrix A, each Geršgorin's disk is separated from the origin and quantities $|a_{ii}| - r_i(A)$ measure that separation. As we already know, the Schur complement of an SDD matrix is, again, SDD, so it makes sense to try to compare separations of the disks for the Schur complement to those of the original matrix. In [59], Liu and Zhang obtained that disk separation of the Schur complement of SDD matrix is greater than that of the original, larger, matrix. Notice that this is a different type of eigenvalue problem for the Schur complement - we do not want just to find the number of eigenvalues in each half-plane, we also want to detect how far from zero they are located. In other words, how far we are from singularity. Let us recall here the short version of the main result from [59].

Theorem 78 ([59]) *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *be an SDD matrix,* $\alpha = \{i_1, i_2, ..., i_k\} \subset N$, $\overline{\alpha} = N \backslash \alpha = \{j_1, j_2, ..., j_l\}$, $k + l = n$. *Denote* $A/\alpha = [a'_{ts}]$. *Then,*

$$|a'_{tt}| - r_t(A/\alpha) \geq |a_{j_t j_t}| - r_{j_t}(A) > 0.$$

As an application of this, in the same paper, bounds for determinants and locations of the eigenvalues are discussed. More precisely, a relation between location (separation) of the eigenvalues of A/α and eigenvalues of $A(\overline{\alpha})$, for SDD matrices with real diagonal entries, is obtained. In the following subsections, we use scaling technique to obtain a generalization of this result.

In the paper [58], further development of results from [59] is made. Estimates of $\alpha 1-$ and $\alpha 2-$dominant degree for Schur complements are obtained. As an application, bounds for eigenvalues of the Schur complement are presented, defined by the entries of the original matrix instead of the entries of SC. It is stated that eigenvalues of SC are contained in Geršgorin's circles of the original matrix, under some conditions. Also, in the same paper, as a greater diagonally dominant degree implies, in general, faster convergence of iterative methods, a Schur-based iteration is designed. In [60], an estimate of Ostrowski dominant degree for SC is presented and it is obtained that eigenvalues of SC are located in Brauer's Ovals of Cassini of the original matrix, under certain conditions.

In the following subsections, we use scaling technique to generalize some of these results.

Results on eigenvalue distribution for diagonal-Schur complements of some special H-matrices can be found in [56].

There is also an interesting paper by J. M. Pena, see [68], on pivoting strategies ensuring that the radii of Geršgorin's circles of Schur complements through Gaussian elimination reduce their lengths. Results are given for some special matrix classes, including the class of M-matrices. A strategy of row diagonal dominance pivoting is discussed. It is stated that, in general, the lengths of the radii of Geršgorin's circles can grow arbitrarily during Gaussian elimination. From this point of view, some results on SC-closure that we presented can be interpreted as pivoting strategies ensuring that matrices obtained in the process are closer to strict diagonal dominance .

In the remainder of Chapter 5, we deal with relations between (localization and separation of) eigenvalues of SC and (localization and separation of) eigenvalues of the corresponding submatrix in the original matrix. Also, we investigate conditions that ensure that eigenvalues of SC are contained in (a subset of) Geršgorin's set for the original matrix.

5.3.1 Vertical eigenvalue bands

In [59], the following result on location of the eigenvalues of the Schur complement is proved.

Theorem 79 ([59]) *Let a matrix $A \in \mathbb{C}^{n,n}$ be an SDD matrix with real diagonal entries, and let α be a proper subset of the index set. Then, A/α and $A(\overline{\alpha})$ have the same number of eigenvalues whose real parts are greater (less) than $w(A)$ (resp. $-w(A)$), where*

$$w(A) = \min_{j \in \overline{\alpha}} \Big(|a_{jj}| - r_j(A) + \min_{i \in \alpha} \frac{|a_{ii}| - r_i(A)}{|a_{ii}|} \sum_{k \in \alpha} |a_{jk}| \Big). \tag{5.7}$$

Using the scaling approach, we obtained the following, more general result, published in the paper [27], a joint work of Lj. Cvetković and the author.

Theorem 80 *Given a nonempty proper subset $S \subseteq N$, let $A \in \mathbb{C}^{n,n}$ be an S-SDD matrix with real diagonal entries, and let α be a proper nonempty subset of the index set. Then, A/α and $A(\overline{\alpha})$ have the same number of eigenvalues whose real parts are greater (less) than $w(W^{-1}AW)$ (resp. $-w(W^{-1}AW)$), where $w(A)$ is defined as in (5.7) and W is any corresponding scaling matrix,*

$$W = diag(w_1, w_2, ..., w_n),$$

with $w_i = \gamma \in (\gamma_1(A), \gamma_2(A))$, as in (3.6), for $i \in S$ and $w_i = 1$ otherwise.

Proof: Since A is an S-SDD matrix with real diagonal entries, and W is the corresponding scaling matrix, we know that $W^{-1}AW$ is an *SDD* matrix (also with real diagonal entries). Then, if α is a proper subset of the index set, we have

$$(W^{-1}AW)/\alpha = W^{-1}(\overline{\alpha})(A/\alpha)W(\overline{\alpha}),$$

which is similar to A/α. Moreover, if $\alpha = S$ or $\alpha = \overline{S}$ this matrix is exactly A/α. Obviously, matrices $(W^{-1}AW)(\overline{\alpha})$ and $A(\overline{\alpha})$ are similar (for any choice of α), so they have the same eigenvalues. Now, we apply Theorem 79 to SDD matrix $W^{-1}AW$, and obtain that A/α and $A(\overline{\alpha})$ have the same number of eigenvalues whose real parts are greater (less) than $w(W^{-1}AW)$ (resp. $-w(W^{-1}AW)$). □

As stated before, the scaling matrix for the fixed S-SDD matrix A is not unique, i.e., the scaling parameter γ can be chosen from the interval $(\gamma_1(A), \gamma_2(A))$, defined in (3.6). In other words, we can transform the given S-SDD matrix to many different SDD matrices, by choosing different values for γ, as long as γ belongs to this interval. In this way we obtain different values for $w(W^{-1}AW)$ for the given S-SDD matrix A, all of them with the separating property. Geometrical explanation of this is that, if the principal submatrix $A(\overline{\alpha})$ has no eigenvalues whose real parts are between the vertical lines $x=-w(W^{-1}AW)$ and $x=w(W^{-1}AW)$, then the Schur complement A/α has no eigenvalues in the band, either.

Now, we could go one step back, and apply this result to SDD matrices. If a matrix A is SDD, then it is also S-SDD for any subset S of the index set N. If we fix any subset S of N and choose γ from the interval $(\gamma_1(A),\gamma_2(A))$, defined in (3.6), we can scale the given SDD matrix to some other, again SDD, matrix and obtain $w_1=w(W^{-1}AW)$ "better" than $w=w(A)$ with the separating property.

Note that a matrix A can be both S-SDD and T-SDD, for different subsets S and T of N and, in that case, we could get two different values for w, i.e. $w(W_S^{-1}AW_S)$ and $w(W_T^{-1}AW_T)$.

The previous result holds for any H-matrix with real diagonal entries, but it is useful in practice only if we are able to determine a scaling matrix in a non-expensive way. Therefore, it can be applied to Ostrowski, DZ, Σ-SDD, Nekrasov or Σ-Nekrasov matrices, as we are able to construct a corresponding scaling matrix for a given (Σ-)Nekrasov matrix using the statement of Theorem 20. The result on vertical bands for SC of Nekrasov matrices was published in the paper [77], where a construction of a scaling matrix for a given Nekrasov matrix was presented.

Example 17 *The given matrix, E, is not SDD, because the last row is not SDD, but it is an Ostrowski matrix. We determine the interval for the scaling parameter γ, as defined in discussion of Theorem 9. It is easy to see that $\gamma=2$ belongs to this interval. For the chosen γ, we scale the matrix E, and obtain the vertical band determined by $w(W^{-1}EW)=3.97796$, given in the Figure 5.1. As there are obviously no eigenvalues of $E(\overline{\alpha})$ in the band, we conclude that there are no eigenvalues of E/α in the band.*

$$E=\begin{bmatrix} 1125 & 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & 1125 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1225 & 1 & -1 & 1 & 2 \\ 1 & 2 & 1 & -25 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & -25 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & -25 & 0 \\ 4 & 1 & 1 & 0 & 0 & 0 & -4 \end{bmatrix}$$

Example 18 *The given matrix, M, is not SDD, but it is S-SDD for $S=\{1,2,3\}$. We take $\alpha=S$, and determine the interval $(\gamma_1(M),\gamma_2(M))$, as defined in (3.6), for parameter γ. It is easy to see that $\gamma=0.00002$ belongs to this interval. For* *the chosen* γ*, we scale the matrix* M*, and, as in Theorem 80, obtain the vertical band (* $w(W^{-1}MW) = 11.9911$ *) given in the Figure 5.2. As there are, obviously, no eigenvalues of* $M(\overline{\alpha})$ *in the band, we conclude that there are no eigenvalues of* M/α *in the band.*

$$M = \begin{bmatrix} 562500 & 500 & 500 & -1 & 1 & 1 & 0 \\ 0 & 562500 & 500 & 1 & 0 & 1 & 1 \\ 500 & 500 & 562500 & 1 & -1 & 1 & 2 \\ 500 & 1000 & 500 & -25 & 0 & 0 & 0 \\ 0 & 1000 & 500 & 0 & -25 & 0 & 0 \\ 500 & 0 & 500 & 0 & 0 & -25 & 0 \\ 0 & 500 & 500 & 0 & 0 & 0 & -12 \end{bmatrix}$$

Now, we give the analogous statement for Nekrasov matrices with numerical examples. The following result is published in the paper [77].

Theorem 81 *Let* $A \in \mathbb{C}^{n,n}$ *be a Nekrasov matrix with nonzero Nekrasov row sums and real diagonal entries, and let* α *be a proper subset of the index set. Then,* A/α *and* $A(\overline{\alpha})$ *have the same number of eigenvalues whose real parts are greater (less) than* $w(D^{-1}AD)$ *(resp.* $-w(D^{-1}AD)$*), where* $w(A)$ *is defined as in (5.7) and* D *is a corresponding scaling matrix.*

Proof: Since A is a Nekrasov matrix with real diagonal entries and D is the corresponding scaling matrix, we know that $D^{-1}AD$ is an SDD matrix (also with real diagonal entries). Then, if α is a proper subset of the index set, we have

$$(D^{-1}AD)/\alpha = D^{-1}(\overline{\alpha})(A/\alpha)D(\overline{\alpha}),$$

which is similar to A/α. Obviously, matrices $(D^{-1}AD)(\overline{\alpha})$ and $A(\overline{\alpha})$ are similar (for any choice of α), so they have the same eigenvalues. Now, we apply Theorem 79 to SDD matrix $D^{-1}AD$, and obtain that A/α and $A(\overline{\alpha})$ have the same number of eigenvalues whose real parts are greater (less) than $w(D^{-1}AD)$ (resp. $-w(D^{-1}AD)$). This completes the proof.□

For practical use, we can construct the scaling matrix D of Theorem 81 in the same way it is done in Theorem 20.

In the same manner, we can determine vertical eigenvalue bounds for the Schur complement of a Σ-Nekrasov matrix - notice that the corresponding scaling matrix is, in that case, constructed as WD, where we could construct W as in explanation following Theorem 27 and D as in Theorem 20.

Example 19 *Consider the matrix*

$$B = \begin{bmatrix} 40 & 8 & 7 & 0 \\ 3 & 3.5 & 0.5 & 0.5 \\ 0 & 5 & 10 & 5 \\ 0 & 0 & 4.5 & 4 \end{bmatrix}.$$

It is a Nekrasov matrix for which, following Theorem 20, we can find one of its corresponding scaling matrices, D,

$$D = \begin{bmatrix} 0.375 & 0 & 0 & 0 \\ 0 & 0.623258 & 0 & 0 \\ 0 & 0 & 0.84623 & 0 \\ 0 & 0 & 0 & 0.976004 \end{bmatrix}.$$

With this choice of D, $D^{-1}AD$ *is an SDD matrix and, for index set* $\alpha = \{1,2\}$, $w(D^{-1}BD) = 0.0983419$. *This means that the Schur complement does not have any eigenvalues in the band shown in the Figure 5.3 (because there are no eigenvalues of* $B(\overline{\alpha})$ *in the band). More precisely,* $\sigma(B(\overline{\alpha})) = \{12.6125, 1.38571\}$, *while* $\sigma(B/\alpha) = \{12.2894, 1.75368\}$.

Example 20 *Let us now illustrate the* Σ*-Nekrasov case. Consider the matrix*

$$C = \begin{bmatrix} 50 & -30 & -10 & 0 \\ -10 & 40 & -10 & -20 \\ -10 & -20 & 50 & -20 \\ -90 & 0 & 0 & 70 \end{bmatrix}.$$

C *is not a Nekrasov matrix, but it is an S-Nekrasov matrix for* $S = \{2,3\}$. *We know from [23] that it can be scaled to a Nekrasov matrix via multiplication from the right by diagonal matrix* $W = diag(1, \gamma, \gamma, 1)$, *where parameter* γ *belongs to the interval defined in (3.28). If we choose* $\gamma = \frac{185}{198}$ *from this interval and then, following Theorem 20, construct a scaling matrix for Nekrasov matrix* CW, *D,*

$$D = \begin{bmatrix} \frac{74}{99} & 0 & 0 & 0 \\ 0 & \frac{2927}{2960} & 0 & 0 \\ 0 & 0 & \frac{2669173}{2697300} & 0 \\ 0 & 0 & 0 & \frac{36371}{37422} \end{bmatrix},$$

what we end up with is an SDD matrix, $D^{-1}W^{-1}CWD$. *Calculation shows that for* $\alpha = \{1,2\}$, $w(D^{-1}W^{-1}CWD) = 1.21469$, *meaning that the vertical band shown in the Figure 5.4 is empty. In other words, there are no eigenvalues of* C/α *in this band because there are no eigenvalues of* $C(\overline{\alpha})$ *in the band.*

5.3.2 Geršgorin-type disks for the Schur complement

In [58], the following result on the dominant degree and the eigenvalue localization for the Schur complement is presented.

Let

$$N_r(A) = \{i \in N \mid |a_{ii}| > r_i(A)\}$$

denote the set of indices for which the corresponding rows of the matrix A are strictly diagonally dominant.

Theorem 82 ([58]) *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $\alpha = \{i_1, i_2, \dots, i_k\} \subseteq N_r(A)$ *and denote* $\overline{\alpha} = \{j_1, j_2, \dots, j_l\}$. *Then, for every eigenvalue* λ *of* A/α*, there exists* $1 \leq t \leq l$*, such that*

$$|\lambda - a_{j_t j_t}| \leq r_{j_t}(A) - w_{j_t} \leq r_{j_t}(A).$$

Here,

$$w_{j_t} = \sum_{u=1}^{k} |a_{j_t i_u}| \frac{|a_{i_u i_u}| - r_{i_u}(A)}{|a_{i_u i_u}|}.$$

This means that the eigenvalues of A/α are contained in the union of those Geršgorin disks for the original matrix A whose indices are in $\overline{\alpha}$.

We can improve this result, using the scaling approach. Let us note that Theorem 82 holds when $\alpha = \{i_1, i_2, \dots, i_k\} \subseteq N_r(A)$, i.e., only if we choose α indices from the set of indices of SDD rows. Therefore, in order to apply this result on a scaled matrix, we must provide that these rows are SDD after diagonal scaling.

We will present here our result on eigenvalue localization of SC for the class of S-SDD matrices, by Geršgorin-type disks based on the entries of the original matrix. It is published in paper [27], which is a joint work of Lj. Cvetković and the author. Again, this result holds for H-matrices in general, but for a practical application, we need a non-expensive construction of a scaling matrix.

Moreover, if we go one step back, to the class of SDD matrices, this result can be applied for any "good" scaling matrix (preserving SDD property in α rows). In that way, the best possible Geršgorin disks (in the sense of scaling) could be found.

Theorem 83 *Let* $A \in \mathbb{C}^{n,n}$ *be an S-SDD matrix, let* $\alpha \subseteq N$*, and let* W *be a corresponding scaling matrix for* A*. Then,*

$$\sigma(A/\alpha) = \sigma((W^{-1}AW)/\alpha) \subseteq \bigcup_{j \in \overline{\alpha}} \Gamma_j(W^{-1}AW).$$

Proof: Let W be any corresponding scaling matrix for A, with γ belonging to the interval (3.6). We have

$$(W^{-1}AW)/\alpha = W^{-1}(\overline{\alpha})(A/\alpha)W(\overline{\alpha}),$$

which is similar to A/α. Therefore, it holds that

$$\sigma(A/\alpha) = \sigma((W^{-1}AW)/\alpha).$$

As $W^{-1}AW$ is an SDD matrix, we can apply Theorem 82 to $W^{-1}AW$. This completes the proof of our statement.□

The benefits from this are the following. First of all, scaling allows us to deal with wider class of matrices. Second, as we will see from the examples, scaled disks give a tighter eigenvalue inclusion area than Geršgorin disks for the original matrix.

Example 21 *Matrix F is S-SDD for $S = \{1,2,3\}$. For $\alpha = S$, we determine the interval for γ as in (3.6), and choose $\gamma = 1/89$. In Figure 5.6, we show the Geršgorin-like set that contains all the eigenvalues of the Schur complement F/α, obtained as in Theorem 83. Notice that in this example, α rows in F are SDD, so one can directly apply Theorem 82 without scaling, see Figure 5.5. But, radii obtained in this way (Figure 5.5) are*

$$(29.2826,\ 18.6993,\ 21.1288,\ 17.6224),$$

while corresponding radii obtained from our Theorem 83 (Figure 5.6) are

$$(5.24719,\ 2.68539,\ 4.1236,\ 5.1236).$$

The spectrum of the Schur complement F/α is

$$\sigma(F/\alpha) = \{-25.0717, -21.0927, -12.3871, -9.89001\}.$$

$$F = \begin{bmatrix} 600 & 50 & 50 & -1 & 1 & 1 & 0 \\ 0 & 650 & 50 & 1 & 0 & 1 & 1 \\ 50 & 50 & 550 & 1 & -1 & 1 & 2 \\ 50 & 100 & 50 & -12 & 1 & 1 & 1 \\ 0 & 100 & 50 & 0 & -21 & 0 & 1 \\ 50 & 0 & 50 & 1 & 1 & -25 & -1 \\ 0 & 50 & 50 & 1 & 2 & 1 & -10 \end{bmatrix}$$

Example 22 *Matrix G is SDD, so, it is S-SDD for any choice of $S \subseteq N$. If we choose $S = \alpha = \{1,2,3\}$, scale the given matrix G to $W^{-1}GW$, with $\gamma = 0.0003$ and then determine Geršgorin-like disks for SC G/α, as in our Theorem 83, see Figure 5.7, our scaled radius for any of the four disks is slightly smaller than the corresponding radius obtained as in Theorem 82. However, both Theorems 82 and 83 produce four disjoint disks, as in Figure 5.7, each containing one eigenvalue of G/α, which means that both results actually shrink the original Geršgorin disks corresponding to $\overline{\alpha}$ indices, made for the original matrix G, see Figure 5.8. Here, the spectrum is $\sigma(G/\alpha) = \{41.0068, -30.9874, 20.9961, -12.0189\}$.*

$$G = \begin{bmatrix} 11125 & 1 & 1 & -1 & 1 & 1 & 0 \\ 0 & 21125 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 31225 & 1 & -1 & 1 & 2 \\ 7 & 1 & 1 & -12 & 1 & 0 & 1 \\ 18 & 0 & 0 & 0 & 21 & 0 & 1 \\ 25 & 0 & 1 & 1 & 1 & -31 & -1 \\ 35 & 0 & 1 & 1 & 0 & 1 & 41 \end{bmatrix}$$

Note that the similar statement holds for matrices satisfying a different condition than S-SDD. Namely, let a matrix $A \in \mathbb{C}^{n,n}$ and $\alpha \subseteq N$ be such that submatrix $A(\alpha)$ is SDD. In the same fashion as above we obtain:

Theorem 84 *Given $A \in \mathbb{C}^{n,n}$ and $\alpha \subseteq N$ such that $A(\alpha)$ is SDD, let $W \in \mathcal{W}^{\alpha}$ be the corresponding scaling matrix for A, with $\gamma > \gamma_1(A)$, defined as in (3.6). Then, the eigenvalue inclusion from Theorem 83 holds.*

Theorem 85 *Let $A \in \mathbb{C}^{n,n}$ be a Nekrasov matrix with nonzero Nekrasov row sums, let $\alpha \subseteq N$, and let D be a corresponding scaling matrix for A. Then,*

$$\sigma(A/\alpha) = \sigma((D^{-1}AD)/\alpha) \subseteq \bigcup_{j \in \overline{\alpha}} \Gamma_j(D^{-1}AD).$$

Proof: Let D be the corresponding scaling matrix for A. We have

$$(D^{-1}AD)/\alpha = D^{-1}(\overline{\alpha})(A/\alpha)D(\overline{\alpha}),$$

which is similar to A/α. Therefore, it holds that

$$\sigma(A/\alpha) = \sigma((D^{-1}AD)/\alpha).$$

As $D^{-1}AD$ is an SDD matrix, we can apply Theorem 82 to the matrix $D^{-1}AD$, which proves our statement.□

In this way, we obtain Geršgorin-like eigenvalue localization area for the Schur complement matrix using only the entries of the original matrix A.

Again, in the same fashion we obtain Geršgorin-like eigenvalue localization area for the Schur complement of a Σ-Nekrasov matrix with scaling matrix WD, where we could construct W as in explanation following Theorem 27 and D as in Theorem 20.

Example 23 *If we consider again the matrix B and its scaling matrix D from Example 19, with $\alpha = \{1,2\}$, we are able to find an eigenvalue localization area for B/α using only the entries of B.*

The localization area obtained by scaling in this way (Figure 5.9) and localization area obtained applying Geršgorin theorem after calculating the Schur complement stand in general position (both sets given together in Figure 5.10). Therefore, not only that it is easier to calculate, but sometimes this preliminary, scaling localization can give us answers that we cannot obtain using Geršgorin theorem after calculating the Schur complement.

In [60] the following result for Ostrowski matrices is proved.

Theorem 86 ([60]) *Let $A \in \mathbb{C}^{n,n}$ be an Ostrowski matrix, let $\alpha = \{i_1, i_2, \dots, i_k\} \subseteq N$, $\overline{\alpha} = \{j_1, j_2, \dots, j_l\}$ and*

$$A_s = A(\alpha \cup \{j_s\}), \ \ 1 \leq s \leq l.$$

Then, for every eigenvalue λ of A/α, there exist $1 \leq s,t \leq l$, $s \neq t$ such that

$$|\lambda - \frac{\det A_t}{\det A(\alpha)}||\lambda - \frac{\det A_s}{\det A(\alpha)}| \leq$$

$$\leq \left[|a_{j_t j_t}| + \max_{v \in N - \{j_t\}} \frac{r_v(A)}{|a_{vv}|} r_{j_t(A)}\right] \left[|a_{j_s j_s}| + \max_{i_w \in \alpha} \frac{r_{i_w}(A)}{|a_{i_w i_w}|} r_{j_s(A)}\right].$$

In other words, there exist $1 \leq s,t \leq l$, $s \neq t$, such that $\lambda \in \mathrm{Oval}_{st}(A)$. Let $\mathrm{OvalSet}(A)$ denote the union of sets $\mathrm{Oval}_{st}(A)$ by $1 \leq s,t \leq l$, $s \neq t$. Then,

$$\sigma(A/\alpha) \subseteq \mathrm{OvalSet}(A).$$

Using the scaling approach, in [28] we obtain localization by ovals for the wider class of Σ-SDD matrices.

Theorem 87 *Let $A \in \mathbb{C}^{n,n}$ be a Σ-SDD matrix, $\alpha \subseteq N$, and W a corresponding scaling matrix for A. Then,*

$$\sigma(A/\alpha) = \sigma((W^{-1}AW)/\alpha) \subseteq \mathit{OvalSet}(W^{-1}AW).$$

We close the section with one more example of eigenvalue localization for the Schur complement of a $PH-$matrix.

Example 24 *Consider, again, the matrix A from Example 7. We know that A is a PM^{π} matrix, with respect to the partition $\pi = \{0,3,6,9\}$ of the index set. We already constructed one diagonal scaling matrix, X, for the given matrix A, in Example 7, as*

$$X = diag(x),$$

where

$$x = [c_1^*,\ c_1^*,\ c_1^*,\ c_2^*,\ c_2^*,\ c_2^*,\ c_3^*,\ c_3^*,\ c_3^*],$$

with

$$c^* = [0.00526575,\ 0.995354,\ 0.0961377]^T.$$

For $\alpha = \{1,2,3,4,5\}$, we now present a vertical band (given in Figure 5.11) determined in the same manner as described in our Theorem 80, and Geršgorin type disks (see Figure 5.12) for A/α, determined as in our Theorem 83. The spectrum of the submatrix in the original matrix A determined by $\overline{\alpha}$ is

$$\sigma(A(\overline{\alpha})) = \{305.535, 134.655, 90.9504, 23.8592\},$$

while the spectrum of the corresponding Schur complement is

$$\sigma(A/\alpha) = \{301.782, 130.943, 67.3272, 19.8034\}.$$

Vertical band in Figure 5.11, determined by $w(X^{-1}AX) = 9.564$, is empty, i.e., we know that there are no eigenvalues of A/α in the band, as there are no eigenvalues of $A(\overline{\alpha})$ in the band.

Notice that, from Theorem 58 and Corollary 2, we know in advance that the resulting Schur complement A/α belongs to the class $PM(2)$, i.e., it is an M-matrix and also a Σ-SDD matrix.

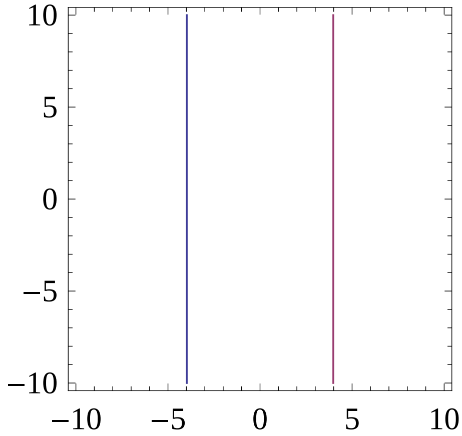

Figure 5.1: Vertical band for SC E/α of Ostrowski matrix E from Example 17

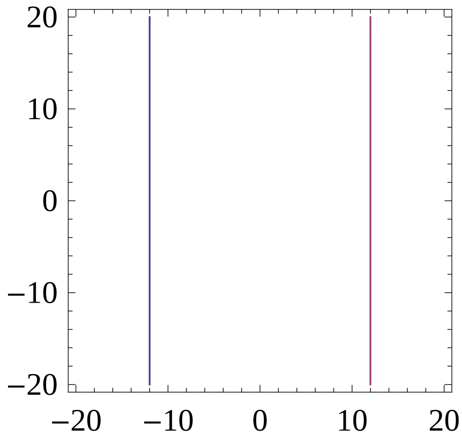

Figure 5.2: Vertical band for SC M/α of Σ-SDD matrix M from Example 18

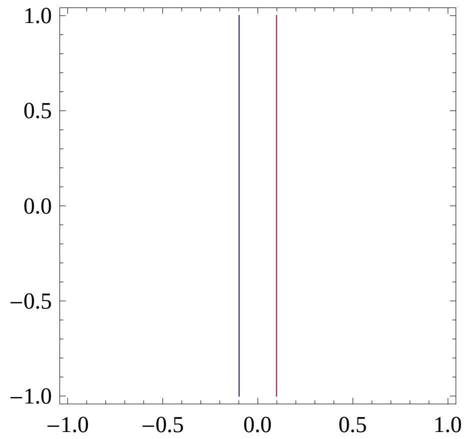

Figure 5.3: Vertical band for SC B/α of Nekrasov matrix B from Example 19

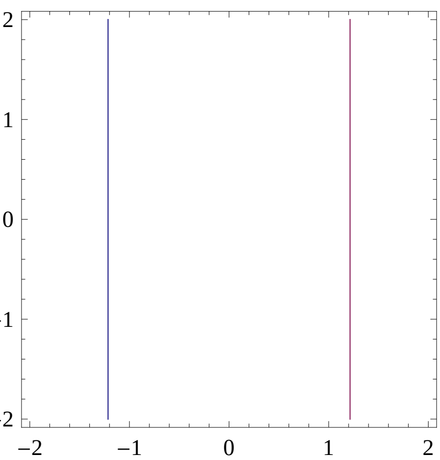

Figure 5.4: Vertical band for SC C/α for Σ-Nekrasov matrix C from Example 20

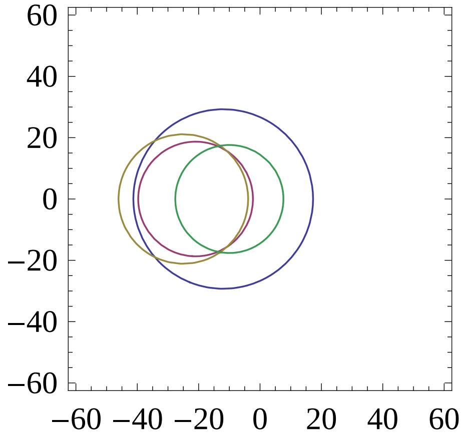

Figure 5.5: Geršgorin-like disks for SC F/α of Σ-SDD matrix F, obtained without scaling, as in Theorem 82, Example 21

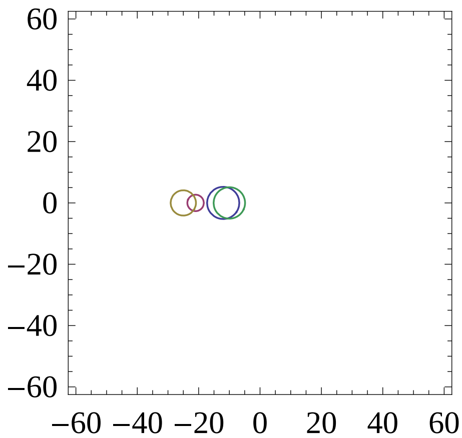

Figure 5.6: Geršgorin-like disks for SC F/α of Σ-SDD matrix F, obtained with scaling, as in our Theorem 83, Example 21

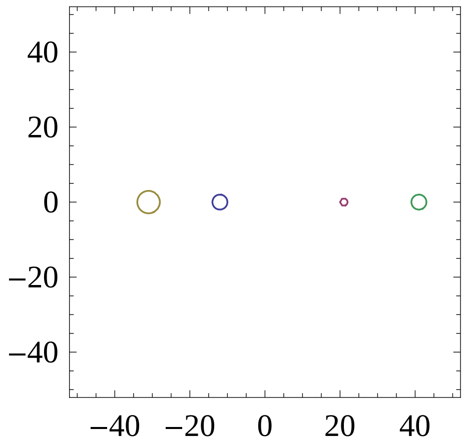

Figure 5.7: Geršgorin-like disks for G/α, determined as in our Theorem 83 from Example 22

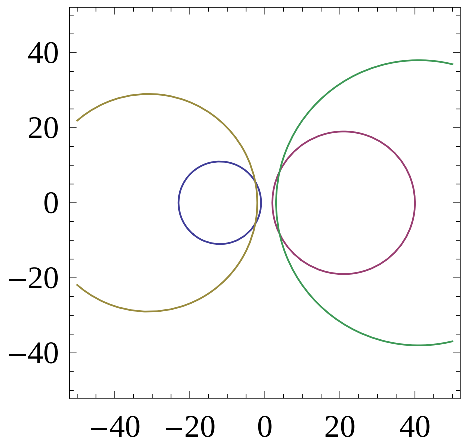

Figure 5.8: Classical Geršgorin disks for the original matrix G, corresponding to $\overline{\alpha}$, from Example 22

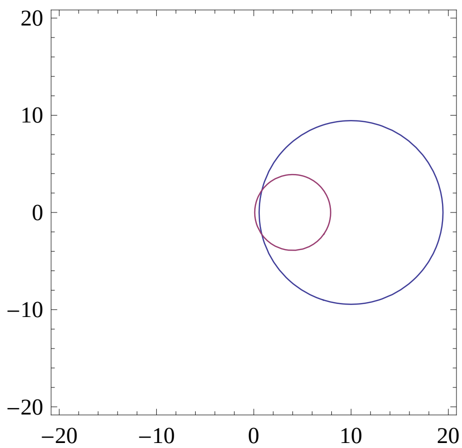

Figure 5.9: Geršgorin-like disks for SC B/α of Nekrasov matrix B determined as in our Theorem 85, Example 23

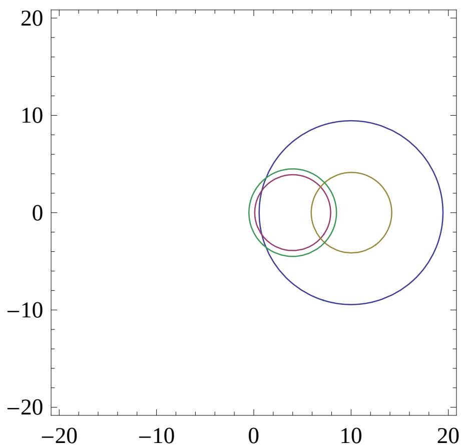

Figure 5.10: Preliminary Geršgorin-like disks obtained as in our Theorem 85, together with Geršgorin's set determined after calculating B/α, Example 23

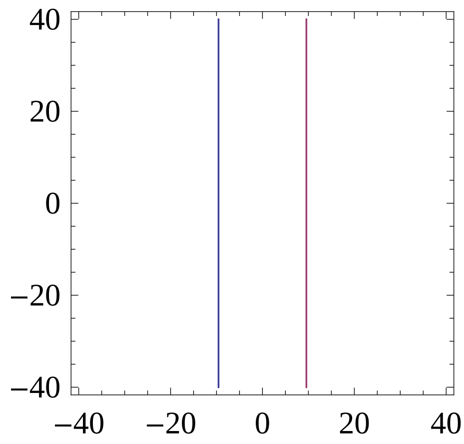

Figure 5.11: Vertical band for A/α, where A is a $PH-$matrix, Example 24

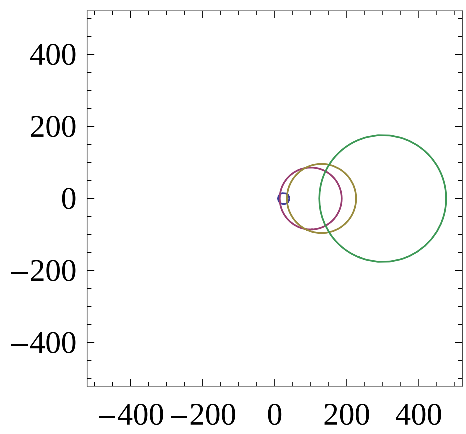

Figure 5.12: Scaled, Geršgorin-like disks for A/α, where A is a $PH-$matrix, Example 24

Chapter 6

Conclusions

The theory of M- and H-matrices has proved to be a source of ideas and useful results in the field of applied linear algebra. Problems such as searching for the (good enough) eigenvalue localization and studying the stability of dynamical systems, or bounding the condition number of a given matrix, can be solved in many cases by the use of tools developed inside H-matrix theory. On the other hand, the interest in Schur complement can be explained by demands in practice. For instance, when dealing with large scale mathematical models, i.e., with matrices (systems) of great dimension, often it is the case that one does not need the complete solution. The model does include and describe relations among many parameters (components), but we are interested only in some of these components. If we make a model of a smaller dimension, involving only components that are the most important, many relations will be ignored. But, if we start with the large model and then use the Schur complement - the smaller matrix is obtained, but the information from the original, large model is still present. This is why properties of the Schur complement and its relation to the given (large) matrix are an interesting topic for researchers in applied linear algebra.

6.1 Contributions

In this thesis, results on different subtypes of H-matrices are given. We presented some new conditions on matrix entries that guarantee nonsingularity and define new subclasses of H-matrices and then, using these new conditions, we defined new upper bounds for the maximum norm of the inverse matrix. Benefits from these new bounds are twofold - first, for some matrices (such as SDD) there exist already some bounds in the literature, but our bounds can be tighter, meaning that they can give a value closer to the exact value of the norm. Second, our bound can be applied for matrices for which there are no other results in the literature on how to estimate the norm of the inverse matrix without calculating the inverse. We also examined relation of some subclasses of H-matrices with the class of strictly diagonally dominant matrices through scaling characterizations and construction of corresponding scaling matrices. When considering the Schur complement related topics, we gave our contribution to the, already very rich, list of matrix properties that are invariant under Schur complement transformation. These properties are transferred from the original, parent matrix, to its Schur complement. In other words, we examined which matrix classes are closed under taking Schur complements. Also, we showed that a preliminary eigenvalue localization for the Schur complement can be obtained even before calculating the Schur complement, only by entries of the original matrix. We considered two types of eigenvalue separation and localization for the Schur complement - vertical bands and Geršgorin-type disks.

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