UNIVERZITET U BEOGRADU

MATEMATIČKI FAKULTET

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Konstrukcija i osobine kontrolnih kartica za stacionarne i nekorelisane podatke

DOKTORSKA DISERTACIJA

BEOGRAD, 2016

UNIVERSITY OF BELGRADE

FACULTY OF MATHEMATICS

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Design and performances of control charts for stationary and uncorrelated data

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DOCTORAL DISSERTATION

BELGRADE, 2016

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Abstract

The subject of this thesis belongs the area of quality control, which represents the practical usage of statistics in following and improving the production process. In 1930 Walter Shewhart started studying quality control, based on control charts, and using statistical principles. Later on, after World War II, Edward Deming took this discipline to Japan, where it flourished. The design and the performance of control charts are the most important problems in this area. The purpose of studying the characteristics of control charts in this thesis is to compare the existing and the suggested control charts.

The thesis is divided into four chapters. The first chapter is introductory and contains motivation and basic definitions related to this subject. In this study it is always assumed that the data are normally distributed, and that the in-control process data are stationary and uncorrelated.

Shewhart control charts and the corresponding limits are constructed in order to meet the given specifications for the quality characteristic that we investigate. Quality control that is applied to a production process always has costs related to the control. The important parameters connected to the cost of quality control are: width of control limits k, the sample size n and the interval between the samples h. In Chapter 2 a new loss function is given, which is connected to the production process and to X−bar quality control chart. Using Matlab program for optimization, values of $\hat{k}$, $\hat{n}$ and $\hat{h}$ are found, which minimize the loss function for given costs. For given values of cost, a non-linear regression model is built using a package Sigma plot and the obtained values are compared to those obtained by numerical optimization.

In Chapter 3, the time series model $Y_i = \lambda X_i + (1 - \lambda)Y_{i-1}$ is investigated, where $0 < \lambda \leq 1$ is a constant, X_i are $N(\mu, \sigma^2)$ distributed. Exponentially Weighted Moving Average (EWMA) control charts for this model are presented, and type I and type II errors are calculated in the case when i is large. For different sample sizes, the new comparison between the optimal design of the X-bar and EWMA control charts for Normally distributed quality characteristic is given, comparing the corresponding cost-loss functions, power functions and average run lengths.

The process of calibration is one of the methods in statistical process control, introduced for improving the quality of the products and for reducing the production costs. In Chapter 4, two new models of non-symmetrical loss function are introduced. Here, the loss function is connected to one product under control (not to the whole sample). Using our program, written in statistical software R, the value which minimizes the expected loss for Shewhart $\bar{X}$ control chart is found. This value is used as the new central target value of the quality characteristic, that is, the production process is calibrated with this new value.

The thesis ends with Conclusions, where the results of the thesis are summarized, and with some open problems to be investigated.

Apstrakt

Tema ove doktorske disertacije pripada oblasti kontrole kvaliteta, koja predstavlja praktično korišćenje statistike prilikom praćenja i poboljšavanja proizvodnog procesa. 1930. godine Volter Šuhart je počeo da proučava kontrolu kvaliteta, baziranu na kontrolnim karticama, koristeći statističke principe. Kasnije, posle Drugog svetskog rata, Edvard Deming odneo je ovu disciplinu u Japan, gde je ona doživela procvat. Konstrukcija i osobine kontrolnih karti su najvažnija pitanja u ovoj oblasti. Cilj izučavanja osobina kontrolnih karti u ovoj tezi je da se uporede postojeće i predložene kontrolne karte.

Teza je podeljena na četiri glave. Prva glava je uvodna i sadrži motivaciju i osnovne definicije koje se odnose na ovu temu. U ovoj tezi uvek se pretpostavlja da su podaci normalno raspodeljeni i da su podaci, kada je proces pod kontrolom, stacionarni i nekorelirani.

Šuhartove kontrolne karte i odgovarajuće granice konstruišu se tako da odgovaraju zadatim specifikacijama za karakteristiku kvaliteta koju ispitujemo. Kontrola kvaliteta koja se primenjuje na proces proizvodnje uvek ima troškove koji su povezani sa kontrolom. Važni parametri koji su povezani troskovima kontrole kvaliteta su: širina kontrolnih granica k, obim uzorka n i interval između uzoraka h. U Glavi 2 data je nova funkcija gubitka, koja je povezana sa proizvodnim procesom i sa kontrolnom karticom. Koristeći program Matlab za optimizaciju, nađene su vrednosti $\hat{k}$, $\hat{n}$ i $\hat{h}$, koje minimizuju funkciju gubitaka pri zadatim troškovima. Za date vrednosti troškova, napravljen je i nelinearan regresioni model korišćenjem paketa Sigma plot i dobijene vrednosti su upoređene sa onima dobijenim numeričkom optimizacijom.

U Glavi 3 izučavan je model vremenskih serija $Y_i = \lambda X_i + (1-\lambda)Y_{i-1}$, gde je $0 < \lambda \le 1$ konstanta, a X_i imaju $N(\mu, \sigma^2)$ raspodelu. Kontrolne karte za vremenske serije sa pokretnim sredinama i eksponencijalnim težinama (EWMA) za ovaj model su predložene i izračunate su greške I i II vrste u slučaju kada je i veliko. Za različite obime uzorka, dato je novo poređenje izmedju optimalnog dizajna za EWMA kontrolne karte, kod kojih karakteristika kvaliteta ima Normalnu raspodelu, i upoređene su odgovarajuće funkcije troška-gubitka, funkcije moći i srednje dužine ciklusa.

Proces kalibracije je jedan od metoda u statističkoj kontroli kvaliteta, koji se uvodi da bi se poboljšao kvalitet proizvoda i da bi se smanjili troškovi proizvodnje. U Glavi 4 predložena su dva nova modela nesimetrične funkcije gubitka. Ovde se funkcija gubitka odnosi na jedan proizvod koji se kontroliše (a ne na ceo uzorak). Koristeći naš program napisan za statistički softver R, nađena je za oba slučaja vrednost koja minimizuje očekivanje funkcije gubitka kod Šuhartove kontrolne karte. Ta vrednost se onda koristi kao nova centralna ciljna vrednost karakteristike kvaliteta, tj. proces se kalibrira sa ovom novom vrednošću.

Teza se završava sa Zaključkom, u kome su sumirani rezultati iz teze i u kome su izloženi neki otvoreni problemi koji će se izučavati.

Acknowledgements

Contents

Chapter 1

Introduction

1.1 Quality characteristics

Quality has become the most important criteria for making a distinction between products and services. Consumers use quality nowadays as the most relevant parameter for choosing which products and services they need. In order to improve competitiveness on the market, companies highlight quality in business strategies and in this way they also secure growth and success.

Quality characteristic represents the key feature of the product that can be measured. It is a random variable that can be continuous (length, diameter, weight, thickness of a product etc.) or discrete (the number of errors or mistakes made in completing a loan application, the number of medical errors made in a hospital etc). The desired value of quality characteristic is called the *nominal* or *target value*, which is usually bounded by interval of values called *specification limits*. It is assumed that specification limits will be sufficiently close to the target value so as to not impact the function or performance of the product if the value of quality characteristic is in that range. The largest allowable value for a quality characteristic is called the *upper specification limit* (USL), and the smallest allowable value for a quality characteristic is called the *lower specification limit* (LSL). Some quality characteristics have specification limits on only one side of the target. A product is called *nonconforming* if the value of quality characteristic lies outside the interval of specification limits [LSL, USL].

1.2 Causes of Quality Variation

Some variability of products always exists, because there are no two products that are exactly the same. For example, the weight of each chocolate bar "Milka" is not exactly 100 gr, some will weight more, others less.

There are two general causes of variability:

- Chance (or common cause): Variability happens randomly and it isn't possible to control it or to remove it.

- Assignable causes (or special causes): This variability in the observed quality characteristic usually arises from three sources: improperly adjusted or controlled machines, operator errors or defective raw material. Variability caused by assignable causes can be identified and we can control it. After doing some research and investigation we can find the cause and eliminate it.

Production process that operates with only chance causes of variability is said to be in statistical control (or in-control state). A process that operates with the presence of assignable causes is said to be an out-of-control process. When a process shifts to the out-of-control state, larger proportion of nonconforming products will be produced.

Quality improvement is used to decrease the variability in processes and products. Statistics and engineering are useful for improving the quality. Moreover, there are techniques, including statistical and analytical tools which are helpful in the analysis of quality problems and for the improvement of the performance of processes. These techniques belongs to time series analysis and depend on the type of the input and output data. For detailed discussions of time series modeling see for example [44], [11]. Important techniques which are used to improve the quality (statistical and engineering quality and important modeling of time series) are to be discussed in what follows.

1.3 Quality control

Quality control is a set of techniques used to secure a required level of quality in a product or service. Statistical quality control (SQC) uses statistical methods to determine when product quality deviates from the specifications. The primary goal of statistical quality control is to maintain and improve the production process through techniques (see [27]) such as

- **Inspection** Inspection is used to examine and test the quality of the product and classify it into a conforming or a nonconforming product using statistical sampling.

- **Statistical Process Control** Statistical process control (SPC) is a primary tool for monitoring and improving quality. The following figure (1.1), (see [35]) shows the steps of the statistical process control procedure
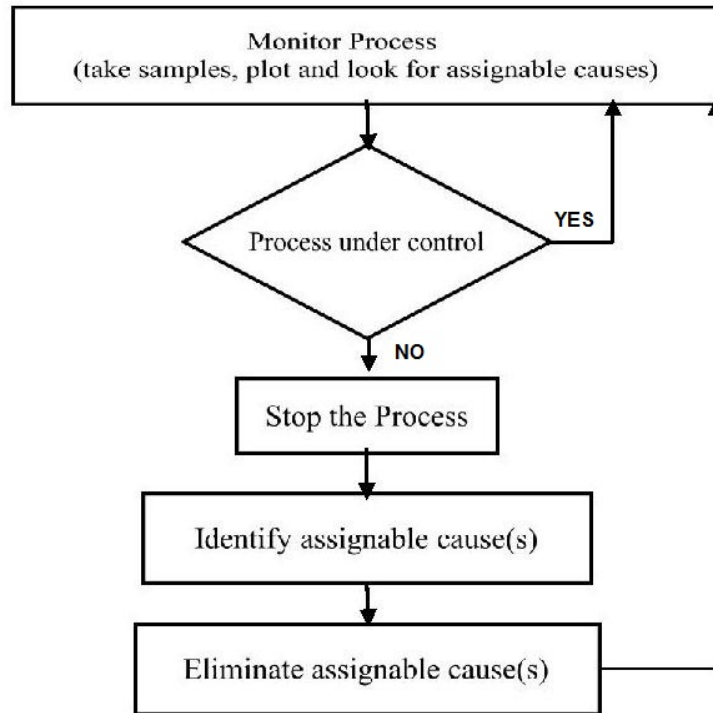
Figure 1.1: Statistical process control procedure.

SPC is one of the greatest technological developments of the twentieth century because it is built on firm basic statistical principles, it is user-friendly, has important influence on quality improvement, and can be implemented on any process.

The statistical control process it usually accompanied by Engineering Quality Control. Engineering Quality Control is made up of operations, management, and engineering actions, which any company uses in order to make sure that the quality of a product meets the necessary target goals and that the variability is kept at the minimum. Engineering process control (EPC) attempts to compensate the causes of deviations by using process adjustment which is usually computed and applied automatically. This type of control is sometimes called Automatic process control (APC). If the process is out of control, the engineer process control (EPC) looks for an assignable cause by following the out-of-control action plan (OCAP), which is associated with statistical process control (SPC) by using the control chart. The following figure (1.2),(see [35]) shows the out-of-control action plan.
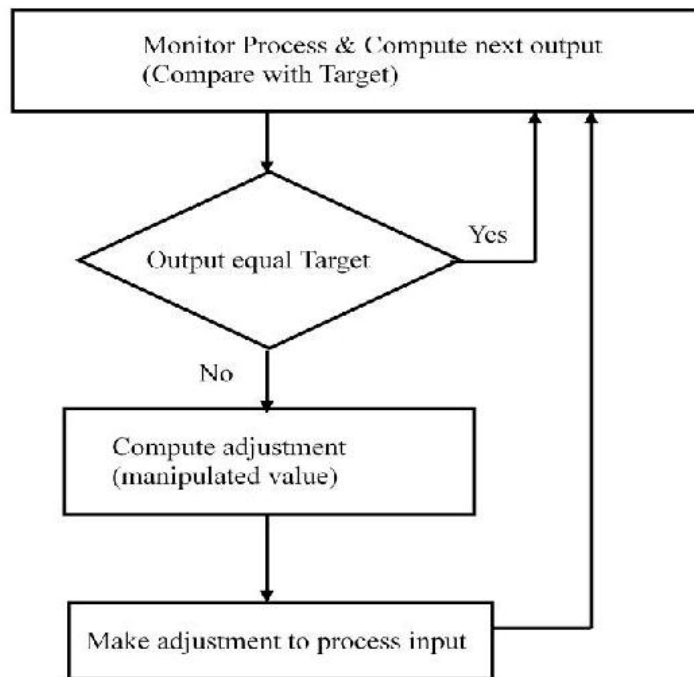
Figure 1.2: Engineering process control.

The product specifications are usually the result of the process design, and the specification limits are usually set by the design engineer. Once we have the specifications, we can assess the quality characteristics. The specifications tell us what kind of quality we want for the components of the product. It should be emphasized that there is no connection between the control limits of the control chart and the specification limits of the process. The process should be both in control and should meet specification limits. As it was explained previously statistical process control and engineering process control have the same objective and that is to ensure that the final result of the process is closest to its target. But this is achieved in different ways, by using different techniques. The different approaches of SPC and EPC are not in competition, but they just have their own points of view on how a manufacturing process works (for example see [44], [11], [49]):

1. SPC highlights process monitoring and detecting faults. This is in contrast with process adjustment when the process is in the statistically in-control state. In this situation changing the process can increase the process variance when we need to decrease it.

2. EPC pushes forward many process adjustment strategies in order to keep the

4

process on the way to its target.

Reasons for using the process control are the following ones:

- We use process control in order to lower the production costs, to reduce the losses and avoid unnecessary actions (see [27]).

- Process control keeps within boundaries the production errors and deviation, as well as the reasons for causing them.

- It is very beneficial to reveal the errors at an early stage of the production.

1.4   Quality control chart

Control charts are tools used to determine whether a manufacturing or business process is in a state of statistical control. Walter Shewhart invented control charts for statistical process control eighty years ago. He suggested a general model for control charts and they are now called Shewhart charts or process-behavior charts. The quality control charts have been used a lot, because they help reduce the cost and they improve the quality of the products. They use statistical methods to control and monitor the production process, find the deviations and find the ways how to fix them. The typical control chart is a graphical display of the measured quality characteristic. We present now a general model of the control chart.

General model for control chart

Every control chart has three lines: central line which represents target value of quality characteristic and two control limits called upper control limit (UCL) and lower control limit (LCL). These control limits help us find whether the data indicates a state of statistical control or not, (see [44], [27]).

Let M be a sample statistic which is a measure of quality. Suppose that M has a normal distribution with mean μ_M and the standard deviation σ_M. The mean of M is equal to the target value $\mu_M = \mu_0$, when the process is in control. Then the central line is $CL = \mu_M$, the upper control limit is $UCL = \mu_M + \sigma_M$, and the lower control limit is $LSL = \mu_M - \sigma_M$. Shewhart control chart is shown in Figure (1.3). The quantity $k\sigma_M$ represents the distance of the control limits from the central line. Parameters of the control chart are the following: the sample size n and the control limit width k. It should be noticed that Shewhart's control charts actually represent tests corresponding to hypothesis testing. Null hypothesis $H_0(\mu_M = \mu_0)$ corresponds

5
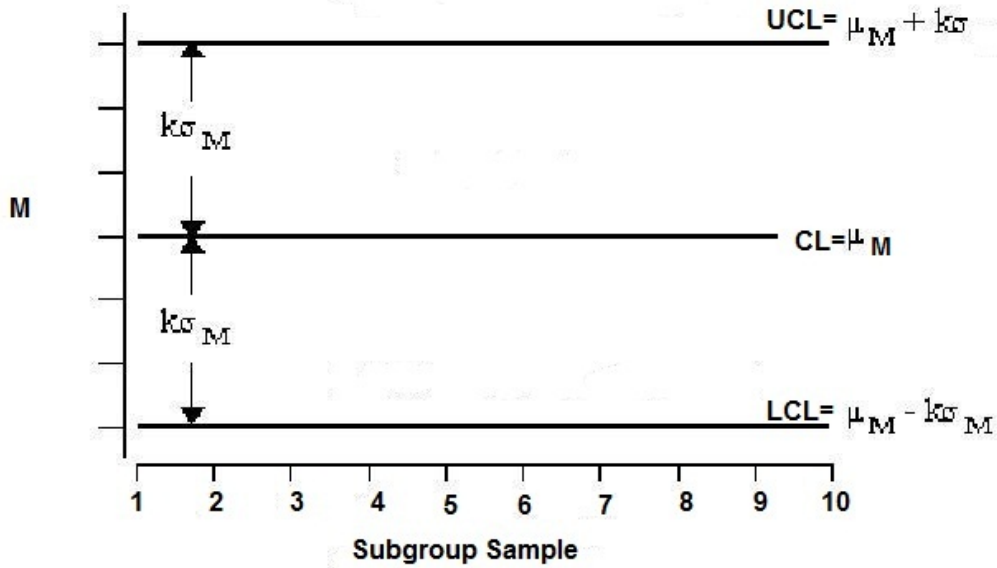
Figure 1.3: Shewhart control chart

to the situation when the production process is in control, while the alternative hypothesis $H_1(\mu_M \neq \mu_0)$ corresponds to the situation when the production process is out of control.

For each sample $i, i = 1, 2, ..m$ we calculate the value of sample statistic M and we denote it by M_i. If the point M_i is within limits UCL and LCL, then the process is in-control state and this leads to the acceptance of the hypothesis $(H_0 : \mu_M = \mu_0)$. But if the point M_i is out of the interval (UCL, LCL), this means that the process is out of control, and we reject the hypothesis $(H_0 : \mu_M \neq \mu_0)$. The result is an acceptance of the alternative hypothesis and it becomes necessary to discover the causes that led to this out-of-control state, and to correct them.

Control charts are classified into two general types: if the quality characteristic can be measured (quantified), then we use the *variable control charts*. If the quality characteristic can not be measured (it is not quantitative), then we use *attribute control charts*.

Variable control charts

The X-bar chart is used for monitoring the process mean. On this chart, we plot the

sample mean. For monitoring the process variability, we use S-chart and R-chart. If we have a sample $(x_1, x_2, ..., x_n)$ of the size n, then on S-chart we represent the sample standard deviation,

$$S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n - 1}}.$$

On R-chart, we represent the sample range, $R = X_{max} - X_{min}$. R-chart is used when the sample size is small ($n < 10$).

When we want to detect smaller process shifts we usually use the *time-weighted charts* (CUSUM, EWMA and MA). On the *cumulative sum chart* (CUSUM)the target level deviations are accumulated. The CUSUM will not increase if there are just the chance causes. If the cumulative sum increases too much, this tells us that the process level changed because of the assignable causes. Therefore, a decision limit is assigned to the cumulative sums, which are similar to the control limits. MA chart is based on a simple, unweighted *moving average* and may be of interest for some purposes. *The exponentially weighted moving average* (EWMA) control chart is very efficient in detecting small shifts in the mean of a process. We will explain this further in Chapter 3.

Attribute control charts

The most frequently used attribute charts are: np-chart, c-chart, u-chart and p-chart. We use np-chart to plot the number of defectives (per batch, per day, per machine). The control limits in this chart are calculated from binominal distribution. On the c-chart, the number of defects (per batch, per day, per machine, per 100 feet of pipe, etc.) is plotted. It is presumed that the quality defects are not frequent and the control limits are computed using Poisson distribution. Defects per unit chart (u-chart) is used to plot the rate of defects. We divide the number of defects by the number of inspected units (the n; e.g., feet of pipe, number of batches). The differences between u-chart and c-chart is that u-chart does not need a constant number of units, and we can use it, for example, when the batches (samples) are not of the same size. We use proportion defective chart (p-chart) in order to plot the fraction of defectives (per any unit of measure, per day, per machine, etc.) just like in the u-chart. The control limits are based on binominal distribution of proportions.

Control chart for Individual Values

Sometimes, the sample size used for process monitoring is $n = 1$, e.g. the sample consists of an individual unit. There are many situations in which control chart for

individual values is appropriate. For example when production rate is low, (meaning data become available slowly) it is inconvenient to allow sample size of $n > 1$ to accumulate before analysis.

1.5 Phase I and Phase II of Control Chart Application

When we use standard control chart, then there are two phases: Phase I and Phase II of applications. They have two different objectives which can be used in many applications of control charting, and it is useful to know the difference between them. (see [44] [23]).

Phase I: A set of data is collected and analyzed all at the same time in a retrospective analysis. Trial control limits are constructed so that we can determine if the process has been in control during the period of data collection, and to check if we can determine reliable control limits for monitoring future production. Usually we do this first when we apply control limits to any process. The primary function of control charts in this phase is to assist operating personnel with putting the process into a statistical control state.

Phase II: This phase can start when we have a "clean" set of process data which are gathered under stable conditions. In Phase II, the control chart is used to monitor the process. The control chart compares the sample statistic for each successive sample, as it is drawn from the process, to the control limits.

For example, in Phase I, we are comparing a collection of m points to a set of control limits computed from these points. Typically $m = 20$ or 25 subgroups are used in phase. Typically in a Phase I it is assumed that the process is initially out of control, so the goal of the analyst is to bring the process into a state of statistical control. Control limits are calculated based on the m subgroups and the data plotted on the control charts. Points outside the control limits are investigated in search of potential assignable causes. If some assignable causes are found, engineering and operating personnel work on eliminating them. Then those points are excluded and a new set of revised control limits are calculated. Next step is to collect new data and compare them to these revised limits. Under the SPC framework, Phase I methods are compared by assessing the probability of the process being out of control, i.e. unstable. In Phase II, the most important thing is to detect quickly the process trends or shifts.

The shifts are usually measured by parameters of the run-length distribution, where the run length is the number of samples taken before an out-of-control signal is given. The average *run-length* (ARL) is frequently used for comparison of competing control chart methods in Phase II. It is usually assumed that there are trends or sustained step shifts in the parameters of the model which was estimated from Phase I. Under the SPC framework, we mainly want to control the false-alarm rate in Phase I and the in-control ARL in Phase II, and to use methods which are proved to be effective in detecting specified trends or shifts in the distribution of the quality variable over time.

1.6 Time series analysis

In order to understand better the types of variability exhibited by the process, we introduce some definitions and properties of time series.

A *time series* is a sequence of observations measured through time. We can make these measurements continuously through time or take them at a discrete set of time points. Conventionally, these two types of series are called *continuous* and *discrete* time series, although the measured variable may be discrete or continuous in both cases. Many sets of data appear as time series: a monthly sequence of the quantity of goods which are shipped from a factory, a weekly series of the number of the road accidents, hourly observations made on the yield of a chemical process, etc. There are many examples of time series in: economics, business, engineering, natural sciences (geophysics, meteorology, etc.), as well as in the social sciences. For a continuous time series, the observed variable is typically a continuous variable recorded continuously, such as a measure of brain activity recorded from an EEG machine. The analysis of such a series starts by sampling (or digitizing) the series at equal intervals of time, in order to obtain a discrete time series. By using this process we lose little or no information as long as the sampling interval is small enough. The models of time series show the relation between the series $\{Z_t, \ t \in T\}$, which is to be modeled, and a white noise generating series $\{a_t\}$. The elements $a_t, a_{t-1}, a_{t-2}, \dots$ may be regarded as a *series of shocks*, which govern the system. It consists of a sequence of uncorrelated random variables with mean zero and constant variance, that is

$$E(a_t) = 0,$$

$$D(a_t) = \sigma_a^2.$$

Since the random variables $\{a_t\}$ are uncorrelated, it follows that their autocovariance function is

$$E(a_t a_s) = \begin{cases} \sigma_a^2, & k = s \\ 0, & k \neq s, \end{cases} \qquad (1.1)$$

where $\{a_t,\ t \in T\}$ is called white noise. We use the notation $a_t \perp (0, \sigma_a^2)$ in order to write down all three characteristics of a_t as time series.

Time series could be stationary or non stationary and we give the definition of stationarity (see [10], [36]). Time series $\{Z_t,\ t \in T\}$ is said to be *weakly stationary* if

1. The expectation of Z_t is constant over the time i.e.

$$E(Z_t) = constant,\ t \in T.$$

2. The variance of Z_t has to be finite, and it is enough that:

$$E(Z_t^2) < \infty,$$

3. The autocovariance function is a function of the difference of arguments, i.e.

$$E[(Z_t - E(Z_t))(Z_s - E(Z_s))] = B(t - s).$$

Time series is *strict stationary* if its probability distribution does not change when shifted in time.

Process behavior

One of the important consideration in control chart usage is the type of variability exhibited by the process. Figure 1.4. represents data from three different processes (see [44]): (a) Stationary and uncorrelated (white noise). (b) Stationary and auto-correlated. (c) Nonstationary.

Figures 1.4.a and 1.4.b illustrate the stationary behavior. By this we mean that the process data vary around a fixed mean in a stable or predictable manner. This is the type of behavior that is produced by an in-control process. The data in Figure 1.4.a are uncorrelated-this type of data represents white noise.

Figure 1.4.b illustrates stationary but autocorrelated process data. Successive observations in these data are dependent - a value above the mean tends to be followed by
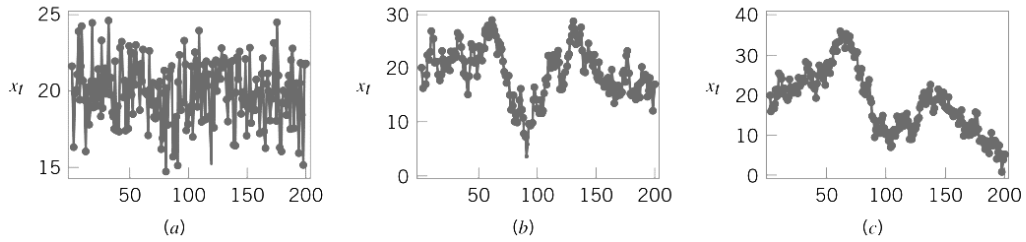
Figure 1.4: Data from three different processes

another value above the mean, whereas a value below the mean is usually followed by another such value. This produces a data series that has a tendency to move in moderately long "runs" on both sides of the mean.

Figure 1.4.c illustrates nonstationary variation. This type of process data occurs frequently in the chemical and process industries. Note that the process is very unstable in that it drifts or "wanders about" without any sense of a stable or fixed mean. In many industrial settings, this type of behavior is stabilized by using engineering process control.

Shewhart control charts are the most effective when the in-control process data look like those given in Figure 1.4.a. In the study of the control charts we always assume that the in-control process data are stationary and uncorrelated.

1.7   Design of control charts

In order to use a control chart it is necessary to determine the value of its parameters. For example X-bar control chart has three parameters : sample size n, sample frequency h and control limits width k. EWMA chart has two parameters: control limits width L and λ. The design of the control chart is defined by the selection of these parameters. We use statistical design and economic design and economic statistical design as criteria for the design of control charts (see [27], [34], [1]).

Originally, only the *first criteria* (statistical criteria) have been used to design control charts. Based on the effects on the statistical properties of the charts, statistical design decides about the parameters. For example we can have pre-selected values of the probabilities of errors, because the control charts are usually designed to minimize the errors. There are two types of errors in the control chart:

Type I error is the situation when a false alarm occurs: although the process is

in-control. the out-of-control state is proclaimed (a point falls outside the control limits and this occurs only because of the natural causes). This makes us look for an assignable cause, when none exists and adds extra cost to the process. The probability of type I error is denoted by α.

Type II error is the opposite situation when the process seems to be in control, although it is out of control. This happens because of the shift in the process mean (a point falls inside the control limits, although some assignable causes are present in the process). If this is the case we do not take any corrective action, while the process is really out of control. That is why the use of control chart to detect process shift is not good enough. Therefore, we need to measure the efficiency of the control chart and for this purpose we will also use the probability of type II error, usually denoted by β, the average run length (ARL), (the average number of samples which we take before the shift is detected) and the average time to signal ATS (the average time necessary until a signal is obtained).

There are also some economic consequences when we use and design the control chart and this is *second criteria*. These economic consequences are, for example, the cost of sampling and checking out-of-control signals, and the cost of eliminating assignable cause and cost of non-conforming units which the consumer will get. Therefore, it is strongly advisable to take into account these consequences when we design a control chart, since the above mentioned qualities affect parameters selection. We need to establish a mathematical model which relates the chart parameter to their economic consequence, and then to select the values of the parameters which optimize a specific objective function. This approach is criticized because it does not take into consideration the statistical properties of the chart.

The third criteria for designing control charts are economic-statistical design of the control chart. In order to satisfy specific statistical economic properties of the design, this approach puts statistical constraints on economic models.

In an economic model, we aim to find the optimal control charts parameters to minimize the process cost by using the economic design of the control charts. This approach does not include the statistical properties, so the combined economic-statistical design should be used as a better approach (see [64]).

The design of control charts was investigated a lot by many authors because it offers material for learning about the performance of control charts. For example, Duncan [26] first suggested an economic model for the optimum economic design of X-bar

control chart. His research dealt with a completely economic model of a Shewhart type control chart, and it included formal optimization methodology in establishing the control chart parameters. Duncan's finding inspired a lot of further research in this area. Lorenzen and Vance [40] used Duncan's original model and turned it into general model which can be applied directly to most types of control charts. Woodall [68] criticized the economic approach for, among other reasons, its possibility of providing a plan that generates an excessive number of false alarms (i.e., looking for assignable causes when none exists). Goel, Jain and Wu [31] came to a practical solution and specified the values of parameters (n, k, h). Serel and Moskowitz [56] proposed an economic model to EWMA (exponentially weighted moving average) control charts which is based on economic or both economic and statistical criteria. Hamoda [34] discussed the optimal economic design for $\bar{x} - 3\sigma$ chart with warning limits. Alarfi [1] discussed the optimal economic design for $\bar{x} - k\sigma$ chart with warning limits. Elfaghihe [27] discussed the optimal economic model of $\bar{x}$-$k\sigma$ chart without warning limits. When we compare the economic and the economic-statistical models, the results showed better statistical performances of the economic-statistical design with non important rise in cost.

The method of economic-statistical design of control charts was developed by Saniga [52]. He proposed economic-statistical model which combines properties of economic and statistical design: it is a method for economic design of control charts that have bounds on type I and type II error probabilities and the average time to signal (ATS) of an expected shift. Economic-statistical design considers economic factors while achieving desirable statistical properties and, therefore, represents the improvement of economic or statistical design.

Chapter 2

Shewhart-type control chart for the mean

In this chapter we construct the X-bar control chart which is designed in order to achieve minimal quality control costs. We create models which can be used for determining the parameters (n, k, h) for different values of variables that rely on the objective function by:

1. Determination of objective function (function of loss by unit of the product) that connects the production process and its characteristics with the X-bar chart, its attributes and the economical consequences of its usage.

2. Defining the values of parameters (n, k, h) which create the objective function as small as possible for different values of incomes.

3. Use of the results obtained in previous step in order to create mathematical methods which will be used to determine the values of the parameters depending on the values of reliable economical variables.

2.1   Construction of the X-bar control chart

The X-bar chart is the most popular control chart. It is used to monitor the average of the process characteristic we are interested in, and we obtain it from the above mentioned principles. The sample statistic is $\bar{X}$ (the sample mean) and the control chart consists of:

1. Measurements of a quality characteristic in samples which are taken from the process at different times.

2. The mean $\bar{x}$ of this characteristic is calculated by using all the samples (e.g., the mean of the means). A central line (CL) of X-bar control chart represents the value of the mean of the quality characteristic. If X is normally distributed

quality characteristic with known mean μ_0 (target value) when the process is in-control, and known variance σ^2, then $CL = \mu_{\bar{x}} = \mu_0$ and $\sigma_{\bar{x}}^2 = \dfrac{\sigma^2}{n}$.

3. Upper and lower control limits (UCL,LCL), which indicate the threshold at which the process output is considered to be statistically 'unlikely', are typically drawn at k standard deviations away from the central line. As a result we have

$$UCL = \mu_0 + k\frac{\sigma}{\sqrt{n}}, \quad LCL = \mu_0 - k\frac{\sigma}{\sqrt{n}}.$$

This chart is called k-sigma X-bar chart. When $k = 2$, the probability that a given sample lies within the control limits is 95.44%. When $k = 3$, the probability is 99.74%. The X-bar chart has three parameters (n, k, h) where n is sample size, k is the width of control limits, h is the sample interval (sampling frequency).

Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ. There are many statistical tests to check out the normality of distribution - χ^2 -test and Kolmogorov-Smirnov, to mention only the two mostly used. In practice for the Normal distribution of a quality characteristic, we usually will not know μ and σ. Therefore, they must be estimated from preliminary samples or subgroups taken when the process is thought to be in control. These estimates should usually be based on at least 20 to 25 samples. Suppose that m samples are available, each containing n observations of the quality characteristic. Typically, n will be small, often either 4, 5, or 6. These small sample sizes usually result from the fact that the sampling and inspection costs, associated with variables measurements, are usually relatively large.

Let $\bar{x}_1, \bar{x}_2, ..., \bar{x}_m$ be the average of each sample. Then the best estimator of m, the process average, is the overall average

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + ... + \bar{x}_m}{m}. \tag{2.1}$$

Thus, $\bar{\bar{x}}$ would be used as the center line on the chart.

To construct the control limits, we need an estimate of the standard deviation σ. We may estimate σ from either the standard deviations or the ranges of the m samples. We will explain both methods, following [44].

Range method

If $x_1, x_2, ..., x_n$ is a sample of size n, then the range of the sample is the difference between the largest and the smallest observations, that is

$$R = x_{max} - x_{min}.$$

Let $R_1, R_2, ..., R_m$ be the ranges of m samples. The average range is

$$\bar{R} = \frac{R_1 + R_2 + ... + R_m}{m}.$$

An unbiased estimator of the standard deviation σ of the Normal distribution is

$$\hat{\sigma} = \frac{\bar{R}}{d_2}, \tag{2.2}$$

where d_2 is a function of the sample size n defined by

$$d_2 = d_2(n) = E(Z_{(n)} - Z_{(1)}), \tag{2.3}$$

with $Z_{(n)}$ and $Z_{(1)}$ the largest term and the smallest term, respectively, in a random $N(0,1)$ sample $Z_1, \ldots, Z_n$.

We may now give the formulas for constructing the control limits on the X-bar chart. They are as follows (where $\hat{\sigma}$ is as above):

$$UCL = \bar{\bar{x}} + \frac{k}{\sqrt{n}}\hat{\sigma}$$

$$CL = \bar{\bar{x}}$$

$$LCL = \bar{\bar{x}} - \frac{k}{\sqrt{n}}\hat{\sigma}. \tag{2.4}$$

Standard deviations method

Suppose that m preliminary samples are available ($x_{i1}, x_{i2}, ..., x_{in}, \ i = 1, ...m$) each of the size n, and let S_i be the standard deviation of the ith sample. That is

$$S_j = \sqrt{\frac{\sum_{i=1}^{n}(x_{ij} - \bar{x}_i)^2}{n - 1}}.$$

Let $S_1, S_2, ..., S_n$ be sample deviations of m samples. The average of standard deviations is

$$\bar{S} = \frac{S_1 + S_2 + ... + S_m}{m},$$

where S_i is the standard deviation of the ith subgroup and m is the number of subgroups. The standard deviation is then estimated from the following equation:

$$\hat{\sigma} = \frac{\bar{S}}{c_4}, \tag{2.5}$$

where c_4 is a function of the sample size n defined by

$$c_4 = c_4(n) = \left(\frac{2}{n-1}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left[\frac{n-1}{2}\right]}. \tag{2.6}$$

We define the control limits of the corresponding X-bar control chart as in (2.4), but now $\hat{\sigma} = \frac{\bar{S}}{c_4}$:

$$UCL = \bar{\bar{x}} + \frac{k}{\sqrt{n}}\hat{\sigma}$$
$$CL = \bar{\bar{x}}$$
$$LCL = \bar{\bar{x}} - \frac{k}{\sqrt{n}}\hat{\sigma}. \tag{2.7}$$

Phase I Application of X-bar control chart

In *phase I control chart* usage, we usually treat the control limits obtained from (2.4) and (2.7) as *trial control limits*, when preliminary samples are used to construct X-bar control chart. They let us determine if the process was in control during the selection of m initial samples. In order to determine whether the process was in control when the preliminary samples were collected, we need to plot the mean values from each sample on the chart and analyze the results.

If all points are inside the control limits, we infer that the process was in control in the past, and we can conclude that the trial control limits are appropriate for controlling the present or future production. It is good to have 20 to 25 samples or subgroups of size n (typically n is between 3 and 5) to compute the trial control limits. It is possible to work with less data but then the control limits are less reliable.

Let us now suppose the values of X-bar plot are out of control when we compare them to the trial control limits. It is obvious that if want the control limits for present or future production to mean something, then they have to be based on data from an in control process. If this is not the case, then we have to revise the trial control limits. Therefore, we examine each of the out-of-control points, in order to find an assignable cause. If we find an assignable cause, then we discard the point and recalculate the trial control limits with only the remaining points. Then these

17

remaining points are reexamined for control. This process continues until all points are in control and when this happens we adopt the trial control limits for current use.

Sometimes we cannot find an assignable cause for a point that is out-of-control. Then we have two options. One is to eliminate the point, pretending that the assignable cause had been found. The other option is to keep the point (or points) assuming the trial control limits are appropriate for current control. If the point really represents an out-of-control condition, the resulting control limits will be too wide. However, if it is just one or two points of the kind, then it will not do any significant damage to the control chart. If future samples still indicate in-control state, then it is probably safe to drop the unexplained points. Sometimes, when the initial sample values are plotted against the trial control limits, many points will be out-of-control.

Clearly, if we arbitrarily drop the out-of-control points, we will not have a satisfying situation, because this approach could ignore a lot of useful information. However, it is not always possible to have success in searching for an assignable cause for each out-of-control point.

As an example of phase I application of the X-bar chart, we use the data of diameter measurements (mm) for automobile engine piston rings (given in Table 2.1, taken from [45]). In the Table 2.1, 25 samples of diameter measurements (mm) for automobile engine piston rings are given, where each sample or subgroup consists of five piston rings. As the variance is unknown, we estimated it with mean sample range. First, we calculate the sample mean and sample range for each of the 25 samples. We get $\bar{\bar{x}} = 74.00118$, $\hat{\sigma} = 0.009171$, where we used sample range to estimate $\hat{\sigma}$ (see (2.2)).

Then, the upper control limit, central line and lower control limit of X-bar control chart are equal, respectively

$$UCL = \bar{\bar{x}} + \frac{k}{\sqrt{n}}\hat{\sigma} = 74.00118 + \frac{3}{\sqrt{5}}(0.009171) = 74.0135$$

$$CL = \bar{\bar{x}} = 74.00118$$

$$LCL = \bar{\bar{x}} - \frac{k}{\sqrt{n}}\hat{\sigma} = 74.00118 - \frac{3}{\sqrt{5}}(0.009171) = 73.9889.$$

The X-bar control chart is shown in the Figure (2.1).

Sample Number,i	Observations					$\bar{x}_i$	$\bar{R}_i$
1	74.030	74.002	74.019	73.992	74.008	74.010	0.038
2	73.995	73.992	74.001	74.011	74.004	74.001	0.019
3	73.988	74.024	74.021	74.005	74.002	74.008	0.036
4	74.002	73.996	73.993	74.015	74.009	74.003	0.022
5	73.992	74.007	74.015	73.989	74.014	74.003	0.026
6	74.009	73.994	73.997	73.985	73.993	73.996	0.024
7	73.995	74.006	73.994	74.000	74.005	74.000	0.012
8	73.985	74.003	73.993	74.015	73.988	73.997	0.030
9	74.008	73.995	74.009	74.005	74.004	74.004	0.014
10	73.998	74.000	73.990	74.007	73.995	73.9987	0.017
11	73.994	73.998	73.994	73.995	73.990	73.994	0.008
12	74.004	74.000	74.007	74.000	73.996	74.001	0.011
13	73.983	74.002	73.998	73.997	74.012	73.998	0.029
14	74.006	73.967	73.994	74.000	73.984	73.990	0.039
15	74.012	74.014	73.998	73.999	74.007	74.006	0.016
16	74.000	73.984	74.005	73.998	73.996	73.997	0.021
17	73.994	74.012	73.986	74.005	74.007	74.001	0.026
18	74.006	74.010	74.018	74.003	74.000	74.007	0.018
19	73.984	74.002	74.003	74.005	73.997	73.998	0.021
20	74.000	74.010	74.013	74.020	74.003	74.0097	0.020
21	73.982	74.001	74.015	74.005	73.996	74.000	0.033
22	74.004	73.999	73.990	74.006	74.009	74.002	0.019
23	74.010	73.989	73.990	74.009	74.014	74.0027	0.025
24	74.015	74.008	73.993	74.000	74.010	74.005	0.022
25	73.982	73.984	73.995	74.017	74.013	73.998	0.035
						$\sum = 1850.029$	$\sum = 0.581$
						$\bar{\bar{x}} = 74.00118$	$\bar{R} = 0.02324$

Table 2.1: The data of diameter in phase I
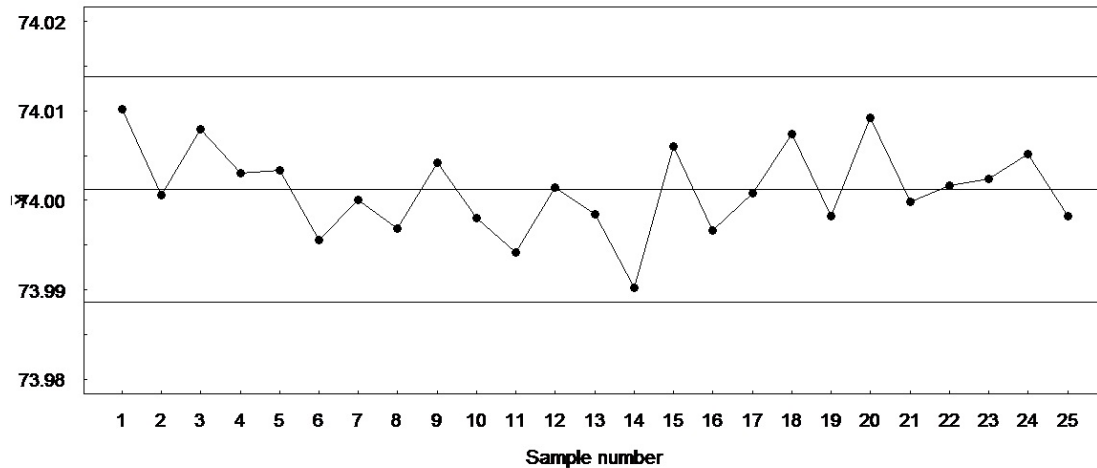
Figure 2.1: X-bar control chart in phase I

As we can see on Figure 2.1, no indication of an out-of-control condition is observed (all sample points fall between the control limits). Therefore, we conclude that the process is in control, and adopt the trial control limits for use in phase II.

Phase II Application of the X-bar chart

When we establish a set of reliable control limits, we use the control chart for monitoring future production. This is the *phase II* of control chart usage. We use additional samples (given in Table 2.2 taken from [44]) for the piston ring process, taken after the initial control charts were established. The X-bar control chart for monitoring process mean is shown in Figure (2.2). We can see that there is indications that the process is out-of-control state (some sample points after the 11th sample are out of control limits).

2.2 Production process

The design of control charts has some statistical and economic consequences, which must be taken into account during the determination of the control chart parameters. The economical design of charts demands formulation of the objective function, which connects economic consequences with parameters from the charts. To determine the objective function (loss function), first we need to make certain assumptions about the behavior of the production process (see [27]).

A production cycle is defined as the interval of time from the start (in the in-control
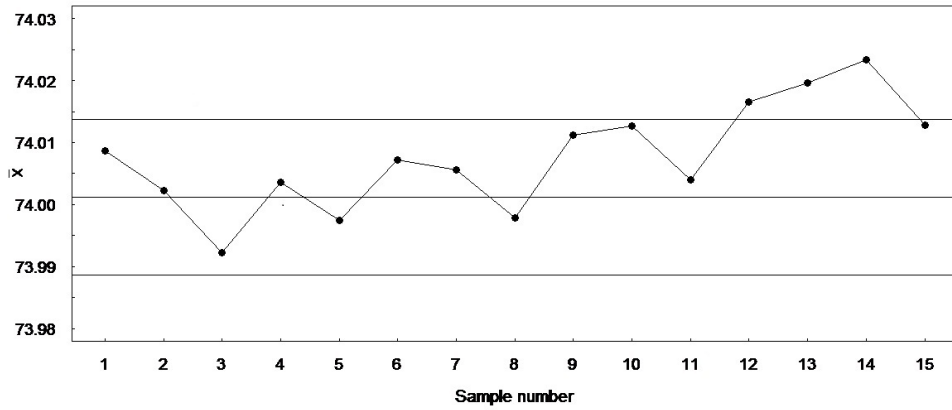
Figure 2.2: X-bar control chart in phase II

state) to the detection and correction of the assignable cause. The cycle consists of four periods:

1. In-control period

2. Out-of-control period

3. The time taken for sampling and interpreting the results

4. The time taken to find the assignable cause (see [5]).

We need to make some assumptions about the process behavior in order to be able to formulate a model for the design of a control chart. These assumptions are given below and they are standard in most models. Process behavior, statistical properties of the chart, control procedure, and economic factors are discussed in the frame of these assumptions. The characteristics and relations for the investigated process are as follows:

I Production has mean of υ items per hour.

II The sample size is n and the sampling interval is h.

III The quality characteristic of interest X has Normal distribution with mean μ and variance σ^2.

IV The production is thought to consist of a series of cycles. Each cycle starts in state I i.e. under control (or in-control state), with mean μ_0 (target level), and ends in state II (out-of-control state) with $\mu \neq \mu_0$. After we detected and corrected the assignable cause, the process returns to the state I again and a

21

Sample Number,i	Observations					$\bar{x}_i$
26	74.012	74.015	74.030	73.986	74.000	74.009
27	73.995	74.010	73.990	74.015	74.001	74.002
28	73.987	73.999	73.985	74.000	73.990	73.992
29	74.008	74.010	74.003	73.991	74.006	74.007
30	74.003	74.000	74.001	73.986	73.997	73.997
31	73.994	74.003	74.015	74.020	74.004	74.007
32	74.008	74.002	74.018	73.995	74.005	74.006
33	74.001	74.004	73.990	73.996	73.998	73.998
34	74.015	74.000	74.016	74.025	74.000	74.011
35	74.030	74.005	74.000	74.016	74.012	74.013
36	74.001	73.990	73.995	74.010	74.024	74.004
37	74.015	74.020	74.024	74.005	74.019	74.017
38	74.035	74.010	74.012	74.015	74.026	74.020
39	74.017	74.013	74.036	74.025	74.026	74.0237
40	74.010	74.005	74.029	74.000	74.020	74.013

Table 2.2: The data of diameter in phase II

new cycle starts and so on. If the assignable cause occurs, it shifts the process to the state II with mean $\mu_1 = \mu_0 - \delta\sigma$ with probability $P(\mu = \mu_1)$ or $\mu_2 = \mu_0 + \delta\sigma$ with probability $P(\mu = \mu_2)$. Here δ is a known shift in process mean.

V Time T (duration of in-control state) is thought to be a random variable that has an exponential distribution with parameter λ. Thus the average duration of the state which is under control in the cycle is $E(T) = \frac{1}{\lambda}$.

VI The states of the process can be recognized by charting only.

Characteristics of the chart

There are two set of characteristics of the chart, statistical and economic (see [27], [35]) .

Statistical characteristic of the chart We draw samples of size n at the interval of h hours. The sample point $\bar{X}_j$ that falls outside the control limits [LCL,UCL] is taken as an indication of the out-of-control state. In case we detect an out-of-control state, then we begin to search for its causes and we need to take suitable steps in order to bring the process back into state I.

The types of errors

There are two types of errors that can happen:

Type-I error (The false alarm): This type of error happens when the chart signals that the process is in state II and actually it is not. The probability of type I error of $\bar{X}$ control chart is given by (see [27]):

$$
\begin{aligned}
\alpha &= P(\bar{X}_j \notin [LCL, UCL] | \mu = \mu_o) \\
&= 1 - P(\bar{X}_j \in [LCL, UCL] | \mu = \mu_o) \\
&= 1 - P\left((\mu_0 - \frac{k\sigma}{\sqrt{n}} \leq \bar{X}_j \leq \mu_0 + \frac{k\sigma}{\sqrt{n}} \right) \\
&= 1 - P[-k \leq Z_j \leq k] \\
&= 1 - [\Phi(k) - \Phi(-k)] = 2\Phi(-k).
\end{aligned} \tag{2.8}
$$

Type-II error (not detecting the real deviations): This error happens when the chart signals that the process is in state I, while it is in state II. The probability of type II error of $\bar{X}$ control chart is given by (see [27]):

$$
\beta = P(\bar{X}_j \in [LCL, UCL] | \mu \neq \mu_o) \tag{2.9}
$$

$$
= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}} \leq \bar{X}_j \leq \mu_0 + \frac{k\sigma}{\sqrt{n}} | \mu \neq \mu_o \right)
$$

$$
= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}} \leq \bar{X}_j \leq \mu_0 + \frac{k\sigma}{\sqrt{n}} | \mu \in \{\mu_1, \mu_2\} \right)
$$

$$
= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}} \leq \bar{X}_j \leq \mu_0 + \frac{k\sigma}{\sqrt{n}} | \mu = \mu_1 \right) P(\mu = \mu_1)
$$

$$
+ P\left(0 - \frac{k\sigma}{\sqrt{n}} \leq \bar{X}_j \leq \mu_0 + \frac{k\sigma}{\sqrt{n}} | \mu = \mu_2 \right) P(\mu = \mu_2).
$$

Put $B_1 = P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}} \leq \bar{X}_j \leq \mu_0 + \frac{k\sigma}{\sqrt{n}} | \mu = \mu_1 \right) P(\mu = \mu_1)$

$$
= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}} \leq \bar{X}_j \leq \mu_0 + \frac{k\sigma}{\sqrt{n}} \right) P(\mu = \mu_1).
$$

We have $\mu_1 = \mu_0 - \delta\sigma$ so B_1 becomes:
$B_1 = P(-k + \delta\sqrt{n} \leq Z_j \leq k + \delta\sqrt{n})P(\mu = \mu_1)$

$$
= [\Phi(k + \delta\sqrt{n}) - \Phi(-k + \delta\sqrt{n})]P(\mu = \mu_1)
$$

Since $\Phi(-k + \delta\sqrt{n}) = 1 - \Phi(k - \delta\sqrt{n})$ and $\Phi(k + \delta\sqrt{n}) = 1 - \Phi(-k - \delta\sqrt{n})$ then

$$B_1 = [\Phi(k - \delta\sqrt{n}) - \Phi(-k - \delta\sqrt{n})]P(\mu = \mu_1).$$

Put $B_2 = P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}} \le \bar{X}_j \le \mu_0 + \frac{k\sigma}{\sqrt{n}} \Big| \mu = \mu_2\right)P(\mu = \mu_2)$

$$= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}} \le \bar{X}_j \le \mu_0 + \frac{k\sigma}{\sqrt{n}}\right)P(\mu = \mu_2).$$

We have $\mu_2 = \mu_0 + \delta\sigma$ and $P(\mu = \mu_2) = 1 - P(\mu = \mu_1)$. Then
$B_2 = P[-k - \delta\sqrt{n} \le Z \le k - \delta\sqrt{n}][1 - P(\mu = \mu_1]$

$$= [\Phi(k - \delta\sqrt{n}) - \Phi(-k - \delta\sqrt{n})][1 - P(\mu = \mu_1)].$$

Because $\beta = B_1 + B_2$, we finally obtain:

$$\beta = \Phi(k - \delta\sqrt{n}) - \Phi(-k - \delta\sqrt{n}). \tag{2.10}$$

It follows that the power is:

$$1 - \beta = 1 - [\Phi(k - \delta\sqrt{n}) - \Phi(-k - \delta\sqrt{n})]$$

$$= \Phi(-k + \delta\sqrt{n}) + \Phi(-k - \delta\sqrt{n}). \tag{2.11}$$

The average run length (ARL)

The average run length (ARL) is another characteristic of the chart which shows the average number of the samples which are taken before the moment we get the first out-of-control signal. Let M be a random variable which represents the number of samples necessary to obtain an out-of-control signal. Then M has geometric distribution with parameter $P(\bar{X} \notin [UCL, LCL]|\mu)$ and the average run is

$$ARL = E[M|\mu] = \sum_{m=1}^{\infty} mP(\bar{X} \notin [UCL, LCL]|\mu)P(\bar{X} \in [LCL, UCL]|\mu)^{m-1}$$

$$= \frac{1}{P(\bar{X} \notin [LCL, UCL]|\mu)}.$$

There are two types of ARL:

ARL_1: In order to obtain the first false alarm, we need to draw an average number of samples and this is ARL_1 - the average run length when the process is in control:

$$ARL_1 = E(M|\mu = \mu_o) = \sum_{m=1}^{\infty} m\alpha(1-\alpha)^{m-1} = \frac{1}{\alpha}, \tag{2.12}$$

where α is the type-I error.

ARL_2: The average run length when the process is out-of-control represents the average number of samples drawn for detecting the shifts:

$$ARL_2 = E(M|\mu \neq \mu_o) = \sum_{m=1}^{\infty} m\beta^{m-1}(1-\beta) = \frac{1}{1-\beta,} \tag{2.13}$$

where β is the type-II error.

The expected number of samples

To determine the expected loss per hour, we first need to determine the expected number of samples for in-control and for out-of-control states and the expected number of false alarms. Below are the definitions of the random variables we use and the related quantities.

Expected number of samples drawn in the state I.

Let S_I be a random variable which represents the number of samples drawn in state I $\{S_I = i\}$, means that the transition to state II happens in the interval $[(ih, (i+1)h)]$:

$$\{S_I = i\} = \{(ih < T \leq (i+1)h)\}.$$

Probability mass function of S_i is

$$P(S_I = i) = P((ih < T \leq (i+1)h)) = \int_{ih}^{(i+1)h} \lambda e^{-\lambda t} dt$$
$$= (1 - e^{-\lambda h})(e^{-\lambda h})^i, \quad i = 0, 1, \dots.$$

This is the probability mass function for geometric distribution with parameter $(1 - e^{-\lambda h})$. Therefore, the expected value of the number of samples drawn in the state I is:

$$\mu_{S_I} = E(S_I) = \frac{e^{-\lambda h}}{1 - e^{-\lambda h}} = \frac{1}{e^{\lambda h} - 1}. \tag{2.14}$$

Expected number of samples drawn in the state II

Let S_{II} be a random variable which represents the number of samples drawn in state II. $\{S_{II} = j\}$ means that the shift in state II is detected at j^{th} sample. The probability mass function for the variable S_{II} is:

$$P(S_{II} = j) = (1 - \beta)\beta^{j-1}, \quad j = 1, 2, 3, \dots$$

This is the probability mass function for geometric distribution with the parameter $(1 - \beta)$. Then the expected value of S_{II} is

$$\mu_{s_{II}} = E(S_{II}) = \frac{1}{1 - \beta}. \tag{2.15}$$

The expected number of samples in cycle

If we suppose that S is a random variable which represents the number of samples in cycle, then $S = S_I + S_{II}$. Let μ_s be the expected number of samples in cycle. Then

$$\mu_s = \mu_{s_I} + \mu_{s_{II}}. \tag{2.16}$$

The expected number of the false alarms

False alarms occur only from sampling in state I. Let F be a random variable that represents the number of false alarms in the cycle. In each sample in state I, the probability of false alarm is α. Therefore $\{F|S_I\}$ is a random variable that has Binomial distribution with parameters (S_I, α) and mean

$$E(F|S_I) = \alpha S_I.$$

The expected number of false alarms in the cycle is:

$$\mu_F = E(F) = E(E(F|S_I)) = E(\alpha S_I) = \alpha E(S_I)$$

$$= \frac{\alpha}{e^{\lambda h} - 1}. \tag{2.17}$$

Economic characteristics of the chart

The most important goal of improving the quality of production is to get the greatest income. When the production process is in in-control state, it is more profitable than when the process is out-of-control. Therefore, the process must be put under control by doing appropriate procedures when there is any warning. Two types of procedures exist and they are executed during the production process, and each of them has economic consequences see [27], [35].

Inspection and sampling procedure

This procedure is done in order to detect a shift. The periodic sampling and testing, as well as the investigation of false alarm are performed. The economic consequence of this procedure is described by the cost and it is as follows.

The cost of inspection and sampling:

It is presumed that c_1^* is the fixed cost per sample, and c_2^* is a variable cost per unit sampled. Then the sampling and inspection costs for sample size n are $c_1^* + c_2^* n$. The expected number of samples in a cycle is denoted μ_s. The expected cycle cost of sampling and inspection is:

$$C_s = (c_1^* + c_2^* n)\mu_s.$$

The cost of false alarms

It is presumed that c_3^* is the cost of a false alarm, which includes the cost for suspension of production during the examination. Then the expected cost of false alarms in the cycle is:
$$C_F = c_3^* \mu_F,$$
where μ_F is expected number of false alarms.

The renewal procedures

These procedures are responsible for the transition from state II to state I. When the cause of the deviation is corrected, the process returns into the controlled stage again and this is related to the following consequences:

Cost of correction

It is presumed that c_4^* is equivalent to the cost for correcting the whole cycle, which includes the cost of suspension of the production process, while the process of correction is in progress.

Revenues from correction

It is presumed that g_1 is the profit of a produced unit when the process is in control,

g_2 is the profit of the produced unit in the out-of-control state. So, $g_1 - g_2$ is the revenue of the transition from a state that is out of control to the state that is in control by the unit. Hourly penalty cost associated with production in the out-of-control state is $C_4 = g_1 - g_2$. The rate of production is v unit of hours, and the average length of the period when the process is in control is the $\frac{1}{\lambda}$ hour. Then the expected benefit from the transition into state I is

$$(g_1 - g_2)\frac{v}{\lambda}.$$

Assuming that b^* is the expected net benefit per renewal, then

$$b^* = (g_1 - g_2)\frac{v}{\lambda} - C_4^*. \tag{2.18}$$

And the average number of units produced in the cycle is $N = \mu_s h v$ by the unit.

2.3   Loss Function

The loss function is $L(n, k, h)$ constructed by combining some characteristics of the production process: If we suppose that P is the expected profit per cycle then

$$P = b^* - C_s - C_F, \tag{2.19}$$

and we define the loss as follows

$$L = -P = C_s + C_F - b^*. \tag{2.20}$$

Then, the expected loss per unit produced, expressed as a function of the chart parameters n, k, h is given by:

$$L(n, k.h) = \frac{L}{N}$$

$$L(n, k, h) = \frac{c_3^*}{hv}\left[\frac{c_1^*}{c_3^*} + \frac{c_2^*}{c_3^*}n - \frac{\frac{b^*}{c_3^*} - \mu_F}{\mu_s}\right]. \tag{2.21}$$

To simplify the optimization procedure for the function (2.21) we define the flowing relative quantities: the relative fixed cost per sample is $C_1 = \frac{c_1^*}{c_3^*} \geq 0$, the relative cost per unit sampled is $C_2 = \frac{c_2^*}{c_3^*} \geq 0$, and the relative benefit renewal is $b = \frac{b^*}{c_3^*} \geq 0$.

Then the loss function can be written as

$$L(n, k, h) = \frac{c_3^*}{hv}\left[C_1 + C_2 n - \frac{b - \mu_F}{\mu_S}\right]. \tag{2.22}$$

Since $\mu_S = \frac{1}{(e^{\lambda h}-1)} + ARL2$ and $\mu_F = \frac{1}{(e^{h\lambda}-1)ARL1}$,

then

$$L(n,k,h) = \frac{c_3^*}{hv}\left[C_1 + C_2 n - \frac{b(e^{h\lambda}-1) - \frac{1}{ARL1}}{1 + ARL2(e^{h\lambda}-1)}\right]. \tag{2.23}$$

The equation 2.23 is the required loss function which has to be minimized in order to determine the parameters (n,k,h).

The values (n^*,k^*,h^*) are the optimum design for the X-bar chart iff

$$L(n^*,k^*,h^*) \leq L(n,k,h) \quad \forall(n,k,h).$$

In order to reduce the number of parameters of the objective function and also to simplify the optimization procedure, we multiply it by $\frac{v}{\lambda c_3^*}$. Put $s = \delta n$, $x = \lambda h$ and $c = \frac{C_2}{\delta^2}$. Then the standardized loss function is given by:

$$L_s(s,k,x) = \frac{1}{x}\left[C_1 + Cs^2 - \frac{b(e^x-1) - \frac{1}{ARL1}}{1 + ARL2(e^x-1)}\right]. \tag{2.24}$$

Is is obvious that the loss function $L(n,k,h)$ attains its minimum at (n^*,k^*,h^*) iff the the standardized loss function $L_s(s,k,x)$ attains its minimum at s^*,k^*,x^*. So the values (n^*,k^*,h^*) are the optimum design for the X-bar chart iff

$$L(s^*,k^*,x^*) \leq L_s(s,k,x) \quad \forall(s,k,x).$$

Numerical minimization

The minimizing problem of the loss function is solved by using MATLAB Optimization: Toolbox for minimization of nonlinear function with constraints for $c_1 = 0$, c in the interval $[0.0001, 1]$, and b in the interval $[10, 860]$. A sample of the output of this program (s^*,k^*,x^*) given in Table (2.3).

Modeling

A non-linear regression models are built by using the a package (Sigma plot) on the results obtained from the standardized loss function (s^*,k^*,x^*). They are dependent variables while (c_1,c,b) are independent variables. In this case when $C \geq 0.067$ and $[0.105 - c]b \leq 0.12638$ we obtain $(s^* = k^* = x^* = 0)$ the following models:

- Model s^*

$$\hat{s}^* = \begin{cases} \dfrac{1.2980b^2 - 1.3342\ln(b^2) - \frac{115090.6604c^2}{\ln(b^2)} - 40756.0351c^2b^2 + 0.0091c\ln(b)}{b^2 + \frac{1323037.3265c^4}{b^2} - 22616.3265c^2b^2 - 790.8770cb^2 - 1.7369\ln(b^2)} - \dfrac{6.5210b}{\ln(b)} + 3.4296 + 0.0091\ln(b) - 17.3746 \\ \\ 0 \end{cases}$$

29

- Model k^*

$$\hat{k}^* = \begin{cases} \frac{1.8569b+2.1325c^C-0.0177ln(b)+8.4195cb}{b(70486c^C-58.84825+1.4296c)} - \frac{8.6777}{ln(b)} \\ 0 \end{cases}$$

- Model x^*

$$\hat{x}^* = \begin{cases} \frac{0.0001c+25.2567bc^2-61.3759c^3b+264477ln(b)-0.0366ln(b)-1699cb+0.1659}{ln(b)+1.2856c^2b^2-0.0190cb^2+24.9502cb)} \frac{77996.3927}{b^4}+0.0217ln(b) \\ 0 \end{cases}$$

In all cases we obtain that $R^2 = 0.9999$ and P-value were for all parameters and coefficients less than the 0.0001.

b	c	s^*	k^*	x^*	L_s
10	0.0401	2.3197	1.5193	0.2441	-7.1111
60	0.0606	2.1796	1.3918	0.1047	-51.3540
110	0.0591	2.2194	1.4248	0.0758	-98.1887
160	0.0701	2.1127	1.3423	0.0654	-144.6392
210	0.0321	2.6057	1.2742	0.0589	-196.6382
260	0.0801	2.0335	1.2784	0.0527	-239.7041
310	.01910	2.8967	1.9686	0.0313	-296.5662
360	0.0371	2.5303	1.6744	0.0360	-341.4782
410	0.0301	2.6526	1.7719	0.0316	-391.6524
460	0.0291	2.6730	1.7881	0.0295	-440.7895
510	0.0471	2.3925	1.5637	0.0324	-485.9551
560	0.0291	2.6751	1.7895	0.0267	-538.7770
610	0.0491	2.3694	1.5450	0.0299	-583.2849
660	0.0551	2.2985	1.4885	0.0297	-631.0765
710	0.0471	2.3963	1.6664	0.0274	-681.5591
760	0.0711	2.1350	1.3583	0.0296	-726.1856
810	0.0651	2.1936	1.4056	0.0280	-776.0662
860	0.0621	2.2253	1.4306	0.0268	-825.5621

Table 2.3: Sample of output (s^*, k^*, x^*) and loss function for given values of b, c

The Table 2.4 contains the estimated values $(\hat{s}^*, \hat{k}^*, \hat{x}^*)$ obtained by using the previous models for some values (c, b).

Note that, by comparing the results in the (Table 2.3) with those in the (Table 2.4) it is obvious that there are no significant differences between actual values (s^*, k^*, x^*) and estimated values $(\hat{s}^*, \hat{k}^*, \hat{x}^*)$. However, some differences have been found, which

b	**c**	$\hat{s}$	$\hat{k}$	$\hat{x}$	L_s
10	0.0401	2.3190	1.5294	0.2415	-7.1111
60	0.0601	2.1867	1.3988	0.1047	-51.3539
110	0.0591	2.2164	1.4262	0.0761	-98.1886
160	0.0701	2.1128	1.3372	0.0658	-144.8491
210	0.0321	2.6021	1.7316	0.0448	-196.6374
260	0.0801	2.0325	1.2706	0.0527	-239.7035
310	0.0191	2.8966	1.9599	0.0305	-296.5597
360	0.0371	2.5267	1.6724	0.0359	-341.4781
410	0.0301	2.6494	1.0661	0.0314	-391.6520
460	0.0291	2.6703	1.7815	0.0294	-440.7891
510	0.0471	2.3907	1.5691	0.0324	-485.9547
560	0.0291	2.6730	1.7830	0.0268	-538.7767
610	0.0491	2.3686	1.5511	0.0299	-583.2845
660	0.0551	2.2994	1.4947	0.0296	-631.0763
710	0.0471	2.3962	1.5726	0.0274	-681.5587
760	0.0711	2.1389	1.3586	0.0295	-726.1854
810	0.0651	2.2918	1.4879	0.0268	-777.7131
860	0.0621	2.2291	1.4351	0.0268	-825.5619

Table 2.4: Estimated the optimal values (s^*, k^*, x^*)

have a slight impact on the standardized loss function.

Algorithms

The next algorithm is used to obtain the approximate optimal sampling interval h^*, the approximate optimal control limits k^*, as well as the approximate optimal sample size n^*. The following values are given:

1. The mean of in-control period $\frac{1}{\lambda}$

2. The cost parameters:

   - b^* benefit per renewal
   - c_1^* fixed cost per sample (in this study $c_1^* = 0$)
   - c_2^* cost per unit sampled
   - c_3^* cost per erroneous inspection

3. Shift parameter δ

b	c	$\hat{h}^*$			h^*		
		$\lambda = 0.05$	$\lambda = 0.5$	$\lambda = 3$	$\lambda = 0.05$	$\lambda = 0.5$	$\lambda = 3$
10	0.0411	4.924	0.1641	0.0821	4.874	0.1625	0.0812
60	0.0391	1.840	0.0613	0.0307	1.846	0.0615	0.0376
110	0.0711	1.596	0.0532	0.0266	1.606	0.0535	0.02677
160	0.0501	1.192	0.0397	0.0199	1.194	0.0398	0.0199
210	0.0351	0.932	0.0311	0.0155	0.924	0.0308	0.0154
260	0.0461	0.906	0.0302	0.0151	0.908	0.0303	0.0151
310	0.0241	0.676	0.0225	0.0113	0.666	0.0222	0.011
360	0.0401	0.736	0.0245	0.0123	0.738	0.0246	0.0123
410	0.0441	0.710	0.0237	0.0118	0.710	0.237	0.0118
460	0.0391	0.606	0.0215	0.0108	0.646	0.0215	0.0108
510	0.0321	0.576	0.0192	0.0996	0.578	0.0193	0.096
560	0.0601	0.662	0.0221	0.0110	0.658	0.0219	0.011
610	0.0311	0.522	0.0174	0.0087	0.526	0.0175	0.088
660	0.0101	0. 342	0.0114	0.0057	0.304	0.0113	0.0057
710	0.0471	0.550	0.0183	0.0092	0.550	0.0183	0.0092
760	0.0611	0.568	0.0189	0.0095	0.568	0.0189	0.0095
810	0.0391	0.484	0.0161	0.0081	0.488	0.0163	0.0081
860	0.0511	0.508	0.0169	0.0085	0.508	0.0169	0.0085

Table 2.5: Approximate values of optimal parameters

4. Relative economical consequences $b = \frac{b^*}{c_3^*}$ and $c = \frac{c_2^*}{c_3^* \delta}$ will be used as known.

a) $\hat{h}^*$ Algorithm

To determine the approximate optimal sample interval $\hat{x}^*$ determine the following:

- Determine the optimal values of $\hat{x}^*$ from x^* model at the specified values of the relative economic consequence using 1-4 above.

- The optimal value ($\hat{h}^*$ the approximate samples interval) is obtained through the relationship $\hat{h}^* = \frac{\hat{x}^*}{\lambda}$.

The result of applying this algorithm for selected values b, c and λ is given in the table (2.5).

b) $\hat{k}^*$ Algorithm

b	c	$\hat{k}^{\star}$	$x^{\star}$
10	0.0391	1.6187	1.6212
110	0.0711	1.3252	1.3186
160	0.0501	1.5177	1.5220
210	0.0351	1.6942	1.6925
260	0.0461	1.5665	1.5708
310	0.0241	1.8684	1.8592
360	0.0401	1.6377	1.6395
410	0.0441	1.5935	1.5975
460	0.0391	1.6521	1.6524
510	0.0321	1.7444	1.7403
560	0.0071	2.3626	2.3830
610	0.0311	1.7601	1.7553
660	0.0101	2.2300	2.2398
710	0.0471	1.5663	1.5626
760	0.0611	1.4374	1.4423
810	0.0391	1.6565	1.6586
860	0.0511	1.5282	1.5354

Table 2.6: The estimated optimal values of control limits $\hat{k}^{\star}$

The optimal values of control limits $\hat{k}^{\star}$ are obtained directly from $x^{\star}$ model at the values of b and c specified using 1-4 above.

The Table (2.6) presents the result of applying $\hat{k}^{\star}$ algorithm for selected value of b and c.

c) $\hat{n}^{\star}$ Algorithm

To determine the approximate optimal sample size $\hat{n}^{\star}$, first the optimal value of $\hat{s}^{\star}$ is obtained from $s^{\star}$ model at the values of b and c specified using 1-4 above. Then the approximate optimal sample size $\hat{n}^{\star}$ is given by:

$$\hat{n}^{\star} = \left[\frac{(\hat{s}^{\star})^2}{\delta^2} \right].$$

The result of applying this algorithm for selected values of b, c and δ is presented in the Table (2.7).

b	c	$\hat{n}^*$			n^*		
		$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$
10	0.0411	21	2	1	21	2	1
6	0.0391	24	3	2	24	3	2
110	0.0711	17	2	1	18	2	1
160	0.0501	22	2	1	22	2	1
210	0.0351	26	3	2	26	3	2
260	0.0461	23	3	1	23	3	1
310	0.0241	31	3	2	31	3	2
360	0.0401	25	3	2	25	3	2
410	0.0441	24	3	1	24	3	1
460	0.0391	25	3	2	25	3	2
510	0.0321	27	3	2	27	3	2
560	0.0601	20	2	1	20	2	1
610	0.0311	28	3	2	28	3	2
660	0.0101	41	5	3	42	5	3
710	0.0471	23	3	1	23	3	1
760	0.0611	20	2	1	20	2	1
810	0.0391	25	3	3	25	3	2
860	0.0511	22	2	1	22	2	1

Table 2.7: Approximate optimal values of control limits

As can be seen, there are different effect on economical consequences for the optimal design of X-bar chart:

- Increasing the per unit (relative) sampling cost (the relative c_2) c_2^* leads to the decrease of the optimum sample size $\hat{n}^*$ and optimum control limits $\hat{k}^*$, but it leads to the increase of the optimum interval between samples $\hat{h}^*$.

- Increasing the benefit per renewal (the relative b) b^* leads to the increase of both the optimum sample size $\hat{n}^*$ and the optimum control limits $\hat{k}^*$, but decreases the optimum interval between samples $\hat{h}^*$.

- Increasing the intensity parameter λ decreases the optimum interval between samples $\hat{n}^*$.

- Increasing of δ decreases the optimum sample size $\hat{n}^*$.

Chapter 3

EWMA-type control charts for the mean

The idea to design the implementation of an the exponential weighted moving average chart is based on the idea of time series moving average model. In this chapter we explain how to construct EWMA chart for monitoring the process mean and we discuss the basic assumptions about the process and method of design of X-bar and EWMA control charts which we use when the quality characteristic has Normal distribution. A model of construction of the chart, together with conclusions, and also sensitivity analysis are discussed.

3.1   Construction of EWMA control chart

Exponentially Weighted Moving Average (EWMA) control chart is one of the control charts that can be used for detecting shifts in the process. Roberts was the first to use it in 1959. It consists of three horizontal lines, central line or mean (CL), upper control limit (UCL) and lower control limit (LCL).

The Shewhart control chart applies statistical test to the process using only the immediate data, while the EWMA chart uses the previous data with weights. The EWMA chart can detect small shifts in the process mean. When λ is near unity or when $\lambda = 1$, then EWMA chart is simillar to, or it is equal to the X-bar chart. Montgomery [44] contributed to this subject and specified the Exponentially Weighted Moving-Average as:

$$Y_i = \lambda X_i + (1 - \lambda)Y_{i-1}. \tag{3.1}$$

Here $0 < \lambda \leq 1$ is constant, X_i are observations, and the starting value of the process is $Y_0 = \mu_0$.

It can happen sometimes that the mean of preliminary data is used as starting value of the EWMA control chart, $Y_0 = \bar{X}$. EWMA Y_i is then the weighted average of all previous sample means.

We may substitute Y_{i-1} on the right hand side of equation 3.1. Then

$$Y_i = \lambda X_i + (1-\lambda)[\lambda X_{i-1} + (1-\lambda)Y_{i-2}]$$

$$= \lambda X_i + \lambda(1-\lambda)X_{i-1} + (1-\lambda)^2 Y_{i-2}.$$

Continuing to substitute recursively Y_{i-j}, $j = 1, 2, 3, .., i-1$ we obtain

$$Y_i = \lambda \sum_{j=0}^{i-1}(1-\lambda)^j X_{j-1} + (1-\lambda)^i Y_0.$$

If the observations X_i are independent random variables with mean μ_0 and variance σ^2, then the mean and the variance of Y_i for EWMA control chart are given by:

$$EY_i = \mu_0,$$

$$\sigma_{Y_i}^2 = \sigma^2(\frac{\lambda}{2-\lambda})[1-(1-\lambda)^{2i}]. \qquad (3.2)$$

In this way, it is possible to construct the EWMA control chart by plotting the Y_i versus the sample number i (or time). Upper Control Limit (UCL), Control Limit (CL) and Lower Control Limit (LCL) for the EWMA control chart are stated below:

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]} \qquad (3.3)$$

$$CL = \mu_0$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]}, \qquad (3.4)$$

where L is the width of the control limits, usually equal 2 or 3, as in the design of Shewhart control chart limits.

In case the sample means fall within the control limits, we can say that the process is in control regarding the mean of process, and we can say that the process is out-of-control if the sample means are out of the control limits.

The term $[1-(1-\lambda)^{2i}]$ approaches unity, when i gets larger. After the EWMA control chart's was used for several time periods, the control limits will get closer to steady-state values.

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \qquad (3.5)$$

$$CL = \mu_0$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}}. \tag{3.6}$$

In a starting period, it is good to use the exact control limits, because this will improve the detection of value that is far from the process target.

EWMA control chart for the mean

We just need to substitute X_i with $\bar{X}_i$ and σ with $\frac{\sigma}{n}$ in the previous equations if the sample size of subgroups is n, $n > 1$. So that

$$Y_i = \lambda\bar{X}_i + (1-\lambda)Y_{i-1}. \tag{3.7}$$

Then the starting value is the overall average, $Y_0 = \bar{\bar{X}}$, and the variance of Y_i for EWMA control chart is given by:

$$\sigma^2_{Y_i} = \frac{\sigma^2}{n}\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2i}]. \tag{3.8}$$

Our assumption is that the quality characteristic is normally $N(\mu_0, \sigma^2)$ distributed, where μ_0 and σ are unknown. Then it is necessary to estimate them from a sample or a subgroup. In this case the limits of EWMA will be

$$UCL = \bar{\bar{X}} + L\frac{\hat{\sigma}}{n}\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]} \tag{3.9}$$

$$CL = \mu_0 = \bar{\bar{X}}$$

$$LCL = \bar{\bar{X}} - L\frac{\hat{\sigma}}{n}\sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]}. \tag{3.10}$$

Since

$$\lim_{i\to\infty}\frac{\sigma^2}{n}\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2i}] = \frac{\sigma^2}{n}\left(\frac{\lambda}{2-\lambda}\right),$$

then control limits after several time periods will get closer to steady-state value values:

$$UCL = \bar{\bar{X}} + L\frac{\hat{\sigma}}{n}\sqrt{\frac{\lambda}{2-\lambda}} \tag{3.11}$$

$$CL = \mu_0 = \bar{\bar{X}}$$

$$LCL = \bar{\bar{X}} - L\frac{\hat{\sigma}}{n}\sqrt{\frac{\lambda}{2-\lambda}}. \tag{3.12}$$

Phase I and phase II of EWMA chart

We use the same data for application of phase I and phase II of EWMA control chart as in the previous chapter. We assume that the EWMA control chart has $\lambda = 0.20$ and $L = 3$. We have $Y_0 = \bar{\bar{X}} = 74.00118$ and $\hat{\sigma} = 0.00917$. Then the first two values of EWMA control chart are:

$$Y_1 = \lambda \bar{X}_1 + (1 - \lambda)Y_0$$

$$= 0.2(74.010) + 0.8(74.00118) = 74.00298$$

and

$$Y_2 = \lambda \bar{X}_2 + (1 - \lambda)Y_1$$

$$= 0.2(74.001) + 0.8(74.00298) = 74.00250$$

The other values of the EWMA statistic, for both cases are computed similarly and summarized in the Table 3.1.

The control limits for period $i = 1$ are found from the formulas

$$UCL = \bar{\bar{X}} + L\frac{\hat{\sigma}}{\sqrt{n}}\sqrt{\frac{\lambda}{2 - \lambda}[1 - (1 - \lambda)^{2i}]}$$

$$UCL = 74.00118 + 3\frac{0.00917}{\sqrt{5}}\sqrt{\frac{0.2}{2 - 0.2}[1 - (1 - 0.2)^{2(1)}]} = 74.00668$$

and

$$LCL = \bar{\bar{X}} - L\frac{\hat{\sigma}}{\sqrt{n}}\sqrt{\frac{\lambda}{2 - \lambda}[1 - (1 - \lambda)^{2i}]}$$

$$LCL = 74.0018 - 3\frac{0.00917}{\sqrt{5}}\sqrt{\frac{0.2}{2 - 0.2}[1 - (1 - 0.2)^{2(1)}]} = 73.99567.$$

For period $i = 2$, the limits are

$$UCL = 74.00118 + 3\frac{0.00917}{\sqrt{5}}\sqrt{\frac{0.2}{2 - 0.2}[1 - (1 - 0.2)^{2(2)}]} = 74.00822$$

$$LCL = 74.00118 - 3\frac{0.00917}{\sqrt{5}}\sqrt{\frac{0.2}{2 - 0.2}[1 - (1 - 0.2)^{2(2)}]} = 73.99413.$$

The control limits vary with $i = 1, 2, ...$ until they stabilize at the steady-state value given by 3.9 and 3.10:

$$UCL = \bar{\bar{X}} + L\frac{\hat{\sigma}}{\sqrt{n}}\sqrt{\frac{\lambda}{2 - \lambda}}$$

$$= 74.00118 + 3\frac{0.00917}{\sqrt{5}}\sqrt{\frac{0.2}{2 - 0.2}} = 74.01034$$

and

$$LCL = \bar{\bar{X}} - L\frac{\hat{\sigma}}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}$$

$$= 74.00118 - 3\frac{0.00917}{\sqrt{5}}\sqrt{\frac{0.2}{2-0.2}} = 73.99201.$$

The corresponding control limits are summarized in the table 3.1.

The EWMA control charts are shown in the Figures 3.1 and 3.2, for the phase I and phase II, respectively. From the first chart, we can conclude that process is in-control and retain estimates we got from preliminary samples. The second EWMA chart, constructed for additional samples, gives indications that the process is out-of-control.
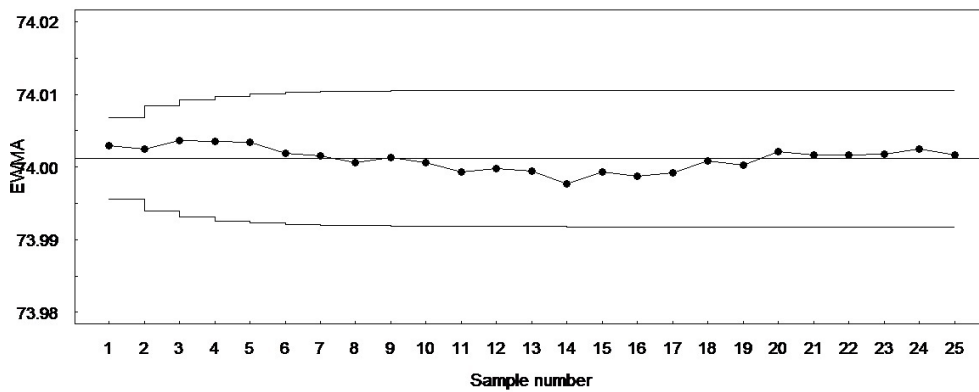

Figure 3.1: The EWMA control chart in phase I

40

| subgroup, i	$\bar{X}_i$	EWMA, Y_i	LCL_{Y_i}	UCL_{Y_i}	subgroup, i	$\bar{X}_i$	EWMA, Y_i	LCL_{Y_i}	UCL_{Y_i}
1	74.010	74.00298	73.99567	74.00668	21	74.000	74.00162	73.99201	74.01035
2	74.001	74.00250	73.99413	74.00822	22	74.002	74.00162	73.99201	74.01035
3	74.008	74.00360	73.99330	74.00905	23	74.002	74.00177	73.99201	74.01035
4	74.003	74.00348	73.99281	74.00954	24	74.005	74.00246	73.99201	74.01035
5	74.003	74.00347	73.99251	74.00984	25	73.998	74.00161	73.99201	74.01035
6	73.996	74.00189	73.99233	74.01003	26	74.009	74.00266	73.99567	74.00668
7	74.000	74.00151	73.99221	74.01014	27	74.002	74.00257	73.99413	74.00822
8	73.997	74.00057	73.99213	74.01022	28	73.992	74.00049	73.99330	74.00905
9	74.004	74.00130	73.99209	74.01026	29	74.004	74.00112	73.99281	74.00954
10	73.998	74.00064	73.99206	74.01029	30	73.997	74.00037	73.99251	74.00984
11	73.994	73.99935	73.99204	74.01031	31	74.007	74.00174	73.99233	74.01003
12	74.001	73.99976	73.99203	74.01033	32	74.006	74.00251	73.99221	74.01014
13	73.998	73.99949	73.99202	74.01033	33	73.998	74.00157	73.99213	74.01022
14	73.990	73.99763	73.99201	74.01034	34	74.011	74.00349	73.99209	74.01026
15	74.006	73.99930	73.99201	74.01034	35	74.013	74.00532	73.99206	74.01029
16	73.997	73.99876	73.99201	74.01034	36	74.004	74.00505	73.99204	74.01031
17	74.001	73.99917	73.99201	74.01034	37	74.017	74.00736	73.99203	74.01033
18	74.007	74.00082	73.99201	74.01034	38	74.02	74.00981	73.99202	74.01033
19	73.998	74.00029	73.99201	74.01035	39	74.023	74.01253	73.99201	74.01034
20	74.009	74.00207	73.99201	74.01035	40	74.013	74.01258	73.99201	74.01034

Table 3.1: The EWMA values Y_i
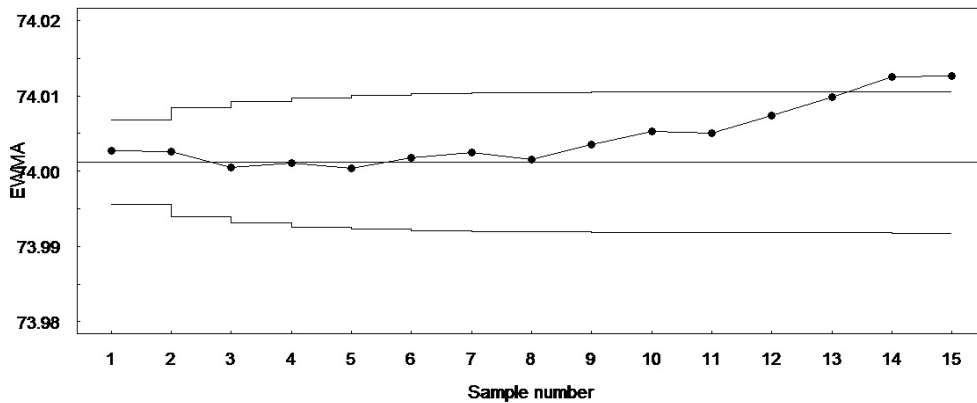
Figure 3.2: The EWMA control chart in phase II

Shewhart control charts are very effective in phase I, because they are easily constructed and interpreted, and because they are effective in detecting both large, sustained shifts in the process parameters and outliers and measurement errors. The types of assignable causes that usually occur in phase I result in fairly large process shifts, which represents exactly the situation in which the Shewhart control chart is most effective. In phase II, it is usually assumed that the process is reasonably stable. Frequently, the assignable causes that occur in phase II only cause smaller process shifts, because (it is hoped) most of the sources of greater variability have been systematically removed during phase I. Now we want to highlight process monitoring, not bringing the process into control. The EWMA control chart is much more likely to be effective in phase II, when smaller shifts occur.

3.2 Basic assumptions and parameters for the design of control charts

n: Sample size

h: Sampling interval

k: Width of control limits.

σ: Standard deviation of observations.

μ_0: In-control process mean.

μ_1: Out-control process mean.

δ: Shifts in process mean in standard deviation units when assignable cause occurs $\left(\delta = \frac{|\mu_1 - \mu_0|}{\sigma}\right)$.

λ: It is assumed that in-control time follows the exponential distribution with mean $\frac{1}{\lambda}$.

τ: Expected time of occurrence of the assignable cause, given it occurs between the $j - th$ and $(j + 1) - st$ samples is

$$\tau = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})}.$$

α: Probability of type I error.

β: Probability of type II error.

ATS: Average time to signal an expected shift, $ATS = \frac{h}{1-\beta}$.

g: The time required to take a sample and interpret the results.

D: The time required to find an assignable cause following an out-of-control signal.

c_1^*: Fixed cost per sample. c_2^*: Variable cost per unit sampled.

C_s: Cost for searching and repairing the assignable cause.

C_F: Cost of investigating a false alarm.

g_1: Profit per hour earned by the process operating in-control.

g_2: Profit per hour earned by the process operating out-of-control $(g_1 > g_2)$.

C_4: Hourly penalty cost associated with production in the out-of-control state.

μ_F: Expected number of false alarms, $\mu_F = \alpha \frac{e^{-h\lambda}}{1-e^{-h\lambda}}$.

$E(T)$: Expected length of a production cycle. The production cycle time T is defined as the interval of time from the start of production (the process is assumed to start in the in-control state), following an adjustment to the detection and elimination of the assignable cause.

$E(C)$: Expected cost per cycle. Total cost C consists of the costs of testing and sampling, the costs associated with investigating an out-of-control signal and with the repair or correction of any assignable causes found, and the costs associated with the production of nonconforming items.

μ_S: Expected number of samples taken within a cycle, $\mu_S = \frac{E(T)}{h}$.

Remark: we derived the errors of type I and II of X-bar chart on the previous chapter, but for EWMA chart we suppose that $[1 - [1 - \lambda]^{2i}] = 1$ when i is large. So the probability of type error is:

- **Type-I error (The false alarm)**. This type of error happens when the chart signals that the process is in state II and actually it is not. Then:

$$\alpha = P(Y_i \notin [LCL, UCL]|\mu = \mu_o)$$

$$= 1 - P(Y_i \in [LCL, UCL]|\mu = \mu_o)$$

$$= 1 - \left(\mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} \le Y_i \le \mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\right)$$

$$= 1 - P[-k \le Z_j \le k]$$

$$= 1 - [\Phi(k) - \Phi(-k)] = 2\Phi(-k). \tag{3.13}$$

- The probabilities of type II error of EWMA control chart given by:

$$\beta = P(Y_i \in [LCL, UCL]|\mu \ne \mu_o) \tag{3.14}$$

$$= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} \le Y_i \le \mu_0 + \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}|\mu \ne \mu_o\right)$$

$$= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} \le Y_i \le \mu_0 + \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}|\mu \in \{\mu_1, \mu_2\}\right)$$

$$= P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} \le Y_i \le \mu_0 + \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}|\mu = \mu_1\right)P(\mu = \mu_1)$$

$$+P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} \le Y_i \le \mu_0 + \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}|\mu = \mu_2\right)P(\mu = \mu_2).$$

Put

$$B_1 = P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} \le Y_i \le \mu_0 + \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}|\mu = \mu_1\right)P(\mu = \mu_1).$$

We have $\mu_1 = \mu_0 - \delta\sigma$ so B_1 becomes:

$$B_1 = P\left(-k + \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \le Z_i \le k + \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}}\right)P(\mu = \mu_1)$$

$$= \left[\Phi\left(k + \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}}\right) - \Phi\left(-k + \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}}\right)\right]P(\mu = \mu_1).$$

Since

$$\Phi\left(-k + \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}}\right) = 1 - \Phi\left(k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}}\right) \text{ and}$$

$$\Phi\left(k + \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}}\right) = 1 - \Phi\left(-k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}}\right), \text{ then}$$

$$B_1 = \left[\Phi\left(k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) - \Phi\left(-k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) \right] P(\mu = \mu_1).$$

Put

$$B_2 = P\left(\mu_0 - \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} \le Y_i \le \mu_0 + \frac{k\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} | \mu = \mu_2 \right) P(\mu = \mu_2).$$

We have $\mu_2 = \mu_0 + \delta\sigma$ and $P(\mu = \mu_2) = 1 - P(\mu = \mu_1)$. Then

$$B_2 = P\left[-k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \le Z_i \le k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right] [1 - P(\mu = \mu_1]$$

$$= \left[\Phi\left(k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) - \Phi\left(-k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) \right] [1 - P(\mu = \mu_1)].$$

Because $\beta = B_1 + B_2$, we finally obtain:

$$\beta = \Phi\left(k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) - \Phi\left(-k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right). \tag{3.15}$$

It follows that the power of determining the real (existing) deviation is:

$$1 - \beta = 1 - \left[\Phi\left(k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) - \Phi\left(-k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) \right]$$

$$= \Phi\left(-k + \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right) + \Phi\left(-k - \delta\sqrt{n}\sqrt{\frac{2-\lambda}{\lambda}} \right). \tag{3.16}$$

The expected cycle time

Each production cycle begins with the process in the in-control state and continues until process monitoring via control charts results in an out-of-control signal. Following an adjustment in which the process is returned to the in-control state, a new cycle begins. As it is shown in Figure 1.3 (see [63]), a cycle is partitioned into five sub-intervals of time, I_a, I_b, I_c, I_d, and I_e.

- I_a: The time from the start of the cycle, until the assignable cause occurs .

- I_b: The time until the next sample is taken from the end of I_a.

- I_c: The time until the chart gives an out-of-control signal from the end of I_b.

- I_d: The time to analyze the sample and the chart from the end of Ic
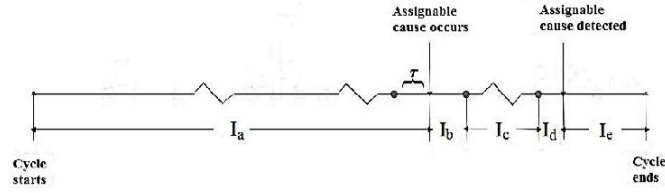
Figure 3.3: Cycle Process

- I_e: The time to discover the assignable cause and repair the process from the end of Id.

Three time points are marked on Figure 3.3: the point 1 represents the last sample before the occurrence of the assignable cause; the point 2 represents the first sample taken after the assignable cause and the point 3 is the moment when lack of control is detected. Process remains in-control state during time interval I_a and it is in the out-control state during intervals I_b, I_c, I_d and I_e. For each part the expected time is calculated as follows:

$$E(I_a) = \frac{1}{\lambda}$$

$$E(I_b) = h - \tau$$

$$E(I_c) = ATS - h$$

$$E(I_d) = gn$$

$$E(Ie) = D.$$

Finally by combining all these expected values we get $E(T)$, which is the expected length of a production cycle. The production cycle time T is defined as the interval of time from the start of production (the process is assumed to start in the in-control state), following an adjustment for detection and elimination of the assignable cause:

$$E(T) = E(I_a) + E(I_b) + E(I_c) + E(I_d) + E(I_e)$$

$$= \frac{1}{\lambda} - \tau + gn + h/(1 - \beta) + D.$$

Then the expected cost per cycle is

$$E(C) = \frac{g_1}{\lambda} + g_2\left(-\tau + gn + \frac{h}{1 - \beta} + D\right) - c_4^* - C_F - (c_1^* + c_2^* n)\mu_s.$$

Loss function

The efficiency of the design of the control chart is presented by the expected loss per hour (loss function), defined as

$$E(L) = g_1 + \frac{E(c)}{E(T)}.$$

Then

$$E(L) = \frac{c_1^* + c_2^* n}{h} + \frac{C_4\left(-\tau + gn + \frac{h}{1-\beta} + D\right) - c_4^* - C_F}{\left(\frac{1}{\lambda} - \tau + gn + \frac{h}{1-\beta} + D\right)}. \tag{3.17}$$

Method of design

The design of a control chart is defined as a design in which the expected loss cost function $E(L)$ is minimized, subject to a constrained minimum value of power and maximum value of the Type I error probability, and ATS of an expected shift. We wish to find sample size, control-limit width, and sampling frequency to

$$minimize \; E(L) \tag{3.18}$$

$$subject \; to$$

$$\alpha \leq \alpha_0, \; p \geq p_0, \; ATS \leq ATS_0,$$

where $p = 1 - \beta$, α_0, p_0 and ATS_0 are, respectively, the power, the desired bound of the Type I error probability, the power and ATS of an expected shift (see [53]).

3.3   Numerical example

This example was presented by Montgomery [45]. There is a manufacturer who produces a nonreturnable glass bottles which are used for packaging a carbonated soft drink beverage. The thickness of the bottle walls is considered to be an important quality characteristic. If the wall isn't thick enough, then the internal pressure during filling may cause the bottle to burst. The manufacturer has for some time applied $\bar{X}$ control charts to monitor the process. These control charts were designed in accordance with statistical criteria. But, because the manufacturer wishes to cut the costs, he wants to design an economically optimum $\bar{X}$ control chart for the process. After analyzing the salaries of quality control technicians and the costs of the test equipment, the estimate cost of taking a sample is $1, whereas the variable cost is approximately $0.01 per bottle. On average, 1 minute $(0.0167h)$ is necessary to measure and record the thickness of the bottle's wall. The process can be subjected

to a couple of types of assignable causes. But when the process goes out-of-control, the magnitude of the shift is equal to about two standard deviations.

Process shift happens randomly with a frequency of approximately one every $20h$. Therefore, the exponential distribution with the parameter $\lambda = 0.05$ is a realistic model of the run length. One hour is usually necessary to investigate an out-of-control signal. To investigate an action signal which ends with the elimination of the assignable cause costs \$25. The investigation of a false alarm costs \$50.

It is estimated by the manufacturer that when we operate in the out-of-control state for one hour the penalty cost is \$100. It was necessary to design a model for the $\bar{X}$ control chart and EWMA chart. (see [26],[45]). Therefore the costs are, $C_1^* = 1, C_2^* = 0.1$, $C_4^* = 25$, $C_3^* = 50$, $\delta = 2$ $g = 0.0167$, $\lambda = 0.05$, $C_4 = 100$, $D = 1$.

Let X be the measurable characteristic. Constrained minimization of expected loss function was done using genetic algorithm ga from R package GA (see [55]).

We selected the following values for bounds on type I error probability, power and ATS an expected shift: $\alpha_0 = 0.05$; $p_0 = 0.9$; $ATS_0 = 2h$. We consider that the quality characteristics of X-bar and EWMA control charts have normal distribution. By solving equations $\alpha = \alpha(k) = 0.05$ and $p = p(k) = 0.9$ using Brent's root-finding method (see [12]), we got values k_1 and k_2. By the constraints 3.18 we got interval of values $[k_1; k_2]$ for parameter k and $[0; 2p(k_1)]$ for parameter h. By constrained optimization of expected loss function, we found the optimal values of parameters k and h. Results of the minimization of expected loss function is given in Tables 1-4, for different values of the shift, λ and sample sizes from 3 to 10.

By looking at the results in Tables 3.2-3.5, we can make following conclusions:

1. The value of k in X-bar chart is less than the value of k in EWMA chart for larger shifts.

2. The value of h in X-bar chart is bigger than the value of h in EWMA chart.

3. At the same value of the shift δ we can see the value of ATS in X-bar chart is bigger than the value of ATS in EWMA chart.

4. Increasing the magnitude of the shift δ increases the power of X-bar chart. Also, we can see that the power of X-bar chart is smaller than the power of EWMA chart.

5. For both charts increasing the sample size n increases the power of the chart at same value of the shift δ.

Sample size	X-bar						EWMA					
	k	h	α	$1-\beta$	ATS	EL	k	h	α	$1-\beta$	ATS	EL
$n=3$	1.92814	0.25889	0.05384	0.14679	1.7648	26.2977	1.94743	0.26753	0.05148	0.74237	0.36038	34.88662
$n=4$	1.91460	0.32321	0.05554	0.18198	1.77607	23.84422	1.95973	0.32982	0.05005	0.85093	0.38759	33.02171
$n=5$	1.93693	0.39047	0.05275	0.20755	1.88136	21.9567	2.05674	0.40117	0.03971	0.90275	0.44439	30.9670
$n=6$	1.93408	0.44424	0.05310	0.23985	1.85212	20.79123	2.38369	0.44661	0.01714	0.90157	0.49537	28.83566
$n=7$	1.94758	0.49845	0.05147	0.26662	1.86951	19.83321	2.68577	0.48427	0.00724	0.90021	0.53794	28.17959
$n=8$	1.92381	0.57685	0.05438	0.30559	1.88766	19.12049	2.95296	0.44137	0.00315	0.90142	0.48964	28.04592
$n=9$	1.93664	0.60087	0.05279	0.33148	1.81267	18.50674	3.21545	0.44069	0.00130	0.90053	0.48937	28.0841
$n=10$	1.92348	0.60591	0.05442	0.36628	1.65423	18.04309	3.45841	0.42619	0.00054	0.90060	0.47323	28.21385

Table 3.2: Design of X-bar and EWMA charts with $\delta = 0.5$

Sample size	X-bar						EWMA					
	k	h	α	$1-\beta$	ATS	EL	k	h	α	$1-\beta$	ATS	EL
$n=3$	1.93373	0.68838	0.05315	0.42021	1.63819	16.61141	3.87156	0.42125	0.00011	0.90735	0.46427	26.7871
$n=4$	1.95978	0.75326	0.05002	0.51608	1.45957	15.24572	4.57662	0.42382	4.73E-06	0.92269	0.45933	26.97943
$n=5$	1.95473	0.85766	0.05061	0.61079	1.40419	14.37982	5.14731	0.42278	2.64E-07	0.94073	0.44942	27.17951
$n=6$	1.95864	0.93081	0.05016	0.68824	1.35246	13.76538	5.97621	0.42398	2.28E-09	0.91501	0.46336	27.37894
$n=7$	1.95833	1.0053	0.05019	0.75409	1.33312	13.3412	5.78633	0.42490	7.19E-09	0.98426	0.4317	27.5773
$n=8$	1.95844	1.09691	0.05018	0.80785	1.35782	13.04493	6.77774	0.42611	1.22E-11	0.95614	0.44565	27.77458
$n=9$	1.95847	1.12839	0.05018	0.85119	1.32567	12.8346	7.24306	0.42727	4.39E-13	0.96054	0.44482	27.97079
$n=10$	1.95976	1.15354	0.05002	0.88542	1.30282	12.68886	6.8625	0.42840	6.77E-12	0.99566	0.43027	28.16594

Table 3.3: Design of X-bar and EWMA charts $\delta = 1$

Sample size	X-bar						EWMA					
	k	h	α	$1-\beta$	ATS	EL	k	h	α	$1-\beta$	ATS	EL
$n=3$	1.95184	1.00085	0.05096	0.74094	1.35078	13.20218	6.01683	0.42035	1.78E-09	0.96225	0.43684	26.77723
$n=4$	1.95973	1.14528	0.05003	0.85089	1.34598	12.49609	6.46209	0.42154	1.03E-10	0.99442	0.42390	26.97893
$n=5$	2.07194	1.05584	0.03827	0.90011	1.17301	11.79137	8.06857	0.42275	7.11E-16	0.97691	0.43274	27.17948
$n=6$	2.38056	0.86103	0.01729	0.90211	0.95446	10.86595	8.29989	0.42384	1.04E-16	0.99676	0.42521	27.37894
$n=7$	2.67863	0.75229	0.00739	0.90147	0.83451	10.37909	8.57585	0.42498	9.84E-18	0.99957	0.42516	27.5773
$n=8$	2.94963	0.69506	0.00318	0.902	0.77058	10.18074	6.96257	0.42612	3.34E-12	1	0.42612	27.77458
$n=9$	3.05307	0.69634	0.00227	0.92604	0.75195	10.1066	7.39433	0.42727	1.42E-13	1	0.42727	27.97079
$n=10$	3.14151	0.70231	0.00168	0.94541	0.74286	10.07373	7.2211	0.42841	5.16E-13	1	0.428409	28.16594

Table 3.4: Design of X-bar and EWMA charts $\delta = 1.5$

Sample size	X-bar							EWMA					
	k	h	α	$1-\beta$	ATS	EL		k	h	α	$1-\beta$	ATS	EL
$n=3$	2.18166	0.99969	0.02913	0.90016	1.11058	11.2513		7.96358	0.42040	1.67E-15	0.99242	0.42361	26.7773
$n=4$	2.71544	0.72664	0.00662	0.90053	0.80691	10.12286		7.76142	0.42153	8.40E-15	1	0.42153	26.97893
$n=5$	3.04719	0.69383	0.00231	0.92291	0.75178	9.83603		8.86822	0.42269	7.43E-19	1	0.42269	27.17948
$n=6$	3.19796	0.70464	0.00138	0.95553	0.73743	9.73648		9.64401	0.42375	5.21E-22	1	0.42375	27.3789
$n=7$	3.34377	0.71197	0.00083	0.97428	0.73077	9.71108		10.56075	0.42498	4.53E-26	1	0.42498	27.5773
$n=8$	3.48366	0.71101	0.00049	0.98512	0.72176	9.72675		11.9069	9.11077	1.09E-32	1	9.11078	60.54219
$n=9$	3.62155	0.71261	0.00029	0.99131	0.71886	9.76583		13.3716	0.4284	8.86E-41	1	0.4284	28.16594
$n=10$	3.75549	0.71413	0.00017	0.9949	0.7178	9.81828		13.37159	0.42840	8.86E-41	1	0.42840	28.16594

Table 3.5: Design of X-bar and EWMA charts with $\delta = 2$

6. At the same value of the shift δ, we can see that the expected value of the loss function in X-bar chart is less than the expected value of loss function of EWMA chart. Increasing the magnitude of the shift δ decreases the loss function in X-bar chart and increasing the magnitude of the shift δ increases the loss function of EWMA chart.

7. At the same value of the shift δ, we can see that increasing the sample size n decreases the loss function of X-bar chart and increases the loss function of EWMA chart.

3.4   Control Chart Performance

There are two commonly used methods for comparing control chart performance. The first method is to determine the charts operating characteristic (OC) curve. The second is to determine the average run length (ARL). Those methods are related with sensitivity analysis of the chart.

This research focuses on increasing the sensitivity of X-bar chart and EWMA chart in order to detect small shifts. The sensitivity of the charts is measured by their power $1 - \beta$ and the average run length (ARL). The power of the chart represents the probability of a shift in the process level, and it is described in the process level and given in the form of an operating characteristic (OC) curve.

The OC Curve: The OC Curve indicates the probability of an observation falling within the control limits according to the state of the process. For example, for the $\bar{X}$ and EWMA charts, we can see the probability of staying within the limits β for particular values of the process mean, as it is presented in the equations(2.11) and (3.16), respectively. The curve is generated by calculating β for different values of μ.

The next five tables have the same value of probability of power X-chart and EWMA chart with different values of the sample size n and shift δ and λ and $L = k = 3$ (the usual three-sigma limits). The OC curves are also presented in the following figures. The figure 3.4 (later used several times for comparison) shows the powers of X-bar chart with different values of sample size n and $k = 3$. It is compared with figures (3.5),(3.7), (3.9), (3.11) and (3.13) which show the EWMA chart with different values of sample size n for the given values λ and $L = 3$.

Sample size	X-bar						EWMA					
	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.5$	$\delta = 2.5$	$\delta = 3$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.5$	$\delta = 2.5$	$\delta = 3$
$n = 3$	0.00543	0.01648	0.04449	0.34388	0.90827	0.98596	0.38369	0.99199	1	1	1	1
$n = 4$	0.00644	0.02278	0.06681	0.50001	0.97725	0.99865	0.54876	0.99941	1	1	1	1
$n = 5$	0.00751	0.02994	0.05275	0.63838	0.9952	0.9999	0.68832	0.99997	1	1	1	1
$n = 6$	0.00863	0.03794	0.05310	0.74993	0.99911	0.99999	0.79511	1	1	1	1	1
$n = 7$	0.00981	0.04677	0.05147	0.83364	0.99985	1	0.87091	1	1	1	1	1
$n = 8$	0.01103	0.0564	0.05438	0.893	0.99998	1	0.9216	1	1	1	1	1
$n = 9$	0.01231	0.06681	0.05279	0.9332	1	1	0.95389	1	1	1	1	1
$n = 10$	0.01365	0.07798	0.05442	0.95937	1	1	0.97364	1	1	1	1	1
$n = 11$	0.01504	0.08985	0.05442	0.97586	1	1	0.9853	1	1	1	1	1
$n = 12$	0.01648	0.10241	0.05442	0.98596	1	1	0.99199	1	1	1	1	1
$n = 13$	0.01797	0.11561	0.05442	0.99199	1	1	0.99572	1	1	1	1	1
$n = 14$	0.01952	0.12942	0.05442	0.99551	1	1	0.99776	1	1	1	1	1
$n = 15$	0.02113	0.14378	0.05442	0.99752	1	1	0.99884	1	1	1	1	1

Table 3.6: The power of X-bar and EWMA charts with $\lambda = 0.05$
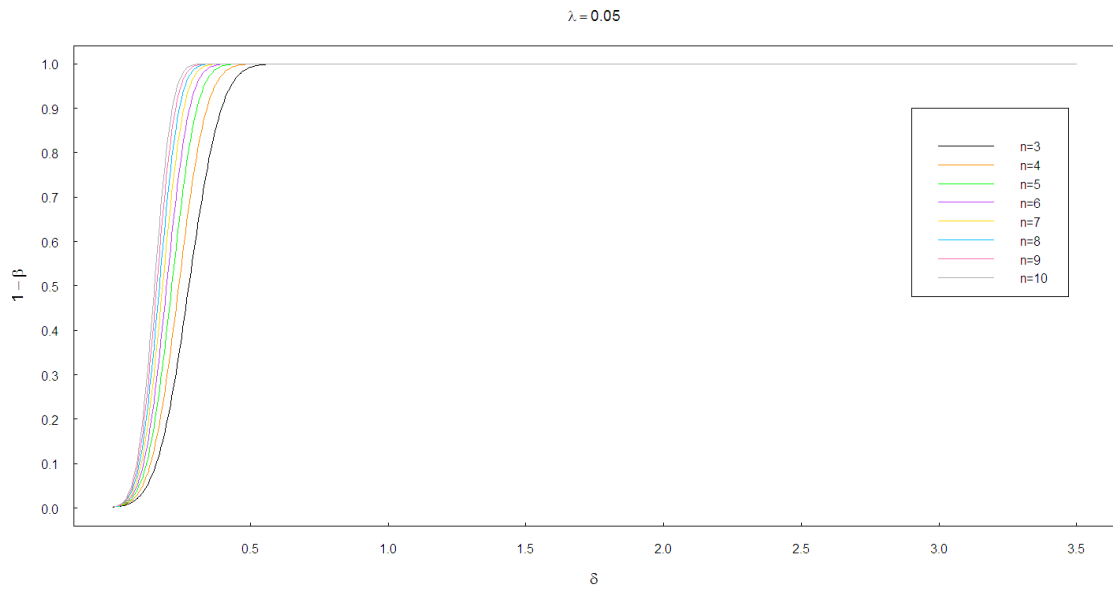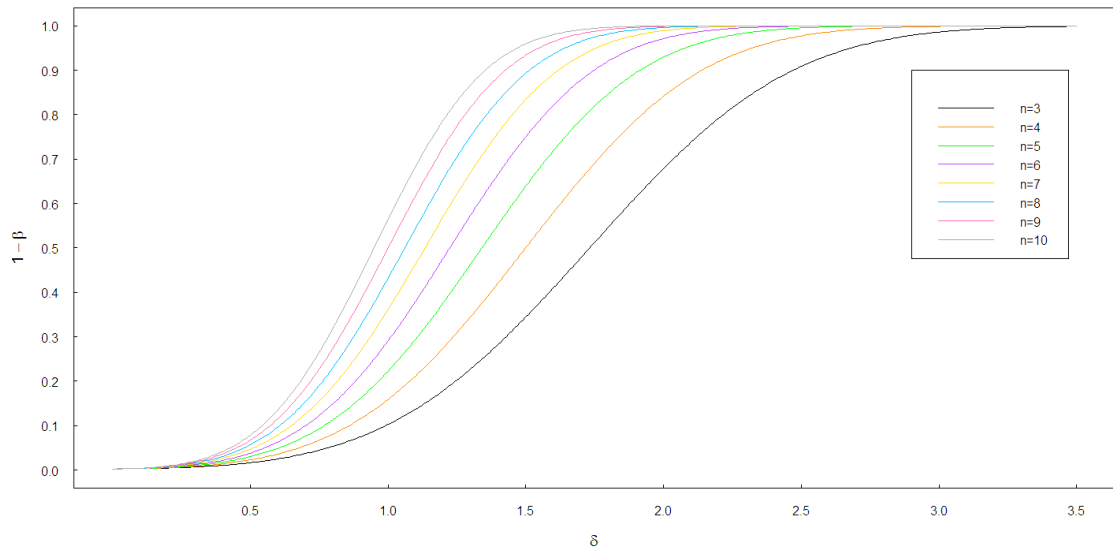
Figure 3.4: The power of $\bar{X}$ chart

Figure 3.5: The power of EWMA chart with $\lambda = 0.05$

Sample size	X-bar						EWMA					
	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.5$	$\delta = 2.5$	$\delta = 3$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.5$	$\delta = 2.5$	$\delta = 3$
$n = 3$	0.00543	0.01648	0.04449	0.34388	0.90827	0.98596	0.13296	0.78081	0.99612	1	1	1
$n = 4$	0.00644	0.02278	0.06681	0.50001	0.97725	0.99865	0.20596	0.91291	0.9998	1	1	1
$n = 5$	0.00751	0.02994	0.09293	0.63838	0.9952	0.9999	0.28662	0.96949	0.99999	1	1	1
$n = 6$	0.00863	0.03794	0.12244	0.74993	0.99911	0.99999	0.37043	0.99032	1	1	1	1
$n = 7$	0.00981	0.04677	0.1549	0.83364	0.99985	1	0.45349	0.99717	1	1	1	1
$n = 8$	0.01103	0.0564	0.18979	0.893	0.99998	1	0.53277	0.99922	1	1	1	1
$n = 9$	0.01231	0.06681	0.22663	0.9332	1	1	0.60611	0.9998	1	1	1	1
$n = 10$	0.01365	0.07798	0.26491	0.95937	1	1	0.67221	0.99995	1	1	1	1
$n = 11$	0.01504	0.08985	0.30415	0.97586	1	1	0.73047	0.99999	1	1	1	1
$n = 12$	0.01648	0.10241	0.34388	0.98596	1	1	0.78081	1	1	1	1	1
$n = 13$	0.01797	0.11561	0.38369	0.99199	1	1	0.82358	1	1	1	1	1
$n = 14$	0.01952	0.12942	0.42319	0.99551	1	1	0.85935	1	1	1	1	1
$n = 15$	0.02113	0.14378	0.46206	0.99752	1	1	0.88886	1	1	1	1	1

Table 3.7: The power of X-bar and EWMA charts with $\lambda = 0.1$
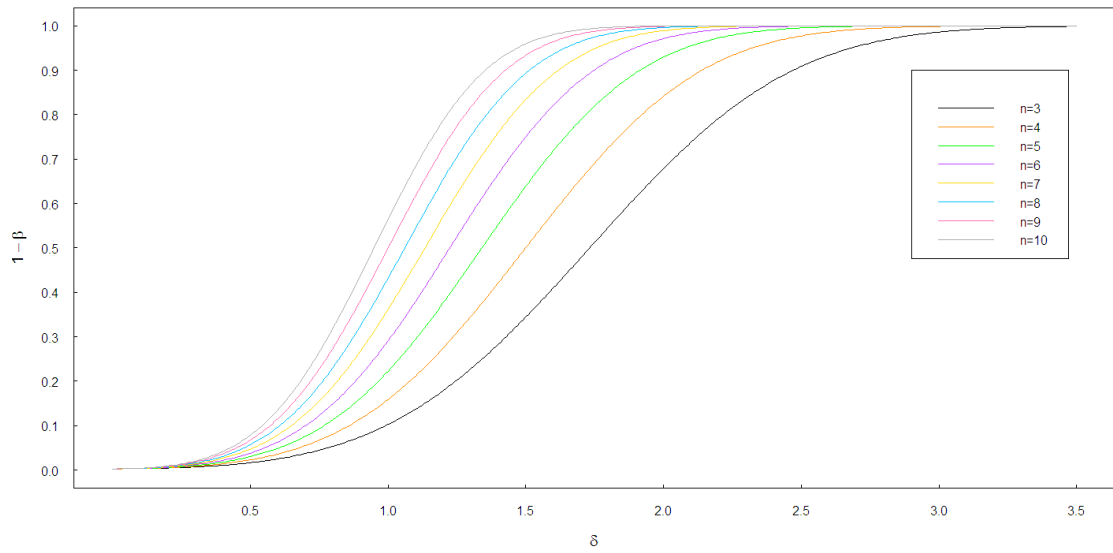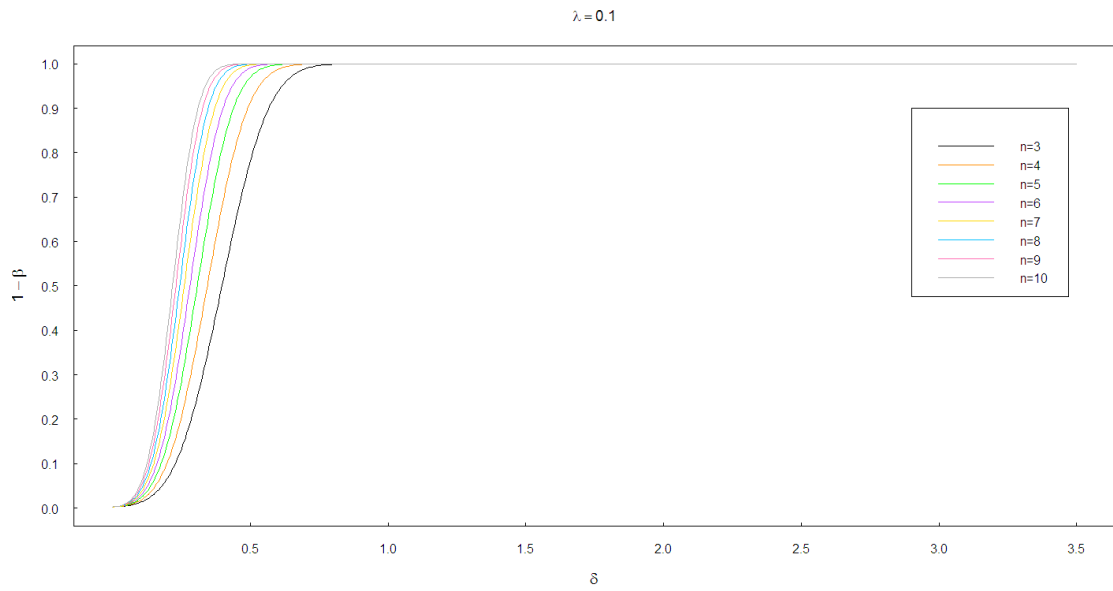
Figure 3.6: The power of $\bar{X}$ chart

Figure 3.7: The power of EWMA chart with $\lambda = 0.1$

Sample size	X-bar						EWMA					
	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$
$n = 3$	0.01648	0.34388	0.67872	0.90827	0.98596	0.9989	0.34388	1	1	1	1	1
$n = 4$	0.02278	0.50001	0.84135	0.97725	0.99865	0.99997	0.50001	1	1	1	1	1
$n = 5$	0.02994	0.63838	0.92951	0.9952	0.9999	1	0.63838	1	1	1	1	1
$n = 6$	0.03794	0.74993	0.97122	0.99911	0.99999	1	0.74993	1	1	1	1	1
$n = 7$	0.04677	0.83364	0.98903	0.99985	1	1	0.83364	1	1	1	1	1
$n = 8$	0.0564	0.893	0.99606	0.99998	1	1	0.893	1	1	1	1	1
$n = 9$	0.06681	0.9332	0.99865	1	1	1	0.9332	1	1	1	1	1
$n = 10$	0.07798	0.95937	0.99956	1	1	1	0.97586	1	1	1	1	1
$n = 11$	0.08985	0.97586	0.99986	1	1	1	0.97586	1	1	1	1	1
$n = 12$	0.10241	0.98596	0.99996	1	1	1	0.98596	1	1	1	1	1
$n = 13$	0.11561	0.99199	0.99999	1	1	1	0.99199	1	1	1	1	1
$n = 14$	0.12942	0.99551	1	1	1	1	0.99551	1	1	1	1	1
$n = 15$	0.14378	0.99752	1	1	1	1	0.99752	1	1	1	1	1

Table 3.8: The power of X-bar and EWMA charts with $\lambda = 0.2$
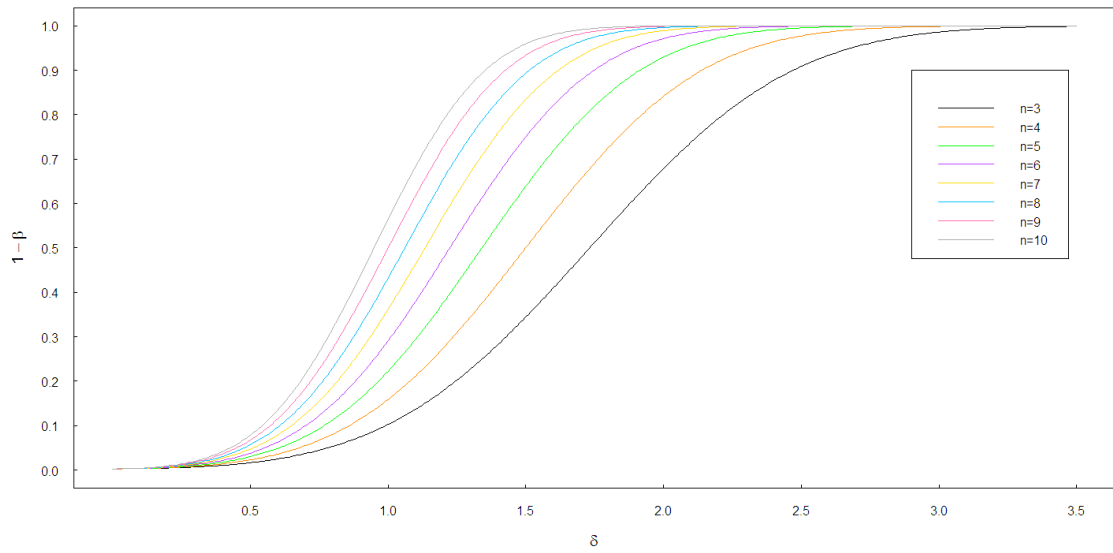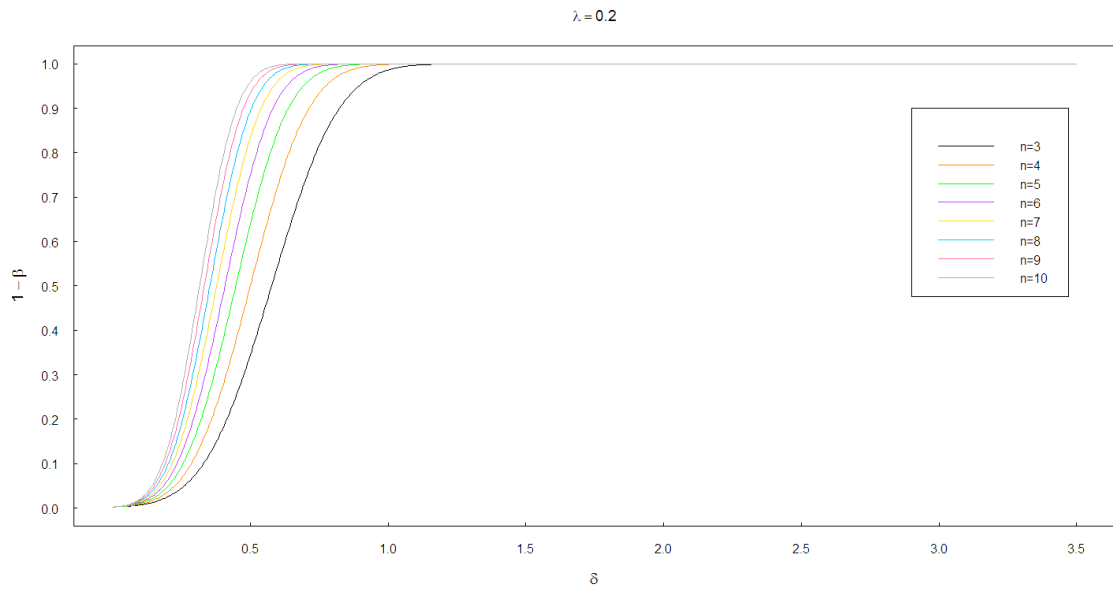
Figure 3.8: The power of $\bar{X}$ chart

Figure 3.9: The power of EWMA chart with $\lambda = 0.2$

Sample size	X-bar						EWMA					
	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$
$n = 3$	0.01648	0.34388	0.67872	0.90827	0.98596	0.9989	0.23926	0.99995	1	1	1	1
$n = 4$	0.02278	0.50001	0.84135	0.97725	0.99865	0.99997	0.36158	1	1	1	1	1
$n = 5$	0.02994	0.63838	0.92951	0.9952	0.9999	1	0.48327	1	1	1	1	1
$n = 6$	0.03794	0.74993	0.97122	0.99911	0.99999	1	0.59499	1	1	1	1	1
$n = 7$	0.04677	0.83364	0.98903	0.99985	1	1	0.69147	1	1	1	1	1
$n = 8$	0.0564	0.893	0.99606	0.99998	1	1	0.77086	1	1	1	1	1
$n = 9$	0.06681	0.9332	0.99865	1	1	1	0.83364	1	1	1	1	1
$n = 10$	0.07798	0.95937	0.99956	1	1	1	0.88166	1	1	1	1	1
$n = 11$	0.08985	0.97586	0.99986	1	1	1	0.91736	1	1	1	1	1
$n = 12$	0.10241	0.98596	0.99996	1	1	1	0.94324	1	1	1	1	1
$n = 13$	0.11561	0.99199	0.99999	1	1	1	0.96161	1	1	1	1	1
$n = 14$	0.12942	0.99551	1	1	1	1	0.9744	1	1	1	1	1
$n = 15$	0.14378	0.99752	1	1	1	1	0.98314	1	1	1	1	1

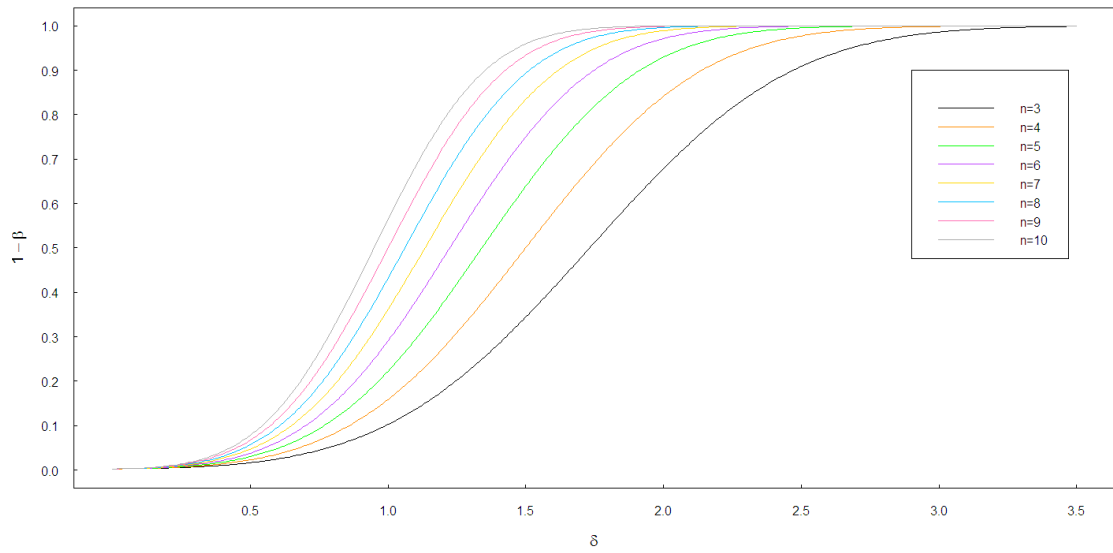Table 3.9: The power of X-bar and EWMA charts with $\lambda = 0.25$
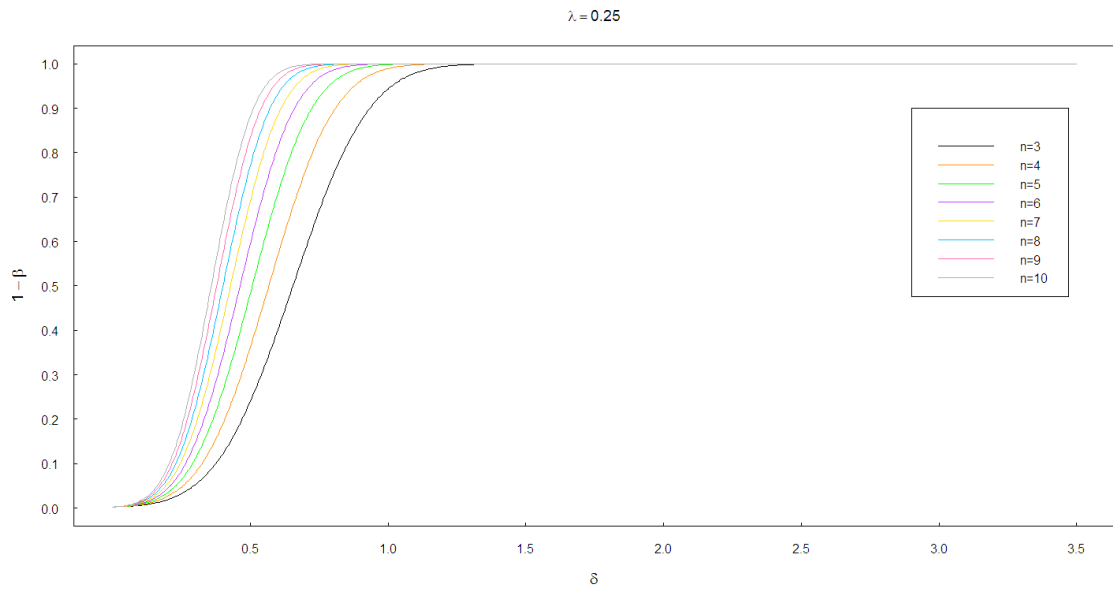
Figure 3.10: The power of $\bar{X}$ chart

Figure 3.11: The power of EWMA chart with $\lambda = 0.25$

Sample size	X-bar						EWMA					
	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$
$n = 3$	0.01648	0.34388	0.67872	0.90827	0.98596	0.9989	0.10241	0.98596	0.99996	1	1	1
$n = 4$	0.02278	0.50001	0.84135	0.97725	0.99865	0.99997	0.15866	0.99865	1	1	1	1
$n = 5$	0.02994	0.63838	0.92951	0.9952	0.9999	1	0.22246	0.9999	1	1	1	1
$n = 6$	0.03794	0.74993	0.97122	0.99911	0.99999	1	0.29099	0.99999	1	1	1	1
$n = 7$	0.04677	0.83364	0.98903	0.99985	1	1	0.36158	1	1	1	1	1
$n = 8$	0.0564	0.893	0.99606	0.99998	1	1	0.4319	1	1	1	1	1
$n = 9$	0.06681	0.9332	0.99865	1	1	1	0.50001	1	1	1	1	1
$n = 10$	0.07798	0.95937	0.99956	1	1	1	0.56447	1	1	1	1	1
$n = 11$	0.08985	0.97586	0.99986	1	1	1	0.62424	1	1	1	1	1
$n = 12$	0.10241	0.98596	0.99996	1	1	1	0.67872	1	1	1	1	1
$n = 13$	0.11561	0.99199	0.99999	1	1	1	0.7276	1	1	1	1	1
$n = 14$	0.12942	0.99551	1	1	1	1	0.77086	1	1	1	1	1
$n = 15$	0.14378	0.99752	1	1	1	1	0.80867	1	1	1	1	1

Table 3.10: The power of X-bar and EWMA charts with $\lambda = 0.4$
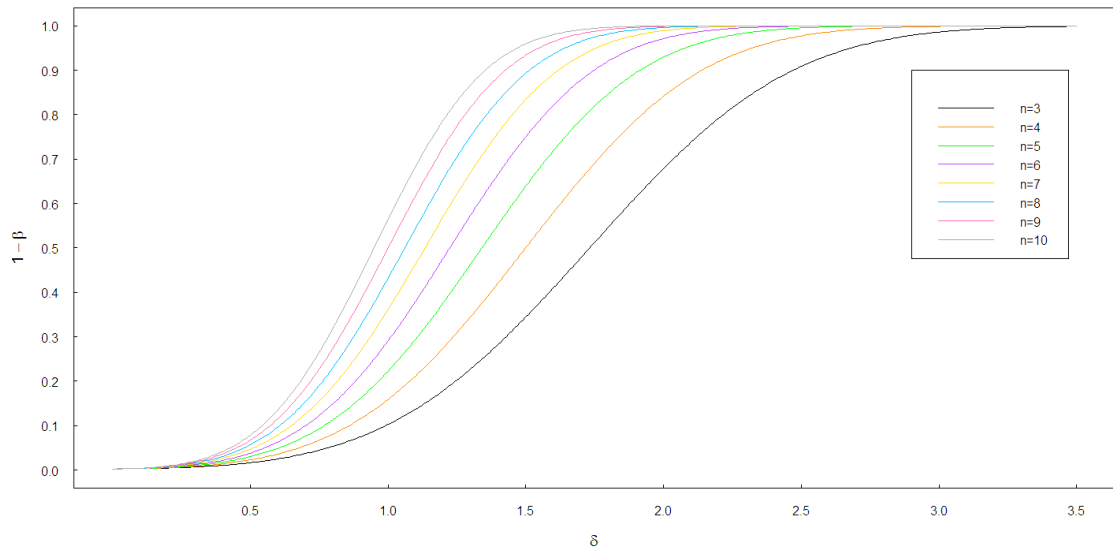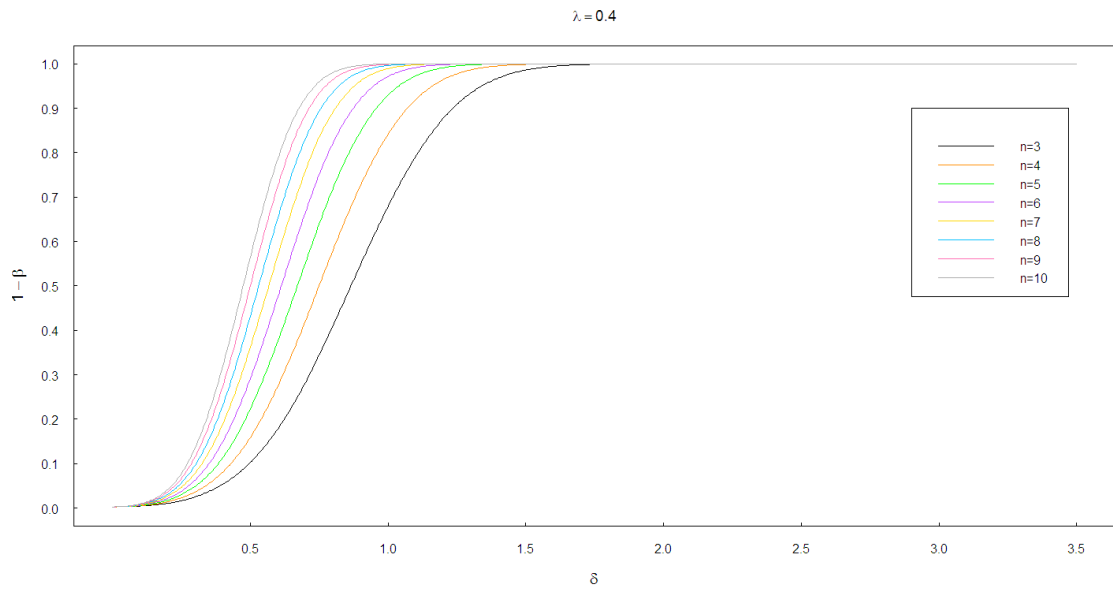
Figure 3.12: The power of $\bar{X}$ chart

Figure 3.13: The power of EWMA chart with $\lambda = 0.4$

By looking at the results in the tables and figures, we can make following conclusions.

- The increase power of charts to detect the deviation increases with the large sample size n. Therefore, detecting of the small variations is done by the use of large sized samples.

- The power of EWMA chart increases with the decrease of the value of λ.

The Average Run Length: The average run length (ARL) is the average number of points plotted on the chart until an out-of-control condition is signaled. It is the expected value of the run length distribution. It is connected to the OC curve as it can be seen from equation (2.12) and (2.13). The ARL is used more often in control chart literature than OC curve values, since it's concept easier to comprehend. The tables below show the ARL values for both charts with different values of λ and δ and $k = L = 3$. We have found that in this case the sample size did not affect ARL1, but decreasing the sample size leads to increasing ARL2 for both charts and changing the shift size produces a change in ARL2; decreasing the λ leads to increasing ARL2 for EWMA chart.

shift in mean	X-bar	EWMA				
δ	$\lambda = 1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.25$	$\lambda = 0.4$
0	370.37	370.37	370.37	370.37	370.37	370.37
0.25	184.16	60.68	31.39	22.48	7.52	2.61
0.5	60.68	9.76	4.18	2.91	1.28	1.01
0.75	22.48	2.91	1.49	1.23	1	1
1	9.76	1.47	1.06	1.01	1	1
1.5	2.91	1.01	1	1	1	1
2	1.1	1	1	1	1	1
2.5	1.1	1	1	1	1	1
3	1.01	1	1	1	1	1
4	1	1	1	1	1	1

Table 3.11: ARL of X-bar and EWMA charts with $n = 3$

shift in mean	X-bar	EWMA				
δ	$\lambda = 1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.25$	$\lambda = 0.4$
0	370.37	370.37	370.37	370.37	370.37	370.37
0.25	133.16	33.4	15.59	10.76	3.49	1.45
0.5	33.4	4.5	2.07	1.57	1.03	1
0.75	10.76	1.57	1.08	1.02	1	1
1	4.5	1.08	1	1	1	1
1.5	1.57	1	1	1	1	1
2	1.10	1	1	1	1	1
2.5	1.08	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1

Table 3.12: ARL of X-bar and EWMA charts with $n = 5$

shift in mean	X-bar	EWMA				
δ	$\lambda = 1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.25$	$\lambda = 0.4$
0	370.37	370.37	370.37	370.37	370.37	370.37
0.25	73.26	12.82	5.5	3.77	1.49	1.03
0.5	12.82	1.77	1.13	1.04	1	1
0.75	3.77	1.04	1	1	1	1
1	1.77	1	1	1	1	1
1.5	1.04	1	1	1	1	1
2	1	1	1	1	1	1
2.5	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1

Table 3.13: ARL of X-bar and EWMA charts with $n = 10$

shift in mean	X-bar	EWMA				
δ	$\lambda = 1$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.25$	$\lambda = 0.4$
0	370.37	370.37	370.37	370.37	370.37	370.37
0.25	47.33	6.96	3.02	2.16	1.13	1
0.5	6.96	1.24	1.01	1	1	1
0.75	2.16	1	1	1	1	1
1	1.24	1	1	1	1	1
1.5	1	1	1	1	1	1
2	1	1	1	1	1	1
2.5	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1

Table 3.14: ARL of X-bar and EWMA charts with $n = 15$

Chapter 4

Optimal Process Calibration for Non-symmetric Loss Functions

The process calibration is one of the methods in statistical process control for improving the quality of the products. We mention again basic notions of statistical process control and present loss functions that can be used in this area. This chapter is concerned with the problem of process calibration for some non-symmetric loss functions. We suggest an optimal calibration policy which can be implemented using our programs written in statistical software R.

In this chapter we consider the problem of the process calibration, i.e. how to set-up a manufacturing process in order to meet specification limits, by which we mean how to minimize the expected loss. In Section 4.1, we introduced traditional method of calibration, which corresponds to the symmetric loss function. However, quite often the lower and the upper specification limits cannot be treated in the same way. For example, it may happen that although one of the specification limits can be exceeded, we obtain a nonconforming product that could be reworked (improved or corrected) which requires some costs. Three models of non-symmetric loss functions and their implementation in statistical software R are considered, respectively, in Sections 4.2 and 4.3. The R code is given at the end of the chapter.

4.1   Symmetric loss function

Let X denote a quality characteristic of the product. We assume that X is normally distributed with mean μ and standard deviation σ, i.e. $X \sim \mathcal{N}(\mu, \sigma^2)$. We suppose that the standard deviation σ is known.

The usual practice is to set-up the mean at the middle distance between the specification limits, i.e. $\mu_0 = \dfrac{LSL + USL}{2}$ and this is legitimate only if the loss does not depend on which specification limit - the lower or upper - is exceeded. In that case

the loss function $L(x)$ is symmetrical,

$$L(x) = \begin{cases} w, & x < LSL \\ 0, & LSL \le x \le USL \\ w, & x > USL \end{cases}$$
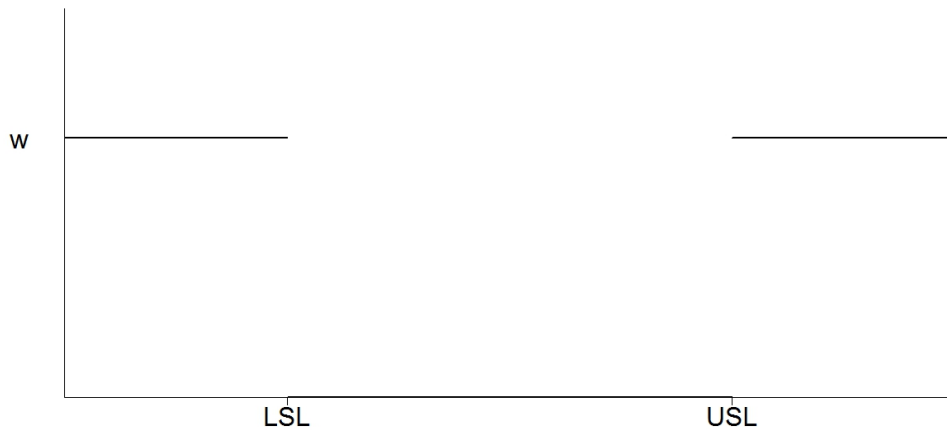
where $w > 0$ (see Figure 4.1).

Figure 4.1: Symmetric loss function

This is a very simple method of process calibration. However, very often the assumption of the symmetrical loss function is not fulfilled in practice. Let us consider the following examples.

Example 4.1.1 The quality characteristic under study is the inner diameter of a hole. Undersized holes can be reworked at extra costs. But the reduction of the over-sized hole would be either impossible or would require much higher costs.

Example 4.1.2 The quality characteristic under study is the outside diameter of the milled item. Over-sized items can be re-grounded at additional costs. But the undersized item could be sold only for scrap.

Example 4.1.3 The quality characteristic under study is the blood glucose level in diabetes patients. If measured value of blood sugar is too low or too high, i.e. below or above allowed limits, then the blood sugar can be stabilized with some extra costs (injection of insulin, food, tablets, etc.). These costs does not have to be the same for too low and too high level of blood sugar.

These examples show that in situations described above, instead of the symmetrical loss function, a non-symmetric loss function should be used.

4.2 Non-symmetric loss function

To find the optimal calibration policy for situations as described in the examples given above we will consider three models of loss functions. First model was proposed by Grzegorzewski and Mrowka [33] and other two models are suggested by us.

First model

The first model of the loss function is given by the formula

$$
L(x) = \begin{cases} w, & x < LSL \\ 0, & LSL \leq x \leq USL \\ z\dfrac{x - USL}{ULR - USL}, & USL < x \leq ULR \\ z, & x > ULR \end{cases}
$$

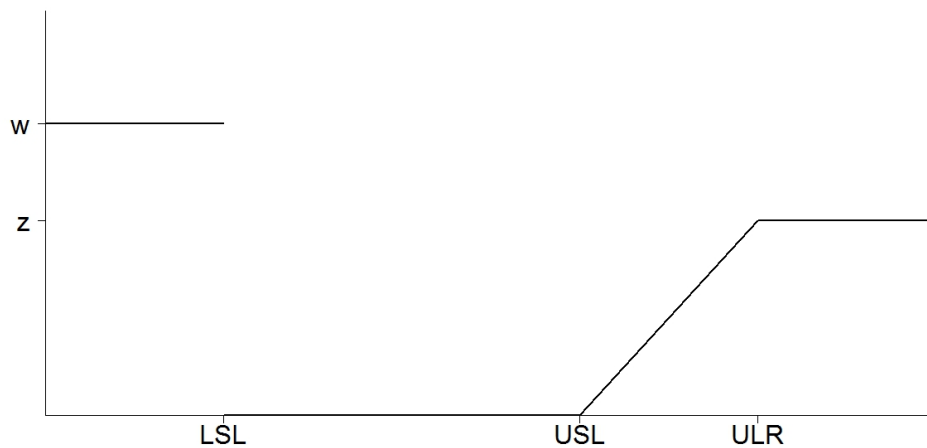where $w > z$ and upper limit for rework $ULR > USL$ (see Fig.4.2).

Figure 4.2: The first model of the loss function

This model could be used in cases when over-sized product items can be reworked with some extra costs (Example 2).

In all models we suppose that the loss function is connected to the quality characeristic X, where X has Normal distribution $N(\mu, \sigma^2)$ and the corresponding density is denoted by $f(x)$. The expected loss function is equal to

$$EL(X) = EL = wP(X < LSL) + zP(X > ULR) + z \int_{USL}^{ULR} \frac{x - USL}{ULR - USL} f(x) \mathrm{d}x.$$

After some transformations we get

$$
\begin{aligned}
EL \quad = \quad & w\Phi\left(\frac{LSL - \mu}{\sigma}\right) + z + z\left(\frac{\mu - USL}{ULR - US} - 1\right)\Phi\left(\frac{ULR - \mu}{\sigma}\right) + \\
+ \quad & z\frac{\mu - USL}{ULR - USL}\Phi\left(\frac{USL - \mu}{\sigma}\right) - \\
- \quad & \sigma\frac{z}{ULR - USL}\left(\phi\left(\frac{ULR - \mu}{\sigma}\right) - \phi\left(\frac{USL - \mu}{\sigma}\right)\right),
\end{aligned}
$$

where Φ and ϕ are, respectively, distribution function and density function of standard Normal distribution $\mathcal{N}(0,1)$. We consider the expected loss as a function of a process mean μ, $EL = EL(\mu)$ and we are searching for such μ in which minimal expected loss occurs. Thus we need

$$
\begin{aligned}
& \frac{\mathrm{d}EL(\mu)}{\mathrm{d}\mu} = \\
= \quad & \frac{z}{ULR - USL}\left[\Phi\left(\frac{ULR - \mu}{\sigma}\right) - \Phi\left(\frac{USL - \mu}{\sigma}\right)\right] - \frac{w}{\sigma}\phi\left(\frac{LSL - \mu}{\sigma}\right) = 0.
\end{aligned}
$$

This equation depends on μ and also LSL, USL, ULR, w, z, σ. To simplify it, we introduce use new parameters:

$$
\begin{aligned}
a \quad & = \quad \frac{ULR - USL}{\sigma}, \\
b \quad & = \quad \frac{USL - LSL}{\sigma}, \\
K \quad & = \quad \frac{w}{z}.
\end{aligned}
$$

The preceding equation reduces to

$$\frac{1}{a}\left[\Phi(a + \Delta) - \Phi(\Delta)\right] - K\phi(-b + \Delta) = 0. \tag{4.1}$$

The optimal calibration is given by

$$\mu = USL - \Delta\sigma, \tag{4.2}$$

where Δ is the correction factor, and it is the solution of the equation 4.1.

For cases when oversized items are reworked with some extra costs (Example 4.1.2), we could consider loss function of this kind.

The following two models could be appropriate in the cases when there is a possibility of rework for both undersized and oversized items (Example 4.1.3).

Second model

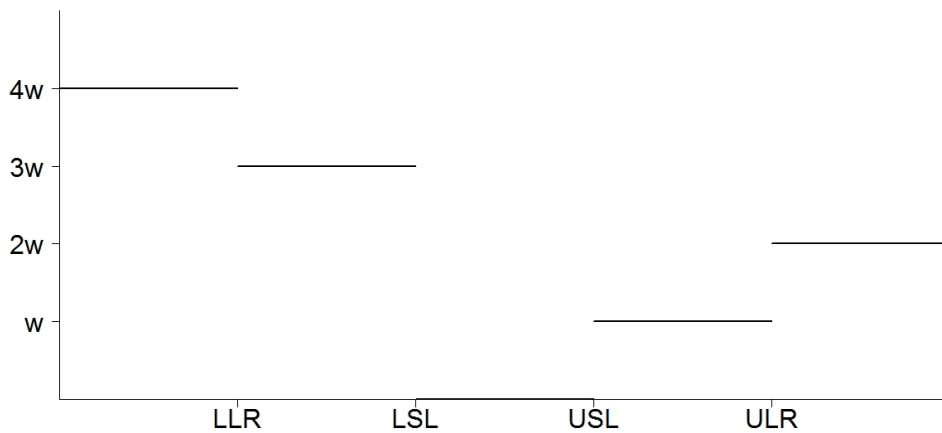Loss function is given by the formula

$$L(x) = \begin{cases} 4w, & x < LLR \\ 3w, & LLR \leq x < LSL \\ 0, & LSL \leq x < USL \\ w, & USL \leq x \leq ULR \\ 2w, & x > ULR \end{cases},$$

where $w > 0$ (see Fig.4.3).

Figure 4.3: The second model of the loss function

In this situation the expected loss is equal to

$$EL(X) = EL = wP(X < LLR) + 3wP(X \leq LSL) - wP(X \leq ULR) - wP(X \leq USL) + 2w.$$

After some transformations we get

$$EL = w\Phi\left(\frac{LLR - \mu}{\sigma}\right) + 3w\Phi\left(\frac{LSL - \mu}{\sigma}\right) - w\Phi\left(\frac{ULR - \mu}{\sigma}\right) - w\Phi\left(\frac{USL - \mu}{\sigma}\right) + 2w.$$

To find the minimal expected loss we need to find μ for which $\frac{\mathrm{d}EL(\mu)}{\mathrm{d}\mu}$ is equal to zero. we have

$$\frac{\mathrm{d}EL(\mu)}{\mathrm{d}\mu} = \frac{w}{\sigma}\left[\phi\left(\frac{LLR-\mu}{\sigma}\right) + 3\phi\left(\frac{LSL-\mu}{\sigma}\right) - \phi\left(\frac{ULR-\mu}{\sigma}\right) - \phi\left(\frac{USL-\mu}{\sigma}\right)\right] = 0.$$

After dividing this equation with $-\frac{w}{\sigma}$ and using the notation

$$a = \frac{ULR-USL}{\sigma} = \frac{LSL-LLR}{\sigma},$$

b and Δ as before, we get

$$\phi(-a-b-\Delta) + 3\phi(-b+\Delta) - \phi(a+\Delta) - \phi(\Delta) = 0. \tag{4.3}$$

Third model

This model is given by the formula

$$L(x) = \begin{cases} w_2, & x < LLR \\ w_1\dfrac{x-LLR}{LSL-LLR} - w_2\dfrac{x-LSL}{LSL-LLR}, & LLR \le x < LSL \\ 0, & LSL \le x \le USL \\ z\dfrac{x-USL}{ULR-USL}, & USL < x \le ULR \\ z, & x > ULR \end{cases}$$

where $w_2 > w_1 > z$ (see Fig.4.4).

In this situation the expected loss is equal to

$$\begin{aligned} EL(X) = EL \quad = \quad & \left[w_2 + \frac{w_2-w_1}{a\sigma}\mu + w_1\frac{LLR}{a\sigma} - w_2\frac{LSL}{a\sigma}\right]\Phi\left(\frac{LLR-\mu}{\sigma}\right) + \\ + \quad & \left[\frac{w_1-w_2}{a\sigma}\mu - w_1\frac{LLR}{a\sigma} + + w_2\frac{LSL}{a\sigma}\right]\Phi\left(\frac{LSL-\mu}{\sigma}\right) + \\ + \quad & \left[-\frac{z}{a\sigma}\mu + z\frac{USL}{a\sigma}\right]\Phi\left(\frac{USL-\mu}{\sigma}\right) + \\ + \quad & \left[\frac{z}{a\sigma}\mu - z\frac{USL}{a\sigma} - z\right]\Phi\left(\frac{ULR-\mu}{\sigma}\right) + \\ + \quad & \frac{w_1-w_2}{a}\phi\left(\frac{LLR-\mu}{\sigma}\right) + \frac{w_2-w_1}{a}\phi\left(\frac{LSL-\mu}{\sigma}\right) + \\ + \quad & \frac{z}{a}\phi\left(\frac{USL-\mu}{\sigma}\right) - \frac{z}{a}\phi\left(\frac{ULR-\mu}{\sigma}\right) + z. \end{aligned}$$

We assumed that

$$a = \frac{ULR-USL}{\sigma} = \frac{LSL-LLR}{\sigma}.$$

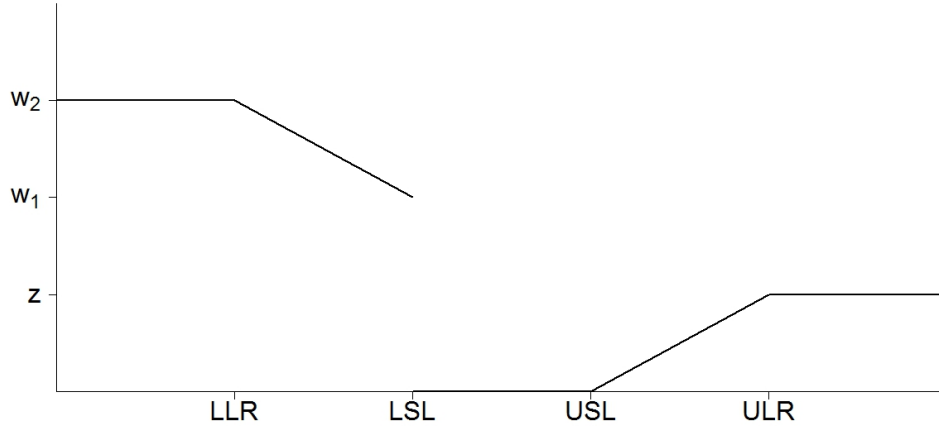Figure 4.4: The third model of the loss function

To find the minimal expected loss we need to find μ which minimizes $EL(\mu)$ as function of μ.

Thus

$$
\begin{aligned}
\frac{\mathrm{d}EL(\mu)}{\mathrm{d}\mu} &= \frac{w_2 - w_1}{a\sigma}\Phi\left(\frac{LLR - \mu}{\sigma}\right) - \frac{w_2 - w_1}{a\sigma}\Phi\left(\frac{LSL - \mu}{\sigma}\right) + \\
&+ \frac{z}{a\sigma}\left[\Phi\left(\frac{ULR - \mu}{\sigma}\right) - \Phi\left(\frac{USL - \mu}{\sigma}\right)\right] - \frac{w_1}{\sigma}\phi\left(\frac{LSL - \mu}{\sigma}\right) + \\
&+ \frac{z}{\sigma}\left[\phi\left(\frac{USL - \mu}{\sigma}\right) - \phi\left(\frac{ULR - \mu}{\sigma}\right)\right] = 0.
\end{aligned}
$$

Using also the notation $b = \frac{USL-LSL}{\sigma}$, $K_1 = \frac{w_1}{z}$ and $K_2 = \frac{w_2}{z}$, we get

$$
\begin{aligned}
\frac{K_2 - K_1}{a}&\left[\Phi(-a - b + \Delta) - \Phi(-b + \Delta)\right] + \frac{\Phi(a + \Delta) - \Phi(\Delta)}{a} - \\
&- K_1\phi(-b + \Delta) + \phi(\Delta) - \phi(a + \Delta) = 0. \tag{4.4}
\end{aligned}
$$

4.3   Implementation in R

To calibrate optimally the process mean, we need to calculate the correction factor Δ. For three considered models, Δ is calculated as a solution of equations (4.1),(4.3) and (4.4), respectively. These equations depend on various parameters: equation (4.1) depends on tree parameters a, b and K; equation (4.3) on two parameters a and b, and the equation (4.4) depends on four parameters a, b, K_1 and K_2. We solve these equations numerically, using Brent's root- finding method (see [12]) im-

plemented in R function *uniroot*.

Correction factor for these three models of non-symmetric loss functions can be found calling our R function *corr.factor*. The code is presented bellow:

```
corr.factor(delta,a,b,K,K1,K2,par)
```

by specifying the values of parameters (by choosing non-zero values for parameters a, b and/or K, $K1$ and $K2$). Parameter *par* represents upper limit for correction function necessary for Brent's method to work properly (by default the value of parameter *par* = 3).

Code of R-function:

```
model1<-function(delta,a,b,K){
1/a*(pnorm(a+delta)-pnorm(delta))-K*dnorm(-b+delta)}


model2<-function(delta,a,b){
dnorm(-a-b+delta)+3*dnorm(-b+delta)-dnorm(a+delta)-dnorm(delta) }



model3<-function(delta,a,b,K1,K2){
(K2-K1)/a*(pnorm(-a-b+delta)-pnorm(-b+delta))+1/a*(pnorm(a+delta)-
pnorm(delta))-K1*dnorm(-b+delta)+dnorm(delta)-dnorm(a+delta) }



corr.factor<-function(delta,a=3/0.1,b=3,K=0,K1=0,K2=0,par=3) {
 if(K==0&&K1==0&K2==0)
   {model.2<-function(delta){model2(delta,a,b)}
    solution<-uniroot(model2,lower=0,upper=par)$root
   }

 else if(K1>0&&K2>0)
  {model.3<-function(delta){model3(delta,a,b,K1,K2)}
   solution<-uniroot(model3,lower=0,upper=par)$root
  }

 else if(K>0)
   {model.1<-function(delta){model1(delta,a,b,K)}
```

```
    solution<-uniroot(model1,lower=0,upper=par)$root
    }

solution }
```

Now, we can find correction factors for considered non-symmetric loss functions for given values of parameters.

First model. Parameters of the first model of non-symmetric loss function a, b and K have the following values: $a = 12$, $b = 6$ and $K = 2$. By calling the function

```
corr.function(a=12,b=6,K=2)
```

we get $\Delta = 2.308$.

Second model. Parameters of the second model of non-symmetric loss function a, b have the following values: $a = 12$, $b = 6$. By calling the function

```
 corr.function(a=12,b=6)
```

we get $\Delta = 2.817$.

Third model. Parameters of the third model of non-symmetric loss function a, b and $K1$ and $K2$ have the following values: $a = 12$, $b = 6$, $K1 = 2$, $K2 = 3$. By calling the function

```
 corr.function(a=12,b=6,K1=2,K2=3)
```

we get $\Delta = 2.887$.

Conclusions

The design and the performance of control charts are the most important problems in Quality Control. In this thesis we studied and compared various characteristics of control charts, together with related costs. We proposed a loss function, which is connected to the production process and to $X-$bar quality control chart. Using Matlab program for optimization, we found values of k, n and h, minimizing the loss function for given costs. These values are compared with those obtained from a non-linear regression model, built using a package Sigma plot, for given values of cost. Production process can be modeled by $Y_i = \lambda X_i + (1 - \lambda)Y_{i-1}$, where $0 < \lambda \leq 1$ is a constant, X_i are $N(\mu, \sigma^2)$ distributed. Exponentially Weighted Moving Average (EWMA) charts for this model are presented, and type I and type II errors are calculated in the case when i is large. For different sample sizes, the new comparison between the optimal design of the X-bar and EWMA control charts for Normally distributed quality characteristic is given, comparing the corresponding cost-loss functions, power functions and average run lengths. The process of calibration is one of the methods in statistical process control, introduced for improving the quality of the products and for reducing the production costs. Two new models of non-symmetrical loss function are introduced, and the value which minimizes the expected loss is found for both cases. The production process is calibrated with this new value.

One direction for further investigation in the area of Quality Control is the case when the quality characteristic does not have symmetrical distribution. The other direction is to study calibration for non-symmetric loss functions, which are not piecewise constant, or piecewise linear, but have some parts with polynomial, or exponential growth.

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   Radovi u međunarodnim časopisima sa SCI liste:

   Economic statistical design of X-bar control chart for non-normal symmetric distribution of quality characteristic, FILOMAT. Vol. 29, No. 10, pp. 2325-2338 (2015) (sa K. Veljković i V. Jevremović).

   Radovi u međunarodnim časopisima:

   One Possible Optimization of Economic design for X-Chart European International Journal of Science and Technology, Vol. 2, No. 5, pp. 45-52 (2013).

Poglavlje u monografiji: ne

Radovi u Zbornicima:

Process Calibration for Some Examples of Non-symmetric Loss Functions. Proceedings of Fourth Mathematical Confernce of the Republic of Srpska, Trebinje, p.141-149. (2014) (sa K. Veljković i V. Jevremović)

Saopš̃tenja na kongresima i konferencijama:

13. Srpski matematički kongres, 22-25. maja 2014. Vrnjačka Banja sa radom publikovanim u FILOMAT-u.

Četvrta matematička konferencija Republike Srpske, 6-7. juni 2014, Trebinje sa radom publikovanim u Zborniku sa konferencije.

Prilog 1.

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broj indeksa 2048/2009

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da je doktorska disertacija pod naslovom

"Konstrukcija i osobine kontrolnih kartica za stacionarne i nekorelisane podatke"

- rezultat sopstvenog istraživačkog rada,

- da predložena disertacija u celini ni u delovima nije bila predložena za dobijanje bilo koje diplome prema studijskim programima drugih visokoškolskih ustanova,

- da su rezultati korektno navedeni i

- da nisam kršio autorska prava i koristio intelektualnu svojinu drugih lica.

Potpis doktoranda

U Beogradu, _____

_____

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Naslov rada: *"Konstrukcija i osobine kontrolnih kartica za stacionarne i nekorelisane podatke"*

Mentor: *redovni Prof. dr.Slobodanka Janković*

Potpisani *Halima Elfaghihe*

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