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Spectral properties of operator matrices on Banach spaces

– doctoral dissertation –

Спектралне особине матрица оператора на Банаховим просторима

– докторска дисертација –

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Word from the author

Being in a position to obtain a PhD diploma, I find myself having a strange feeling: in spite of realizing that I am at the brick of obtaining a higher academic degree, I feel like I am at the beginning of some new study level and that the study process will never end. On the one hand, this is illogical, since doctoral studies are the third and the final level of academy studies. On the other hand, this makes sense, since all researchers who marked they spot in science can confirm that a mathematician must accept the fact of longlife studying. One way or another, I feel this is just a start of something completely new and demanding, and that the "combat" with mathematical problems begins right on.

During the writing of my thesis, my main hope was that humanity will one day benefite from my results. I cannot be certain that this will become reality, but I witness that my thoughts during the writing process were nobel and filled with hope. If anyone ever finds I way to utilize my results in a way that this planet becomes a better place for life, then I would know that my mission in writing the thesis is accomplished.

I would now like to dedicate a paragraph or two to those who have influenced on my life and on this research, directly or indirectly. It would be impossible to provide concrete names of all my colleagues who were there for me in any possible way. In spite of this fact, I cannot resist to single out some of them.

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Резиме рада

У овој докторској дисертацији уопштавамо многе познате резултате везане за горње троугаоне матрице оператора. Наш задатак је да прикажемо резултате карактеризације за разне типове инвертибилности таквих матрица оператора, који ће онда дати одговарајуће резултате пертурбација и неке резултате за проблем ,,попуњавања рупа”.

Уопштења која вршимо спроводимо у два правца. Прво, уопштавамо многе познате резултате изражене за матрице оператора реда 2 на матрице оператора које су произвољног реда $n \geq 2$. Друго, резултате који су дати у контексту директне тополошке суме сепарабилних Хилбертових простора проширујемо тако да важе на директној тополошкој суми произвољних Банахових простора.

У дисертацији, приказана су два нова метода помоћу којих можемо истраживати спектралне особине матрица оператора. Први метод се односи на проширење технике утапања Банахових простора уведене од стране Драгана С. Ђорђевића у [12] на горње троугаоне матрице оператора произвољне димензије, а други метод је адаптација технике свођења на апсурд из [54], [55] на матрице оператора које делују на директној суми простора који не морају бити сепарабилни. Ове идеје представљају оригиналан научни допринос аутора, и приказане су у радовима [44]–[48].

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Сажетак

Ова докторска дисертација има за циљ да прикаже разне резултате везане за многе типове инвертибилности блок матрица оператора чији су неки елементи дати, а остали нису. Блок матрице оператора које ћемо ми проучавати јесу горње троугаоне матрице оператора, тј. матрице код којих се испод главне дијагонале налазе нула оператори. За такве матрице оператора увек ћемо претпостављати да су им дијагонални елементи дати, а елементи изнад главне дијагонале нису познати. Подразумева се да овакве матрице оператора делују на директној тополошкој суми Банахових простора, као што је прецизирано у наставку. Наш задатак је да прикажемо резултате карактеризације за разне типове инвертибилности таквих оператора, који ће онда дати одговарајуће резултате пертурбација и неке резултате за проблем ,,попуњавања рупа".

До сада, истраживања ове врсте била су углавном предузимана у контексту сепарабилних Хилбертових простора. Стога, пребројиве ортогоналне базе су често коришћене у тим истраживањима. Аутор ове дисертације одабрао је другачији пут. Уместо да користи линеарне базе Банахових простора које не морају бити пребројиве, аутор је радије одабрао рад са одговарајућим утапањима између оних потпростора Банахових простора који имају тополошки комплемент. Осим тога, стручњаци у овој области су до сада углавном испитивали случај 2×2 горње троугаоних матрица оператора, док је аутор ове дисертације проучавао горње троугаоне матрице произвољне димензије, На овај начин, техника утапања Банахових простора уведена од стране Драгана С. Ђорђевића у [12] је уопштена на горње троугаоне матрице оператора произвољне димензије, а технике из [54], [55] су прилагођене на матрице оператора које делују на директној суми простора који не морају бити сепарабилни. Ове идеје представљају оригиналан научни допринос аутора, и приказане су у радовима [44]–[46].

Дисертација је подељена у неколико глава, од којих је свака подељена у поглавља и потпоглавља. Све дефиниције, теореме,

итд. су нумерисане континуално.

Прва глава је уводног карактера. У њој се приказује историјска позадина везана за нашу тему, и повезујемо тему рада са осталим деловима теоријске математике. Такође уводимо и ознаке које ће се користити током писања дисертације. Завршавамо ову главу са неким уводним резултатима који се тичу нашег рада.
У другој глави приказујемо разне резултате везане за инвертибилност 2×2 матрица оператора. Резултати ове главе су одвећ познати у литератури. Најпре дајемо резултате за сепарабилне Хилбертове просторе, а затим проширујемо ове резултате на Банахове просторе који не морају бити сепарабилни. Резултати који користе сепарабилност су се појавили пре резултата који не користе сепарабилност. Аутор је добио велики део резултата ове главе као специјалне случајеве општијих истраживања у главама које следе.

Главни задатак треће главе је приказивање резултата претходне главе прилагођене за 3×3 матрице оператора. Дајемо неке резултате који користе сепарабилност, и неке који не подразумевају овај јак услов. Резултати који користе сепарабилност били су познати у литератури, а они који не користе добијени су од стране аутора и његовог ментора у [48].

Четврта глава је најбитнија глава. У њој су искључиво дати резултати који су оригинални научни допринос аутора. На тај начин, приказани су резултати из референци [44]-[46]. Истражујемо карактеризације (левог, десног) спектра, (левог, десног) Фредхолмовог спектра и левог/десног Вејловог спектра горњих троугаоних матрица димензије $n \geq 2$.

Пета глава служи да прикаже једну примену резултата из претходних глава. Наиме, решавамо проблем ,,попуњавања рупа" за различите матричне димензије $n \in \mathbb{N}$ и у различитим окружењима векторских простора. Приказани су неки познати резултати, највише у Хилбертовим просторима, али су дати и неки оригинални резултати аутора из [47].

Последња глава је додатак на претходне главе. У овој глави се бавимо блок матрицама оператора које нису горње троугаоне.

Abstract

This doctoral dissertation has as its aim to present various results related to different types of invertibility of block operator matrices, whose some of the entries are known, and the others are unknown. Block matrices in question are upper triangular operator matrices, that is matrices whose entries below the main diagonal are zero operators. We will always assume that diagonal elements of such matrices are given, while elements above the main diagonal are not. It is understood that all operator matrices in question act on a direct topological sum of Banach spaces, as it is precised in the sequel. Our goal is to present characterization results for different types of invertibility of such operators, which then yield appropriate perturbation results and some "filling in holes" results.

So far, investigations of this kind were mainly undertaken in the context of separable Hilbert spaces. Thus, countable orthogonal bases were frequently used in such research. The author of this dissertation has taken a different path. Instead of using linear bases of Banach spaces that need not be countable, the present author has rather worked with appropriate embedding mappings between certain subspaces of Banach spaces that have a topological complement. Moreover, specialist in this area have usually examined the case of 2×2 upper triangular operator matrices, while the author of this dissertation examines upper triangular operator matrices of an arbitrary dimension. In this way, the technique of Banach space embeddings introduced by Dragan S. Djordjević in [12] is generalized to upper triangular operator matrices of an arbitrary dimension, and techniques from [54],[55] are adapted for operator matrices acting on a direct sum of spaces which need not be separable. The latter ideas represent the original scientific contribution of the present author, and they can be found in papers [44]–[48].

Dissertation is divided into several chapters, each of which is divided into sections and subsections. All definitions, theorems, etc. that appear are numbered continuously.

The first chapter is an introductory one. In this chapter we present some historical background related to this topic, and we connect it to other areas

of pure mathematics. We also provide notation that will be used throughout this text. We end this chapter with some preliminary results regarding our work.

The second chapter presents various results on invertibility for 2×2 operator matrices. Results in this chapter are already known in the literature. We first present results for separable Hilbert spaces, and afterwards generalize these to Banach spaces for which separability is not assumed. The former appeared chronologically before the latter ones. The present author has obtained much of the results of this chapter as special cases of more general investigations in chapters to follow.

Main task of third chapter is to present results of the previous one adapted to 3×3 operator matrices. We present some of the results that assume separability, and some that do not use this strong assumption. The former were already known in the literature, while the latter were obtained by the present author and his PhD advisor in [48].

The fourth chapter is our main chapter. In this chapter one can find results that are new scientific contribution of the author. In that way, results from references [44]–[46] are presented. We investigate characterizations of (left, right) spectrum, (left, right) Fredholm and left/right Weyl spectrum for upper triangular operator matrices having dimension $n \geq 2$.

The fifth chapter ought to present an application of results from the previous chapters. Namely, we solve the "filling in holes" problem for different operator matrix dimensions $n \in \mathbb{N}$ and in different linear space settings. Some of the known results are presented, mainly in the setting of Hilbert spaces, but some new results that are the original work of the present author from [47] are also presented.

The last chapter contains an appendix to previous results. It is devoted to invertibility of block operator matrices that are not upper triangular.

Chapter 1

Introduction

1.1 Notation and main tasks

Let $X, Y, X_1, ..., X_n$ be arbitrary Banach spaces. We use notation $\mathcal{B}(X,Y)$ for the collection of all linear and bounded mappings from X to Y. Usually, spaces X and Y are linear spaces of functions, and in that case each $T \in \mathcal{B}(X,Y)$ is traditionally called an operator. Particularly, $\mathcal{B}(X) = \mathcal{B}(X,X)$. If $T \in \mathcal{B}(X)$, then by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ we denote the kernel and the range space of T. Those sets are subspaces of X and Y, respectively, and $\mathcal{N}(T)$ is closed.

If $T \in \mathcal{B}(X,Y)$, then its dual operator is $T' \in \mathcal{B}(Y',X')$ defined by $T'f(x) := f(Tx),\ f \in Y'$. We shall use some properties of dual operators on several occasions. For example, it is known that $\|T\| = \|T'\|$ and $T \mapsto T'$ is an isometric isomorphism of $\mathcal{B}(X,Y)$ into $\mathcal{B}(X',Y')$. For other basic features of dual operators we recommend [17].

For $U \subseteq X$ we define set $U^\circ \subseteq X'$, and for $V \subseteq X'$ we define set ${}^\circ V \subseteq X$ as

$$U^\circ := \{f \in X' : \ f \restriction_U = 0\}.$$

$${}^\circ V := \{x \in X : \ f(x) = 0 \ for\ every\ f \in V\}.$$

U° and ${}^\circ V$ are called the left and right annihilator of U and V, respectively. Above all interesting features that hold for annihilators, we point out only a few of them, see [49]:

$$\overline{\mathcal{R}(A)} = ^{\circ}[\mathcal{N}(A')],\ \overline{\mathcal{R}(A')} = \mathcal{N}(A)^{\circ},\ \mathcal{R}(A)^{\circ} = \mathcal{N}(A'), \ldots \tag{1.1.1}$$

where $A \in \mathcal{B}(X,Y)$.

For a subset K of $\mathbb{C}$ we use $acc(K), int(K)$ and ∂K, respectively, to denote the set of all points of accumulation of K, the interior of K and the boundary of K.

Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$ be given. We denote by $T_n^d(A)$ an $n \times n$ partial upper triangular operator matrix of the form

$$T_n^d(A) = \begin{bmatrix} D_1 & A_{12} & A_{13} & \ldots & A_{1,n-1} & A_{1n} \\ 0 & D_2 & A_{23} & \ldots & A_{2,n-1} & A_{2n} \\ 0 & 0 & D_3 & \ldots & A_{3,n-1} & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & D_{n-1} & A_{n-1,n} \\ 0 & 0 & 0 & \ldots & 0 & D_n \end{bmatrix} \in \mathcal{B}(X_1 \oplus X_2 \oplus \cdots \oplus X_n), \tag{1.1.2}$$

where $A := (A_{12},\ A_{13}, \ldots,\ A_{ij}, \ldots,\ A_{n-1,n})$ is an operator tuple consisting of unknown variables $A_{ij} \in \mathcal{B}(X_j, X_i)$, $1 \le i < j \le n,\ n \ge 2$. For convenience, we denote by $\mathcal{B}_n$ the collection of all such tuples A. Sum $X_1 \oplus \cdots \oplus X_n$ appearing in (1.1.2) is a direct topological sum of Banach spaces. We highly recommend article [42] as literature for properties of topological sums of more than two subspaces.

There are several questions that we are interested in:

Question 1. Can we find an appropriate characterization for Fredholmness, Weylness, etc. for $T_n^d(A)$, in terms of Fredholmness, Weylness, etc. of its diagonal entries D_i?

Question 2. What can we say about $\bigcap_{A \in \mathcal{B}_n} \sigma_*(T_n^d(A))$, where σ_* is one of the well known spectra that is examined in Fredholm theory?

Question 3. Under what conditions the equality $\sigma_*(T_n^d(A)) = \bigcup_{i=1}^{n} \sigma_*(D_i)$ holds for different types of spectra σ_*?

Here we give complete answers to each of the above questions.

We use $\mathcal{G}_l(X)$ and $\mathcal{G}_r(X)$ to denote the sets of all left and right invertible operators on X, respectively. The set of all invertible operators on X is denoted by $\mathcal{G}(X) = \mathcal{G}_l(X) \cap \mathcal{G}_r(X)$. We list some elementary notions from Fredholm theory (see [61]). Let $T \in \mathcal{B}(X)$, and put $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim X/\mathcal{R}(T)$. Quantities α and β are called the nullity and the deficiency of T, respectively, and in the case where at least one of them is finite we define $\text{ind}(T) = \alpha(T) - \beta(T)$ to be the index of T. Notice that $\text{ind}(T)$ may be $\pm\infty$ or integer. The following lemma enlights the reason for using such terminology in this article.

Lemma 1.1.1. *Let $T \in \mathcal{B}(X)$. The following equivalences hold:*

$$T \textit{ is left invertible } \Leftrightarrow \alpha(T) = 0 \textit{ and } \mathcal{R}(T) \textit{ is closed and complemented;}$$

$$T \textit{ is right invertible } \Leftrightarrow \beta(T) = 0 \textit{ and } \mathcal{N}(T) \textit{ is complemented.}$$

The following statement due to T. Kato [33] holds.

Lemma 1.1.2. *Let $T \in \mathcal{B}(X)$. If $\beta(T) < \infty$, then $\mathcal{R}(T)$ is closed in X.*

Families of left and right Fredholm operators, respectively, are defined as [12]

$$\Phi_l(X) = \{T \in \mathcal{B}(X) : \ \alpha(T) < \infty \textit{ and } \mathcal{R}(T) \textit{ is closed and complemented}\}$$

and

$$\Phi_r(X) = \{T \in \mathcal{B}(X) : \beta(T) < \infty \textit{ and } \mathcal{N}(T) \textit{ is complemented}\}.$$

The set of Fredholm operators is

$$\Phi(X) = \Phi_l(X) \cap \Phi_r(X) = \{T \in \mathcal{B}(X) : \alpha(T) < \infty \textit{ and } \beta(T) < \infty\}.$$

Families of left and right Weyl operators, respectively, are defined as

$$\Phi_l^-(X) = \{T \in \Phi_l(X) : \ \mathrm{ind}(T) \le 0\}$$

and

$$\Phi_r^+(X) = \{T \in \Phi_r(X) : \ \mathrm{ind}(T) \ge 0\}.$$

The set of Weyl operators is

$$\Phi_0(X) = \Phi_l^-(X) \cap \Phi_r^+(X) = \{T \in \Phi(X) : \ \mathrm{ind}(T) = 0\}.$$

Next, we also define the families of upper and lower semi-Fredholm operators, respectively, as [61]

$$\Phi_+(X) = \{T \in \mathcal{B}(X) : \ \alpha(T) < \infty \ \textit{and} \ \mathcal{R}(T) \ \textit{is closed}\}$$

and

$$\Phi_-(X) = \{T \in \mathcal{B}(X) : \beta(T) < \infty\}.$$

Put

$$\Phi_+^-(X) = \{T \in \Phi_+(X) : \ \mathrm{ind}(T) \le 0\}$$

and

$$\Phi_-^+(X) = \{T \in \Phi_-(X) : \ \mathrm{ind}(T) \ge 0\}.$$

These are the collections of upper and lower semi-Weyl operators, respectively.

Remark 1.1.3. *If X is a Hilbert space, then it is clear that*

$$\Phi_l(X) = \Phi_+(X), \quad \Phi_r(X) = \Phi_-(X), \quad \Phi_l^-(X) = \Phi_+^-(X), \quad \Phi_r^+(X) = \Phi_-^+(X).$$

For $T \in \mathcal{B}(X)$ consider the following inclusions: $\{0\} \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2) \subseteq \cdots$ and $X \supseteq \mathcal{R}(T) \supseteq \mathcal{R}(T^2) \supseteq \cdots$. The ascent of T, denoted by $asc(T)$, is defined as the least k (if it exists) for which $N(T^k) = N(T^{k+1})$ holds. If such k does not exist, then we say that the ascent of A is equal to infinity. The descent of T, denoted by $des(T)$, is defined as the least k (if it exists) for which $R(T^k) = R(T^{k+1})$ is satisfied. If such k does not exist, then we say that the descent of A is equal to infinity. If the ascent and the descent of T are finite, then they are equal ([14]). The Drazin inverse of $T \in \mathcal{B}(X)$ is the unique operator $T^D \in \mathcal{B}(X)$ satisfying $T^{k+1}T^D = T^k$, $T^D T T^D = T$ and

$TT^D = T^D T$ for some nonnegative integer k. The least k in the previous definition is known as the Drazin index of T. It is well-known that T^D exists if and only if $p = asc(T) = des(T) < \infty$. In this case the Drazin index of T is equal to p ([14]). The set of Browder operators on X is defined as $\mathbb{B}(X) = \{T \in \Phi(X) : \ asc(T) = des(T) < \infty\} = \{T \in \Phi(X) : \ T^D \ exists\} = \{T \in \Phi(X) : \ 0 \notin acc \ \sigma(T)\}$.

Corresponding spectra of an operator $T \in \mathcal{B}(X)$ are defined as follows:
-left spectrum: $\sigma_l(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}_l(X)\}$;
-right spectrum: $\sigma_r(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}_r(X)\}$;
-spectrum: $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}(X)\}$;
-left essential spectrum: $\sigma_{le}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_l(X)\}$;
-right essential spectrum: $\sigma_{re}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_r(X)\}$;
-essential spectrum: $\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(X)\}$;
-left Weyl spectrum: $\sigma_{lw}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_l^-(X)\}$;
-right Weyl spectrum: $\sigma_{rw}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_r^+(X)\}$;
-Weyl spectrum: $\sigma_w(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_0(X)\}$;
-Drazin spectrum: $\sigma_d(T) = \{\lambda \in \mathbb{C} : \lambda - T \ is \ not \ Drazin \ invertible\}$;
-Browder spectrum: $\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda - T \ is \ not \ Browder \ invertible\}$.

We write $\rho_l(T), \rho_r(T), \rho(T), \rho_{le}(T), \rho_{re}(T), \rho_e(T), \rho_{lw}(T), \rho_{rw}(T), \rho_w(T)$, $\rho_d(T), \rho_b(T)$ for the corresponding complements of the sets above, respectively.

Five more types of spectra will also appear, namely:
-point spectrum: $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ \lambda - T \ is \ not \ one-one\}$;
-approximate point spectrum: $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda - T \ is \ not \ bounded \ below\}$;
-residual spectrum: $\sigma_r(T) = \{\lambda \in \mathbb{C} : \ T - \lambda \ is \ not \ one-one \ and \ \overline{\mathcal{R}(\lambda - T)} \neq X\}$;
-defect spectrum: $\sigma_\delta(T) = \{\lambda \in \mathbb{C} : \ \lambda - T \ is \ not \ surjective\}$;
-Moore-Penrose spectrum: $\sigma_m(T) = \{\lambda \in \mathbb{C} : \ \mathcal{R}(\lambda - T) \ is \ not \ closed\}$.

1.2 Preliminaries

If $X_1,...,X_n$ are Hilbert spaces, one easily verifies that $T_n^d(A)$ given by (1.1.2) has the adjoint operator matrix $T_n^d(A)^*$ given by

$$T_n^d(A)^* = \begin{bmatrix} D_1^* & 0 & 0 & \dots & 0 & 0 \\ A_{12}^* & D_2^* & 0 & \dots & 0 & 0 \\ A_{13}^* & A_{23}^* & D_3^* & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{1,n-1}^* & A_{2,n-1}^* & A_{3,n-1}^* & \dots & D_{n-1}^* & 0 \\ A_{1n}^* & A_{2n}^* & A_{3n}^* & \dots & A_{n-1,n}^* & D_n^* \end{bmatrix} \in \mathcal{B}(X_1^* \oplus X_2^* \oplus \cdots \oplus X_n^*). \tag{1.2.1}$$

The following lemma imposes a connection between T and its adjoint operator T^* in terms of nullity and deficiency of T. This claim will be crucial at some points.

Lemma 1.2.1. *Let X be a Hilbert space and $T \in \mathcal{B}(X)$. Then the following holds:*
(a) $\alpha(T) = \beta(T^*), \beta(T) = \alpha(T^*)$;
(b) $T \in \Phi_l(X)$ *if and only if* $T^* \in \Phi_r(X^*)$;
(c) $T \in \Phi_r(X)$ *if and only if* $T^* \in \Phi_l(X^*)$;
(d) $\mathrm{ind}(T^*) = -\mathrm{ind}(T)$.

The following statement is well known in the literature.

Lemma 1.2.2. *Let X be a Hilbert space and $T \in \mathcal{B}(X)$. Then $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^*)$ is closed.*

We emphasize the fact that our results are ought to hold in arbitrary Banach spaces, not just the separable ones. In order to prove the main theorems which concern perturbation of various spectra of $T_n^d(A)$, we introduce a concept that will compensate loss of separability: the notion of embedded spaces. To our knowledge, this condition was first used in this context by D. S. Djordjević in 2002.

Definition 1.2.3. *([12, Definition 2.2]) We say that X can be embedded in Y and write $X \preceq Y$ if there exists a left invertible operator $J : X \to Y$.*

Remark 1.2.4. *Obviously, $X \preceq Y$ if and only if there exists a right invertible operator $J_1 : Y \to X$.*

If X and Y are Hilbert spaces, than $X \preceq Y$ if and only if $\dim X \leq \dim Y$. Here, $\dim X$ stands for the orthogonal dimension of X.

In order to prove results about the left (right) Weyl spectrum, we shall need another variant of Definition 1.2.3.

Definition 1.2.5. *([12, Definition 4.2]) We say that X can be essentially embedded in Y and write $X \prec Y$ if and only if:*
(a) $X \preceq Y$;
(b) for every $T \in \mathcal{B}(X, Y)$, $Y/\mathcal{R}(T)$ is an infinite dimensional linear space.

Remark 1.2.6. *If X and Y are Hilbert spaces, than $X \prec Y$ if and only if $\dim X < \dim Y$ and Y is infinite dimensional, where $\dim X$ is the orthogonal dimension of X.*

In order to prove results about the essential spectrum, we need the following notion.

Definition 1.2.7. *[12, Definition 2.2] We say that X and Y are isomorphic up to a finite dimensional subspace, if one of the following two statements hold:*

1) *There exists a bounded below operator $J_1 : X \to Y$ so that $\dim Y/J_1(X) < \infty$, or*

2) *There exists a bounded below operator $J_2 : Y \to X$ so that $\dim X/J_2(Y) < \infty$.*

Characterization of the previous notion is proved in [12]. It goes as follows:

Lemma 1.2.8. *Let M, N be finite dimensional spaces. If $M \oplus X \cong N \oplus Y$, then X and Y are isomorphic up to a finite dimensional subspace. Particularly, if $\dim M = \dim N$, then $X \cong Y$.*

Remark 1.2.9. *If X and Y are Hilbert spaces, then X and Y are isomorphic up to a finite dimensional subspace if and only if $X \cong Y$ or both X, Y are finite dimensional.*

One important difference between Hilbert and Banach spaces is that closed subspace of a Hilbert space is always complemented ($\mathcal{H} = M \oplus M^{\perp}$).

This is not true for the case of Banach spaces. Since we would like to prove our results by decomposing Banach spaces in an appropriate way, we shall use a well known notion of inner regular operators.

We say that an operator $T \in \mathcal{B}(X,Y)$ is inner regular if and only if there is $\widehat{T} \in \mathcal{B}(Y,X)$ such that $T\widehat{T}T = T$ holds. In that case we say $\widehat{T}$ is inner generalized inverse of T. Notice that existence of such $\widehat{T}$ does not imply its uniqueness. In the sequel, instead of "inner regular", we only write "regular" for short. We consider the appropriate spectrum: $\sigma_g(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not regular}\}$. One can prove the following characterization:

Theorem 1.2.10. *([14, Corollary 1.1.5]) $T \in \mathcal{B}(X,Y)$ is regular if and only if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are closed and complemented subspaces of X and Y, respectively.*

It is important to highlight that operators in sets $\mathcal{G}_l(X)$, $\mathcal{G}_r(X)$, $\Phi_l(X)$, $\Phi_r(X)$ are regular operators.
Moreover, it is easily proved that, following upper terminology, $T\widehat{T}$ and $\widehat{T}T$ are both projections, and so we have decompositions (see [14, Theorem 1.1.3])

$$X = \mathcal{N}(T) \oplus \mathcal{R}(\widehat{T}T), \quad Y = \mathcal{N}(T\widehat{T}) \oplus \mathcal{R}(T). \tag{1.2.2}$$

We provide one more auxiliary lemma.

Lemma 1.2.11. *([55]) Let $T_n^d(A) \in \mathcal{B}(X_1 \oplus \cdots \oplus X_n)$. Then:*

(i) $\sigma_{le}(D_1) \subseteq \sigma_{le}(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_{le}(D_k)$;

(ii) $\sigma_{re}(D_n) \subseteq \sigma_{re}(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_{re}(D_k)$;

(iii) $\sigma_{le}(D_1) \cup \sigma_{re}(D_n) \subseteq \sigma_e(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_e(D_k)$;

(iv) $\sigma_{lw}(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_{lw}(D_k)$;

(v) $\sigma_{rw}(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_{rw}(D_k)$;

(vi) $\sigma_l(D_1) \subseteq \sigma_l(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_l(D_k)$;

(vii) $\sigma_r(D_n) \subseteq \sigma_r(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_r(D_k)$;

(viii) $\sigma_l(D_1) \cup \sigma_r(D_n) \subseteq \sigma(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma(D_k)$.

Definition 1.2.12. *([8]) Calkin algebra over X is the quotient algebra $\mathcal{B}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the collection of all compact operators on X. Natural homomorphism of $\mathcal{B}(X)$ onto $\mathcal{C}(X)$ is called the Calkin homomorphism of X.*

This concept is used to prove Theorem 2.1.8.

Definition 1.2.13. *[19] We say that $T \in \mathcal{B}(X)$ has the single valued extension property (SVEP for short) at $\lambda \in \mathbb{C}$ if for every open neighborhood U of λ, the only solution of the equation $(T-u)f(u) = 0$ that is analytic on U is the constant function $f = 0$.*

We will use this concept in the third chapter.

1.3 Historical background

Block operator matrices arise in various areas of mathematics and its applications: in systems theory as Hamiltonians (see [10]), in saddle point problems in non-linear analysis (see [5]), in evolution problems as linearizations of second order Cauchy problems (see [18]), and as linear operators describing coupled systems of partial differential equations. Such systems occur widely in magnetohydrodynamics (see [39]) and quantum mechanics (see [50]). In all these applications, the spectral properties of the corresponding block operator matrices are of vital importance, as they govern for instance the time evolution and hence the stability of the underlying physical systems. One can see some other applications of this topic in [17, Chapter VIII]. Moreover, reference [52] is highly recommended as a well written treatise on this subject.

In the last few decades considerable attention has been devoted to the study of spectral properties of operator matrices, having in mind their governing importance in various areas of mathematics. One has soon realized that one way for successful work on problems arising in spectral theory, is to see operator matrices as entries of smaller blocks. Block operator matrices, and especially upper triangular operator matrices, have been extensively studied by numerous authors (see [6], [7], [12], [16], [25], [35], [36], [54], [55], references therein and many others...). The reason for this lies in the fact that if an operator T is acting on a direct sum of Banach spaces, it takes the upper triangular form under condition that certain number of those spaces is invariant for T.

Development of this topic began in the last century, and is of great importance ever since. In the beginning, authors have only considered the case of 2×2 operator matrices. Pioneering work in that direction was the article of Du and Pan from 1994 ([16]) treating the usual spectrum. Han et al. have generalized their result to Banach spaces ([25]), and Lee has proved some facts concerning the Weyl spectrum ([35]). Afterwards, Djordjević in 2002 gave some characterizations for 2×2 upper triangular operators to be Fredholm, Weyl, and Browder ([12]). After that, many authors have explored various properties of 2×2 block operators in a connection with intersection of spectra, Weyl and Browder type theorems, etc. (see for example [36], [6]).

First article treating operator matrices of dimension 3 appeared only a few years ago (2015). It is article [58], in which authors characterize invertibility of $T_3^d(A)$ on separable Hilbert spaces. Their result is ought to be generalized to arbitrary Hilbert spaces by the author of this thesis and his PhD advisor in [48]. It is interesting that investigation of this particular case ($n = 3$) had not begun before the investigation of general case $n \geq 3$.

Investigation of spectral properties of general $n \times n$ operators began no sooner than 2005, when Benhida el al. published article [4]. Next, Zguitti published article [59] investigating Drazin spectrum. Huang et al. continued his work in 2016 by investigating properties of the point, residual, and continuous spectrum of $n \times n$ matrix operators ([29]). Fredholm and Weyl spectrum of such operators have been studied by Wu and Huang in [54], [55] only a few years ago, and this thesis is concerned with generalizing their results from separable Hilbert to arbitrary Banach spaces.

We mention that there has been some interest in block operators with unbounded entries lately, see [2], [40], [43], but we shall not pursue this point any further. We also mention that it is possible to replace the setting of Hilbert spaces with C^*−algebras, see recent article [32].

Chapter 2

Case $n = 2$

In this chapter we provide statements related to different types of invertibility of $T_n^d(A)$ when $n = 2$. Historically, this is the first case studied in a connection with this topic. We shall use the notation $M_C := T_2^d(A)$, where $C = A_{12}$. In other words, let M_C be operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{B}(X_1 \oplus X_2), \tag{2.0.1}$$

where $A \in \mathcal{B}(X_1), B \in \mathcal{B}(X_2)$ are given operators and $C \in \mathcal{B}(X_2, X_1)$ is unknown. First, we consider invertibility of M_C, and afterwards we continue with Fredholm, Weyl, Browder, and Drazin invertibility of M_C.

2.1 Invertible completions of M_C

2.1.1 Separable Hilbert space setting

In this subsection we present an early result due to Du and Pan [16, Theorem 2]. In fact, [16] is the first article dealing with invertibility properties of block operator matrices in this context. We assume here that X_1 and X_2 are separable Hilbert spaces.

Theorem 2.1.1. *[16, Theorem 2] For a given pair of operators (A, B), there exist $C \in \mathcal{B}(X_2, X_1)$ such that M_C is invertible if and only if A is left invertible, B is right invertible, and $\alpha(B) = \beta(A)$.*

Previous Theorem obviosly implies the following:

Corollary 2.1.2. *For a given pair (A, B) of operators we have*

$$\bigcap_{C \in \mathcal{B}(X_2, X_1)} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \ \alpha(B-\lambda) \neq \beta(A-\lambda)\}.$$

A simple example will show that inclusion $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B)$ may be proper.

Example 2.1.3. *[16] If $\{g_i\}_{i=1}^{\infty}$ is an orthonormal basis of X_2, define an operator B_0 by*

$$\begin{cases} B_0 g_1 = 0, \\ B_0 g_i = g_{i-1}, \quad i = 2, 3, \dots \end{cases}$$

If $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis of X_1, define an operator A_0 by $A_0 f_i = f_{i+1}$, $i = 1, 2, \dots$ and an operator C_0 by $C_0 = (\cdot, g_1) f_1$ from X_2 to X_1. Then it is easy to see that $\sigma(A_0) = \sigma(B_0) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\}$. But, in this case, M_{C_0} is a unitary operator, $\sigma(M_{C_0}) \subseteq \{\lambda : \ |\lambda|\}$, so the inclusion $\sigma(M_{C_0}) \subseteq \sigma(A) \cup \sigma(B)$ is proper.

2.1.2 Non-separable Banach space setting

Next, we generalize result from subsection 2.1.1 to the setting of arbitrary Banach spaces. We specially emphasize that we do not need separability in order to prove results to follow. This subsection follows the article of Han, Lee, and Lee [25]. These authors have exploited in a very elegant way so called ghost of an index theorem due to Harte. The latter goes as follows.

Theorem 2.1.4. *[26],[27] Let $A \in \mathcal{B}(X, Y), B \in \mathcal{B}(Y, Z)$ be operators with closed range, where X, Y, Z are Banach spaces. Then the following relation holds:*

$$\mathcal{N}(A) \times \mathcal{N}(B) \times Z/\mathcal{R}(AB) \cong \mathcal{N}(AB) \times Y/\mathcal{R}(A) \times Z/\mathcal{R}(B).$$

Now, we are able to state and prove the main result of this subsection. In what follows, X_1 and X_2 are arbitrary Banach spaces.

Theorem 2.1.5. *[25] Let operators (A, B) be given. There exist $C \in \mathcal{B}(X_2, X_1)$ such that M_C is invertible if and only if:*

(i) A is left invertible;
(ii) B is right invertible;
(iii) $X/\mathcal{R}(A) \cong \mathcal{N}(B)$.

Perturbation result immediately follows.

Corollary 2.1.6. *For a given pair (A,B) of operators we have*

$$\bigcap_{C \in \mathcal{B}(X_2,X_1)} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \ \mathcal{N}(B-\lambda) \ncong X_2/\mathcal{R}(A-\lambda)\}.$$

The following two corollaries are also immediate results from Theorem 2.1.5.

Corollary 2.1.7. *For a given pair (A,B) of operators we have*

$$(\sigma(A) \cup \sigma(B)) \setminus (\sigma(A) \cap \sigma(B)) \subseteq \sigma(M_C) \subseteq \sigma(A) \cup \sigma(B)$$

for every $C \in \mathcal{B}(X_2,X_1)$.

Corollary 2.1.8. *If M_C is Fredholm and if either A or B are Fredholm, then A and B are both Fredholm with*

$$ind M_C = ind A + ind B.$$

Equality in Corollary 2.1.8 is called "the snake lemma". From this we can also see that if M_C is Weyl, and if either A or B is Fredholm, then A is Weyl if and only if B is Weyl.

2.2 Various completions of M_C

In this section we provide results related to (left, right) invertibility, Fredholm, Weyl and Drazin invertibility of M_C. Unless different is said, we assume X, Y to be arbitrary Banach spaces. Results in this section are from article [12].

2.2.1 Fredholm completions of M_C

We state the following result.

Theorem 2.2.1. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ be given and consider the following statements*

1) $M_C \in \Phi(X \oplus Y)$ *for some* $C \in \mathcal{B}(Y, X)$.

2)

2.1) $A \in \Phi_l(X)$;

2.2) $B \in \Phi_r(Y)$;

2.3) $\mathcal{N}(B)$ *and* $X/\mathcal{R}(A)$ *are isomorphic up to a finite dimensional subspace.*

Then 1) $\Longleftrightarrow$ 2).

We get the following consequence.

Corollary 2.2.2. *For given $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ the following holds:*

$$\bigcap_{C \in \mathcal{B}(X,Y)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}(A, B),$$

where

$$\mathcal{W}(A,B) = \{\lambda \in \mathbb{C} : \mathcal{N}(B-\lambda) \text{ and } X/\mathcal{R}(A-\lambda) \ are\ not\ isomorphic \\ up\ to\ a\ finite\ dimensional\ subspace\}.$$

2.2.2 Weyl completions of M_C

We consider the Weyl spectrum of M_C.

Theorem 2.2.3. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ be given and consider the statements:*

1) $M_C \in \Phi_0(X \oplus Y)$ *for some* $C \in \mathcal{B}(Y, X)$.

2) $A \in \Phi_l(X)$, $B \in \Phi_r(Y)$, $\mathcal{N}(A) \oplus \mathcal{N}(B) \cong X/\overline{\mathcal{R}(A)} \oplus Y/\overline{\mathcal{R}(B)}$.

Then 1) $\Longleftrightarrow$ 2).

As a corollary we get the following result.

Corollary 2.2.4. *For given $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ the following holds:*

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_w(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}_0(A, B),$$

where

$$\mathcal{W}_0(A,B) = \{\lambda \in \mathbb{C} : \mathcal{N}(A-\lambda) \oplus \mathcal{N}(B-\lambda) \text{ is not isomorphic to } X/\overline{R(A-\lambda)} \oplus Y/\overline{\mathcal{R}(B-\lambda)}\}.$$

2.2.3 Browder completions of M_C

We formulate the result for the Browder spectrum.

Corollary 2.2.5. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ be given. Consider the following statements:*

1) $A \in \Phi_l(X)$; $B \in \Phi_r(Y)$; $\mathcal{N}(B)$ and $X/\overline{\mathcal{R}(A)}$ are isomorphic up to a finite dimensional subspace; A and B are Drazin invertible.

2) $M_C \in \mathcal{B}(X \oplus Y)$ for some $C \in \mathcal{B}(Y,X)$.

Then 1) $\Longrightarrow$ 2).

Moreover, if $0 \notin \operatorname{acc}(\sigma(A) \cup \sigma(B))$, then 1) $\Longleftrightarrow$ 2).

We have more details concerning the perturbation of the Browder spectrum.

Theorem 2.2.6. *If $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, then*

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_b(M_C) \subset \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}(A,B) \cup \mathcal{W}_1(A,B), \tag{2.2.1}$$

where $\mathcal{W}(A,B)$ is defined in Corollary 2.2.1 and

$$\mathcal{W}_1(A,B) = \{\lambda \in \mathbb{C} : \text{ one of } A-\lambda \text{ or } B-\lambda \text{ is not Drazin invertible}\}.$$

If $\operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(B) = \emptyset$, then the equality holds in (2.2.1).

If $\sigma_a(A) = \sigma(A)$ and $\sigma_d(B) = \sigma(B)$, then the equality holds in (2.2.1).

If $\sigma(A) \cup \sigma(B)$ does not have interior points, then the equality holds in (2.2.1).

2.2.4 Right and left Fredholm completions of M_C

We formulate the following statement.

Lemma 2.2.7. *For given $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y,X)$, the following inclusion holds:*

$$\sigma_{re}(M_C) \subset \sigma_{re}(A) \cup \sigma_{re}(B).$$

Particularly, if $A \in \Phi_r(X)$ and $B \in \Phi_r(Y)$, then $M_C \in \Phi(X \oplus Y)$ for every $C \in \mathcal{B}(Y,X)$.

The main result of this subsection follows.

Theorem 2.2.8. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ be given operators. Consider the following statements:*

1) $B \in \Phi_r(X)$ and [$A \in \Phi_r(X)$ or ($\mathcal{R}(A)$ is closed and complemented in X and $X/\overline{\mathcal{R}(A)} \preceq \mathcal{N}(B)$)].

2) $M_C \in \Phi_r(X \oplus Y)$ for some $C \in \mathcal{B}(Y,X)$.

3) $B \in \Phi_r(Y)$ and [$A \in \Phi_r(X)$, or $\mathcal{R}(A)$ is not closed, or $\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$ does not hold].

Then 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3).

As a corollary we get the following result.

Corollary 2.2.9. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ be given. Then*

$$\begin{aligned}
&\sigma_{re}(B) \cup \{\lambda \in \sigma_{re}(A) : R(A-\lambda) \text{ is closed and } \mathcal{N}(B-\lambda) \prec X/\mathcal{R}(A-\lambda)\} \\
&\subset \bigcap_{C \in \mathcal{B}(Y,X)} \sigma_{re}(M_C) \\
&\subset \sigma_{re}(B) \cup \{\lambda \in \sigma_{re}(A) : \mathcal{R}(A-\lambda) \text{ is not closed and complemented}\} \\
&\qquad \cup \{\lambda \in \sigma_{re}(A) : X/\overline{\mathcal{R}(A-\lambda)} \preceq \mathcal{N}(B-\lambda) \text{ does not hold}\}.
\end{aligned}$$

Analogously, we can prove similar results for the left Fredholm spectrum.

Theorem 2.2.10. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ be given operators and consider the following statements:*

1) $A \in \Phi_l(X)$ and [$B \in \Phi_l(Y)$, or ($\mathcal{R}(B)$ and $\mathcal{N}(B)$ are closed and complemented subspaces of Y and $\mathcal{N}(B) \preceq X/\overline{\mathcal{R}(A)}$)].

2) $M_C \in \Phi_l(X \oplus Y)$ for some $C \in \mathcal{B}(Y,X)$.

3) $A \in \Phi_l(X)$ *and [*$B \in \Phi_l(Y)$*, or* $\mathcal{R}(B)$ *is not closed, or* $\mathcal{R}(A)^\circ \prec \mathcal{N}(B)'$ *does not hold)].*

Then 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3).

The following result concerning the perturbation of the left Fredholm spectrum holds.

Corollary 2.2.11. *Let* $A \in \mathcal{B}(X)$*,* $B \in \mathcal{B}(Y)$ *be given. Then*

$$\begin{aligned}
&\sigma_{le}(A) \cup \{\lambda \in \sigma_{le}(B) : \mathcal{R}(B-\lambda) \text{ is closed and } \mathcal{R}(A-\lambda)^\circ \prec \mathcal{N}(B-\lambda)'\} \\
&\subset \bigcap_{C \in \mathcal{B}(Y,X)} \sigma_{le}(M_C) \\
&\subset \sigma_{le}(A) \cup \{\lambda \in \sigma_{le}(B) : \mathcal{R}(B-\lambda) \text{ and } \mathcal{N}(B-\lambda) \\
&\qquad\qquad \text{are not closed and complemented}\} \\
&\qquad \cup \{\lambda \in \sigma_{le}(B) : \mathcal{N}(B-\lambda) \preceq X/\overline{\mathcal{R}(A-\lambda)} \text{ does not hold}\}.
\end{aligned}$$

Finally, we get the result for perturbations of the Fredholm spectrum for Hilbert space operators. This result can also be obtained from Corollary 2.2.1.

Corollary 2.2.12. *Let* $X \oplus Y$ *be the orthogonal sum of infinite dimensional Hilbert spaces. Then*

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}_2(A,B),$$

where

$$\begin{aligned}
\mathcal{W}_2(A,B) = \{\lambda \in \mathbb{C} : &\dim \mathcal{N}(B-\lambda) \neq \dim \mathcal{R}(A-\lambda)^\perp \\
&\text{and at least one of these spaces is infinite dimensional}\}.
\end{aligned}$$

2.2.5 Left and right completions of M_C

We begin with the following statement.

Lemma 2.2.13. *Let* $A \in \mathcal{B}(X)$*,* $B \in \mathcal{B}(Y)$ *be given. Then the inclusion*

$$\sigma_l(M_C) \subset \sigma_l(A) \cup \sigma_l(B)$$

holds for every $C \in \mathcal{B}(Y,X)$. *Particularly, if* A, B *are left invertible, then* M_C *is left invertible for every* $C \in \mathcal{B}(Y,X)$.

For the left invertibility of an operator matrix we can state the following result.

Theorem 2.2.14. *Let* $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ *be given. Consider the following statements:*

1) $A \in \mathcal{G}_l(X)$, $\mathcal{N}(B) \preceq X/\overline{\mathcal{R}(A)}$ *and* B *is inner regular.*

2) $M_C \in \mathcal{G}_l(X \oplus Y)$ *for some* $C \in \mathcal{B}(Y,X)$.

3) $A \in \mathcal{G}_l(X)$ *and* $X/\overline{\mathcal{R}(A)} \prec \mathcal{N}(B)$ *does not hold.*

Then 1) $\Longrightarrow$ 2).

Moreover, if X, Y *are infinite dimensional Hilbert spaces, and* $Z = X \oplus Y$ *is the orthogonal sum, then* 2) $\Longrightarrow$ 3).

As a corollary we get the following result.

Corollary 2.2.15. *Let* $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ *be given. Then the following inclusion holds:*

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_l(M_C) \subset \sigma_l(A) \cup \sigma_g(B) \cup \{\lambda \in \mathbb{C} : \mathcal{N}(B-\lambda) \preceq X/\overline{\mathcal{R}(A-\lambda)} \text{ does not hold}\}.$$

If $X \oplus Y$ *is the orthogonal sum of infinite dimensional Hilbert spaces* X *and* Y, *then*

$$\sigma_l(A) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{R}(A)^\perp < \dim \mathcal{N}(B-\lambda)\} \subset \bigcap_{C \in \mathcal{B}(Y,X)} \sigma_l(M_C).$$

Analogously, we can prove a similar result concerning the right spectrum and right invertibility of M_C.

Theorem 2.2.16. *Let* $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ *be given operators, and consider statements:*

1) $B \in \mathcal{G}_r(Y)$, $X/\overline{\mathcal{R}(A)} \preceq \mathcal{N}(B)$, A *is inner regular.*

2) $M_C \in \mathcal{G}_r(X \oplus Y)$ *for some* $C \in \mathcal{B}(Y,X)$.

3) $B \in \mathcal{G}_r(Y)$, *and* $\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$ *does not hold.*

Then 1) $\Longrightarrow$ 2).

If $X \oplus Y$ *is the orthogonal sum of infinite dimensional Hilbert spaces, then* 2) $\Longrightarrow$ 3).

As a corollary we get the following result.

Corollary 2.2.17. *For given* $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ *the following inclusion holds:*

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_r(M_C) \subset \sigma_r(B) \cup \sigma_g(A) \cup \{\lambda \in \mathbb{C} : X/\overline{\mathcal{R}(A-\lambda)} \preceq \mathcal{N}(B-\lambda) \text{ does not hold}\}.$$

Moreover, if $X \oplus Y$ *is the orthogonal sum of infinite dimensional Hilbert spaces, then*

$$\sigma_r(B) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(B-\lambda) < \dim \mathcal{R}(A-\lambda)^{\perp}\} \subset \bigcap_{C \in \mathcal{B}(K,H)} \sigma_r(M_C).$$

As a corollary, we get the following main result.

Corollary 2.2.18. *Let* $X \oplus Y$ *be the orthogonal sum of infinite dimensional Hilbert spaces. For given* $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ *the following equality holds:*

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(B-\lambda) \neq \dim \mathcal{R}(A-\lambda)^{\perp}\}.$$

Chapter 3

Case $n = 3$

In this chapter we provide statements related to different types of invertibility of $T_n^d(A)$ when $n = 3$. Historically, this is the case that has not been studied until a few years ago. We shall use the notation $M_{D,E,F} := T_3^d(A)$, where $D = A_{12}, E = A_{13}, F = A_{23}$. In other words, let $M_{D,E,F}$ be operator matrix

$$M_{D,E,F} = \begin{bmatrix} A & D & E \\ 0 & B & F \\ 0 & 0 & C \end{bmatrix} \in \mathcal{B}(X_1 \oplus X_2 \oplus X_3), \tag{3.0.1}$$

where $A \in \mathcal{B}(X_1), B \in \mathcal{B}(X_2), C \in \mathcal{B}(X_3)$ are known operators, and $D \in \mathcal{B}(X_2, X_1)$, $E \in \mathcal{B}(X_3, X_1), F \in \mathcal{B}(X_3, X_2)$ are unknown. First, we consider invertibility of $M_{D,E,F}$ if underlying spaces are separable Hilbert, and afterwards we give an extension to arbitrary Banach spaces case.

3.1 Separable Hilbert space setting

In this short section assume that X_1, X_2, X_3 are separable Hilbert spaces. This section is based on article [58]. We start with an obvious auxiliary result.

Lemma 3.1.1. *Given triple (A, B, C), $M_{D,E,F} - \lambda$ is one - one for all $D \in \mathcal{B}(X_2, X_1), E \in \mathcal{B}(X_3, X_1), F \in \mathcal{B}(X_3, X_2)$ if and only if $A - \lambda, B - \lambda, C - \lambda$ are all one - one.*

We give the main theorem in the perturbation form.

Theorem 3.1.2. *Let triple* (A, B, C) *be given. Then*

$$\begin{aligned}\bigcap_{D,E,F} \sigma(M_{D,E,F}) = \sigma_l(A) \cup \sigma_\delta(B) \cup \{\lambda \in \sigma_p(B) : \ \alpha(B-\lambda) > \beta(A-\lambda)\} \\ \cup\{\lambda \in \sigma_\delta(B) : \ \beta(B-\lambda) > \alpha(C-\lambda)\} \\ \cup\{\lambda \in \sigma_m(B) : \ \min\{\alpha(C-\lambda), \beta(A-\lambda)\} < \infty\} \\ \cup\{\lambda \in \mathbb{C} : \ \alpha(B-\lambda) + \alpha(C-\lambda) \neq \beta(A-\lambda) + \beta(B-\lambda)\}.\end{aligned} \tag{3.1.1}$$

Proof. See [58]. □

Now, the following extension of Theorem 2.1.1 to matrix dimension 3 follows at once.

Corollary 3.1.3. *Let triple* (A, B, C) *be given. There exist* $D \in \mathcal{B}(X_2, X_1)$, $E \in \mathcal{B}(X_3, X_1), F \in \mathcal{B}(X_3, X_2)$ *so that* $M_{D,E,F}$ *is invertible if and only if:*
(i) A *is left invertible;*
(ii) C *is right invertible;*
(iii) $\begin{cases} \alpha(B) \leq \beta(A), \ \beta(B) \leq \alpha(C), \ \alpha(B) + \alpha(C) = \beta(A) + \beta(B), & \mathcal{R}(B) \text{ closed}, \\ \alpha(C) = \beta(A) = \infty, & \mathcal{R}(B) \text{ not closed}. \end{cases}$

3.2 Non-separable Banach space setting

Assume now that X_1, X_2, X_3 are arbitrary Banach spaces. This section is based on article [48]. In [25] authors exploited decomposition properties of inner regular operators (see (0.2) in [25]), and we pursue such an idea. Quick reminder: the class of inner regular operators consists of operators $T \in \mathcal{B}(X, Y)$ that can be expressed in the form $T = TT'T$ for some $T' \in \mathcal{B}(Y, X)$. It is known that $T \in \mathcal{B}(X, Y)$ is inner regular if and only if its kernel and range are closed and complemented subspaces [14, 1.1.5. Corollary].

In the sequel, we will find a huge benefit of the following matrix decomposition:

$$M_{D,E,F} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & C \end{pmatrix}\begin{pmatrix} I & 0 & E \\ 0 & I & F \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & D & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{3.2.1}$$

Notice that the second and the fourth factor in (3.2.1) are invertible matrices for all $D \in \mathcal{B}(Y,X), E \in \mathcal{B}(Z,X), F \in \mathcal{B}(Z,Y)$.

The following two lemmas will be used several times in proofs of our results.

Lemma 3.2.1. *Let $S,T \in \mathcal{B}(X)$. If T is invertible, then:*
(a) $\mathcal{R}(TS) \cong \mathcal{R}(S)$ and $\mathcal{R}(ST) = \mathcal{R}(S)$;
(b) $\mathcal{N}(ST) \cong \mathcal{N}(S)$ and $\mathcal{N}(TS) = \mathcal{N}(S)$.

Lemma 3.2.2. *Consider $M_{D,E,F}$ and its diagonal operators A,B,C. If any three of those four operators are invertible for all $D \in \mathcal{B}(Y,X), E \in \mathcal{B}(Z,X), F \in \mathcal{B}(Z,Y)$, then the fourth is invertible as well.*

Proof. This is obvious from (3.2.1). □

The following lemma is well known in the literature (see for example [12, Lemma 2.3]).

Lemma 3.2.3. *If X,Y,Z are Banach spaces then*

$$X \times Y \cong X \times Z \ \wedge \ \dim X < \infty \Rightarrow \ Y \cong Z.$$

First we prove the following theorem which will yield our main result as a consequence. We employ the notion of embedded Banach spaces [12]. In this section we provide conditions for invertibility of $M_{D,E,F}$ We will make use of the following definition introduced in [12]: a Banach space X can be embedded in a Banach space Y, denoted by $X \preceq Y$, provided that there exists a left invertible operator $A \in \mathcal{B}(X,Y)$. Then, it is obvious that $X \cong Y$ if and only if $X \preceq Y$ and $Y \preceq X$. If X,Y are Hilbert spaces and $\dim_h X$ is the orthogonal dimension of X, then $X \preceq Y$ if and only if $\dim_h X \leq \dim_h Y$.

If U is a closed subspace of a Banach space V, we will use the following notation for the quotient space: $\dfrac{V}{U} = V/U$.

We prove the following auxilliary result.

Lemma 3.2.4. *Let X be a Banach space and let X_1,X_2 be closed subspaces of*

X such that $X = X_1 \oplus X_2$. If $T \in \mathcal{B}(X)$ such that

$$\mathcal{R}(T) = \left\{ \begin{pmatrix} T_1 u_1 \\ 0 \end{pmatrix} \in \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} : u_1 \in \mathcal{D}(T_1) \right\}$$

for some bounded linear operator T_1 with domain $\mathcal{D}(T_1)$, then $X_2 \preceq X/\mathcal{R}(T)$.

Proof. Notice that

$$X/\mathcal{R}(T) = \left\{ \begin{pmatrix} x_1 + T_1 u_1 \\ x_2 \end{pmatrix} : x_1 \in X_1, x_2 \in X_2, u_1 \in \mathcal{D}(T_1) \right\}.$$

For $x_1 \in X_1$ and $x_2 \in X_2$, define $K : X_2 \to X/\mathcal{R}(T)$ and $K' : X/\mathcal{R}(T) \to X_2$ as follows:

$$Kx_2 = \left\{ \begin{pmatrix} T_1 u_1 \\ x_2 \end{pmatrix} : u_1 \in \mathcal{D}(T_1) \right\}$$

and

$$K' \left\{ \begin{pmatrix} x_1 + T_1 u_1 \\ x_2 \end{pmatrix} : u_1 \in \mathcal{D}(T_1) \right\} = x_2.$$

We see that $K'K = I_{X_2}$ (and KK' is not necessarily equal to $I_{X/\mathcal{R}(T)}$). K and K' are obviously continuous. □

We prove the following theorem.

Theorem 3.2.5. *Let X, Y, Z be Banach spaces, and let $B \in \mathcal{B}(Y)$ be regular, $A \in \mathcal{B}(X)$ and $C \in \mathcal{B}(Z)$. Consider the following statements:*

1) a) *A is left invertible and C is right invertible;*

   b) *$\mathcal{N}(B) \preceq X/\mathcal{R}(A)$ and $Y/\mathcal{R}(B) \preceq \mathcal{N}(C)$;*

   c) $\dfrac{X/\mathcal{R}(A)}{\mathcal{R}(J_1)} \cong \dfrac{\mathcal{N}(C)}{\mathcal{R}(J_2)}$ *for left invertible operators $J_1 : \mathcal{N}(B) \to X/\mathcal{R}(A)$ and $J_2 : Y/\mathcal{R}(B) \to \mathcal{N}(C)$ which realize relations $\preceq$ in* 1) b)*.*

2) *There exist $D \in \mathcal{B}(Y,X), E \in \mathcal{B}(Z,X), F \in \mathcal{B}(Z,Y)$ such that $M_{D,E,F}$ is invertible.*

3) a) *A is left invertible and C is right invertible;*

   b) *$\mathcal{N}(B) \preceq X/\mathcal{R}(A)$ and $Y/\mathcal{R}(B) \preceq \mathcal{N}(C)$;*

Then 1) $\Rightarrow$ 2) $\Rightarrow$ 3).

Proof. 1) $\Longrightarrow$ 2): Suppose that 1) holds. By 1) a) there exist closed subspaces: X_1 of X, Y_1 and Y_2 of Y, and Z_1 of Z such that:

$$X = X_1 \oplus \mathcal{R}(A), \quad Y = Y_1 \oplus \mathcal{R}(B) = Y_2 \oplus \mathcal{N}(B), \quad Z = Z_1 \oplus \mathcal{N}(C).$$

Consequently,

$$X/\mathcal{R}(A) \cong X_1, \quad Y/\mathcal{R}(B) \cong Y_1, \quad Y/\mathcal{N}(B) \cong Y_2, \quad Z/\mathcal{N}(C) \cong Z_1.$$

The condition 1) b) implies the existence of left invertible operators $J_1 : \mathcal{N}(B) \to X_1$ and $J_2 : Y_1 \to \mathcal{N}(C)$. Consider their invertible reductions $J_1 : \mathcal{N}(B) \to \mathcal{R}(J_1)$ and $J_2 : Y_1 \to \mathcal{R}(J_2)$, which are denoted by the same symbols. There exist closed subspaces $\mathcal{R}(J_1)'$ and $\mathcal{R}(J_2)'$ such that

$$X_1 = \mathcal{R}(J_1)' \oplus \mathcal{R}(J_1), \quad \mathcal{N}(C) = \mathcal{R}(J_2)' \oplus \mathcal{R}(J_2).$$

By 1) c) there exists an isomorphism $J : \mathcal{R}(J_2)' \to \mathcal{R}(J_1)'$.

Define

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & J_1 \end{pmatrix} : \begin{pmatrix} Y_2 \\ \mathcal{N}(B) \end{pmatrix} = Y \to X = \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(J_1)' \\ \mathcal{R}(J_1) \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & J_2^{-1} \end{pmatrix} : \begin{pmatrix} Z_1 \\ \mathcal{R}(J_2)' \\ \mathcal{R}(J_2) \end{pmatrix} = Z \to Y = \begin{pmatrix} \mathcal{R}(B) \\ Y_1 \end{pmatrix},$$

and

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} Z_1 \\ \mathcal{R}(J_2)' \\ \mathcal{R}(J_2) \end{pmatrix} = Z \to X = \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(J_1)' \\ \mathcal{R}(J_1) \end{pmatrix}.$$

Since J_1, J_2^{-1} and J are isomorphisms between appropriate subspaces, it is obvious that $D \in \mathcal{B}(Y, X), E \in \mathcal{B}(Z, X), F \in \mathcal{B}(Z, Y)$.

To prove that $M_{D,E,F}$ is invertible, notice the following. We have

$$\begin{aligned}\mathcal{R}(M_{D,E,F}) &= \mathcal{R}\begin{pmatrix}A\\0\\0\end{pmatrix}+\mathcal{R}\begin{pmatrix}D\\B\\0\end{pmatrix}+\mathcal{R}\begin{pmatrix}E\\F\\C\end{pmatrix}\\ &= \Big(\mathcal{R}(A)+\mathcal{R}(J_1)+\mathcal{R}(J_1)'\Big)+\Big(\mathcal{R}(B)+Y_1\Big)+\mathcal{R}(C)=X\oplus Y\oplus Z,\end{aligned}$$

so $M_{D,E,F}$ is onto.

Moreover, if $w=\begin{pmatrix}x\\y\\z\end{pmatrix}\in X\oplus Y\oplus Z$ and $M_{D,E,F}w=0$, we have

$$Ax+Dy+Ez=0,\quad By+Fz=0,\quad Cz=0.$$

From $Cz=0$ we get $z\in\mathcal{N}(C)=\mathcal{R}(J_2)'\oplus\mathcal{R}(J_2)$. We know that $By\in\mathcal{R}(B)$ and $Fz\in Y_1$. Thus, from $By+Fz=0$ we get $By=0$ and $Fz=0$. Hence, $y\in\mathcal{N}(B)$ and $z\in\mathcal{R}(J_2)'$. We have $Ax\in\mathcal{R}(A)$, $Dy\in D(\mathcal{N}(B))=J_1(\mathcal{N}(B))=\mathcal{R}(J_1)$ and $Ez\in E(\mathcal{R}(J_2)')=J(\mathcal{R}(J_2)')=\mathcal{R}(J_1)'$. Hence, from $Ax+Dy+Ez=0$ we conclude $Ax=0$, $Dy=J_1y=0$ and $Jz=0$, implying that $x=0$, $y=0$ and $z=0$. Thus, $M_{D,E,F}$ is one-to-one.

2) $\Longrightarrow$ 3): Assume that $M_{D,E,F}$ is invertible for some D, E and F defined on appropriate domains. Consider factorization (3.2.1) to conclude that A is left invertible and C is right invertible, thus the condition 3) a) follows.

Denote the product of the first two factors in (3.2.1) by S, the product of the last three factors by T, i.e.

$$S=\begin{pmatrix}I&0&E\\0&I&F\\0&0&C\end{pmatrix},\quad T=\begin{pmatrix}A&D&0\\0&B&0\\0&0&I\end{pmatrix}.$$

Now, we apply Theorem 2.1.4 and obtain

$$\mathcal{N}(S)\times\mathcal{N}(T)\times\begin{pmatrix}X\\Y\\Z\end{pmatrix}/\mathcal{R}(ST)\cong\mathcal{N}(ST)\times\begin{pmatrix}X\\Y\\Z\end{pmatrix}/\mathcal{R}(S)\times\begin{pmatrix}X\\Y\\Z\end{pmatrix}/\mathcal{R}(T).$$

Since $M_{D,E,F}=ST$ is invertible, we know that S is right invertible and T is left invertible. Thus, using Lemma 3.2.1, we have

$$\mathcal{N}(C) \cong \mathcal{N}(S) \cong \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} / \mathcal{R}(T). \tag{3.2.2}$$

Since (again) $Y = Y_1 \oplus \mathcal{R}(B) = Y_2 \oplus \mathcal{N}(B)$ and $X = X_1 \oplus \mathcal{R}(A)$, we have

$$B = \begin{pmatrix} 0 & 0 \\ B_1 & 0 \end{pmatrix} : \begin{pmatrix} Y_2 \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} Y_1 \\ \mathcal{R}(B) \end{pmatrix} \qquad (B_1 : Y_2 \to \mathcal{R}(B) \text{ is invertible})$$

and

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} : \begin{pmatrix} Y_2 \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} X_1 \\ \mathcal{R}(A) \end{pmatrix}.$$

Now,

$$\begin{aligned} \mathcal{R}(T) &= \mathcal{R}\begin{pmatrix} A & D & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{pmatrix} \\ &= \left\{ \begin{pmatrix} D_{11}u + D_{12}v \\ Ax + D_{21}u + D_{22}v \\ 0 \\ B_1 u \\ z \end{pmatrix} \in \begin{pmatrix} X_1 \\ \mathcal{R}(A) \\ Y_1 \\ \mathcal{R}(B) \\ Z \end{pmatrix} : x \in X, u \in Y_2, v \in \mathcal{N}(B), z \in Z \right\}. \end{aligned}$$

From Lemma 3.2.4 we know that

$$Y/\mathcal{R}(B) \cong Y_1 \preceq \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} / \mathcal{R}(T) \cong \mathcal{N}(C).$$

If we denote the product of the first three factors in (3.2.1) by S', and the product of the last two by T', i.e.

$$S' = \begin{pmatrix} I & 0 & E \\ 0 & B & F \\ 0 & 0 & C \end{pmatrix}, \quad T' = \begin{pmatrix} A & D & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

we know that $M_{D,E,F} = S'T'$ is invertible, S' is right invertible and T' is left invertible. Thus,

$$\mathcal{N}(S') \times \mathcal{N}(T') \cong \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} / \mathcal{R}(S') \times \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} / \mathcal{R}(T').$$

and consequently

$$\mathcal{N}(S') \cong \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} / \mathcal{R}(T') \tag{3.2.3}$$

If $x \in X$, $y \in Y$ and $z \in Z$, notice that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{N}(S')$ if and only if

$$x + Ez = 0, \ By + Fz = 0, \ Cz = 0.$$

For $x \in X$, $y \in Y$ and $z \in Z$ define $L : \mathcal{N}(B) \to \mathcal{N}(S')$ and $L' : \mathcal{N}(S') \to \mathcal{N}(B)$ as

$$\mathcal{N}(B) \ni y \mapsto Ly = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \quad \mathcal{N}(S') \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto L' \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y.$$

L and L' are obviously continuous. We see that $L'L = I_{\mathcal{N}(B)}$ (and LL' is not necessarily equal to $I_{\mathcal{N}(S')}$). Thus, $\mathcal{N}(B) \preceq \mathcal{N}(S')$.

We have

$$\mathcal{R}(T') = \left\{ \begin{pmatrix} Au + Dv \\ v \\ w \end{pmatrix} : u \in X, v \in Y, w \in Z \right\}$$

and

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \left\{ \begin{pmatrix} x + Au + Dv \\ y + v \\ z + w \end{pmatrix} : x, u \in X, y, v \in Y, y, w \in Z \right\}.$$

For $x \in X, y \in Y, z \in Z$ define $M : \begin{pmatrix} X \\ Y \\ Y \end{pmatrix} / \mathcal{R}(T') \to Y/\mathcal{R}(A)$ and $M' : X/\mathcal{R}(A) \to$

$\begin{pmatrix} X \\ Y \\ Y \end{pmatrix}/\mathcal{R}(T')$ as follows:

$$M\left\{\begin{pmatrix} x+Au+Dv \\ y+v \\ y+w \end{pmatrix} : u\in X, v\in Y, w\in Z\right\} = \{x+Au : u\in X\},$$

$$M'\{x+Au : u\in X\} = \left\{\begin{pmatrix} x+Au \\ v \\ w \end{pmatrix} : u\in X, v\in Y, w\in Z\right\}.$$

Then M, M' are continuous, $M'M = I_{(X\oplus Y\oplus Z)/\mathcal{R}(T')}$, but $MM' = I_{X/\mathcal{R}(A)}$ does not necessarily hold. Thus, $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}/\mathcal{R}(T') \preceq X/\mathcal{R}(A)$.

Finaly, we obtain $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$. □

We prove the following result for Hilbert space operators.

Theorem 3.2.6. *Let X, Y, Z be Hilbert spaces, $A \in \mathcal{B}(X)$ is left invertible, $B \in \mathcal{B}(Y)$ is inner regular, $C \in \mathcal{B}(Z)$ is right invertible,*

$$\dim_h \mathcal{N}(B) \le \dim_h X/\mathcal{R}(A) \quad and \quad \dim_h Y/\mathcal{R}(B) \le \dim_h \mathcal{N}(C).$$

Then the following statements are equivalent;

1) $\dfrac{X/\mathcal{R}(A)}{\mathcal{R}(J_1)} \cong \dfrac{\mathcal{N}(C)}{\mathcal{R}(J_2)}$ *for some left invertible operators $J_1 : \mathcal{N}(B) \to X/\mathcal{R}(A)$ and $J_2 : Y/\mathcal{R}(B) \to \mathcal{N}(C)$.*

2) $\mathcal{N}(B) \times \mathcal{N}(C) \cong X/\mathcal{R}(A) \times Y/\mathcal{R}(B)$.

Proof. It is enough to prove implication 2) $\Longrightarrow$ 1). Suppose that 2) holds. Left invertible operators $J_1 : \mathcal{N}(B) \to X/\mathcal{R}(A)$ and $J_2 : Y/\mathcal{R}(B) \to \mathcal{N}(C)$ exist by the main assumption of this theorem. We have to prove that J_1 and J_2 can be adjusted such that 1) is also satisfied.

We consider several cases and subcases.

Case I. $\dim_h \mathcal{N}(B) < \dim_h X/\mathcal{R}(A)$ and $\dim_h \mathcal{N}(C) \le \dim_h X/\mathcal{R}(A)$.

Subcase I.1. $X/\mathcal{R}(A)$ is infinite dimensional.

Since

$$\dim_h Y/\mathcal{R}(B) \leq \dim_h \mathcal{N}(C) \leq \dim_h X/\mathcal{R}(A),$$

by 2) it follows that $\dim_h \mathcal{N}(C) = \dim_h X/\mathcal{R}(A)$. Then

$$\dim_h J_1(\mathcal{N}(B)) = \dim_h \mathcal{N}(B) < \dim_h X/\mathcal{R}(A).$$

Thus

$$\frac{X/\mathcal{R}(A)}{\mathcal{R}(J_1)} \cong X/\mathcal{R}(A).$$

Since

$$\dim_h \mathcal{R}(J_2) = \dim_h Y/\mathcal{R}(B) \leq \dim_h X/\mathcal{R}(A) = \dim_h \mathcal{N}(C),$$

we conclude that J_2 can be adjusted such that

$$\frac{\mathcal{N}(C)}{\mathcal{R}(J_2)} \cong \mathcal{N}(C) \cong X/\mathcal{R}(A) \cong \frac{X/\mathcal{R}(A)}{\mathcal{R}(J_1)}.$$

Thus, 1) holds.

Subcase I.2. $X/\mathcal{R}(A)$ is finite dimensional.

Let

$$k = \dim_h \mathcal{N}(B), \quad l = \dim_h \mathcal{N}(C), \quad m = \dim_h X/\mathcal{R}(A), \quad n = \dim Y/\mathcal{R}(B).$$

We have

$$k < m, \quad n \leq l \leq m, \quad k + l = m + n,$$

all these quantities are finite, and we get

$$0 < m - k = l - n,$$

which is 1) in finite dimensions.

Case II. $\dim_h \mathcal{N}(B) < \dim_h X/\mathcal{R}(A) < \dim_h \mathcal{N}(C)$.

Subcase II.1. $\mathcal{N}(C)$ is infinite dimensional.

We get that

$$\mathcal{N}(B)\times\mathcal{N}(C)\cong\mathcal{N}(C) \text{ and } \dim_h Y/\mathcal{R}(B)=\dim_h\mathcal{N}(C).$$

Since $\dim_h X/\mathcal{R}(A)<\dim_h\mathcal{N}(C)$, for every left invertible $J_1:\mathcal{N}(B)\to X/\mathcal{R}(A)$ is is possible to adjust some left invertible $J_2:Y/\mathcal{R}(B)\to\mathcal{N}(C)$ such that

$$\frac{X/\mathcal{R}(A)}{\mathcal{R}(J_1)}\cong\frac{\mathcal{N}(C)}{\mathcal{R}(J_2)}$$

holds.

Subcase II.2. $\mathcal{N}(C)$ is finite dimensional.

Keep k,l,m,n the same as in Subcase I.2. We get

$$k<m<l,\quad n\leq l,\quad l+l=m+n,$$

implying that all these quantities are finite and

$$0<m-k=l-n,$$

which is again 1) in finite dimensions.

Case III. $\dim_h\mathcal{N}(B)=\dim_h X/\mathcal{R}(A)$ and $\dim_h\mathcal{N}(C)\leq\dim_h X/\mathcal{R}(A)$.

Subcase III.1. $X/\mathcal{R}(A)$ is infinite dimensional.

From

$$\begin{aligned}\dim_h J_2(Y/\mathcal{R}(B))=\dim_h Y/\mathcal{R}(B)&\leq\dim_h\mathcal{N}(C)\leq\dim_h X/\mathcal{R}(A)\\&=\dim_h\mathcal{N}(B)\end{aligned}$$

we get that for every left invertible $J_2:Y/\mathcal{R}(B)\to\mathcal{N}(C)$ we can find a left invertible $J_1:\mathcal{N}(B)\to X/\mathcal{R}(A)$ such that

$$\frac{X/\mathcal{R}(A)}{\mathcal{R}(J_1)}\cong\frac{\mathcal{N}(C)}{\mathcal{R}(J_2)}.$$

Subcase III.2. $X/\mathcal{R}(A)$ is finite dimensional.

This is proved in the same way as in the previous finite dimensional subcases.

Case IV. $\dim_h \mathcal{N}(B) = \dim_h X/\mathcal{R}(A) < \dim_h \mathcal{N}(C)$.

Subcase IV.1. $\mathcal{N}(C)$ is infinite dimensional.

We get

$$\mathcal{N}(C) \cong \mathcal{N}(B) \times \mathcal{N}(C) \cong X/\mathcal{R}(A) \times Y/\mathcal{R}(B),$$

implying $\mathcal{N}(C) \cong R/\mathcal{R}(B)$. Thus, for every left invertible $J_1 : \mathcal{N}(B) \to X/\mathcal{R}(A)$ we can adjust a left invertible $J_2 : Y/\mathcal{R}(B) \to \mathcal{N}(C)$ such that

$$\frac{X/\mathcal{R}(A)}{\mathcal{R}(J_1)} \cong \frac{\mathcal{N}(C)}{\mathcal{R}(J_2)}.$$

Subcase IV.2. $\mathcal{N}(C)$ is finite dimensional.

Again, this is a routine. $\square$

Chapter 4

General case $n \geq 3$

In this chapter we provide statements related to different types of invertibility of $T_n^d(A)$ when $n \geq 3$ is arbitrary. This is the case that the present author has studied the most. Let us remind ourselves, if $D_i \in \mathcal{B}(X_i),\ 1 \leq i \leq n$ are given operators,

$$T_n^d(A) = \begin{bmatrix} D_1 & A_{12} & A_{13} & \dots & A_{1,n-1} & A_{1n} \\ 0 & D_2 & A_{23} & \dots & A_{2,n-1} & A_{2n} \\ 0 & 0 & D_3 & \dots & A_{3,n-1} & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & D_{n-1} & A_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & D_n \end{bmatrix} \in \mathcal{B}(X_1 \oplus X_2 \oplus \cdots \oplus X_n), \quad (4.0.1)$$

where $A := (A_{12},\ A_{13}, \dots,\ A_{ij}, \dots,\ A_{n-1,n})$ is an operator tuple consisting of unknown variables $A_{ij} \in \mathcal{B}(X_j, X_i)$, $1 \leq i < j \leq n,\ n \geq 2$. For convenience, we denote by $\mathcal{B}_n$ the collection of all such tuples A.

In this chapter we aim to generalize results of two preceding chapters to the case where operator matrix $T_n^d(A)$ is with arbitrary dimension $n \geq 3$. Our method strongly relies on results from references [54], [55]. We provide results related to (left, right) spectrum, (left, right) Fredholm and left/right Weyl spectrum.

4.1 Invertible completions of $T_n^d(A)$

If Y is a complemented subspace of X, we denote a topological complement of Y in X by Y_1. Specially, in the rest of this section, if $Y = \mathcal{N}(T)$ ($Y = \overline{\mathcal{R}(T)}$), we use $\mathcal{M}(T)$ ($\mathcal{K}(T)$) to denote its topological complement in X.

We start with a result which deals with left invertibility of $T_n^d(A)$.

Theorem 4.1.1. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. Assume that D_s, $2 \leq s \leq n-1$, are regular operators. Consider the following statements:*

(i) (a) $D_1 \in \mathcal{G}_l(X_1)$;
(b) D_n is regular and $\mathcal{N}(D_i) \preceq \mathcal{K}(D_{i-1})$ for every $2 \leq i \leq n$;

(ii) There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \mathcal{G}_l(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_1 \in \mathcal{G}_l(X_1)$;
(b) $\bigoplus_{s=1}^{i-1} \mathcal{K}(D_s) \prec \mathcal{N}(D_i)$ does not hold for $2 \leq i \leq n$.

Then $(i) \Rightarrow (ii)$.
If $X_1, ..., X_n$ are infinite dimensional Hilbert spaces, then $(ii) \Rightarrow (iii)$.

Proof. $(i) \Rightarrow (ii)$

In this case it holds $\alpha(D_1) = 0$, $\mathcal{R}(D_s)$ is closed for all $1 \leq s \leq n$ and $\mathcal{N}(D_i) \preceq \mathcal{K}(D_{i-1})$ for every $2 \leq i \leq n$. By Lemma 1.1.1 we need to find $A \in \mathcal{B}_n$ such that $\alpha(T_n^d(A)) = 0$ and $\mathcal{R}(T_n^d(A))$ is closed and complemented. We choose $A = (A_{ij})_{1 \leq i < j \leq n}$ so that $A_{ij} = 0$ if $j - i \neq 1$, that is we place all nonzero operators of tuple A on the superdiagonal. It remains to define A_{ij} for $j = i+1$, $1 \leq i < n$. First notice that $A_{i,i+1} : X_{i+1} \to X_i$. Since all of diagonal entries have closed range, we know that $X_{i+1} = \mathcal{N}(D_{i+1}) \oplus \mathcal{M}(D_{i+1})$, $X_i = \mathcal{K}(D_i) \oplus \mathcal{R}(D_i)$, and we have $\mathcal{N}(D_{i+1}) \preceq \mathcal{K}(D_i)$. It follows that there is a left invertible operator $J_i : \mathcal{N}(D_{i+1}) \to \mathcal{K}(D_i)$. We put $A_{i,i+1} = \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix}$: $\begin{bmatrix} \mathcal{N}(D_{i+1}) \\ \mathcal{M}(D_{i+1}) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}(D_i) \\ \mathcal{R}(D_i) \end{bmatrix}$, and we implement this procedure for all $1 \leq i \leq n-1$. Notice that $\mathcal{R}(A_{i,i+1})$ is contained in a subspace which is complementary to $\mathcal{R}(D_i)$ for each $1 \leq i \leq n-1$.

Now we have chosen our A, we show that $\mathcal{N}(T_n^d(A)) \cong \mathcal{N}(D_1)$, implying $\alpha(T_n^d(A)) = \alpha(D_1) = 0$. Let us put $T_n^d(A)x = 0$, where $x = x_1 + \cdots + x_n \in X_1 \oplus \cdots \oplus X_n$. The previous equality is then equivalent to the following system of equations

$$\begin{bmatrix} D_1x_1 + A_{12}x_2 \\ D_2x_2 + A_{23}x_3 \\ \vdots \\ D_{n-1}x_{n-1} + A_{n-1,n}x_n \\ D_nx_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Last equation gives $x_n \in \mathcal{N}(D_n)$. Since $\mathcal{R}(A_{s,s+1})$ is contained in a subspace which is complementary to $\mathcal{R}(D_s)$ for all $1 \leq s \leq n-1$, we have $D_sx_s = A_{s,s+1}x_{x+1} = 0$ for all $1 \leq s \leq n-1$. That is, $x_i \in \mathcal{N}(D_i)$ for every $1 \leq i \leq n$, and $J_sx_{s+1} = 0$ for every $1 \leq s \leq n-1$. Due to left invertibility of J_s we get $x_s = 0$ for $2 \leq s \leq n$, which proves the claim. Therefore, $\alpha(T_n^d(A)) = \alpha(D_1) = 0$.

Next, we show that $\mathcal{R}(T_n^d(A))$ is closed and complemented. Left invertibility of J_i's implies the existence of closed subspaces U_i of $\mathcal{K}(D_i)$ such that $\mathcal{K}(D_i) = \mathcal{R}(J_i) \oplus U_i$, $1 \leq i \leq n-1$ (Lemma 1.1.1). It means that $X_1 \oplus X_2 \oplus \cdots \oplus X_n = \mathcal{R}(D_1) \oplus \mathcal{R}(J_1) \oplus U_1 \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(J_2) \oplus U_2 \oplus \cdots \oplus \mathcal{R}(D_{n-1}) \oplus \mathcal{R}(J_{n-1}) \oplus U_{n-1} \oplus \mathcal{R}(D_n) \oplus \mathcal{K}(D_n)$. It is not hard to see that $\mathcal{R}(T_n^d(A)) = \mathcal{R}(D_1) \oplus \mathcal{R}(J_1) \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(J_2) \oplus \cdots \oplus \mathcal{R}(D_{n-1}) \oplus \mathcal{R}(J_{n-1}) \oplus \mathcal{R}(D_n)$. Comparing these equalities, one easily sees that $\mathcal{R}(T_n^d(A))$ is closed and complemented (this follows from [42, Theorem 3.6] as well).

$(ii) \Rightarrow (iii)$

Assume that $T_n^d(A)$ is left invertible and $X_1, ..., X_n$ are Hilbert spaces. Then $D_1 \in \mathcal{G}_l(X_1)$ (Lemma 1.2.11). Assume that $(iii)(b)$ fails. Then there exists some $j \in \{2, ..., n\}$ such that $\alpha(D_j) > \sum_{s=1}^{j-1} \beta(D_s)$.

We use a method similar to that in [54],[55]. We know that for each $A \in \mathcal{B}_n$, the operator matrix $T_n^d(A)$ as an operator from $X_1 \oplus \mathcal{N}(D_2)^\perp \oplus \mathcal{N}(D_2) \oplus \mathcal{N}(D_3)^\perp \oplus \mathcal{N}(D_3) \oplus \cdots \oplus \mathcal{N}(D_n)^\perp \oplus \mathcal{N}(D_n)$ into $\mathcal{R}(D_1) \oplus \mathcal{R}(D_1)^\perp \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(D_2)^\perp \oplus \cdots \oplus \mathcal{R}(D_{n-1}) \oplus \mathcal{R}(D_{n-1})^\perp \oplus X_n$ admits the following block representation

$$T_n^d(A) = \begin{bmatrix} D_1^{(1)} & A_{12}^{(1)} & A_{12}^{(2)} & A_{13}^{(1)} & A_{13}^{(2)} & \dots & A_{1n}^{(1)} & A_{1n}^{(2)} \\ 0 & A_{12}^{(3)} & A_{12}^{(4)} & A_{13}^{(3)} & A_{13}^{(4)} & \dots & A_{1n}^{(3)} & A_{1n}^{(4)} \\ 0 & D_2^{(1)} & 0 & A_{23}^{(1)} & A_{23}^{(2)} & \dots & A_{2n}^{(1)} & A_{2n}^{(2)} \\ 0 & 0 & 0 & A_{23}^{(3)} & A_{23}^{(4)} & \dots & A_{2n}^{(3)} & A_{2n}^{(4)} \\ 0 & 0 & 0 & D_3^{(1)} & 0 & \dots & A_{3n}^{(1)} & A_{3n}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & \dots & A_{3n}^{(3)} & A_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & A_{n-1,n}^{(1)} & A_{n-1,n}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & \dots & A_{n-1,n}^{(3)} & A_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & \dots & D_n^{(1)} & 0 \end{bmatrix}. \tag{4.1.1}$$

Notice that $D_s^{(1)}$, $1 \leq s \leq n-1$ are invertible, and $D_n^{(1)}$ is injective. Therefore, there exist invertible operator matrices U and V such that

$$UT_n^d(A)V = \begin{bmatrix} D_1^{(1)} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & A_{12}^{(4)} & 0 & A_{13}^{(4)} & \dots & A_{1n}^{(3)} & A_{1n}^{(4)} \\ 0 & D_2^{(1)} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{23}^{(4)} & \dots & A_{2n}^{(3)} & A_{2n}^{(4)} \\ 0 & 0 & 0 & D_3^{(1)} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & A_{3n}^{(3)} & A_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & A_{n-1,n}^{(3)} & A_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & \dots & D_n^{(1)} & 0 \end{bmatrix} \tag{4.1.2}$$

We will explain the construction of matrices U and V in more details. It is known that elementary transformations of a matrix can be carried out by multiplying the matrix with elementary matrices. In that way, since $D_1^{(1)}$, $D_2^{(1)}$,...$D_{n-1}^{(1)}$ are invertible, by multiplying the matrix $T_n^d(A)$ with suitable elementary matrices from the left, we ,,destroy" operators $A_{ij}^{(1)}$ and $A_{ij}^{(3)}$, where $1 \leq i, j \leq n-1$. The product of those matrices is our matrix U. Now, analogously, we multiply $T_n^d(A)$ with suitable elementary matrices from the right in order to ,,destroy" operators $A_{ij}^{(2)}$; the product of those matrices equals matrix V.

Note that $A_{ij}^{(3)}$ and $A_{ij}^{(4)}$ in (4.2.6) are not the original ones from (4.2.5) in general, but we still use them for convenience. Now, it is obvious that if (4.2.6) is left invertible, then since $D_n^{(1)}$ is injective,

$$\begin{bmatrix} A_{12}^{(4)} & A_{13}^{(4)} & A_{14}^{(4)} & \dots & A_{1n}^{(4)} \\ 0 & A_{23}^{(4)} & A_{24}^{(4)} & \dots & A_{2n}^{(4)} \\ 0 & 0 & A_{34}^{(4)} & \dots & A_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{n-1,n}^{(4)} \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D_2) \\ \mathcal{N}(D_3) \\ \mathcal{N}(D_4) \\ \vdots \\ \mathcal{N}(D_n) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(D_1)^\perp \\ \mathcal{R}(D_2)^\perp \\ \mathcal{R}(D_3)^\perp \\ \vdots \\ \mathcal{R}(D_{n-1})^\perp \end{bmatrix} \tag{4.1.3}$$

is injective. Since $\alpha(D_j) > \sum_{s=1}^{j-1} \beta(D_s)$ it follows that

$$\begin{bmatrix} A_{1j}^{(4)} \\ A_{2j}^{(4)} \\ A_{3j}^{(4)} \\ \vdots \\ A_{j-1,j}^{(4)} \end{bmatrix} : \mathcal{N}(D_j) \to \begin{bmatrix} \mathcal{R}(D_1)^\perp \\ \mathcal{R}(D_2)^\perp \\ \vdots \\ \mathcal{R}(D_{j-1})^\perp \end{bmatrix}$$

is not injective, and hence operator defined in (4.2.7) is not injective for every $A \in \mathcal{B}_n$. Contradiction. This proves the desired. □

Remark 4.1.2. *Notice the validity of Theorem 4.1.1 without assuming separability of $X_1, \dots, X_n$.*

Corollary 4.1.3. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s - \lambda$, $2 \le s \le n-1$, $\lambda \in \mathbb{C}$ are regular operators. Then*

$$\bigcap_{A \in \mathcal{B}_n} \sigma_l(T_n^d(A)) \subseteq \sigma_l(D_1) \cup \left(\bigcup_{k=2}^{n} \Delta'_k \right) \cup \Delta'',$$

where

$$\Delta'_k := \left\{ \lambda \in \mathbb{C} : \ \mathcal{N}(D_k - \lambda) \preceq \mathcal{K}(D_{k-1} - \lambda) \text{ does not hold} \right\}, \ 2 \le k \le n,$$

$$\Delta'' = \left\{ \lambda \in \mathbb{C} : \ D_n - \lambda \text{ is not regular} \right\}.$$

If $X_1, ..., X_n$ are infinite dimensional Hilbert spaces, then

$$\sigma_l(D_1) \cup \left(\bigcup_{k=2}^{n} \Delta_k \right) \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_l(T_n^d(A)),$$

where

$$\Delta_k := \left\{ \lambda \in \mathbb{C} : \bigoplus_{s=1}^{k-1} \mathcal{K}(D_s - \lambda) \prec \mathcal{N}(D_k - \lambda) \text{ holds} \right\}, \ 2 \leq k \leq n.$$

Remark 4.1.4. *Obviously, $\Delta_k \subseteq \Delta'_k$ for $2 \leq k \leq n$ holds.*

If $n = 2$, we recover a result from [12].

Theorem 4.1.5. *([12, Theorem 5.2]) Let $D_1 \in \mathcal{B}(X_1), D_2 \in \mathcal{B}(X_2)$. Consider the following statements:*

(i) (a) *$D_1 \in \mathcal{G}_l(X_1)$;*
(b) *D_2 is regular;*
(c) *$\mathcal{N}(D_2) \preceq \mathcal{K}(D_1)$;*

(ii) *There exists $A \in \mathcal{B}_2$ such that $T_2^d(A) \in \mathcal{G}_l(X_1 \oplus X_2)$;*

(iii) (a) *$D_1 \in \mathcal{G}_l(X_1)$;*
(b) *$\mathcal{K}(D_1) \prec \mathcal{N}(D_2)$ does not hold.*

Then $(i) \Rightarrow (ii)$.
If X_1, X_2 are infinite dimensional Hilbert spaces, then $(ii) \Rightarrow (iii)$.

Corollary 4.1.6. *([12, Corollary 5.3]) Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Then*

$$\bigcap_{A \in \mathcal{B}_2} \sigma_l(T_2^d(A)) \subseteq \sigma_l(D_1) \cup \Delta'_2 \cup \Delta'',$$

where

$$\Delta'_2 := \left\{ \lambda \in \mathbb{C} : \ \mathcal{N}(D_2 - \lambda) \preceq \mathcal{K}(D_1 - \lambda) \text{ does not hold} \right\},$$

$$\Delta'' = \left\{ \lambda \in \mathbb{C} : \ D_2 - \lambda \text{ is not regular} \right\}.$$

If X_1, X_2 are infinite dimensional Hilbert spaces, then

$$\sigma_l(D_1) \cup \Delta_2 \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_l(T_2^d(A)),$$

where

$$\Delta_2 := \Big\{\lambda \in \mathbb{C}:\ \mathcal{K}(D_1-\lambda) \prec \mathcal{N}(D_2-\lambda)\ holds\Big\}.$$

Remark 4.1.7. *One might conjecture that the left invertible $T_2^d(A)$ must have D_2 with closed range. However, this is not the case. See [54, Lemma 2] and [31, Example 3].*

Notice that Theorem 4.1.5 is a correct version of [60, Theorem 2.1]. There are several remarks concerning Theorem 2.1 in [60]. First of all, in the notation of [60], condition $(i)(b)$ of Theorem 4.1.5 is omitted in [60, Theorem 2.1], which is an oversight. Without that condition direction $(ii) \Rightarrow (i)$ in [60, Theorem 2.1] need not hold. Namely, the choice of Q in the proof of part $(ii) \Rightarrow (iv)$ implies $\mathcal{R}(M_Q) = X \oplus \mathcal{R}(B)$, and for $\mathcal{R}(M_Q)$ to be closed (Lemma 1.1.1) we must assume that $\mathcal{R}(B)$ is closed. Furthermore, if $\mathcal{R}(B)$ is closed, notice that condition ($\beta(A) = \infty$ or ($B \in \Phi_+(\mathcal{K})$ and $\alpha(B) \le \beta(A)$)) in [60, Theorem 2.1] is equivalent to a simple condition $\alpha(B) \le \beta(A)$, which is condition $(i)(c)$ in Theorem 4.1.5 interpreted in the setting of Hilbert spaces. Similar reasoning holds for [60, Theorem 2.2].

Now, we provide results dealing with right invertibility of $T_n^d(A)$.

Theorem 4.1.8. *Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2),\dots,D_n \in \mathcal{B}(X_n)$. Assume that $D_s,\ 2 \le s \le n-1$ are regular operators. Consider the following statements:*
(i) (a) $D_n \in \mathcal{G}_r(X_n)$;
(b) D_1 is regular and $\mathcal{K}(D_i) \preceq \mathcal{N}(D_{i+1})$ for every $1 \le i \le n-1$;

(ii) There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \mathcal{G}_r(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_n \in \mathcal{G}_r(X_n)$;
(b) $\bigoplus_{s=i+1}^{n} \mathcal{N}(D_s) \prec \mathcal{K}(D_i)$ does not hold for $1 \le i \le n-1$.

Then $(i) \Rightarrow (ii)$.
If $X_1,\dots,X_n$ are infinite dimensional Hilbert spaces, then $(ii) \Rightarrow (iii)$.

Proof. $(i) \Rightarrow (ii)$

In this case it holds $\beta(D_n) = 0$, $\mathcal{R}(D_s)$ is closed for all $1 \le s \le n$ and $\mathcal{K}(D_i) \preceq \mathcal{N}(D_{i+1})$ for every $1 \le i \le n-1$. By Lemma 1.1.1, we need to find

$A \in \mathcal{B}_n$ such that $\beta(T_n^d(A)) = 0$ and $\mathcal{N}(T_n^d(A))$ is closed and complemented. We choose $A = (A_{ij})_{1\leq i<j\leq n}$ so that $A_{ij} = 0$ if $j - i \neq 1$, that is we place all nonzero operators of tuple A on the superdiagonal. It remains to define A_{ij} for $j = i+1$, $1 \leq i < n$. First notice that $A_{i,i+1} : X_{i+1} \to X_i$. Since all of diagonal entries have closed and complemented range and kernel, we know that $X_{i+1} = \mathcal{N}(D_{i+1}) \oplus \mathcal{M}(D_{i+1})$, $X_i = \mathcal{K}(D_i) \oplus \mathcal{R}(D_i)$, and we have $\mathcal{K}(D_i) \leq \mathcal{N}(D_{i+1})$. It follows that there is a right invertible operator $J_i : \mathcal{N}(D_{i+1}) \to \mathcal{K}(D_i)$. We put $A_{i,i+1} = \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D_{i+1}) \\ \mathcal{M}(D_{i+1}) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}(D_i) \\ \mathcal{R}(D_i) \end{bmatrix}$, and we implement this procedure for all $1 \leq i \leq n-1$.

Notice that $\mathcal{R}(A_{i,i+1}) = \mathcal{K}(D_i)$ for each $1 \leq i \leq n-1$. Therefore, it is immediate that $\mathcal{R}(T_n^d(A)) = \mathcal{R}(D_1) \oplus \mathcal{R}(A_{12}) \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(A_{23}) \oplus \cdots \oplus \mathcal{R}(D_{n-1}) \oplus \mathcal{R}(A_{n-1,n}) \oplus \mathcal{R}(D_n)$ is equal to $X_1 \oplus \cdots \oplus X_n$, that is $T_n^d(A)$ is surjective.

Now we show that $T_n^d(A)$ has a complemented kernel. First, by Lemma 1.1.1, there exist closed subspaces V_{i+1} of $\mathcal{N}(D_{i+1})$ such that $\mathcal{N}(D_{i+1}) = \mathcal{N}(J_i) \oplus V_{i+1}$, $1 \leq i \leq n-1$. It means that $X_1 \oplus X_2 \oplus \cdots \oplus X_n = \mathcal{N}(D_1) \oplus \mathcal{M}(D_1) \oplus \mathcal{N}(D_2) \oplus \mathcal{N}(J_1) \oplus V_2 \oplus \cdots \oplus \mathcal{N}(D_n) \oplus \mathcal{N}(J_{n-1}) \oplus V_n$. Second, direct computation shows that $\mathcal{N}(T_n^d(A)) \cong \mathcal{N}(D_1) \oplus \mathcal{N}(J_1) \oplus \cdots \oplus \mathcal{N}(J_{n-1})$. Comparing these equalities, and consulting Theorem 3.6 from [42], we conclude that $\mathcal{N}(T_n^d(A))$ is closed and complemented.

$(ii) \Rightarrow (iii)$

This implication follows directly from part $(ii) \Rightarrow (iii)$ of Theorem 4.1.1 by employing dual relations $\mathcal{N}(T) = \mathcal{R}(T^*)^\perp$, $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$. □

Corollary 4.1.9. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. Assume that $D_s - \lambda$, $2 \leq s \leq n-1$, $\lambda \in \mathbb{C}$ are regular operators. Then*

$$\bigcap_{A \in \mathcal{B}_n} \sigma_r(T_n^d(A)) \subseteq \sigma_r(D_n) \cup \left(\bigcup_{k=2}^{n-1} \Delta'_k \right) \cup \Delta'',$$

where

$$\Delta_k' := \Big\{\lambda \in \mathbb{C} : \ \mathcal{K}(D_k - \lambda) \preceq \mathcal{N}(D_{k+1} - \lambda) \ \textit{does not hold}\Big\}, \ 1 \le k \le n-1,$$

$$\Delta'' := \Big\{\lambda \in \mathbb{C} : \ D_1 - \lambda \ \textit{is not regular}\Big\}.$$

If $X_1, \ldots, X_n$ are infinite dimensional Hilbert spaces, then

$$\sigma_r(D_n) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k\Big) \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_r(T_n^d(A)),$$

where

$$\Delta_k = \Big\{\lambda \in \mathbb{C} : \ \bigoplus_{s=k+1}^{n} \mathcal{N}(D_s - \lambda) \prec \mathcal{K}(D_k - \lambda) \ \textit{holds}\Big\}, \ 1 \le k \le n-1.$$

Remark 4.1.10. *Obviously, $\Delta_k \subseteq \Delta_k'$ for $1 \le k \le n-1$ holds.*

If $n = 2$, we recover more results from [12].

Theorem 4.1.11. *([12, Theorem 5.4]) Let $D_1 \in \mathcal{B}(X_1), D_2 \in \mathcal{B}(X_2)$. Consider the following statements:*

(i) (a) $D_2 \in \mathcal{G}_r(X_2)$;
  (b) D_1 *is regular;*
  (c) $\mathcal{K}(D_1) \preceq \mathcal{N}(D_2)$;

(ii) *There exists $A \in \mathcal{B}_2$ such that $T_2^d(A) \in \mathcal{G}_r(X_1 \oplus X_2)$;*

(iii) (a) $D_2 \in \mathcal{G}_r(X_2)$;
  (b) $\mathcal{N}(D_2) \prec \mathcal{K}(D_1)$ *does not hold.*

Then $(i) \Rightarrow (ii)$.
If X_1, X_2 are infinite dimensional Hilbert spaces, then $(ii) \Rightarrow (iii)$.

Corollary 4.1.12. *([12, Corollary 5.5]) Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Then*

$$\bigcap_{A \in \mathcal{B}_2} \sigma_r(T_2^d(A)) \subseteq \sigma_r(D_2) \cup \Delta_1' \cup \Delta'',$$

where

$$\Delta_1' := \Big\{\lambda \in \mathbb{C} : \ \mathcal{K}(D_1 - \lambda) \preceq \mathcal{N}(D_2 - \lambda) \ \textit{does not hold}\Big\},$$

$$\Delta'' := \Big\{\lambda \in \mathbb{C} : \ D_1 - \lambda \ \textit{is not regular}\Big\}.$$

If X_1, X_2 are infinite dimensional Hilbert spaces, then

$$\sigma_r(D_2) \cup \Delta_1 \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_r(T_2^d(A)),$$

where

$$\Delta_1 := \Big\{\lambda \in \mathbb{C} : \ \mathcal{N}(D_2 - \lambda) \prec \mathcal{K}(D_1 - \lambda) \ \textit{holds}\Big\}.$$

We finish our investigations with results regarding invertibility of $T_n^d(A)$.

Theorem 4.1.13. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. Assume that all D_s, $2 \leq s \leq n-1$, are inner regular operators. Consider the following statements:*

(i) (a) *$D_1 \in \mathcal{G}_l(X_1)$ and $D_n \in \mathcal{G}_r(X_n)$;*
(b) *$\mathcal{N}(D_{i+1}) \cong \mathcal{K}(D_i)$ for $1 \leq i \leq n-1$;*

(ii) *There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \mathcal{G}(X_1 \oplus \cdots \oplus X_n)$;*

(iii) (a) *$D_1 \in \mathcal{G}_l(X_1)$ and $D_n \in \mathcal{G}_r(X_n)$;*
(b) *$\bigoplus_{s=1}^{i-1} \mathcal{K}(D_s) \prec \mathcal{N}(D_i)$ does not hold for $2 \leq i \leq n$ and $\bigoplus_{s=i+1}^{n} \mathcal{N}(D_s) \prec \mathcal{K}(D_i)$ does not hold for $1 \leq i \leq n-1$.*

Then $(i) \Rightarrow (ii)$.
If $X_1, \dots, X_n$ are infinite dimensional Hilbert spaces, then $(ii) \Rightarrow (iii)$.

Proof. $(ii) \Rightarrow (iii)$

Let $T_n^d(A)$ be invertible for some $A \in \mathcal{B}_n$. Then $T_n^d(A)$ is both left and right invertible, and so Theorems 4.1.1 and 4.1.8 yield the desired.

$(i) \Rightarrow (ii)$

We find $A \in \mathcal{B}_n$ such that $\alpha(T_n^d(A)) = 0$ and $\mathcal{R}(T_n^d(A)) = X_1 \oplus \cdots \oplus X_n$. We choose $A = (A_{ij})_{1 \leq i < j \leq n}$ so that $A_{ij} = 0$ if $j - i \neq 1$, that is we place all nonzero operators of tuple A on the superdiagonal. It remains to define A_{ij} for $j = i+1$, $1 \leq i < n$. First notice that $A_{i,i+1} : X_{i+1} \to X_i$. Since all of diagonal entries have closed ranges, we know that $X_{i+1} = \mathcal{N}(D_{i+1}) \oplus \mathcal{M}(D_{i+1})$, $X_i = \mathcal{K}(D_i) \oplus \mathcal{R}(D_i)$, and we have $\alpha(D_{i+1}) = \beta(D_i)$. It follows that there is an invertible $J_i : \mathcal{N}(D_{i+1}) \to \mathcal{K}(D_i)$. We put $A_{i,i+1} = \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D_{i+1}) \\ \mathcal{M}(D_{i+1}) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}(D_i) \\ \mathcal{R}(D_i) \end{bmatrix}$, and we implement this procedure for all $1 \le i \le n-1$.

Notice that $\mathcal{R}(A_{i,i+1}) = \mathcal{K}(D_i)$ for each $1 \le i \le n-1$. Thus, we prove that $T_n^d(A)$ is surjective in the same way as in the proof of Theorem 4.1.8.

Next, we are able to show that $\mathcal{N}(T_n^d(A)) \cong \mathcal{N}(D_1)$, implying $\alpha(T_n^d(A)) = \alpha(D_1) = 0$. This is proved in the same way as in the proof of Theorem 4.1.1. □

Corollary 4.1.14. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. Assume that all $D_s - \lambda$, $2 \le s \le n-1$, $\lambda \in \mathbb{C}$ are regular operators. Then*

$$\bigcap_{A \in \mathcal{B}_n} \sigma(T_n^d(A)) \subseteq \sigma_l(D_1) \cup \sigma_r(D_n) \cup \left(\bigcup_{k=1}^{n-1} \Delta_k' \right),$$

where

$$\Delta_k' := \left\{ \lambda \in \mathbb{C} : \ \mathcal{N}(D_{k+1} - \lambda) \cong \mathcal{K}(D_k - \lambda) \ \textit{does not hold} \right\}, \ 1 \le k \le n-1.$$

If $X_1, \dots, X_n$ are infinite dimensional Hilbert spaces, then

$$\sigma_l(D_1) \cup \sigma_r(D_n) \cup \left(\bigcup_{k=2}^{n-1} \Delta_k \right) \cup \Delta_n \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma(T_n^d(A)),$$

where

$$\Delta_k = \left\{ \lambda \in \mathbb{C} : \bigoplus_{s=1}^{k-1} \mathcal{K}(D_s - \lambda) \prec \mathcal{N}(D_k - \lambda) \ \textit{holds} \right\} \cup$$
$$\left\{ \lambda \in \mathbb{C} : \bigoplus_{s=k+1}^{n} \mathcal{N}(D_s - \lambda) \prec \mathcal{K}(D_k - \lambda) \ \textit{holds} \right\}, \quad 2 \le k \le n-1,$$

$$\Delta_n = \left\{\lambda \in \mathbb{C} : \bigoplus_{s=1}^{n-1} \mathcal{K}(D_s - \lambda) \prec \mathcal{N}(D_n - \lambda) \text{ } holds\right\} \cup$$
$$\left\{\lambda \in \mathbb{C} : \bigoplus_{s=2}^{n} \mathcal{N}(D_s - \lambda) \prec \mathcal{K}(D_1 - \lambda) \text{ } holds\right\}.$$

Remark 4.1.15. *Obviously,* $\left(\bigcup_{k=2}^{n-1} \Delta_k\right) \cup \Delta_n \subseteq \left(\bigcup_{k=1}^{n-1} \Delta'_k\right)$ *holds.*

If we put $n = 2$ we get:

Theorem 4.1.16. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. *Consider the following statements:*
(i) (a) $D_1 \in \mathcal{G}_l(X_1)$ *and* $D_2 \in \mathcal{G}_r(X_2)$;
(b) $\mathcal{N}(D_2) \cong \mathcal{K}(D_1)$;

(ii) *There exists* $A \in \mathcal{B}_2$ *such that* $T_2^d(A) \in \mathcal{G}(X_1 \oplus X_2)$;

(iii) (a) $D_1 \in \mathcal{G}_l(X_1)$ *and* $D_2 \in \mathcal{G}_r(X_2)$;
(b) $\mathcal{K}(D_1) \prec \mathcal{N}(D_2)$ *does not hold and* $\mathcal{N}(D_2) \prec \mathcal{K}(D_1)$ *does not hold.*

Then $(i) \Rightarrow (ii)$.
If X_1, X_2 *are infinite dimensional Hilbert spaces, then* $(ii) \Rightarrow (iii)$.

Corollary 4.1.17. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. *Then*

$$\bigcap_{A \in \mathcal{B}_2} \sigma(T_2^d(A)) \subseteq \sigma_l(D_1) \cup \sigma_r(D_2) \cup \Delta',$$

where
$$\Delta' := \left\{\lambda \in \mathbb{C} : \mathcal{N}(D_2 - \lambda) \cong \mathcal{K}(D_1 - \lambda) \text{ } does \text{ } not \text{ } hold\right\}.$$

If X_1, X_2 *are infinite dimensional Hilbert spaces, then*

$$\sigma_l(D_1) \cup \sigma_r(D_2) \cup \Delta \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma(T_2^d(A)),$$

where
$$\Delta := \left\{\lambda \in \mathbb{C} : \mathcal{K}(D_1 - \lambda) \prec \mathcal{N}(D_2 - \lambda) \text{ } holds\right\} \cup$$
$$\left\{\lambda \in \mathbb{C} : \mathcal{N}(D_2 - \lambda) \prec \mathcal{K}(D_1 - \lambda) \text{ } holds\right\}.$$

Theorem 4.1.16 interpreted in the setting of Hilbert spaces is a special case of [25, Theorem 2]. Notice that Han et al. ([25]) have proved the equivalence $(i) \Leftrightarrow (ii)$ of Theorem 4.1.16 in arbitrary Banach spaces. Corollary 4.1.17 recovers a result of Du and Pan ([16, Theorem 2]). Notice, however, that in [16] separability was used, while our statement is separability-free.

4.2 Weylness of $T_n^d(A)$

In this section we first provide results assuming separability, and afterwards extend results to the case without separability.

4.2.1 Separability case

Assume that $X_1,...,X_n$ are infinite dimensional separable Hilbert spaces. This subsection bases on article [45]. We begin with a result concerning upper semi-Weyl invertibility of $T_n^d(A)$.

Theorem 4.2.1. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2),...,D_n \in \mathcal{B}(X_n)$ be given. Consider the following conditions:*

(i) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\Big(D_s \in \Phi_+(X_s)$ *for* $2 \le s \le n$ *and* $\sum\limits_{s=1}^{n} \alpha(D_s) \le \sum\limits_{s=1}^{n} \beta(D_s)\Big)$

or

$\Big(\beta(D_j) = \infty$ *for some* $j \in \{1,...,n-1\}$, $\alpha(D_s) < \infty$ *for* $2 \le s \le j$ *and* $\mathcal{R}(D_s)$ *is closed for* $2 \le s \le n\Big)$;

(ii) *There exists* $A \in \mathcal{B}_n$ *such that* $T_n^d(A) \in \Phi_+^-(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\Big(D_s \in \Phi_+(X_s)$ *for* $2 \le s \le n$ *and* $\sum\limits_{s=1}^{n} \alpha(D_s) \le \sum\limits_{s=1}^{n} \beta(D_s)\Big)$

or

$\Big(\beta(D_j) = \infty$ *for some* $j \in \{1,...,n-1\}$, $\alpha(D_s) < \infty$ *for* $2 \le s \le j\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.2.2. *If $j = 1$ in $(i)(b)$ or $(iii)(b)$, part "$\alpha(D_s) < \infty$ for $2 \le s \le j$" is omitted there.*

Remark 4.2.3. *Notice the similarity between sufficent condition (i) and necessary condition (iii): parts (i)(a) and (iii)(a) are the same, while (i)(b) and (iii)(b) differ in "$\mathcal{R}(D_s)$ is closed for $2 \leq s \leq n$" solely.*

Proof. $(ii) \Rightarrow (iii)$

Suppose that $T_n^d(A)$ is upper semi-Weyl. Then $T_n^d(A)$ is upper semi-Fredholm, implying $D_1 \in \Phi_+(X_1)$ (Lemma 1.2.11). Suppose that $(iii)(b)$ is not true. We have two possibilities. First, suppose that for $2 \leq s \leq n$ we have $\beta(D_s) < \infty$. It means (Theorem 1.1.2) that $\mathcal{R}(D_s)$ is closed for $1 \leq s \leq n$. Again, we have two possibilities: either there exists some $i \in \{2, ..., n\}$ with $\alpha(D_i) = \infty$, or we have $\sum\limits_{s=1}^{n} \alpha(D_s) > \sum\limits_{s=1}^{n} \beta(D_s)$.

Assume $\alpha(D_i) = \infty$ for some $i \in \{2, ..., n\}$. We use a method from [58]. We know that for each $A \in \mathcal{B}_n$, operator $T_n^d(A)$ regarded as an operator from $\mathcal{N}(D_1)^\perp \oplus \mathcal{N}(D_1) \oplus \mathcal{N}(D_2)^\perp \oplus \mathcal{N}(D_2) \oplus \mathcal{N}(D_3)^\perp \oplus \mathcal{N}(D_3) \oplus \cdots \oplus \mathcal{N}(D_n)^\perp \oplus \mathcal{N}(D_n)$ into $\mathcal{R}(D_1) \oplus \mathcal{R}(D_1)^\perp \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(D_2)^\perp \oplus \cdots \oplus \mathcal{R}(D_{n-1}) \oplus \mathcal{R}(D_{n-1})^\perp \oplus \mathcal{R}(D_n) \oplus \mathcal{R}(D_n)^\perp$ has the following block representation

$$T_n^d(A) = \begin{bmatrix} D_1^{(1)} & 0 & A_{12}^{(1)} & A_{12}^{(2)} & A_{13}^{(1)} & A_{13}^{(2)} & \ldots & A_{1n}^{(1)} & A_{1n}^{(2)} \\ 0 & 0 & A_{12}^{(3)} & A_{12}^{(4)} & A_{13}^{(3)} & A_{13}^{(4)} & \ldots & A_{1n}^{(3)} & A_{1n}^{(4)} \\ 0 & 0 & D_2^{(1)} & 0 & A_{23}^{(1)} & A_{23}^{(2)} & \ldots & A_{2n}^{(1)} & A_{2n}^{(2)} \\ 0 & 0 & 0 & 0 & A_{23}^{(3)} & A_{23}^{(4)} & \ldots & A_{2n}^{(3)} & A_{2n}^{(4)} \\ 0 & 0 & 0 & 0 & D_3^{(1)} & 0 & \ldots & A_{3n}^{(1)} & A_{3n}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & A_{3n}^{(3)} & A_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & A_{n-1,n}^{(1)} & A_{n-1,n}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & A_{n-1,n}^{(3)} & A_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & D_n^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{4.2.1}$$

Evidently, $D_1^{(1)}$, $D_2^{(1)}, ..., D_n^{(1)}$ from (4.2.1) are invertible. Hence, there exist invertible operator matrices U and V so that

$$UT_n^d(A)V = \begin{bmatrix} D_1^{(1)} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & A_{12}^{(4)} & 0 & A_{13}^{(4)} & \dots & 0 & A_{1n}^{(4)} \\ 0 & 0 & D_2^{(1)} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{23}^{(4)} & \dots & 0 & A_{2n}^{(4)} \\ 0 & 0 & 0 & 0 & D_3^{(1)} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & A_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & A_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & D_n^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{4.2.2}$$

Operators $A_{ij}^{(4)}$ in (4.2.1) and (4.2.2) are not the same, but we will keep the same notation for simplicity. Next, it is clear that (4.2.2) is upper semi-Weyl if and only if

$$\begin{bmatrix} 0 & A_{12}^{(4)} & A_{13}^{(4)} & A_{14}^{(4)} & \dots & A_{1n}^{(4)} \\ 0 & 0 & A_{23}^{(4)} & A_{24}^{(4)} & \dots & A_{2n}^{(4)} \\ 0 & 0 & 0 & A_{34}^{(4)} & \dots & A_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D_1) \\ \mathcal{N}(D_2) \\ \mathcal{N}(D_3) \\ \mathcal{N}(D_4) \\ \vdots \\ \mathcal{N}(D_n) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(D_1)^\perp \\ \mathcal{R}(D_2)^\perp \\ \mathcal{R}(D_3)^\perp \\ \vdots \\ \mathcal{R}(D_{n-1})^\perp \\ \mathcal{R}(D_n)^\perp \end{bmatrix} \tag{4.2.3}$$

is upper semi-Weyl. Since $\sum_{s=1}^{i-1} \beta(D_s) < \infty$ and $\alpha(D_i) = \infty$, it follows that

$$\alpha\left(\begin{bmatrix} A_{1i}^{(4)} \\ A_{2i}^{(4)} \\ A_{3i}^{(4)} \\ \vdots \\ A_{i-1,i}^{(4)} \end{bmatrix}\right) = \infty,$$

and hence operator defined in (4.2.3) is not upper semi-Weyl for every $A \in \mathcal{B}_n$. This proves the desired.

Assume next that $\alpha(D_s) < \infty$ for $2 \leq s \leq n$. Then $\sum_{s=1}^{n} \alpha(D_s) > \sum_{s=1}^{n} \beta(D_s)$, and for each $A \in \mathcal{B}_n$, $T_n^d(A)$ has representation as (4.2.1), and we use (4.2.2) and (4.2.3) again. Since D_s, $1 \leq s \leq n$ are upper semi-Fredholm, then $T_n^d(A)$ is upper semi-Weyl if and only if (4.2.3) is upper semi-Weyl. But $\sum_{s=1}^{n} \beta(D_s) < \sum_{s=1}^{n} \alpha(D_s)$ implies (4.2.3) is not upper semi-Weyl for every $A \in \mathcal{B}_n$. Contradiction.

Second option is that there is $j \in \{2, \ldots, n\}$ with $\beta(D_j) = \infty$, and assume we have found the smallest such j. Then $\beta(D_s) < \infty$ for $1 \leq s \leq j-1$, hence $\mathcal{R}(D_s)$ is closed for $1 \leq s \leq j-1$. Now, $\alpha(D_s) < \infty$ for $2 \leq s \leq j-1$ is not possible, otherwise $(iii)(b)$ would be true. Finally, $\alpha(D_j) = \infty$ for some $j \in \{2, \ldots, j-1\}$ and be proceed with (4.2.1), (4.2.2), (4.2.3) applied to $T_{j-1}^d(A)$.

$(i) \Rightarrow (ii)$

Assume that $D_1 \in \Phi_+(X_1)$ and $(i)(b)$ holds. If $D_s \in \Phi_+(X_j)$ for $2 \leq s \leq n$ and $\sum_{s=1}^{n} \alpha(D_s) \leq \sum_{s=1}^{n} \beta(D_s)$, we choose trivially $A = \mathbf{0}$ and $T_n^d(A)$ is upper semi-Weyl.

Suppose that $\beta(D_j) = \infty$ for some $j \in \{1, \ldots, n-1\}$, $\alpha(D_s) < \infty$ for $2 \leq s \leq j$ and $\mathcal{R}(D_s)$ is closed for all $1 \leq s \leq n$. Assume that $\{f_s^{(k)}\}_{s=1}^{\infty}$, $\{e_s^{(1)}\}_{s=1}^{\infty}$, $\{e_s^{(2)}\}_{s=1}^{\infty}, \ldots, \{e_s^{(n-1)}\}_{s=1}^{\infty}$ are orthogonal bases of $\mathcal{R}(D_k)^{\perp}$, $X_2, \ldots, X_n$, respectively. We have two cases. Again, we adopt a method from [58].

Case 1: $\beta(D_1) = \infty$

In this case it holds $\alpha(D_1) < \infty$, $\mathcal{R}(D_s)$ is closed for all $1 \leq s \leq n$ and $\beta(D_1) = \infty$. We find $A \in \mathcal{B}_n$ such that $\alpha(T_n^d(A)) < \infty$ and $\mathcal{R}(T_n^d(A))$ is closed. We choose $A = (A_{ij})_{1 \leq i < j \leq n}$ so that $A_{ij} = 0$ if $i > 1$, that is we place all nonzero operators of tuple A in the first row. It remains to define A_{1s} for

$2 \le s \le n$. We put

$$\begin{aligned} A_{12}(e_s^{(1)}) &= f_{ns}^{(1)}, \quad s = 1,2,...; \\ A_{13}(e_s^{(2)}) &= f_{ns+1}^{(1)}, \quad s = 1,2,...; \\ &\cdots \\ A_{1n}(e_s^{(n-1)}) &= f_{ns+n-2}^{(1)}, \quad s = 1,2,.... \end{aligned}$$

Now we have chosen our $A = (A_{ij})$, it is easy to show that $\mathcal{N}(T_n^d(A)) = \mathcal{N}(D_1) \oplus \{\mathbf{0}\} \oplus \cdots \oplus \{\mathbf{0}\}$. Therefore, $\alpha(T_n^d(A)) = \alpha(D_1) < \infty$.

Secondly, we show that $\mathcal{R}(T_n^d(A))$ is closed and $\beta(T_n^d(A)) = \infty$. Since $\mathcal{R}(D_s)$ is closed for all $1 \le s \le n$, it will follow that $\mathcal{R}(T_n^d(A))$ is closed if we prove that $\mathcal{R}(A_{1s})$ is closed for $2 \le s \le n$. But, since we are in the setting of separable Hilbert spaces, with regards to definition of A_{1s}'s, the former is obvious. We have that $T_n^d(A)$ is upper semi-Fredholm, and since $\beta(T_n^d(A)) = \beta(D_1) = \infty$ due to definition of A_{1s}'s, we find that $T_n^d(A)$ is upper semi-Weyl.

Case 2: $\beta(D_k) = \infty$ for some $k \in \{2,...,n-1\}$

In this case it holds $\alpha(D_s) < \infty$, $1 \le s \le k$, $\mathcal{R}(D_s)$ is closed for all $1 \le s \le n$ and $\beta(D_k) = \infty$. We find $A \in \mathcal{B}_n$ such that $\alpha(T_n^d(A)) < \infty$ and $\mathcal{R}(T_n^d(A))$ is closed. We choose $A = (A_{ij})_{1 \le i < j \le n}$ so that $A_{ij} = 0$ if $i \ne k$, that is we place all nonzero operators of tuple A in the k-th row. It remains to define A_{ks} for $k+1 \le s \le n$. We put

$$\begin{aligned} A_{k,k+1}(e_s^{(k)}) &= f_{ns}^{(k)}, \quad s = 1,2,...; \\ A_{k,k+2}(e_s^{(k+1)}) &= f_{ns+1}^{(k)}, \quad s = 1,2,...; \\ &\cdots \\ A_{kn}(e_s^{(n-1)}) &= f_{ns+n-k-1}^{(k)}, \quad s = 1,2,.... \end{aligned}$$

Now we have chosen our $A = (A_{ij})$, it is easy to show that $\mathcal{N}(T_n^d(A)) = \mathcal{N}(D_1) \oplus \cdots \oplus \mathcal{N}(D_k) \oplus \{\mathbf{0}\} \oplus \cdots \oplus \{\mathbf{0}\}$. Therefore, $\alpha(T_n^d(A)) \le \alpha(D_1) + \cdots + \alpha(D_k) < \infty$.

Secondly, we show that $\mathcal{R}(T_n^d(A))$ is closed and $\beta(T_n^d(A)) = \infty$. Since $\mathcal{R}(D_s)$ is closed for all $1 \leq s \leq n$, it will follow that $\mathcal{R}(T_n^d(A))$ is closed if we prove that $\mathcal{R}(A_{ks})$ is closed for $k+1 \leq s \leq n$. But, since we are in the setting of separable Hilbert spaces, with regards to definition of A_{ks}'s, the former is obvious. We have that $T_n^d(A)$ is upper semi-Fredholm, and since $\beta(T_n^d(A)) = \beta(D_k) = \infty$ due to definition of A_{ks}'s, we find that $T_n^d(A)$ is upper semi-Weyl.

□

Remark 4.2.4. *Notice the validity of part* $(ii) \Rightarrow (iii)$ *without assuming separability of* $X_1, ..., X_n$.

Next corollary is immediate from Theorem 4.2.1.

Corollary 4.2.5. *([58, Theorem 2.5], corrected version) Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. *Then*

$$\sigma_{le}(D_1) \cup \left(\bigcup_{k=2}^{n+1} \Delta_k\right) \subseteq$$

$$\bigcap_{A \in \mathcal{B}_n} \sigma_{lw}(T_n^d(A)) \subseteq$$

$$\sigma_{le}(D_1) \cup \left(\bigcup_{k=2}^{n+1} \Delta_k\right) \cup \left(\bigcup_{k=2}^{n} \Delta'_k\right),$$

where

$$\Delta_k := \left\{\lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) = \infty \ and \ \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty\right\}, \ 2 \leq k \leq n,$$

$$\Delta_{n+1} := \left\{\lambda \in \mathbb{C} : \ \sum_{s=1}^{n} \beta(D_s - \lambda) < \sum_{s=1}^{n} \alpha(D_s - \lambda)\right\},$$

$$\Delta'_k := \left\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_k - \lambda) \ is \ not \ closed\right\}, \ 2 \leq k \leq n.$$

Remark 4.2.6. *One should also spot a difference between collections* Δ_k, $2 \leq k \leq n$, *in Corollary 4.2.5 and in [58, Theorem 2.5]. This difference is implied* *by the existence of sets Δ'_k, $2 \le k \le n$, in the formulation of Corollary 4.2.5. Our estimates are better in a sense that Δ_k in Corollary 4.2.5 is a subset of Δ_k from [58] for every $2 \le k \le n$.*

Previous statements for $n = 2$ become very simple, as shown in the sequel.

Theorem 4.2.7. *([60, Theorem 2.5], corrected version) Let $D_1 \in \mathcal{B}(X_1)$ and $D_2 \in \mathcal{B}(X_2)$. Consider the following statements:*

(i) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\Big(D_2 \in \Phi_+(X_2)$ *and* $\alpha(D_1)+\alpha(D_2) \le \beta(D_1)+\beta(D_2)\Big)$

or

$\Big(\beta(D_1) = \infty$ *and* $\mathcal{R}(D_2)$ *is closed*$\Big)$;

(ii) *There exists* $A \in \mathcal{B}_2$ *such that* $T_2^d(A) \in \Phi_+^-(X_1 \oplus X_2)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\Big(D_2 \in \Phi_+(X_2)$ *and* $\alpha(D_1)+\alpha(D_2) \le \beta(D_1)+\beta(D_2)\Big)$

or

$\beta(D_1) = \infty$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Notice that Theorem 4.2.7 is a corrected version of [60, Theorem 2.5]. Condition '$\mathcal{R}(D_2)$ is closed' in $(i)(b)$ is omitted in [60], which is an oversight. Without that condition we can not prove that $\mathcal{R}(T_2^d(A))$ is closed and therefore direction $(ii) \Rightarrow (i)$ in [60, Theorem 2.5] would not hold.

Corollary 4.2.8. *([60, Corollary 2.7], corrected version) Let $D_1 \in \mathcal{B}(X_1)$ and $D_2 \in \mathcal{B}(X_2)$. Then*

$$\sigma_{le}(D_1) \cup \Delta \cup \Delta' \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_{lw}(T_2^d(A)) \subseteq \sigma_{le}(D_1) \cup \Delta \cup \Delta' \cup \Delta'',$$

where

$$\Delta := \Big\{\lambda \in \mathbb{C}:\ \alpha(D_2-\lambda) = \infty \ \textit{and}\ \beta(D_1-\lambda) < \infty\Big\},$$

$$\Delta' := \Big\{\lambda \in \mathbb{C}:\ \beta(D_1-\lambda)+\beta(D_2-\lambda) < \alpha(D_1-\lambda)+\alpha(D_2-\lambda)\Big\},$$

$$\Delta'' := \{\lambda \in \mathbb{C}:\ \mathcal{R}(D_2-\lambda) \ \textit{is not closed}\}.$$

Now we list statements dealing with the lower semi-Weyl spectrum.

Theorem 4.2.9. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$ be given. Consider the following conditions:*
(i) (a) $D_n \in \Phi_-(X_n)$;

(b) $\Big(D_s \in \Phi_-(X_s)$ *for* $1 \leq s \leq n-1$ *and* $\sum_{s=1}^{n} \beta(D_s) \leq \sum_{s=1}^{n} \alpha(D_s)\Big)$

or

$\Big(\alpha(D_j) = \infty$ *for some* $j \in \{2, ..., n\}$, $\beta(D_s) < \infty$ *for* $j \leq s \leq n-1$ *and* $\mathcal{R}(D_s)$ *is closed for* $1 \leq s \leq n-1\Big)$;

(ii) *There exists* $A \in \mathcal{B}_n$ *such that* $T_n^d(A) \in \Phi_-^+(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_n \in \Phi_-(X_n)$;

(b) $\Big(D_s \in \Phi_-(X_s)$ *for* $1 \leq s \leq n-1$ *and* $\sum_{s=1}^{n} \beta(D_s) \leq \sum_{s=1}^{n} \alpha(D_s)\Big)$

or

$\Big(\alpha(D_j) = \infty$ *for some* $j \in \{2, ..., n\}$, $\beta(D_s) < \infty$ *for* $j \leq s \leq n-1\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.2.10. *If $j = n$ in $(i)(b)$ or $(iii)(b)$, part "$\beta(D_s) < \infty$ for $j \leq s \leq n-1$" is omitted there.*

Remark 4.2.11. *Notice the similarity between sufficent condition (i) and necessary condition (iii): parts (i)(a) and (iii)(a) are the same, while (i)(b) and (iii)(b) differ in "$\mathcal{R}(D_s)$ is closed for $1 \leq s \leq n-1$" solely.*

Proof. This easily follows from the statement of Theorem 4.2.1 by duality argument, putting into use Lemmas 1.2.1 and 1.2.2. □

Corollary 4.2.12. *([58, Theorem 2.6], corrected version) Let $D_1 \in \mathcal{B}(X_1)$ and $D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. Then*

$$\sigma_{re}(D_n) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k\Big) \cup \Delta_{n+1} \subseteq$$
$$\bigcap_{A \in \mathcal{B}_n} \sigma_{rw}(T_n^d(A)) \subseteq$$
$$\sigma_{re}(D_n) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k\Big) \cup \Delta_{n+1} \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k'\Big),$$

where

$$\Delta_k := \left\{\lambda \in \mathbb{C} : \ \beta(D_k - \lambda) = \infty \ and \ \sum_{s=k+1}^{n} \alpha(D_s - \lambda) < \infty\right\}, \ 1 \leq k \leq n-1,$$

$$\Delta_{n+1} := \left\{\lambda \in \mathbb{C} : \ \sum_{s=1}^{n} \alpha(D_s - \lambda) < \sum_{s=1}^{n} \beta(D_s - \lambda)\right\},$$

$$\Delta'_k := \left\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_k - \lambda) \ is \ not \ closed\right\}, \ 1 \leq k \leq n-1.$$

Remark 4.2.13. *Again, existence of sets Δ'_k, $1 \leq k \leq n-1$ in the statement of Corollary 4.2.12 implies a difference between definitions of collections Δ_k, $1 \leq k \leq n-1$ in Corollary 4.2.12 and in [58, Theorem 2.6].*

If we put $n = 2$ we get:

Theorem 4.2.14. *([60, Theorem 2.6], corrected version) Let $D_1 \in \mathcal{B}(X_1)$ and $D_2 \in \mathcal{B}(X_2)$. Consider the following conditions:*

(i) (a) $D_2 \in \Phi_-(X_2)$;

(b) $\Big(D_1 \in \Phi_-(X_1)$ and $\alpha(D_1) + \alpha(D_2) \geq \beta(D_1) + \beta(D_2)\Big)$

or

$\Big(\alpha(D_2) = \infty$ and $\mathcal{R}(D_1)$ is closed$\Big)$;

(ii) There exists $A \in \mathcal{B}_2$ such that $T_2^d(A) \in \Phi_-^+(X_1 \oplus X_2)$;

(iii) (a) $D_2 \in \Phi_-(X_2)$;

(c) $\Big(D_1 \in \Phi_-(X_1)$ and $\alpha(D_1) + \alpha(D_2) \geq \beta(D_1) + \beta(D_2)\Big)$

or

$\alpha(D_2) = \infty$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.2.15. *([60, Corollary 2.8], corrected version) Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Then*

$$\sigma_{re}(D_2) \cup \Delta \cup \Delta' \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_{rw}(T_2^d(A)) \subseteq \sigma_{re}(D_2) \cup \Delta \cup \Delta' \cup \Delta'',$$

where

$$\Delta := \left\{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) = \infty \ and \ \alpha(D_2 - \lambda) < \infty\right\},$$

$$\Delta' := \Big\{\lambda \in \mathbb{C} : \ \alpha(D_1 - \lambda) + \alpha(D_2 - \lambda) < \beta(D_1 - \lambda) + \beta(D_2 - \lambda)\Big\},$$

$$\Delta'' := \{\lambda \in \mathbb{C} : \ \mathcal{R}(D_1 - \lambda) \text{ is not closed}\}.$$

4.2.2 Nonseparable spaces

In this subsection we assume $X_1, ..., X_n$ to be arbitrary infinite dimensional Hilbert spaces. This subsection bases on results from [44]. We generalize results of [7],[60] from $n = 2$ to an arbitrary dimension of upper triangular operators, and we pose perturbation results of [58] without assuming separability of underlying spaces.

We start with a result which deals with the upper Weyl spectrum of $T_n^d(A)$.

Theorem 4.2.16. *Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. Consider the following statements:*
(i) (a) $D_1 \in \Phi_+(X_1)$;
(b) $\mathcal{R}(D_s)$ is closed for $2 \leq s \leq n$ and

$$\Big(\alpha(D_s) \leq \beta(D_{s-1}) \quad for\ 2 \leq s \leq n, \quad \sum_{s=1}^{n} \beta(D_s) = \infty\Big) \tag{4.2.4}$$

or $\Big(D_s \in \Phi_+(X_s)$ for $2 \leq s \leq n$ and $\sum\limits_{s=1}^{n} \alpha(D_s) \leq \sum\limits_{s=1}^{n} \beta(D_s)\Big)$;

(ii) There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \Phi_+^-(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$;
(b) $\Big(\beta(D_j) = \infty$ for some $j \in \{1, ..., n\}$ and $\alpha(D_s) < \infty$ for $2 \leq s \leq j\Big)$ or $\Big(D_s \in \Phi_+(X_s)$ for $2 \leq s \leq n$ and $\sum\limits_{s=1}^{n} \alpha(D_s) \leq \sum\limits_{s=1}^{n} \beta(D_s)\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.2.17. *If $j = 1$ in $(iii)(b)$, we simply omit condition "$\alpha(D_s) < \infty$ for $2 \leq s \leq j$" there.*

Proof: $(ii) \Rightarrow (iii)$

Assume that $T_n^d(A)$ is upper Weyl. Then $T_n^d(A)$ is upper Fredholm, hence $D_1 \in \Phi_+(X_1)$ (Lemma 1.2.11). Assume that $(iii)(b)$ fails. We have two possibilities. On the one hand, assume that for $2 \le s \le n$ we have $\beta(D_s) < \infty$. It means (Theorem 1.1.2) that $\mathcal{R}(D_s)$ is closed for $1 \le s \le n$. Again, we have two possibilities. Either there exists some $i \in \{2, \dots, n\}$ with $\alpha(D_i) = \infty$, or we have $\sum\limits_{s=1}^{n} \alpha(D_s) > \sum\limits_{s=1}^{n} \beta(D_s)$.

First suppose $\alpha(D_i) = \infty$ for some $i \in \{2, \dots, n\}$. We use a method from [58]. We know that for each $A \in \mathcal{B}_n$, operator matrix $T_n^d(A)$ as an operator from $\mathcal{N}(D_1)^\perp \oplus \mathcal{N}(D_1) \oplus \mathcal{N}(D_2)^\perp \oplus \mathcal{N}(D_2) \oplus \mathcal{N}(D_3)^\perp \oplus \mathcal{N}(D_3) \oplus \cdots \oplus \mathcal{N}(D_n)^\perp \oplus \mathcal{N}(D_n)$ into $\mathcal{R}(D_1) \oplus \mathcal{R}(D_1)^\perp \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(D_2)^\perp \oplus \cdots \oplus \mathcal{R}(D_{n-1}) \oplus \mathcal{R}(D_{n-1})^\perp \oplus \mathcal{R}(D_n) \oplus \mathcal{R}(D_n)^\perp$ admits the following block representation

$$T_n^d(A) = \begin{bmatrix} D_1^{(1)} & 0 & A_{12}^{(1)} & A_{12}^{(2)} & A_{13}^{(1)} & A_{13}^{(2)} & \dots & A_{1n}^{(1)} & A_{1n}^{(2)} \\ 0 & 0 & A_{12}^{(3)} & A_{12}^{(4)} & A_{13}^{(3)} & A_{13}^{(4)} & \dots & A_{1n}^{(3)} & A_{1n}^{(4)} \\ 0 & 0 & D_2^{(1)} & 0 & A_{23}^{(1)} & A_{23}^{(2)} & \dots & A_{2n}^{(1)} & A_{2n}^{(2)} \\ 0 & 0 & 0 & 0 & A_{23}^{(3)} & A_{23}^{(4)} & \dots & A_{2n}^{(3)} & A_{2n}^{(4)} \\ 0 & 0 & 0 & 0 & D_3^{(1)} & 0 & \dots & A_{3n}^{(1)} & A_{3n}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & A_{3n}^{(3)} & A_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & A_{n-1,n}^{(1)} & A_{n-1,n}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & A_{n-1,n}^{(3)} & A_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & D_n^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \qquad (4.2.5)$$

Obviously, $D_1^{(1)}, D_2^{(1)}, \dots, D_n^{(1)}$ from (4.2.5) are invertible. Hence, there exist invertible operator matrices U and V so that

$$UT_n^d(A)V = \begin{bmatrix} D_1^{(1)} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & B_{12}^{(4)} & 0 & B_{13}^{(4)} & \dots & 0 & B_{1n}^{(4)} \\ 0 & 0 & D_2^{(1)} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{23}^{(4)} & \dots & 0 & B_{2n}^{(4)} \\ 0 & 0 & 0 & 0 & D_3^{(1)} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & B_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & B_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & D_n^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{4.2.6}$$

Next, it is clear that (4.2.6) is upper Weyl if and only if

$$\begin{bmatrix} 0 & B_{12}^{(4)} & B_{13}^{(4)} & B_{14}^{(4)} & \dots & B_{1n}^{(4)} \\ 0 & 0 & B_{23}^{(4)} & B_{24}^{(4)} & \dots & B_{2n}^{(4)} \\ 0 & 0 & 0 & B_{34}^{(4)} & \dots & B_{3n}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & B_{n-1,n}^{(4)} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D_1) \\ \mathcal{N}(D_2) \\ \mathcal{N}(D_3) \\ \mathcal{N}(D_4) \\ \vdots \\ \mathcal{N}(D_n) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(D_1)^\perp \\ \mathcal{R}(D_2)^\perp \\ \mathcal{R}(D_3)^\perp \\ \vdots \\ \mathcal{R}(D_{n-1})^\perp \\ \mathcal{R}(D_n)^\perp \end{bmatrix} \tag{4.2.7}$$

is upper Weyl. Since $\sum_{s=1}^{i-1} \beta(D_s) < \infty$ and $\alpha(D_i) = \infty$, it follows that

$$\alpha\left(\begin{bmatrix} B_{1i}^{(4)} \\ B_{2i}^{(4)} \\ B_{3i}^{(4)} \\ \vdots \\ B_{i-1,i}^{(4)} \end{bmatrix}\right) = \infty,$$

and hence operator defined in (4.2.7) is not upper Weyl for every $A \in \mathcal{B}_n$. This proves the desired.

Now assume $\alpha(D_s) < \infty$ for $2 \leq s \leq n$. Then we have $\sum_{s=1}^{n} \alpha(D_s) > \sum_{s=1}^{n} \beta(D_s)$, and for each $A \in \mathcal{B}_n$, $T_n^d(A)$ has representation as (4.2.5), and we still use

(4.2.6) and (4.2.7). Since D_s, $1 \leq s \leq n$ are upper Fredholm, we conclude that $T_n^d(A)$ is upper Weyl if and only if (4.2.7) is upper Weyl. From $\sum_{s=1}^{n} \beta(D_s) < \sum_{s=1}^{n} \alpha(D_s)$, we know (4.2.7) is not upper Weyl for every $A \in \mathcal{B}_n$.

On the other hand, assume that there is $j \in \{2, \ldots, n\}$ with $\beta(D_j) = \infty$, and assume we have chosen the smallest such j. In that case $\beta(D_s) < \infty$ for $1 \leq s \leq j-1$, hence $\mathcal{R}(D_s)$ is closed for $1 \leq s \leq j-1$. Now, we easily conclude it is impossible that $\alpha(D_s) < \infty$ for $2 \leq s \leq j-1$, otherwise $(iii)(b)$ would not fail. Therefore, $\alpha(D_j) = \infty$ for some $j \in \{2, \ldots, j-1\}$ and be proceed with (4.2.5), (4.2.6), (4.2.7) applied to $T_{j-1}^d(A)$.

$(i) \Rightarrow (ii)$

If $D_s \in \Phi_+(X_s)$ for $2 \leq s \leq n$ and $\sum_{s=1}^{n} \alpha(D_s) \leq \sum_{s=1}^{n} \beta(D_s)$, we trivially choose $A = (A_{ij}) = \mathbf{0}$. Assume that this is not the case. Otherwise, it holds $\alpha(D_1) < \infty$, $\mathcal{R}(D_s)$ is closed for all $1 \leq s \leq n$ and (4.2.4) holds. We find $A \in \mathcal{B}_n$ such that $\alpha(T_n^d(A)) < \infty$ and $\mathcal{R}(T_n^d(A))$ is closed. We choose $A = (A_{ij})_{1 \leq i < j \leq n}$ so that $A_{ij} = 0$ if $j - i \neq 1$, that is we place all nonzero operators of tuple A on the superdiagonal. It remains to define A_{ij} for $j - i = 1$, $1 \leq i < j \leq n$. First notice that $A_{i,i+1} : X_{i+1} \to X_i$. Since all of diagonal entries have closed ranges, we know that $X_{i+1} = \mathcal{N}(D_{i+1}) \oplus \mathcal{N}(D_{i+1})^\perp$, $X_i = \mathcal{R}(D_i)^\perp \oplus \mathcal{R}(D_i)$, and from assumption (4.2.4) we get $\alpha(D_{i+1}) \leq \beta(D_i)$. It follows that there is a left invertible operator $J_i : \mathcal{N}(D_{i+1}) \to \mathcal{R}(D_i)^\perp$. We put $A_{i,i+1} = \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix}$: $\begin{bmatrix} \mathcal{N}(D_{i+1}) \\ \mathcal{N}(D_{i+1})^\perp \end{bmatrix} = X_{i+1} \to X_i = \begin{bmatrix} \mathcal{R}(D_i)^\perp \\ \mathcal{R}(D_i) \end{bmatrix}$, and we implement this procedure for all $1 \leq i \leq n-1$. Notice that $\mathcal{R}(D_i)$ is complemented to $\mathcal{R}(A_{i,i+1})$ for each $1 \leq i \leq n-1$.

Now we have chosen our A, we show that $\mathcal{N}(T_n^d(A)) \cong \mathcal{N}(D_1)$, implying $\alpha(T_n^d(A)) = \alpha(D_1) < \infty$. Let us put $T_n^d(A)x = 0$, where $x = x_1 + \cdots + x_n \in X_1 \oplus \cdots \oplus X_n$. The previous equality is then equivalent to the following system of equations

$$\begin{bmatrix} D_1x_1 + A_{12}x_2 \\ D_2x_2 + A_{23}x_3 \\ \vdots \\ D_{n-1}x_{n-1} + A_{n-1,n}x_n \\ D_nx_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

The last equation gives $x_n \in \mathcal{N}(D_n)$. Since $\mathcal{R}(D_s)$ is complemented to $\mathcal{R}(A_{s,s+1})$ for all $1 \leq s \leq n-1$, we have $D_sx_s = A_{s,s+1}x_{x+1} = 0$ for all $1 \leq s \leq n-1$. That is, $x_i \in \mathcal{N}(D_i)$ for every $1 \leq i \leq n$, and $J_sx_{s+1} = 0$ for every $1 \leq s \leq n-1$. Due to left invertibility of J_s we get $x_s = 0$ for $2 \leq s \leq n$, which proves the claim. Therefore, $\alpha(T_n^d(A)) = \alpha(D_1) < \infty$.

Secondly, we show that $\mathcal{R}(T_n^d(A))$ is closed. It is not hard to see that

$$\begin{aligned}\mathcal{R}(T_n^d(A)) = \mathcal{R}(D_1) \oplus \mathcal{R}(J_1) \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(J_2) \oplus \cdots \oplus \mathcal{R}(D_{n-1}) \oplus \\ \mathcal{R}(J_{n-1}) \oplus \mathcal{R}(D_n).\end{aligned} \tag{4.2.8}$$

Furthermore, due to left invertibility of J_i's, there exist closed subspaces U_i of $\mathcal{R}(D_i)^\perp$ such that $\mathcal{R}(D_i)^\perp = \mathcal{R}(J_i) \oplus U_i$, $1 \leq i \leq n-1$. It means that

$$\begin{aligned}X_1 \oplus X_2 \oplus \cdots \oplus X_n = \mathcal{R}(D_1) \oplus \mathcal{R}(J_1) \oplus U_1 \oplus \mathcal{R}(D_2) \oplus \mathcal{R}(J_2) \oplus U_2 \oplus \cdots \oplus \\ \mathcal{R}(D_{n-1}) \oplus \mathcal{R}(J_{n-1}) \oplus U_{n-1} \oplus \mathcal{R}(D_n) \oplus \mathcal{R}(D_n)^\perp.\end{aligned} \tag{4.2.9}$$

Comparing equalities (4.2.8) and (4.2.9), we conclude that $\mathcal{R}(T_n^d(A))$ is closed.

We have proved that $T_n^d(A)$ is upper Fredholm. Notice that $\beta(T_n^d(A)) = \dim(U_1) + \dim(U_2) + \cdots + \dim(U_{n-1}) + \beta(D_n)$. Now, with respect to (4.2.4), either $\beta(D_n) = \infty$ or we can choose at least one J_i such that its codimension is infinite, that is $\dim U_i = \infty$, $i \in \{1, \ldots, n-1\}$. In either case we get $\beta(T_n^d(A)) = \infty$ and it follows that $T_n^d(A)$ is upper Weyl. $\square$

Corollary 4.2.18. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. Then*

$$\sigma_{le}(D_1) \cup \Big(\bigcup_{k=2}^{n+1} \Delta_k \Big) \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_{lw}(T_n^d(A)) \subseteq \sigma_{le}(D_1) \cup \Big(\bigcup_{k=2}^{n+1} \Delta'_k \Big) \cup \Big(\bigcup_{k=2}^{n} \Delta''_k \Big),$$

where

$$\Delta_k := \Big\{ \lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) = \infty \ and \ \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty \Big\}, \ 2 \le k \le n,$$

$$\Delta_{n+1} := \Big\{ \lambda \in \mathbb{C} : \ \sum_{s=1}^{n} \beta(D_s - \lambda) < \sum_{s=1}^{n} \alpha(D_s - \lambda) \Big\},$$

$$\Delta'_k := \{ \lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) > \beta(D_{k-1} - \lambda) \}, \quad 2 \le k \le n,$$

$$\Delta'_{n+1} := \Delta_{n+1},$$

$$\Delta''_k := \Big\{ \lambda \in \mathbb{C} : \ \mathcal{R}(D_k - \lambda) \ is \ not \ closed \Big\}, \ 2 \le k \le n.$$

Remark 4.2.19. *Obviously, $\Delta_k \subseteq \Delta'_k$ for $2 \le k \le n+1$.*

Theorem 4.2.20. *Let $D_1 \in \mathcal{B}(X_1), D_2 \in \mathcal{B}(X_2)$. Consider the following statements:*

(i) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\Big(\alpha(D_2) \le \beta(D_1)$, $\beta(D_1) + \beta(D_2) = \infty$ and $\mathcal{R}(D_2)$ is closed$\Big)$ or $\Big(D_2 \in \Phi_+(X_2)$ and $\alpha(D_1) + \alpha(D_2) \le \beta(D_1) + \beta(D_2)\Big)$;

(ii) There exists $A \in \mathcal{B}_2$ such that $T_2^d(A) \in \Phi_+^-(X_1 \oplus X_2)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\Big(\beta(D_1) = \infty$ or $(\beta(D_2) = \infty$ and $\alpha(D_1) < \infty)\Big)$ or $\Big(D_2 \in \Phi_+(X_2)$ and $\alpha(D_1) + \alpha(D_2) \le \beta(D_1) + \beta(D_2)\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.2.21. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Then*

$$\sigma_{le}(D_1) \cup \Delta_2 \cup \Delta_3 \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_{lw}(T_2^d(A)) \subseteq \sigma_{le}(D_1) \cup \Delta_2' \cup \Delta_3 \cup \Delta_2'',$$

where

$$\Delta_2 := \Big\{\lambda \in \mathbb{C} : \ \alpha(D_2 - \lambda) = \infty \ and \ \beta(D_1 - \lambda) < \infty\Big\},$$

$$\Delta_3 := \Big\{\lambda \in \mathbb{C} : \ \alpha(D_1 - \lambda) + \alpha(D_2 - \lambda) > \beta(D_1 - \lambda) + \beta(D_2 - \lambda)\Big\},$$

$$\Delta_2' := \{\lambda \in \mathbb{C} : \ \alpha(D_2 - \lambda) \geq \beta(D_1 - \lambda)\},$$

$$\Delta_2'' := \Big\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_2 - \lambda) \ is \ not \ closed\Big\}.$$

Remark 4.2.22. *Notice that $\Delta_2 \subseteq \Delta_2'$.*

Statements concerning the lower Weyl spectrum of $T_n^d(A)$ we get by duality.

Theorem 4.2.23. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. Consider the following statements:*
(i) (a) $D_n \in \Phi_-(X_n)$;
(b) $\mathcal{R}(D_s)$ is closed for $1 \leq s \leq n-1$ and

$$\Big(\beta(D_s) \leq \alpha(D_{s+1}) \quad for \ 1 \leq s \leq n-1, \quad \sum_{s=1}^{n} \alpha(D_s) = \infty\Big) \tag{4.2.10}$$

or $\Big(D_s \in \Phi_-(X_s)$ for $1 \leq s \leq n-1$ and $\sum_{s=1}^{n} \beta(D_s) \leq \sum_{s=1}^{n} \alpha(D_s)\Big)$;

(ii) There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \Phi_-^+(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_n \in \Phi_-(X_n)$;
(b) $\Big(\alpha(D_j) = \infty$ for some $j \in \{2, \ldots, n\}$ and $\beta(D_s) < \infty$ for $j \leq s \leq n-1\Big)$ or $\Big(D_s \in \Phi_-(X_s)$ for $1 \leq s \leq n-1$ and $\sum_{s=1}^{n} \beta(D_s) \leq \sum_{s=1}^{n} \alpha(D_s)\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.2.24. *If $j=n$ in $(iii)(b)$, we simply omit condition "$\beta(D_s)<\infty$ for $j\le s\le n-1$".*

Proof. The result immediately follows from Theorem 4.2.16, having in mind the statements of Lemma 1.2.1 and Lemma 1.2.2. □

Corollary 4.2.25. *Let $D_1\in\mathcal{B}(X_1),\ D_2\in\mathcal{B}(X_2),\dots,D_n\in\mathcal{B}(X_n)$. Then*

$$\sigma_{re}(D_n)\cup\Big(\bigcup_{k=1}^{n-1}\Delta_k\Big)\cup\Delta_{n+1}\subseteq$$

$$\bigcap_{A\in\mathcal{B}_n}\sigma_{rw}(T_n^d(A))\subseteq$$

$$\sigma_{re}(D_n)\cup\Big(\bigcup_{k=1}^{n-1}\Delta_k'\Big)\cup\Delta_{n+1}\cup\Big(\bigcup_{k=1}^{n-1}\Delta_k''\Big),$$

where

$$\Delta_k:=\Big\{\lambda\in\mathbb{C}:\ \beta(D_k-\lambda)=\infty\ and\ \sum_{s=k+1}^{n}\alpha(D_s-\lambda)<\infty\Big\},\ 1\le k\le n-1,$$

$$\Delta_{n+1}:=\Big\{\lambda\in\mathbb{C}:\ \sum_{s=1}^{n}\alpha(D_s-\lambda)<\sum_{s=1}^{n}\beta(D_s-\lambda)\Big\},$$

$$\Delta_k':=\{\lambda\in\mathbb{C}:\ \beta(D_k-\lambda)>\alpha(D_{k+1}-\lambda)\},\quad 1\le k\le n-1,$$

$$\Delta_k'':=\Big\{\lambda\in\mathbb{C}:\ \mathcal{R}(D_k-\lambda)\ is\ not\ closed\Big\},\ 2\le k\le n-1.$$

Remark 4.2.26. *Obviously, $\Delta_k\subseteq\Delta_k'$ for $1\le k\le n-1$.*

Theorem 4.2.27. *Let $D_1\in\mathcal{B}(X_1),\ D_2\in\mathcal{B}(X_2)$. Consider the following statements:*

(i) (a) $D_2\in\Phi_-(X_2)$;

(b) $\Big(\beta(D_1)\le\alpha(D_2),\ \alpha(D_1)+\alpha(D_2)=\infty$ and $\mathcal{R}(D_1)$ is closed$\Big)$ or $\Big(D_1\in\Phi_-(X_1)$ and $\beta(D_1)+\beta(D_2)\le\alpha(D_1)+\alpha(D_2)\Big)$;

(ii) There exists $A\in\mathcal{B}_2$ such that $T_2^d(A)\in\Phi_-^+(X_1\oplus X_2)$;

(iii) (a) $D_2\in\Phi_-(X_2)$;

(b) $\Big(\alpha(D_2) = \infty$ *or* $(\alpha(D_1) = \infty$ *and* $\beta(D_2) < \infty)\Big)$ *or* $\Big(D_1 \in \Phi_-(X_1)$ *and* $\beta(D_1) + \beta(D_2) \leq \alpha(D_1) + \alpha(D_2)\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.2.28. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_{re}(D_2) \cup \Delta_1 \cup \Delta_3 \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_{rw}(T_2^d(A)) \subseteq \sigma_{re}(D_2) \cup \Delta_1' \cup \Delta_3 \cup \Delta_1'',$$

where

$$\Delta_1 := \Big\{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) = \infty \text{ and } \alpha(D_2 - \lambda) < \infty\Big\},$$

$$\Delta_3 := \Big\{\lambda \in \mathbb{C} : \ \alpha(D_1 - \lambda) + \alpha(D_2 - \lambda) < \beta(D_1 - \lambda) + \beta(D_2 - \lambda)\Big\},$$

$$\Delta_1' := \{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) \geq \alpha(D_2 - \lambda)\},$$

$$\Delta_1'' := \Big\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_1 - \lambda) \text{ is not closed}\Big\}.$$

Remark 4.2.29. *Notice that* $\Delta_1 \subseteq \Delta_1'$.

4.3 Fredholmness of $T_n^d(A)$

In this section we provide statements related to the Fredholmness of $T_n^d(A)$. One can notice that these statements are quite similar to the ones presented in the previous section. Their proofs are also similar, and so we omit them. First we consider separability case, and afterwards we generalize those results.

4.3.1 Separability case

Assume that $X_1, ..., X_n$ are separable Hilbert cases. This subsection bases on results from [45]. We start with a result which deals with upper semi-Fredholm invertibility of $T_n^d(A)$.

Theorem 4.3.1. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$ *be given. Consider the following conditions:*

(i) (a) $D_1 \in \Phi_+(X_1)$;

(b) $D_s \in \Phi_+(X_s)$ *for* $2 \le s \le n$

or

$\Big(\beta(D_j) = \infty$ *for some* $j \in \{1,\dots,n-1\}$, $\alpha(D_s) < \infty$ *for* $2 \le s \le j$ *and* $\mathcal{R}(D_s)$ *is closed for* $2 \le s \le n\Big)$;

(ii) *There exists* $A \in \mathcal{B}_n$ *such that* $T_n^d(A) \in \Phi_+(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$;

(b) $D_s \in \Phi_+(X_s)$ *for* $2 \le s \le n$

or

$\Big(\beta(D_j) = \infty$ *for some* $j \in \{1,\dots,n-1\}$, $\alpha(D_s) < \infty$ *for* $2 \le s \le j\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.3.2. *If* $j = 1$ *in* $(i)(b)$ *or* $(iii)(b)$, *part "*$\alpha(D_s) < \infty$ *for* $2 \le s \le j$*" is omitted there.*

Remark 4.3.3. *Notice the similarity between sufficent condition (i) and necessary condition (iii): parts (i)(a) and (iii)(a) are the same, while (i)(b) and (iii)(b) differ in "*$\mathcal{R}(D_s)$ *is closed for* $2 \le s \le n$*" solely.*

Remark 4.3.4. *Again, we have the validity of part* $(ii) \Rightarrow (iii)$ *without assuming separability of* $X_1,\dots,X_n$.

Corollary 4.3.5. *([55, Theorem 1], corrected version)*
Let $D_1 \in \mathcal{B}(X_1), D_2 \in \mathcal{B}(X_2),\dots,D_n \in \mathcal{B}(X_n)$. *Then*

$$\sigma_{le}(D_1) \cup \Big(\bigcup_{k=2}^{n} \Delta_k\Big) \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_{le}(T_n^d(A)) \subseteq \sigma_{le}(D_1) \cup \Big(\bigcup_{k=2}^{n} \Delta_k\Big) \cup \Big(\bigcup_{k=2}^{n} \Delta_k'\Big),$$

where

$$\Delta_k := \Big\{\lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) = \infty \ \textit{and} \ \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty\Big\}, \ 2 \le k \le n,$$

$$\Delta'_k := \Big\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_k - \lambda) \ is\ not\ closed\Big\},\ 2 \leq k \leq n.$$

Remark 4.3.6. *Notice a difference between definitions of sets Δ_k, $2 \leq k \leq n$, in Corollary 4.3.5 and in [55, Theorem 1].*

If we put $n = 2$ we get:

Theorem 4.3.7. *([60, Theorem 2.10], corrected version) Let $D_1 \in \mathcal{B}(X_1)$ and $D_2 \in \mathcal{B}(X_2)$. Consider the following statements:*
(i) (a) $D_1 \in \Phi_+(X_1)$;
(b) $D_2 \in \Phi_+(X_2)$ or $\Big(\beta(D_1) = \infty$ and $\mathcal{R}(D_2)$ is closed$\Big)$.
(ii) There exists $A \in \mathcal{B}_2$ such that $T_2^d(A) \in \Phi_+(X_1 \oplus X_2)$;
(iii) (a) $D_1 \in \Phi_+(X_1)$;
(b) $D_2 \in \Phi_+(X_2)$ or $\beta(D_1) = \infty$.
Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.3.8. *([60, Corollary 2.12], corrected version) Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Then*

$$\sigma_{le}(D_1) \cup \Delta \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_{le}(T_2^d(A)) \subseteq \sigma_{le}(D_1) \cup \Delta \cup \Delta',$$

where

$$\Delta := \Big\{\lambda \in \mathbb{C} : \ \alpha(D_2 - \lambda) = \infty \ and \ \beta(D_1 - \lambda) < \infty\Big\},$$

$$\Delta' := \{\lambda \in \mathbb{C} : \ \mathcal{R}(D_2 - \lambda) \ is\ not\ closed\}.$$

Now we list statements dealing with the lower semi-Fredholm spectrum.

Theorem 4.3.9. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$ be given. Consider the following conditions:*
(i) (a) $D_n \in \Phi_-(X_n)$;
(b) $D_s \in \Phi_-(X_s)$ for $1 \leq s \leq n-1$
or
$\Big(\alpha(D_j) = \infty$ for some $j \in \{2, \dots, n\}$, $\beta(D_s) < \infty$ for $j \leq s \leq n-1$ and $\mathcal{R}(D_s)$ is closed for $1 \leq s \leq n-1\Big)$;

(ii) There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \Phi_-(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_n \in \Phi_-(X_n)$;
(b) $D_s \in \Phi_-(X_s)$ *for* $1 \le s \le n-1$
or

$$\Big(\alpha(D_j) = \infty \text{ for some } j \in \{2, \ldots, n\},\ \beta(D_s) < \infty \text{ for } j \le s \le n-1\Big).$$

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.3.10. *If* $j = n$ *in* $(i)(b)$ *or* $(iii)(b)$, *part "*$\beta(D_s) < \infty$ *for* $j \le s \le n-1$*" is omitted there.*

Remark 4.3.11. *Notice the similarity between sufficent condition (i) and necessary condition (iii): parts (i)(a) and (iii)(a) are the same, while (i)(b) and (iii)(b) differ in "*$\mathcal{R}(D_s)$ *is closed for* $1 \le s \le n-1$*" solely.*

Corollary 4.3.12. *([55, Theorem 2], corrected version)*

$$\sigma_{re}(D_n) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k\Big) \subseteq$$
$$\bigcap_{A \in \mathcal{B}_n} \sigma_{re}(T_n^d(A)) \subseteq$$
$$\sigma_{re}(D_n) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k\Big) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k'\Big),$$

where

$$\Delta_k := \Big\{\lambda \in \mathbb{C} :\ \beta(D_k - \lambda) = \infty \text{ and } \sum_{s=k+1}^{n} \alpha(D_s - \lambda) < \infty\Big\},\ 1 \le k \le n-1,$$

$$\Delta_k' := \Big\{\lambda \in \mathbb{C} :\ \mathcal{R}(D_k - \lambda) \text{ is not closed}\Big\},\ 1 \le k \le n-1.$$

Remark 4.3.13. *Again we have a difference between definitions of the sets* Δ_k, $1 \le k \le n-1$ *in Corollary 4.3.12 and in [55, Theorem 2].*

Theorem 4.3.14. *([60, Theorem 2.11], corrected version) Let* $D_1 \in \mathcal{B}(X_1), D_2 \in \mathcal{B}(X_2)$. *Consider the following conditions:*

(i) (a) $D_2 \in \Phi_-(X_2)$;
(b) $D_1 \in \Phi_-(X_1)$ *or* $\Big(\alpha(D_2) = \infty$ *and* $\mathcal{R}(D_1)$ *is closed*$\Big)$;

(ii) *There exists* $A \in \mathcal{B}_2$ *such that* $T_2^d(A) \in \Phi_-(X_1 \oplus X_2)$;

(iii) *(a)* $D_2 \in \Phi_-(X_2)$;
(c) $D_1 \in \Phi_-(X_1)$ *or* $\alpha(D_2) = \infty$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.3.15. *([60, Corollary 2.13], corrected version) Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_{re}(D_2) \cup \Delta \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_{re}(T_2^d(A)) \subseteq \sigma_{re}(D_2) \cup \Delta \cup \Delta',$$

where

$$\Delta := \Big\{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) = \infty \ and \ \alpha(D_2 - \lambda) < \infty\Big\},$$
$$\Delta' := \{\lambda \in \mathbb{C} : \ \mathcal{R}(D_1 - \lambda) \ is \ not \ closed\}.$$

And this is a result about Fredholm invertibility of $T_n^d(A)$.

Theorem 4.3.16. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. *Consider the following statements:*
(i) *(a)* $D_1 \in \Phi_+(X_1)$ *and* $D_n \in \Phi_-(X_n)$;
(b) $\Big(D_j \in \Phi_+(X_j)$ *for* $2 \leq j \leq n$ *and* $D_k \in \Phi_-(X_k)$ *for* $1 \leq k \leq n-1\Big)$
or
$\Big(\beta(D_j) = \infty$ *for some* $j \in \{1, \dots, n-1\}$, $\alpha(D_j) < \infty$, $\alpha(D_k) = \infty$ *for some* $k \in \{2, \dots, n\}$, $k > j$, $\beta(D_k) < \infty$, $\alpha(D_s), \beta(D_s) < \infty$ *for* $1 \leq s \leq j-1$ *and* $k+1 \leq s \leq n$, *and* $\mathcal{R}(D_s)$ *is closed for* $2 \leq s \leq n-1\Big)$

(ii) *There exists* $A \in \mathcal{B}_n$ *such that* $T_n^d(A) \in \Phi(X_1 \oplus \cdots \oplus X_n)$;

(iii) *(a)* $D_1 \in \Phi_+(X_1)$ *and* $D_n \in \Phi_-(X_n)$;
(b) $\Big(D_j \in \Phi_+(X_j)$ *for* $2 \leq j \leq n$ *and* $D_k \in \Phi_-(X_k)$ *for* $1 \leq k \leq n-1\Big)$
or
$\Big(\beta(D_j) = \infty$ *for some* $j \in \{1, \dots, n-1\}$ *and* $\alpha(D_s) < \infty$ *for* $2 \leq s \leq j$, $\alpha(D_k) = \infty$ *for some* $k \in \{2, \dots, n\}$, *and* $\beta(D_s) < \infty$ *for* $k \leq s \leq n-1$, $k > j\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.3.17. *If* $j = 1$ *and/or* $k = n$ *in* $(i)(b)$, *condition that is ought to hold for* $1 \leq s \leq j-1$ *and/or* $k+1 \leq s \leq n$ *is omitted there.*

If $j = 1$ and/or $k = n$ in $(iii)(b)$, condition that is ought to hold for $2 \le s \le j$ and/or $k \le s \le n-1$ is omitted there.
If $n = 2$, condition "$\mathcal{R}(D_s)$ is closed for $2 \le s \le n-1$" is omitted in $(i)(b)$.

Remark 4.3.18. *Notice the similarity between sufficent condition (i) and necessary condition (iii): again, parts (i)(a) and (iii)(a) are the same, while (i)(b) and (iii)(b) differ only slightly.*

Proof. $(ii) \Rightarrow (iii)$

Let $T_n^d(A)$ be Fredholm for some $A \in \mathcal{B}_n$. Then $T_n^d(A)$ is both upper and lower semi-Fredholm, and so by employing Theorems 4.3.1 and 4.3.9 we easily get the desired.

$(i) \Rightarrow (ii)$

Let conditions $(i)(a)$ and $(i)(b)$ hold. If $D_j \in \Phi_+(X_j)$ for $2 \le j \le n$ and $D_k \in \Phi_-(X_k)$ for $1 \le k \le n-1$, then all D_i's are Fredholm, and so we trivially choose $A = \mathbf{0}$. Assume the validity of a lengthy condition expressed in $(i)(b)$. Then, one easily checks that one of the Cases 1 or 2 in the proof of [55, Theorem 3] holds, and so we get $A \in \mathcal{B}_n$ so that $T_n^d(A) \in \Phi(X_1 \oplus \cdots \oplus X_n)$ as described there. $\square$

Corollary 4.3.19. *([55, Theorem 3], corrected version) Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. Then*

$$\sigma_{le}(D_1) \cup \sigma_{re}(D_n) \cup \left(\bigcup_{k=2}^{n-1} \Delta_k\right) \cup \Delta_n \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_e(T_n^d(A)) \subseteq \sigma_{le}(D_1) \cup \sigma_{re}(D_n) \cup \left(\bigcup_{k=2}^{n-1} \Delta_k\right) \cup \Delta_n \cup \left(\bigcup_{k=2}^{n-1} \Delta'_k\right),$$

where

$$\Delta_k = \{\lambda \in \mathbb{C} : \alpha(D_k - \lambda) = \infty \text{ and } \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty\} \cup$$
$$\{\lambda \in \mathbb{C} : \beta(D_k - \lambda) = \infty \text{ and } \sum_{s=k+1}^{n} \alpha(D_s - \lambda) < \infty\}, \quad 2 \leq k \leq n-1,$$

$$\Delta_n = \{\lambda \in \mathbb{C} : \alpha(D_n - \lambda) = \infty \text{ and } \sum_{s=1}^{n-1} \beta(D_s - \lambda) < \infty\} \cup$$
$$\{\lambda \in \mathbb{C} : \beta(D_1 - \lambda) = \infty \text{ and } \sum_{s=2}^{n} \alpha(D_s - \lambda) < \infty\},$$

$$\Delta'_k := \Big\{\lambda \in \mathbb{C} : \mathcal{R}(D_k - \lambda) \text{ is not closed}\Big\}, \ 2 \leq k \leq n-1,$$

Remark 4.3.20. *Again, due to the presence of sets Δ'_k, $2 \leq k \leq n-1$, we have a difference between definitions of collections Δ_k, $2 \leq k \leq n-1$, in Corollary 4.3.19 and in [55, Theorem 3].*

We get some interesting results for $n = 2$ that seem new in the literature.

Theorem 4.3.21. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Consider the following statements:*
(i) There exists $A \in \mathcal{B}_2$ such that $T_2^d(A) \in \Phi(X_1 \oplus X_2)$;

(ii) (a) $D_1 \in \Phi_+(X_1)$ and $D_2 \in \Phi_-(X_2)$;
(b) $\beta(D_1) = \alpha(D_2) = \infty$ or $\Big(D_2 \in \Phi_+(X_2)$ and $D_1 \in \Phi_-(X_1)\Big)$.

Then $(i) \Leftrightarrow (ii)$.

Corollary 4.3.22. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Then*

$$\bigcap_{A \in \mathcal{B}_2} \sigma_e(T_2^d(A)) = \sigma_{le}(D_1) \cup \sigma_{re}(D_2) \cup \Delta,$$

where

$$\Delta = \{\lambda \in \mathbb{C} : \alpha(D_2 - \lambda) = \infty \text{ and } \beta(D_1 - \lambda) < \infty\} \cup$$
$$\{\lambda \in \mathbb{C} : \beta(D_1 - \lambda) = \infty \text{ and } \alpha(D_2 - \lambda) < \infty\}.$$

4.3.2 Nonseparable spaces

We now assume $X_1,...,X_n$ to be arbitrary infinite dimensional Hilbert spaces. This subsection bases on article [44]. We generalize results of [6],[60] from $n = 2$ to an arbitrary dimension of upper triangular operators, and we pose perturbation results of [55] without assuming separability of underlying spaces. Proofs of theorems to follow are very similar to proofs of theorems from subsection 4.2.2, and so we omit them.

We start with a result which deals with the upper Fredholm spectrum of $T_n^d(A)$.

Theorem 4.3.23. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2),...,D_n \in \mathcal{B}(X_n)$. Consider the following statements:*
(i) (a) $D_1 \in \Phi_+(X_1)$;
(b) $\mathcal{R}(D_s)$ is closed for $2 \le s \le n$ and

$$\alpha(D_s) \le \beta(D_{s-1}) \quad for\ 2 \le s \le n \tag{4.3.1}$$

or $D_s \in \Phi_+(X_s)$ for $2 \le s \le n$;

(ii) There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \Phi_+(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$;
(b) $\Big(\beta(D_j) = \infty$ for some $j \in \{1,...,n-1\}$ and $\alpha(D_s) < \infty$ for $2 \le s \le j\Big)$ or $D_s \in \Phi_+(X_s)$ for $2 \le s \le n$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.3.24. *If $j = 1$ in $(iii)(b)$, we simply omit condition "$\alpha(D_s) < \infty$ for $2 \le s \le j$" there.*

Corollary 4.3.25. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. *Then*

$$\sigma_{le}(D_1) \cup \Big(\bigcup_{k=2}^{n} \Delta_k\Big) \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_{le}(T_n^d(A)) \subseteq \sigma_{le}(D_1) \cup \Big(\bigcup_{k=2}^{n} (\Delta_k' \cap \Delta')\Big) \cup \Big(\bigcup_{k=2}^{n} \Delta_k''\Big),$$

where

$$\Delta_k := \Big\{\lambda \in \mathbb{C}:\ \alpha(D_k - \lambda) = \infty \ \text{and} \ \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty\Big\},\ 2 \leq k \leq n,$$

$$\Delta_k' := \{\lambda \in \mathbb{C}:\ \alpha(D_k - \lambda) > \beta(D_{k-1} - \lambda)\}, \quad 2 \leq k \leq n,$$

$$\Delta' := \Big\{\lambda \in \mathbb{C}:\ \sum_{s=2}^{n} \alpha(D_s - \lambda) = \infty\Big\},$$

$$\Delta_k'' := \Big\{\lambda \in \mathbb{C}:\ \mathcal{R}(D_k - \lambda) \ \text{is not closed}\Big\},\ 2 \leq k \leq n.$$

Remark 4.3.26. *Obviously,* $\Delta_k \subseteq \Delta_k' \cap \Delta'$ *for* $2 \leq k \leq n$.

Theorem 4.3.27. *Let* $D_1 \in \mathcal{B}(X_1), D_2 \in \mathcal{B}(X_2)$. *Consider the following statements:*

(i) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\Big(\alpha(D_2) \leq \beta(D_1)$ *and* $\mathcal{R}(D_2)$ *is closed*$\Big)$ *or* $D_2 \in \Phi_+(X_2)$;

(ii) *There exists* $A \in \mathcal{B}_2$ *such that* $T_2^d(A) \in \Phi_+(X_1 \oplus X_2)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$;

(b) $\beta(D_1) = \infty$ *or* $D_2 \in \Phi_+(X_2)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.3.28. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_{le}(D_1) \cup \Delta_2 \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_{le}(T_2^d(A)) \subseteq \sigma_{le}(D_1) \cup \Delta_2' \cup \Delta_2'',$$

where

$$\Delta_2 := \Big\{\lambda \in \mathbb{C} : \ \alpha(D_2-\lambda) = \infty \ and \ \beta(D_1-\lambda) < \infty\Big\},$$

$$\Delta_2' := \{\lambda \in \mathbb{C} : \ \alpha(D_2-\lambda) \geq \beta(D_1-\lambda)\},$$

$$\Delta_2'' := \Big\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_2-\lambda) \ is \ not \ closed\Big\}.$$

Remark 4.3.29. *Notice that* $\Delta_2 \subseteq \Delta_2'$.

Statements concerning the lower Fredholm spectrum of $T_n^d(A)$ we get by duality.

Theorem 4.3.30. *Let* $D_1 \in \mathcal{B}(X_1), \ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. *Consider the following statements:*
(i) (a) $D_n \in \Phi_-(X_n)$;
(b) $\mathcal{R}(D_s)$ *is closed for* $1 \leq s \leq n-1$ *and*

$$\beta(D_s) \leq \alpha(D_{s+1}) \quad for \ 1 \leq s \leq n-1 \tag{4.3.2}$$

or $D_s \in \Phi_-(X_s)$ *for* $1 \leq s \leq n-1$;

(ii) *There exists* $A \in \mathcal{B}_n$ *such that* $T_n^d(A) \in \Phi_-(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_n \in \Phi_-(X_n)$;
(b) $\Big(\alpha(D_j) = \infty$ *for some* $j \in \{2,...,n\}$ *and* $\beta(D_s) < \infty$ *for* $j \leq s \leq n-1\Big)$ *or* $D_s \in \Phi_-(X_s)$ *for* $1 \leq s \leq n-1$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.3.31. *If* $j = n$ *in* $(iii)(b)$, *we simply omit condition "*$\beta(D_s) < \infty$ *for* $j \leq s \leq n-1$*" there.*

Corollary 4.3.32. *Let* $D_1 \in \mathcal{B}(X_1), \ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. *Then*

$$\sigma_{re}(D_n) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k\Big) \subseteq$$

$$\bigcap_{A \in \mathcal{B}_n} \sigma_{re}(T_n^d(A)) \subseteq$$

$$\sigma_{re}(D_n) \cup \Big(\bigcup_{k=1}^{n-1} (\Delta_k' \cap \Delta'\Big) \cup \Big(\bigcup_{k=1}^{n-1} \Delta_k''\Big),$$

where

$$\Delta_k := \left\{\lambda \in \mathbb{C} : \ \beta(D_k - \lambda) = \infty \ and \ \sum_{s=k+1}^{n} \alpha(D_s - \lambda) < \infty\right\}, \ 1 \leq k \leq n-1,$$

$$\Delta'_k := \{\lambda \in \mathbb{C} : \ \beta(D_k - \lambda) > \alpha(D_{k+1} - \lambda)\}, \quad 1 \leq k \leq n-1,$$

$$\Delta' := \{\lambda \in \mathbb{C} : \ \sum_{s=1}^{n-1} \beta(D_s - \lambda) = \infty\},$$

$$\Delta''_k := \left\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_k - \lambda) \ is \ not \ closed\right\}, \ 2 \leq k \leq n-1.$$

Remark 4.3.33. $\Delta_k \subseteq \Delta'_k \cap \Delta'$ *for* $1 \leq k \leq n-1$.

Theorem 4.3.34. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. *Consider the following statements:*

(i) (a) $D_2 \in \Phi_-(X_2)$;

(b) $\left(\beta(D_1) \leq \alpha(D_2) \ and \ \mathcal{R}(D_1) \ is \ closed\right)$ *or* $D_1 \in \Phi_-(X_1)$;

(ii) *There exists* $A \in \mathcal{B}_2$ *such that* $T_2^d(A) \in \Phi_-(X_1 \oplus X_2)$;

(iii) (a) $D_2 \in \Phi_-(X_2)$;

(b) $\alpha(D_2) = \infty$ *or* $D_1 \in \Phi_-(X_1)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.3.35. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_{re}(D_2) \cup \Delta_1 \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_{re}(T_2^d(A)) \subseteq \sigma_{re}(D_2) \cup \Delta'_1 \cup \Delta''_1,$$

where

$$\Delta_1 := \left\{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) = \infty \ and \ \alpha(D_2 - \lambda) < \infty\right\},$$

$$\Delta'_1 := \{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) \geq \alpha(D_2 - \lambda)\},$$

$$\Delta''_1 := \left\{\lambda \in \mathbb{C} : \ \mathcal{R}(D_1 - \lambda) \ is \ not \ closed\right\}.$$

Remark 4.3.36. *Notice that* $\Delta_1 \subseteq \Delta'_1$.

Last topic is the class $\Phi(X_1 \oplus \cdots \oplus X_n)$ and its corresponding essential spectrum.

Theorem 4.3.37. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. Consider the following statements:*
(i) (a) $D_1 \in \Phi_+(X_1)$ and $D_n \in \Phi_-(X_n)$;
(b) $\Big(\mathcal{R}(D_s)$ is closed for $2 \le s \le n-1$ and $\big(\alpha(D_s) = \beta(D_{s-1})$ for $2 \le s \le n$ or $\alpha(D_s) \le \beta(D_{s-1}) < \infty$ for $2 \le s \le n\big)\Big)$ or $\Big(D_j \in \Phi_+(X_j)$ for $2 \le j \le n$ and $D_k \in \Phi_-(X_k)$ for $1 \le k \le n-1\Big)$;

(ii) There exists $A \in \mathcal{B}_n$ such that $T_n^d(A) \in \Phi(X_1 \oplus \cdots \oplus X_n)$;

(iii) (a) $D_1 \in \Phi_+(X_1)$ and $D_n \in \Phi_-(X_n)$;
(b) $\Big(\beta(D_j) = \infty$ for some $j \in \{1, \dots, n-1\}$ and $\alpha(D_s) < \infty$ for $2 \le s \le j$, $\alpha(D_k) = \infty$ for some $k \in \{2, \dots, n\}$, and $\beta(D_s) < \infty$ for $k \le s \le n-1$, $k > j\Big)$ or $\Big(D_j \in \Phi_+(X_j)$ for $2 \le j \le n$ and $D_k \in \Phi_-(X_k)$ for $1 \le k \le n-1\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Remark 4.3.38. *If $j = 1$ and/or $k = n$ in $(iii)(b)$, condition that is ought to hold for $2 \le s \le j$ and/or $k \le s \le n-1$ is omitted there.*

Proof. $(ii) \Rightarrow (iii)$

Let $T_n^d(A)$ be Fredholm for some $A \in \mathcal{B}_n$. Then $T_n^d(A)$ is both left and lower Fredholm, and so by employing Theorems 4.3.23 and 4.3.30 we easily get the desired.

$(i) \Rightarrow (ii)$

If $D_j \in \Phi_+(X_j)$ for $2 \le j \le n$ and $D_k \in \Phi_-(X_k)$ for $1 \le k \le n-1$ choose trivially $A = \mathbf{0}$. Otherwise, this part follows the argument as seen in the proof of Theorem 4.2.16. Namely, assumptions of $(i)(b)$ ensure the existence of left invertible J_i's, and so we choose $A = (A_{ij})$ as shown there. We shall again have $\alpha(T_n^d(A)) = \alpha(D_1) < \infty$, and due to our assumptions we can choose all U_i's to be finite dimensional. Therefore, $\beta(T_n^d(A)) = \dim U_1 + \cdots + \dim U_{n-1} + \beta(D_n) < \infty$, having in mind that $D_n \in \Phi_-(X_n)$. $\square$

Corollary 4.3.39. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. Then*

$$\sigma_{le}(D_1) \cup \sigma_{re}(D_n) \cup \Big(\bigcup_{k=2}^{n-1} \Delta_k\Big) \cup \Delta_n \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_e(T_n^d(A)) \subseteq$$
$$\sigma_{le}(D_1) \cup \sigma_{re}(D_n) \cup \Big(\Big(\bigcup_{k=2}^{n} \delta_k'\Big) \cap \Big(\bigcup_{k=2}^{n} \Delta_k'\Big)\Big) \cup \Big(\bigcup_{k=2}^{n-1} \Delta_k''\Big),$$

where

$$\Delta_k = \Big\{\lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) = \infty \ and \ \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty\Big\} \cup$$
$$\Big\{\lambda \in \mathbb{C} : \ \beta(D_k - \lambda) = \infty \ and \ \sum_{s=k+1}^{n} \alpha(D_s - \lambda) < \infty\Big\}, \quad 2 \leq k \leq n-1,$$

$$\Delta_n = \Big\{\lambda \in \mathbb{C} : \ \alpha(D_n - \lambda) = \infty \ and \ \sum_{s=1}^{n-1} \beta(D_s - \lambda) < \infty\Big\} \cup$$
$$\Big\{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) = \infty \ and \ \sum_{s=2}^{n} \alpha(D_s - \lambda) < \infty\Big\},$$

$$\delta_k' := \{\lambda \in \mathbb{C} : \ \beta(D_{k-1} - \lambda) = \infty \ or \ \alpha(D_k - \lambda) > \beta(D_{k-1} - \lambda)\}, \ 2 \leq k \leq n,$$
$$\Delta_k' := \{\lambda \in \mathbb{C} : \ \alpha(D_s - \lambda) \neq \beta(D_{s-1} - \lambda)\}, \ 2 \leq k \leq n,$$
$$\Delta_k'' := \{\lambda \in \mathbb{C} : \ \mathcal{R}(D_s - \lambda) \ is \ not \ closed\}, \ 2 \leq k \leq n-1.$$

Remark 4.3.40. *Obviously, $\Delta_k \subseteq \Delta_k' \cap \delta_k'$ for each $2 \leq k \leq n$.*

Theorem 4.3.41. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2)$. Consider the following statements:*

(i) (a) $D_1 \in \Phi_+(X_1)$ and $D_2 \in \Phi_-(X_2)$;

(b) $\Big(\alpha(D_2) = \beta(D_1)$ or $\alpha(D_2) \leq \beta(D_1) < \infty\Big)$ or $\Big(D_2 \in \Phi_+(X_2)$ and $D_1 \in \Phi_-(X_1)\Big)$.

(ii) There exists $A \in \mathcal{B}_2$ such that $T_2^d(A) \in \Phi(X_1 \oplus X_2)$.

(iii) (a) $D_1 \in \Phi_+(X_1)$ and $D_2 \in \Phi_-(X_2)$;

(b) $\Big(\alpha(D_2) = \beta(D_1) = \infty\Big)$ *or* $\Big(D_2 \in \Phi_+(X_2)$ *and* $D_1 \in \Phi_-(X_1)\Big)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Corollary 4.3.42. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_{le}(D_1) \cup \sigma_{re}(D_2) \cup \Delta \subseteq \bigcap_{A \in \mathcal{B}_2} \sigma_e(T_2^d(A)) \subseteq \sigma_{le}(D_1) \cup \sigma_{re}(D_2) \cup \Delta',$$

where

$$\Delta = \{\lambda \in \mathbb{C} :\ \alpha(D_2 - \lambda) = \infty \text{ and } \beta(D_1 - \lambda) < \infty\} \cup \{\lambda \in \mathbb{C} :\ \beta(D_1 - \lambda) = \infty \text{ and } \alpha(D_2 - \lambda) < \infty\},$$

$$\Delta' = \{\lambda \in \mathbb{C} :\ \alpha(D_2 - \lambda) \neq \beta(D_1 - \lambda)\} \cap \{\lambda \in \mathbb{C} :\ \beta(D_1 - \lambda) = \infty \text{ or } \alpha(D_2 - \lambda) > \beta(D_1 - \lambda)\}.$$

Remark 4.3.43. *Notice that* $\Delta \subseteq \Delta'$.

Chapter 5

Filling in holes problem

In this chapter we use results from previous chapters in order to deal with the filling in holes problem for the operator matrices. In case $n = 2$, this problem can be formulated as follows. Consider operator M_C from Chapter 2. Then, in general, the following is true:

$$\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup W.$$

The filling in holes problem has as its task desrcribing set W. It usually turns out that this set is a union of some of the holes in $\sigma(M_C)$, which explains the name of this problem. We present this problem in several stages, first for $n = 2$, then for $n = 3$, and afterwards for general $n \geq 3$. The main tool in succeding sections will be the concept of polynomially convex hull. Denote by $Poly_1$ the collection of all complex polynomials of one variable.

Definition 5.0.1. *[11] Let $K \subseteq \mathbb{C}$ be compact. The polynomial hull of K is defined as*

$$Hull(K) = \{\lambda \in \mathbb{C} : \ |p(z)| \leq \sup_{z \in K} |p(z)| \ for \ every \ p \in Poly_1\}.$$

Obviously, $K \subseteq Hull(K)$, and if $K = Hull(K)$ we say that K is *polynomially convex.* Notice also that if $K_1 \subseteq K_2$, then $Hull(K_1) \subseteq Hull(K_2)$. The most important for us is the following relation between polynomial hulls and holes.

Theorem 5.0.2. *[11] If $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \setminus Hull(K)$ is equal to the* *unbounded component of* $\mathbb{C} \setminus K$. *Hence,*

$$Hull(K) = K \cup \text{"holes in } K\text{"}.$$

5.1 Case $n = 2$

In this case the filling in holes problem was successfully solved about 20 years ago by Han, Lee, and Lee [25], their work being done on arbitrary Banach spaces. We present their results. Assume that X_1, X_2 are arbitrary Banach spaces.

From [25, Corollary 4] we see that, in perturbing a nilpotent matrix $\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$ to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\sigma(M_C)$ shrinks from $\sigma(A) \cup \sigma(B)$. How much of $\sigma(A) \cup \sigma(B)$ survives? The following theorem provides a clue.

Theorem 5.1.1. *For a given pair* (A, B) *of operators we have*

$$\eta(\sigma(M_C)) = \eta(\sigma(A) \cup \sigma(B)) \quad for\ every\ C \in \mathcal{B}(X_2, X_1), \tag{5.1.1}$$

where $\eta(\cdot)$ *denotes the "polynomially convex hull".*

Proof. See [25]. □

The following corollary says that the passage from $\sigma(A) \cup \sigma(B)$ to $\sigma(M_C)$ is the punching of some open sets in $\sigma(A) \cap \sigma(B)$.

Corollary 5.1.2. *For a given pair* (A, B) *of operators we have*

$$\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup W,$$

where W *is the union of certain of the holes in* $\sigma(M_C)$ *which happen to be subsets of* $\sigma(A) \cap \sigma(B)$.

Proof. See [25]. □

The following is a generalization of [20, Problem 72].

Corollary 5.1.3. *If $\sigma(A) \cap \sigma(B)$ has no interior points, then*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad for\ every\ C \in \mathcal{B}(X_2, X_1).$$

In particular, if either $A \in \mathcal{B}(X_1)$ or $B \in \mathcal{B}(X_2)$ is a compact operator, then the previous equality holds.

We now consider another case in which equality in Corollary 5.1.3 holds. To do this write, for $T \in \mathcal{B}(X)$,

$$\rho_\sigma^l(T) = \sigma(T) \setminus \sigma_l(T) \quad and \quad \rho_\sigma^t(T) = \sigma(T) \setminus \sigma_r(T)$$

Thus by Corollary 5.1.3 and Theorem 2.1.5 we can see that holes in $\sigma(M_C)$ should lie in $\rho_\sigma^l(A) \cap \rho_\sigma^r(B)$. Thus we have:

Corollary 5.1.4. *If $\rho_\sigma^l(A) \cap \rho_\sigma^r(B) = \emptyset$ then*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad for\ every\ C \in \mathcal{B}(X_2, X_1).$$

We conclude with an application of Corollary 5.1.4.

Corollary 5.1.5. *Suppose X_1 and X_2 are Hilbert spaces. If either $A \in \mathcal{B}(X_1)$ is cohyponormal or $B \in \mathcal{B}(X_2)$ is hyponormal, then*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad for\ every\ C \in \mathcal{B}(X_2, X_1).$$

Proof. See [25]. □

5.2 Case $n = 3$

In this case the filling in holes problem was successfully solved a few years ago by Alatancang et al. [58], their work being done on separable Hilbert spaces. We present their results. Assume that X_1, X_2, X_3 are separable Hilbert spaces. Now we show that

$$\sigma(A) \cup \sigma(B) \cup \sigma(C) = \sigma(M_{D,E,F}) \cup W,$$

where W is the union of certain gaps in $\sigma(M_{D,E,F})$ which are subsets of $(\sigma(A) \cap \sigma(B)) \cup (\sigma(A) \cap \sigma(C)) \cup (\sigma(B) \cap \sigma(C))$. We obtain a necessary and sufficent condition for the relation $\sigma(M_{D,E,F}) = \sigma(A) \cup \sigma(B) \cup \sigma(C)$ to hold for any $D \in \mathcal{B}(X_2, X_1), E \in \mathcal{B}(X_3, X_1), F \in \mathcal{B}(X_3, X_2)$.

Theorem 5.2.1. *Let triple (A, B, C) be given. Then*

$$\sigma(A) \cup \sigma(B) \cup \sigma(C) = \sigma(M_{D,E,F}) \cup W,$$

where W is the union of some gaps in $\sigma(M_{D,E,F})$, which are subsets of $(\sigma(A) \cap \sigma(B)) \cup (\sigma(A) \cap \sigma(C)) \cup (\sigma(B) \cap \sigma(C))$.

Proof. See [58]. □

In Preliminary section we have defined point and residual spectrum of an operator. Now we define some of their parts. The following subdivisions are closely related to the relevant space decomposition and are useful when studying spectral properties of operators. Let $T \in \mathcal{B}(X)$. Then:

$\sigma_{p,1}(T) = \{\lambda \in \mathbb{C} : \ \lambda \in \sigma_p(T) : \ \mathcal{R}(\lambda - T) = X\}$;
$\sigma_{p,2}(T) = \{\lambda \in \mathbb{C} : \ \lambda \in \sigma_p(T) : \ \overline{\mathcal{R}(\lambda - T)} = X \ and \ \mathcal{R}(\lambda - T) \neq X\}$;
$\sigma_{p,3}(T) = \{\lambda \in \mathbb{C} : \ \lambda \in \sigma_p(T) : \ \overline{\mathcal{R}(\lambda - T)} \neq X \ and \ \mathcal{R}(\lambda - T) \ is \ closed\}$;
$\sigma_{p,4}(T) = \{\lambda \in \mathbb{C} : \ \lambda \in \sigma_p(T) : \ \overline{\mathcal{R}(\lambda - T)} \neq X \ and \ \mathcal{R}(T - \lambda) \ is \ not \ closed\}$;
$\sigma_{r,1}(T) = \{\lambda \in \mathbb{C} : \ \lambda \in \sigma_r(T) : \ \mathcal{R}(\lambda - T) \ is \ closed\}$;
$\sigma_{r,2}(T) = \{\lambda \in \mathbb{C} : \ \lambda \in \sigma_r(T) : \ \mathcal{R}(\lambda - T) \ is \ not \ closed\}$;

Corollary 5.2.2. *Let triple (A, B, C) be given. Then*

$$\sigma(M_{D,E,F}) = \sigma(A) \cup \sigma(B) \cup \sigma(C)$$

for any $D \in \mathcal{B}(X_2, X_1), E \in \mathcal{B}(X_3, X_1), F \in \mathcal{B}(X_3, X_2)$ if and only if the following conditions hold.
(i) If $\lambda \in \rho(C)$, then one of the following statements (a)-(b) is satisfied:

(a) $\lambda \in \sigma_{r,1}(A) \setminus \sigma_\delta(B)$ implies $\alpha(B - \lambda) = 0$ or $\alpha(B - \lambda) \neq \beta(A - \lambda)$);

(b) $\lambda \in \sigma_{p,1}(B) \setminus \sigma_l(A)$ implies $\beta(A - \lambda) = 0$ or $\alpha(B - \lambda) \neq \beta(A - \lambda)$.

(ii) If $\lambda \in \rho(A)$, then one of the following statements $(a) - (b)$ is satisfied:

(a) $\lambda \in \sigma_{r,1}(B) \setminus \sigma_\delta(C)$ implies $\alpha(C - \lambda) = 0$ or $\alpha(C - \lambda) \neq \beta(B - \lambda)$;

(b) $\lambda \in \sigma_{p,1}(C) \setminus \sigma_l(B)$ *implies* $beta(B-\lambda) = 0$ *or* $\alpha(C-\lambda) \neq \beta(B-\lambda)$.

(iii) If $\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,1}(C)$, *then one of the following statements (a)-(e) is satisfied:*

(a) $\lambda \in \sigma_{p,1}(B)$ *implies* $\alpha(B-\lambda) + \alpha(C-\lambda) \neq \beta(A-\lambda)$;

(b) $\lambda \in \sigma_{r,1}(B)$ *implies* $\alpha(C-\lambda) \neq \beta(A-\lambda) + \beta(B-\lambda)$;

(c) $\lambda \in \sigma_{p,3}(B)$ *implies* $\alpha(B-\lambda) > \beta(A-\lambda)$ *or* $\alpha(C-\lambda) < \beta(B-\lambda)$ *or* $\alpha(B-\lambda) + \alpha(C-\lambda) \neq \beta(A-\lambda) + \beta(B-\lambda)$;

(d) $\lambda \in \sigma_m(B)$ *implies* $\min\{\alpha(C-\lambda), \beta(A-\lambda)\} < \infty$;

(e) $\lambda \in \rho(B)$ *implies* $\alpha(C-\lambda) \neq \beta(A-\lambda)$.

Proof. See [58]. □

Corollary 5.2.3. *Let triple* (A, B, C) *be given. Then*

$$\sigma(M_{D,E,F}) = \sigma(A) \cup \sigma(B) \cup \sigma(C)$$

for all $D \in \mathcal{B}(X_2, X_1), E \in \mathcal{B}(X_3, X_1), F \in \mathcal{B}(X_3, X_2)$ *if one of the following assumptions is satisfied:*
(i) A^* *and* C *have the single valued extension property (SVEP) (see [15]);*
(ii) A *is cohypernormal, and* C *is hypernormal (see [25]);*
(iii) B *and* C *are hypernormal;*
(iv) A *and* B *are cohypernormal;*
(v) A^* *and* B^* *have the SVEP;*
(vi) B *and* C *have the SVEP.*

Proof. See [58]. □

Corollary 5.2.4. *Let triple* (A, B, C) *be given. then*

$$\sigma(M_{D,E,F}) = \sigma(A) \cup \sigma(B) \cup \sigma(C)$$

for any $D \in \mathcal{B}(X_2, X_1), E \in \mathcal{B}(X_3, X_1), F \in \mathcal{B}(X_3, X_2)$ *if one of the following assumptions is satisfied:*
(i) $(\sigma(A) \cap \sigma(B)) \cup (\sigma(A) \cap \sigma(C)) \cup (\sigma(B) \cap \sigma(C))$ *has no interior points (see*

[25]);
(ii) A is cohypernormal, and $\sigma(B)\cap\sigma(C)$ *has no interior points.*
(iii) C is hypernormal, and $\sigma(A)\cap\sigma(B)$ *has no interior points.*
(iv) A^* *has the SVEP, and* $\sigma(B)\cap\sigma(C)$ *has no interior points.*
(v) C has the SVEP, and $\sigma(A)\cap\sigma(B)$ *has no interior points.*
(vi) Any two of the operators A, B, C *are compact.*

Proof. See [58]. □

5.3 General case $n \geq 3$

In this section, $X_1,...,X_n$ are infinite dimensional Hilbert spaces. Occasionally, we will need an assumption that the former are separable, in which case we shall emphasize this fact. This section is based on article [47].

5.3.1 The Weyl spectrum

In this subsection we generalize results from [55, Section 3] to arbitrary Hilbert spaces. We report that Corollaries 3.3 and 3.8 in [55] do not hold with the equivalence: 'only if' part is not valid. The reason for this is that the proofs of these corollaries summon [55, Theorems 2.5, 2.6] which do not hold with an equality (see [44, Corollaries 2.3, 2.10] for corrected versions). Corollaries 4, 8 and 12 from [54, Section 3] are not valid for analogous reasons. In the sequel we provide correct forms of these statements.

Theorem 5.3.1. *([44, Corollary 2.3]) Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. *Then*

$$\sigma_{le}(D_1) \cup \left(\bigcup_{k=2}^{n+1} \delta_k\right) \subseteq \bigcap_{A\in\mathcal{B}_n} \sigma_{lw}(T_n^d(A)), \tag{5.3.1}$$

where

$$\delta_k := \left\{\lambda \in \mathbb{C}:\ \alpha(D_k - \lambda) = \infty \text{ and } \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty\right\},\ 2 \leq k \leq n,$$

$$\delta_{n+1} := \left\{\lambda \in \mathbb{C}:\ \sum_{s=1}^{n} \beta(D_s - \lambda) < \sum_{s=1}^{n} \alpha(D_s - \lambda)\right\}.$$

Theorem 5.3.2. *([45, Corollary 2.5]) Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. *If* $X_1, \dots, X_n$ *are separable and* $\mathcal{R}(D_s - \lambda)$ *are closed for* $2 \leq s \leq n$, $\lambda \in \mathbb{C}$, *then*

$$\sigma_{le}(D_1) \cup \left(\bigcup_{k=2}^{n+1} \delta_k \right) = \bigcap_{A \in \mathcal{B}_n} \sigma_{lw}(T_n^d(A)), \tag{5.3.2}$$

where δ_k, $2 \leq k \leq n+1$, *are defined as in Theorem 5.3.1.*

Now we are able to prove the following generalization to arbitrary Hilbert spaces of [55, Theorem 3.1].

Theorem 5.3.3. *Let* $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. *Then*

$$\bigcup_{k=1}^{n} \sigma_{lw}(D_k) = \sigma_{lw}(T_n^d(A)) \cup \Delta_1 \cup \Delta_2 \tag{5.3.3}$$

holds for every $A \in \mathcal{B}_n$, *where*

$$\Delta_1 = \bigcup_{k=2}^{n} \{\lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) = \infty, \ \alpha(D_s - \lambda) < \infty \ for \ 2 \leq s \leq k-1 \ and$$
$$\sum_{s=1}^{k-1} \beta(D_s - \lambda) = \infty\} \cap \rho_{le}(D_1) \cap \{\lambda \in \mathbb{C} : \ \sum_{s=1}^{n} \beta(D_s - \lambda) \geq \sum_{s=1}^{n} \alpha(D_s - \lambda)\},$$

$$\Delta_2 = \bigcup_{k=1}^{n} \Big\{ \lambda \in \mathbb{C} : \ \alpha(D_s - \lambda) < \infty \ for \ all \ 1 \leq s \leq n, \sum_{s=1}^{n} \beta(D_s - \lambda) \geq$$
$$\sum_{s=1}^{n} \alpha(D_s - \lambda) \ and \ \Big(\alpha(D_k - \lambda) > \beta(D_k - \lambda) \ or \ \mathcal{R}(D_k - \lambda) \ is \ not \ closed \Big) \Big\}.$$

Remark 5.3.4. *Condition '*$\alpha(D_s - \lambda) < \infty \ for \ 2 \leq s \leq k-1$*' in* Δ_1 *is omitted when* $k = 2$.

Proof. Obviously, $\Delta_1 \cup \Delta_2 \subseteq \bigcup_{k=1}^{n} \sigma_{lw}(D_k)$, and $\sigma_{lw}(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_{lw}(D_k)$ according to Lemma 1.2.11. Assume that $\lambda \in \bigcup_{k=1}^{n} \sigma_{lw}(D_k) \setminus \sigma_{lw}(T_n^d(A))$. Then by Theorem 5.3.1 we get that λ does not belong to the left side of (5.3.1), which together with observation $\lambda \in \bigcup_{k=1}^{n} \sigma_{lw}(D_k)$ easily gives $\lambda \in \Delta_1 \cup \Delta_2$. □

Now, we can give a sufficient condition for the stability of the left Weyl spectrum.

Corollary 5.3.5. *Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n} \sigma_{lw}(D_k) = \sigma_{lw}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ if

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 are defined as in Theorem 5.3.3.

If we summon the separability condition, then we are able to state the following.

Corollary 5.3.6. *([55, Theorem 3.1], corrected version)*
Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. If $X_1, ..., X_n$ are separable and $\mathcal{R}(D_s - \lambda),\ 2 \le s \le n,\ \lambda \in \mathbb{C}$ are closed, then

$$\bigcup_{k=1}^{n} \sigma_{lw}(D_k) = \sigma_{lw}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ if and only if

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 are defined as in Theorem 5.3.3.

Proof. Sufficiency is clear, and necessity follows from Theorem 5.3.3 and Theorem 5.3.2. □

By duality, we obtain results related to the stability of the right Weyl spectrum. We begin with the following generalization of [55, Theorem 3.6] to arbitrary Hilbert spaces.

Theorem 5.3.7. *Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n} \sigma_{rw}(D_k) = \sigma_{rw}(T_n^d(A)) \cup \Delta_1 \cup \Delta_2$$

holds for every $A \in \mathcal{B}_n$*, where*

$$\Delta_1 = \bigcup_{k=1}^{n-1} \{\lambda \in \mathbb{C} : \ \beta(D_k - \lambda) = \infty, \ \beta(D_s - \lambda) < \infty \ for \ k+1 \leq s \leq n-1 \ and$$

$$\sum_{s=k+1}^{n} \alpha(D_s - \lambda) = \infty\} \cap \rho_{re}(D_n) \cap \{\lambda \in \mathbb{C} : \ \sum_{s=1}^{n} \alpha(D_s - \lambda) \geq \sum_{s=1}^{n} \beta(D_s - \lambda)\},$$

$$\Delta_2 = \bigcup_{k=1}^{n} \{\lambda \in \mathbb{C} : \ \beta(D_s - \lambda) < \infty \ for \ all \ 1 \leq s \leq n, \ \sum_{s=1}^{n} \alpha(D_s - \lambda) \geq$$

$$\sum_{s=1}^{n} \beta(D_s - \lambda) \ and \ \beta(D_k - \lambda) > \alpha(D_k - \lambda)\}.$$

Remark 5.3.8. *Condition '*$\beta(D_s - \lambda) < \infty \ for \ k+1 \leq s \leq n-1$*' in* Δ_1 *is omitted when* $k = n-1$.

Now, we can give a sufficient condition for the stability of the right Weyl spectrum.

Corollary 5.3.9. *Let* $D_1 \in \mathcal{B}(X_1), \ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$*. Then*

$$\bigcup_{k=1}^{n} \sigma_{rw}(D_k) = \sigma_{rw}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ *if*

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 *are defined as in Theorem 5.3.7.*

If we include the separability assumption, we obtain characterization for the stability of the right Weyl spectrum.

Corollary 5.3.10. *([55, Theorem 3.6], corrected version) Let* $D_1 \in \mathcal{B}(X_1), \ D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$*. If* $X_1, ..., X_n$ *are separable and* $\mathcal{R}(D_s - \lambda), \ 1 \leq s \leq n-1, \ \lambda \in \mathbb{C}$ *are closed, then*

$$\bigcup_{k=1}^{n} \sigma_{rw}(D_k) = \sigma_{rw}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ *if and only if*

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 *are defined as in Theorem 5.3.7.*

5.3.2 The Fredholm spectrum

In this subsection we generalize results from [54, Section 3] to arbitrary Hilbert spaces. We prove statements related to left Fredholm invertibility, and then by duality obtain corresponding statements related to right Fredholm invertibility. Finally, we finish this subsection with investigation of the essential spectra.

We start with the following two known results.

Theorem 5.3.11. *([44, Corollary 3.3]) Let* $D_1 \in \mathcal{B}(X_1)$*,* $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$*. Then*

$$\sigma_{le}(D_1) \cup \left(\bigcup_{k=2}^{n} \delta_k \right) \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_{le}(T_n^d(A)), \tag{5.3.4}$$

where

$$\delta_k := \left\{ \lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) = \infty \ \text{and} \ \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty \right\}, \ 2 \le k \le n.$$

Theorem 5.3.12. *([45, Corollary 2.20])*
Let $D_1 \in \mathcal{B}(X_1)$*,* $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$*. If* $X_1, \dots, X_n$ *are separable and* $\mathcal{R}(D_s - \lambda)$*,* $2 \le s \le n$*,* $\lambda \in \mathbb{C}$ *are closed, then*

$$\sigma_{le}(D_1) \cup \left(\bigcup_{k=2}^{n} \delta_k \right) = \bigcap_{A \in \mathcal{B}_n} \sigma_{le}(T_n^d(A)), \tag{5.3.5}$$

where δ_k*,* $2 \le k \le n$*, are defined as in Theorem 5.3.11.*

Now, we generalize [54, Theorem 4] to arbitrary Hilbert spaces.

Theorem 5.3.13. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n} \sigma_{le}(D_k) = \sigma_{le}(T_n^d(A)) \cup \Delta_1 \cup \Delta_2 \tag{5.3.6}$$

holds for every $A \in \mathcal{B}_n$, where

$$\Delta_1 = \bigcup_{k=2}^{n} \{\lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) = \infty, \ \alpha(D_s - \lambda) < \infty \ for \ 2 \leq s \leq k-1 \\ and \ \sum_{s=1}^{k-1} \beta(D_s - \lambda) = \infty\} \cap \rho_{le}(D_1),$$

$$\Delta_2 = \bigcup_{k=1}^{n} \{\lambda \in \mathbb{C} : \ \alpha(D_s - \lambda) < \infty \ for \ all \ s = 1, \dots, n \ and \\ \mathcal{R}(D_k - \lambda) \ is \ not \ closed\}.$$

Remark 5.3.14. *Condition '$\alpha(D_s - \lambda) < \infty \ for \ 2 \leq s \leq k-1$' in Δ_1 is omitted when $k = 2$.*

Proof. Obviously, $\Delta_1 \cup \Delta_2 \subseteq \bigcup_{k=1}^{n} \sigma_{le}(D_k)$, and $\sigma_{le}(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_{le}(D_k)$ according to Lemma 1.2.11. Assume that $\lambda \in \bigcup_{k=1}^{n} \sigma_{le}(D_k) \setminus \sigma_{le}(T_n^d(A))$. Then by Theorem 5.3.11 we get that λ does not belong to the left side of (5.3.5), which together with observation $\lambda \in \bigcup_{k=1}^{n} \sigma_{le}(D_k)$ easily gives $\lambda \in \Delta_1 \cup \Delta_2$. □

Now, we can give sufficient condition for the stability of the left Fredholm spectrum.

Corollary 5.3.15. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n} \sigma_{le}(D_k) = \sigma_{le}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ if

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 are defined as in Theorem 5.3.13.

Corollary 5.3.16. *([54, Corollary 4], corrected version)*
Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. If $X_1, \ldots, X_n$ are separable and $\mathcal{R}(D_s - \lambda),\ 2 \leq s \leq n,\ \lambda \in \mathbb{C}$ are closed, then

$$\bigcup_{k=1}^{n} \sigma_{le}(D_k) = \sigma_{le}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ if and only if

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 are defined as in Theorem 5.3.13.

Proof. Sufficiency is obvious, and necessity follows from Theorem 5.3.13 and Theorem 5.3.12. □

We provide the following results for the right Fredholm spectrum. First we generalize [54, Theorem 5] to arbitrary Hilbert spaces.

Theorem 5.3.17. *Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n} \sigma_{re}(D_k) = \sigma_{re}(T_n^d(A)) \cup \Delta_1 \cup \Delta_2$$

holds for every $A \in \mathcal{B}_n$, where

$$\Delta_1 = \bigcup_{k=1}^{n-1} \{\lambda \in \mathbb{C} :\ \beta(D_k - \lambda) = \infty,\ \beta(D_s - \lambda) < \infty\ for\ k+1 \leq s \leq n-1\ and\ \sum_{s=k+1}^{n} \alpha(D_s - \lambda) = \infty\} \cap \rho_{re}(D_n),$$

$$\Delta_2 = \{\lambda \in \mathbb{C} :\ \beta(D_k - \lambda) < \infty\ for\ all\ 1 \leq k \leq n\}.$$

Remark 5.3.18. *Condition '$\beta(D_s - \lambda) < \infty\ for\ k+1 \leq s \leq n-1$' in Δ_1 is omitted when $k = n-1$.*

Sufficient condition for the stability of the right Fredholm spectrum follows.

Corollary 5.3.19. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2),...,D_n \in \mathcal{B}(X_n)$. *Then*

$$\bigcup_{k=1}^{n} \sigma_{re}(D_k) = \sigma_{re}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ *if*

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 *are defined as in Theorem 5.3.17.*

Let us summon separability next.

Corollary 5.3.20. *([54, Corollary 8], corrected version)*
Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2),...,D_n \in \mathcal{B}(X_n)$. *Assume that* $X_1,...,X_n$ *are separable and* $\mathcal{R}(D_s - \lambda),\ 1 \leq s \leq n-1,\ \lambda \in \mathbb{C}$ *are closed. Then*

$$\bigcup_{k=1}^{n} \sigma_{re}(D_k) = \sigma_{re}(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ *if and only if*

$$\Delta_1 \cup \Delta_2 = \emptyset,$$

where Δ_1, Δ_2 *are defined as in Theorem 5.3.17.*

To end this section, we provide statements dealing with the essential spectrum. We begin with

Theorem 5.3.21. *([44, Corollary 3.17])*
Let $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2),...,D_n \in \mathcal{B}(X_n)$. *Then*

$$\sigma_{le}(D_1) \cup \sigma_{re}(D_n) \cup \left(\bigcup_{k=2}^{n-1} \delta_k\right) \cup \delta_n \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_e(T_n^d(A)) \tag{5.3.7}$$

where

$$\delta_k = \left\{\lambda \in \mathbb{C}:\ \alpha(D_k - \lambda) = \infty \text{ and } \sum_{s=1}^{k-1} \beta(D_s - \lambda) < \infty\right\} \cup$$

$$\left\{\lambda \in \mathbb{C}:\ \beta(D_k - \lambda) = \infty \text{ and } \sum_{s=k+1}^{n} \alpha(D_s - \lambda) < \infty\right\}, \quad 2 \leq k \leq n-1,$$

$$\delta_n = \Big\{\lambda \in \mathbb{C} : \ \alpha(D_n-\lambda)=\infty \ and \ \sum_{s=1}^{n-1}\beta(D_s-\lambda)<\infty\Big\}\cup$$
$$\Big\{\lambda \in \mathbb{C} : \ \beta(D_1-\lambda)=\infty \ and \ \sum_{s=2}^{n}\alpha(D_s-\lambda)<\infty\Big\},$$

Theorem 5.3.22. *([45, Corollary 2.34])*
Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2),\dots,D_n \in \mathcal{B}(X_n)$. If $X_1,\dots,X_n$ are separable and $\mathcal{R}(D_s-\lambda)$, $2\le s\le n-1$, $\lambda\in\mathbb{C}$, are closed, then

$$\sigma_{le}(D_1)\cup\sigma_{re}(D_n)\cup\Big(\bigcup_{k=2}^{n-1}\delta_k\Big)\cup\delta_n = \bigcap_{A\in\mathcal{B}_n}\sigma_e(T_n^d(A)), \tag{5.3.8}$$

where δ_k, $2\le k\le n$, are defined as in Theorem 5.3.21.

Theorem 5.3.23. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2),\dots,D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n}\sigma_e(D_k)=\sigma_e(T_n^d(A))\cup\Delta$$

holds for every $A\in\mathcal{B}_n$, where

$$\Delta=(\Delta_1\cup\Delta_2)\cap\rho_{le}(D_1)\cap\rho_{re}(D_n),$$

$$\Delta_1=\bigcup_{k=2}^{n-1}\Big\{\lambda\in\mathbb{C}: \ \Big(\alpha(D_k-\lambda)=\sum_{s=1}^{k-1}\beta(D_s-\lambda)=\infty \ and \ \alpha(D_s-\lambda)<\infty$$
$$for \ 2\le s\le k-1\Big) \ or \ \Big(\beta(D_k-\lambda)=\sum_{s=k+1}^{n}\alpha(D_s-\lambda)=\infty \ and \ \beta(D_s-\lambda)<\infty$$
$$for \ k+1\le s\le n-1\Big)\Big\},$$

$$\Delta_2=\Big\{\lambda\in\mathbb{C}: \ \Big(\alpha(D_n-\lambda)=\sum_{s=1}^{n-1}\beta(D_s-\lambda)=\infty \ and \ \alpha(D_s-\lambda)<\infty$$
$$for \ 2\le s\le n-1\Big) \ or \ \Big(\beta(D_1-\lambda)=\sum_{s=2}^{n}\alpha(D_s-\lambda)=\infty \ for \ 2\le s\le n-1$$
$$and \ \beta(D_s-\lambda)<\infty \ for \ 2\le s\le n-1\Big)\Big\}.$$

Proof. Obviously, $\Delta \subseteq \bigcup_{k=1}^{n} \sigma_e(D_k)$, and $\sigma_e(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_e(D_k)$ according to Lemma 1.2.11. Assume that $\lambda \in \bigcup_{k=1}^{n} \sigma_e(D_k) \setminus \sigma_e(T_n^d(A))$. Then by Theorem 5.3.21 we get that λ does not belong to the left side of (5.3.8), which together with observation $\lambda \in \bigcup_{k=1}^{n} \sigma_e(D_k)$ easily gives $\lambda \in \Delta$. □

Now we can give sufficient condition for the stability of the essential spectrum.

Corollary 5.3.24. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n} \sigma_e(D_k) = \sigma_e(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ if

$$\Delta = \emptyset,$$

where Δ is defined as in Theorem 5.3.23.

Corollary 5.3.25. *([54, Corollary 12], corrected version)*
Let $D_1 \in \mathcal{B}(X_1)$ and $D_2 \in \mathcal{B}(X_2), ..., D_n \in \mathcal{B}(X_n)$. Assume that $X_1, ..., X_n$ are separable and $\mathcal{R}(D_s - \lambda)$, $2 \leq s \leq n-1$, $\lambda \in \mathbb{C}$ are closed. Then

$$\bigcup_{k=1}^{n} \sigma_e(D_k) = \sigma_e(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ if and only if

$$\Delta = \emptyset,$$

where Δ is defined as in Theorem 5.3.23.

Proof. Sufficiency is obvious, and necessity follows from Theorem 5.3.23 and Theorem 5.3.22. □

Statements related to the Fredholm spectrum become especially elegant when $n = 2$.

Theorem 5.3.26. *([3, Corollary 3.2]) Let* $D_1 \in \mathcal{B}(X_1)$*,* $D_2 \in \mathcal{B}(X_2)$*. Then*

$$\sigma_e(D_1) \cup \sigma_e(D_2) = \sigma_e(T_2^d(A)) \cup \Big(\{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) = \alpha(D_2 - \lambda) = \infty\} \cap \rho_{le}(D_1) \cap \rho_{re}(D_2) \Big)$$

holds for every $A \in \mathcal{B}_2$*.*

Corollary 5.3.27. *Let* $D_1 \in \mathcal{B}(X_1)$*,* $D_2 \in \mathcal{B}(X_2)$*. Then*

$$\sigma_e(D_1) \cup \sigma_e(D_2) = \sigma_e(T_2^d(A))$$

holds for every $A \in \mathcal{B}_2$ *if*

$$\{\lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) = \alpha(D_2 - \lambda) = \infty\} \cap \rho_{le}(D_1) \cap \rho_{re}(D_2) = \emptyset.$$

Thus, we recover Remark 3 from [54, Section 3].

5.3.3 The Spectrum

We begin with results related to the left and the right spectrum, and afterwards conclude with the spectrum.

Theorem 5.3.28. *([46, Corollary 2.3]) Let* $D_1 \in \mathcal{B}(X_1)$*,* $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$*. Then*

$$\sigma_l(D_1) \cup \Big(\bigcup_{k=2}^{n} \Delta_k \Big) \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma_l(T_n^d(A)) \tag{5.3.9}$$

where

$$\Delta_k := \Big\{ \lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) > \sum_{s=1}^{k-1} \beta(D_s - \lambda) \Big\}, \ 2 \le k \le n,$$

Theorem 5.3.29. *Let* $D_1 \in \mathcal{B}(X_1)$*,* $D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$*. Then*

$$\bigcup_{k=1}^{n} \sigma_l(D_k) = \sigma_l(T_n^d(A)) \cup \Delta \tag{5.3.10}$$

holds for every $A \in \mathcal{B}_n$, *where*

$$\Delta = \bigcup_{k=2}^{n} \{\lambda \in \sigma_l(D_k) : \ \alpha(D_k - \lambda) \leq \sum_{s=1}^{k-1} \beta(D_s - \lambda)\}.$$

Proof. Obviously, $\Delta \subseteq \bigcup_{k=1}^{n} \sigma_l(D_k)$, and $\sigma_l(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma_l(D_k)$ according to Lemma 1.2.11. Assume that $\lambda \in \bigcup_{k=1}^{n} \sigma_l(D_k) \setminus \sigma_l(T_n^d(A))$. Then by Theorem 5.3.28 we get that λ does not belong to the left side of (5.3.9), which together with observation $\lambda \in \bigcup_{k=1}^{n} \sigma_l(D_k)$ easily gives $\lambda \in \Delta$. □

Corollary 5.3.30. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. *Then*

$$\bigcup_{k=1}^{n} \sigma_l(D_k) = \sigma_l(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ *if*

$$\Delta = \emptyset,$$

where Δ *is defined as in Theorem 5.3.29.*

If we put $n = 2$ we get:

Theorem 5.3.31. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_l(D_1) \cup \sigma_l(D_2) = \sigma_l(T_n^d(A)) \cup \{\lambda \in \sigma_l(D_2) : \ \alpha(D_2 - \lambda) \leq \beta(D_1 - \lambda)\} \quad (5.3.11)$$

holds for every $A \in \mathcal{B}_2$.

Corollary 5.3.32. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Assume that* X_1, X_2 *are infinite dimensional Hilbert spaces. Then*

$$\sigma_l(D_1) \cup \sigma_l(D_2) = \sigma_l(T_2^d(A))$$

holds for every $A \in \mathcal{B}_2$ *if*

$$\{\lambda \in \sigma_l(D_2) : \ \alpha(D_2 - \lambda) \leq \beta(D_1 - \lambda)\} = \emptyset.$$

Using duality, we obtain results related to the right spectrum.

Theorem 5.3.33. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. *Then*

$$\bigcup_{k=1}^{n} \sigma_r(D_k) = \sigma_r(T_n^d(A)) \cup \Delta \tag{5.3.12}$$

holds for every $A \in \mathcal{B}_n$, *where*

$$\Delta = \bigcup_{k=1}^{n-1} \{\lambda \in \sigma_r(D_k) :\ \beta(D_k - \lambda) \leq \sum_{s=k+1}^{n} \alpha(D_s - \lambda)\},$$

Corollary 5.3.34. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. *Then*

$$\bigcup_{k=1}^{n} \sigma_r(D_k) = \sigma_r(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ *if*

$$\Delta = \emptyset,$$

where Δ *is defined as in Theorem 5.3.33.*

Special case $n = 2$ gives:

Theorem 5.3.35. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_r(D_1) \cup \sigma_r(D_2) = \sigma_r(T_2^d(A)) \cup \{\lambda \in \sigma_r(D_1) :\ \beta(D_1 - \lambda) \leq \alpha(D_2 - \lambda)\}$$

holds for every $A \in \mathcal{B}_2$.

Corollary 5.3.36. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma_r(D_1) \cup \sigma_r(D_2) = \sigma_r(T_2^d(A))$$

holds for every $A \in \mathcal{B}_2$ *if*

$$\{\lambda \in \sigma_r(D_1) :\ \beta(D_1 - \lambda) \leq \alpha(D_2 - \lambda)\}.$$

We finish our investigations with results related to the spectrum of $T_n^d(A)$. First we recall:

Theorem 5.3.37. *([46, Corollary 2.14])*

Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. Then

$$\sigma_l(D_1) \cup \sigma_r(D_n) \cup \left(\bigcup_{k=2}^{n-1} \delta_k \right) \cup \delta_n \subseteq \bigcap_{A \in \mathcal{B}_n} \sigma(T_n^d(A)), \tag{5.3.13}$$

where

$$\delta_k = \left\{ \lambda \in \mathbb{C} : \ \alpha(D_k - \lambda) > \sum_{s=1}^{k-1} \beta(D_s - \lambda) \right\} \cup$$

$$\left\{ \lambda \in \mathbb{C} : \ \beta(D_k - \lambda) > \sum_{s=k+1}^{n} \alpha(D_s - \lambda) \right\}, \quad 2 \leq k \leq n-1,$$

$$\delta_n = \left\{ \lambda \in \mathbb{C} : \ \alpha(D_n - \lambda) > \sum_{s=1}^{n-1} \beta(D_s - \lambda) \right\} \cup$$

$$\left\{ \lambda \in \mathbb{C} : \ \beta(D_1 - \lambda) > \sum_{s=2}^{n} \alpha(D_s - \lambda) \right\}.$$

Theorem 5.3.38. *Let $D_1 \in \mathcal{B}(X_1)$, $D_2 \in \mathcal{B}(X_2), \ldots, D_n \in \mathcal{B}(X_n)$. Then*

$$\bigcup_{k=1}^{n} \sigma(D_k) = \sigma(T_n^d(A)) \cup \Delta$$

holds for every $A \in \mathcal{B}_n$, where

$$\Delta = \bigcup_{k=2}^{n-1} \left\{ \lambda \in \mathbb{C} : \ \left(\beta(D_k - \lambda) \leq \sum_{s=k+1}^{n} \alpha(D_s - \lambda) \ and \ \lambda \in \sigma_r(D_k) \right) \right.$$

$$\left. or \ \left(\alpha(D_k - \lambda) \leq \sum_{s=1}^{k-1} \beta(D_s - \lambda \right) \ and \ \lambda \in \sigma_l(D_k) \right) \right\} \bigcup$$

$$\left\{ \lambda \in \mathbb{C} : \ 0 < \beta(D_1 - \lambda) \leq \sum_{s=2}^{n} \alpha(D_s - \lambda) \ or \ 0 < \alpha(D_n - \lambda) \leq \sum_{s=1}^{n-1} \beta(D_s - \lambda) \right\}.$$

Proof. Obviously, $\Delta \subseteq \bigcup_{k=1}^{n} \sigma(D_k)$ and $\sigma(T_n^d(A)) \subseteq \bigcup_{k=1}^{n} \sigma(D_k)$ according to Lemma 1.2.11. Assume that $\lambda \in \bigcup_{k=1}^{n} \sigma(D_k) \setminus \sigma(T_n^d(A))$. Then by Theorem 5.3.37 we get that λ does not belong to the left side of (5.3.13), which together with observation $\lambda \in \bigcup_{k=1}^{n} \sigma(D_k)$ easily gives $\lambda \in \Delta$. □

Corollary 5.3.39. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2), \dots, D_n \in \mathcal{B}(X_n)$. *Then*

$$\bigcup_{k=1}^{n} \sigma(D_k) = \sigma(T_n^d(A))$$

holds for every $A \in \mathcal{B}_n$ *if*

$$\Delta = \emptyset,$$

where Δ *is defined as in Theorem 5.3.38.*

Statements related to the spectrum become especially elegant when $n = 2$.

Theorem 5.3.40. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma(D_1) \cup \sigma(D_2) = \sigma(T_2^d(A)) \cup \Big\{\lambda \in \mathbb{C} :\ 0 < \beta(D_1 - \lambda) \le \alpha(D_2 - \lambda) \ or \ 0 < \alpha(D_2 - \lambda) \le \beta(D_1 - \lambda)\Big\}.$$

holds for every $A \in \mathcal{B}_2$.

Observe that Theorem 5.3.40 is a more precise version of [25, Corollary 7] in the Hilbert space setting. Namely, in [25], authors prove that a passage from $\sigma(D_1) \cup \sigma(D_2)$ to $\sigma(T_2^d(A))$ is accomplished by filling some holes in $\sigma(T_2^d(A))$ which happen to be subsets of $\sigma(D_1) \cap \sigma(D_2)$. Notice, however, that in Theorem 5.3.40, we have specified the form of these holes. To our best knowledge, this has not been done so far.

Corollary 5.3.41. *Let* $D_1 \in \mathcal{B}(X_1),\ D_2 \in \mathcal{B}(X_2)$. *Then*

$$\sigma(D_1) \cup \sigma(D_2) = \sigma(T_2^d(A))$$

holds for every $A \in \mathcal{B}_2$ *if*

$$\Big\{\lambda \in \mathbb{C} :\ 0 < \beta(D_1 - \lambda) \le \alpha(D_2 - \lambda) \ or \ 0 < \alpha(D_2 - \lambda) \le \beta(D_1 - \lambda)\Big\} = \emptyset.$$

Chapter 6

Block operator matrices that are not upper triangular

In this chapter we present some additional results regarding spectra of block operator matrices that were not the main interest of the author of this dissertation.

6.1 Historical overview

In the last 30 years, experts in spectral theory have examined spectral properties of block operator matrices that need not be upper triangular. We will observe two types of such matrices.

Let $A \in \mathcal{B}(X_1),\ B \in \mathcal{B}(X_2, X_1)$. Denote by

$$M_{D,C} = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathcal{B}(X_1 \oplus X_2),$$

where $D \in \mathcal{B}(X_1, X_2), C \in \mathcal{B}(X_2)$ are unknown operators.

Next, let $A \in \mathcal{B}(X_1),\ B \in \mathcal{B}(X_2, X_1),\ C \in \mathcal{B}(X_2)$. Denote by

$$M_X = \begin{bmatrix} A & B \\ X & C \end{bmatrix} \in \mathcal{B}(X_1 \oplus X_2),$$

where $X \in \mathcal{B}(X_1, X_2)$ is unknown. In other words, $M_X = M_{X,C}$ with C known and X unknown.

We will give a brief historical review regarding investigations of spectral properties of $M_{D,C}$ and M_X.

We know from Preliminary section that the pioneering work regarding spectral properties of M_C was the article of Du and Pan from 1994 [16]. Soon after, in 1995, Takahashi [51] gave neccessary and sufficient conditions for the invertibility of operator matrix M_X. His result was unfairly neglected till 2009, when Chen and Hai characterized semi-Fredholm invertibility of M_X in [21]. Year after, in 2010, they published an article which characterized (left, right) invertibility of M_X [22]. Next, in 2017, Hai and Zhang investigated Fredholm invertibility of M_X [24], and after them, in 2018, Wu et al. characterized (left, right) Weyl invertibility of M_X [56]. All articles mentioned in this paragraph used the setting of separable Hilbert spaces and have not been generalized to the setting of arbitrary Hilbert or Banach spaces till now.

First results related to the completion of operator $M_{D,C}$ date back to 2010 when Chen and Hai published their work [9]. In this article they gave some results regarding perturbations of the (left, right) spectra of $M_{D,C}$ on separable Hilbert spaces. M. Kolundžija extended some of their work to arbitrary Banach spaces [34], and two years after Chen and Hai generalized their own results from 2010 to the Banach space setting [23]. Next article we ought to mention is [13], in which right and left Fredholm completions of $M_{D,C}$ are discussed. Finally, Weylness of $M_{D,C}$ has been characterized in 2019 [57].

We also mention a recent result of Huang et al. [30] in which authors characterize invertibility of block operator matrices in terms of row operators.

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Biography

Nikola Sarajlija was born in Novi Sad, Serbia, on December 19th in 1995. He graduated from elementary school "Jovan Dučić", Petrovaradin, in 2010, as a bearer of "Vuk Karadžić" award and many awards in mathematics competitions. He graduated from grammar school "Jovan Jovanović Zmaj", Novi Sad in 2014, where he was in class of mathematically gifted students, again as a bearer of "Vuk Karadžić" award.

On fall 2014 he started Bachelor studies in Pure Mathematics at Faculty of Sciences and Mathematics, University of Novi Sad, and graduated from the University in 2017, as an Honoring student. On fall 2017, Nikola started Master studies in Pure mathematics at Faculty of Sciences and Mathematics, University of Novi Sad, and graduated in 2019 with thesis title: "Topological approach to distribution theory with applications to some partial differential equations" (in Serbian). During his Bachelor and Master studies, Nikola was funded by national stipends of the Republic of Serbia. In 2019 he won an award given by Mathematical Institute of the Serbian Academy of Science and Arts for having the best master thesis in the field of mathematics, mechanics.

On fall 2019, Nikola started PhD School of Mathematics, module analysis, at the same faculty. He completed his PhD exams and finals in September 2022. He has published two scientific papers in the field of Functional analysis and Operator theory. So far, he has given two talks in scientific congresses.

From 2019 till now, Nikola was involved in teaching activities at Faculty of Sciences and Mathematics, University of Novi Sad, Serbia, working as a research trainee, involved in the project founded by the Ministry of Education, Science and Technological Development of the Republic of Serbia under Grant No. 451-03-68/2022-14/200125.

List of publications:

1. N. Sarajlija, *Fredholmness and Weylness of operator matrices*, Filomat 36:8 (2022), 2507-2518;
   IF=0.988, SCIe, M22

2. N. Sarajlija, *Perturbing the spectrum of operator* $T_n^d(A)$, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas (RACSAM), 117 (2023), no. 1, Paper No. 10, 12 pp.
   IF2021=2.276 SCIe, M21a

List of conferences and talks:

1. Kongres mladih matematičara u Novom Sadu 2019, Novi Sad.

2. CPMMI 2022, Novi Pazar, talk title: Perturbing the spectrum of operator $T_n^d(A)$.

3. Kongres mladih matematičara u Novom Sadu 2022, Novi Sad

Овај Образац чини саставни део докторске дисертације, односно докторског уметничког пројекта који се брани на Универзитету у Новом Саду. Попуњен Образац укоричити иза текста докторске дисертације, односно докторског уметничког пројекта.

План третмана података

Назив пројекта/истраживања
Спектралне особине матрица оператора на Банаховим просторима (Spectral properties of operator matrices on Banach spaces)
Назив институције/институција у оквиру којих се спроводи истраживање
Природно – математички факултет, Универзитет у Новом Саду
Назив програма у оквиру ког се реализује истраживање
1. Опис података
1.1 Врста студије *У овој студији нису прикупљани подаци.*
2. Прикупљање података
3. Третман података и пратећа документација
4. Безбедност података и заштита поверљивих информација
5. Доступност података
6. Улоге и одговорност