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MAKER–BREAKER GAMES ON GRAPHS

DOCTORAL DISSERTATION

MEJKER–BREJKER IGRE NA GRAFOVIMA

DOKTORSKA DISERTACIJA

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Contents

Rezime

Tema istraživanja ove disertacije su igre tipa Mejker–Brejker u kojima učestvuju dva igrača, Mejker i Brejker, koji naizmjenično uzimaju slobodne grane/čvorove datog grafa. Bavimo se Voker–Brejker igrama koje se igraju na skupu grana grafa K_n. Voker, u ulozi Mejkera, je ograničen da uzima svoje grane kao da se šeta kroz graf, dok Brejker može da uzme bilo koju slobodnu granu grafa. Fokus je na dvije standardne igre - igri povezanosti, gdje Voker ima za cilj da napravi pokrivajuće stablo grafa K_n i igri Hamiltonove konture, gdje Voker ima za cilj da napravi Hamiltonovu konturu. Brejker pobjeđuje ako spriječi Vokera u ostvarenju njegovog cilja. Pokazaćemo da Voker sa biasom 2 može da pobijedi u obje igre čak i ako igra protiv Brejkera čiji je bias b reda $n/\ln n$. Potom razmatramo $(1:1)$ VokerMejker–VokerBrejker igre na K_n, gdje oba igrača, i Mejker i Brejker, moraju da biraju grane koje su dio šetnje u njihovom grafu s ciljem određivanja brze pobjedniče strategije VokerMejkera u igri povezanosti i igri Hamiltonove konture. Konačno, istražujemo Mejker–Brejker igre totalne dominacije koje se igraju na skupu čvorova datog grafa. Dva igrača, Dominator i Stoler naizmjenično uzimaju slobodne čvorove datog grafa. Stoler je Mejker i pobjeđuje ako uspije da uzme sve susjede nekog čvora. Dominator je Brejker i pobjeđuje ako čvorovi koje uzme dok kraja igre formiraju skup totalne dominacije. Za određene klase povezanih kubnih grafova reda $n \geq 6$, dajemo karakterizaciju onih grafova na kojima Dominator pobjeđuje i onih na kojima Stoler pobjeđuje.

Abstract

The topic of this thesis are different variants of Maker–Breaker positional game, where two players Maker and Breaker alternatively take turns in claiming unclaimed edges/vertices of a given graph. We consider Walker–Breaker game, played on the edge set of the graph K_n. Walker, playing the role of Maker is restricted to claim her edges according to a walk, while Breaker can claim any unclaimed edge per move. The focus is on two standard games - the Connectivity game, where Walker has the goal to build a spanning tree on K_n, and the Hamilton Cycle game, where Walker has the goal to build a Hamilton cycle on K_n. We show that Walker with bias 2 can win both games even when playing against Breaker whose bias b is of the order of magnitude $n/\ln n$. Next, we consider $(1:1)$ WalkerMaker–WalkerBreaker game on $E(K_n)$, where both Maker and Breaker are walkers and we are interested in seeing how fast WalkerMaker can build spanning tree and Hamilton cycle.
Finally, we study Maker–Breaker total domination game played on the vertex set of a given graph. Two players, Dominator and Staller, alternately take turns in claiming unclaimed vertices of the graph. Staller is Maker and wins if she can claim an open neighbourhood of a vertex. Dominator is Breaker and wins if he manages to claim a total dominating set of a graph. For certain connected cubic graphs on $n \geq 6$ vertices, we give the characterization of those graphs which are Dominator's win and those which are Staller's win.

Preface

We study different kinds of Maker–Breaker positional games. The theory of positional games is a relatively young field of combinatorics whose task is to develop a broad mathematical framework for different games of perfect information played by two players alternately. The positional game is described through the board of the game (finite set X), the family of winning set ($\mathcal{F} \subseteq 2^X$) and the winning condition. In the Maker–Breaker games players have opposite goals. Maker wins if she can claim all elements of some winning set. Breaker wins if he manages to claim at least one element from each winning set. No draw is possible.

The first game we consider is the so-called Walker–Breaker game, a variant of Maker–Breaker game, recently introduced by Espig, Frieze, Krivelevich, and Pegden [45]. The game is played by two players Walker (playing the role of Maker) and Breaker who alternately choose unclaimed edges of the graph K_n. Breaker can claim any unclaimed edge from the graph while Walker is restricted to claim her edges according to a walk, that is, an edge claimed by her must be incident with the vertex in which she has finished her previous move. We are interested in $(2 : b)$ Walker–Breaker games where Walker claims two edges per move and Breaker claims b edges per move. The focus is on two standard and well-known games: the Connectivity game – where the winning sets are spanning trees of K_n, and the Hamilton Cycle game – where the winning sets are all Hamilton cycles of K_n. We are curious to see what is the largest value of b for which Walker can build a spanning tree and Hamilton cycle thus answering the question of Clemens and Tran [34].
Next, we investigate a question of Espig et al. [45] – what happens if Breaker is also a walker, and consider unbiased WalkerMaker–

WalkerBreaker Connectivity game and Hamilton Cycle game with a goal of determining how fast WalkerMaker can build corresponding spanning structure.
Finally, we consider Maker–Breaker total domination game, introduced by Gledel, Henning, Iršič, and Klavžar [59]. In this game, players alternate in claiming unclaimed **vertices** of a given graph. The players are called Staller and Dominator, according to the roles they have in the game. Winning sets are open neighbourhoods of all vertices in a given graph. Staller is Maker and wins if she manages to isolate a vertex from the graph, i.e. to claim all the vertices in its open neighbourhood. On the other hand, Dominator is Breaker and wins if the vertices he claimed during the game form a total dominating set. Being that this type of game is an easy Dominator's win on the complete graphs, it is interesting to play the game on some given graphs that are not complete. As suggested by Gledel et al. [59] we focus on the characterization of the connected cubic graphs (i.e. graphs whose all vertices have degree three) on which the Dominator wins and those graphs on which Staller wins.

The thesis is organized in the following way.

In **Chapter 1** we introduce positional games and give an overview of Maker–Breaker games. We introduce some basic concepts and give the terminology used throughout the thesis.

In **Chapter 2** we state the main results and theorems that will be proven in this thesis.

In **Chapter 3** we study $(2 : b)$ Walker–Breaker games and we prove that the maximum value of the parameter b that allows Walker to win the Connectivity and Hamilton Cycle game is of order $\frac{n}{\ln n}$.

The results of this chapter are submitted for publication as:

- J. Forcan and M. Mikalački, Spanning structures in Walker–Breaker game [53].

In **Chapter 4** we study the fast winning strategies of WalkerMaker in the unbiased WalkerMaker–WalkerBreaker games.

The results of this chapter are published as:

- J. Forcan and M. Mikalački, On the WalkerMaker–WalkerBreaker games, Discrete Applied Mathematics, (2020) [55].

In **Chapter 5** we consider Maker–Breaker total domination game on connected cubic graphs.

The results of this chapter are submitted for publication as:

- J. Forcan and M. Mikalački, Maker–Breaker total domination game on cubic graphs [54].

Novi Sad, 2021. Jovana Forcan

Chapter 1

Introduction

Some of the most interesting mathematical problems involve combinatorial games. The Combinatorial Game Theory studies strategies of two-player games of perfect information and no chance moves. The development of modern Combinatorial Game Theory can be attributed to two publications: "On Numbers and Games" [35] by Conway from 1976 and "Winning Ways for Your Mathematical Plays" [19] by Berlekamp, Conway, and Guy originally published in 1982.

The branch of combinatorial games, not covered by Conway's theory is the theory of *positional games*. The theory of positional games relies on the combinatorial arguments of various kinds and it is deeply connected with Ramsey's theory, Extreme Graph Theory and Probability Theory. These games include popular Tic-Tac-Toe and Hex games, but also the abstract games played on graphs and hypergraphs. The pioneering results for the beginning of the study of the theory of positional games are Hales-Jewett theorem [64] which is considered as the fundamental result in Ramsey theory and Erdős-Selfridge Criterion [44] which uses potential functions to analyse the games providing the first derandomization argument which is a central concept in the theory of algorithms.

The further development of the theory of position games is made by József Beck whose monograph [9] covers many aspects of positional games and where the author shows "how to escape from the combinatorial chaos via the fake probabilistic method, a game-theoretic adaptation of the probabilistic method in combinatorics", [9]. The recent monograph [67] of Hefetz,

Krivelevich, Stojaković and Szabó also provides a thorough introduction to the theory of positional games and presents recent developments in this field.
Positional game is a hypergraph $(X, \mathcal{F})$, where X is a set, usually finite, and $\mathcal{F} \subseteq 2^X$. The game is played by two players who alternately claim unoccupied elements of a set X until all elements are claimed. Set X is called the *board* of the game and $\mathcal{F}$ – the *family of winning sets.* Two additional parameters configure in the game: positive integers a and b which define the *bias* of the game. In the biased $(a : b)$ game, the first player claims a elements per move and the second player claims b elements per move. If $a = b = 1$, the game is called *unbiased.* The winning condition is usually defined in one of the following three ways:

- Under the *strong* win convention, the winner is the first player to occupy all the elements of some set $F \in \mathcal{F}$. If the board X is exhausted and none of the players has won, the game is a draw.
- In the *weak* (or *Maker–Breaker*) convention, the goals of the players are opposite: one player (*Maker*) has a goal to occupy a winning set, while the other player (*Breaker*) tries to prevent her. Maker wins if she manages to claim all elements from some $F \in \mathcal{F}$. Otherwise, Breaker wins. Draws are impossible.
- In *Avoider–Enforces* convention, which is the *misère* version of the Maker–Breaker convention, one player (*Avoider*) tries to avoid occupying a winning set, while the other (*Enforcer*) tries to force her to do so.

The two largest classes into which the positional games can be divided are strong games and weak games. An example of a well-known strong game is Tic-Tac-Toe (Noughts and Crosses) which is played on 3×3 grid square. The board X consists of nine elements, and the family $\mathcal{F}$ consists of eight winning sets (all rows, columns, and diagonals of the grid are included).
The first thing that comes to mind when analysing deterministic games is to use a brute force computer search. In case of Tic-Tac-Toe it can be applied, but in general this is not an option as there are too many cases and possibilities to consider and analyse. Thus, in practice, this is not feasible in a reasonable time for any computer. So, when analysing these games,

it would be useful to have some general tools. The following argument for strong games confirms and proves that being the first player is always an advantage.

Theorem 1.1. *(Strategy stealing argument, [9]) In the strong positional game $(X, \mathcal{F})$, the first player can ensure at least a draw.*

Proof. Indeed, if the second player (Player II) would have a winning strategy (a book of instructions telling him how to answer each move), the basic idea is to see what happens if the first player (Player I) steals that winning strategy from Player II. Player I plays her first move arbitrarily and from now on she ignores it. Then, after the first move of Player II, Player I imagines that she is the second player and uses stolen strategy against her opponent. If, at some point in the game, Player I needs to claim the element she has already claimed in her first move, then she plays arbitrarily. It is important to observe that playing an extra move can not harm a player. So, by playing according to the stolen strategy, Player I will win before her opponent. A contradiction. □

So, if the game is played optimally by both players, then there are two possible outcomes of the strong game: the first player's win, and a draw. For certain games, Ramsey type argument can be used to prove that draw is not possible, and therefore in these games, the first player is a winner. The argument asserts that if a hypergraph $\mathcal{F}$ is non-2-colorable, then the first player has a winning strategy in the strong game over $\mathcal{F}$. Indeed, if every two-coloring of elements of X (where colors represent players' moves) gives a monochromatic winning set, the draw is not an option. So, by strategy stealing argument, the first player wins.
The strategy stealing argument and Ramsey type argument are currently the only general tools for strong positional games. Although these tools are very powerful, they tell us nothing about how a winning strategy for the first player should look like. An explicit winning strategy is known for a few natural strong games played on the edge set of a given graph, such as Perfect Matching, Hamilton Cycle [47] and k-vertex connectivity game [48].
The strong games are very difficult to analyse. The reason is in the fact that they are not hypergraph monotone, which means that adding another edge to the game hypergraph can change the outcome of the game (see

[10]). Given it is very difficult to analyse strong games, and yet the second player can hope for a draw only, the weak games are introduced.

1.1 Maker–Breaker games

Maker–Breaker games, played by two players, Maker and Breaker, can be considered as the relaxation of strong games, and sometimes they are called the *weak games*. If Maker has a winning strategy in the game over the hypergraph $\mathcal{F}$, then $\mathcal{F}$ is called Maker's win, otherwise, $\mathcal{F}$ is called Breaker's win. Maker's win is also called *Weak Win*. This is because the first player in the corresponding strong game cannot always apply the Maker's winning strategy to ensure winning. Maker, unlike the first player in the corresponding strong game, needs to occupy a winning set to win, but not necessarily first. Breaker's winning strategy is called *Strong Draw* since the second player can use Breaker's winning strategy to ensure his draw in the corresponding strong game, [8].
An example of Maker–Breaker game is the popular Hex game. The game is played on a rhombus of hexagons of size $n \times n$ (traditionally it is played on 11×11 board). Maker is assigned a pair of opposite sides of red color and Breaker is assigned a pair of opposite sides of blue color. Each player's goal is to connect the opposite sides of the board by coloring, in each move, one of the uncolored hexagons in his/her own color. By looking at the description, it may seem that Hex belongs to the class of strong games. However, Hex is not a strong game because players' winning sets are different. The Hex Theorem of J. Nash [57] stating that all red/blue colorings of the hex board must result in a path connecting opposite sides of the rhombus makes the traditional game of Hex a Maker–Breaker game. So, in the Maker–Breaker setup, the winning sets are all paths between two opposite red sides of the board. Maker wins if by the end of the game she owns one of these paths. Breaker wins if he blocks connecting red sides by building his own path between the blue sides.

In the default setup of the Maker–Breaker game, Maker is the first player. When providing a winning strategy for Maker in some games, that strategy should work for every possible scenario, even if Breaker plays first. Since being the first player in Maker–Breaker games is always an

advantage, then if Maker can win as the second player in a game, she can also win as the first player in the same game [67]. Indeed, if Maker has a winning strategy S as the second player in $(X, \mathcal{F})$ game she can adapt it as a winning strategy in the same game when she is the first player. She can play her first move arbitrarily and then she can pretend that she is the second player and apply strategy S. If at some point of the game strategy S tells Maker to play the move which she already played as her first move, she takes another arbitrary element from X. By induction, it can be proven that at any point after Maker's first move, Maker's set of claimed elements contains those played according to the strategy S plus exactly one extra move.
The same is true for Breaker, that is, if he has a winning strategy in the game over $\mathcal{F}$ as the second player then he has a winning strategy in the same game as the first player.
The following remarkable result gives a simple and useful criterion that guarantees Breaker's winning strategy on a hypergraph $\mathcal{F}$.

Theorem 1.2. *(Erdős-Selfridge Criterion, [44]) Let $\mathcal{F}$ be a hypergraph. Then,*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2} \Rightarrow \mathcal{F} \textit{ is Breaker's win.}$$

If Breaker is the first player, then $\sum_{A \in \mathcal{F}} 2^{-|A|} < 1$ is enough to ensure his win.

The theorem gives a condition that is not hard to check and when it is satisfied it provides an efficient algorithm for Breaker's win. If the game-hypergraph is k-uniform (i.e. all winning sets are of order k), then by Erdős-Selfridge theorem Breaker wins if $|\mathcal{F}| < 2^{k-1}$. The potential-based strategies play a key role in determining the breaking points for many games, (see [7]).
A general criterion for Maker's win is given by Beck [9].

Theorem 1.3. *[9] Let $(X, \mathcal{F})$ be a positional game. Let $\Delta_2(\mathcal{F})$ denote the max-pair degree of $\mathcal{F}$, that is* $\max\{|\{A \in \mathcal{F} : \{u, v\} \subseteq A\}| : u, v \in X\}$. *If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} > \frac{1}{8} \Delta_2(\mathcal{F}) |X|,$$

then Maker has a winning strategy in $(1:1)$ *game* $(X, \mathcal{F})$.

It is very common to play Maker–Breaker games on the edges of a graph $G = (V, E)$ with $|V| = n$. In this case, the board of the game is $E(G)$ and the winning sets are all edge sets of subgraphs of G which possess some given graph property. The well-studied positional games played on a given graph G with n vertices are the *Perfect Matching game* $\mathcal{M}_n$ – the winning sets are all sets containing $\lfloor n/2 \rfloor$ independent edges of G, the *Connectivity game* $\mathcal{C}_n$ – the wining sets are all spanning trees of G, the *Hamilton Cycle game* $\mathcal{H}_n$ – the winning sets are edges of all Hamilton cycles of G, the *min-degree-c game* $\mathcal{D}_n^c$ – the winning sets are the subgraphs of G of positive minimum degree c and *k-vertex connectivity game* $\mathcal{C}_n^k$ - the winning sets are all spanning k-vertex connected subgraphs of G.

Usually Maker–Breaker games are played on the complete graph K_n, i.e. when $X = E(K_n)$. Lehman [88] showed that in the $(1:1)$ Connectivity game on $E(K_n)$ Maker can build a spanning tree in exactly $n-1$ moves, which is clearly, the fastest possible. The research on the Hamilton Cycle game has a long history. First, Chvátal and Erdős in [27] proved that Maker can win in Hamilton Cycle game on K_n for large enough n. Later in [94] Papaioannou proved that Maker wins the game for all $n \geq 600$. In the same paper, he conjectured that the smallest n for which Maker can win is 8. Hefetz and Stich [77] further improved the bound of 600 by showing that Maker wins for all $n \geq 29$. Finally, in [103] Stojaković and Trkulja proved that no matter of who starts the game, Maker can win the game if and only if $n \geq 8$ and resolved the long-standing conjecture of Papaioannou from [94].

In the Maker–Breaker games in which it is not hard to determine the identity of the winner, a more interesting question to ask is how fast can the winner win the game. This was first studied by Hefetz et al. in [72], and for the Hamilton Cycle game, they showed that the minimum number of moves needed for Maker to win, denoted by $\tau(H_n)$, is $n+1 \leq \tau(\mathcal{H}_n) \leq n+2$, for sufficiently large n. Hefetz and Stich in [77] proved that $\tau(\mathcal{H}_n) = n+1$, which is optimal.
Fast winning strategies are also studied for other unbiased games on K_n. For example, Maker can build a perfect matching on $E(K_n)$ in $n/2+1$

moves for even n, and in $n/2$ moves for odd n [72]. From these results, it was not hard to obtain that in min-degree-1 game Maker wins in $\lfloor n/2 \rfloor + 1$ moves [72]. One more example in which Maker can win fast, within $n+1$ moves, is the T-game, where T is a given spanning tree with the bounded maximum degree, studied by Clemens et al. in [29].
Determining how fast Maker can win and how to win fast is important for studying other positional games. Sometimes the winning strategy of a player requires that the player builds some structure fast before proceeding to some other task. Fast winning strategies of Maker turned out to be very useful for the analysis of strong positional games, (see [33, 47, 48]).

Since many unbiased Maker–Breaker games on K_n are in favor of Maker, it is natural to give Breaker more power. One way to do this is to allow him to claim $b > 1$ edges per move.

1.1.1 Biased Maker–Breaker games

Motivated by the easy win of Maker, Chvátal and Erdős in [27] introduced and studied $(1 : b)$ Maker–Breaker games where Breaker is allowed to claim more that one edge per move. They noted that $(1 : b)$ Maker–Breaker games are biased monotone. This means that if Breaker wins in some $(1 : b)$ Maker–Breaker game $(X, \mathcal{F})$, then he also wins $(1 : b+1)$ game $(X, \mathcal{F})$. Therefore, there exists a unique positive integer $b_{\mathcal{F}}$, called the *threshold bias* of $(X, \mathcal{F})$, such that the $(1 : b)$ game $(X, \mathcal{F})$ is a Maker's win if and only if $b \leq b_{\mathcal{F}}$ where $\mathcal{F} \neq \emptyset$ and $\min\{|A| : A \in \mathcal{F}\} \geq 2$.

Two general criteria for biased Maker–Breaker games

The biased version of the Erdős-Selfridge Theorem due to Beck [5] gives general criteria for Breaker's win.

Theorem 1.4. *(Generalized Erdős-Selfridge criterion, [5]) If*

$$\sum_{A \in \mathcal{F}} (1+b)^{-|A|/a} < \frac{1}{1+b}$$

then Breaker has a winning strategy in the $(a : b)$ Maker–Breaker game $(X, \mathcal{F})$ as the second player. If Breaker is the first player then $\sum_{A \in \mathcal{F}} (1 + b)^{-|A|/a} < 1$ is enough to ensure his win.

The sufficient condition for Maker's win in biased games is also given by Beck in [5].

Theorem 1.5. *(Maker's winning condition, [5]) If*

$$\sum_{A\in\mathcal{F}} \left(\frac{a+b}{a}\right)^{-|A|} > \frac{a^2b^2}{(a+b)^3}\cdot\Delta_2(\mathcal{F})\cdot|X|$$

then Maker (as the first player) has a winning strategy in the $(a:b)$ *game* $(X,\mathcal{F})$*, where* $\Delta_2(\mathcal{F}) = \max\{|\{A\in\mathcal{F} : \{u,v\}\subseteq A\}| : u,v\in X, u\neq v\}$.

The threshold bias for some Maker–Breaker games

In the $(1:b)$ Maker–Breaker game the main goal is to determine the threshold bias of the game, especially for the natural games such as the Connectivity game, the Hamilton Cycle game, etc. To guess the threshold bias, Erdős suggested the heuristic approach which has become known as the "probabilistic intuition" or the "Erdős Paradigm". According to the paradigm, the threshold bias in $(1:b)$ Maker–Breaker game $(X,\mathcal{F})$ in which both players Maker and Breaker play according to their optimal strategy should be approximately the same as in the corresponding game where both players play randomly. So, if both players play randomly on $(E(K_n),\mathcal{F})$, then Maker's graph M is distributed according to the well-known random graph model $\mathbb{G}(n,m)$ where $m\approx\frac{n^2}{2(b+1)}$. According to the probabilistic intuition Maker wins the game asymptotically almost surely if M contains some winning set. Otherwise, Breaker wins. To estimate the threshold bias b we should look at the threshold m for which the corresponding combinatorial property starts appearing almost surely in $\mathbb{G}(n,m)$. For example, RandomMaker wins in the Connectivity game almost surely if and only if $m > (1/2+o(1))n\ln n$, so the threshold bias of RandomBreaker's win in the random game is almost surely $(1/2+o(1))n\ln n$, [43].
In [79], Komlós and Szemerédi proved that if $m=\frac{n}{2}(\log n+\log\log n+\omega(n))$, then $\mathbb{G}(n,m)$ is almost surely Hamiltonian and it is known that if $m=\frac{n}{2}(\log n+\log\log n-\omega(n))$, then $\mathbb{G}(n,m)$ almost surely non-Hamiltonian. This means that the threshold bias for the Hamilton Cycle game should be of order $n/\log n$.

Maker's random graph can be also modeled by Erdős-Rényi random graph $\mathbb{G}(n,p)$ (a graph on n vertices where each edge of K_n is included independently with probability p) with $p = 1/(b+1)$. One of the indicators that probabilistic intuition predicts the outcome of a game well is to show that the threshold bias in the $(1:b)$ game $(E(K_n),\mathcal{F})$ is equal the reciprocal of the threshold probability for Maker's win in the game on $\mathbb{G}(n,p)$ with winning sets $\mathcal{F}$.

The threshold bias for $(1 : b)$ Maker–Breaker games is studied in many papers. In [27], Chvátal and Erdős conjectured that the threshold bias of the $(1 : b)$ Connectivity game is of order $n/\ln n$. In the same paper they showed that the threshold bias is between $(1/4-\varepsilon)n/\ln n$ and $(1+\varepsilon)n/\ln n$ for any $\varepsilon > 0$. To prove the upper bound they provided Breaker with a strategy to isolate a vertex from Maker's graph. After building a large clique that does not contain any Maker's edge, Breaker's goal is to claim all the remaining edges incident to some vertex of this clique. A slightly better lower bound is proved latter by Beck [5] who improved a constant factor to $\log 2$. In his proof Beck applied the biased version of the Erdős–Selfridge Theorem and *building via blocking technique* by which Maker instead of trying to build a spanning tree, blocks every cut.

Regarding Hamilton Cycle game, Chvátal and Erdős in [27] proved that there is a function $b(n)$ such that for sufficiently large n Maker wins the $(1 : b(n))$ game $\mathcal{H}_n$ on $E(K_n)$ if $b(n) \geq 1$. They conjectured that Maker can win if $b(n) \to \infty$ as $n \to \infty$. This is verified by Bollobás and Papaioannou [20], who proved that Maker can build a Hamilton cycle even if Breaker's bias is as large as $O\left(\frac{\ln n}{\ln\ln n}\right)$. In [6], Beck gave the explicit winning strategy for Maker in the $(1 : b)$ Hamilton Cycle game where $b \leq \left(\frac{\log_2}{27} - o(1)\right)\frac{n}{\log n}$ for large enough n. In this way, he established that the order of magnitude of the threshold bias in the Hamilton Cycle game is $n/\log n$, supporting probabilistic intuition. Beck's result was improved by Krivelevich and Szabó in [84], who showed that the threshold bias for the Hamilton Cycle game is at least $(\ln 2 - o(1))n/\ln n$.

For a long time, the asymptotic determination of the threshold bias for the Connectivity and Hamilton Cycle game was the open problem. It is stated in [84] that the main reason for that is "the inability of current techniques to deal with the min-degree-1 game", that is to determine the smallest bias of Breaker for which he can isolate a vertex in Maker's graph. This obstacle has finally been overcome in [58], where Gebauer and Szabó showed that the threshold bias for the Connectivity game and min-degree-c game K_n is asymptotically equal to $n/\ln n$. These two results both support probabilistic intuition. Next, in [81] Krivelevich proved that the threshold bias for the Hamilton Cycle game is asymptotically equal to $n/\ln n$ as well and resolved the long-standing conjecture.

Doubly biased Maker–Breaker games

For many biased $(a : b)$ Maker–Breaker games played on $E(K_n)$ the identity of the winner is known for $a = 1$ and almost all values of b. Unlike the $(1 : b)$ games, the $(a : b)$ games, where $a > 1$, are less studied, but they are also very important. There are examples of games where just a slight change in bias can change the outcome of the game. One such example is the diameter-2 game, where the board of the game is $E(K_n)$ and the winning sets are all subgraphs of K_n with the diameter at most 2. It is shown that this game is a Breaker's win for $a = b = 1$, [3]. Increasing biases by one, the situation changes. When the game is played in the $(2 : 2)$ setup, Maker is a winner, as it is shown in [3]. Maker–Breaker games, in which biases of Maker and Breaker are, both, larger than one, are often referred to as the *doubly biased games*. Similarly as for the $(1 : b)$ Maker–Breaker games, the *generalized threshold bias* for the $(a : b)$ games $(X, \mathcal{F})$ where $a \geq 1$, denoted by $b_{\mathcal{F}}(a)$, is the unique positive integer such that the $(a : b)$ game $(X, \mathcal{F})$ is Maker's win if and only if $b \leq b_{\mathcal{F}}(a)$. In [76, 89] the generalized threshold bias for the Connectivity game and Hamilton Cycle game is estimated for every a. Further examples of the $(a : b)$ games, where $a > 1$, can be found in [3, 9].

Winning fast in biased Maker–Breaker games on K_n

The concept of biased games on K_n and fast winning strategies is combined in several papers. In [49], Ferber et al. considered T-game and proved that Maker can win within $n + o(n)$ moves. How fast Maker can win in the $(1 : b)$ Perfect Matching game and Hamilton Cycle game played on $E(K_n)$

is studied by Mikalački and Stojaković in [90]. They showed that the shortest duration of the $(1:b)$ Maker–Breaker Perfect matching game is between $\frac{n}{2}+\frac{b}{4}$ and $\frac{n}{2}+O(b\ln n)$ moves for $b \leq \frac{\delta n}{100\ln n}$ where $\delta > 0$ is a small constant, while the shortest duration of Hamilton Cycle game is between $n+\frac{b}{2}$ and $n+O(b^2\ln^5 n)$ moves for $b \leq \delta\sqrt{\frac{n}{\ln^5 n}}$, where $\delta > 0$ is a small constant.
Fast winning strategies in the biased $(a:a)$ games on K_n are studied by Clemens and Mikalački in [33].

1.1.2 The variants of Maker–Breaker games

Maker–Breaker games on random boards

Another approach proposed to compensate the advantage of Maker in the unbiased games is to reduce the number of winning sets by making the base graph sparser and play on the random board, introduced by Stojaković and Szabó in [102]. Given a positional game $(X, \mathcal{F})$ and probability p, the game on the random board $(X_p, \mathcal{F}_p)$ is a probability space of games where X_p is obtained from X by removing elements independently with probability $1-p$ and $\mathcal{F}_p = \{A \in \mathcal{F}_p : A \subseteq X_p\}$.
By decreasing p it gets harder for Maker to win, as "being a Maker's win in $\mathcal{F}$" is an increasing graph property [101]. So, it makes sense to search for the *threshold probability* $p_{\mathcal{F}}$ for the family of the games $\{\mathcal{F}_n : n \in \mathbb{N}\}$, which can be defined as the probability for which an almost sure Breaker's win turns into an almost sure Maker's win, [66].
For the Connectivity and Perfect Matching game, Stojaković and Szabó proved in [102] that the threshold probability is equal to $\frac{\log n}{n}$. For the Hamilton Cycle game, it was shown in [102] that the threshold probability is between $\frac{\log n}{n}$ and $\frac{\log n}{\sqrt{n}}$ and it was conjectured there that it is $\Theta(\frac{\log n}{n})$ which is verified in [100]. Using a different approach, in [70] Hefetz et al. proved that property of Maker's winning in the Hamilton Cycle game has a sharp threshold at $(1+o(1))\log n/n$.
The unbiased H-game on the random boards, where Maker has a goal to claim all the edges of a copy of a fixed graph H, is considered in [91, 93, 102]. The threshold probability for the case when H is K_k, where $k \geq 4$, is determined by Müller and Stojaković in [91] to be $p_{\mathcal{K}_k} = \Theta\left(n^{-2/(k+1)}\right)$. They gave the lower bound that matches the upper

bound on the threshold probability given in [102].
It is interesting to notice that the threshold probability for Maker's win in the unbiased K_k-game, $k \geq 4$, played on the random board has the same order of magnitude as the inverse of the threshold bias in the corresponding $(1 : b)$ game played on $E(K_n)$. The threshold bias for this game is determined by Bednarska and Łuczak in [11]. This reciprocal connection between the threshold probability and threshold bias is found for the first time in [102] for the Perfect Matching and Connectivity game, but it is discovered that the connection does not hold for the K_3-game, since it is obtained $p_{\mathcal{K}_3} = n^{-5/9}$ in [102], while the threshold bias is $b_{\mathcal{K}_3} = \Theta(n^{1/2})$, as it is shown in [27]. The hitting time version of this result, which provided a better understanding of the game, is given in [91] where the authors proved that Maker wins the K_3-game asymptotically almost surely in the moment of appearance of $K_5 - e$, and that typically happens at $p = n^{-5/9}$.
Considering the relationship between the three thresholds: the threshold bias in the $(1 : b)$ game $E((K_n), \mathcal{F})$, the threshold probability for Maker's win in the $(1 : 1)$ game on $\mathbb{G}(n, p)$, and the threshold probability for appearance of combinatiorial property in the $(1 : b)$ game on $\mathbb{G}(n, p)$, the following is known. For the Connectivity and Hamilton Cycle game the total agreement is achieved between these three thresholds. For the K_k-game, for $k \geq 4$, the agreement is achieved between $p_{\mathcal{K}_k}$ and $b_{\mathcal{K}_k}$ as it holds $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}} = 1/b_{\mathcal{K}_k}$, while the threshold probability for appearance of K_k in $\mathbb{G}(n, p)$ is $n^{-\frac{2}{k-1}}$. For K_3-game all three parameters are different: they are $n^{1/2}$, $n^{-5/9}$ and n^{-1}, respectively.

In [32], Clemens and Mikalački studied T_k-tournament game where Maker has a goal to create a copy of a given tournament T_k, (i.e. complete graph on k vertices where each edge has an orientation) by the end of the game. It is shown in [32] that the threshold probability for winning in unbiased T_k-tournament game on the random graph $\mathbb{G}(n, p)$ is $n^{-\frac{2}{k+1}}$ for $k \geq 4$, and that the threshold bias in the $(1 : b)$ game on $E(K_n)$ is $\Theta(n^{\frac{2}{k+1}})$ for $k \geq 3$. The results support the probabilistic intuition for k-clique game, $k \geq 4$, no matter whether it is required to orient the edges or not. The authors in [32] also proved that for acyclic tournament T_3 the threshold probability is $n^{-5/9}$ as well as in the K_3-game, while for the cyclic tournament T_3 the threshold probability is $n^{-\frac{8}{15}}$ which is closer to the $1/b_{\mathcal{K}_3}$.

For the general undirected graph H, different from a triangle or a tree, Nenadov, Steger and Stojaković in [93] showed that the threshold for the H-game is determined by the maximum 2-density of the graph H, which corresponds to the reciprocal order of the threshold bias $b_{\mathcal{H}}$ that is found by Bednarska and Łuczak in [11]. The lower bound for the threshold probability for this game can be obtained almost directly from the general Ramsey-type result of Rödl and Ruciński [98] and the strategy stealing argument.
In [102] the authors were interested to find the smallest bias $b^p_{\mathcal{F}}$ such that Breaker can win the $(1 : b^p_{\mathcal{F}})$ game $(X_p, \mathcal{F}_p)$ almost surely. For the Connectivity game $\mathcal{C}_n$ and Perfect Matching game $\mathcal{M}_n$ they obtained $b^p_{\mathcal{C}_n} = \Theta(pb_{\mathcal{C}_n})$ and $b^p_{\mathcal{M}_n} = \Theta(pb_{\mathcal{M}_n})$ for $p \geq C_1\frac{1}{b_{\mathcal{C}_n}}$ and $p \geq C_2\frac{1}{b_{\mathcal{M}_n}}$, respectively, for some constants C_1 and C_2. They conjectured that the Hamilton Cycle game behaves in the same way as the Connectivity game and the Perfect Matching game. Ferber et al. in [46] resolved the conjecture by proving the stronger statement which says that for $p = \omega(\ln n/n)$, random graph $G \sim \mathbb{G}(n, p)$ is typically such that $\frac{np}{\ln n}$ is the asymptotic threshold bias for the games $\mathcal{M}_n, \mathcal{H}_n$ and $\mathcal{C}^k_n$.

Fast winning strategies of various Maker–Breaker games played on the edge set of a random graph were studied in [30]. Clemens et al. in [30] proved that for $p = \frac{\ln n^K}{n}$, where $K > 100$, the graph $G \sim \mathbb{G}(n, p)$ is is typically such that Maker can win games $\mathcal{M}_n$, $\mathcal{H}_n$ and $\mathcal{C}^k_n$ asymptotically as fast as possible, i.e. within $n/2 + o(n)$, $n + o(n)$, and $kn/2 + o(n)$ moves, respectively.

Hitting time results for various Maker–Breaker games are established in [16] where Ben-Shimon et al. showed that with high probability, Maker wins the Hamilton Cycle game, k-vertex connectivity game, and Perfect Matching game exactly at the time the random graph process first reaches minimum degree 4, $2k$ and 2, respectively.

Random versions of Maker–Breaker games

It turns out that Erdős Paradigm holds for many Maker–Breaker games. Therefore, Krivelevich and Kronenberg in [82] proposed the new direction and studied the biased games in which only one player is the clever player while the other one plays randomly. They studied randomized versions of Maker–Breaker games which are played in the same way as the ordinary Maker–Breaker games except that one player uses deterministic optimal strategy, and the other player plays randomly. In the $(1 : b)$ random-Breaker game Maker claims one element per move and plays according to her optimal strategy while Breaker plays randomly and claims b elements per move. In the $(m : 1)$ random-Maker game Breaker uses his best strategy and claims one element per move and Maker is a random player who claims $m > 1$ elements per move. Several classical Maker–Breaker games are analysed in [82], such as the Hamilton Cycle game, the Perfect Matching game, and the k-vertex connectivity game played on $E(K_n)$.
It is proven in [82] that for these random-Breaker games Maker typically wins in Hamilton Cycle game, Perfect Matching game and k-vertex connectivity game, if $b \leq (1-\varepsilon)\frac{n}{2}$, $b \leq (1-\varepsilon)n$ and $b \leq (1-\varepsilon)\frac{n}{k}$, respectively, for every $\varepsilon > 0$. For random-Maker games, it is proven that the maximal value of m that allows Breaker to win is of order $\ln \ln n$.
In [61] Groschwitz and Szabó considered random-Maker game and provided sharp threshold bias for $\mathcal{D}_n^k$, $\mathcal{H}_n$ and $\mathcal{C}_n$. A sharp threshold bias for the Connectivity game, Perfect Matching game and the Hamilton Cycle game in random-Breaker setup is obtained in [62] and it is shown that Maker can win fast wasting only logarithmically many moves.

Other examples of Maker–Breaker games that involve randomness are p-random-turn Maker–Breaker games, studied in [50, 95]. In these games, before each turn, a biased coin is being tossed and Maker plays this turn with probability p independently of all other turns.

A quantitative version of a Maker–Breaker type game

The Toucher–Isolator game is a quantitative version of a Maker–Breaker type game, recently introduced by Dowden, Kang, Mikalački and Stojaković [41]. The board of the game is the edge set of a given graph G. Two players, Toucher and Isolator alternately claim edges of a given graph. Toucher has a goal to maximize the number of vertices that are incident to at least one of her chosen edges and Isolator tries to minimize the number of vertices that are so touched. In [41] authors analyse the number of untouched vertices $u(G)$ at the end of the game when both players play optimally. They focused on some classes of graphs, such as cycles, paths, trees, and k-regular graphs and also gave results for general graphs. This type of Maker–Breaker game is further considered in [23, 96, 97].

1.1.3 Games related to Maker–Breaker games

A misère version of a Maker–Breaker game

Avoider–Enforcer games are a misère version of the Maker–Breaker games. The winning condition in these games is the opposite of the winning condition of the Maker–Breaker game over the same hypergraph. Enforcer wins if he forces Avoider to occupy a winning set, which is for this type of game sometimes referred to as the *losing set*, and Avoider wins if she manages to avoid occupying it. Two different sets of the rules can be defined for Avoider–Enforcer games - the *strict* game, where in each move players claim exactly the number of elements given by their biases, and the *monotone* game, where players claim at least the number of elements given by their biases. Avoider–Enforcer games played under the strict rules are not bias monotone. So, to overcome the non-monotonicity of strict Avoider-Enforcer games, Hefetz et al. [69] introduced the monotone variant of Avoider–Enforcer games. Both strict and monotone Avoider–Enforcer games have been intensively studied over the years (see [4, 13, 28, 51, 63, 68, 69, 71, 73]).

Waiter–Client and Client–Waiter games

Waiter–Client game (also called Picker–Chooser game) is the positional game in which Waiter has the same goal as Enforcer and Client has the same goal as Avoider in Avoider–Enforcer game, but the rules of taking elements are different. In the biased $(a : b)$ Waiter–Client game, the first player Waiter offers the second player, called Client, $a + b$ previously unclaimed elements of the board. Client claims a of these elements

and then Waiter takes the remaining b elements. If in the last round there are only $1 \leq t < a + b$ remaining elements, then Client will choose $\max\{0, t-b\}$ elements and Waiter will choose $\min\{t, b\}$ elements. Waiter's goal is to force Client to claim all elements of some winning set and Client tries to avoid it. Waiter–Client games are also bias monotone in Waiter's bias b.
Client–Waiter games (also called Chooser–Picker game) are related to Maker–Breaker games. Client wins if he manages to claim all elements of some winning set. Otherwise, Waiter wins. So, Client and Waiter have the same goals as Maker and Breaker respectively, but the process of selecting elements in Client–Waiter games is different than in Maker–Breaker games, as Client can only choose edges out of $a + b$ edges offered by Waiter.
Waiter–Client and Client–Waiter games are studied for the first time by Beck in [7]. The further development of these games can be found in [12, 14, 15, 31, 36, 37, 38, 39, 74, 75, 85].

In the next two sections, we focus on Walker–Breaker games and Maker–Breaker domination games, which are another two variants of Maker–Breaker games. Since these two types of Maker–Breaker game are in focus in this thesis, in the following we also discuss the motivation for studying these games.

1.2 Maker–Breaker games with constraints

Maker–Breaker games, played on the edge set of a given graph G, in which Maker is constrained to choose her edges according to a walk or path in G are called Walker–Breaker games. These games are recently introduced by Espig, Frieze, Krivelevich, and Pegden in [45]. In her first move, Walker (playing the role of Maker) can choose any vertex to be her starting position. In every other round, when it is her turn to play, she needs to claim an edge not previously claimed by Breaker which is incident with the vertex in which she finished her previous move. The other endpoint of the claimed edge becomes her new position. On the other hand, Breaker plays in the usual way, that is, in each move he claims an unclaimed edge of the board. Since Breaker has no restrictions on the way he chooses his edges, these games increase Breaker's power and make up for Maker's advantage in the unbiased Maker–Breaker games.

It is not hard to see that Breaker can easily isolate a vertex from Walker's graph in the unbiased Walker–Breaker game. This can be done in the following way: after Walker's first move, Breaker can fix some vertex v (still untouched by Walker) and in each round, he can claim edges between this vertex and Walker's current position. In this way, he can prevent Walker from visiting vertex v. This implies that Walker is not able to make any spanning structure. So, the first natural question to consider is how many vertices of a given graph Walker can visit. This question was studied by Espig, Frieze, Krivelevich, and Pegden in [45] for different variants of Walker–Breaker games. They proved that if b is a constant, Walker can visit $n-2b+1$ vertices in the $(1:b)$ Walker–Breaker game on K_n. In [45] a variant of the game is also studied, where Walker is restricted to choose her edges according to a path, that is, she is not allowed to use an edge more than once. It was proven there that the longest path PathWalker can create in the unbiased game on $E(K_n)$ has $n-2$ vertices, while in $(1:b)$ game, where $b>1$, the largest path made by PathWalker contains $n-\Theta(\ln n)$ vertices. Several interesting questions for the further development of these games were proposed in [45], such as how large a cycle can Walker make under the various conditions, how many edges Walker can visit under various game conditions, how large a clique she can make, and also what happens if Breaker is a walker as well. The first of these questions was considered by Clemens and Tran in [34]. They analysed how long a cycle Walker can create in the unbiased game and for which biases b Walker has a chance to create a cycle of given constant length. They proved that the length of the largest cycle that Walker can create in the unbiased game is $n-2$, while in the biased $(1:b)$ game Walker can create a cycle of length $n-O(b)$ where $b \leq \frac{n}{\ln^2 n}$.

Since Walker can not hope to make a spanning structure for any $b \geq 1$, an interesting question is what happens when Walker's bias changes, and whether the situation changes with the increase of Walker's bias by 1. As suggested in [34] another interesting problem is to consider the doubly biased $(2:b)$ Walker–Breaker games in order to determine the largest bias b for which Walker has a strategy to create a spanning tree and the largest bias b for which she can create a Hamilton cycle.

Furthermore, it is an intriguing question to resolve Maker's problem of creating a spanning structure in the unbiased Walker–Breaker game. To help Maker, one can restrict Breaker's selection of edges in the same way as for Maker in the Walker–Breaker games. In this case, both players would be walkers, that is each player would have to claim her/his edges according to a walk, i.e. when a player is at some vertex v, she/he could only choose edges incident with v not previously claimed by the opponent. This type of games will be called WalkerMaker–WalkerBreaker games.

1.3 Maker–Breaker (total) domination game

Maker–Breaker domination game (or MBD game for short) was introduced for the first time by Duchêne, Gledel, Parreau, and Renault in [42]. The game is played on the vertex set of a given graph. The players are called *Dominator* and *Staller*, according to the roles they have in the game and to be consistent with the *domination game*, introduced by Brešar, Klavžar, and Rall in [24] and further studied in [25, 40, 92, 99, 105]. In the domination game, Dominator and Staller alternate in choosing an unclaimed *vertex* from a given graph G. Dominator's aim is to dominate a graph in as few steps as possible and Staller wants to delay the process for as long as possible, [24].

As already mentioned, in the MBD game played on the graph $G = (V, E)$ the board X of the game is set V. It seems natural to define the family of winning sets $\mathcal{F}$ as the family of all the dominating sets of G. However, in that case, it is hard to control the sizes of the winning sets and thus to apply some general tools such as Erdős-Selfridge Theorem. To overcome these difficulties one can consider the reverse version of the game in which Staller becomes Maker and Dominator takes the role of Breaker. So, the winning sets are closed neighbourhoods of all vertices in G. In the principal paper on this topic [42], the focus was on determining which player has a winning strategy, while in [60] Gledel, Iršič, and Klavžar studied the minimum number of moves needed for Dominator to win provided that he has a winning strategy.

As a natural extension to the Maker–Breaker domination game, Gledel, Henning, Iršič, and Klavžar introduced the Maker–Breaker total domination game (or MBTD game, for short) [59]. The board of the game is the vertex set of a given graph and the winning sets are open neighbourhoods of all vertices in the graph. Dominator wins if he can claim a total dominating set, that is a set T such that every vertex of a graph has a neighbour in T. Staller wins if she manages to claim an open neighbourhood of a vertex in the graph. In [59] authors were interested in determining the outcome of the games played on grids and some Cartesian products of paths and cycles. They also classify cacti (connected graphs in which any two simple cycles have at most one vertex in common) with reference to the outcome of the game. The classification of cubic graphs (the graphs whose all vertices have degree three) with reference to the outcome of the MBTD game seems not to be so easy. It is shown in [59] that there are infinitely many connected cubic graphs in which Staller wins and that no minimum degree condition is sufficient to guarantee that Dominator wins when Staller starts the game. So, this opens the question of studying the MBTD games on connected cubic graphs and to characterize those graphs that are Dominator's win and those that are Staller's win, as it is suggested in [59].

1.4 General notation and terminology

The notation in this thesis is standard and follows that of [22]. Specifically, we use the following.

A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges. Graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. Graph H is isomorphic to G, denoted by $H \cong G$ if there is a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. The mapping ϕ is called an isomorphism between G and H.

The *order* of graph G is the number of vertices in G which we denote by $v(G) = |V(G)|$, and the *size* of graph G is the number of edges in G which we denote by $e(G) = |E(G)|$. We write uv for an unordered pair $\{u, v\}$.

If vertex u is an endpoint of edge e, then u and e are *incident*. Two vertices which are incident with a common edge are *adjacent* and two edges which are incident with a common vertex are also adjacent. Two distinct adjacent vertices are *neighbours*.

Given a graph G and two disjoint sets $A, B \subseteq V(G)$, let

$N(A,B) = \{b \in B : \exists a \in A, ab \in E(G)\}$ be the set of neighbours of the vertices of A in B. We abbreviate $N(\{v\},B)$ to $N(v,B)$ for some $v \in V(G) \setminus B$. Let $d_G(v,B) = |N(v,B)|$ denote the *degree* of vertex v in G toward vertices from B. The open neighbourhood of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G and the closed neighbourhood of a vertex $v \in V(G)$ is defined as $N_G[v] = N_G(v) \cup \{v\}$. Let $d_G(v) = |N_G(v)|$ denote a degree of v in G. A graph is *simple* if it has no loops (loop - an edge with identical ends) or parallel edges (edges with the same pair of ends).

A *clique* of a graph is a set of mutually adjacent vertices.

An *independent set* (or stable set) in a graph is a set of pairwise nonadjacent vertices.

A *complete graph* is a graph in which any two vertices are adjacent. A complete graph with n vertices is denoted by K_n. A *diamond* is the complete graph on four vertices minus one edge. A *bipartite graph* is a graph whose vertices can be divided into two disjoint and independent sets X and Y such that every edge has one end in X and one end in Y. Partition (X,Y) is called a bipartition of the graph. Bipartite graph with bipartition (X,Y) is denoted by $G[X,Y]$. If every vertex in X is joined to every vertex in Y, then G is called a *complete bipartite graph.* A complete bipartite graph with vertex classes of order m and n is denoted by $K_{m,n}$. A *claw* is the complete bipartite graph $K_{1,3}$.

A *walk* in a graph G is an alternating sequence, not necessarily distinct, of vertices and edges $v_0, e_1, v_1, ..., v_{l-1}, e_l, v_l$ such that $e_i = v_{i-1}v_i$ for each $1 \leq i \leq l$. A *path* is a walk whose vertices and edges are distinct. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle such that two vertices are adjacent if they are consecutive in a cyclic sequence. A path or cycle which contains every vertex of a graph is called a *Hamilton path* or *Hamilton cycle*, respectively of the graph. The *length* of a path or a cycle is the number of its edges.

Graph G is *connected* if each pair of vertices in G belongs to a path. Otherwise, G is *disconnected.*

A graph with no cycle is *acyclic.* A *tree* is a connected acyclic graph. A spanning subgraph of G is a subgraph of G with vertex set $V(G)$. A *spanning tree* is a spanning subgraph that is a tree. Graph G is r-regular if every vertex $v \in V(G)$ has degree r. A 3-regular graph is

called *cubic graph*. Generalized Petersen graph is a cubic graph which is defined as follows: Let k and n be positive integers, with $n > 2k$. The generalized Petersen graph $GP(n, k)$ is the simple graph with vertices $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$, and edges $x_i x_{i+1}, y_i y_{i+k}, x_i y_i$, $1 \leq i \leq n$, indices being taken modulo n.
The Cartesian product $G \Box H$ of graphs G and H is the graph with a vertex set $V(G \Box H) = V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.
The circular ladder graph (or prism graph) CL_n is the Cartesian product of a cycle of length $n \geq 3$ and an edge, that is, $CL_n = C_n \Box P_2$.
A n-prism graph is equivalent to the generalized Petersen graph $GP(n, 1)$.

Let n be a positive integer and let $0 \leq p := p(n) \leq 1$. The Erdős-Rényi model $\mathbb{G}(n, p)$ is a random subgraph G of K_n, constructed by retaining each edge of K_n in G independently at random with probability p. We say that graph $G \sim \mathbb{G}(n, p)$ possesses a graph property $\mathcal{P}$ (i.e. a family of graphs which is closed under isomorphisms) *asymptotically almost surely*, or a.a.s., for brevity, if the probability that $\mathbb{G}(n, p)$ possesses $\mathcal{P}$ tends to 1 as n goes to infinity. We use the approximation $\ln n \leq \sum_{i=1}^{n} \frac{1}{i} \leq \ln n + 1$, where $\ln n$ stands for natural logarithm throughout the thesis.
The model $\mathbb{G}(n, m)$ consists of all graphs with n vertices and $0 \leq m \leq \binom{n}{2}$ edges in which the graphs have the same probability. It has $\binom{\binom{n}{2}}{m}$ elements and every elements occur with the probability $\binom{\binom{n}{2}}{m}^{-1}$. Random graph models are studied in [21, 78].

Chapter 2

Results

In this chapter, we state the main results of this thesis.

2.1 Doubly biased Walker–Breaker games

In Chapter 3 we study the $(2 : b)$ Walker–Breaker games on K_n. We are interested in determining the threshold bias for two standard games: Connectivity game and Hamilton Cycle game. As it is stated earlier Walker is not able to create a spanning tree or Hamilton cycle in Walker–Breaker game on K_n even if she plays against the Breaker with bias 1. We show that the outcome of the game changes if we increase Walker's bias by just 1. More precisely, we answer the next two question raised in [34]:

Question 2.1 ([34], Problem 6.4)**.** *What is the largest bias b for which Walker has a strategy to create a spanning tree of K_n in the $(2 : b)$ Walker–Breaker game on K_n?*

Question 2.2 ([34], Problem 6.5)**.** *Is there a constant $c > 0$ such that Walker has a strategy to occupy a Hamilton cycle of K_n in the $(2 : \frac{cn}{\ln n})$ Walker–Breaker game on K_n?*

To answer Question 2.1 we need the following two theorems. The first one gives the lower bound for the threshold bias in the $(2 : b)$ Walker–Breaker Connectivity game.

Theorem 2.3. *For every $0 < \varepsilon < \frac{1}{4}$ and every large enough n, Walker has a strategy to win in the biased $(2 : b)$ Walker–Breaker Connectivity game played on K_n, provided that $b \leq \left(\frac{1}{4} - \varepsilon\right) \frac{n}{\ln n}$.*

Theorem 2.4 provides the upper bound for the threshold bias in the $(2 : b)$ Connectivity game.

Theorem 2.4. *For every $\varepsilon > 0$ and $b \geq (1+\varepsilon)\frac{n}{\ln n}$, Breaker has a strategy to win in the $(2 : b)$ Walker–Breaker Connectivity game on K_n, for large enough n.*

The following theorem answers Question 2.2 and gives the lower bound for the threshold bias in the $(2 : b)$ Walker–Breaker Hamilton Cycle game.

Theorem 2.5. *There exists a constant $\alpha > 0$ for which for every large enough n and $b \leq \alpha\frac{n}{\ln n}$, Walker has a winning strategy in the $(2 : b)$ Hamilton Cycle game played on K_n.*

2.2 WalkerMaker–WalkerBreaker games

In Chapter 4 we study Maker–Breaker games on K_n in which both players are walkers. Since it is impossible for Maker to create a spanning structure in the $(1 : 1)$ Walker–Breaker game we are interested in seeing how the situation changes if we also restrict the way Breaker moves, i.e. if Breaker is also a walker. So, we consider $(1 : 1)$ WalkerMaker–WalkerBreaker game (*WMaker–WBreaker* games for brevity) with the goal of finding fast winning strategy of WMaker in the Connectivity game and Hamilton Cycle game. We prove the following theorems:

Theorem 2.6. *In the $(1 : 1)$ WMaker–WBreaker Connectivity game on $E(K_n)$, WMaker has a strategy to win in at most $n + 1$ moves.*

Theorem 2.7. *In the $(1 : 1)$ WMaker–WBreaker Hamilton Cycle game on $E(K_n)$, WMaker has a strategy to win in at most $n + 6$ moves.*

We also look at WBreaker's possibilities to postpone WMaker's win in the Connectivity game.

Theorem 2.8. *In the $(1 : 1)$ WMaker–WBreaker Connectivity game on $E(K_n)$, WBreaker, as the second player, has a strategy to postpone WMaker's win by at least n moves.*

2.3 MBTD game on cubic graphs

In Chapter 5 we study MBTD game on cubic graphs. We are interested in the characterization of connected cubic graphs that are Dominator's win and those that are Staller's win. To do so, we use the following classification of cubic graphs. In a cubic graph on $n \geq 6$ vertices each vertex has only three possibilities [80]:

1. it lies in two triangles (Figure 2.1(a))
2. it lies in one triangle (Figure 2.1(b))
3. it lies in zero triangles (Figure 2.1(c)).

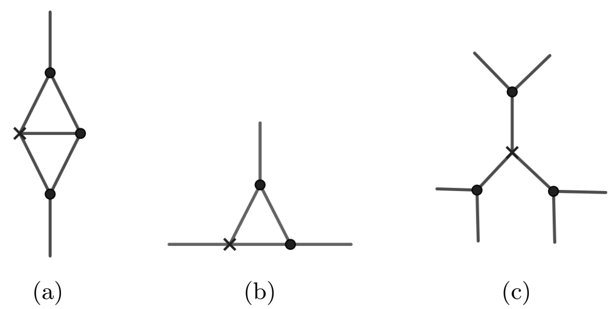


Figure 2.1: Different possible locations for vertices in cubic graph of order $n \geq 6$. Vertex can lie in (a) two triangles (b) one triangle (c) no triangles.

Therefore, cubic graphs can be classified according to the number of vertices of type 1 (being in two triangles), type 2 (being in one triangle), and type 3 (being in no triangle). Let T_1, T_2 and T_3 denote the number of vertices of type 1, type 2 and type 3, respectively. These three numbers are related by the following formulas [80]:

$$T_1 = 2k_1, \quad T_2 = T_1 + 3k_2, \quad T_1 + T_2 + T_3 = n,$$

where k_1 and k_2 are nonnegative integers.
If the cubic graph contains vertices of type 1, then this means that G contains at least one diamond. Hereinafter, when we say a triangle we

refer to an induced K_3 which is not part of a diamond.
Taking into consideration the possible types of cubic graphs, we prove the following theorems.

Theorem 2.9. *Let G be a cubic graph on $n \geq 6$ which is the union of vertex-disjoint diamonds. Then, the MBTD game on G is Dominator's win.*

Theorem 2.10. *Let G be a cubic graph on $n \geq 6$ vertices in which every vertex lies in exactly one triangle, that is, G is the union of vertex-disjoint triangles. If $n = 6$, the graph G is Dominators'win. Otherwise, the graph G is Staller's win.*

Theorem 2.11. *Let G be a cubic graph on $n \geq 6$ vertices which is the union of vertex-disjoint triangles and diamonds. Then, there are only two types of such a graph on which Dominator wins, but only, as the first player. In all other cases, the graph G is Staller's win.*

Another type of graph interesting to us is the Generalized Petersen graphs. They drew lots of attention since their definition, and have already been examined in the domination game. Namely, in [59], the authors showed that the prism $P_2 \Box C_n$, which is equivalent to the Generalized Petersen graph $GP(n, 1)$, for $n \geq 3$ is Dominator's win. On the other hand, graph $GP(5, 2)$ is proven to be Staller's win also in [59]. We consider the graphs $GP(n, 2)$, for all $n \geq 6$, and give the following characterization.

Theorem 2.12. *MBTD game on generalized Petersen graph $GP(n, 2)$ for $n \geq 6$ is Staller's win.*

We are also curious about MBTD games on cubic bipartite graphs and the union of bipartite graphs and prove the following.

Theorem 2.13. *A cubic bipartite graph is Dominator's win.*

Finally, we consider cubic graphs that are union of vertex disjoint claws and prove the following.

Theorem 2.14. *Let G be a connected cubic graph on $n \geq 6$ vertices formed as the union of $k \geq 2$ vertex-disjoint claws. For $k = 2$, G is Dominator's win. For $k \geq 3$, the graph G is Staller's win.*

There has been some research on this topic previously, although not under the name MBTD game. Namely, in his PhD thesis, [86], Kutz was considering the Maker–Breaker games on almost-disjoint hypergraphs of rank three (edges with up to three vertices intersecting in at most one vertex), where the players alternately claim vertices of a given hypergraph and where Maker is the first player. In an almost disjoint hypergraph of rank three, it can be decided in polynomial time whether Maker or Breaker wins, as shown in [86].

Looking at the winning sets of Staller in the MBTD game which will be considered in this thesis, we would have a hypergraph whose all edges have three vertices (3-uniform hypergraph), but the key difference is that majority of the graphs that we consider are not almost-disjoint, as a lot of intersections exist among the edges. We also look at some graphs whose hypergraph would have almost-disjoint hyperedges (e.g. the union of at least four vertex-disjoint $K_{1,3}$). In this case, the analysis from [86] would require searching through all possible pairs of first moves, and then applying some reductions when Staller is the first player. However, to be able to classify these graphs, we provide more applicable and explicit winning strategy for the players.

Chapter 3

Doubly biased Walker–Breaker games

In this chapter we prove the theorems 2.3, 2.4, 2.5. We consider $(2 : b)$ Walker–Breaker Connectivity and Hamilton Cycle game. By proving Theorem 2.3 we establish the lower bound for the threshold bias in $(2 : b)$ Walker–Breaker Connectivity game. To prove this theorem we provide Walker with the strategy which allows her to build a spanning tree for every $b \leq \left(\frac{1}{4} - \varepsilon\right) \frac{n}{\ln n}$ where $0 < \varepsilon < \frac{1}{4}$.
In every round, Walker's goal is to visit a vertex, still untouched by her, which has the maximum degree in Breaker's graph. When analysing Walker's strategy we will use the well-known Box game, introduced by Chvátal and Erdős in [27].

To obtain the upper bound for the threshold bias in the $(2 : b)$ Walker–Breaker Connectivity game we need to prove that Breaker can isolate a vertex from Walker's graph for every $b \geq (1 + \varepsilon)\frac{n}{\ln n}$ and $\varepsilon > 0$. As in the $(1 : b)$ Maker–Breaker Connectivity game, Breaker's strategy will consist of building a clique which does not contain any Walker's edge, and then Breaker's aim will be to isolate one vertex from that clique by claiming all remaining free edges incident with that vertex.

To prove Theorem 2.5 we use the approach of connecting $(2 : b)$ Walker–Breaker games and local resilience in random graphs. The

approach is proposed by Ferber et al. in [52] for studying $(1 : b)$ Maker–Breaker games and its adjustment for studying $(1 : b)$ Walker–Breaker games are made by Clemens and Tran in [34]. The approach presented in [52] is based on the argument given by Bednarska and Łuczak in [11], who investigated the relationship between results in Maker–Breaker games played on graphs and threshold properties for random graphs to obtain the threshold bias for H-game. They provided Maker with the random strategy for which they showed that is almost optimal for her. Maker's strategy is to choose edges of K_n uniformly at random which has not been claimed by her so far. In the case that Maker claims an edge that already belongs to Breaker, this edge is declared as the failure, and Maker losses her move. So, at the end of the game, the Maker's graph will not be exactly the random graph. Bednarska and Łuczak [11] showed that failure edges represent a small fraction of the total number of edges claimed by Maker and after removing them, her final graph will contain a copy of H with positive probability. This is related to the resilience of random graphs with respect to the property "containing a copy of H", [52].
The systematic study of the graph resilience is initiated in [104] by Sudakov and Vu and since then this field received a lot of attention (see [2, 17, 26, 56, 18, 83, 87]).

To prove that Walker can win in the $(2 : b)$ Walker–Breaker Hamilton Cycle game for $b \leq c\frac{n}{\ln n}$ for some $c > 0$ and large enough n, we adapt approach presented in [52] and provide Walker with the strategy which will be partly deterministic and partly random. We want to ensure that Walker generates a random graph that asymptotically almost surely satisfies the property "being Hamiltonian".

In Section 3.1 we give additional theory (definitions, tools, auxiliary statements) needed for proving theorems 2.3 and 2.4, 2.5. In Section 3.2 we prove Theorem 2.3 and Theorem 2.4, and in Section 3.3 we present the proof of Theorem 2.5. In Section 3.4 we give some concluding remarks.

At any given moment during this game, we denote the graph spanned by Walker's edges by W and the graph spanned by Breaker's edges by B. For some vertex v we say that it is *visited* by a player if he/she has claimed *at least one* edge incident with v. A vertex is *isolated/unvisited* if no edge incident to it is claimed. The edges in $E(G) \setminus (W \cup B)$ are called *free*. By $U \subseteq V(K_n)$ we denote the set of vertices, not yet visited by Walker, which is dynamically maintained throughout the game. At the beginning of the game, we have $U := V(K_n)$. Unless otherwise stated, we assume that Breaker starts the game, i.e. one *round* in the game consists of a move by Breaker followed by a move of Walker.

3.1 Preliminaries

To analyse players' strategy in Theorem 2.3 and Theorem 2.5, we use the Box game introduced by Chvátal and Erdős in [27]. The game is very helpful in describing strategies whose goal is to bound the degrees in the opponent's graph.

Maker–Breaker Box game. The rules are as follows. Let $\mathcal{F} = \{A_1, ..., A_k\}$ be a hypergraph where the sets A_i's are pairwise disjoint and $||A_i| - |A_j|| \leq 1$ for every $1 \leq i, j \leq k$. Let $X = \bigcup_{i=1}^{k} A_i$ such that $|X| = t$. The Box game $B(k, t, a, 1)$ is played by two players, *BoxMaker* and *BoxBreaker*, who take turns in claiming elements of the board X. In each round, BoxMaker claims a unclaimed elements from X per move and BoxBreaker claims one unclaimed element per move. Since sets $A_1, A_2, ..., A_k$ represent boxes, where for every $1 \leq i \leq k$, the box A_i contains $|A_i|$ balls, the players' moves can be also described in the following way: in each move BoxMaker removes a balls from these boxes and BoxBreaker destroys one box for BoxMaker.
BoxMaker wins if and only if she succeeds to claim all elements of some box (that is if she succeeds to empty one of the boxes) before it is destroyed by BoxBreaker. BoxBreaker wins if he succeeds to claim at least one element from each box, that is if he destroys all boxes.

Chvátal and Erdős in [27] gave the criterion for BoxMaker's win in $B(k, t, a, 1)$. They defined the following recursive function:

$$f(k, a) := \begin{cases} 0, & k = 1 \\ \left\lfloor \frac{k(f(k-1,a)+a)}{k-1} \right\rfloor, & k \geq 2 \end{cases}$$

The value of $f(k,a)$ can be approximated as

$$(a-1)k\sum_{i=1}^{k-1}\frac{1}{i} \leq f(k,a) \leq ak\sum_{i=1}^{k-1}\frac{1}{i}.$$

Theorem 3.1. *([27], **the Box game criterion**) Let a, k and t be positive integers. BoxMaker has a winning strategy in $B(k,t,a,1)$ if and only if $t \leq f(k,a)$.*

For BoxBreaker it is important to always destroy the box of the smallest size in each move. This is because of the following: if in some moment one box A_i, $1 \leq i \leq k$, has $x \leq a$ elements and BoxBreaker decides to destroy some other box with more than a elements, then BoxMaker in her next move can claim all the remaining elements from A_i and wins. The optimal strategy for BoxMaker is to balance the sizes of all boxes, because, if some box is too small compared to others, then BoxBreaker will surely destroy this box in his next move. So, BoxMaker should in fact ignore these boxes and balance the rest.

The proof of Theorem 3.1 given in [27] contained an error which is fixed by Hamidoune and Las Vergnas in [65]. Hamidoune and Las Vergnas in [65] also gave more general result. They considered $(a : b)$ Box game, $b > 1$, for all values $|A_1|, |A_2|, ..., |A_k|$, where boxes do not have to be almost equal.

To prove Theorem 2.5 and answer Question 2.2, we need the following related to local resilience and random graphs.

Definition 3.2. *[34] For $n \in \mathbb{N}$, let $\mathcal{P} = \mathcal{P}(n)$ be some graph property that is monotone increasing, and let $0 \leq \varepsilon, p = p(n) \leq 1$. Then $\mathcal{P}$ is said to be (p,ε)-resilient if a random graph $G \sim \mathbb{G}(n,p)$ a.a.s. has the following property: For every $R \subseteq G$ with $d_R(v) \leq \varepsilon d_G(v)$ for every $v \in V(G)$ it holds that $G \setminus R \in \mathcal{P}$.*

The Hamiltonicity is one of the central research concepts in graph theory. Since the problem of determining whether a given graph contains a Hamilton cycle is NP-complete, finding sufficient conditions for Hamiltonicity became one of the most important questions. In [104], Sudakov

and Vu proposed to study resilience of random graphs with respect to the Hamiltonicity. They proved that $\mathbb{G}(n,p)$ is $(p, 1/2+o(1))$-resilient with the respect to Hamiltonicity for $p > \log^4 n/n$ and conjectured that this holds for $pn/\log n \to \infty$.
The following theorem provides a good bound on the local resilience of a random graph with respect to Hamiltonicity and resolves the conjecture of Sudakov and Vu [104].

Theorem 3.3. *[87] For every positive $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that for $p \geq \frac{C\ln n}{n}$, a graph $G \sim \mathbb{G}(n,p)$ is a.a.s. such that the following holds. Suppose that H is a subgraph of G for which $G' = G - H$ has minimum degree at least $(1/2+\varepsilon)np$, then G' is Hamiltonian.*

The proof of Theorem 2.5 will follow from Theorem 3.3 and the following statement, which is the key ingredient. We will prove both of them in Section 3.3.

Theorem 3.4. *For every constant $0 < \varepsilon \leq 1/100$ and a sufficiently large integer n the following holds. Suppose that $\frac{10\ln n}{\varepsilon n} \leq p < 1$ and $\mathcal{P}$ is a monotone increasing graph property which is (p,ε)-resilient. Then in the $\left(2 : \frac{\varepsilon}{60p}\right)$ game on K_n Walker has a strategy to create a graph that satisfies $\mathcal{P}$.*

To prove Theorem 3.4 we will use an auxiliary *MinBox* game motivated by the study of the degree game [58]. The *MinBox*(n, D, α, b) game is a Maker–Breaker game played on n disjoint boxes $F_1, ..., F_n$, each of order at least D. Maker claims 1 element and Breaker claims b elements in each round. Maker wins the game if she succeeds to claim at least $\alpha|F_i|$ elements in each box F_i, $1 \leq i \leq n$.
The number of elements in box F claimed so far by Maker and Breaker are denoted by $w_M(F)$ and $w_B(F)$, respectively. The box F is *free* if there are elements in it still not claimed by any of the players. If $w_M(F) < \alpha|F|$, then F is an *active* box. For each box F we set the *danger* value to be $\text{dang}(F) := w_B(F) - b \cdot w_M(F)$.
We also need the following upper bound on the danger value.

Theorem 3.5. *[52] Let $n, b, D \in \mathbb{N}$, let $0 < \alpha < 1$ be a real number, and consider the game MinBox(n, D, α, b). Assume that Maker plays as* *follows: In each turn, she chooses an arbitrary free active box with maximum danger, and then she claims one free element from this box. Then, proceeding according to this strategy,*

$$\mathrm{dang}(F) \leq b(\ln n + 1)$$

is maintained for every active box F throughout the game.

3.2 Connectivity game

Proof of Theorem 2.3. First, we describe Walker's winning strategy and then prove that during the game she can follow it.

Walker's strategy. In the first round Walker visits three vertices. She identifies two vertices v_0 and v_1 with the largest degrees in Breaker's graph. Let $d_B(v_0) \geq d_B(v_1)$. She starts her move in vertex v_0 and then, if $v_0v_1 \in E(B)$, she finds a vertex $u \in U$ such that the edges v_0u and uv_1 are free, and claims them. Otherwise, if $v_0v_1 \notin E(B)$, she claims v_0v_1 and then from v_1 moves to some $u' \in U$ such that $d_B(u') = \max\{d_B(u) : u \in U\}$ (ties broken arbitrarily) and v_1u' is free.
In every round $r \geq 2$ Walker visits at least one vertex from U. After Breaker's move, Walker identifies a vertex $a \in U$ such that $d_B(a) = \max\{d_B(u) : u \in U\}$ (ties broken arbitrarily). Then Walker checks if there is some vertex $y \in U$ such that edges wy and ya are free, where w is Walker's current position, and she claims these two edges wy and ya. If no such vertex $y \in U$ exists, then Walker finds an arbitrary vertex $y' \in V(K_n)$, which could be already visited by Walker, such that edges wy' and $y'a$ are free. She claims these two edges.

Assuming Walker can follow this strategy, she plays at most $n - 2$ rounds.

In the rest of this proof, we will show that Walker can follow the proposed strategy.
First, we are going to consider the maximum degree of unvisited vertices in Breaker's graph B. We can analyse Walker's strategy through an

auxiliary Box game, where she takes the role of BoxBreaker. In the Box game each box represents all free edges adjacent to some vertex $u \in U$. At the beginning of the game, the number of boxes is n and the number of elements in each box is $n-1$. Boxes are not disjoint since for any $u \in U$, vertices from $N(u)$ can also belong to U. So, each edge of the original game belongs to two of these boxes. BoxBreaker can pretend that the boxes are disjoint and that BoxMaker claims $2b$ elements in every move. So, we look at the Box game $Box(n, n(n-1), 2b, 1)$. We estimate the number of the elements that BoxMaker could claim within at most $n-2$ rounds. This gives us the maximum degree of vertices from U in Breaker's graph B.
After $n-2$ rounds, the number of elements that BoxMaker could claim is at least

$$\frac{2b}{n} + \frac{2b}{n-1} + \ldots + \frac{2b}{3} \leq 2b(\ln n + 1) - \left(\frac{2b}{2} + \frac{2b}{1}\right) = 2b \ln n - b. \quad (3.1)$$

Now, we are going to prove that Walker can follow her strategy. The proof goes by induction on the number of rounds. After Breaker's first move we have that $d_B(v_0) + d_B(v_1) \leq b + 1$, so it is obvious that Walker can visit vertices v_0 and v_1. Suppose that Walker already played $k \leq n-3$ rounds and visited at least $k+2$ vertices. Suppose that Walker finished this round at some vertex w and at the end of this round $d_B(w) \leq 2b \ln n - b$.
Breaker could have claimed all b edges incident with w, in his $(k+1)^{\text{st}}$ move, so $d_B(w) \leq 2b \ln n$. According to (3.1), after Breaker's move in round $k+1$, some vertex $a \in U$ can have degree at most $2b \ln n - b$ in B. Walker finds a vertex $y' \in V(K_n)$ such that edges wy' and $y'a$ are free, with preference that $y' \in U$. Such a vertex exists since

$$d_B(w) + d_B(a) \leq 4b \ln n - b < n - 2$$

So, Walker is able to play her move in $(k+1)^{st}$ round. □

To prove Theorem 2.4 we need to provide Breaker with a strategy which will enable him to isolate a vertex from Walker's graph for given bias $b \geq (1+\varepsilon)\frac{n}{\ln n}$. For that, we rely on the strategy of Breaker in the $(1 : b)$ Maker–Breaker Connectivity game [27], where Breaker first makes a clique in his graph and then isolates one of the vertices from that clique in

Maker's graph. Looking from Breaker's point of view, in the Connectivity game, Walker claiming two edges per move can achieve the same as Maker claiming one edge per move. Therefore, in order to win in the $(2 : b)$ Walker–Breaker Connectivity game Breaker can apply the same strategy as Breaker in the $(1 : b)$ Maker–Breaker Connectivity game [27].

Proof of Theorem 2.4. Suppose that Walker begins the game. Breaker's winning strategy is divided into two stages.

Stage 1. Breaker builds a clique C of order $m = \lfloor \frac{b}{2} \rfloor$ such that all vertices from C are isolated in Walker's graph.

Stage 2. Breaker isolates one of the vertices from C in Walker's graph.

Now we are going to prove that Breaker can follow his strategy.

Stage 1. Breaker will play at most $b/2$ moves. Suppose that in round $i-1$, where $i \leq b/2$, Breaker built a clique C_{i-1}, such that all its vertices are isolated in Walker's graph. After Walker's move in round i, Walker's graph contains at most $2i$ edges and at most $2i + 1$ vertices. Since $i < n/2 - 2$, there are at least two vertices u and v outside the Breaker's clique which are not incident with Walker's edges.
Then Breaker can claim the edge uv and $2(i-1)$ edges joining uv to $V(C_{i-1})$. In this way he creates a clique C_i of order $|C_{i-1}| + 2$. In the round $i+1$, Walker can visit only one vertex from C_i. After Walker visits some $c \in C_i$, Breaker's graph still contains a clique C' isolated in Walker graph with $V(C') = C_i \setminus \{c\}$.

Stage 2. Let C be the Breaker's clique of order $m = |C|$ after Stage 1. Let $c_1, c_2, ..., c_m \in C$. To isolate some vertex $c_i \in C$ Breaker needs to claim $n - m$ edges $c_i u$, $u \in V(K_n) \setminus V(C)$. In each round Walker can visit at most one vertex from C, so she will need to play at most m rounds in Stage 2 to visit all vertices from C. We can use an auxiliary Box Game $Box(m, m \cdot (n-m), b, 1)$ to show that Breaker can isolate a vertex from his clique in Walker's graph in at most m moves. Breaker is the BoxMaker who claims b elements per move. Walker, whom Breaker sees as BoxBreaker,

can claim an element in at most one unvisited box per move.
After $m-1$ rounds in Stage 2 BoxMaker could claim at least

$$\begin{aligned}\frac{b}{m}+\frac{b}{m-1}+\cdots+\frac{b}{m-(m-2)} &= b\sum_{k=1}^{m}\frac{1}{k}-\frac{b}{1}\geq b\ln m-b\\ &= b\ln\left\lfloor\frac{b}{2}\right\rfloor-b>n>n-m\end{aligned}$$

elements from some box.

It follows that there is some $c_i \in C$ which is still unvisited by Walker, such that $d_B(c_i) > m-1+(n-m) = n-1$. This means that Breaker is able to isolate a vertex in Walker's graph and thus he wins in the $(2:b)$ Walker–Breaker Connectivity game. □

3.3 Hamilton Cycle game

In this section we prove Theorem 3.4 and Theorem 2.5. The proof of Theorem 3.4 follows very closely to the proofs of Theorem 1.5 in [52] and Theorem 2.4 in [34].

Proof of Theorem 3.4. We show that Walker has a strategy to build a graph that satisfies property $\mathcal{P}$. Walker's strategy will be partly deterministic and partly random.
In the random part of the strategy, Walker generates a random graph $H \sim \mathbb{G}(n,p)$ on the vertex set $V(K_n)$, by tossing a biased coin on each edge of K_n (even if this edge already belongs to $E(B)$), independently at random, which succeeds with probability p. When Walker tosses a coin for an edge e, we say that she *exposes* the edge e. For each vertex $v \in V(K_n)$ we consider the set $U_v \subseteq N(v, V(K_n))$ which contains those vertices u for which the edge vu is still not exposed. At the beginning of the game, $U_v = N(v, V(K_n))$ for all $v \in V(K_n)$.

To decide for which edges she needs to toss a coin, Walker identifies an *exposure vertex* v (the way of choosing the exposure vertex will be explained later). If her current position is different from v, she needs to play her move deterministically. That is, she finds two edges wy and

yv, where w is her current position and y is some vertex from $V(K_n)$, such that $wy, yv \notin E(B)$. She claims these edges and after that move, $wy, yv \in E(W)$, where by W we denote a graph spanned by all Walker's edges.
Once she comes to the exposure vertex v, she starts tossing her coin for edges incident with v with the second endpoint in U_v in the arbitrary order, until she has a first success or until all edges incident with v are exposed. Every edge e on which Walker has success when tossing her coin is included in H. If this edge e is free, Walker claims it. If the exposure failed to reveal a new edge in H, she declares her move a failure of type I. If she has success on an edge, but that edge belongs to $E(B)$, she declares her move a failure of type II.
Let G' be a graph containing all the edges in $H \cap W$.

We need to prove that $G' \in \mathcal{P}$. In order to do this we need to show that following her strategy, Walker will achieve that a.a.s. $d_{G'}(v) \geq (1-\varepsilon)d_H(v)$ holds for each $v \in V(K_n)$, where $0 < \varepsilon \leq 1/100$. Since H is random, the degree of v in H can be determined by using Chernoff's inequality [1]. We have

$$\mathbb{P}\left[Bin(n-1,p) < \frac{99}{100}(n-1)p\right] = o\left(\frac{1}{n}\right)$$

for $p \geq \frac{10 \ln n}{\varepsilon n}$. Thus, by the union bound, it holds that a.a.s.

$$d_H(v) \geq \frac{99}{100}(n-1)p$$

for all vertices in $V(K_n)$.
So, if we prove that the number of failures of type II is relatively small, at most $\varepsilon d_H(v)$, we get that $d_{G'}(v) \geq (1-\varepsilon)d_H(v)$ for every $v \in V(K_n)$. Then, by the assumption on $\mathcal{P}$, we know that G' a.a.s. satisfies property $\mathcal{P}$.
Let $f_I(v)$ and $f_{II}(v)$ denote the number of failures of type I and type II, respectively, for the exposure vertex v.
To keep the number of failures of type II small enough, Walker simulates an auxiliary $MinBox(n, 4n, p/2, 4b)$ game in which she takes the role of Maker. To each vertex $v \in V(K_n)$ Walker assigns the box F_v of size $4n$ at the beginning of the game. For each box F the danger value is defined by

$\text{dang}(F) = w_B(F) - 4b \cdot w_M(F)$.
We describe Walker's strategy in detail.

Walker's strategy. Walker's strategy is divided into two stages.

Stage 1. After every Breaker's move, she updates the simulated $MinBox(n, 4n, p/2, 4b)$ game, in the following way: for each of b edges pq that Breaker claimed in his move, Walker assumes that he claimed one free element from F_p and one from F_q.
In the first round, after Walker identifies the exposure vertex, say v, she starts her move in some vertex $v_0 \in V(K_n)$, $v_0 \neq v$, and then finds some vertex $v_1 \in V(K_n)$, such that edges v_0v_1 and v_1v are free. This is possible since $d_B(v_0) + d_B(v) \leq b + 1$. Maker claims an element of F_v in the $MinBox(n, 4n, p/2, 4b)$ game. In the second round, Walker starts the exposure process on the edges vv', $v' \in U_v$, that is, proceeds with Case 2 (see case distinction below).
In every other round $r \geq 3$, Walker plays in the following way. Denote by w Walker's current position and suppose that it is her turn to make a move. First, she updates the simulated $MinBox(n, 4n, p/2, 4b)$ game, as described above. After that, she checks whether an exposure vertex exists and proceeds with the case distinction below. Otherwise, she finds a vertex v such that in the simulated $MinBox(n, 4n, p/2, 4b)$ game F_v is a free active box of the largest danger. If no such box exists, then Walker proceeds to Stage 2. Otherwise, she declares the vertex v as the new exposure vertex, Maker claims an element of F_v in the $MinBox(n, 4n, p/2, 4b)$ game and then in the real game Walker proceeds with the following cases.

Case 1. $w \neq v$. Walker finds a vertex $y \in V(K_n)$ such that edges wy and yv are free or belong to $E(W)$, where v is the new exposure vertex. Then, she moves to vertex v using these edges. If these edges were free and Walker claimed them, then these edges are now part of the Walker's graph W. Walker proceeds with Case 2.

Case 2. Vertex in which Walker is currently positioned is the exposure vertex. Let $\sigma : [|U_v|] \to U_v$ be an arbitrary permutation on U_v. She starts tossing a biased coin for vertices in U_v, independently at random with probability of success p, according to the ordering of σ.

2a. If this coin tossing brings no success, she increases the value of $f_I(v)$ by 1 and in the simulated game $MinBox(n, 4n, p/2, 4b)$ Maker claims $2pn-1$ additional free elements from F_v or all remaining free elements if their number is less than $2pn-1$. She updates $U_v = \emptyset$ and removes v from all other $U_{\sigma(i)}$ for each $i \leq |U_v|$. In the real game Walker moves along some edge which is in $E(W)$ and then returns to v by using the same edge.

2b. Suppose that first success happen at the k^{th} coin toss. Walker declares that $v\sigma(k)$ is an edge of H.

- If the edge $v\sigma(k)$ is free, Walker claims this edge and from now on $v\sigma(k) \in E(W)$. She moves along this edge one more time in order to return to vertex v. Also, this edge is included in G'. Walker removes v from $U_{\sigma(i)}$ and $\sigma(i)$ from U_v, for all $i \leq k$. Maker claims one free element from box $F_{\sigma(k)}$.
- If the edge $v\sigma(k)$ already belongs to $E(W)$, Walker moves along this edge twice. Also, this edge becomes part of graph G'. Walker removes v from $U_{\sigma(i)}$ and $\sigma(i)$ from U_v, for all $i \leq k$. Maker also claims one free element from box $F_{\sigma(k)}$.
- If the edge $v\sigma(k)$ belongs to Breaker, then the exposure is a failure of type II. She increments $f_{II}(v)$ and $f_{II}(\sigma(k))$ by 1. She also updates $U_v := U_v \setminus \{\sigma(i) : i \leq k\}$ and $U_{\sigma(i)} := U_{\sigma(i)} \setminus \{v\}$ for each $i \leq k$. To make her move, Walker uses an arbitrary edge vu from her graph and returns to v by using the same edge.

At the end of Walker's move in Case 2, the vertex v is not exposure vertex any more.

Stage 2. Walker tosses her coin on every unexposed edge $uv \in E(K_n)$. In case of success, she declares a failure of type II for both vertices u and v.

Observation 3.6. *At any point of Stage 1, there can be at most one exposure vertex.*

Claim 3.7. *During Stage 1, Breaker claims at most $4b$ elements in the simulated MinBox game between two consecutive moves of Maker.*

Proof. Suppose that Breaker finished his move in round t and now it is Walker's turn to make her move in this round. Suppose that in the previous round, $t-1$, Walker played according to her strategy from Case 2. Let w be Walker's current position. Walker identifies a free active box F_v which has the largest danger. Maker claims an element from F_v. If $w = v$, Walker will start her exposure process on the edges vv' with $v' \in U_v$ in round t and then in the following round, $t+1$, she will again identify a new exposure vertex and Maker will claim an element from the corresponding box. In this case between two Maker's moves, Breaker claims b edges, that is, $2b$ elements from all boxes. If $w \neq v$, Walker needs to play her move deterministically in order to move from w to v and then in round $t+1$ she will start her exposure process. After she identifies the new exposure vertex in round $t+2$, Maker will claim an element from the corresponding box. In this case between two Maker's moves (in rounds t and $t+2$), Breaker claims $2b$ edges, that is, $4b$ elements from all boxes. $\square$

Claim 3.8. *At any point during Stage 1, we have $w_M(F_v) < (1+2p)n$ and $w_B(F_v) < n$ for every box F_v in the simulated game. In particular, $w_M(F_v) + w_B(F_v) < 4n$, thus no box is ever exhausted of free elements.*

Proof. According to Walker's strategy, the number $w_M(F_v)$ increases by one every time vertex v is the exposure vertex or when coin tossing brings success on edge vv', where v' is exposure vertex. There can be at most $n-1$ exposure processes in which Walker can toss a coin on an edge that is incident with v. So, both cases together can happen at most $n-1$ times. Also, when Walker declares the failure of type I, $w_M(F_v)$ increases by at most $2pn-1$. So, we have

$$w_M(F_v) < n + f_I(v) \cdot 2pn.$$

We claim that failure of type I can happen at most once. This is true, because after the first failure of type I on v, when Maker receives at most $2pn-1$ additional free elements from F_v, the box F_v is not active any more. So, Maker will never play on vertex v again. Therefore, $w_M(F_v) < n + 2pn = (1+2p)n$.

During Stage 1, we have $w_B(F_v) < n$, because Breaker claims an element of F_v in the simulated game $MinBox(n, 4n, p/2, 4b)$ if and only if in the real game he claims an edge incident with v. Therefore, $w_M(F_v)+w_B(F_v) < 4n$, as stated. $\square$

Claim 3.9. *For every vertex $v \in V(K_n)$, F_v becomes inactive before $d_B(v) \geq \frac{\varepsilon(n-1)}{5}$.*

Proof. Assume that F_v is an active box such that $w_B(F_v) = d_B(v) \geq \frac{\varepsilon(n-1)}{5}$. Since $w_B(F_v) - 4b \cdot w_M(F_v) \leq 4b(\ln n + 1)$, according to Theorem 3.5, it follows that $w_M(F_v) \geq \frac{w_B(Fv)}{4b} - (\ln n + 1)$. With $b = \frac{\varepsilon}{60p}$ we have $w_M(F_v) \geq 3p(n-1) - (\ln n + 1) > 2pn$, where $p \geq \frac{10 \ln n}{\varepsilon n}$. This is a contradiction because F_v is active. $\square$

Claim 3.10. *Walker is able to move from her current position to the new exposure vertex.*

Proof. Let w be Walker's current position at the beginning of some round t and let v be the new exposure vertex. This means that at the beginning of round $t-1$, the box F_w was active and we had $d_B(w) < \frac{\varepsilon(n-1)}{5}$. If F_w is no longer active at the end of round $t-1$, then after Breaker's move in round t we have $d_B(w) < \frac{\varepsilon(n-1)}{5} + b$. We need to show that Walker can find a vertex $y \in V(K_n)$ such that edges wy and yv are not in $E(B)$. Since F_v is a free active box and taking into consideration the value of b, we have

$$d_B(w) + d_B(v) < \frac{2\varepsilon(n-1)}{5} + b < n - 2$$

and so Walker is able to move to v. $\square$

Claim 3.11. *For every vertex $v \in V(K_n)$ we have that a.a.s. F_v is active, for as long as $U_v \neq \emptyset$. In particular, at the end of Stage 1 all edges of K_n are exposed a.a.s.*

Proof. Suppose that there is a vertex v such that F_v is not an active box and $U_v \neq \emptyset$. Since F_v is not an active box it holds that $w_M(F_v) \geq \frac{p}{2}|F_v|$. Also, since $U_v \neq \emptyset$, we have that $f_I(v) = 0$. Maker could increase $w_M(v)$ at the moment when v became the exposure vertex, or when Walker had success on edge vv' where v' is the exposure vertex. Consider the case when v was the exposure vertex. Since $f_I(v) = 0$ it means that in the exposure process Walker had success on at least $\frac{p}{2}|F_v| - 1 = 2np - 1$ edges incident with v. Also, every time coin tossing brought success for an edge incident with v, the degree of vertex v increased in H by one. It follows that $d_H(v) \geq \frac{p}{2}|F_v| - 1 > 2(n-1)p$. By using Chernoff's inequality [1], we have

$$\mathbb{P}[Bin(n-1,p) \geq 2(n-1)p] < e^{-(n-1)p/3} = o\left(\frac{1}{n}\right).$$

Applying the union bound, it follows that with probability $1 - o(1)$, there exists no such vertex.
Now, suppose that at the beginning of Stage 2 there is an edge $uv \in E(K_n)$ which is not exposed. This means that $U_v \neq \emptyset$. So, F_v is an active box and we have that $w_M(F_v) < 2pn$. Since F_v is active it also holds that $w_B(F_v) < \frac{\varepsilon(n-1)}{5}$, according to Claim 3.9. Therefore, since $w_M(F_v) + w_B(F_v) < |F_v|$, the box F_v is free. But this is not possible at the beginning of Stage 2. A contradiction. □

Claim 3.12. *For every vertex $v \in V(K_n)$, a.a.s. we have $f_{II}(v) \leq \frac{9}{10}\varepsilon(n-1)p$.*

Proof. Failures of type II happen in Stage 1 in case when Walker has success on edge which is in $E(B)$. During Stage 1, by Claim 3.9, for as long as the box F_v is active, for some $v \in K_n$, we have $d_B(v) < \frac{\varepsilon(n-1)}{5}$. So, for every $v \in V(K_n)$ there is a non-negative integer $m \leq \frac{\varepsilon(n-1)}{5}$ such that $f_{II}(v)$ is dominated by $Bin(m,p)$. Applying a Chernoff's argument [1] with $p \geq \frac{10 \ln n}{\varepsilon n}$ we obtain

$$\mathbb{P}\left[Bin(m,p) \geq \frac{9}{10}\varepsilon(n-1)p\right] \leq \left(\frac{e\varepsilon(n-1)p/5}{\frac{9}{10}\varepsilon(n-1)p}\right)^{\frac{9}{10}\varepsilon(n-1)p} = o\left(\frac{1}{n}\right).$$

Thus, a.a.s. $f_{II}(v) \leq \frac{9}{10}\varepsilon(n-1)p$ for all $v \in V(K_n)$. □

According to Claim 3.11, Walker never played Stage 2 since she exposed all the edges of K_n by the end of Stage 1. By Claim 3.12 we know that for each vertex v coin tossing has brought success for at most

$$\frac{9}{10}\varepsilon(n-1)p \leq \frac{90}{99}\varepsilon d_H(v)$$

edges which were claimed by Breaker. So, it follows that for each vertex $v \in V(K_n)$ we have

$$d_{G'}(v) \geq d_H(v) - \frac{90}{99}\varepsilon d_H(v) \geq (1-\varepsilon)d_H(v).$$

This completes the proof of Theorem 3.4. □

Proof of Theorem 2.5. When we know that Theorem 3.4 holds, the proof of this theorem is almost the same as the proof of Theorem 1.6 in [52].
Let $C = C\left(\frac{1}{6}\right)$, $p = \frac{c\ln n}{n}$ where $c = \max\{C, 1000\}$, and let $G \sim \mathbb{G}(n,p)$. Note that the property $\mathcal{P} :=$ "being Hamiltonian" is $(p, \frac{1}{6})$-resilient for $p \geq \frac{c\ln n}{n}$.
Applying Chernoff's inequality [1], we obtain

$$\mathbb{P}\left[d_G(v) < \frac{5}{6}np\right] < e^{-\frac{np}{72}} = o\left(\frac{1}{n}\right).$$

Thus, by the union bound, it holds that a.a.s. $\delta(G) \geq \frac{5}{6}np$.
Let $R \subseteq G$ be a subgraph such that $d_R(v) \leq \frac{1}{6}d_G(v)$. For $R' = G - R$ we have

$$d_{R'}(v) \geq \frac{5}{6}d_G(v) \geq \frac{25}{36}np > \frac{2}{3}np = (1/2 + 1/6)np.$$

Theorem 3.3 implies that graph R' is Hamiltonian.
According to Theorem 3.4, Walker can create a graph $G' \in \mathcal{P}$ in the $\left(2 : \frac{1}{360p}\right)$ game on K_n. For $p = \frac{c\ln n}{n}$ it follows that Walker has a winning strategy in $\left(2 : \frac{n}{360c\ln n}\right)$ Walker–Breaker Hamilton Cycle game. Setting $\alpha = \frac{1}{360c}$ completes the proof. □

3.4 Concluding remarks

We saw that when Walker's bias is 2, she can win both the Connectivity and the Hamilton Cycle game. From theorems 2.3, 2.5 and 2.4 it follows that the threshold bias in the $(2 : b)$ Walker–Breaker Connectivity game and Hamilton Cycle game is of order of magnitude $n/\ln n$ which is the same order of magnitude as in the corresponding $(1 : b)$ Maker–Breaker games.

Analysing other games. Now, we wonder how the situation changes in other games involving spanning structures, when Walker's bias is 2. For example, it is not hard to show that for $b = o(\sqrt{n})$ Walker can win in the $(2 : b)$ Pancyclicity game, that is, she can build a graph which consists of cycles of any given length $3 \leq l \leq n$. Indeed, since for $p = \omega(n^{-1/2})$ the property $\mathcal{P}$:="being pancyclic" is $(p, 1/2 + o(1))$-resilient (see Theorem 1.1 in [83]), by applying Theorem 3.4 with p and $\mathcal{P}$ we obtain that Walker has the winning strategy in the $\left(2 : \frac{1}{180p}\right)$ Pancyclicity game on K_n.
It would be interesting to consider k-vertex connectivity game, for $k \geq 2$, on K_n to determine the largest value of b for which Walker can win the $(2 : b)$ Walker–Breaker k-vertex connectivity game on K_n.

Different board. Another question that comes naturally is what happens if we change the board to be the edge set of a general graph G or some sparse graph. How many vertices could Walker visit then in both unbiased and biased games?

Chapter 4

WalkerMaker–WalkerBreaker games

We study WalkerMaker–WalkerBreaker (or WMaker–WBreaker, for short) games on K_n and prove theorems 2.6, 2.7 and 2.8. WMaker–WBreaker game is played by two players WalkerMaker (WMaker) and WalkerBreaker (WBreaker) who alternately claim edges of a graph that are not chosen by the opponent, and both players have the constraint to claim edges according to a walk. Unlike the standard Maker–Breaker Connectivity game on K_n, where Maker wins in the optimal number of moves, $n-1$, Theorem 2.6 says that Maker, as a walker, will need to play at most two moves more than it is optimal. Considering WBreaker's side of the game, Theorem 2.8 states that WMaker as the first player needs to play at least n moves.

In the Hamilton Cycle game, it becomes more challenging for WMaker to win fast. Theorem 2.7 shows that she needs at most five more moves than is the case in the standard Maker–Breaker Hamilton Cycle game.

This chapter is organized in the following way: in Section 4.1. we describe WMaker's strategy and prove Theorem 2.6 and Theorem 2.7, in Section 4.2 we prove Theorem 2.8 and in Section 4.3 we give some concluding remarks.

At any point of the game, let M and B denote the graphs spanned by edges of WMaker, respectively WBreaker, claimed so far. We use

U to denote the set of vertices that are still unvisited by WMaker, i.e. $U = V(K_n) \setminus V(M)$. For some vertex v we say that it is *visited* by a player if he/she has claimed *at least one* edge incident with v. A vertex is *isolated/unvisited* if no edge incident to it is claimed.

4.1 Building spanning structure fast

In both Connectivity and Hamilton Cycle game, WMaker's strategy will be first to create a path of length $n-4$ ensuring that the vertex she visits for the first time has a very small degree in WBreaker's graph, less or equal than 6, and that after each round the following holds:

- WMaker is positioned at a vertex which is incident with at most one WBreaker's edge that has the second endpoint in U,
- every WBreaker's edge is incident with a vertex already visited by WMaker, and
- at most two vertices that are unvisited by WMaker can belong to WBreaker's graph.

In the Connectivity game, in the second phase of her strategy, WMaker needs to find a way to add three remaining untouched vertices to her tree. In the Hamilton Cycle game, the second part of WMaker's strategy will be to close a cycle of length $n-2$ or $n-1$, and in the last, third, phase of her strategy she will embed remaining vertices in the cycle by claiming edges between a vertex she wants to build in the cycle, and two vertices which are connected by an edge in the cycle.

Let us define the following strategy $\mathcal{S}$, which WMaker will use in the first part of both of the games in order to win. WBreaker starts the game.

Strategy $\mathcal{S}$ For her starting vertex, WMaker chooses the vertex v_1, in which WBreaker has finished his first move, and claims an edge v_1u such that $d_B(u) = 0$ (ties are broken arbitrarily). In every other round WMaker checks if there exists an edge $e \in E(B)$, $e = pq$, s.t. $p, q \in U$, and from her current position w claims wp, or wq, whichever is free. If both wp and wq are free she chooses wp if $d_B(p) > d_B(q)$, and wq, if $d_B(q) > d_B(p)$ (ties are broken arbitrarily). If no such edge exists, WMaker from her current position w claims a free edge wu such that $u \in U$ and $d_B(u) = \max\{d_B(v) : v \in U\}$, ties broken arbitrarily, for as long as $|U| \geq 3$. If all free edges wu are such that $d_B(u) = 0$ for all $u \in U$, then WMaker claims an arbitrary free edge wu.

First we will show that WMaker can play according to the strategy $\mathcal{S}$, and then show that the rest of the following statements hold, which will be used in proving Theorem 2.6 and Theorem 2.7.

Lemma 4.1. *In the* $(1:1)$ *WMaker–WBreaker game on* $E(K_n)$*, WMaker can follow strategy* $\mathcal{S}$ *for as long as* $|U| > 2$*.*

Proof. In the first round, we know that WMaker for her starting position chooses the vertex v_1 in which WBreaker has finished his first move and claims the edge v_1u, $u \in U, d_B(u) = 0$. So, obviously WMaker can follow the strategy $\mathcal{S}$ in the first round. The rest of the proof goes by induction on the number of rounds k. Now we show for $k = 2$. In the second round WBreaker needs to claim an edge v_1x, where $x \in U \setminus \{v_0\}$ or go along the edge v_1v_0 again. Suppose first that he claimed v_1x. As WMaker is positioned at u, she can claim either of the edges uv_0 and ux, as both edges are free and it holds that $d_B(v_0) = d_B(x) = 1$ and for all $y \in U \setminus \{x, v_0\}$ it holds that $d_B(y) = 0$. Otherwise, if he goes along v_0v_1 again, WMaker claims uv_0 as that is the only vertex in U such that $d_B(v_0) = 1$. So, she plays according to $\mathcal{S}$ in the second round. Assume that she can follow the strategy $\mathcal{S}$ for $2 \leq k \leq n-4$ rounds. Now assume that in round $k+1 \leq n-3$ WBreaker claimed an edge b_1b_2, and now it is WMaker's turn to play.
Suppose first that $b_1, b_2 \notin V(M)$. Denote WMaker's current position by w and suppose that WMaker is not able to visit either b_1 or b_2 in round $k+1$. It follows that $wb_1, wb_2 \in E(B)$. Suppose that WBreaker claimed these edges in the following order: b_1w, wb_2, b_2b_1. This means that, in round $t \leq k$ in which WBreaker claimed edge b_1w, WMaker moved to some vertex different from b_1 and w, and that contradicts the induction hypothesis that WMaker followed the strategy in round $t \leq k$.
Now suppose that at least one of $\{b_1, b_2\}$ is in $V(M)$, and suppose it is $b_1 \in V(M)$. Let w denote WMaker's current position. As $k+1 \leq$

$n-3$, it holds that $|U| \geq 3$. We will show that WMaker is always able to claim a free edge incident to w whose other endpoint is in U. To show that, we will prove that $d_B(w, U) \leq 2$ holds before WMaker's move. Suppose, for a contradiction that $d_B(w, U) > 2$ before WMaker's move in this round. Suppose $d_B(w, U) = 3$. This implies that there exist three edges wa, wb, wc, such that $\{a, b, c\} \subseteq U$ and suppose that they were claimed in this order. Note that wc might be the edge $b_1 b_2$. As WMaker moved to w only in round k, and being that WBreaker is also a walker and needs at least 4 moves to claim these 3 edges, the edge wa was claimed by WBreaker in some round $r < k$. Since WMaker did not visit any of the vertices $\{w, a\}$ in that round, this leads to a contradiction with the induction hypothesis that WMaker could follow her strategy in round r. □

Directly from the strategy $\mathcal{S}$ and the previous lemma, we obtain the following corollary, which will be extensively used in the rest of the paper.

Corollary 4.2. *In the* $(1:1)$ *WMaker–WBreaker game on* $E(K_n)$ *strategy* $\mathcal{S}$ *guarantees WMaker that after each round, every WBreaker's edge is incident with some vertex* $v \in V(M)$*.*

Corollary 4.3. *In the* $(1:1)$ *WMaker–WBreaker game on* $E(K_n)$*, as long as* $|U| > 2$*, strategy* $\mathcal{S}$ *guarantees that after each round WMaker is positioned at some vertex* w *such that* $d_B(w, U) \leq 1$*.*

Proof. This already holds after the first round. Suppose that after some round $i > 1$ WMaker is at vertex w such that $d_B(w, U) = 2$, that is $wu, wu' \in E(B)$, for some $u, u' \in U$ and suppose that WBreaker claimed wu before wu'. Assume also that it is again WBreaker's turn and he claims some edge incident with $u' \in U$ in round $i+1$.
This contradicts Corollary 4.2, because when WBreaker claimed wu in round $i-1$, WMaker visited some vertex different from w and u, and after that round, wu was WBreaker's edge not incident with $V(M)$. It follows that vertex in which WMaker is positioned at the end of each round can have degree $d_B(w, U) \leq 1$. □

Corollary 4.4. *In the* $(1:1)$ *WMaker–WBreaker game on* $E(K_n)$*, the strategy* $\mathcal{S}$ *guarantees WMaker that, as long as* $|U| > 2$*, after WBreaker's* *move (and before WMaker's move) in some round $i \geq 2$, vertex w in which WMaker finished her previous move, can have degree $d_B(w, U) \leq 2$. If $d_B(w, U) = 2$, then WBreaker finished his move in round $i - 1$ at vertex w.*

Lemma 4.5. *In the $(1 : 1)$ WMaker–WBreaker game on $E(K_n)$, strategy $\mathcal{S}$ guarantees WMaker that, as long as $|U| > 2$, after each round there can be at most 2 vertices from U belonging to $V(B)$.*

Proof. The proof goes by induction on the number of rounds k. After the first round, there is only one vertex from U visited by WBreaker. This is the vertex that WBreaker chose for his starting position. Also, after the second round, no matter how WBreaker plays, applying strategy $\mathcal{S}$, there can be at most one vertex in $U \cap V(B)$. Suppose that after $2 < k < n - 4$ rounds, there were at most two vertices from U visited by WBreaker. Assume that WBreaker played his move in round $k + 1$ and now it is WMaker's turn to play her move in round $k + 1$. Suppose that there are at least 3 vertices in $U \cap V(B)$. Being that WBreaker is a walker, he can visit one additional vertex from U in his move. So, there can be at most 3 vertices in $U \cap V(B)$. If WMaker cannot visit any of them in round $k + 1$, it means that all three of them are adjacent to her current position at vertex w, and so $d_B(w, U) = 3$. As WBreaker claimed only one edge in his $(k + 1)^{st}$ move, it must be an edge incident to w, and we obtain a contradiction to Corollary 4.3 after round k. □

Corollary 4.6. *In the $(1 : 1)$ WMaker–WBreaker game on $E(K_n)$, WMaker can build a path P of length $n - 3$ (with $n - 2$ vertices) in $n - 3$ moves by playing according to strategy $\mathcal{S}$.*

Proof. Suppose that WMaker already built a path P of length $n - 4$ ($v(P) = n - 3$). Let $U = \{u_1, u_2, u_3\}$. Let vertex w be WMaker's current position. If $wu_i \in E(B)$ for every $i \in \{1, 2, 3\}$, this means that after WBreaker's move in this round we have $d_B(w, U) = 3$ and this contradicts Corollary 4.4. □

Lemma 4.7. *In the $(1 : 1)$ WMaker–WBreaker game on $E(K_n)$, strategy $\mathcal{S}$ guarantees WMaker that for every vertex $x \in U$, $d_B(x) \leq 6$ holds at the moment when WMaker visits it for the first time.*

Proof. Assume that WBreaker touched vertex x for the first time in some round i using the edge ax. We will show that $d_B(x) \leq 6$ at the moment WMaker visits it. We analyse the following cases:

Case 1 WMaker was already positioned at vertex a at the beginning of round i, and after WBreaker claimed ax, there can be at most 2 additional vertices u_1, u_2 from U visited by WBreaker before WMaker's move in round i, according to Lemma 4.5.
In our analysis, we will assume that both vertices $u_1, u_2 \in U$ are in $V(B)$. When only one of vertices $u_1, u_2 \in U$ is in $V(B)$ or none of them belong to $V(B)$, the analysis is similar, but much simpler.

Case 2 WMaker's current position is at some vertex w and she visits a for the first time in round i. Beside the vertex $x \in U$ there can be at most one vertex from U visited by WBreaker after round i, according to Lemma 4.5. Denote this vertex with u'. In our analysis we will assume that there exists such vertex $u' \in U \cap V(B)$. Otherwise, the analysis is similar, but much simpler.

Case 3 WMaker is at some vertex $w \neq a$ at the beginning of round i and $a \in V(M)$. There can be at most 2 additional vertices $u_1, u_2 \in U \cap V(B)$ before WMaker's move in round i, according to Lemma 4.5.
In our analysis, we will assume that both vertices $u_1, u_2 \in U$ are in $V(B)$. When only one of the vertices $u_1, u_2 \in U$ is in $V(B)$ the analysis is similar but much simpler. If none of these two vertices belong to $V(B)$, WMaker moves from w to x in round i, which completes the analysis.

In the following we analyse all three cases separately.

Case 1 By Corollary 4.4 it is not possible that all three edges ax, au_1, au_2 are in $E(B)$, and so WMaker can move to some u_i, $i \in \{1, 2\}$, say u_1.
If WBreaker in round $i + 1$ moves to vertex $b \in V(M)$, then, if $d_B(x) > d_B(u_2)$, following $\mathcal{S}$, WMaker moves to x from u_1. Otherwise, she moves to u_2 (suppose she moves to u_2 even if $d_B(u_2) = d_B(x)$). The edge $u_1u_2 \notin E(B)$, otherwise we will have a contradiction with Corollary 4.2 before round i. Also, edges $u_1x, u_2x \notin E(B)$, since WBreaker visited x for the

first time in round i by claiming ax. In the following round, $i+2$, WMaker claims u_2x applying strategy $\mathcal{S}$, because after WBreaker's move in this round there will be no edges with both endpoints in $E(B)$, or vertices from U whose degree in B is larger than $d_B(x) = 2$.
However, if WBreaker claims the edge xb, in round $i+1$, for some $b \in U, b \notin \{u_1, u_2\}$, WMaker must move from u_1 to b or x. Since $d_B(x) = 2$ and $d_B(b) = 1$, the strategy $\mathcal{S}$ will tell her to choose the edge u_1x. This is as $u_1x, u_1b \notin E(B)$ (otherwise it would contradict Corollary 4.2 before round i).
If WBreaker chose edge xu_1 in round $i+1$, WMaker moves from u_1 to u_2 ($u_1u_2 \notin E(B)$ due to Corollary 4.2), and in round $i+2$ claims u_2x.
If WBreaker chose edge xu_2 in round $i+1$, then, if $d_B(x) > d_B(u_2)$, following $\mathcal{S}$, WMaker moves to x. Otherwise, she moves to u_2 (suppose she moves to u_2 even if $d_B(u_2) = d_B(x)$). Now for all $u \in U \setminus \{x\}$, $d_B(u) = 0$. In round $i+2$, WMaker claims u_2m_1 for some $m_1 \in U$, $d_B(m_1) = 0$. If in round $i+3$, WMaker is not able to claim m_1x, this means that:

i. WBreaker returned to x along u_2x in round $i+2$ and in round $i+3$ he claimed xm_1. Then, WMaker moves from m_1 to some $m_2 \in U$ in round $i+3$. So, in round $i+4$, she will be able to claim m_2x. In that moment we would have $d_B(x) = 3$ because in this round WBreaker could have either returned to x along m_1x or claimed m_1v, for some $v \in V(K_n), v \neq x$ (and $d_B(v) < d_B(x)$). The edge m_2x is free in the moment when WMaker wants to claim it. Otherwise, we will have a contradiction to Corollary 4.2 before round $i+3$.

ii. WBreaker claimed edges u_2y_1 and y_1y_2 for some $y_1, y_2 \in U$, in rounds $i+2$ and $i+3$, respectively. Since $y_1y_2 \in E(B)$ is not incident with $V(M)$, WMaker must visit y_1 or y_2 from m_1 in round $i+3$. Corollary 4.2 implies that edges $y_1m_1, y_2m_1 \notin E(B)$. Since $d_B(y_1) > d_B(y_2)$, WMaker moves to y_1. If WBreaker moves to some vertex $v \notin U$ or to $v = x$, WMaker can claim y_1x (and $d_B(x) \leq 3$). Otherwise, if WBreaker claims y_2y_3 for some $y_3 \in U$, in round $i+4$, strategy $\mathcal{S}$ will tell WMaker to claim y_1y_3, because edge y_2y_3 is not incident with $V(M)$. If WBreaker claims y_3v' for some $v' \neq x$, WMaker visits x along the edge y_3x. Otherwise, if WBreaker claims y_3x, WMaker claims y_3m_2, for some $m_2 \in U$ (and it holds that $d_B(m_2) = 0$). At

that point $d_B(x) = 3$.
If $d_B(x) = 4$ in round $i+6$, when WBreaker claimed xm_2, WMaker moves from m_2 to y_2, and afterwards, in round $i+7$, she moves from y_2 to x, where $d_B(x) = 4$. Since WBreaker finished his move in round $i+6$ at vertex m_2, he is not able to prevent WMaker from visiting x in round $i+7$. If WBreaker claimed xu, $u \neq m_2$, in round $i+6$, WMaker can visit x in this round by moving along the edge m_2x.

Case 2 WMaker is at vertex a at the beginning of round $i+1$. If WBreaker claims xb, where $b \in V(M)$, then WMaker moves to $u' \in U$ and in the following round claims $u'x$. This edge is free, otherwise it would contradict Corollary 4.2 before round i.
From now on, suppose that WBreaker claimed xb, where $b \in U$ in round $i+1$. By $\mathcal{S}$, WMaker must claim the edge ab. Also, Corollary 4.2 implies that $ab \notin E(B)$.
Consider the following situations:

i. WBreaker claims edges bc and cx, $c \in U$, in rounds $i+2$ and $i+3$, respectively.

   a) Let $c \neq u'$. Then WMaker claims bu', $u' \in U$ in round $i+2$. In the following round, WMaker is able to claim $u'x$ or $u'c$ (due to Corollary 4.2 before round i), but she will move to vertex x, because $3 = d_B(x) > d_B(c) = 2$.
   In case $b = u'$, WMaker first moves from b to some $m_1 \in U$ in round $i+2$, and then visits x along the edge m_1x in round $i+3$.

   b) If $c = u'$, then WMaker moves from b to some $m_1 \in U$ in round $i+2$. Since $d_B(c) \geq d_B(x)$, after WBreaker claims cx in round $i+3$, suppose that WMaker claims m_1c (even if $d_B(c) = d_B(x)$, as otherwise WMaker claims m_1x and that completes the argument). Now for all $u \in U \setminus \{x\}$, it holds that $d_B(u) = 0$. In the next round suppose that WBreaker claims xy_1. If $y_1 \in V(M)$, WMaker moves from c to some $y \in U$, in round $i+4$, and then claims yx in round $i+5$, which completes the analysis (at that moment $d_B(x) = 4$).
   If $y_1 \in U$, WMaker moves from c to y_1 and then she claims

y_1m_2 for some $m_2 \in U$ in round $i+5$. If WMaker is not able to visit x in round $i+6$, this means one of the following:

b.1) WBreaker returned to x along the edge y_1x in round $i+5$ and then he claimed xm_2 in round $i+6$. So, WMaker needs to move to some $m_3 \in U$ in round $i+6$ and in the following round she is able to visit x by claiming the edge m_3x, where $d_B(x) = 5$. WBreaker is not able to prevent WMaker from visiting x in round $i+7$ since he finished his previous move at vertex m_2.

b.2) WBreaker claimed y_1y_2 and y_2y_3, for some $y_2, y_3 \in U$, in rounds $i+5$ and $i+6$, respectively. Since y_2y_3 is not incident with $V(M)$, WMaker needs to move from m_2 to y_2 or y_3 in round $i+6$ (both edges are free to claim at that moment, as otherwise it would contradict Corollary 4.2 before round $i+5$). Since $2 = d_B(y_2) > d_B(y_3) = 1$, she moves to y_2, as it is illustrated in Figure 4.1.

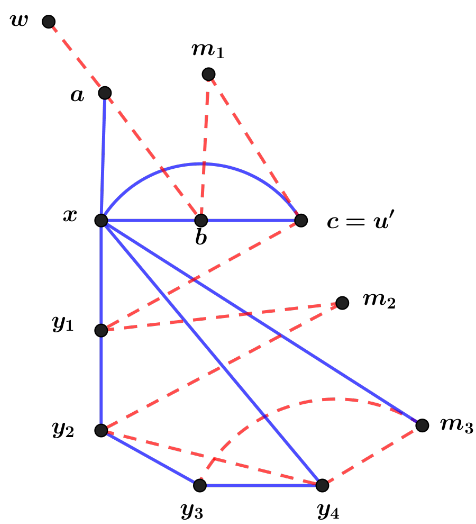


Figure 4.1: WMaker's and WBreaker's moves in Case 2.i.b.2) when $c = u'$. Dashed lines show WMaker's moves and solid lines represent WBreaker's moves.

If WMaker is not able to visit x in round $i+7$, this means that there is, again, an edge in $E(B)$ not incident with $V(M)$.

That is, WBreaker claimed y_3y_4 for some $y_4 \in U$. So, WMaker moves to y_4 (note that $d_B(y_4) = 1$, so WMaker can make this move). In round $i+8$, WBreaker can make degree 5 at vertex x by claiming y_4x and in this way he prevents WMaker from visiting x. Then, WMaker claims y_4m_3 for some $m_3 \in U$, again it holds that $d_B(m_3) = 0$ and $d_B(y_3) = 2$, but the edge y_4y_3 is already taken by WBreaker. If she is not able to move to x in round $i+9$ because WBreaker claimed xm_3 (and at this moment $d_B(x) = 6$), WMaker moves to y_3 and in the following round, $i+10$, she claims y_3x. WBreaker is not able to prevent WMaker from visiting x, because he finished his move at vertex m_3 in round $i+9$.

ii. WBreaker claims bc and cy_1, for $c, y_1 \in U$, in rounds $i+2$ and $i+3$, respectively. In this case, WMaker first moves from b to some $u \in U$. Then, according to the strategy $\mathcal{S}$, she needs to move to c, since cy_1 is not incident with $V(M)$ and $d_B(c) > d_B(y_1)$. If $b \neq u'$ and $c \neq u'$, then $u = u'$. If $b = u'$, WMaker already visited b. Also, if $c = u'$, WMaker already visited c.
If in the next round WBreaker claims y_1v, for some $v \in V(M)$, WMaker will be able to claim cx. Otherwise, if WBreaker claims edge y_1y_2 for some $y_2 \in U$, in round $i+4$, WMaker needs to move from c to vertex y_2 (this is possible as $d_B(y_2) = 1$ and its only neighbour in B is y_1). If afterwards WBreaker claims y_2v' for some $v' \neq x$, then WMaker visits x along the edge y_2x, where $d_B(x) = 2$. Otherwise, if WBreaker moves from y_2 to x and $d_B(x) = 3$, then WMaker must move to some $m \in U$ (at that point $d_B(m) = 0$). In round $i+6$, it is possible that $d_B(x) = 4$ if WBreaker claims either the edge xm or xy, for some vertex $y \neq m$. If WBreaker claims xm, WMaker claims my_1 and in the following round, $i+7$, WMaker claims y_1x. Whatever WBreaker plays in round $i+7$, he will not be able to prevent WMaker from visiting x. Otherwise, if WBreaker claimed $xy, y \neq m$, in round $i+6$, WMaker visits x in this round.

iii. If WBreaker returned to x along the edge bx in round $i+2$, then WMaker moves to some $m_1 \in U$ along the edge bm_1 (if $b \neq u'$ WMaker moves to $m_1 = u'$). In round $i+3$, WBreaker can claim

xm_1 and then $d_B(x) = 3$ (otherwise, WMaker claims m_1x and then $d_B(x) = 2$). In this case, WMaker must move to some $m_2 \in U$ ($m_2m_1 \notin E(B)$, otherwise it is a contradiction to Corollary 4.2 after round $i+1$). Whatever WBreaker plays in round $i+4$, he will not be able to prevent WMaker from moving to vertex x along the edge m_2x, because WBreaker finished his move in round $i+3$ at the vertex m_1.

Case 3 According to Corollary 4.3, $d_B(w, U) \leq 1$ after round $i-1$. Since WBreaker claimed ax in round i, WMaker is able to move from w to some u_i, say u_1, and to x, because $d_B(x) = 1$ and x is adjacent only to a in B. If $d_B(u_1) > d_B(x)$, she needs to move to u_1. The edges $xu_1, xu_2, u_1u_2 \notin E(B)$ due to Corollary 4.2. Otherwise, she moves to x which completes the analysis.
In round $i+1$, suppose that WBreaker claims xb. If $b = u_2$, WMaker moves to $b = u_2$ since $d_B(b) > d_B(x)$. The further analysis is similar to Case 1 so we skip details.
If $b = u_1$, then WMaker moves to u_2 and in the following round, $i+2$, she moves to x (this is possible as $xu_2 \notin E(B)$). At that moment $d_B(x) = 2$.
If $b \in U$ and $b \notin \{u_1, u_2\}$, then WMaker needs to move from u_1 to x or b because xb is not incident with $V(M)$. Since $d_B(x) = 2$ and $d_B(b) = 1$, she moves to x.
If $b \in V(M)$, WMaker first moves to u_2, if $d_B(u_2) \geq d_B(x)$ (suppose that she moves to u_2 even if $d_B(u_2) = d_B(x)$), and then claims u_2x. Otherwise, she visits x in round $i+1$ along the edge u_1x. In both cases at the moment when WMaker visits x, $d_B(x) = 2$. □

Finally, we need the following two lemmas.

Lemma 4.8. *In the* $(1:1)$ *WMaker–WBreaker game on* $E(K_n)$*, strategy* $\mathcal{S}$ *guarantees that if WBreaker visits a vertex* $u \in U$ *in some round* $2 \leq i \leq n-6$ *from a vertex* $m \in V(M)$*, different from her current position, WMaker will visit* u *at latest in round* $i+2$*.*

Proof. Suppose that WMaker is at some vertex w at the beginning of round i. Suppose that before WBreaker visits u arriving from m, he arrives to m (in round $i-1$) from some vertex b on his path. By Lemma 4.5 at that point there are at most two vertices in $U \cap V(B)$, say p and q (note that

one of them can be u).
First suppose that both of them exist. If WMaker couldn't visit any of them from her current position, say y, in round $i-1$, that means that one of the $\{p, q\}$ is m (due to Corollary 4.4), which is not the case. So, suppose that WMaker visits q in that round and $q \neq u$, i.e. $w = q$. Due to Corollary 4.2, the edge $wp \notin E(B)$. After WBreaker's move in round i, if $p = u$, then WMaker visits u which completes the claim. Suppose that $u \neq p$. Due to Corollary 4.2 $wp, wu \notin E(B)$. If $d_B(p) < d_B(u)$, then WMaker visits u in her i^{th} move and the proof is complete. Otherwise $d_B(p) \geq d_B(u)$ and suppose that WMaker visits p in her i^{th} move even if $d_B(p) = d_B(q)$. Note that now for all $v \in U \setminus \{u\}$ it holds that $d_B(v) = 0$. If WMaker cannot visit u in her $(i+1)^{\text{st}}$ move, this means that WBreaker claimed the edge up in round $i+1$, and WMaker visits some vertex $t \in U$. Whichever vertex WBreaker visits in round $i+2$, strategy $\mathcal{S}$ will tell WMaker to visit u in this round.
Now, suppose that there is only one vertex $p \in U \cap V(B)$, when WBreaker arrives at m in round $i-1$. If WMaker cannot visit p in round $i-1$, then w is a vertex from U such that $d_B(w) = 0$. After WBreaker's i^{th} move, suppose that WMaker visits p by playing by $\mathcal{S}$ (otherwise the claim is satisfied). If WMaker cannot visit u in her $(i+1)^{\text{st}}$ move, this means that WBreaker claimed the edge up in round $i+1$, and WMaker visits some vertex $t \in U$, and $d_B(t) = 0$ and afterwards she visits u in round $i+2$. This completes the claim.
If $U \cap V(B) = \emptyset$ after WBreaker's move in round $i-1$, then WMaker can visit u in round i along the edge wu. $\square$

Lemma 4.9. *Let $3 \leq i \leq n-6$ and V_M denote $V(M)$ before WMaker's move in round $i-1$. In the $(1:1)$ WMaker–WBreaker game on $E(K_n)$, strategy $\mathcal{S}$ guarantees that if WBreaker visits a vertex $u \in U$ in some round i and then a vertex $m \in V_M$ in round $i+1$, WMaker will visit u at latest in round $i+2$.*

Proof. Suppose WMaker is at some vertex w before her i^{th} move. WBreaker visits u in round i from some vertex b. We distinguish between the two cases:

1. $b \in V(M)$. If $b \neq w$, WMaker will visit u at latest in round $i+2$, due to Lemma 4.8;

Otherwise, according to Lemma 4.5, there can be at most 2 other vertices p, q in $U \cap V(B)$ when u is visited for the first time.
Suppose that both vertices exist. The edge $pq \notin E(B)$ due to Corollary 4.2. As $d_B(w, U) \leq 2$ before her move in round i (Corollary 4.4), she can visit one of the vertices in $\{p, q\}$. Suppose that the strategy $\mathcal{S}$ tells her to visit q. In round $i+1$, WBreaker moves to some $m \in V_M$, according to the assumption of the lemma. Still, there are 2 vertices in $U \cap V(B)$, p and u and $pu \notin E(B)$. If $d_B(u) > d_B(p)$, then WMaker visits u which completes the proof. Otherwise, $d_B(p) \geq d_B(u)$ and suppose WMaker visits p in round $i+1$ (even if $d_B(p) = d_B(u)$). No matter how WBreaker plays his $(i+2)^{\text{nd}}$ move, WMaker visits u by playing according to the strategy $\mathcal{S}$ in round $i+2$.
Suppose that only one other vertex, say p, exists in $U \cap V(B)$ after WBreaker's move in round i, that is $\{p, u\} \in U \cap V(B)$. If $d_B(w, U) = 2$, then, in round i, WMaker claims an edge wt, for some $t \in U$ and $d_B(t) = 0$ (note that $d_B(v) = 0$ for all $v \in U \setminus \{p, u\}$). After that, the analysis is same as above, applied to $t = q$.
If $d_B(w, U) = 1$, WMaker claims wp in round i. In the following round, WMaker can visit u, as WBreaker moves to $m \in V_M$ and due to Corollary 4.2 the edge $pu \notin E(B)$.

2. Let $b \in U$. There can be at most one more vertex, say p, in $U \cap V(B)$. Since WMaker is at vertex w at the beginning of round i and WBreaker is at vertex b, we have $d_B(w, U) \leq 1$, due to Corollary 4.3 and Corollary 4.4. The edge $bp \notin E(B)$ due to Corollary 4.2. WMaker needs to visit u or b, according to the strategy $\mathcal{S}$. If she visits u, this completes the proof. Otherwise, suppose she visits b. In the following round, $i+1$, as WBreaker moves from u to some vertex $m \in V_M$, WMaker claims bp in this round. No matter how WBreaker plays in round $i+2$, he will not prevent WMaker from visiting u in this round by claiming the edge pu. If only $b, u \in U \cap V(B)$ after WBreaker's move in round i, then the analysis is almost the same as the previous analysis with $t = p$, for some $t \in U$, such that $d_B(t) = 0$ before WMaker's move in round $i+1$.

□

In the following we prove Theorem 2.6 and Theorem 2.7.

WMaker–WBreaker Connectivity game

Proof of Theorem 2.6. At the beginning of the game, all vertices are isolated in WMaker's graph and $U = V(K_n)$. The strategy of WMaker in this game is divided into following two stages.

Stage 1 In this stage WMaker builds a path P of length $n-4$ in $n-4$ rounds, by playing according to the strategy $\mathcal{S}$, which is possible due to Corollary 4.6.

Stage 2 During the course of this stage WMaker visits the three remaining vertices in at most 5 additional moves. At the beginning of this stage suppose that WMaker is at vertex w and $U = \{u_1, u_2, u_3\}$. Assume that it is WMaker's turn to play her move in round $n-3$. Corollary 4.4 implies that $d_B(w, U) \leq 2$.
First, suppose that vertex w is such that $d_B(w, U) \leq 1$. Let $wu_1 \in E(B)$. Since WMaker visited w in round $n-4$ this means that WBreaker must have claimed wu_1 in this round. Otherwise, if WBreaker claimed this edge earlier, then we would have a contradiction to Corollary 4.2 before round $n-4$. If WBreaker finished his move in round $n-4$ at vertex u_1, then in his $(n-3)^{\text{rd}}$ move he could claim u_1u_i, for some $i \in \{2, 3\}$. Suppose that $u_1u_2 \in E(B)$ and WBreaker is at u_2. This edge could not exist earlier, because it would be a contradiction to Corollary 4.2. Also, the edges $u_2u_3, u_1u_3 \notin E(B)$ because of Corollary 4.2. In round $n-3$, WMaker claims the edge wu_3 and in the following round she moves to u_1. WBreaker is not able to prevent WMaker from claiming u_3u_1 because he finished his $(n-3)^{\text{rd}}$ move at the vertex u_2.
In round $n-3$, when WMaker visited u_3 for the first time, we had $d_B(u_3) \leq 6$ (according to Lemma 4.7). Since u_1, u_2 were still unvisited by WMaker in round $n-3$, we have $d_B(u_1), d_B(u_2) \leq 8$, in round $n-1$. Since $d_B(u_1, V(P)) + d_B(u_2, V(P)) < v(P)$, there exists a vertex $v \in V(P)$ such that u_1v and vu_2 are free. She claims u_1v in round $n-1$. If WMaker is not able to claim vu_2 in round n, this means that WBreaker finished his move in the previous round at vertex u_2, so he was able to prevent WMaker from visiting u_2 by claiming u_2v in his n^{th} move. Then WMaker

moves to some $v' \in V(P)$ such that edges vv' and $v'u_2$ are free. We need to prove that such vertex v' exists.
Let $P' = P \setminus \{v, u_1\}$.
Since $d_B(u_2) \leq 6$ in round $n-3$, we have $d_B(u_2) \leq 8$ before WMaker's move in round n. So, if $d_B(v, V(P')) + d_B(u_2, V(P')) \geq v(P') = n-3$, it follows that $d_B(v, V(P')) \geq n - 11$. To make such a large degree at vertex v, WBreaker needed at least $2(n-11)-2$ moves because he is also a walker. Since he played exactly n moves, this is not possible.
So, in round n WMaker claims vv' and, in the last round, $n+1$, she moves to u_2. WBreaker is not able to prevent WMaker from claiming $v'u_2$, because he finished his move in round n at vertex v.

Let $d_B(w, U) = 2$ before WMaker's move in round $n-3$ and let $wu_1, wu_2 \in E(B)$. From Corollary 4.3 we know that WBreaker has moved to u_1 or u_2 from vertex w in his last move, because at the end of round $n-4$ when WMaker came to w, we had $d_B(w, U) \leq 1$. Assume that WBreaker is at vertex u_2. Edges $u_1u_2, u_2u_3, u_1u_3 \notin E(B)$. Otherwise, this would mean that WBreaker claimed some of these edges in some round before round $n-3$ and we would have a contradiction to Corollary 4.2.
WMaker claims wu_3 in round $n-3$. If in the following round WBreaker moves to u_3, WMaker claims u_3u_1 and then u_1u_2 in round $n-1$. WBreaker is not able to prevent WMaker from claiming u_1u_2 because he finished his move in round $n-2$ at vertex u_3.
If WBreaker moves to u_1 in round $n-2$, WMaker claims u_3u_2. In round $n-1$ WMaker identifies a vertex $v \in V(P)$ such that edges u_2v and vu_1 are free. Similarly as above we can prove that such vertex v exists. WMaker claims u_2v in round $n-1$ and in round n, she claims vu_1. Since WBreaker must move from u_1 in round $n-1$, he can not prevent WMaker from claiming vu_1 in the last round n.
Otherwise, WMaker can visit the remaining two vertices in two moves. $\square$

WMaker–WBreaker Hamilton Cycle game

Proof of Theorem 2.7. First, we describe WMaker's strategy and then prove that she can follow it. At the beginning of the game $U = V(K_n)$.

Stage 1 In the first $n-4$ rounds WMaker builds a path P of length $n-4$ (with $n-3$ vertices) by playing according to the strategy $\mathcal{S}$.

Stage 2 In the next at most 4 rounds, WMaker closes the cycle of length $n-2$ or $n-1$.
Denote by v_1 the vertex in which WMaker starts the game. In round $n-3$, WMaker from her current position moves to vertex $u_i \in U$, $i \in \{1,2,3\}$ which is not incident with v_1 in B. If WMaker can claim the edge u_iv_1 in her following move, then she claims it and creates a cycle of length $n-2$. Otherwise, she moves to u_j along the edge u_iu_j, where $i,j \in \{1,2,3\}$ and $i \neq j$, in round $n-2$.
If the edge u_jv_1 is free after WBreaker's move in round $n-1$, WMaker claims this edge and closes the cycle of length $n-1$.
Otherwise, she finds a vertex $v \in V(P)$ such that edges u_jv and vv_1 are free. WMaker first claims the edge u_jv and then in the following round, n, she claims vv_1 and thus closes the cycle of length $n-1$.

Stage 3 Depending on how Stage 2 ended, WMaker completes the Hamilton cycle in at most 8 rounds. The details follow in the analysis of this stage.

We now prove that WMaker can follow her strategy.

Stage 1 Corollary 4.6 implies that WMaker can follow her strategy in Stage 1 and build the path P of length $n-4$, thus visiting $n-3$ vertices in $n-4$ moves.

Stage 2 At the beginning of round $n-3$, WMaker is positioned at vertex x and $U=\{u_1,u_2,u_3\}$. We know that WMaker started the game at the vertex v_1 and we consider several cases.

Case 1 WBreaker is not positioned at vertex v_1 at the beginning of WMaker's move in round $n-3$.

Case 1.a Suppose that $d_B(x,U)=2$ before WMaker's $(n-3)^{\text{rd}}$ move. Let $u_1x, u_2x \in E(B)$. From Corollary 4.4 we know that $u_3x \notin E(B)$, because after WBreaker's move (and before WMaker's move) in each round we can have $d_B(x, U) \leq 2$. Also, WBreaker must be positioned at u_1 or u_2, that is, one of the edges, u_1x or u_2x is the edge which WBreaker claimed in his last move. Otherwise, we would have a contradiction to Corollary 4.3 and Corollary 4.4. Suppose that WBreaker finished his $(n-3)^{\text{rd}}$ move at vertex u_1.

Claim 4.10. *For all $i \in \{1, 2, 3\}$, $v_1u_i \notin E(B)$, before WMaker's $(n-3)^{\text{rd}}$ move.*

Proof. Suppose that at least one of these three edges is in WBreaker's graph.
Based on the assumption of Case 1.a, we know that WBreaker claimed the edge u_2x in his $(n-4)^{\text{th}}$ move, and the edge xu_1 in his $(n-3)^{\text{rd}}$ move. In round $n-5$ he moved to u_2 from some vertex. Assume that this vertex is v_1. So, $u_2v_1 \in E(B)$. Following the proof of Lemma 4.8 line by line with $m = v_1$, WMaker would visit u_2 in round $n-4$. A contradiction, as she visited x in this round. Note that if the edge v_1u_2 existed earlier, the analysis is either the same or if WBreaker claimed u_2v_1, by moving from u_2 to v_1, in some round $k \leq n-7$, then by Lemma 4.9, WMaker could visit u_2 at latest in round $k+1 \leq n-6$, which is a contradiction.
If $u_1v_1 \in E(B)$, then it follows that WBreaker could claim it, at latest, in round $n-6$. By similar analysis as above, we can conclude that this is also not possible.
Now suppose that $v_1u_3 \in E(B)$. If WBreaker in some round $k \leq n-7$ moved from u_3 to v_1, then Lemma 4.9 implies that WMaker would visit u_3 at latest in round $n-6$. A contradiction. Otherwise, WBreaker in some round $k \leq n-8$ moved from his current position p to vertex v_1 and then in round $k+1$ he claimed v_1u_3. Applying Lemma 4.8 with $m = v_1$, WMaker visits u_3 at latest in round $k+3 \leq n-5$. A contradiction. □

Claim 4.10 gives that in her $(n-3)^{\text{rd}}$ move, WMaker can claim the edge xu_3 and in the following round, $n-2$, she can close a cycle of length $n-2$, by claiming the edge u_3v_1.

Case 1.b Suppose that $d_B(x, U) = 1$ before WMaker's move in round $n-3$. Let $xu_1 \in E(B)$. Since WMaker visited x in round $n-4$, this means that WBreaker claimed xu_1 in round $n-4$, as otherwise this would

contradict Corollary 4.2 before round $n-4$. If WBreaker finished $(n-4)^{\text{th}}$ move in x, after his move in round $n-3$ we can have Case 1.a. which we already considered. If WBreaker moved to some v on WMaker's path $v \neq v_1$, then WMaker could close a cycle of length $n-2$ in the next two moves.
Assume that WBreaker finished his $(n-4)^{\text{th}}$ move in u_1.

Claim 4.11. *After round $n-4$, $u_i v_1 \notin E(B)$ for each $i \in \{1,2,3\}$.*

Proof. Suppose that $\exists i \in \{1,2,3\}$ such that $v_1 u_i \in E(B)$. Since WBreaker moved from x to u_1 in round $n-4$, it follows that in round $n-5$, he came from some vertex p to x. We know that $p \notin \{u_1, u_2, u_3\}$ following $\mathcal{S}$ and due to Corollary 4.2. Thus, it could happen that WBreaker claimed $u_2 p$ (or $u_1 p$ or $u_3 p$) in round $n-6$ and in round $n-7$, he claimed $v_1 u_2$ (or $v_1 u_1$ or $v_1 u_3$). Let $v_1 u_2, u_2 p \in E(B)$. By Lemma 4.8 with $m = v_1$, WMaker visits u_2 at latest in round $n-5$. A contradiction.
If WBreaker moved from u_i, $i \in \{1,2,3\}$ to v_1 in some round $k \leq n-6$, then by Lemma 4.9, WMaker could visit u_i at latest in round $k+1$. A contradiction. Thus, after round $n-4$, all edges $u_1 v_1, u_2 v_1, u_3 v_1$ are free. $\square$

If WBreaker, in his $(n-3)^{\text{rd}}$ move, claims $u_1 u_2$ (respectively to $u_1 u_3$), then WMaker moves from x to u_3 (or u_2). In the following round, $n-2$, she claims $u_3 v_1$ (or $u_2 v_1$), which is free according to Claim 4.11, and closes the cycle of length $n-2$.
If WBreaker, in his $(n-3)^{\text{rd}}$ move, claims $u_1 v_1$, then WMaker moves to u_2 or u_3. Suppose WMaker claimed $x u_2$. In the following round WBreaker can claim $v_1 u_2$. So, WMaker is not able to close the cycle of length $n-2$ in round $n-2$. In that case, she moves to u_3 along $u_2 u_3$ ($u_2 u_3 \notin E(B)$ as this contradicts Corollary 4.2 before round $n-3$). In round $n-1$, WMaker moves from u_3 to v_1 and makes a cycle of length $n-1$. WBreaker cannot block her because he finished his $(n-2)^{\text{nd}}$ move at vertex u_2. The analysis is the same if WMaker claimed $x u_3$ in round $n-3$.

Case 1.c Suppose that $d_B(x, U) = 0$ before WMaker's move in round $n-3$.

Claim 4.12. *Before WMaker's $(n-3)^{\mathrm{rd}}$ move, there can be at most two vertices from U adjacent to v_1 in B.*

Proof. If $u_1v_1, u_2v_1, u_3v_1 \in E(B)$, then WBreaker spent at least 4 moves to do this. Assume that he claimed edges in this order: u_1v_1, v_1u_2, u_2v_1 and v_1u_3. Thus, he moved from u_1 to v_1 in round $k \leq n-6$. Applying Lemma 4.9 we conclude that WMaker can visit u_1 at latest in round $k+1 \leq n-5$. A contradiction. Similarly, if WBreaker first visited u_1 coming from vertex $v_1 \in V(M)$ in round $k \leq n-7$, then by Lemma 4.8 with $m = v_1$, WMaker visits u_1 at latest in round $k+2 \leq n-5$. A contradiction. □

Claim 4.12 implies that there can be at most two vertices from U, adjacent to v_1 in B. If there are exactly 2 vertices, say u_1, u_2 adjacent to v_1, then in the similar way as in the proof of Claim 4.12, we can show that WBreaker finished his last move, in round $n-3$, in vertex u_1 or u_2.

In her $(n-3)^{\mathrm{rd}}$ move, WMaker claims xu_3. Whatever WBreaker plays in round $n-2$, he will not be able to prevent WMaker from claiming u_3v_1 in this round. Thus, WMaker closes the cycle of length $n-2$ in round $n-2$.

Case 2 Suppose that WBreaker is at vertex v_1 after his move in round $n-3$.

Claim 4.13. *It is not possible that at the same time $u_iv_1, u_jv_1 \in E(B)$ for some $i, j \in \{1, 2, 3\}$, $i \neq j$ and that WBreaker is at v_1 after his move in round $n-3$.*

Proof. If the assertion of the claim was true, it would mean that WBreaker spent at least three moves claiming edges in the following order: u_iv_1, v_1u_j and u_jv_1, for some $i, j \in \{1, 2, 3\}, i \neq j$. This is not possible because of the following. Let $i = 1$ and $j = 2$. Suppose that WBreaker came to vertex u_1 in round $k \leq n-6$ from some vertex p which was on his path because WBreaker is a walker, and then to a vertex $v_1 \in V(M)$ in round $k+1$. By Lemma 4.9, WMaker visits u_1 at latest in round $k+2 \leq n-4$. A contradiction.
If WBreaker came from v_1 to u_1 in some round $k \leq n-6$, then by

Lemma 4.8 with $m = v_1$, then WMaker would visit u_1 at latest in round $k+2 \leq n-4$. A contradiction. $\square$

Claim 4.13 implies that there can be at most one edge $v_1u_i \in E(B)$ for $i \in \{1,2,3\}$. Thus, suppose that edge $v_1u_2 \in E(B)$, that is, WBreaker came from u_2 to v_1. WMaker is at vertex x. We know that $d_B(x,U) \leq 1$ because of Corollary 4.3 and because in his last move WBreaker moved to v_1. Suppose the edge xu_1 is free. WMaker claims it. If edge v_1u_1 is free at the beginning of WMaker's $(n-2)^{\text{nd}}$ move, then WMaker claims it and closes the cycle of length $n-2$. Otherwise, this means that WBreaker claimed this edge in his $(n-2)^{\text{nd}}$ move (Claim 4.13). In this case, WMaker moves to u_3. The edge u_1u_3 must be free due to Corollary 4.2 after round $n-4$. In round $n-1$ WBreaker is not able to prevent WMaker from claiming the edge u_3v_1 because he finished his previous move at vertex u_1. So, the cycle of length $n-1$ is created in WMaker's graph.
Therefore, WMaker is able to create a cycle of length $n-2$ or $n-1$.

Stage 3 Depending on how Stage 2 ended we analyse two cases.

Case 1 Suppose that WMaker created a cycle C of length $n-2$. She played exactly $n-2$ rounds. WMaker's current position is at vertex v_1. Denote by v_{n-2} the vertex which was last visited by WMaker in round $n-3$. Let $U = \{u_1, u_2\}$. In round $n-1$ WMaker returns from v_1 to vertex v_{n-2}. Lemma 4.7 guarantees that $d_B(u_i) \leq 6$ for all $i \in \{1,2,3\}$ in round $n-3$ (in which WMaker visited v_{n-2}) and so in that moment $d_B(u_i, C) \leq 6$ for $i \in \{1,2\}$. It follows that, after WBreaker's move in round n, vertices u_1, u_2, v_{n-2} can have degree at most 8 in B towards cycle C. Since $v(C) = n-2$, by pigeonhole principle, there are 3 consecutive vertices w_1, w_2, w_3 on cycle C such that there are no WBreaker's edges between $\{w_1, w_2, w_3\}$ and $\{u_1, u_2, v_{n-2}\}$. Since n is large enough, there are at least 4 such triples (with at least 6 vertices if triples are not disjoint). If one of these 6 vertices, say t, is such that $d_B(t) \geq (n+c)/4$, for some constant $c \geq 0$, WMaker will choose another triple not containing such a vertex. Taking into consideration that both players are walkers, there can be at most one vertex in $V(K_n)$, with such a large degree. Otherwise, this would mean that WBreaker played more than n moves.
WMaker first claims the edge $v_{n-2}w_2$ in round n. If in the following round,

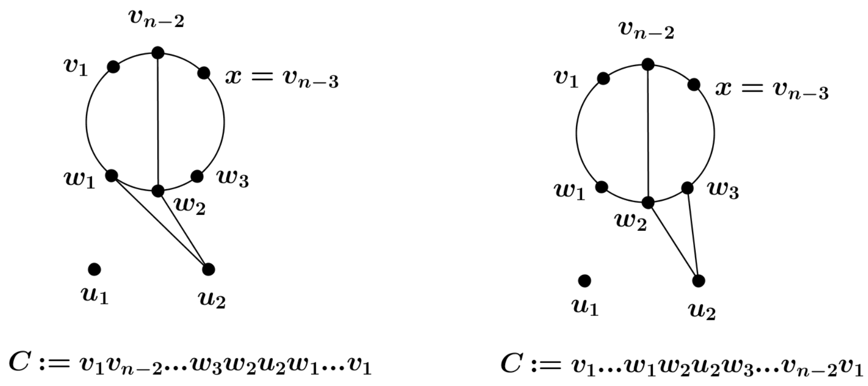


Figure 4.2: WMaker's cycle C of the length $n-1$ after round $n+2$. Left (right) figure illustrates the case when WMaker claimed u_2w_1 (u_2w_3) in round $n+2$.

WBreaker claimed the edge w_iu_1 for some $i \in \{1,2,3\}$, then WMaker chooses edge w_2u_2 (otherwise, if he claimed w_iu_2, we just interchange the vertices u_1 and u_2) and in round $n+2$ closes the cycle C of length $n-1$ by claiming either the edge u_2w_1 or u_2w_3, as both are free. This is illustrated on Figure 4.2. If WBreaker did not claim any of edges w_iu_1, w_iu_2, for $i \in \{1,2,3\}$, in round $n+1$, then WMaker moves from w_2 to either of these two vertices u_1, u_2. In round $n+2$, she moves from chosen vertex (u_1 or u_2) to w_1 or w_2, because WBreaker could not claim both edges u_iw_1 and u_iw_2 in round $n+2$, where u_i, $i \in \{1,2\}$, is the vertex which WMaker chose in the previous round.
Let $U = \{u_1\}$. Suppose that WMaker finished her last move at w_1. Consider the following cases.

a) WBreaker finished his $(n+3)^{\text{rd}}$ move at vertex u_1. Then WMaker returns from w_1 to vertex u_2 which now belongs to C. By similar reasoning as above, knowing that $d_B(u_1,C) \leq 11$ and $d_B(u_2,C) \leq 11$, by pigeonhole principle, we conclude that there are three vertices y_1, y_2, y_3 on C such that there are no Breaker's edges between $\{y_1,y_2,y_3\}$ and $\{u_1,u_2\}$. Therefore, WMaker in round $n+4$ claims u_2y_2 and then in rounds $n+5$ and $n+6$ claims y_2u_1, u_1y_1 (or u_1y_3), respectively.

b) WBreaker finished his $(n+3)^{\text{rd}}$ move at some vertex $v \neq u_1$. By pigeonhole principle, as $v(C) = n-1$ and $d_B(w_1) < n/4$ and $d_B(u_1) \leq 10$, WMaker can find three consecutive vertices y_1, y_2, y_3 on C such that there are no Breaker's edges between $\{y_1, y_2, y_3\}$ and $\{w_1, u_1\}$. She first claims w_1y_2 and then y_2u_1 and u_1y_1 (or u_1y_3), and completes the Hamilton cycle in round $n+5$. This is possible by using the same argument as above.

Case 2 Suppose that in Stage 2 WMaker created a cycle C of length $n-1$ (in at most n rounds). Denote by v_{n-1} the vertex from U that was last visited by WMaker. In the last round of Stage 2, WMaker moved from v_{n-1} to v_1. Let $U = \{u\}$.
WMaker first moves from v_1 to v_{n-1} in round $k \leq n+1$ ($k = n+1$ if WMaker finished her cycle in round n of previous stage).
If WBreaker finished his move in round $k+1 \leq n+2$ at some vertex different from u, then WMaker finds three consecutive vertices y_1, y_2, y_3 on C, such that there are no edges between $\{y_1, y_2, y_3\}$ and $\{u, v_{n-1}\}$ in B. Since $d_B(v_{n-1}), d_B(u) \leq 9$ in round $k+1 \leq n+2$ and since $v(C) = n-1$, by pigeonhole principle, such vertices y_1, y_2, y_3 exist. She first claims $v_{n-1}y_2$ and in the following round the edge y_2u. In the last round, $k+3 \leq n+4$, she claims uy_1 or uy_3 because WBreaker could not claim both edges.
If WBreaker finished his move in round $k+1 \leq n+2$ at vertex u, then WMaker moves from v_{n-1} to v_{n-2} in this round. In round $k+2 \leq n+3$ WBreaker must move from u. In this round WMaker finds three consecutive vertices y_1, y_2, y_3 on C, such that there are no edges between $\{y_1, y_2, y_3\}$ and $\{u, v_{n-2}\}$ in B. These vertices exist as WMaker visited vertex v_{n-2} in round $n-3$ and in that moment we had $d_B(v_{n-2}), d_B(u) \leq 6$ (Lemma 4.7). Since $d_B(v_{n-2}), d_B(u) \leq 9$ in round $k+2 \leq n+3$ and since $v(C) = n-1$, by pigeonhole principle, such vertices y_1, y_2, y_3 exist. WMaker first claims the edge $v_{n-2}y_2$. In the following round WMaker claims y_2u and in the final round, $k+4 \leq n+5$, she completes the Hamilton cycle by claiming edge uy_1 or uy_3. $\square$

4.2 WBreaker's strategy

In this section we prove Theorem 2.8. We provide WBreaker with a strategy which allows him to postpone WMaker's winning by at least n moves. Here, we suppose that WMaker starts the game and $U = V(K_n)$.

Proof. WBreaker plays arbitrarily until $|U| = 3$. To be able to visit $n-3$ vertices, WMaker needs to play at least $n-4$ moves. Let $u_1, u_2, u_3 \in U$ after round $k \geq n-4$.
If in round $k+1 \geq n-3$ WMaker moves to some vertex $v \neq u_i$, $i \in \{1, 2, 3\}$, then she will need at least 3 more moves to visit u_1, u_2, u_3, which satisfies the claim.
Suppose that in round $k+1 \geq n-3$ WMaker moves to some u_i, $i \in \{1, 2, 3\}$. WBreaker moves to u_j, $j \neq i$. WBreaker is able to move to u_j since $u_j \in U$ and there is no WMaker's edge between WBreaker's current position and vertex u_j.
Without loss of generality, suppose that WMaker has moved to u_1 and WBreaker to u_2. If WMaker in round $k+2$ moves to one of $\{u_2, u_3\}$, WBreaker claims the edge u_2u_3. As $u_2u_3 \in E(B)$ from her current position, WMaker is not able to visit the remaining isolated vertex in her graph in round $k+3 \geq n-1$, so she needs to make at least one additional move to touch the remaining vertex. If WMaker moves to some vertex $v \neq u_i$, $i \in \{2, 3\}$ in round $k+2 \geq n-2$, then she will need at least two more moves to visit u_2, u_3.
It follows that WMaker needs at least n moves to win in the Connectivity game. $\square$

4.3 Concluding remarks

According to the theorems 2.6 and 2.8 WMaker needs t, $n \leq t \leq n+1$ moves to make a spanning tree. From Theorem 2.7 it follows that she needs at most $n+6$ moves to create a Hamilton cycle. As WMaker needs at least n move to make a spanning tree, it follows that WMaker can not create Hamilton cycle in less that $n+1$ moves.
Note that $b = 1$ is the largest WBreaker's bias for which WMaker can win in the Connectivity game and Hamilton Cycle game. For $b = 2$ WBreaker can isolate a vertex in WMaker's graph in the following way: in each round

he can use one move to return to the fixed vertex along the previously claimed edge, and the other to claim the edge between this particular vertex and WMaker's current position.

Chapter 5

MBTD game on cubic graphs

We study Maker–Breaker total domination game on cubic graphs and prove the theorems 2.9, 2.10, 2.11, 2.12, 2.13 and 2.14. Recall that the MBTD game is played by two players Dominator and Staller. Dominator wants to build a total dominating set, and Staller tries to prevent him. To determine which connected cubic graphs are Dominator's win and which are Staller's win, we classify cubic graphs according to the number of vertices lying in two triangles, in one triangle and zero triangles.

This chapter is organized in the following way. In Section 5.1 we give some additional notation and statements needed for proving stated theorems. In Section 5.2 we present the proofs of theorems 2.9, 2.10, 2.11, 2.12, 2.13 and 2.14. Concluding remarks we give in Section 5.3.

5.1 Preliminaries

We say that a vertex $x \in V(G)$ is adjacent to some triangle $Y \subseteq G$, where $V(Y) = \{y_1, y_2, y_3\}$, if $xy_i \in E(G)$, for some $i \in \{1, 2, 3\}$. Also, we say that two triangles $X \subseteq G$ and $Y \subseteq G$, with the vertex sets $V(X) = \{x_1, x_2, x_3\}$ and $V(Y) = \{y_1, y_2, y_3\}$, are adjacent if $x_i y_j \in E(G)$ for some $i, j \in \{1, 2, 3\}$.
Assume that the MBTD game is in progress. We denote by $d_1, d_2, ...$ the sequence of vertices chosen by Dominator and by $s_1, s_2, ...$ the sequence of vertices chosen by Staller. At any given moment during this game, we denote the set of vertices claimed by Dominator by $\mathfrak{D}$ and the set of

vertices claimed by Staller by $\mathfrak{S}$. As in [60], we say that the game is the D-game if Dominator is the first to play, i.e. one *round* consists of a move by Dominator followed by a move of Staller. In the S-game, one round consists of a move by Staller followed by a move of Dominator. We say that the vertex v is *isolated* by Staller if all neighbours of v are claimed by Staller. Following the notation from [59] we say that a graph G is

- $\mathcal{D}$, if Dominator wins the game
- $\mathcal{S}$, if Staller wins the game.

We point out some basic properties of the MBTD games given in [59].

Proposition 5.1. *([59], Corollary 2.2(ii)) Let G be a graph. Let $V_1, ..., V_k$ a partition of $V(G)$ such that V_i, $i \in [k] := \{1, ..., k\}$, induces a graph on which Dominator wins the MBTD game, then Dominator wins MBTD game on G.*

Proposition 5.2. *([59], Proposition 2.4) Dominator wins in MBTD game on cycle C_4.*

5.1.1 Traps

While playing against the Dominator, Staller's best option to prevent him from winning is to make different kinds of traps. In the following, we define these traps.

Consider the MBTD game on graph G. Let $v \in V(G)$ and let $u_1, u_2, u_3 \in N_G(v)$. Let u_2 and u_3 be free vertices. Let $u_1 \in \mathfrak{S}$ and suppose that it is Staller's turn to make her move. If Staller claims u_2 (or u_3), she creates a *trap* for Dominator, that is, Staller forces Dominator to claim u_3 (or u_2) as otherwise she isolates v.

Double trap. We say that Staller creates a *double trap* $u - v$ in the MBTD game on G, where $u, v \in V(G)$ are free vertices if after Staller's move Dominator is forced to claim both vertices u and v. Since Dominator can not claim two vertices in one move, in her next move Staller will claim either u or v and isolate either a neighbour of u or a neighbour of v.
If Staller creates a double trap, Dominator loses the game.

Diamond trap. Suppose that the MBTD game on the connected cubic graph G is in progress. Let $Z \subseteq G$ be a diamond with the vertex set $V(Z) = \{z_1, z_2, z_3, z_4\}$ and the edge set $E(Z) = \{z_1z_2, z_2z_3, z_3z_4, z_4z_1, z_2z_4\}$ and suppose that all vertices from $V(Z)$ are free. By claiming z_1 (or z_3) Staller creates a *diamond trap* on Z. That is, she forces Dominator to claim a vertex from $V(Z) \setminus \{z_1\}$ (or $V(Z) \setminus \{z_3\}$), as otherwise, in her next move Staller will claim z_3 (or z_1) and create a double trap $z_2 - z_4$.

Vertex-diamond trap. Suppose that the MBTD game on the connected cubic graph G is in progress. Consider subgraph $G' \subseteq G$ with the vertex set $V(G') = \{v, y_1, y_2, z_1, z_2, z_3, z_4\}$ where the vertices z_1, z_2, z_3 and z_4 form a diamond Z with the edge set $E(Z) = \{z_1z_2, z_2z_3, z_3z_4, z_4z_1, z_2z_4\}$. Let $E(G') = E(Z) \cup \{vy_1, vy_2, vz_1\}$. Let y_2, z_1, z_2, z_3, z_4 be free vertices. Suppose that $y_1 \in \mathfrak{S}$ and it is Staller's turn to make her move. If Staller claims z_1 she creates a *vertex-diamond trap* $y_2 - Z$. That is, she forces Dominator to claim y_2, as otherwise, Staller can isolate v in her next move. Also, Staller has created a diamond trap on Z which forces Dominator to claim $V(Z) \setminus \{z_1\}$. In any case, Dominator will lose the game.

5.1.2 Pairing strategy for Dominator.

In the MBTD game played on certain graphs Dominator, to win, will use the pairing strategy. This means that the subset of the board of the game can be partitioned into pairs such that every winning set (i.e. open neighbourhood of a vertex in the graph) contains one of the pairs. When Staller claims an element from some pair, Dominator will respond by claiming the other element from that pair.

5.2 Graphs from $\mathcal{D}$ and $\mathcal{S}$

First we consider MBTD game on a connected cubic graph which is the union of vertex-disjoint diamonds and prove Theorem 2.9.

The proof of Theorem 2.9. The vertex set $V(G)$ can be partitioned into 4-sets, each containing a C_4. So, by Proposition 5.2 and Proposition 5.1

Dominator wins on C_4, and therefore on diamond. □

Definition 5.3. *Suppose that the MBTD game on the connected cubic graph G on $n \geq 6$ vertices is in progress.*

1. *By G_1 denote a induced subgraph of G with the vertex set $V(G_1) = \{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$ where the vertices u_1, u_2, u_3 form a triangle U and the vertices v_1, v_2, v_3 form a triangle V. Let $E(G_1) = E(U) \cup E(V) \cup \{u_0u_1, v_0v_1\}$. The subgraph is illustrated in Figure 5.1(a).*

2. *By G_2 denote a subgraph of G with the vertex set $\{x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3, z_1, z_2, z_3, v\}$ where the vertices x_1, x_2, x_3 from a triangle X, u_1, u_2, u_3 form a triangle U, v_1, v_2, v_3 form a triangle V, and z_1, z_2, z_3 form a triangle Z. Let $E(G_2) = E(X) \cup E(U) \cup E(V) \cup E(Z) \cup \{v_3v\}$. The subgraph is illustrated in Figure 5.1(b).*

3. *By G_3 denote a subgraph of G which contains a triangle U with the vertex set $\{u_1, u_2, u_3\}$, and a diamond Z with the vertex set $\{z_1, z_2, z_3, z_4\}$ and the edge set $E(Z) = \{z_1z_2, z_2z_3, z_3z_4, z_4z_1, z_2z_4\}$. Let $E(G_3) = E(U) \cup E(Z) \cup \{u_2z_1\}$. The subgraph is illustrated in Figure 5.1(c).*

4. *By G_4 denote a subgraph of G which contains a triangle U with the vertex set $\{u_1, u_2, u_3\}$ and two diamonds, a diamond Y with the vertex set $\{y_1, y_2, y_3, y_4\}$ and the edge set $E(Y) = \{y_1y_2, y_2y_3, y_3y_4, y_2y_4\}$, and a diamond Z with the vertex set $\{z_1, z_2, z_3, z_4\}$ and the edge set $E(Z) = \{z_1z_2, z_2z_3, z_3z_4, z_4z_1, z_2z_4\}$. Let $E(G_4) = E(U) \cup E(Y) \cup E(Z) \cup \{u_2y_1, u_3z_1\}$. The subgraph is illustrated in Figure 5.1(d).*

To prove Theorem 2.10 and Theorem 2.11 we will need the following lemmas.

Lemma 5.4. *Consider the D-game on G_1. If $d_1 = u_0$, then G_1 is $\mathcal{S}$. Also, Staller wins S-game on G_1.*

Proof. We have $d_1 = u_0$. Then, $s_1 = v_2$. Consider the following cases:

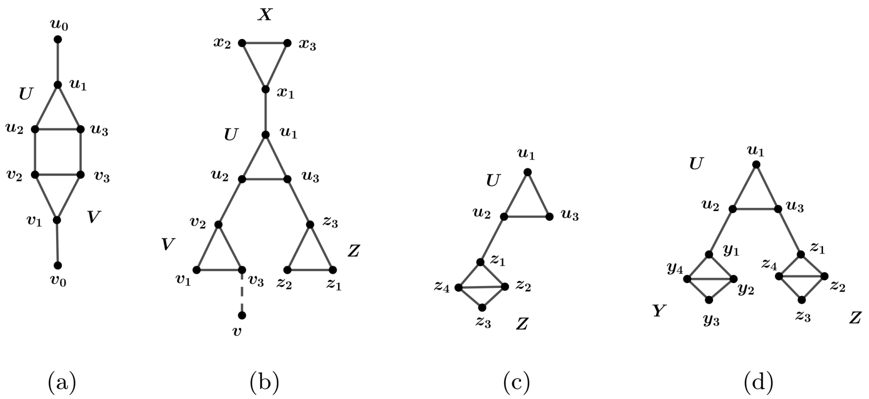


Figure 5.1: Subgraph (a) G_1 (b) G_2 (Vertex v can belong to $V(Z)\cup V(X)$) (c) G_3 (d) G_4.

Case 1. $d_2 \in \{u_1, u_3, v_1\}$. Then, $s_2 = v_3$ which forces $d_3 = v_0$. In her third move, if $d_2 = u_1$, Staller claims v_1 and creates a double trap $u_2 - u_3$, if $d_2 = u_3$, Staller claims u_2 and creates a double trap $u_1 - v_1$, and if $d_2 = v_1$, Staller claims u_1 Staller and creates a double trap $u_3 - u_2$.

Case 2. $d_2 \in \{u_2, v_3, v_0\}$. Then, by playing $s_2 = u_3$ Staller creates a double trap $u_1 - v_1$. In her third move Staller isolates u_2 or v_3.

In the S-game, Staller can pretend that she is the second player and $d_1 = u_0$ and win the game. □

Lemma 5.5. *Consider the MBTD game on G_2. Let $u_1 \in \mathfrak{S}$ and suppose that at least the vertices $u_2, u_3, v_1, v_2, v_3, v, z_3$ are free. Suppose that it is Staller's turn to make her move. Then, Staller wins.*

Proof. Staller plays in the following way: $s_1 = u_2$ which forces $d_1 = z_3$ and $s_2 = v_2$ which forces $d_2 = u_3$. By playing $s_3 = v_1$ Staller creates a double trap $v_3 - v$. In her next move Staller isolates either v_2 or v_3 by claiming v_3 or v. □

Lemma 5.6. *Consider the MBTD game on G_3. Let $u_1 \in \mathfrak{S}$ and suppose that at least the vertices u_3, z_1, z_2, z_3, z_4 are free. Suppose that it is Staller's turn to make her move. Then, Staller wins.*

Proof. Staller plays $s_1 = z_1$ and creates a vertex-diamond trap $u_3 - Z$. Dominator can not win. □

Lemma 5.7. *Staller wins the S-game on G_4.*

Proof. Consider the S-game on G_4. Staller plays in the following way: $s_1 = y_1$ which forces $d_1 \in \{y_2, y_3, y_4\}$ (a diamond trap on Y), as otherwise Staller will claim y_3 in her second move and then in her third move she can isolate y_2 or y_4. Next, $s_2 = u_1$ which forces $d_2 = u_3$. By playing $s_3 = z_1$ Staller creates a vertex-diamond trap $u_2 - Z$. □

Lemma 5.8. *Consider the MBTD game on the connected cubic graph η illustrated in Figure 5.2. In the D-game, if $d_1 \in \{h_1, h_3\}$, Dominator wins. Otherwise, Staller wins as the second player. In the S-game on η Staller wins.*

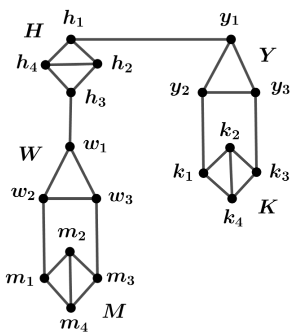


Figure 5.2: Graph η.

Proof. Consider the D-game and let $d_1 \in \{h_1, h_3\}$. W.l.o.g. let $d_1 = h_1$. To dominate vertices from $V(Y) \cup V(K)$ Dominator plays in the following way:

Case 1. If Staller's first move on $V(Y) \cup V(K)$ is y_1, Dominator responds with k_1. To cover the remaining vertices from $V(Y) \cup V(K)$, Dominator will use the pairing strategy on pairs (y_2, k_3) and (k_2, k_4). If Staller claims y_3, Dominator will claim an arbitrary free vertex among $\{y_2, k_2, k_3, k_4\}$.

Case 2. If Staller's first move on $V(Y) \cup V(K)$ is a vertex from $\{y_2, y_3\}$, Dominator responds with y_1. To dominate the remaining vertices from $V(Y) \cup V(K)$, Dominator will use the pairing strategy on pairs (k_1, k_3) and (k_2, k_4).

Case 3. If Staller's first move on $V(Y) \cup V(K)$ is k_1 (or k_3), Dominator responds with k_3 (or k_1). To cover the remaining vertices from $V(Y) \cup V(K)$, Dominator will use the pairing strategy where the pairs are (y_1, y_3) (or (y_1, y_2)) and (k_2, k_4). If Staller claims y_2 (or y_3), Dominator will claim an arbitrary free vertex from one of the pairs.

To cover vertices from $V(H) \cup V(W) \cup V(M)$ Dominator plays in the following way. First, he makes a pairing (h_2, h_4) and (m_2, m_4), and applies a pairing strategy there. So, suppose, $h_2, m_2 \in \mathfrak{D}$. It is enough to consider the following cases.

Case 1. If Staller's first move on $V(H) \cup V(W) \cup V(M)$ is h_3, Dominator responds with w_1. In order to cover vertices from $V(W) \cup V(M)$ Dominator will use the pairing strategy on pairs (w_2, w_3) and (m_1, m_3).

Case 2. If Staller's first move on $V(H) \cup V(W) \cup V(M)$ is w_1 (or w_2), Dominator responds with m_1 (or m_3). In order to cover vertices from $V(H) \cup V(W) \cup V(M)$ Dominator will use the pairing strategy on pairs (w_2, m_3) (or (w_1, m_1)) and (w_3, h_3).

Case 3. If Staller's first move on $V(H) \cup V(W) \cup V(M)$ is w_3, Dominator responds with m_1. In order to cover vertices from $V(H) \cup V(W) \cup V(M)$ Dominator will use the pairing strategy on pairs (w_2, h_3) and (w_1, m_3).

Case 4. If Staller's first move on $V(H) \cup V(W) \cup V(M)$ is m_1 (or m_3), Dominator responds with m_3 (or m_1). In order to cover vertices from $V(H) \cup V(W) \cup V(M)$ Dominator will use the pairing strategy on pairs (w_1, w_3) and (w_2, h_3) (or (w_1, w_2) and (w_3, h_3)).

Next, suppose that $d_1 \in \{h_2, h_4\}$. W.l.o.g. let $d_1 = h_2$. Staller plays in the following way: $s_1 = k_1$ which forces $d_2 \in V(K) \setminus \{k_1\}$ (a diamond trap), $s_2 = y_3$ which forces $d_3 = y_1$, $s_3 = h_1$ which forces $d_4 = y_2$ and $s_4 = h_3$ which forces $d_5 = h_4$. Next, $s_5 = w_1$. Afterwards,

- if $d_6 = w_2$ (or $d_6 = w_3$), then $s_6 = m_1$ (or $s_6 = m_3$) and Staller creates a vertex-diamond trap $w_3 - M$ (or $w_2 - M$).
- if $d_6 \in \{m_1, m_2, m_4\}$ (or $d_6 = m_3$), then $s_6 = w_2$ (or $s_6 = w_3$) and Staller creates a double trap $m_3 - w_3$ (or $m_1 - w_2$). In her next move Staller isolates either w_3 or w_1 (or, w_2 or w_1).

Next, suppose that $d_1 \notin V(H)$. W.l.o.g. let $d_1 \in V(Y) \cup V(K)$. Then, Staller plays on the subgraph G_4 with the vertex set $V(W) \cup V(H) \cup V(M)$. By Lemma 5.7, Staller wins. In the the S-game, Staller uses the same strategy. $\square$

Lemma 5.9. *Consider the MBTD game on the connected cubic graph ω illustrated in Figure 5.3, where the chain of diamonds adjacent to A can consist of one or more diamonds. In the D-game, if $d_1 = a_1$ Dominator wins, if $d_1 \in V(D_1)$ Staller wins. In the S-game on ω Staller wins.*

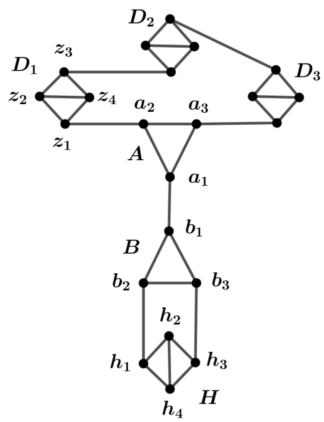


Figure 5.3: One example of graph ω.
Chain of diamonds consists of three diamonds.

Proof. We first look at the D-game on ω. In his first move Dominator claimed a_1. When Staller plays on a diamond which is different from H, Dominator will apply a pairing strategy on that diamond where one pair consists of two opposite vertices of that diamond and remaining two

vertices in diamond form the other pair. Since each diamond contains a 4-cycle, Dominator is able to cover all vertices from these diamonds, by Proposition 5.2.
Dominator also makes a pairing (a_2, a_3) and (h_2, h_4), and apply the pairing strategy when Staller plays there. Let $a_2, h_2 \in \mathfrak{D}$.

- If Staller's first move on $V(B) \cup V(H)$ is b_1, Dominator responds with h_1. If Staller in her next move claims a vertex from $\{b_2, b_3, h_3\}$, Dominator claims the free vertex from $\{b_2, h_3\}$ and in this way he covers all vertices from the graph.

- If Staller's first move on $V(B) \cup V(H)$ is b_2 (or b_3), Dominator responds with h_1 (or h_3). If Staller in her next move claims a vertex from $\{b_1, h_3, b_3\}$ (or $\{b_1, h_1, b_2\}$), Dominator claims the free vertex from $\{b_1, h_3\}$ (or $\{b_1, h_1\}$) and he covers all vertices.

- If Staller's first move on $V(B) \cup V(H)$ is h_1 (or h_3), Dominator responds with h_3 (or h_1). In his next move Dominator claims a free vertex from $\{b_1, b_3\}$ (or $\{b_1, b_2\}$).

Let $d_1 \in V(D_1)$. Then, Staller, in her first move plays $s_1 = b_1$. If $d_2 \in V(A) \cup V(B) \cup V(D_i)$, then Staller can create $b_3 - H$ trap or $b_2 - H$ trap by claiming h_1 if $d_2 = b_2$, or h_3 if $d_2 = b_3$. So, Dominator needs to play his second move on H. If $d_2 \in \{h_1, h_2, h_4\}$ (or $d_2 = h_3$), then $s_2 = b_2$ (or $s_2 = b_3$) forcing $d_3 = h_3$ (or $d_3 = h_1$). Next, $s_3 = a_1$ forcing $d_4 = b_3$ (or $d_4 = b_2$).
If chain of diamonds consist only of one diamond, say D_1, then if $d_1 \in \{z_1, z_2, z_4\}$ Staller plays $s_4 = a_2$ creating $z_3 - a_3$ trap, and if $d_1 = z_3$ Staller plays $s_4 = a_3$ creating $z_1 - a_2$. Otherwise, if chain has more than one diamond, then Staller claims a vertex from a diamond incident with a_3 and creates vertex-diamond trap.

Consider the S-game. Staller plays in the following way: $s_1 = h_1$ which forces $d_1 \in V(H) \setminus \{h_1\}$ (a diamond trap), $s_2 = b_1$ which forces $d_2 = b_3$, $s_3 = z_1$ which forces $d_3 \in V(D_1) \setminus \{z_1\}$ (a diamond trap) and $s_4 = a_3$. Staller creates a double trap $a_1 - a_2$. $\square$

In the following we look at the graph on n vertices that consists on vertex-disjoint triangles and prove that if $n = 6$ Dominator wins in the S-game while for every $n > 6$ Staller wins even as the second player.

The proof of Theorem 2.10. First, let $n = 6$. Consider the S-game. Let U be a triangle with the vertex set $\{u_1, u_2, u_3\}$ and let V be a triangle with the vertex set $\{v_1, v_2, v_3\}$. Let $u_i v_i \in E(G)$ for every $i \in \{1, 2, 3\}$. W.l.o.g. suppose that Staller in her first move chooses a vertex u_1. Then Dominator will choose a vertex from the opposite triangle V which is not adjacent to u_1, say a vertex v_2. In her second move Staller needs to claim u_2, as otherwise Dominator will win after his second move. If $s_2 = u_2$, then $d_2 = v_3$. In his third move Dominator will claim a free vertex from $\{v_1, u_3\}$ and win. One of these two vertices must be free after Staller's third move.

Let $n > 6$. Since the graph is cubic, the number of vertices needs to be even, so we have the even number of triangles. Consider the D-game on graph G. Suppose that in his first move Dominator claims some vertex a_1 which belongs to a triangle A with the vertex set $V(A) = \{a_1, a_2, a_3\}$.

Case 1 Let $a_1 b_1 \in E(G)$ where b_1 is a vertex of some triangle B with the vertex set $V(B) = \{b_1, b_2, b_3\}$ and there is only one edge between A and B. We consider the following subcases.

1.i. Triangle B is adjacent to one more triangle, say Y with the vertex set $V(Y) = \{y_1, y_2, y_3\}$. So, there are two edges between B and Y, say $b_2 y_2$ and $b_3 y_3$. Let $y_1' \in N_G(y_1)$ for some $y_1' \in V(G) \setminus \{y_2, y_3\}$. Consider a subgraph induced by $\{a_1, b_1, b_2, b_3, y_1, y_2, y_3, y_1'\}$ where a_1 is claimed by Dominator and now it is Staller's turn to make her move. By Lemma 5.4 it follows that Staller wins.

1.ii. Triangle B is adjacent to two more triangles, say Y with the vertex set $V(Y) = \{y_1, y_2, y_3\}$, and W with the vertex set $V(W) = \{w_1, w_2, w_3\}$. Let $b_2 w_2, b_3 y_3 \in E(G)$. Let $w_1' \in N_G(w_1)$ for some $w_1' \in V(G) \setminus \{w_2, w_3\}$ and let $w_3' \in N_G(w_3)$ for some $w_3' \in V(G) \setminus \{w_1, w_2\}$.
In her first move Staller plays $s_1 = b_1$. The rest of the Staller's strategy depends on Dominator's second move. So, we analyse the following cases.

1.ii.1. $d_2 \in V(A)$.
Then, $s_2 = b_2$ which forces $d_3 = y_3$, $s_3 = w_2$ which forces $d_4 = b_3$. If $w'_1 = a_i$ (or $w'_3 = a_i$), for some $i \in \{2, 3\}$, and a_i is claimed by Dominator in his second move, then $s_4 = w_1$ (or w_3). In this way Staller creates a double trap $w_3 - w'_3$ (or $w_1 - w'_1$). In her next move Staller isolates either w_2 or w_3 (or, either w_2 or w_1). Otherwise, Staller can claim any of the vertices w_1, w_3 in her fourth move and then play in the same way as above, i.e. she creates a double trap and wins in the following move.

1.ii.2. $d_2 \in V(B)$.
W.l.o.g. let $d_2 = b_3$.

1.ii.2.1. Triangle Y is adjacent to two more triangles, say K with the vertex set $V(K) = \{k_1, k_2, k_3\}$ and $H = \{h_1, h_2, h_3\}$. Let $y_2k_2 \in E(G)$ and $y_1h_1 \in E(G)$. Then, $s_2 = y_3$ which forces $d_3 = b_2$, $s_3 = y_1$ which forces $d_4 = k_2$ and $s_4 = h_1$ which forces $d_5 = y_2$. Next, $s_5 = h_2$ and Staller creates a double trap $h_3 - h'_3$, where $h'_3 \in N_G(h_3)$ for some $h'_3 \in V(G) \setminus \{h_1, h_2\}$. In her next move Staller isolates either h_1 or h_3.
The statement holds also if $h'_3 \in (V(K) \setminus \{k_2\}) \cup (V(W) \setminus \{w_2\}) \cup (V(A) \setminus \{a_1\})$.
It could be the case that one of these triangles K, H is the triangle W. The statement also holds in this case.
If $K = A$, the statement also holds. If $H = A$, the proof is very similar, but simpler, as the Staller wins in her fifth move.

1.ii.2.2. Triangle Y is adjacent to one more triangle, say $K \neq A$ with the vertex set $V(K) = \{k_1, k_2, k_3\}$. Let $y_1k_1, y_2k_2 \in E(G)$. Assume that $k'_3 \in N_G(k_3)$ for some $k'_3 \in V(G) \setminus \{k_1, k_2\}$. Since the graph induced by $\{b_3, y_1, y_2, y_3, k_1, k_2, k_3, k'_3\}$ is a variant of graph G_1 where $b_3 \in \mathfrak{D}$. According to Lemma 5.4, Staller wins.
If $K = W$, the statement also holds.

1.ii.2.3. Triangle Y is adjacent to triangle A and there are two edges between them, say y_2a_2, y_1a_3.
Then, Staller plays $s_2 = y_3$ which forces $d_3 = b_2$. By playing $s_3 = a_2$

Staller creates a double trap $a_3 - y_1$. In her next move Staller isolates either a_1 or y_2.

1.ii.3. $d_2 \in V(Y) \cup V(W)$. W.l.o.g. let $d_2 \in V(Y)$.
Let $w'_1 \in N_G(w_1)$ and let $w'_3 \in N_G(w_3)$. Consider the following subcases.

1.ii.3.1. $d_2 \neq y_3$. W.l.o.g. let $d_2 = y_1$.
Then, $s_2 = w_2$ which forces $d_3 = b_3$ and $s_3 = b_2$ which forces $d_4 = y_3$. It is enough to consider the case if one of the vertices w'_1 and w'_3 is the vertex y_1. Let, for example, $w'_1 = y_1$. Then, $s_4 = w_1$ and Staller creates a double trap $w_3 - w'_3$. In her next move Staller isolates either w_2 or w_3.
Staller plays in the same way if $y_1 \notin \{w'_1, w'_3\}$.

1.ii.3.2. $d_2 = y_3$.
Then, $s_2 = w_2$ which forces $d_3 = b_3$, $s_3 = w_1$ which forces $d_4 = w'_3$ and $s_4 = w_3$. Staller creates a double trap $b_2 - w'_1$. In her next move Staller isolates either w_2 or w_1.

Case 2 Let C be a triangle with the vertex set $V(C) = \{c_1, c_2, c_3\}$ such that $a_1c_1, a_2c_2 \in E(G)$. Suppose that C is adjacent to some triangle B with the vertex set $V(B) = \{b_1, b_2, b_3\}$. If $b_ia_3 \notin E(G)$, for all $i \in \{1, 2, 3\}$, then the analysis from Case 1 can be applied on B. Otherwise, consider another triangle T adjacent to B and apply the adjusted analysis from Case 1 on triangle T and its neighbours.
According to the analysed cases it follows that the graph G is $\mathcal{S}$. □

Next, we consider the connected cubic graph that consists of vertex-disjoint triangles and diamonds, and prove Theorem 2.11.

The proof of Theorem 2.11. Consider the following cases.

Case 1 Let $d_1 = a_1 \in V(A)$ where A is a triangle with the vertex set $V(A) = \{a_1, a_2, a_3\}$. Consider the following cases.

1.i. The vertex a_1 is adjacent to a diamond H with the vertex set $V(H) = \{h_1, h_2, h_3, h_4\}$ where $E(H) = \{h_1h_2, h_2h_3, h_3h_4, h_4h_1, h_2h_4\}$. Let $a_1h_1 \in E(G)$. Consider the following subcases.

1.i.1. Vertex a_2 is adjacent to a diamond D_1 different from H and vertex a_3 is adjacent to a diamond D_2 (which can be equal to one of the diamonds D_1, H).
Since the number of triangles must be even there exists at least one more triangle, say X, with the vertex set $V(X) = \{x_1, x_2, x_3\}$ such that one of the cases 1.i.1.a., 1.i.1.b., 1.i.1.c., 1.i.1.d., 1.i.1.e. and 1.i.1.f. from Figure 5.4 holds.

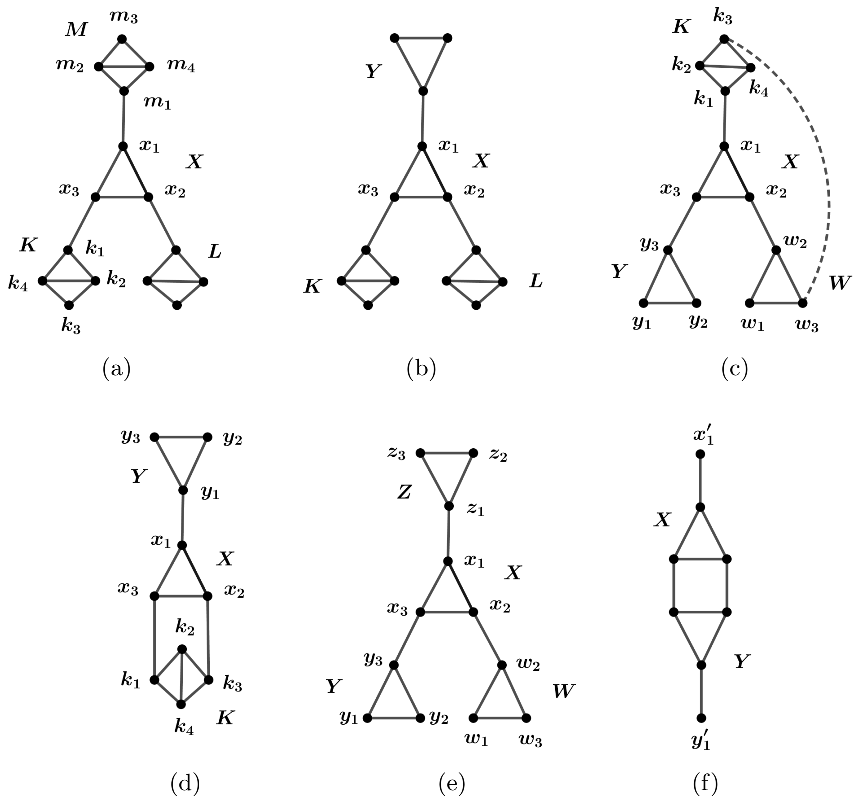


Figure 5.4: Subgraph (a) Case 1.i.1.a. (b) Case 1.i.1.b. (c) Case 1.i.1.c. (d) Case 1.i.1.d. (e) Case 1.i.1.e. (f) Case 1.i.1.f.

For cases 1.i.1.a. and 1.i.1.b, consider S-game on the subgraph G_4

which vertex set is a union of $V(X)$ and vertex sets of two diamonds that are adjacent to X. By Lemma 5.7, Staller wins.

1.i.1.c. In case that one of the triangles Y and W is adjacent to k_3, then, w.l.o.g. suppose that triangle W is adjacent to k_3. Then, Staller plays in the following way:
$s_1 = k_1$ which forces $d_2 \in V(K) \setminus \{k_1\}$ (a diamond trap), $s_2 = x_2$ which forces $d_3 = x_3$ and $s_3 = y_3$ which forces $d_4 = x_1$. Triangle Y can be adjacent to some diamonds and/or triangles. The vertices of these diamonds and/or triangles together with $V(Y)$ form one of the subgraphs G_1, G_2 or G_3. According to the lemmas 5.4, 5.5 and 5.6, Staller wins.

1.i.1.d. Staller plays in the following way:
$s_1 = k_1$ which forces $d_2 \in V(K) \setminus \{k_1\}$ (a diamond trap), $s_2 = x_2$ which forces $d_3 = x_1$ and $s_3 = y_1$ which forces $d_4 = x_3$. Triangle Y can be adjacent to some diamonds and/or triangles. Vertices of these diamonds and/or triangles together with $V(Y)$ form one of the subgraphs G_1, G_2 or G_3. According to the lemmas 5.4, 5.5 and 5.6, Staller wins.

1.i.1.e. If the triangles X, Y, Z, W are adjacent only to triangles, then we can use the analysis from the proof of Theorem 2.10 Case 1 where $X = B, Z = A$ and $V(Z) \not\subseteq \mathfrak{D}$. Otherwise, if at least one of the triangles are adjacent to diamond then we can have a subgraph from Figure 5.4 (b), (c) or (d) for which we can apply an analysis from the corresponding Case 1.i.1.b, 1.i.1.c or 1.i.1.d.

1.i.1.f. Consider the S-game on the subgraph G_1 with the vertex set $V(X) \cup V(Y) \cup \{x_1', y_1'\}$. By Lemma 5.4, Staller wins.

1.i.2. The vertices a_2 and a_3 are adjacent to the same triangle, say B, where $a_2b_2, a_3b_3 \in E(G)$.

1.i.2.a. B is adjacent to a triangle Y with the vertex set $V(Y) = \{y_1, y_2, y_3\}$. Let $b_1y_1 \in E(G)$. Then, there exists at least one more triangle, X such that one of the cases from Figure 5.4 can occur, so adjusted analysis from Case 1.i.1 can be applied.

1.i.2.b. B is adjacent to a diamond K with the vertex set $V(K) = \{k_1, k_2, k_3, k_4\}$ where $E(K) = \{k_1k_2, k_2k_3, k_3k_4, k_4k_1, k_2k_4\}$. Let $b_1k_1 \in E(G)$. Then, $s_1 = k_1$ forces $d_2 = V(K) \setminus \{k_1\}$, $s_2 = b_2$ forces $d_3 = b_3$, $s_3 = a_3$ forces $d_4 = b_1$ and $s_4 = h_1$ creates a vertex-diamond trap $a_2 - H$.

1.i.2.c. B is adjacent to a diamond H, that is, $b_1h_3 \in E(G)$. Since induced subgraph with the vertex set $V(A) \cup V(B) \cup V(H)$ is a connected cubic graph, it follows that this graph is the graph G on 10 vertices. In her first move Staller plays $s_1 = b_1$. Then

- if $d_2 = h_1$, Staller plays $s_2 = b_3$ which forces $d_3 = a_2$. Next, $s_3 = b_2$ and Staller creates a double trap $a_3 - h_3$. In her next move she isolates either b_3 or b_1.

- if $d_2 \in V(H)\setminus\{h_1\}$, Staller plays $s_2 = a_2$ which forces $d_3 = b_3$. Next, $s_3 = a_3$ and Staller creates a double trap $h_1 - b_2$. In her next move she isolates either a_1 or b_3.

- if $d_2 = a_2$ (or $d_2 = a_3$), Staller plays $s_2 = b_2$ (or $s_2 = b_3$) which forces $d_3 = a_3$ (or $d_3 = a_2$). Next, $s_3 = h_3$ and Staller creates a vertex-diamond trap $b_3 - H$ (or $b_2 - H$).

- if $d_2 = b_2$ ($d_2 = b_3$), Staller plays $s_2 = a_2$ (or $s_2 = a_3$) which forces $d_3 = b_3$ (or $d_3 = b_2$). Next, $s_3 = h_1$ and Staller creates a vertex-diamond trap $a_3 - H$ (or $a_2 - H$).
  It follows that the graph G is $\mathcal{S}$.

1.i.3. The vertex a_2 is adjacent to a triangle, say T with the vertex set $V(T) = \{t_1, t_2, t_3\}$ and the vertex a_3 is adjacent to a triangle, say B with the vertex set $\{b_1, b_2, b_3\}$. Let $a_2t_2, a_3b_3 \in E(G)$.
There exists at least one more triangle, say X with the vertex set $V(X) = \{x_1, x_2, x_3\}$. If X is not adjacent to any of B, T, then one of the cases from Figure 5.4 can occur, so adjusted analysis from Case 1.i.1 can be applied.
If X is adjacent to a triangle B or T and there are two edges between them, then we can use Lemma 5.4, as we have subgraph G_1.
Suppose that X is adjacent to at least one of the triangles B and T and there is only one edge between X and that triangle. Let

$x_1b_1 \in E(G)$. Staller plays in the following way: $s_1 = h_1$ forcing $d_2 \in V(H) \setminus \{h_1\}$, $s_2 = a_3$ forcing $d_3 = a_2$, $s_3 = b_2$ forcing $d_4 = b_1$ and $s_4 = x_1$ forcing $d_5 = b_3$.
If none of x_2 and x_3 is not adjacent to H then triangles and/or diamonds adjacent to X together with $V(X)$ can form one of the subgraphs G_1, G_2 or G_3 and according to the lemmas 5.4, 5.5, and 5.6, Staller wins the game on these subgraphs.
If x_2 is adjacent to H and x_3 is adjacent to some other diamond K, then we can use subgraph G_3 with $V(G_3) = V(X) \cup V(K)$, and according to Lemma 5.6, Staller wins.
Otherwise, suppose that x_2 is adjacent to H and x_3 is adjacent to triangle, say Y with the vertex set $V(Y) = \{y_1, y_2, y_3\}$ where $x_3y_3 \in E(G)$. Then, $s_5 = y_3$ forcing $d_6 = x_2$. Triangle Y can be adjacent to some diamonds and/or triangles. The vertices of these triangles and/or diamonds different from A and H, which are adjacent to Y, together with $V(Y)$ and their neighbours form one of the subgraphs G_1, G_2 or G_3. According to the lemmas 5.4, 5.5, or 5.6, Staller wins the game on these subgraphs.
If $x_3t_3 \in E(G)$, then $s_5 = t_3$ forces $d_6 = x_2$. If t_1 is adjacent to a diamond then consider subgraph G_3 and according to Lemma 5.6, Staller wins. Let t_1 be adjacent to a triangle W with the vertex set $V(W) = \{w_1, w_2, w_3\}$ and let $t_1w_1 \in E(G)$. Then Staller plays $s_6 = w_1$ which forces $d_7 = t_2$. Triangle W can be adjacent to triangles and/or diamonds different from A, X, H. Vertices of these triangles and/or diamonds together with W and their neighbours can form one of the subgraphs G_1, G_2 or G_3. So, according to lemmas 5.4, 5.5 or 5.6, Staller wins.

1.i.4. Let a_2 be adjacent to a diamond and a_3 to a triangle B with the vertex set $V(B) = \{b_1, b_2, b_3\}$ and let $a_3b_3 \in E(G)$.
If b_1 and b_2 are adjacent to two different diamonds, then consider subgraph G_4 and according to Lemma 5.7 Staller wins.
If b_1 and b_2 are adjacent to the same diamond K with the vertex set $V(K) = \{k_1, k_2, k_3, k_4\}$ and the edge set $E(K) = \{k_1k_2, k_2k_3, k_3k_4, k_4k_1, k_2k_4\}$. Let $b_1k_1, b_2k_3 \in E(G)$. Then, Staller plays in the following way: $s_1 = h_1$ forcing $d_2 \in V(H)\setminus\{h_1\}$, $s_2 = a_3$ forcing $d_3 = a_2$ and $s_3 = b_1$ forcing $d_4 = b_2$. By playing $s_4 = k_3$

Staller creates $b_3 - K$ trap.
Otherwise, consider some triangle X and the very similar analysis from previous Case 1.i.3 could be used.

So, the graph G is $\mathcal{S}$.

1.ii. The vertex a_1 is adjacent to a triangle, say B with the vertex set $V(B) = \{b_1, b_2, b_3\}$. Let $a_1b_1 \in E(G)$. Consider the following subcases.

1.ii.1 The vertex b_2 is adjacent to some diamond, say H, and the vertex b_3 is adjacent to a diamond, say K. Consider the S-game on the subgraph G_4 with the vertex set $V(B) \cup V(H) \cup V(K)$. By Lemma 5.7, Staller wins. So, G is $\mathcal{S}$.

1.ii.2 The vertices b_2 and b_3 are adjacent to the same diamond, say H with the vertex set $V(H) = \{h_1, h_2, h_3, h_4\}$ and the edge set $E(H) = \{h_1h_2, h_2h_3, h_3h_4, h_4h_1, h_2h_4\}$, where $b_2h_1, b_3h_3 \in E(G)$.

1.ii.2.a. If there are no more triangles in the graph G, then the vertices a_2 and a_3 are adjacent to the chain of diamonds. We have a graph ω from Figure 5.3. According to Lemma 5.9, Dominator wins as the first player.

1.ii.2.b. Otherwise, suppose that graph contains more triangles. If both a_2 and a_3 are adjacent to some diamonds which do not form a chain of diamonds, then consider some triangle X which could not be adjacent either to A or B. One of the cases from Figure 5.4 has to occur, so adjusted analysis from Case 1.i.1 can be applied.

1.ii.2.c. At least one of a_2, a_3 are adjacent to some triangle. Let Y be a triangle with the vertex set $V(Y) = \{y_1, y_2, y_3\}$ such that $a_2y_2 \in E(G)$. If Y is adjacent to some diamond K with the vertex set $V(K) = \{k_1, k_2, k_3, k_4\}$ and the edge set $E(K) = \{k_1k_2, k_2k_3, k_3k_4, k_4k_1, k_2k_4\}$, so there is at least one edge between Y and K, say $y_1k_1 \in E(G)$. Staller plays in the following way: $s_1 = k_1$ forcing $d_2 \in V(K) \setminus \{k_1\}$, $s_2 = y_3$ forcing $d_3 = y_2$, $s_3 = a_2$ forcing $d_4 = y_1$, $s_4 = b_1$ forcing $d_5 = a_3$ and $s_5 = h_1$, so Staller creates $b_3 - H$ trap.
If y_1 and y_3 are adjacent to the same triangle, then consider subgraph G_1 of a given graph, so according to Lemma 5.4, Staller wins.

Otherwise, suppose that y_1 is adjacent to a triangle W with the vertex set $V(W) = \{w_1, w_2, w_3\}$ and let $y_1w_1 \in E(G)$, and y_3 is adjacent to a triangle T. If there is no edge between A and W, or if $T = A$, then Staller plays in the following way: $s_1 = h_1$ which forces $d_2 \in V(H) \setminus \{h_1\}$, $s_2 = b_1$ which forces $d_3 = b_3$, $s_3 = a_2$ which forces $d_4 = a_3$, $s_4 = y_3$ which forces $d_5 = y_1$ and $s_5 = w_1$ which forces $d_6 = y_2$. Triangle W can be adjacent to some triangles and/or diamonds different from A, B and H. Vertices of these diamonds and/or triangles together with $V(W)$ and their neighbours form one of the subgraphs G_1, G_2 or G_3. According to the lemmas 5.4, 5.5 and 5.6, Staller wins.
Otherwise, if there is an edge between A and W, then the analysis above can be applied on triangle T and its neighbours, instead of W.

1.ii.3 The vertex b_2 is adjacent to a diamond, say H with the vertex set $V(H) = \{h_1, h_2, h_3, h_4\}$, where $b_2h_1 \in E(G)$ and vertex b_3 is adjacent to a triangle, say W, with the vertex set $V(W) = \{w_1, w_2, w_3\}$, where $b_3w_3 \in E(G)$. If W is the part of a subgraph G_1, that is W is adjacent to one more triangle and there are two edges between them, then by Lemma 5.4, Staller wins. Otherwise, Staller plays in the following way:
$s_1 = h_1$ which forces $d_2 \in V(H) \setminus \{h_1\}$ (a diamond trap), $s_2 = b_1$ which forces $d_3 = b_3$, $s_3 = w_3$ which forces $d_4 = b_2$. Triangle W can be adjacent to diamonds and/or triangles.

1.ii.3.a. If these diamonds and/or triangles are different from A and H, then vertices of these diamonds and/or triangles together with $V(W)$ and their neighbours form one of the subgraphs G_2 or G_3. According to the lemmas 5.5 and 5.6, Staller wins.

1.ii.3.b. If at least one of the vertices w_1, w_2 is adjacent to A, e.g. let $w_2a_2 \in E(G)$, then by playing $s_4 = a_2$ Staller creates a double trap $a_3 - w_1$. In her next move Staller isolates either a_1 or w_2.

1.ii.3.c. If w_2 is adjacent to some diamond K with the vertex set $V(K) = \{k_1, k_2, k_3, k_4\}$, where $w_2k_1 \in E(G)$, then Staller plays $s_4 = k_1$ and creates a vertex-diamond trap $w_1 - K$.

1.ii.3.d. If $w_1h_3 \in E(G)$ and w_2 is adjacent to some triangle different from A, say R, with the vertex set $V(R) = \{r_1, r_2, r_3\}$ where $w_2r_2 \in E(G)$. Then, Staller plays $s_4 = r_2$ and forces $d_5 = w_1$. Next,

- if there is at least one edge between R and A, say a_3r_3, then Staller plays $s_5 = a_3$ and creates a double trap $a_2 - r_1$. In her next move she isolates a_1 or r_3.

- if there are no edges between R and A, then R is adjacent to other diamonds and/or triangles. The vertices of these diamonds and/or triangles together with $V(R)$ form one of the subgraphs G_1, G_2 or G_3. According to the lemmas 5.4, 5.5 or 5.6, Staller wins.
So, G is $\mathcal{S}$.

1.ii.4 The vertex b_2 is adjacent to a triangle, say W, with the vertex set $V(W) = \{w_1, w_2, w_3\}$, where $b_2w_2 \in E(G)$ and the vertex b_3 is adjacent to a triangle, say Y, with the vertex set $V(Y) = \{y_1, y_2, y_3\}$, where $b_3y_3 \in E(G)$.
Depending of the type of the neighbours of triangle W we can use the adjusted analysis from the proof of Theorem 2.10 for the Case 1.ii or analysis from the previous Case 1.ii.3. So, the graph G is $\mathcal{S}$.

1.iii. The vertex a_1 is adjacent with triangle B with the vertex set $V(B) = \{b_1, b_2, b_3\}$ and there are two edges between them, $a_1b_1, a_2b_2 \in E(G)$. If there exists a diamond, say H adjacent to B or connected with B by a path of triangles (see Figure 5.5), then Staller for her first move plays h_1, and then follows the strategy illustrated on Figure 5.5. She creates $a_3 - b_3$ trap.

Otherwise, the adjusted analysis from Case 2 in the proof of Theorem 2.10 can be applied to prove that Staller wins.

Case 2 $d_1 \in V(Z)$ where Z is a diamond with the vertex set $V(Z) = \{z_1, z_2, z_3, z_4\}$ and the edge set $E(Z) = \{z_1z_2, z_2z_3, z_3z_4, z_4z_1, z_2z_4\}$. Suppose that $d_1 \in \{z_1, z_2\}$. Consider the following subcases.

2.i. If both z_1 and z_3 are adjacent to some diamonds then there exists a triangle X adjacent to some triangles or diamonds different from Z, such

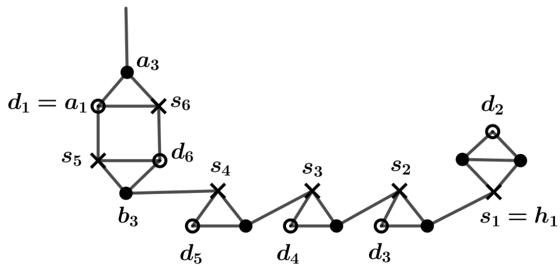


Figure 5.5: Possible situation for Case 1.iii.

that one of cases from Figure 5.4 can occur. So, adjusted analysis from Case 1.i.1 can be applied.

2.ii. Suppose that z_1 is adjacent to a diamond and z_3 is adjacent to a triangle, say B. Adjusted analysis from Case 1.ii can be applied on B to prove that Staller wins.

2.iii. Suppose that z_1 is adjacent to a triangle A with the vertex set $V(A) = \{a_1, a_2, a_2\}$ and let $z_1a_3 \in E(G)$ and z_3 is adjacent to a diamond.
If a_1 is adjacent to a diamond H and a_2 is adjacent to a diamond K, then we can consider subgraph G_4 on which, by Lemma 5.7, Staller wins.
If a_1 and a_2 are adjacent to the same diamond, then we can find some triangle X adjacent to some triangles and/or diamonds different from A and Z. So, one case from Figure 5.4 can occur and adjusted analysis from Case 1.i.1 can be applied.
If triangle A is adjacent to some triangle B with the vertex set $V(B) = \{b_1, b_2, b_3\}$ and are two edges between A and B then we can consider subgraph G_1 and by Lemma 5.4 Staller wins. Otherwise, if there is only one edge between A and B, then, we can use adjusted analysis from Case 1.ii.

2.iv. The vertex z_1 is adjacent to a triangle, say A, with the vertex set $V(A) = \{a_1, a_2, a_3\}$, where $z_1a_1 \in E(G)$ and the vertex z_3 is adjacent to a triangle, say B, with the vertex set $V(B) = \{b_1, b_2, b_3\}$, where $z_3b_1 \in E(G)$. If there are two edges between A and B, let $a_2b_2, a_3b_3 \in E(G)$. Consider the MBTD game on the subraph G_1 with the vertex set $V(A) \cup V(B) \cup \{z_1, z_3\}$. By Lemma 5.4, Staller wins. So, G is $\mathcal{S}$.

Next, suppose that there is one edge between A and B, and let $a_3b_3 \in E(G)$.
If graph G contains only these two triangles A and B, then suppose that triangle A is adjacent to a diamond, say K, with the vertex set $V(K) = \{k_1, k_2, k_3, k_4\}$ and $E(K) = \{k_1k_2, k_2k_3, k_3k_4, k_4k_1, k_2k_4\}$. Let $a_2k_1 \in E(G)$. We differentiate between the following cases:

- If $b_2k_3 \in E(G)$, Staller plays in the following way: $s_1 = a_1$ which forces Dominator to claim a vertex from $V(K) \cup \{a_3\}$, as otherwise if Staller claims k_1 in her second move she will create a vertex-diamond trap $a_3 - K$.
  If $d_2 = a_3$, then $s_2 = b_3$ which forces $d_3 = a_2$, $s_3 = b_2$ which forces $d_4 = z_3$. Next, $s_4 = k_3$ and Staller creates a vertex-diamond trap $b_1 - K$.
  Otherwise, if $d_2 = k_1$ (or $d_2 \in \{k_2, k_3, k_4\}$), then $s_2 = b_3$ which forces $d_3 = a_2$, $s_3 = b_2$ which forces $d_4 = z_3$. Next, $s_4 = b_1$ (or $s_4 = a_3$) and Staller creates a double trap $a_3 - k_3$ (or $k_1 - b_1$). In her next move Staller isolates either b_3 or b_2 (or, a_2 or b_3). So, G is $\mathcal{S}$.

- Triangle B is not adjacent to a diamond K. Then, there exists at least one more diamond, say M different from K with the vertex set $V(M) = \{m_1, m_2, m_3, m_4\}$ adjacent to B, where $b_2m_1 \in E(G)$. Staller plays in the following way:
  $s_1 = m_1$ which forces $d_2 \in V(M) \setminus \{m_1\}$ (a diamond trap), $s_2 = b_3$ which forces $d_3 = b_1$ and $s_3 = a_1$ which forces $d_4 = a_2$. Next, $s_4 = k_1$ and Staller creates a vertex-diamond trap $a_3 - K$. Dominator can not win. So, G is $\mathcal{S}$.

Otherwise, graph G contains at least four triangles. Consider some triangle X different from A and B. One of the cases from Figure 5.4 must hold and adjusted analysis from Case 1.i.1 can be applied. So, G is $\mathcal{S}$.

Next, suppose there are no edges between A and B.
If graph G contains only these two triangles A and B, then consider the following

- Let K with the vertex set $V(K) = \{k_1, k_2, k_3, k_4\}$ be a diamond adjacent to A where $a_2k_1 \in E(G)$, and let M with the vertex set

$V(M) = \{m_1, m_2, m_3, m_4\}$ be a diamond adjacent to B, where $b_2m_1 \in E(G)$. If $a_3k_3, b_3m_3 \in E(G)$, then we have the graph η (see Figure 5.2). By Lemma 5.8, if $d_1 \in \{z_1, z_3\}$, Dominator wins in the D-game. Otherwise, Staller wins.

- Otherwise, at least one of the triangles A, B is adjacent to two more diamonds (different from Z). Let K and L be two diamonds with the vertex sets $V(K) = \{k_1, k_2, k_3, k_4\}$ and $V(L) = \{l_1, l_2, l_3, l_4\}$, respectively, such that $a_2k_1, a_3l_1 \in E(G)$. Staller plays on subgraph G_4 with the vertex set $V(A) \cup V(K) \cup V(L)$. By Lemma 5.7, Staller wins. Statement also holds if $L = M$. So, G is $\mathcal{S}$.

Otherwise, graph G contains at least four triangles. Consider some triangle X different from A and B. One of the cases from Figure 5.4 must holds and adjusted analysis from Case 1.i.1 can be applied. So, G is $\mathcal{S}$.

2.v. Vertices z_1 and z_3 are adjacent to the same triangle, say A. Then, we can have situations from Figure 5.6. If we have Case 2.v.(a), then there exists a triangle X such that one of the cases from Figure 5.4 must holds and adjusted analysis from Case 1.i.1 can be applied.
If it is Case 2.v.(b), then according to Lemma 5.9, since $d_1 \in \{z_1, z_2\}$, Staller wins.
For Case 2.v.(c) consider subgraph G_4 on $V(B) \cup V(H) \cup V(K)$. By Lemma 5.7, Staller wins. For Case 2.v.(d) we can apply adjusted analysis from Case 1.ii.3.
For Case 2.v.(e) we can apply adjusted analysis from Case 1.ii.4. $\square$

Before we give the proof for Theorem 2.12, we give the winning strategy Dominator in the MBTD game on $\mathcal{G}_1 = GP(n, 1)$ where $n \geq 3$. Note that it is already proven in [59] that $GP(n, 1)$ is $\mathcal{D}$ (precisely, the authors considered the prism $P_2 \square C_n$, which is equivalent to $GP(n, 1)$). Here we give a shorter proof for that.

Claim 5.10. *MBTD game on $GP(n, 1)$, $n \geq 3$ is $\mathcal{D}$.*

Proof. Let $V(\mathcal{G}_1) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and let $E(\mathcal{G}_1) = \{u_{i-1}u_i, v_{i-1}v_i | i \in \{2, ..., n\}\} \cup \{u_1u_n, v_1v_n\} \cup \{u_iv_i | i \in \{1, ..., n\}\}$.
If n is even, then $V(\mathcal{G}_1)$ can be partitioned into 4-sets, each inducing a C_4. So, by Proposition 5.2, Dominator wins.

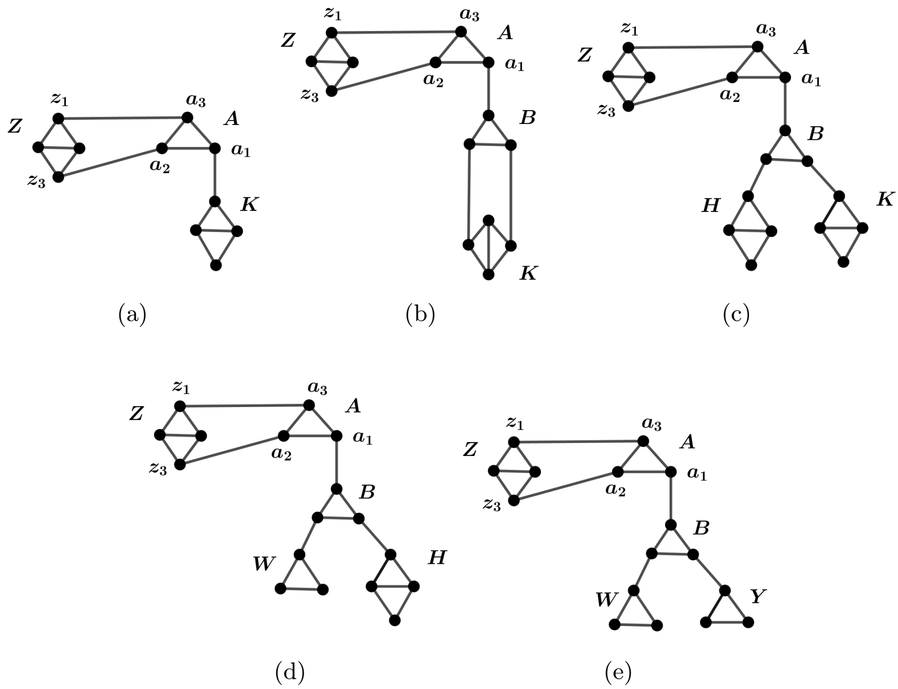


Figure 5.6: Possible situations for Case 2.v.
(a) Case 2.v.a. (b) Case 2.v.b. (c) Case 2.v.c. (d) Case 2.v.d. (e) Case 2.v.e.

Let n be odd. Then, Dominator uses the pairing strategy where the pairs are (u_i, v_{i-1}) for $i \in \{2, ..., n\}$ and (u_1, v_n). We need to prove that this is his winning strategy. Suppose that at some point of the game we have a situation that Staller's set contains vertices u_i, u_{i+2}, v_{i+1}. This means that vertex u_{i+1} stays uncovered by Dominator. This is not possible, because when Staller claimed vertex u_{i+2} (or v_{i+1}), Dominator, according to his strategy, must claim vertex v_{i+1} (or u_{i+2}) and in this way he covers vertex u_{i+1}. A contradiction. $\square$

The proof of Theorem 2.12. Consider Generalized Petersen graph $\mathcal{G}_2 = GP(n, 2)$.
Let $n = 6$. Suppose that in his first move Dominator claims some vertex l which belongs to internal polygon (see Figure 5.7(a)). Staller responds

with $s_1 = u$. We consider the following cases:

Case 1. $d_2 \in \{z, t, v, w, y\}$.
Then, $s_2 = t_2$ which forces $d_3 = t_1$. By playing $s_3 = r_1$ Staller creates a double trap $r - r_2$. In her next move Staller isolates either r_2 or r.

Case 2. $d_2 \in \{t_1, t_2\}$.
Then, $s_2 = z$ which forces $d_3 = w$. By playing $s_3 = r_2$ Staller creates a double trap $r - r_1$. By claiming r or r_1 in her next move, Staller will isolate either r_1 or r.

Case 3. $d_2 \in \{r, r_1, r_2\}$.
Then, $s_2 = z$ which forces $d_3 = w$. By playing $s_3 = t_2$ Staller creates a double trap $y - t_1$. In her next move Staller isolates either l or t.

Next, suppose that Dominator plays his first move on the external polygon. Let $d_1 = y$. Then, Staller responds with $s_1 = u$. If $d_2 \in \{z, t, v, w, l\}$ or $d_2 \in \{t_1, t_2\}$, then Staller can use the same strategy as in Case 1 or Case 2, respectively.
Otherwise, if $d_2 \in \{r, r_1, r_2\}$, then Staller plays in the following way: $s_2 = t_1$ which forces $d_3 = t_2$. Next, by playing $s_3 = w$, Staller creates $l - z$ trap. By claiming z or l in her next move Staller will isolate v or y.

Let $n = 7$. Due to symmetries of the graph, the vertex l can be the first Dominator's move (see Figure 5.7(b)). Staller responds with $s_1 = u$. We consider the following cases:

Case 1. $d_2 \in \{t, r, w, z, y_2\}$.
Then, $s_2 = t_1$ which forces $d_3 = t_2$ and $s_3 = r_1$ which forces $d_4 = r_2$. Next, $s_4 = v$ and Staller creates a double trap $y_3 - y_1$. In her next move Staller isolates either w or z.

Case 2. $d_2 \in \{v, t_1, t_2, y_3\}$.
Then, $s_2 = w$ which forces $d_3 = z$. By playing $s_3 = r_2$ Staller creates a double trap $r_1 - y_1$. In her next move Staller isolates either r or y_3.

Case 3. $d_2 \in \{r_1, r_2, y_1\}$.
Then, $s_2 = w$ which forces $d_3 = z$, $s_3 = t$ which forces $d_4 = y_2$ and

$s_4 = r$ which forces $d_5 = v$. Next, $s_5 = t_2$ and Staller creates a double trap $y_3 - t_1$. In her next move Staller isolates either r_2 or t.

Let $n = 8$ and suppose that in his first move Dominator claims a vertex l which belongs to internal polygon (see Figure 5.7(c)). Staller responds with $s_1 = u$. We consider the following cases:

Case 1. $d_2 \in \{v, r, t_1, t_2, y_1, y_2, y_5, \}$.
Then, $s_2 = r_2$ which forces $d_3 = r_1$ and $s_3 = t$ which forces $d_4 = y_3$. Next, by playing $s_4 = w$ Staller creates a double trap $y_4 - z$. By claiming z or y_4 in her fifth move, Staller isolates either v or t_1.

Case 2. $d_2 \in \{w, t, y_3, y_4, z\}$.
Then, $s_2 = t_1$ which forces $d_3 = t_2$ and $s_3 = r_1$ which forces $d_4 = r_2$. Next, by playing $s_4 = v$ Staller creates a double trap $y_1 - y_5$. By claiming y_5 or y_1 in her fifth move, Staller isolates either w or z.

Case 3. $d_2 \in \{r_1, r_2\}$.
Then, $s_2 = t_2$ which forces $d_3 = t_1$, $s_3 = r$ which forces $d_4 = y_2$ and $s_4 = t$ which forces $d_5 = v$. Next, by playing $s_5 = w$ Staller creates a double trap $y_4 - z$. By claiming z or y_4 in her sixth move, Staller isolates v or t_1.

Next, suppose that Dominator plays his first move on the external polygon. Let $d_1 = y_2$. Then, Staller responds with $s_1 = u$. If $d_2 \in \{v, r, t_1, t_2, y_1, y_2, y_5, \}$ or if $d_2 \in \{w, t, y_3, y_4, z\}$, then Staller can use the same strategy as in Case 1 or Case 2, respectively. If $d_2 \in \{r_1, r_2\}$, then Staller plays in the following way: $s_2 = w$ which forces $d_3 = z$, $s_3 = y_4$ which forces $d_4 = t$. Next, by playing $s_4 = t_2$ Staller creates $l - t_1$ trap. By playing l or t_1 in her next move Staller will isolate either y_3 or t.

Consider Generalized Petersen graph $\mathcal{G}_2 = GP(n, 2)$ where $n \geq 9$. After Dominator's first move, Staller can find subgraph $\tau \subseteq \mathcal{G}_2$ such that $d_1 \notin V(\tau)$. Suppose that in his first move Dominator claims some vertex l which belongs to internal polygon (see Figure 5.8(a)). Consider subgraph $\tau \subseteq \mathcal{G}_2$ with the vertex set $\{u, v, w, z, t, t_1, t_2, r_1, r_2, y_1, y_2, y_3, y_4, y_5, y_6\}$ where the vertices u and v are at distance 4 from the vertex l on the

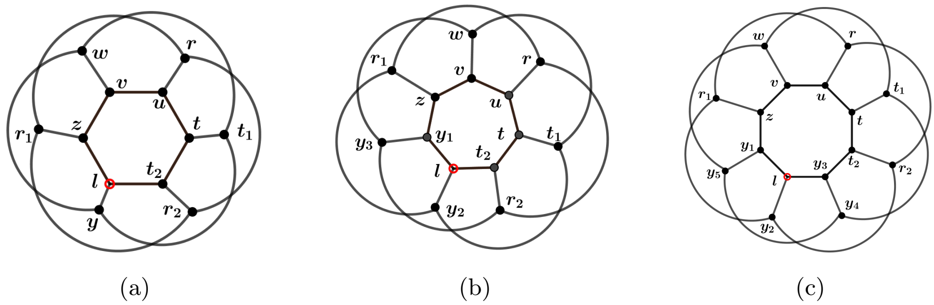


Figure 5.7: (a) Generalized Petersen graph $GP(6,2)$ (b) Generalized Petersen graph $GP(7,2)$ (c) Generalized Petersen graph $GP(8,2)$.

internal polygon. The subgraph τ is illustrated in Figure 5.8(b). In her first move Staller claims $s_1 = u$. It is enough to consider the cases when $d_2 \in \{v, w, t, t_1, t_2, y_4, y_5, y_6\}$.

Case 1. $d_2 \in \{w, t, y_4, y_5\}$.
Then, $s_2 = t_1$ which forces $d_3 = t_2$ and $s_3 = r_1$ which forces $d_4 = r_2$. Next, by playing $s_4 = v$ Staller creates a double trap $y_1 - y_6$. In her next move Staller isolates either w or z by claiming y_6 or y_1.

Case 2. $d_2 = \{v, t_1, t_2, y_6\}$.
Then, $s_2 = r_2$ which forces $d_3 = r_1$ and $s_3 = t$ which forces $d_4 = y_4$. Next, by playing $s_4 = w$ Staller creates a double trap $y_5 - z$. In her next move Staller isolates either t_1 or v by claiming y_5 or z.

□

In the following we prove Theorem 2.13.

The proof of Theorem 2.13. Consider the cubic bipartite graph on n vertices with the vertex set $\{u_1, ..., u_{n/2}, v_1, ..., v_{n/2}\}$. Let $U = \{u_1, ..., u_{n/2}\}$ and $V = \{v_1, ..., v_{n/2}\}$ be a bipartition of the graph. Add an edge from each u_i to v_i, v_{i+i} and v_{i+2} (with indices modulo $n/2$).

W.l.o.g. suppose that $s_1 = u_i \in U$, for some $i \in \{1, 2, ..., n/2\}$. Then $d_1 = u_{i-1} \in U$, modulo $n/2$. Note that every two vertices u_{i-1} and u_i

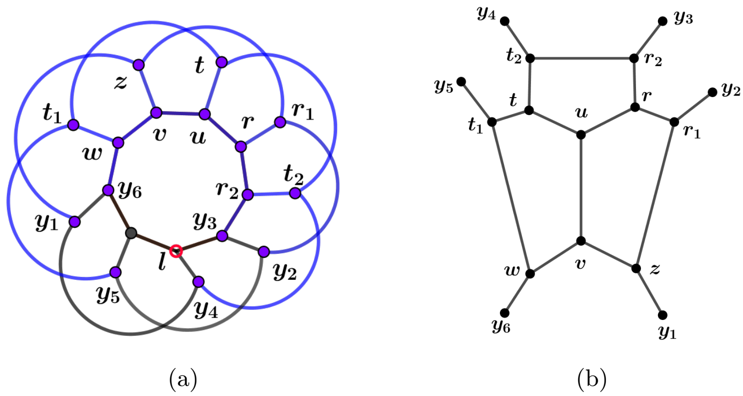


Figure 5.8: (a) Generalized Petersen graph $GP(9, 2)$ (b) subgraph τ.

from U have two common neighbours in V, v_i and v_{i+1} (and every two $v_{i-1}, v_i \in V$ have two common neighbours in U, u_{i-1}, u_{i-2}). In every other round $r \geq 2$, Dominator plays in the following way. If Staller claims a vertex which is a common neighbour of two vertices, say u_{k-1} and u_k such that for example, $u_{k-1} \in \mathfrak{D}$ and $u_k \in \mathfrak{S}$, then Dominator responds by claiming the other common neighbour of these two vertices. Otherwise, if Staller claimed some vertex u_l (or v_l) which is not common neighbour of any two vertices, $x, y \in U$ or V, such that, for example, $x \in \mathfrak{D}$ and $y \in \mathfrak{S}$ (or vice versa), then Dominator claims a free vertex u_{l-1} or u_{l+1}, with preference u_{l-1} (or, v_{l-1} or v_{l+1} with preference v_{l-1}) modulo $n/2$. If Dominator can not find such a free vertex, he claims an arbitrary free vertex from the graph with the preference that a vertex is a neighbour of vertex which is claimed by him earlier in the game.

We prove that this is a winning strategy for Dominator. Suppose that $v_i, v_{i+1}, v_{i+2} \in \mathfrak{S}$ for some $i \in \{1, 2..., n/2\}$ modulo $n/2$, that is, Staller isolated vertex u_i. This means that when Staller claimed v_i, Dominator responded with v_{i-1}, but then when Staller claimed v_{i+1} (or v_{i+2}), according to his strategy Dominator had to take v_{i+2} (or v_{i+1}). A contradiction. $\square$

Finally, we consider MBTD game on the connected cubic graph which is disjoint union of claws and prove Theorem 2.14.

The proof of Theorem 2.14. The graph G is a connected cubic graph on $4k$ vertices formed with $k \geq 2$ disjoint claws $\mathcal{C}_i$, $i \in \{1, ..., k\}$. Let $V(\mathcal{C}_i) = \{x_i, y_i, z_i, t_i\}$, where t_i is a center of $\mathcal{C}_i$, for every $i \in \{1, ..., k\}$.
First, suppose that $k = 2$. Let $E(G) = E(G[\mathcal{C}_1]) \cup E(G[\mathcal{C}_2]) \cup \{x_1x_2, y_1y_2, z_1z_2, x_1y_2, y_1z_2, z_1x_2\}$.
The graph can be partitioned into two 4-sets, $\{x_1, t_1, y_1, y_2\}$ and $\{z_1, z_2, t_2, x_2\}$ each inducing a C_4 (see Figure 5.9(a)). By Proposition 5.2 and Proposition 5.1, Dominator wins.
Let $k = 3$. Let $E(G) = E(G[\mathcal{C}_1]) \cup E(G[\mathcal{C}_2]) \cup E(G[\mathcal{C}_3]) \cup \{x_{i-1}x_i, y_{i-1}y_i, z_{i-1}z_i | i \in \{2, 3\}\} \cup \{x_1x_3, y_1y_3, z_1z_3\}$.
It is enough to consider the case when $d_1 \in V(\mathcal{C}_1)$. The cases when $d_1 \in V(\mathcal{C}_2)$ or $d_1 \in V(\mathcal{C}_3)$ are symmetric.

Case 1. $d_1 = x_1$.
Then, $s_1 = t_1$. After Dominator's second move either all vertices from $\mathcal{C}_2$ are free or all vertices from $\mathcal{C}_3$ are free. Suppose that all vertices from $\mathcal{C}_3$ are free. Also, at least two of the vertices x_2, y_2, z_2 must be free. Suppose that x_2 and z_2 are free. Then, $s_2 = z_3$ which forces $d_3 = z_2$. By $s_3 = x_3$ Staller creates a double trap $x_2 - y_3$. In her next move Staller isolates either x_1 or t_3.
If $d_1 \in \{y_1, z_1\}$, the proof is very similar.

Case 2. $d_1 = t_1$.
Then, $s_1 = t_2$.

Case 2.1. $d_2 = t_3$. Then, $s_2 = z_3$ which forces $d_3 = z_1$. By playing $s_3 = x_3$, Staller creates a double trap $x_1 - y_3$. In her fourth move Staller isolates either x_2 or t_3.

Case 2.2. $d_2 \in \{x_i, y_i, z_i\}$, $i \in \{1, 2, 3\}$.
Let $d_2 = x_i$. If $i = 1$, then Staller will make her next move on $\mathcal{C}_3$ and she will force Dominator to play his next move on $\mathcal{C}_1$, if $i = 3$, Staller will make her next move on $\mathcal{C}_1$ and force Dominator to play on $\mathcal{C}_3$. If $i = 2$, then she can make her next move either on $\mathcal{C}_1$ or $\mathcal{C}_3$.
Suppose that $d_2 = x_1$. Then, $s_2 = z_3$ which forces $d_3 = z_1$. By playing $s_3 = y_3$ Staller creates a double trap $y_1 - x_3$. In her

fourth move Staller isolates either y_2 or t_3.
The proof is very similar if $d_2 = y_i$ or $d_2 = z_i$, $i \in \{1, 2, 3\}$.

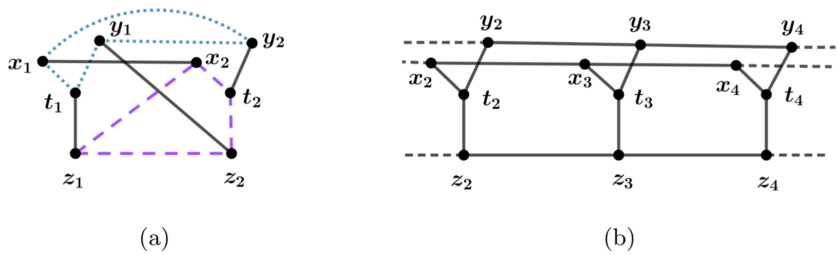


Figure 5.9: (a) Two claws (b) Three consecutive claws.

Let $k \geq 4$. After Dominator's first move, Staller can find three consecutive claws $\mathcal{C}_{i-1}$, $\mathcal{C}_i$ $\mathcal{C}_{i+1}$ such that all vertices from these three claws are free. Suppose that these three claws are $\mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$. Staller will play on a subgraph with the vertex set $V(\mathcal{C}_2), V(\mathcal{C}_3) \cup V(\mathcal{C}_4)$, and the edge set $E(G[\mathcal{C}_2]) \cup E(G[\mathcal{C}_3]) \cup E(G[\mathcal{C}_4]) \cup \{x_{i-1}x_i, y_{i-1}y_i, z_{i-1}z_i | i \in \{3, 4\}\}$ (Figure 5.9(b)). In her first move Staller plays $s_1 = t_3$.
If $d_2 \in V(\mathcal{C}_2)$, then Staller will make her next move on $\mathcal{C}_4$ and she will force Dominator to play his next move on $\mathcal{C}_2$, if $d_2 \in V(\mathcal{C}_4)$, Staller will make her next move on $\mathcal{C}_2$ and force Dominator to play on $\mathcal{C}_4$. If $d_2 \in V(\mathcal{C}_3)$, then she can make her moves either on $\mathcal{C}_2$ or $\mathcal{C}_4$.
Suppose that $d_2 \in V(\mathcal{C}_2) \cup V(\mathcal{C}_3)$.
Let $d_2 = x_2$ or $d_2 = t_2$. Then, $s_2 = z_4$ which forces $d_3 = z_2$. By playing $s_3 = y_4$ Staller creates a double trap $y_2 - x_4$. In her next move she isolates either y_3 or t_4.
The cases when $d_2 \in \{y_2, z_2\}$ are symmetric. If $d_2 \in \{x_3, y_3, z_3\}$, Staller can apply the same strategy. □

Remark 5.11. *Note that if in the D-game on the connected cubic graph G on $n \geq 6$ vertices after Dominator's first move Staller can find at least one of the subgraphs G_1, G_4, τ, or subgraph which consists of three consecutive connected claws as in Figure 5.9(b), such that all vertices from that subgraph are free, then the graph G is $\mathcal{S}$.*

5.3 Concluding remarks

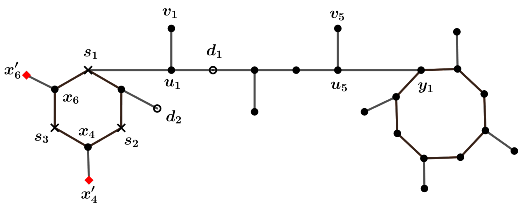
We considered several types of connected cubic graphs in the MBTD game and determined which are $\mathcal{D}$ and which are $\mathcal{S}$. We saw that Dominator wins in the game on the connected cubic graph if all vertices are of type 1 (each vertex lies on a diamond). When the graph is a disjoint union of triangles, then if the number of triangles in graph is 2, graph is $\mathcal{D}$, otherwise the graph is $\mathcal{S}$. In the game when some vertices are of type 1 and some are of type 2, then Dominator can win in the game on the graph ω, and on the graph η, but only as the first player. In all other cases, Staller wins. Regarding graphs where all vertices are of type 3 (each vertex lies in zero triangles), we know that a cubic bipartite graph, graph $GP(n,1)$, $n \geq 3$, and graph $GP(5,2)$ are $\mathcal{D}$, while $GP(n,2)$, where $n \geq 6$ is $\mathcal{S}$. If the graph is a union of k vertex-disjoint claws, Dominator wins only if $k = 2$, while in all other cases Staller wins.

In order to determine the outcome of the game, we have focused on finding a representative subgraph of the given graph. Related to this, another interesting type of subgraph of the cubic graph G on which Staller wins as the first player in the MBTD game is the following graph, which will be denoted by Q.

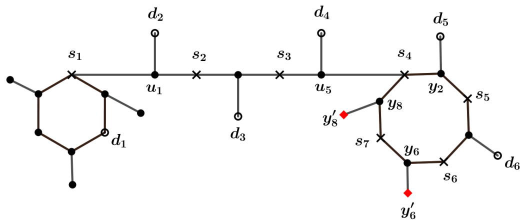
Graph Q. Let Q be a subgraph of the graph G which consists of two even cycles C_l and C_m with the vertex sets $V(C_l) = \{x_1, x_2, ..., x_l\}$ and $V(C_m) = \{y_1, y_2, ..., y_m\}$, respectively, where $E(C_l) = \{x_i x_{i+1} | i \in \{1, 2..., l-1\}\} \cup \{x_l x_1\}$ and $E(C_m) = \{y_i y_{i+1} | i \in \{1, 2..., m-1\}\} \cup \{y_m y_1\}$ for $l, m \geq 4$. Suppose that between C_l and C_m there is a path P with the vertex set $V(P) = \{u_1, u_2, ..., u_t\}$, where $t \geq 1$ is odd. Suppose that for each $k \in \{1, 3, 5, ..., t\}$ there exist a different vertex v_k, such that $u_k v_k \in E(G)$. Let $x_1 u_1, y_1 u_t \in E(G)$. If $l > 4$, then for every two $x_{i+2}, x_{i+4} \in C_l$, for $i \in \{0, ..., l-2\}$ there exists a different neighbour $x'_{i+2} \in N_G(x_{i+2})$ and $x'_{i+4} \in N_G(x_{i+4})$. Also, if $m > 4$, then for every two $y_{i+2}, y_{i+4} \in C_m$, for $i \in \{0, ..., m-2\}$ there exists a different neighbour $y'_{i+2} \in N_G(y_{i+2})$ and $y'_{i+4} \in N_G(y_{i+4})$.

Staller's strategy in the S-game is illustrated in Figure 5.10(a) and

5.10(b) where $l = 6$, $m = 8$ and $t = 5$.
If x_1 lies on a diamond, then in his first move Dominator is forced to claim a vertex from that diamond. In the following, Staller uses the strategy illustrated in Figure 5.10(b).



(a)



(b)

Figure 5.10: Staller's strategy in the S-game on Q when (a) $d_1 \in V(P) \cup V(C_8)$ (b) $d_2 \in V(C_6)$.
Vertices claimed by Dominator are denoted by cycles and vertices claimed by Staller by crosses. Traps are denoted by red squares.

As we can see from this paper, finding a suitable subgraph makes it easier to determine the winner of the game and helps in characterization of cubic connected graphs. However, we have not covered all connected cubic graphs, so there are still open problems related to this topic. Therefore, it would be interesting to find some other subgraphs that could contribute to expanding the class of cubic connected graphs for which the winner is known in MBTD game.

Biased games. We are curious to know what will happen in the *biased* setup of MBTD game. Given two positive integers, a and b, representing the *biases* of Staller and Dominator, respectively, in the biased $(a : b)$ MBTD game, Staller claims exactly a and Dominator claims exactly b elements of the board in each move. Now, if the biases of the players are the same, i.e. *fair* $(a : a)$ *game*, for $a \geq 2$, we wonder whether the outcome of the games change compared to the outcome of the $(1 : 1)$ MBTD games that were previously studied.
Finally, we wonder how the situation changes if biased non-fair $(a : b)$ MBTD games are played, i.e. the games in which $a \neq b$.

Prošireni izvod

Teorija kombinatornih igara bavi se istraživanjem igara u kojima učestvuju dva igrača sa potpunom informacijom i bez elemenata slučajnosti. Za razvoj moderne teorije kombinatornih igara zaslužan je Džon Konvej (*John Conway*) koji je u knjigama „O brojevima i igrama” [35] iz 1976. i „Pobjednički način za vaše matematičke igre” [19] (zajedno sa Elvinom Berlekampom (*Elwyn Berlekamp*) i Ričardom Gajom (*Richard Guy*)) iz 1982. postavio njene temelje, analizirao veliki broj igara i predstavio značajne koncepte. Konvejeva teorija obuhvata igre kao što su Nim koje su zasnovane na algebarskim argumentima i pojmu dekompozicije. Igrama koje ne obuhvata Konvejeva teorija, bavi se grana kombinatorike koja se naziva *teorija pozicionih igara*. U pozicione igre spadaju popularne igre kao što su Iks-Oks, Heks, ali i apstraktne igre na grafovima i hipegrafovima. Za početak sistematskog proučavanja teorije pozicionih igara na grafovima i hipergrafovima uzima se Hejls-Džuit teorema (*Hales–Jewett theorem*) [64] iz 1963. godine, koja se smatra temeljem moderne Remzijeve teorije, i Erdoš-Selfridž kriterij (*Erdős-Selfridge criterion*) [44] koji predstavlja centralni koncept u teoriji algoritama.
Jožef Bek (*József Beck*) zaslužan je za dalji razvoj ove oblasti. Teoriju pozicionih igara oblikovao je u koherentnu kombinatornu disciplinu i u svojoj monografiji [9] pokrio je mnoge njene aspekte. Monografija autora Hefeca (*Hefetz*), Kriveleviča (*Krivelevich*), Stojakovića i Saboa (*Szabó*) [67] takođe pruža detaljan uvod u teoriju pozicionih igara i predstavlja rezultate novijih istraživanja u ovoj oblasti.

Poziciona igra je hipegraf $(X, \mathcal{F})$ gdje je X skup, uglavnom konačan, a $\mathcal{F} \subseteq 2^X$. U igri učestvuju dva igrača koji naizmjenično uzimaju slobodne elemente skupa X. Igra traje sve dok se i posljednji element iz X ne zauzme. Skup X se naziva *tabla igre*, a $\mathcal{F}$ *familija pobjedničkih skupova.* U igri konfigurišu još dva parametra, pozitivni cijeli brojevi a i b koji definišu bias igre. U $(a : b)$ igri, prvi igrač uzima a elemenata po potezu, a drugi igrač uzima b elemenata po potezu. U igrama bez biasa važi da je $a = b = 1$. U zavisnosti od pravila koja određuju koji igrač je pobjednik, pozicione igre možemo podijeliti u nekoliko kategorija, a dvije osnovne katagorije su: *jake igre* i *slabe igre*. U jakim igrama prvi igrač koji uzme sve elemente nekog skupa $F \in \mathcal{F}$ je pobjednik. Igra je neriješena, ako u trenutku kada na tabli ne ostane nijedan element, nijedan igrač nije ostvario zadati cilj. Igra Iks-Oks spada u jake igre i igra se na 3×3 kvadratnoj mreži. Tabla igre se sastoji od 9 elemenata, a familija $\mathcal{F}$ se sastoji od 8 pobjedničkih skupova koji uključuju sve redove, sve kolone i dijagonale mreže.

Kada analiziramo determinističke igre prvo na što pomislimo je primjena komjuterskog algoritma grube pretrage. Iako je to teorijski moguće, u praksi nije izvodljivo u nekom razumnom vremenu, s obzirom da je u nekim igrama potrebno ispitati i analizirati veliki broj mogućnosti i slučajeva. Stoga, bilo bi poželjno imati neke generalne alate i algoritme koji nam mogu pomoći za analizu ovih igara.
Argument *krađe strategije* je moćan alat koji potvrđuje i dokazuje intuiciju da biti prvi igrač je uvijek prednost.

Teorema 5.1. *(Argument krađe strategije, [9]) U jakoj pozicinoj igri, prvi igrač može da obezbijedi bar neriješen rezultat.*

Za neke igre argument Remzijevog tipa se može iskoristiti da se dokaže da ishod neriješeno nije moguć i da prvi igrač pobjeđuje. Argument tvrdi da ako hipergraf $\mathcal{F}$ nije 2-obojiv, onda prvi igrač ima pobjedničku strategiju u jakoj igri nad hipergrafom $\mathcal{F}$. Argument krađe strategije i argument Remzijevog tipa su trenutno jedini generalni alati za jake pozicione igre. Oba alata, iako veoma moćna, ništa ne govore o tome kako pobjednička strategija prvog igrača treba da izgleda. Eksplicitna pobjednička strategija poznata je samo za nekoliko jakih igara kao što je igra savršenog mečinga, igra Hamiltonove konture [47] i igra k-povezanosti [48].
Jake igre je veoma teško analizirati. Razlog je činjenica da jake igre nisu hipergraf monotone, što znači da dodavanje grane u hipegraf igre može

da promijeni ishod igre (vidjeti [10]). Takođe, u jakim igrama drugi igrač se jedino može boriti za neriješen ishod. Ovo prirodno vodi do koncepta Mejker–Brejker igara koje su poznate još pod nazivom slabe igre.

Mejker–Brejker igre

Mejker–Brejker igre igraju dva igrača Mejker i Brejker koji imaju suprotne ciljeve. Mejker pobjeđuje ako uzme sve elemente nekog pobjedničkog skupa, a Brejker pobjeđuje u suprotnom, tj. ako spriječi Mejkera u ostvarenju njegovog cilja. Ove igre su relaksacija jakih igara, pa se često nazivaju slabim igrama. Primjer Mejker–Brejker igre je popularna igra Heks koja se igra na tabli u obliku romba koja se sastoji od $n \times n$ šestouglova (tradicionalno se igra na tabli 11×11). Mejkeru se dodjeljuje par suprotnih strana romba crvene boje, a Brejkeru par suprotnih strana romba plave boje. Igrači naizmjenično boje svojom bojom po jedan neobojen šestougao. Cilj svakog igrača je da napravi put od uzastopnih šestouglova između svojih strana romba. Heks nije jaka igra jer igrači imaju različite pobjedničke skupove. Heks teorema (*Hex Theorem*) Džona Neša (*John Nash*) [57] tvrdi da svako crveno/plavo bojenje table daje put koji povezuje dvije suprotne strane romba. Dakle, heks igra se ne može završiti neriješeno, pa je možemo posmatrati kao Mejker–Brejker igru tako što za pobjedničke skupove uzmemo sve puteve između crvenih strana romba. Mejker pobjeđuje ako do kraja igre osvoji jedan od tih puteva. Brejker pobjeđuje ako blokira Mejkera tako što će napraviti svoj put između plavih strana romba.

Kao što za jake igre, tako i za slabe igre važi da prvi igrač ima prednost. Stoga, ako Mejker pobjeđuje u igri kao drugi igrač, onda on pobjeđuje u istoj igri i kao prvi igrač. Isto važi i za Brejkera, [67].
Sljedeći rezultat daje jednostavan kriterij koji garantuje pobjedničku strategiju Brejkera na hipergrafu $\mathcal{F}$.

Teorema 5.2. *(Erdoš-Selfridž kriterij, [44]) Neka je $\mathcal{F}$ hipergraf. Onda,*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2} \Rightarrow \text{ Brejker pobjeđuje.}$$

Ako je Brejker prvi igrač, onda je uslov $\sum_{A\in\mathcal{F}} 2^{-|A|} < 1$ dovoljan da mu osigura pobjedu.

Ako je hipegraf igre k-uniforman (tj. svi pobjednički skupovi su reda k), onda prema teoremi 5.2, Brejker pobjeđuje ako je $|\mathcal{F}| < 2^{k-1}$. Uopšteni kriterij za Mejkerovu pobjedu dao je Bek u [9].

Teorema 5.3. *[9] Neka je $(X, \mathcal{F})$ poziciona igra. Neka je $\Delta_2(\mathcal{F}) = \max\{|\{A \in \mathcal{F} : \{u, v\} \subseteq A| : u, v \in X\}$ Ako je*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} > \frac{1}{8}\Delta_2|X|,$$

onda Mejker ima pobjedničku strategiju u $(1:1)$ igri $(X, \mathcal{F})$.

Mejker–Brejker igre se uglavnom igraju na skupu grana nekog grafa $G(V, E)$, tj. $X = E(G)$, gdje su pobjednički skupovi svi skupovi grana podgrafova od G koji posjeduju neko svojstvo grafa. Najčešće razmatrane Mejker–Brejker igre su igre koje se igraju na tabli $E(K_n)$, kao što su *igra savršenog mečinga* - pobjednički skupovi su svi skupovi koji sadrže $\lfloor n/2 \rfloor$ nezavisnih grana grafa K_n, *igra povezanosti* - pobjednički skupovi su pokrivajuća stabla od K_n, *igra Hamiltonove konture* - pobjednički skupovi su grane Hamiltonovih ciklusa od K_n, *igra najmanjeg stepena* - pobjednički skupovi su svi podgrafovi od K_n pozitivnog minimalnog stepena c i *igra k-povezanosti* - pobjednički skupovi su k-povezani podgrafovi od K_n.
Prvi koji je istraživao igru povezanosti na tabli $E(K_n)$ bio je Leman (*Lehman*) [88] koji je pokazao da u igri bez bijasa Mejker može da pobijedi za $n-1$ poteza što je i najkraće moguće vrijeme za pobjedu u ovoj igri.
Istraživanje igre Hamiltonove konture ima dugu istoriju. U svom radu [27] iz 1978. godine, Hvatal (*Chvátal*) i Erdoš (*Erdős*) su razmatrali $(1:1)$ igru Hamiltonove konture na tabli $E(K_n)$ gdje su pokazali da Mejker pobjeđuje za dovoljno veliko n. Kasnije je Papajoanau (*Papaioannou*) u [94] pokazao da Mejker pobjeđuje za svako $n \geq 600$. U istom radu postavio je hipotezu da je minimalan broj n za koji Mejker, kao prvi igrač, pobjeđuje u $(1:1)$ igri Hamiltonove konture jednak 8. Hefec (*Hefetz*) i Štih (*Stich*) su u [77] unaprijedili rezultat iz [94] pokazavši da Mejker pobjeđuje za svako $n \geq 9$. Konačno, u [103], dokazano je da nezavisno od

toga ko je prvi igrač, Mejker pobjeđuje ako i samo ako je $n \geq 8$.

U Mejker–Brejker igrama u kojima nije teško odrediti koji igrač je pobjednik, zanimljivije je istražiti pitanje koliko brzo igrač sa pobjedničkom strategijom može da pobijedi. Brze pobjedničke strategije Mejkera u $(1:1)$ igri Hamiltonove konture proučavali su Hefec i autori u [72] koji su pokazali da Mejkeru treba bar $n+1$, a najviše $n+2$ poteza da pobijedi. Da je optimalan broj poteza $n+1$ dokazali su Hefec i Štih u [77].
Brze pobjedniče strategije proučavane i za druge Mejker–Brejker igre bez biasa, kao što su igra savršenog mečinga, igra k-povezanosti, T-igra (vidjeti [29, 48, 72]).

Kako bi se nadoknadila Mejkerova prednost u igrama bez biasa, jedna od mogućnosti je da se Brejkeru dozvoli da uzima više od jedne grane po potezu, tj. da se analiziraju $(1:b)$ Mejker–Brejker igre.

Mejker–Brejker igre sa biasom

Motivisani brzom pobjedom Mejkera u igrama bez biasa, Hvatal i Erdoš su u [27] istraživali $(1:b)$ igre u kojima Brejker po potezu uzima $b>1$ grana. Primijetili su da su $(1:b)$ Mejker–Brejker igre *bias monotone*, što znači da ako Brejker pobjeđuje u nekoj $(1:b)$ Mejker–Brejker igri $(X, \mathcal{F})$, onda on takođe pobjeđuje i u $(1:b+1)$ igri $(X, \mathcal{F})$. Dakle, postoji jedinstven pozitivan cio broj $b_{\mathcal{F}}$ takav da Mejker pobjeđuje u $(1:b)$ igri $(X, \mathcal{F})$ ako i samo ako je $b \leq b_{\mathcal{F}}$, gdje je $\mathcal{F} \neq \emptyset$ i $\min\{|A| : A \in \mathcal{F}\} \geq 2$. Ovaj broj naziva se *granični* ili *kritični bias* igre $(X, \mathcal{F})$.

Dva kriterija za pobjedu u Mejker–Brejker igrama sa biasom

Generalni kriterij za Brejkerovu pobjedu u igrama sa biasom dao je Bek [5]. Njegov kriterij je uopštenje teoreme 5.2.

Teorema 5.4. *(Proširena Erdoš-Selfridž teorema, [5]) Ako je*

$$\sum_{A \in \mathcal{F}} (1+b)^{-|A|/a} < \frac{1}{1+b}$$

onda Brejker, kao drugi igrač, ima pobjedničku strategiju u $(a : b)$ *Mejker–Brejker igri* $(X, \mathcal{F})$. *Ako je Brejker prvi igrač onda je uslov* $\sum_{A \in \mathcal{F}}(1 + b)^{-|A|/a} < 1$ *dovoljan da mu osigura pobjedu.*

Sljedeća teorema daje dovoljan uslov za Mejkerovu pobjedu u igrama sa biasom.

Teorema 5.5. *(Uopšteni kriterijum za Mejkerovu pobjedu, [5]) Ako je*

$$\sum_{A \in \mathcal{F}} \left(\frac{a+b}{a}\right)^{-|A|} > \frac{a^2 b^2}{(a+b)^3} \cdot \Delta_2(\mathcal{F}) \cdot |X|,$$

onda Mejker kao prvi igrač ima pobjedničku strategiju u $(a : b)$ *igri* $(X, \mathcal{F})$, *gdje je* $\Delta_2(\mathcal{F}) = \max\{|\{A \in \mathcal{F} : \{u, v\} \subseteq A\}| : u, v \in X, u \neq v\}$.

Granični bias u Mejker–Brejker igrama

U $(1 : b)$ Mejker–Brejker igri osnovni zadatak je odrediti granični bias igre. Granični bias proučavan je u mnogim radovima. U [27] Hvatal i Erdoš su pokazali da je za igru povezanosti granični bias između $(1/4 - \varepsilon)n/\ln n$ i $(1 + \varepsilon)n/\ln n$ za svako $\varepsilon > 0$. Donju granicu unaprijedio je Bek u [5].
Pitanje određivanje graničnog biasa za igru Hamiltonove konture bio je dugo otvoren problem koji je istraživan u [20, 27, 84] i vjerovalo se da je granični bias reda $\ln n/n$. Problem je riješio Krivelevič (*Krivelevich*) u [81] čiji rezultat je potvrdio hipotezu. Rješenju problema pomagao je rezultat Gebauer (*Gebauer*) i Saboa (*Szabó*) iz [58] gdje je data strategija za Mejkera koja mu omogućuje da pobijedi u igri minimalnog stepena protiv biasa reda $n/\ln n$. Određivanje graničnog biasa za igru minimalnog stepena bio je bitan korak da se odredi granični bias za igru Hamiltonove konture.
Koncept asimetričnih igara kombinovan sa brzim pobjedničkim strategijama istraživali su Ferber (*Ferber*), Hefec (*Hefetz*) i Krivelevič (*Krivelevich*) u [49], i Mikalački i Stojaković u [90].

Mejker–Brejker igre sa dvostrukim biasom

Za mnoge asimetrične $(a : b)$ Mejker–Brejker igre identitet pobjednika je poznat za slučaj kada je $a = 1$. Međutim, postoje igre u kojima mala promjena Mejkerovog i Brejkerovog biasa može promijeniti ishod igre. Primjer

jedne takve igre je diametar-2 igra gdje je tabla igre $E(K_n)$, a pobjednički skupovi su sva pokrivajuća stabla od K_n diametra 2. U $(1:1)$ igri poznato je da Brejker pobjeđuje. Povećanjem Mejkerovog i Brejkerovog biasa za 1, situacija se mijenja i Mejker je pobjednik u $(2:2)$ što je pokazano u [3].
Igre u kojima su biasi Mejkera i Brejkera oba veća od 1 se često nazivaju *igre sa dvostrukum biasom.* Kako su i $(a:b)$ bias monotone, može se definisati *opšti granični bias* igre kao jedinstven pozitivan cio broj takav da u $(a:b)$ igri $(X,\mathcal{F})$ Mejker pobjeđuje ako i samo ako $b \le b_{\mathcal{F}}(a)$. Opšti granični bias za igru povezanosti i igru Hamiltonove konture procijenjen je za svako $a > 1$ u [76, 89]. Brze pobjedničke strategije u fer igrama sa dvostrukim biasom istraživali su Klemens (*Clemens*) i Mikalački u [33].

Različite varijante Mejker–Brejker igara

Postoje različite varijante i modifikacije Mejker–Brejker igara. U radovima [61, 62, 82] proučavane su Mejker–Brejker igre na K_n u kojima jedan od igrača igra nasumično, dok drugi igrač igra prema optimalnoj strategiji. U radovima [50, 95] proučavane su Mejker–Brejker igre gdje se prije svakog poteza baca novčić i Mejker igra svoj potez sa vjerovatnoćom p nezavisno od drugih poteza.
U [102] Stojaković i Sabo predložili su novi pristup koji nadoknađuje Mejkerovu prednost u igrama bez biasa. Oni su posmatrali Mejker–Brejker igre na tabli koja se dobija uklanjanjem elemenata table nezavisno sa vjerovatnoćom $1-p$ za dato $0 < p < 1$. Slučajni graf $\mathbb{G}(n,p)$ se dobija bacanjem novčića za svaku granu grafa K_n nezavisno sa vjerovatnoćom p kako bi se odredilo da li grana treba da bude element table X. Postavlja se pitanje određivanja granične vjerovatnoće za postojanje Mejkerove strategije kojom bi osvojio pobjednički skup u igri koja se igra na tabli slučajnog grafa $\mathbb{G}(n,p)$. Mejker–Brejker igre na slučajnim grafovima istraživane su u [30, 32, 66, 70, 91, 93, 101, 100, 102].

Tačer–Izolator (*Toucher–Isolator*) igre su varijanta Mejker–Brejker igara koja se igra na skupu grana datog grafa G. Prvi igrač (Tačer) ima za cilj da dotakne što veći broj čvorova grafa, a drugi igrač (Izolator) pokušava da minimizuje taj broj. Ove igre prvi su predložili Dauden (*Dowden*), Kang (*Kang*), Mikalački i Stojaković u [41], a dalje su proučavane u [23, 96, 97].

Avojder–Enforser igre (*Avoider–Enforcer*) su igre u kojima igrači, Avojder i Enforser, imaju suprotne ciljeve od onih koje imaju Mejker i Brejker. Naime, Avojder pokušava da izbjegne osvajanje pobjedničkog skupa, dok Enforser pokušava da ga natjera na to. Postoje dvije verzije ove igre, *striktna* - gdje svaki igrač uzima po potezu tačan broj elementa određen njegovim biasom, i *monotona* - gdje igrači uzimaju bar onoliko elemenata po potezu koliko je određeno njihovim biasom. Ove igre proučavane su u mnogim radovima (vidjeti [4, 13, 28, 51, 63, 68, 69, 71, 73]).

Vejter–Klijent (*Waiter–Client*) i Klijent–Vejter (*Client–Waiter*) igre su blisko povezane se Mejker–Brejker i Avojder–Enforser igrama. Glavna razlika su pravila po kojima igrači biraju elemente. U $(a : b)$ igri obje vrste Vejter nudi Klijentu $a + b$ slobodnih elemenata, Klijent bira a, a preostalih b uzima Vejter. Ako je u posljednjoj rundi ostalo $1 \leq t \leq a + b$ elementa, onda Klijent uzima $\max\{0, t - b\}$ elementa, a Vejter uzima $\min\{t, b\}$. U Vejter–Klijent igrama, Vejterov cilj je da natjera Klijenta da uzme sve elemente nekog pobjedničkog skupa, dok Klijent pokušava da izbjegne osvajanje pobjedničkog skupa. U Klijent–Vejter igrama, Klijent pobjeđuje ako osvoji sve elemente nekog pobjedničkog skupa, dok u suprotnom Vejter pobjeđuje. Vejter–Klijent i Klijent–Vejter igre je prvi proučavao Bek [7]. Dalja istraživanja rađena su u [12, 14, 15, 31, 36, 37, 38, 39, 74, 75, 85].

Varijante Mejker–Brejekr igara koje su od posebnog interesa za ovu disertaciju su Voker–Brejker igre (*Walker–Breaker games*) i Mejker–Brejker igre totalne dominacije (*Maker–Breaker total domination games*).

Mejker–Brejker igre sa ograničenjima

Mejker–Brejker igre koje se igraju na skupu grana datog grafa G gdje Mejker mora da bira grane kao da se šeta kroz graf, odnosno grana koju bira u trenutnom potezu mora biti incidentna sa čvorom u kojem je završio svoj prethodni potez, nazivaju se Voker–Brejker igre. Ove igre uveli su nedavno Espig (*Espig*), Friz (*Frieze*), Krivelevič (*Krivelevich*) i Pegden (*Pegden*) u [45].
Nije teško primijetiti da Brejker jednostavno može da izoluje čvor iz

Vokerovog grafa. Nakon prvog Vokerovog poteza Brejker fiksira čvor koji Voker nije dodirnuo u svom prvom potezu, a potom u svakom narednom potezu uzima grane između tog fiksiranog čvora i Vokerove trenutne pozicije.
Kako Voker nije u mogućnosti da dodirne sve čvorove grafa, postavlja se pitanje koliko najviše čvorova Voker može da dodirne. Ovo pitanje razmatrano je u [45] i pokazano je da najveći broj čvorova koje Voker može da dodirne u $(1 : b)$ igri je $n - 2b + 1$ gdje je b konstanta. U svom radu [45], autori su predložili mnogo interesantnih pitanja za dalji razvoj ovih igara. Jedno od tih pitanja istraživali su Klemens (*Clemens*) i Tran (*Tran*) u [34], gdje su pokazali da u igri bez biasa Voker može da napravi ciklus dužine $n - 2$, dok u igri sa biasom dužina najvećeg ciklusa koji Voker može da napravi je $n - O(b)$ gdje je $b \leq \frac{n}{\ln^2 n}$.

Pošto Voker ne može da napravi pokrivajuću strukturu ni za jedno $b \geq 1$, postavlja se pitanje da li se situacija mijenja ako se Vokerov bias poveća za 1 ili ako su oba igrača ograničena da biraju grane kao da se šetaju kroz graf.

Mejker–Brejker igre (totalne) dominacije

Mejker–Brejker igre dominacije su prvi put proučavali Dušen (*Duchêne*), Gledel (*Gledel*), Paro (*Parreau*) i Reno (*Renault*) u [42]. Igra se igra na skupu čvorova datog grafa G, a igrači se zovu Dominator (*Dominator*) i Stoler (*Staller*) prema ulogama koje imaju u igri. U Mejker–Brejker igri dominacije, koja igra na datom grafu $G(V, E)$, tabla igre je $X = V(G)$, a familija pobjednički skupovi su zatvorena susjedstva svih čvorova iz datog grafa. Stoler je Mejker i pobjeđuje ako uspije da osvoji neki čvor v i sve njegove susjede, a Dominator je Brejker i pobjeđuje ako čvorovi koje uzme u toku igre formiraju dominirajući skup. U radu [42] autori su istraživali koji igrač ima pobjedničku strategiju, a u radu [60] fokus je bio na istraživaju minimalnog broja poteza koji su potrebni da Dominator pobijedi u igri za koju ima pobjedničku strategiju.
Mejker–Brejker igre totalne dominacije (MBTD igre) uveli su Gledel (*Gledel*), Hening (*Henning*), Iršič (*Iršič*) i Klavžar (*Klavžar*) u [59]. Pobjednički skupovi su otvorena susjedstva svih čvorova datog grafa. Stoler

pobjeđuje ako uspije da uzme sve susjede nekog čvora, a Dominator pobjeđuje ako čvorovi koje je uzeo tokom igre formiraju totalni dominirajući skup.
U [59] autori su istraživali ishod u MBTD igrama koje se igraju na mrežama i grafovima koje su Dekartov proizvod puteva i ciklusa. Klasifikovali su kaktus grafove (povezane grafove u kojima svaka dva ciklusa imaju najviše jedan zajednički čvor) u zavisnosti od ishoda igre. U istom radu [59] autori su primjetili da klasifikacija kubnih grafova u zavisnosti od ishoda MBTD igre nije jednostavan zadatak s obzirom da postoji beskonačno mnogo povezanih kubnih grafova u kojima Stoler pobjeđuje, a uslov minimalnog stepena grafa nije dovoljan da garantuje pobjedu Dominatora u slučaju kada je Stoler prvi igrač.
Ovo otvara pitanje da se klasifikuju povezani kubni grafovi na kojima Dominator pobjeđuje i oni povezani kubni grafovi na kojima Stoler pobjeđuje u igri totalne dominacije, kao što je sugerisano u [59].

Rezultati

Voker–Brejker igre sa dvostrukim biasom

U glavi 3 istražujemo $(2:b)$ Voker–Brejker igre na K_n kako bismo odredili granični bias u igri povezanosti i igri Hamiltonove konture. Kao što je ranije istaknuto, u Voker–Brejker igrama na K_n, Voker ne može da napravi pokrivajuću strukturu kada igra sa biasom 1, jer Brejker može jednostavno da izoluje čvor iz Vokerovog grafa. Pokazujemo da se situacija mijenja sa povećanjem Vokerovog biasa za 1. Dajemo odgovore na sljedeća pitanja koja su predložena u [34]:

Pitanje 5.6 ([34], Problem 6.4)**.** *Koja je najveća vrijednost biasa b za koji Voker ima strategiju da napravi pokrivajuće stablo u $(2:b)$ Voker–Brejker igri na K_n?*

Pitanje 5.7 ([34], Problem 6.5)**.** *Da li postoji konstanta $c>0$ takva da Voker ima strategiju da napravi Hamiltonovu konturu u $\left(2:\frac{cn}{\ln n}\right)$ Voker–Brejker igri na K_n?*

Da bi se mogao dati odgovor na pitanje 5.6 potrebne su sljedeće dvije teoreme. Prva teorema daje donju granicu za granični bias u $(2:b)$ Voker–Brejker igri povezanosti.

Teorema 5.8. *Za svako $0 < \varepsilon < \frac{1}{4}$ i dovoljno veliko n, Voker ima strategiju da pobijedi u $(2 : b)$ Voker–Brejker igri povezanosti na K_n za $b \leq \left(\frac{1}{4} - \varepsilon\right) \frac{n}{\ln n}$.*

Teorema 5.9 daje daje gornju granicu za granični bias u $(2 : b)$ Voker–Brejker igri povezanosti.

Teorema 5.9. *Za svako $\varepsilon > 0$ i $b \geq (1 + \varepsilon)\frac{n}{\ln n}$, Brejker ima strategiju da pobijedi u $(2 : b)$ Voker–Brejker igri povezanosti na K_n, za dovoljno veliko n.*

Sljedeća teorema daje donju granicu za granični bias u $(2 : b)$ Voker–Brejker igri Hamiltonove konture.

Teorema 5.10. *Postoji konstanta $\alpha > 0$ takva da za svako dovoljno veliko n i $b \leq \alpha\frac{n}{\ln n}$ Voker ima strategiju da pobijedi $(2 : b)$ Voker–Brejker igri Hamiltnove konture na K_n.*

VokerMejker–VokerBrejker igre bez biasa

U glavi 4 istražujemo VokerMejker–VokerBrejker igre (ili VMejker–VBrejker igre, kraće). Podsjetimo se da su to Mejker–Brejker igre u kojima su oba igrača ograničeni po pitanju izbora grana, tj. i Mejker i Brejker moraju da biraju grane kao da se šetaju kroz graf. Fokusiraćemo se na VokerMejker–VokerBrejker igre bez biasa sa ciljem pronalaska brze pobjedničke strategije za VokerMejkera u igri povezanosti i igri Hamiltonove konture. Dokazujemo sljedeće teoreme:

Teorema 5.11. *U $(1 : 1)$ VMejker–VBrejker igri povezanosti na $E(K_n)$, VMejker ima strategiju da pobijedi za najviše $n + 1$ poteza.*

Teorema 5.12. *U $(1 : 1)$ VMejker–VBrejker igri Hamiltonove konture na $E(K_n)$, VMejker ima strategiju da pobijedi za najviše $n + 6$ poteza.*

Takođe, razmatramo koliko dugo VBrejker, kao drugi igrač, može da prolongira VMejkerovu pobjedu.

Teorema 5.13. *U $(1 : 1)$ VMejker–VBrejker igri povezanosti na $E(K_n)$, VBrejker, kao drugi igrač, ima strategiju da odloži VMejkerovu pobjedu za bar n poteza.*

MBTD igra na kubnim grafovima

U glavi 5 istražujemo Mejker–Brejker igre totalne dominacije na kubnim grafovima. Bavimo se karakterizacijom povezanih kubnih grafova na kojima Dominator pobjeđuje i onih povezanim kubnim grafovima na kojima Stoler pobjeđuje. U kubnim grafovima sa $n \geq 6$ čvorova, za svaki čvor važi jedna od sljedeće tri mogućnosti, [80]:

tip 1. čvor pripada tačno dvama trouglovima

tip 2. čvor pripada jednom trouglu

tip 3. čvor ne pripada nijednom trouglu.

Dakle, kubni grafovi se mogu klasifikovati prema broju čvorova tipa 1, tipa 2 i tipa 3. Ako sa T_1, T_2 i T_3 označimo broj čvorova tipa 1, tipa 2 i tipa 3, redom, onda su ovi brojevi povezani sljedećom formulom, [80]:

$$T_1 = 2k_1, \quad T_2 = T_1 + 3k_2, \quad T_1 + T_2 + T_3 = n,$$

gdje su k_1 i k_2 nenegativni cijeli brojevi, a n označava broj čvorova u grafu. Ako kubni graf sadrži čvor tipa 1, onda taj graf sadrži bar jedan dijamant (graf K_4 bez jedne grane). U daljem, pod trouglom podrazumijevaćemo indukovan graf K_3 koji nije dio dijamanta.
Uzimajući u razmatrane navedene tipove kubnih grafova, dokazujemo sljedeće teoreme.

Teorema 5.14. *Neka je G povezan kubni graf sa $n \geq 6$ čvorova koji je unija čvorno-disjunktnih dijamanata. Tada, Dominator pobjeđuje u MBTD igri na G.*

Teorema 5.15. *Neka je G povezan kubni graf sa $n \geq 6$ čvorova takav da svaki čvor pripada tačno jednom trouglu, tj. G je unija čvorno-disjunktnih trouglova. Ako je $n = 6$, Dominator pobjeđuje u MBTD igri. Ako je $n > 6$, Stoler pobjeđuje.*

Teorema 5.16. *Neka je G povezan kubni graf sa $n \geq 6$ čvorova koji je unija čvorno-disjunktnih trouglova i dijamanata. Tada, postoje samo dva tipa takvih grafova na kojima Dominator pobjeđuje. U svim ostalim slučajevim, Stoler pobjeđuje.*

Uopšteni Petersenovi grafovi privukli su dosta pažnje još od njihove definicije. U [59] autori su pokazali da u MBTD igri na grafu $P_2 \square C_n$, za $n \geq 3$, Dominator pobjeđuje. Ovaj graf je ekvivalentan uopštenom Petersenom grafu $GP(n, 1)$. Za graf $GP(5, 2)$ dokazano je u [59] da Stoler pobjeđuje. Sljedeća teorema daje karakterizaciju grafova $GP(n, 2)$ za svako $n \geq 6$.

Teorema 5.17. *U MBTD igra na $GP(n, 2)$ za $n \geq 6$ Stoler pobjeđuje.*

Istražujemo i MBTD igre na kubnim bipartitnim grafovima i dokazujemo sljedeću teoremu.

Teorema 5.18. *U MBTD igri na kubnom bipartitnom grafu Dominator pobjeđuje.*

Konačno, posmatramo MBTD igru na grafu G koji je unija čvorno-disjunktnih $K_{1,3}$ i dokazujemo sljedeću teoremu.

Teorema 5.19. *Neka je G povezna kubni graf sa $n \geq 6$ čvorova formiran kao unija $k \geq 2$ čvorno-disjunktnih $K_{1,3}$. Za $k = 2$, Dominator pobjeđuje. Za $k \geq 3$, Stoler pobjeđuje.*

Bibliography

[1] N. Alon and J. Spencer. *The Probabilistic Method*. Wiley-Interscience Series in Discrete Mathematics and Optimization, 2008. 3rd Edition.

[2] J. Balogh, B. Csaba, and W. Samotij. Local resilience of almost spanning trees in random graphs. *Random Structures & Algorithms*, 38(1-2):121–139, 2011.

[3] J. Balogh, R. Martin, and A. Pluhár. The Diameter Game. *Random Structures & Algorithms*, 35(3):369–389, 2009.

[4] J. Barát and M. Stojaković. On Winning Fast in Avoider–Enforcer Games. *The Electronic Journal of Combinatorics*, 17:#R56, 2010.

[5] J. Beck. Remarks on positional games. I. *Acta Mathematica Academiae Scientiarum Hungarica*, 40:65–71, 1982.

[6] J. Beck. Random Graphs and Positional Games on the Complete Graph. In M. Karoński and A. Ruciński, editors, *Random Graphs '83*, volume 118 of *North-Holland Mathematics Studies*, pages 7 – 13. North-Holland, 1985.

[7] J. Beck. Positional Games and the Second Moment Method. *Combinatorica*, 22:169–216, 2002.

[8] J. Beck. Ramsey games. *Discrete Mathematics*, 249(1-3):3 – 30, 2002.

[9] J. Beck. *Combinatorial Games: Tic-Tac-Toe Theory*. Cambridge University Press, 2008.

[10] J. Beck. *Inevitable Randomness in Discrete Mathematics.* University Lecture Series 49, American Mathematical Society, Providence, RI, 2009.

[11] M. Bednarska and T. Łuczak. Biased Positional Games for Which Random Strategies are Nearly Optimal. *Combinatorica*, 20:477–488, 2000.

[12] M. Bednarska-Bzdęga. On weight function methods in Chooser–Picker games. *Theoretical Computer Science*, 475:21 – 33, 2013.

[13] M. Bednarska-Bzdęga. Avoider–Forcer Games on Hypergraphs with Small Rank. *The Electronic Journal of Combinatorics*, 21(1):#P1.2, 2014.

[14] M. Bednarska-Bzdęga, D. Hefetz, M. Krivelevich, and T. Łuczak. Manipulative Waiters with Probabilistic Intuition. *Combinatorics, Probability and Computing*, 25(6):823–849, 2016.

[15] M. Bednarska-Bzdęga, D. Hefetz, and T. Łuczak. Picker–Chooser fixed graph games. *Journal of Combinatorial Theory, Series B*, 119:122 – 154, 2016.

[16] S. Ben-Shimon, A. Ferber, D. Hefetz, and M. Krivelevich. Hitting time results for Maker–Breaker games. *Random Structures & Algorithms*, 41(1):23–46, 2012.

[17] S. Ben-Shimon, M. Krivelevich, and B. Sudakov. Local Resilience and Hamiltonicity Maker–Breaker Games in Random Regular Graphs. *Combinatorics, Probability and Computing*, 20(2):173–211, 2011.

[18] S. Ben-Shimon, M. Krivelevich, and B. Sudakov. On the Resilience of Hamiltonicity and Optimal Packing of Hamilton Cycles in Random Graphs. *SIAM Journal on Discrete Mathematics*, 25(3):1176–1193, 2011.

[19] R. Berlekamp, E, J. Conway, and R. Guy. *Winning ways for your mathematical plays.* Academic press, London, 1982.

[20] B. Bollobás and A. Papaioannou. A biased Hamiltonian game. *Congressus Numerantium*, 35:105–115, 1982.

[21] B. Bollobás. *Random Graphs.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2001.

[22] J. Bondy and U. Murty. *Graph Theory.* Springer, New York, 2008.

[23] S. Boriboon and T. Kittipassorn. A strategy for Isolator in the Toucher–Isolator game on trees, 2020. arXiv preprint, arXiv:2005.01931v2.

[24] B. Brešar, S. Klavžar, and D. F. Rall. Domination Game and an Imagination Strategy. *SIAM Journal on Discrete Mathematics*, 24(3):979–991, 2010.

[25] C. Bujtás. On the game domination number of graphs with given minimum degree. *The Electronic Journal of Combinatorics*, 22(3):P3.29, 2015.

[26] J. Böttcher, Y. Kohayakawa, and A. Taraz. Almost spanning subgraphs of random graphs after adversarial edge removal. *Electronic Notes in Discrete Mathematics*, 22:335–340, 2009.

[27] V. Chvátal and P. Erdős. Biased Positional Games. *Annals of Discrete Mathematics*, 2:221 – 229, 1978.

[28] D. Clemens, J. Ehrenmüller, Y. Person, and T. Tran. Keeping Avoider's Graph Almost Acyclic. *The Electronic Journal of Combinatorics*, 22(1):#P1.60, 2015.

[29] D. Clemens, A. Ferber, R. Glebov, D. Hefetz, and A. Liebenau. Building Spanning Trees Quickly in Maker–Breaker Games. *SIAM Journal on Discrete Mathematics*, 29(3):1683–1705, 2015.

[30] D. Clemens, A. Ferber, M. Krivelevich, and A. Liebenau. Fast Strategies in Maker–Breaker Games Played on Random Boards. *Combinatorics, Probability and Computing*, 21(6):897–915, 2012.

[31] D. Clemens, P. Gupta, F. Hamann, A. Haupt, M. Mikalački, and Y. Mogge. Fast Strategies in Waiter–Client Games. *The Electronic Journal of Combinatorics*, 27(3):#P3.57, 2020.

[32] D. Clemens and M. Mikalački. A Remark on the Tournament Game. *The Electronic Journal of Combinatorics*, 22(3):#P3.42, 2015.

[33] D. Clemens and M. Mikalački. How fast can Maker win in fair biased games? *Discrete Mathematics*, 341(1):51 – 66, 2018.

[34] D. Clemens and T. Tran. Creating cycles in Walker–Breaker games. *Discrete Mathematics*, 339(8):2113–2126, 2016.

[35] J. Conway. *On Numbers and Games*. Academic press, London, 1976.

[36] A. Csernenszky. The Picker–Chooser diameter game. *Theoretical Computer Science*, 411:3757–3762, 2010.

[37] A. Csernenszky, R. Martin, and A. Pluhár. On the Complexity of Chooser–Picker Positional Games. *Integers*, 12(3):427–444, 2012.

[38] A. Csernenszky, C. I. Mándity, and A. Pluhár. On Chooser–Picker positional games. *Discrete Mathematics*, 309(16):5141 – 5146, 2009.

[39] O. Dean and M. Krivelevich. Client–Waiter Games on Complete and Random Graphs. *The Electronic Journal of Combinatorics*, 23(4):#P4.38, 2016.

[40] P. Dorbec, G. Košmrlj, and G. Renault. The domination game played on unions of graphs. *Discrete Mathematics*, 338(1):71–79, 2015.

[41] C. Dowden, M. Kang, M. Mikalački, and M. Stojaković. The Toucher–Isolator game. *The Electronic Journal of Combinatorics*, 26(4):#P4.6, 2019.

[42] E. Duchêne, V. Gledel, A. Parreau, and G. Renault. Maker–Breaker domination game. *Discrete Mathematics*, 343(9):111955, 2020.

[43] P. Erdős and A. Rényi. On Random Graphs I. *Publicationes Mathematicae Debrecen*, 6:290–297, 1959.

[44] P. Erdős and J. Selfridge. On a combinatorial game. *Journal of Combinatorial Theory, Series A*, 14(3):298 – 301, 1973.

[45] L. Espig, A. Frieze, W. Pegden, and M. Krivelevich. Walker–Breaker Games. *SIAM Journal on Discrete Mathematics*, 29(3):1476–1485, 2015.

[46] A. Ferber, R. Glebov, M. Krivelevich, and A. Naor. Biased games on random boards. *Random Structures & Algorithms*, 46(4):651–676, 2015.

[47] A. Ferber and D. Hefetz. Winning Strong Games through Fast Strategies for Weak Games. *The Electronic Journal of Combinatorics*, 18(1):#P144, 2011.

[48] A. Ferber and D. Hefetz. Weak and strong k-connectivity games. *European Journal of Combinatorics*, 35:169 – 183, 2014. Selected Papers of EuroComb'11.

[49] A. Ferber, D. Hefetz, and M. Krivelevich. Fast embedding of spanning trees in biased Maker–Breaker games. *European Journal of Combinatorics*, 33(6):1086–1099, 2012.

[50] A. Ferber, M. Krivelevich, and G. Kronenberg. Efficient Winning Strategies in Random-Turn Maker–Breaker Games. *Journal of Graph Theory*, 85(2):446–465, 2017.

[51] A. Ferber, M. Krivelevich, and A. Naor. Avoider–Enforcer games played on edge disjoint hypergraphs. *Discrete Mathematics*, 313(24):2932–2941, 2013.

[52] A. Ferber, M. Krivelevich, and H. Naves. Generating random graphs in biased Maker–Breaker games. *Random Structures & Algorithms*, 47(4):615–634, 2015.

[53] J. Forcan and M. Mikalački. Spanning structures in Walker–Breaker games, 2019. arXiv preprint, arXiv:1907.08436v2.

[54] J. Forcan and M. Mikalački. Maker–Breaker total domination games on cubic graphs, 2020. arXiv preprint, arXiv:2010.03448.

[55] J. Forcan and M. Mikalački. On the WalkerMaker–WalkerBreaker games. *Discrete Applied Mathematics*, 279:69–79, 2020.

[56] A. Frieze and M. Krivelevich. On two Hamilton cycle problems in random graphs. *Israel Journal of Mathematics*, 166:221–234, 2008.

[57] D. Gale. The Game of Hex and the Brouwer Fixed-Point Theorem. *American Mathematical Monthly*, 86(10):818–827, 1979.

[58] H. Gebauer and T. Szabó. Asymptotic Random Graph Intuition for the Biased Connectivity Game. *Random Structures & Algorithms*, 35(4):431–443, 2009.

[59] V. Gledel, M. Henning, V. Iršič, and S. Klavžar. Maker–Breaker total domination game. *Discrete Applied Mathematics*, 282:96–107, 2020.

[60] V. Gledel, V. Iršič, and S. Klavžar. Maker–Breaker Domination Number. *Bulletin of the Malaysian Mathematical Sciences Society*, 42(4):1773–1789, 2019.

[61] J. Groschwitz and T. Szabó. Sharp Thresholds for Half-Random Games I. *Random Structures & Algorithms*, 49(4):766–794, 2016.

[62] J. Groschwitz and T. Szabó. Sharp Thresholds for Half-Random Games II. *Graphs and Combinatorics*, 33(2):387–401, 2017.

[63] A. Grzesik, M. Mikalački, Z. Nagy, A. Naor, B. Patkós, and F. Skerman. Avoider–Enforcer star games. *Discrete Mathematics and Theoretical Computer Science*, 17(1):145–160, 2015.

[64] A. Hales and R. Jewett. Regularity and Positional Games. *Transactions of the American Mathematical Society*, 106(2):222–229, 1963.

[65] Y. O. Hamidoune and M. Las Vergnas. A solution to the Box Game. *Discrete Mathematics*, 65(2):157–171, 1987.

[66] D. Hefetz. *Positional games on graphs.* PhD thesis, Tel Aviv University, 2007.

[67] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. *Positional Games*. Springer Basel, Basel, 2014.

[68] D. Hefetz, M. Krivelevich, M. Stojakovic, and T. Szabó. Planarity, Colorability, and Minor Games. *SIAM Journal of Discrete Mathematics*, 22(1):194–212, 2008.

[69] D. Hefetz, M. Krivelevich, M. Stojakovic, and T. Szabó. Avoider–Enforcer: The Rules of the Game. *Journal of Combinatorial Theory, Series A*, 117(2):152–163, 2010.

[70] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. A Sharp Threshold for the Hamilton Cycle Maker–Breaker Game. *Random Structures & Algorithms*, 34(1):112 – 122, 2009.

[71] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Fast Winning Strategies in Avoider–Enforcer Games. *Graphs and Combinatorics*, 25(4):533–544, 2009.

[72] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Fast winning strategies in Maker–Breaker games. *Journal of Combinatorial Theory, Series B*, 99(1):39 – 47, 2009.

[73] D. Hefetz, M. Krivelevich, and T. Szabó. Avoider–Enforcer Games. *Journal of Combinatorial Theory, Series A*, 114(5):840–853, 2007.

[74] D. Hefetz, M. Krivelevich, and W. Tan. Waiter–Client and Client–Waiter planarity, colorability and minor games. *Discrete Mathematics*, 339(5):1525–1536, 2016.

[75] D. Hefetz, M. Krivelevich, and W. Tan. Waiter–Client and Client–Waiter Hamiltonicity games on random graphs. *European Journal of Combinatorics*, 63:26–43, 2017.

[76] D. Hefetz, M. Mikalački, and M. Stojaković. Doubly Biased Maker–Breaker Connectivity Game. *The Electronic Journal of Combinatorics*, 19(1):#P61, 2012.

[77] D. Hefetz and S. Stich. On Two Problems Regarding the Hamiltonian Cycle Game. *The Electronic Journal of Combinatorics*, 16(1):#R28, 2009.

[78] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs.* John Wiley & Sons, Ltd, 2000.

[79] J. Komlós and E. Szemerédi. Limit distribution for the existence of hamiltonian cycles in a random graph. *Discrete Mathematics*, 43(1):55 – 63, 1983.

[80] R. Korfhage. *Discrete Computational Structures.* Academic Press, New York, 1984. 2nd Edition.

[81] M. Krivelevich. The critical bias for the Hamiltonicity game is $(1 + o(1))n/lnn$. *Journal of the American Mathematical Society*, 24(1):125–131, 2011.

[82] M. Krivelevich and G. Kronenberg. Random-Player Maker–Breaker games. *The Electron Journal of Combinatorics*, 22(4):#P4.9, 2015.

[83] M. Krivelevich, C. Lee, and B. Sudakov. Resilient Pancyclicity of Random and Pseudorandom Graphs. *SIAM Journal on Discrete Mathematics*, 24(1):1–16, 2010.

[84] M. Krivelevich and T. Szabó. Biased Positional Games and Small Hypergraphs with Large Covers. *The Electronic Journal of Combinatorics*, 15:#R70, 2008.

[85] M. Krivelevich and N. Trumer. Waiter–Client Maximum Degree Game, 2018. arXiv preprint, arXiv:1807.11109.

[86] M. Kutz. *The Angel Problem, Positional Games, and Digraph Roots.* PhD thesis, Freie Universität Berlin, 2004.

[87] C. Lee and B. Sudakov. Dirac's theorem for random graphs. *Random Structures & Algorithms*, 41(3):293–305, 2012.

[88] A. Lehman. A Solution of the Shannon Switching Game. *Journal of the Society for Industrial and Applied Mathematics*, 12(4):687–725, 1964.

[89] M. Mikalački. *Positional games on graphs.* PhD thesis, University of Novi Sad, 2013.

[90] M. Mikalački and M. Stojaković. Fast strategies in biased Maker–Breaker games. *Discrete Mathematics & Theoretical Computer Science*, 20(2), 2018.

[91] T. Müller and M. Stojaković. A threshold for the Maker–Breaker clique game. *Random Structures & Algorithms*, 45(2):318–341, 2014.

[92] M. J. Nadjafi-Arani, M. Siggers, and H. Soltani. Characterisation of forests with trivial game domination numbers. *Journal of Combinatorial Optimization*, 32(3):800–811, 2016.

[93] R. Nenadov, A. Steger, and M. Stojaković. On the threshold for the Maker–Breaker H-game. *Random Structures & Algorithms*, 49(3):558–578, 2016.

[94] A. Papaioannou. A Hamiltonian game. *Annals of Discrete Mathematics*, 13:171–178, 1982.

[95] Y. Peres, O. Schramm, S. Sheffield, and D. Wilson. Random-Turn Hex and Other Selection Games. *American Mathematical Monthly*, 114(5):373–387, 2007.

[96] E. Raty. An Achievement Game on a Cycle, 2019. arXiv preprint, arXiv:1907.11152.

[97] E. Raty. The Toucher-Isolator Game on Trees, 2020. arXiv preprint, arXiv:2001.10498.

[98] V. Rödl and A. Ruciński. Threshold Functions for Ramsey Properties. *Journal of the American Mathematical Society*, 8(4):917–942, 1995.

[99] S. Schmidt. The 3/5-conjecture for weakly $S(K_{1,3})$-free forests. *Discrete Mathematics*, 339(11):2767–2774, 2016.

[100] M. Stojaković. *Games on Graphs*. PhD thesis, ETH Zürich, 2005.

[101] M. Stojaković. Games on Graphs. In *Hernandez N., Jäschke R., Croitoru M. (eds) Graph-Based Representation and Reasoning. ICCS 2014. Lecture Notes in Computer Science*, volume 8577. Springer, Cham, 2014.

[102] M. Stojaković and T. Szabó. Positional games on random graphs. *Random Structures & Algorithms*, 26(1-2):204–223, 2005.

[103] M. Stojaković and N. Trkulja. Hamiltonian Maker–Breaker Games on Small Graphs, 2018. arXiv preprint, arXiv:1708.07579v3.

[104] B. Sudakov and V. Vu. Local Resilience of Graphs. *Random Structures & Algorithms*, 33(4):409–433, 2008.

[105] K. Xu, X. Li, and S. Klavžar. On graphs with largest possible game domination number. *Discrete Mathematics*, 341(6):1768–1777, 2018.

Short biography

Jovana Forcan (maiden Janković) was born in Sarajevo, Bosnia and Herzegovina, on November 13, 1991. She received the B.Sc. degree in Mathematics and Computer Science from the University of East Sarajevo, Faculty of Philosophy, East Sarajevo, Bosnia and Herzegovina, in 2014, and the M.Sc. degree in Informatics from the University of East Sarajevo, Faculty of Philosophy, East Sarajevo, Bosnia and Herzegovina, in 2016, and the M.Sc. degree in Mathematics from the University of Banja Luka, Faculty of Natural Sciences and Mathematics, Banja Luka, Bosnia and Herzegovina, in 2016.
She received the scholarship Fund Dr. Milan Jelic, in the category of successful students in the academic years 2012/2013, 2013/2014 and 2015/2016, and scholarships of the Ministry of Education and Culture of the Republic of Srpska in the academic years 2011/2012, 2014/2015.
She started PhD studies of Theoretical Computer Science at Faculty of Sciences, University of Novi Sad, in 2016. Within two years she passed all exams with the average grade of 10.00. During her PhD studies, she participated in many conferences, workshops, and summer schools.
Currently, she is a Senior Research and Teaching Assistant at the University of East Sarajevo, Faculty of Philosophy, East Sarajevo, Bosnia and Herzegovina.

Novi Sad, 2021 Jovana Forcan

Овај Образац чини саставни део докторске дисертације, односно докторског уметничког пројекта који се брани на Универзитету у Новом Саду. Попуњен Образац укоричити иза текста докторске дисертације, односно докторског уметничког пројекта.

План третмана података

Назив пројекта/истраживања
Мејкер-Брејкер игре на графовима/Maker-Breaker games on graphs
Назив институције/институција у оквиру којих се спроводи истраживање
Универзитет у Новом Саду, Природно-математички факултет, Департман за математику и информатику
Назив програма у оквиру ког се реализује истраживање
Истраживање се реализује у оквиру израде докторске дисерације на студијском програму Докторске академске студије Информатике.
1. Опис података
1.1 Врста студије *Укратко описати тип студије у оквиру које се подаци прикупљају* Студија се бави позиционим играма типа Мејкер-Брејкер (Maker-Breaker). Студија је теоријског типа у оквиру које се развијају нове теореме и њихови докази. 1.2 Врсте података квалитативни

1.3. Начин прикупљања података

а) текст, навести врсту: научне публикације

1.3 Формат података, употребљене скале, количина података

Претходно објављени научни радови у пдф формату.

1.3.1 Употребљени софтвер и формат датотеке:

PDF фајл, датотека: .pdf

1.3.2. Број записа (код квантитативних података)

а) број варијабли -

б) број мерења (испитаника, процена, снимака и сл.) -

1.3.3. Поновљена мерења

Не.

Уколико је одговор да, одговорити на следећа питања:

а) временски размак између поновљених мера је__________________

б) варијабле које се више пута мере односе се на__________________

в) нове верзије фајлова који садрже поновљена мерења су именоване као ____

Напомене: ____

Да ли формати и софтвер омогућавају дељење и дугорочну валидност података?

Да.

2. Прикупљање података

2.1 Методологија за прикупљање/генерисање података

2.1.1. У оквиру ког истраживачког нацрта су подаци прикупљени?

анализа текста из претходно објављене литературе

2.1.2 Навести врсте мерних инструмената или стандарде података специфичних за одређену научну дисциплину (ако постоје).

-

2.2 Квалитет података и стандарди

-

2.2.1. Третман недостајућих података

а) Да ли матрица садржи недостајуће податке? Не

Ако је одговор да, одговорити на следећа питања:

а) Колики је број недостајућих података? ________________________

б) Да ли се кориснику матрице препоручује замена недостајућих података? Да Не

в) Ако је одговор да, навести сугестије за третман замене недостајућих података ____

2.2.2. На који начин је контролисан квалитет података? Описати

-

2.2.3. На који начин је извршена контрола уноса података у матрицу?

-

3. Третман података и пратећа документација

3.1. Третман и чување података

3.1.1. Подаци ће бити депоновани у Заједнички портал свих докторских дисертација и извештаја комисија о њиховој оцени на универзитетима у Србији (NaRDUS) и у репозиторијуму докторских дисертација Универзитета у Новом Саду (CRIS).

*3.1.2. URL адреса*____

*3.1.3. DOI*____

3.1.4. Да ли ће подаци бити у отвореном приступу?

а) Да

*Ако је одговор не, навести разлог*______________________________________

3.1.5. Подаци неће бити депоновани у репозиторијум, али ће бити чувани.

Образложење

Докторска дисертација ће бити депонована у репозиторијуму дисертација Универзитета у Новом Саду.

3.2 Метаподаци и документација података

3.2.1. Који стандард за метаподатке ће бити примењен?-

3.2.1. Навести метаподатке на основу којих су подаци депоновани у репозиторијум.

-

Ако је потребно, навести методе које се користе за преузимање података, аналитичке и процедуралне информације, њихово кодирање, детаљне описе варијабли, записа итд.

3.3 Стратегија и стандарди за чување података

3.3.1. До ког периода ће подаци бити чувани у репозиторијуму?
Неограничено

3.3.2. Да ли ће подаци бити депоновани под шифром? Не

3.3.3. Да ли ће шифра бити доступна одређеном кругу истраживача? Не

3.3.4. Да ли се подаци морају уклонити из отвореног приступа после извесног времена?

Не

Образложити

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4. Безбедност података и заштита поверљивих информација

Овај одељак МОРА бити попуњен ако ваши подаци укључују личне податке који се односе на учеснике у истраживању. За друга истраживања треба такође размотрити заштиту и сигурност података.

4.1 Формални стандарди за сигурност информација/података

Истраживачи који спроводе испитивања с људима морају да се придржавају Закона о заштити података о личности (*https://www.paragraf.rs/propisi/zakon_o_zastiti_podataka_o_licnosti.html*) и одговарајућег институционалног кодекса о академском интегритету.

-

4.1.2. Да ли је истраживање одобрено од стране етичке комисије? Да Не

-

Ако је одговор Да, навести датум и назив етичке комисије која је одобрила истраживање

-

4.1.2. Да ли подаци укључују личне податке учесника у истраживању? Да Не

Ако је одговор да, наведите на који начин сте осигурали поверљивост и сигурност информација везаних за испитанике:

а) Подаци нису у отвореном приступу

б) Подаци су анонимизирани

ц) Остало, навести шта

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5. Доступност података

5.1. Подаци ће бити

јавно доступни

Ако су подаци доступни само уском кругу истраживача, навести под којим условима могу да их користе:

-

Ако су подаци доступни само уском кругу истраживача, навести на који начин могу приступити подацима:

-

5.4. Навести лиценцу под којом ће прикупљени подаци бити архивирани.

Ауторство – некомерцијално – делити под истим условима

6. Улоге и одговорност

6.1. Навести име и презиме и мејл адресу власника (аутора) података

Јована Форцан dmi.jovana.jankovic@student.pmf.uns.ac.rs

Мирјана Микалачки mirjana.mikalacki@dmi.uns.ac.rs

6.2. Навести име и презиме и мејл адресу особе која одржава матрицу с подацима

-

6.3. Навести име и презиме и мејл адресу особе која омогућује приступ подацима другим истраживачима

Јована Форцан dmi.jovana.jankovic@student.pmf.uns.ac.rs

Мирјана Микалачки mirjana.mikalacki@dmi.uns.ac.rs