UNIVERSITY OF NIŠ
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ASYMPTOTIC REPRESENTATION OF SOLUTIONS OF NONLINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH REGULARLY VARYING COEFFICIENTS

DOCTORAL DISSERTATION

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ASIMPTOTSKA REPREZENTACIJA REŠENJA NELINEARNIH DIFERENCIJALNIH I DIFERENCNIH JEDNAČINA SA PRAVILNO PROMENLJIVIM KOEFICIJENTIMA

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Ментор:	др Јелена Манојловић, редовни професор, Природно-математички факултет, Универзитет у Нишу
Наслов:	Асимптотска репрезентација решења нелинеарних диференцијалних и диференцних једначина са правилно променљивим коефицијентима
Резиме:	У овој дисертацији разматране су диференцијалне једначине четвртог реда, диференцне једначине другог реда и циклични системи диференцних једначина другог реда. Наиме, под претпоставком да су коефицијенти диференцијалне једначине четвртог реда типа Емден-Фаулера уопштене правилно променљиве функције, дата је комплетна информација о постојању свих могућих укљештених правилно променљивих решења, као и о њиховом прецизном асимптотском понашању у бесконачности. У оквиру теорије правилно променљивих низова разматрана је диференцна једначина другог реда типа Томас-Ферми. Строго растућа и строго опадајућа решења ове једначине су детаљло испитана. Одређени су потребни и довољни услови за постојање ових решења као и њихове асимптотске репрезентације. Добијени резултати омогућили су да буде представљена комплетна структура скупа правилно променљивих решења. Као природно уопштење диференцних једначина другог реда разматрани су циклични системи диференцних једначина. Дата је комплетна асимптотска анализа понашања свих позитивних решења. Конкретно, асимптотско понашање укљештених, као и строго растућих и строго опадајућих решења, посматрано је под претпоставком да су коефицијенти система правилно променљиви низови и њихове прецизне асимтотске формуле су одређене. Такође, добијени су и услови за егзистенцију свих типова позитивних решења посматраних система.
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Редни број, **РБР**:	
Идентификациони број, **ИБР**:	
Тип документације, **ТД**:	монографска
Тип записа, **ТЗ**:	текстуални
Врста рада, **ВР**:	докторска дисертација
Аутор, **АУ**:	Александра Б. Капешић
Ментор, **МН**:	Јелена В. Манојловић
Наслов рада, **НР**:	Асимптотска репрезентација решења нелинеарних диференцијалних и диференцних једначина са правилно променљивим коефицијентима
Језик публикације, **ЈП:**	енглески
Језик извода, **ЈИ**:	енглески
Земља публиковања, **ЗП**:	Србија
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		У оквиру теорије правилно променљивих низова разматрана је диференцна једначина другог реда типа Томас-Ферми. Строго растућа и строго опадајућа решења ове једначине су детаљло испитана. Одређени су потребни и довољни услови за постојање ових решења као и њихове асимптотске репрезентације. Добијени резултати омогућили су да буде представљена комплетна структура скупа правилно променљивих решења. Као природно уопштење диференцних једначина другог реда разматрани су циклични системи диференцних једначина. Дата је комплетна асимптотска анализа понашања свих позитивних решења. Конкретно, асимптотско понашање укљештених, као и строго растућих и строго опадајућих решења, посматрано је под претпоставком да су коефицијенти система правилно променљиви низови и њихове прецизне асимтотске формуле су одређене. Такође, добијени су и услови за егзистенцију свих типова позитивних решења посматраних система.
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Contents

Preface

The study of differential equations is a broad field in both theoretical and applied mathematics, physics, engineering, biology, chemistry and other sciences. Differential equations play an important role in modeling almost all physical, technical or biological processes from celestial motion through bridge design to interaction between neurons. Differential equations such as those used to solve real-life problems do not necessarily have to be directly solvable. Instead, sometimes it is enough to just know the properties of the solution such as periodicity, stability oscillatory, asymptotic behavior of non-oscillatory solutions and so on. An area that deals with this type of research is known as qualitative theory of differential equations. On the other hand, in the last fifty years, the application of difference equations in solving many problems in statistics, engineering and science in general has experienced expansion. The development of high-speed digital computer technology has motivated the application of difference equations to ordinary and partial differential equations. Apart from this, difference equations are very useful for analyzing electrical, mechanical, thermal and other systems, the behavior of electric-wave filters and other filters, insulator strings, crankshafts of multi-cylinder engines etc.

One of the most studied second-order nonlinear differential equations is

$$(EF) \qquad x''(t) + q(t)|x(t)|^{\lambda-1}x(t) = 0, \quad \lambda \neq 1.$$

This equation is known as the Emden-Fowler type equation. When the coefficient q is negative, the mentioned equation is also called the Thomas-Fermi equation. In fact, studies of polytropic and isothermal gas spheres in the state of gravitational equilibrium, the electron distribution in heavy atoms and electrostatic potential in the spherically and cylindrically symmetric combustion products plasma volume, led to the appearance of differential equations $x''(t) \pm t^{\delta}x(t)^{\lambda} = 0$. New problems in nuclear physics, for example, problems related to nuclear matter in neutron stars, can be solved using a model based on the solution of the Thomas-Fermi equation. The classic model is given in the form

$$x''(t) = \sqrt{\frac{x^3(t)}{t}}$$

and with the boundary conditions $x(0) = 1$, $x(\infty) = 0$. This equation describes a spherically symmetric charge distribution for a multi-electron atom.

As the generalization of the equation (EF) many authors studied equations

$$(A) \qquad \left(p(t)|x'(t)|^{\alpha-1}x'(t)\right)' + q(t)|x(t)|^{\beta-1}x(t) = 0,$$

and

$$(E_1) \qquad \left(p(t)|x''(t)|^{\alpha-1}x''(t)\right)'' + q(t)|x(t)|^{\beta-1}x(t) = 0.$$

Properties such as existence, uniqueness and continuity of the solution, oscillatory and nonoscillatory behavior of solutions of equation (A) have been studied in monographs [31, 66, 103] as well as in papers [12, 13, 17, 18, 22, 30, 58, 59, 67, 104–107, 121, 133–135]. Oscillation theory for fourth-order equations of the type (E_1) was first developed by Wu [137]. These results have been further developed and enriched with information about the asymptotic behavior of nonoscillatory solutions of (E_1) in the series of papers in [57, 60, 78, 82, 86, 108, 109, 138].

Along with the differential equations (A), (E_1) discrete counterpart of these equations

$$(B) \qquad \Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) + q(n)|x(n+1)|^{\beta-1}x(n+1) = 0,$$

and

$$(D) \qquad \Delta^2(p(n)|\Delta^2 x(n)|^{\alpha-1}\Delta x(n)) + q(n)|x(n+2)|^{\beta-1}x(n+2) = 0,$$

has attracted many researchers, see e.g. [14–16, 19–21, 23, 24, 123, 124, 127, 136] for second-order equations and [3, 5, 6, 27–29, 87, 125, 126, 128–131, 139] for fourth-order equations and monographs [1, 2].

Along with qualitative study of second-order and fourth-order nonlinear difference equations, nonlinear one-dimensional and two-dimensional difference systems were also studied in [4, 55, 56, 83, 84, 94, 95]. In the mid nineties, there was a significant interest in symmetric systems (see [9, 110–112, 117] and references therein). By modifying certain parameters, general systems were obtained, often called close-to-symmetric systems (see [118, 119]). Multidimensional extensions of symmetric and close-to-symmetric systems are cyclic systems of difference equations. Nonlinear cyclic systems of second–order difference equations

$$(SE) \quad \Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) + q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1) = 0, \ \ i = \overline{1,N}, x_{N+1} = x_1,$$

can be considered as natural generalization of second-order and fourth-order nonlinear difference equations (B) and (D). However, asymptotic properties of solutions of this type of difference systems have not been studied so far in the existing literature.

When studying the asymptotic behavior of solutions of nonlinear differential or difference equations, the solutions are first classified on the basis of their behavior in infinity, thus dividing the set of solutions into disjoint sets, whereby the necessary and sufficient conditions under which it is not empty are determined for each of them. This problem is easily solved for certain sets, however, there are sets for which this cannot be done and only necessary or only sufficient conditions can be determined. For such a solution, the problem of determining precise asymptotic formulas is almost impossible to solve in the general case, i.e., assuming that the coefficients are continuous functions when it comes to equations (A) or (E_1) or that the coefficients are arbitrary sequences when it comes to their discrete analogues. The recent development of asymptotic analysis of differential and difference equations indicates that the problem can be solved by using the theory of regularly varying functions and sequences.

The theory of regularly varying functions originated in 1930. when the concept of a regularly varying function was introduced by Karamata in [65]. Further development was done by Avakumović, Bojanić, Tomić, Marić, as well as Bingham, Goldie, Seneta, de Haan and many others (see monographs [10, 42, 43, 116]). Avakumović was the first to consider the application of this theory in the asymptotic analysis of differential equations (see [8]). With the appearance of the papers of Marić and Tomić [89–93] and Marić's monograph [88], the increasing application of this theory by other authors began, providing important contributions to the understanding of the asymptotic expansion for special classes of nonoscillatory solutions of linear and nonlinear differential equations and systems of differential equations. Second-order differential equation were studied in this framework e.g. in [68–72,80,81,85], for fourth-order differential equations see e.g. [73,74,79] and for study of systems of differential equation in this framework see [46–51, 54, 100, 101, 113].

That the class of regularly varying functions in the sense of Karamata is not suitable for describing the asymptotic behavior of positive solutions of the self-adjoint linear second-order differential equation $(p(t)x'(t))' + q(t)x(t) = 0$, in compare with the linear second-order differential equation $x''(t) + q(t)x(t) = 0$, was first observed by Jaroš and Kusano in [45]. The problem was solved by properly generalizing the class of regularly varying functions in the sense of Karamata. In fact, the definitions and basic properties of generalized regularly varying functions are given in [45] and applied in the asymptotic analysis of the self-adjoint linear second-order differential equation. This theory is also applied in papers [32,52,76,77,102] considering asymptotic behavior of intermediate solutions of equations (A) and (E_1) under different assumptions for regularly varying coefficient p. In this Ph.D. thesis, we proceed further in this direction by applying the theory of generalized regularly varying functions for the asymptotic analysis of solutions of the fourth-order differential equation (E_1) under certain integral condition.

On the other hand, the theory of regularly varying sequences, often called Karamata sequences (see [64]), was developed during the seventies by Galambos, Seneta and Bojanić in [11,41]. However, until the appearance of the paper of Matucci and Rehak [96], the connection between regularly varying sequences and difference equations was not considered. In this paper, as well as in the following ones [97, 99, 114, 115], the theory of regularly varying sequences is further developed and applied in the asymptotic analysis of linear and half-linear difference equations of the second-order, giving necessary and sufficient conditions for the existence of regularly varying solutions of these equations. After this, further development of the discrete theory of regular variation, as well as its application to nonlinear difference equations of type Emden-Fowler type, can be found in [7]. Another goal of this Ph.D. thesis is to further develop the theory of regularly varying sequences with application to second-order difference equations of type (B) as well as to cyclic systems of difference equations (SE).

Along with the theory of regularly varying functions and sequences, the theory of rapidly varying functions and sequences has also been introduced by de Haan [43] in 1970. In the continuous case, there are many results that consider the relationship between rapidly varying functions and the behavior of linear and nonlinear differential equations (see [25, 26, 31, 39, 40, 53, 122]). For some properties of rapidly varying sequences we refer to [34], while rapidly varying solutions of linear and half-linear second-order difference equations were studied by Rehak and Matucci in [96,98]. Also, papers of Djurčić, Elez, Kočinac and Žižović (see [33, 35–38]) showed the connection between Karamata theory and the theory of rapidly varying sequences along with selection principles theory, game theory and Ramsey theory, indicating the wide application of regularly and rapidly varying sequences.

The dissertation is organized into four chapters, followed by a bibliography and a biography of the author.

The first chapter is of an introductory character. First, the basic concepts and theorems that will be used further are presented. Then, the basics of the theory of regularly varying functions and sequences, are given (sections 1.2 and 1.3), which gives a framework within which equations and systems will be considered.

In the second chapter, the Emden-Fowler differential equation of the fourth-order (E_1) is considered. The equation (E_1) will be considered under certain integral condition for the coefficient p, under which it was already considered by Kusano and Tanigawa in [82]. Assuming that coefficients are continuous functions, they determined the necessary and sufficient conditions for the existence of four types of primitive solutions. However, for two types of intermediate solutions, only sufficient conditions for the existence were given. Assuming that the coefficients of the equation are generalized regularly varying functions, we will not only give the necessary and sufficient conditions for the intermediate solutions to exist, but we will also determine their precise asymptotic formula. *Main results presented in sections 2.3, 2.4, 2.5 are original and published in the paper [132].*

In the third chapter, we will study a discrete analogue of the equation (A) , that is, the equation (B) assuming that the coefficient q is negative. The boundary asymptotic behavior of decreasing solutions plays an important role in the discretization of certain elliptic problems with free boundaries. This is one of the reasons that indicate the necessity for a detailed analysis of these solutions. However, for sublinear case i.e. $\beta < \alpha$ there are no results in the existing literature for the existence of strongly decreasing solutions, while for the existence of strongly increasing only sufficient conditions are known. Therefore, we will limit ourselves to equations whose coefficients are regularly varying sequences and examine in detail the strongly increasing and strongly decreasing solutions. Necessary and sufficient conditions for the existence of these solutions as well as their asymptotic representation formulas, will be established, which will allow us to present the complete structure of a set of regularly varying solutions. *Results presented in sections 3.3, 3.4, 3.5 and 3.6 are original and published in the papers [62,63].*

The fourth chapter is dedicated to the complete analyze of positive solutions of the cyclic systems of type (SE) for both positive and negative q_i. For each of the systems, the classification of solutions according to behavior at infinite was done first. After that, the necessary and sufficient conditions for the existence of primitive solutions were determined. These solutions behave asymptotically as a constant function or as a constant function multiplied by another suitable function. To examine the intermediate solutions of the system (SE) when q_i are positive and strongly increasing, strongly decreasing solutions of the system (SE) when q_i are negative, we will assume that the coefficients of those systems are regularly varying sequences. As in the previous chapter, necessary and sufficient conditions for the existence of all possible types of these solutions, as well as their asymptotic representation formulas, will be established. *The whole chapter is based on the original results, among which results presented in Section 4.3 was published in [61].*

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Chapter 1

Introduction

The most practical solution is a good theory.
Albert Einstein (1879 - 1955)

Primary purpose of this chapter is to present some basic definitions and theorems used throughout the thesis. Fixed point theorems and theory of regular variation, as our main tools, are presented in the following sections.

1.1 Basic notations, definitions and theorems

In the beginning, we first define the asymptotic equivalence relation and the dominance relation that will be used through the thesis.

Definition 1.1.1 *Let $f(t)$ and $g(t)$ be two positive functions. The asymptotic equivalence relation $\sim$ of functions f and g is defined as*

$$f(t) \sim g(t),\ t \to \infty \ \Leftrightarrow\ \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

Definition 1.1.2 *Let $f(t)$ and $g(t)$ be two positive functions. The dominance relation $\prec$ between functions f and g is defined as*

$$f(t) \prec g(t),\ t \to \infty \ \Leftrightarrow\ \lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty.$$

To better understand one of our main tools -fixed point theorems, we provide the following basic definitions.

Definition 1.1.3 *A subset S of a normed space X is called convex if, for any $x, y \in S$, $\lambda x + (1-\lambda)y \in S$ for all $\lambda \in [0,1]$.*

Definition 1.1.4 *A subset S of a Banach space X is said to be compact if every sequence of elements of S has a subsequence which converges to an element of S. Set E is relatively compact (or precompact) if its closure is compact.*

Definition 1.1.5 *Let X and Y be two metric spaces, and $\mathcal{F}$ a family of functions from X to Y. The family F is equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in \mathcal{F}$ and all x such that $d(x_0, x) < \delta$. The family is equicontinuous on X if it is equicontinuous at each point of X. The family $\mathcal{F}$ is uniformly equicontinuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in \mathcal{F}$ and all $x_1, x_2 \in \mathcal{F}$ such that $d(x_1, x_2) < \delta$.*

Definition 1.1.6 *The family $\mathcal{F}$ of functions from $C([a,b], \mathbb{R})$ is uniformly bounded on $[a,b]$ if there exists a positive real number K so that $|f(t)| \leq K$ for all $t \in [a,b]$ and all $f \in \mathcal{F}$.*

Fixed point techniques will be used to prove the existence of solutions of equations and systems under consideration. In fact, throughout the thesis next two fixed point theorems will be used.

Theorem 1.1.1 (Knaster-Tarski fixed point theorem) *Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $\mathcal{F} : M \to M$ be an increasing mapping, i.e. $x \geq y$ implies $\mathcal{F}x \geq \mathcal{F}y$. Then $\mathcal{F}$ has a fixed point in M.*

Theorem 1.1.2 (Schauder-Tychonoff fixed point theorem) *Let S be closed, convex, nonempty subset of a locally convex topological vector space X. Let T be a continuous mapping from S to itself, such that TS is relatively compact. Then T has a fixed point.*

To prove that appropriately constructed operator T from the previous theorem is continuous, we will apply the following theorem.

Theorem 1.1.3 (Lebesgue dominated convergence theorem) *Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measurable set S, such that $\lim_{n\to\infty} f_n(x) = f(x)$, almost everywhere on S. Also, let $g(x)$ be an integrable on S, such that $|f_n(x)| \leq g(x)$ almost everywhere on S and for all n. Then*

$$\lim_{n\to\infty} \int_S f_n(x)dx = \int_S f(x)dx.$$

When it comes to main results for differential equations, we use the preceding theorem, while in the proofs of main results for difference equations, we use its discrete analogue.

Theorem 1.1.4 *Let $\{a^{(m)}(k)\}$ be a double real sequence, $a^{(m)}(k) \geq 0$ for $m, k \in \mathbb{N}$ such that $\lim_{m\to\infty} a^{(m)}(k) = A(k)$, for every $k \in \mathbb{N}$. Assume that the series $\sum_{k=1}^{\infty} a^{(m)}(k)$ is totally convergent, that is, there exists a sequence $\{\alpha(k)\}$ such that $a^{(m)}(k) \leq \alpha(k)$ with $\sum_{k=1}^{\infty} \alpha(k) < \infty$. Then, the series $\sum_{k=1}^{\infty} A(k)$ converges and*

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} a^{(m)}(k) = \sum_{k=1}^{\infty} A(k).$$

To apply the Schauder-Tychonoff fixed point theorem, relatively compactness of the set TS must be verified and for that purpose the following statement will be used.

Theorem 1.1.5 (Arzela-Ascoli theorem) *The set S of continuous functions from $C([a,b],\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

As before, in the discrete case we need a discrete version of the Arzela-Ascoli theorem.

Theorem 1.1.6 *A bounded, uniformly Cauchy subset Ω of l^{∞} is relatively compact.*

Our main tools to show the the existence of regularly varying solutions are, besides fixed point theory and theory of regularly varying functions and sequences, presented in the following sections, the generalized L'Hospital rule (see [44]) and Stolz-Cesaro theorem (see [120]).

Theorem 1.1.7 *Let $f, g \in C^1[T,\infty)$. Let*

$$\lim_{t\to\infty} g(t) = \infty \quad \text{and} \quad g'(t) > 0 \quad \text{for all large } t. \tag{1.1.1}$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \leq \liminf_{t\to\infty} \frac{f(t)}{g(t)} \leq \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leq \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

If we replace (1.1.1) *with the condition*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \text{and} \quad g'(t) < 0 \quad \text{for all large } t,$$

then the same conclusion holds.

We recall two variants of the Stolz-Cesaro theorem.

Theorem 1.1.8 *If $f = \{f(n)\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n\to\infty} f(n) = \infty$, then for any sequence $g = \{g(n)\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)} \le \liminf_{n\to\infty} \frac{f(n)}{g(n)} \le \limsup_{n\to\infty} \frac{f(n)}{g(n)} \le \limsup_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)}.$$

In particular, if the sequence $\{\Delta f(n)/\Delta g(n)\}$ has a limit, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)}. \tag{1.1.2}$$

Theorem 1.1.9 *Let $f = \{f(n)\}$, $g = \{g(n)\}$ be sequences of positive real numbers, such that*
(i) $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$;
(ii) the sequence g is strictly monotone;
(iii) the sequence $\{\Delta f(n)/\Delta g(n)\}$ has a limit.
Then, a sequence $\{f(n)/g(n)\}$ is convergent and (1.1.2) *holds.*

1.2 Regularly varying functions

The concept of regular variation was introduced by one of the most frequently cited Serbian mathematicians, Jovan Karamata (1902-1967), in 1930 (see [65]). The application of the theory of regular variation is quite wide, and therefore it can be considered as a chapter of mathematical analysis. The appearance of a monograph of Marić [88] initiated a large number of researchers to apply the theory of regular variation in the study of differential equations of the second or higher orders and some systems, functional differential equations, difference and dynamic ones, as well as some partial differential ones.

We recall that the following definition introduces the set of regularly varying functions of index $\rho \in \mathbb{R}$.

Definition 1.2.1 *A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \textit{for all } \lambda > 0.$$

The totality of all regularly varying functions of the index ρ is denoted by $\mathcal{RV}(\rho)$. In the particular case when $\rho = 0$, we use the notation $\mathcal{SV}$ instead of $\mathcal{RV}(0)$ and refer to members of $\mathcal{SV}$ as *slowly varying functions*. Any function $f \in \mathcal{RV}(\rho)$ is written as $f(t) = t^{\rho} g(t)$ with $g \in SV$, and so the class $\mathcal{SV}$ of slowly varying functions is of fundamental importance in the theory of regular variation. If

$$\lim_{t\to\infty} \frac{f(t)}{t^{\rho}} = \lim_{t\to\infty} g(t) = \text{const} > 0$$

then f is said to be a *trivial* regularly varying function of index ρ and it is denoted by $f \in tr - \mathcal{RV}(\rho)$. Otherwise, $f(t)$ is said to be a *nontrivial* regularly varying function of index ρ and it is denoted by $f \in ntr - \mathcal{RV}(\rho)$. For a complete exposition of regular variation theory and its application to various branches of mathematical analysis, we suggest looking at N.H. Bingham et al. [10] and E. Seneta [116].

We give some examples of regularly varying function. Trivially, function with positive limits (at infinity), in particular, positive constants are slowly varying. Of course, the simplest non-trivial example is $l(x) = \log x$. The iterates $\log\log x$, which will be denoted with $\log_2 x$, $\log_k x = \log\log_{k-1} x$ are also slowly varying as are powers of $\log_k x$, rational functions with positive coefficients formed from the $\log_k x$. Non-logarithmic examples are given by

$$l(x) = \exp\left\{\prod_{k=1}^{N} \left(\log_k x\right)^{\alpha_k}\right\}, \quad 0 < \alpha_k < 1,\ k = \overline{1, N},$$

and

$$l(x) = \exp\left\{\frac{\log x}{\log_2 x}\right\}.$$

An example of a slowly varying function which infinity oscillate, i.e. for which hold

$$\liminf_{x\to\infty} l(x) = 0, \quad \limsup_{x\to\infty} l(x) = \infty$$

is

$$l(x) = \exp\left\{(\log x)^{\frac{1}{3}} \cos\left(\log x\right)^{\frac{1}{3}}\right\}.$$

One of the most important theorem from theory of regular variation in the research of differential equations is *Karamata's integration theorem* which gives information about the asymptotic behavior of the integral of a regularly varying function.

Theorem 1.2.1 (Karamata's integration theorem) *Let $l \in \mathcal{SV}$. Then,*

(i) If $\alpha > -1$,

$$\int_a^t s^{\alpha} l(s)\, ds \sim \frac{t^{\alpha+1}\, l(t)}{\alpha+1}, \quad t \to \infty;$$

(ii) If $\alpha < -1$,

$$\int_t^{\infty} s^{\alpha}\, l(s)\, ds \sim -\frac{t^{\alpha+1}\, l(t)}{\alpha+1}, \quad t \to \infty;$$

(iii) If $\alpha = -1$,

$$L^\star(t) = \int_a^t s^{-1}\, l(s)\, ds,\ \ L^\star \in \mathcal{SV}$$

and

$$L_\star(t) = \int_t^\infty s^{-1}\, l(s)\, ds,\ \ L_\star \in \mathcal{SV}.$$

In order to properly describe the possible asymptotic behavior of nonoscillatory solutions of the self-adjoint second-order linear differential equation

$$(p(t)x'(t))' + q(t)x(t) = 0,$$

which are essentially affected by the function p, Jaroš and Kusano introduced in [45] the class of generalized Karamata functions with the following definition.

Let $R(t)$ be a positive function which is continuously differentiable on (a,∞) and satisfies $R'(t) > 0,\ t > a$ and $\lim\limits_{t\to\infty} R(t) = \infty$.

Definition 1.2.2 *A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be regularly varying of index $\rho \in \mathbb{R}$ with respect to R if $f \circ R^{-1}$ is defined for all large t and is regularly varying function of index ρ in the sense of Karamata, where R^{-1} denotes the inverse function of R.*

The symbol $\mathcal{RV}_R(\rho)$ is used to denote the totality of regularly varying functions of index $\rho \in \mathbb{R}$ with respect to R the symbol $\mathcal{SV}_R$ is often used for $\mathcal{RV}_R(0)$.

It is easy to see that for generalised regularly varying function hold similar properties. Namely, if $f \in \mathcal{RV}_R(\rho)$, then $f(t) = R(t)^\rho\, g(t),\ g \in \mathcal{SV}_R$. If

$$\lim_{t\to\infty} \frac{f(t)}{R(t)^\rho} = \lim_{t\to\infty} g(t) = \text{const} > 0$$

then f is said to be a *trivial* regularly varying function of index ρ with respect to R and it is denoted by $f \in tr - \mathcal{RV}_R(\rho)$. Otherwise, f is said to be a *nontrivial* regularly varying function of index ρ with respect to R and it is denoted by $f \in ntr - \mathcal{RV}_R(\rho)$. Also, from Definition 1.2.2 it follows that $f \in \mathcal{RV}_R(\rho)$ if and only if it is written in the form $f(t) = g(R(t)),\ g \in \mathcal{RV}(\rho)$. It is clear that $\mathcal{RV}(\rho) = \mathcal{RV}_t(\rho)$. We emphasize that there exists a function which is regularly varying in generalized sense, but is not regularly varying in the sense of Karamata, so that, roughly speaking, the class of generalized Karamata functions is larger than that of classical Karamata functions. Indeed, if we denote

$$\exp_n t = \exp(\exp_{n-1} t), \quad \exp_0 t = t,$$

then

$$2 + \sin(\exp_2 t) \notin SV, \quad 2 + \sin(\exp_2 t) \in SV_{\exp_5 t}.$$

Let we present here some elementary properties of generalized regularly varying functions.

Proposition 1.2.1 *(i) If $g_1 \in \mathcal{RV}_R(\sigma_1)$, then $g_1^\alpha \in \mathcal{RV}_R(\alpha\sigma_1)$ for any $\alpha \in \mathbb{R}$.*

(ii) If $g_i \in \mathcal{RV}_R(\sigma_i)$, $i = 1, 2$, then $g_1 + g_2 \in \mathcal{RV}_R(\sigma)$, $\sigma = \max(\sigma_1, \sigma_2)$.

(iii) If $g_i \in \mathcal{RV}_R(\sigma_i)$, $i = 1, 2$, then $g_1 g_2 \in \mathcal{RV}_R(\sigma_1 + \sigma_2)$.

(iv) If $g_i \in \mathcal{RV}_R(\sigma_i)$, $i = 1, 2$ and $g_2 \to \infty$ as $t \to \infty$, then $g_1 \circ g_2 \in \mathcal{RV}_R(\sigma_1\sigma_2)$.

(v) If $l \in \mathcal{SV}_R$, then for any $\varepsilon > 0$,

$$\lim_{t\to\infty} R(t)^{\varepsilon} l(t) = \infty, \qquad \lim_{t\to\infty} R(t)^{-\varepsilon} l(t) = 0.$$

Here, also, we present a fundamental result (see [45]), called *Generalized Karamata integration theorem*, which will be used throughout the Chapter 2 and play a central role in establishing main results for solutions of differential equations of fourth-order.

Theorem 1.2.2 (Generalized Karamata integration theorem) *Let $f \in \mathcal{SV}_R$. Then,*

(i) *If $\alpha > -1$,*

$$\int_a^t R'(s)R(s)^{\alpha} f(s)\, ds \sim \frac{R(t)^{\alpha+1} f(t)}{\alpha + 1}, \quad t \to \infty;$$

(ii) *If $\alpha < -1$,*

$$\int_t^{\infty} R'(s)\, R(s)^{\alpha}\, f(s)\, ds \sim -\frac{R(t)^{\alpha+1} f(t)}{\alpha + 1}, \quad t \to \infty;$$

(iii) *If $\alpha = -1$,*

$$r^{\star}(t) = \int_a^t R'(s)\, R(s)^{-1} f(s)\, ds, \;\; r^{\star} \in \mathcal{SV}_R$$

and

$$r_{\star}(t) = \int_t^{\infty} R'(s)\, R(s)^{-1} f(s)\, ds, \;\; r_{\star} \in \mathcal{SV}_R.$$

1.3 Regularly varying sequences

Let turn our attention now to regularly varying sequences. There are two main approaches in the basic theory of regularly varying sequences: the approach due to Karamata [64], based on a definition that can be understood as a direct discrete counterpart of simple and elegant continuous definition (see Definition 1.2.1), and the approach due to Galambos and Seneta, based on purely sequential definition.

Definition 1.3.1 (Karamata [64]) A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{k\to\infty} \frac{y([\lambda\, k])}{y(k)} = \lambda^{\rho} \quad \textit{for} \quad \forall\lambda > 0,$$

where $[u]$ denotes the integer part of u.

Definition 1.3.2 (Galambos and Seneta [41]) A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha(k)\}$ satisfying

$$\lim_{k\to\infty} \frac{y(k)}{\alpha(k)} = C,\ 0 < C < \infty \qquad \lim_{k\to\infty} k\,\frac{\Delta\alpha(k-1)}{\alpha(k)} = \rho\,.$$

If $\rho = 0$, then y is said to be *slowly varying*. The totality of regularly varying sequences of the index ρ and slowly varying sequences denoted, respectively, by $\mathcal{RV}(\rho)$ and $\mathcal{SV}$.

Bojanić and Seneta have shown in [11] that Definition 1.3.1 and Definition 1.3.2 are equivalent.

The concept of normalized regularly varying sequences was introduced by Matucci and Rehak in [97], where they also offered a modification of Definition 1.3.2, i.e. they proved that the second limit in Definition 1.3.2 can be replaced with

$$\lim_{k\to\infty} k\,\frac{\Delta\alpha(k)}{\alpha(k)} = \rho\,.$$

Definition 1.3.3 A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *normalized regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y(k)}{y(k)} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence.*

In what follows, $\mathcal{NRV}(\rho)$ and $\mathcal{NSV}$ will be used to denote the set of all normalized regularly varying sequences of the index ρ and the set of all normalized slowly varying sequences.

Typical examples are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^{\rho}\log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}\,.$$

In order to present results for a system of difference equations, we need to define a regularly varying vector $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, where ${}^{\mathbb{N}}\mathbb{R} = \{f \mid f : \mathbb{N} \to \mathbb{R}\}$.

Definition 1.3.4 *A vector* $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, $\mathbf{x} = (\{x_1(n)\}, \ldots, \{x_N(n)\})$ *is said to be regularly varying of index* $(\rho_1, \rho_2, \ldots, \rho_N)$ *if* $x_i = \{x_1(n)\} \in \mathcal{RV}(\rho_i)$ *for* $i = \overline{1, N}$. *If all* ρ_i *are positive (or negative), then* $\mathbf{x}$ *is called regularly varying vector sequence of positive (or negative) index* $(\rho_1, \rho_2, \ldots, \rho_N)$. *The set of all regularly varying vectors of index* $(\rho_1, \rho_2, \ldots, \rho_N)$ *is denoted by* $\mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$.

A various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying was established (see [11, 41, 96, 97]) and consequently, each one of them may be used to define regularly varying sequence. The one that is the most important is the following Representation theorem (see [11, Theorem 3]), while some other representation formula for regularly varying sequences was established in [97, Lemma 1].

Theorem 1.3.1 (Representation theorem) *A positive sequence* $\{y(k)\}, k \in \mathbb{N}$ *is said to be regularly varying of index* $\rho \in \mathbb{R}$ *if and only if there exists sequences* $\{c(k)\}$ *and* $\{\delta(k)\}$ *such that*

$$\lim_{k\to\infty} c(k) = c_0 \in (0, \infty) \quad \textit{and} \quad \lim_{k\to\infty} \delta(k) = 0,$$

and

$$y(k) = c(k)\, k^{\rho}\, \exp\left(\sum_{i=1}^{k} \frac{\delta(i)}{i} \right).$$

In [11] very useful embedding theorem was proved, which gives the possibility of using the continuous theory in developing a theory of regularly varying sequences. However, as noticed in [11], such development is not generally close and sometimes far from simple imitation of arguments for regularly varying functions.

Theorem 1.3.2 (Embedding Theorem) *If* $y = \{y(n)\}$ *is regularly varying sequence of index* $\rho \in \mathbb{R}$, *then function* $Y(t)$ *defined on* $[0, \infty)$ *by* $Y(t) = y([t])$ *is a regularly varying function of index* ρ. *Conversely, if* $Y(t)$ *is a regularly varying function on* $[0, \infty)$ *of index* ρ, *then a sequence* $\{y(k)\}$, $y(k) = Y(k)$, $k \in \mathbb{N}$ *is regularly varying of index* ρ.

Next, we state some important properties of $\mathcal{RV}$ sequences useful for the development of the asymptotic behavior of solutions of (E) in the subsequent section (for more properties and proofs see [11, 96]).

Theorem 1.3.3 *Following properties hold:*

(i) $y \in \mathcal{RV}(\rho)$ *if and only if* $y(k) = k^{\rho}\, l(k)$, *where* $l = \{l(k)\} \in \mathcal{SV}$.

(ii) Let $x \in \mathcal{RV}(\rho_1)$ *and* $y \in \mathcal{RV}(\rho_2)$. *Then,* $xy \in \mathcal{RV}(\rho_1 + \rho_2)$, $x + y \in \mathcal{RV}(\rho)$, $\rho = \max\{\rho_1, \rho_2\}$ *and* $1/x \in \mathcal{RV}(-\rho_1)$.

(iii) *If* $y \in \mathcal{RV}(\rho)$, *then* $\lim_{k\to\infty} \dfrac{y(k+1)}{y(k)} = 1$.

(iv) *If* $l \in \mathcal{SV}$ *and* $l(k) \sim L(k)$, $k \to \infty$, *then,* $L \in \mathcal{SV}$.

(v) *If* $l \in \mathcal{SV}$, *then for any* $\varepsilon > 0$,

$$\lim_{k\to\infty} k^{\varepsilon} l(k) = \infty, \qquad \lim_{k\to\infty} k^{-\varepsilon} l(k) = 0,$$

i.e. if $y \in \mathcal{RV}(\rho)$, *then*

$$\lim_{k\to\infty} k^{-\sigma} y(k) = \infty, \qquad \text{for every } \sigma < \rho$$

and

$$\lim_{k\to\infty} k^{-\mu} y(k) = 0, \qquad \text{for every } \mu > \rho\,.$$

(vi) $y \in \mathcal{RV}(\rho)$ *if and only if for every* $\sigma < \rho$ *and for every* $\nu > \rho$

$$\max_{1\le k\le n}\Big(k^{-\sigma} y(k)\Big) \sim n^{-\sigma} y(n) \quad \text{and} \quad \inf_{k\ge n}\Big(k^{-\sigma} y(k)\Big) \sim n^{-\sigma} y(n) \quad \text{as } n \to \infty\,,$$

$$\min_{1\le k\le n}\Big(k^{-\nu} y(k)\Big) \sim n^{-\nu} y(n) \quad \text{and} \quad \sup_{k\ge n}\Big(k^{-\nu} y(k)\Big) \sim n^{-\nu} y(n) \quad \text{as } n \to \infty\,.$$

(vii) *If* $y \in \mathcal{NRV}(\rho)$, *then* $\{n^{-\sigma} y(n)\}$ *is eventually increasing for each* $\sigma < \rho$ *and* $\{n^{-\mu} y(n)\}$ *is eventually decreasing for each* $\mu > \rho$.

In view of the statement (*i*) of the previous theorem, if for $y \in \mathcal{RV}(\rho)$

$$\lim_{k\to\infty} \frac{y(k)}{k^{\rho}} = \lim_{k\to\infty} l(k) = \text{const} > 0,$$

then $y = \{y(n)\}$ is said to be a *trivial regularly varying sequence of the index* ρ and is denoted by $y \in tr - \mathcal{RV}(\rho)$. Otherwise, y is said to be a *nontrivial regularly varying sequence of the index* ρ, denoted by $y \in ntr - \mathcal{RV}(\rho)$.

The next theorem can be found in [7] for normalized regularly varying sequences, but it clearly holds for all regularly varying sequences because its proof is based on the Mean Value Theorem and property (*iii*) from Theorem 1.3.3, which holds for all $\mathcal{RV}$ sequences (not only for $\mathcal{NRV}$).

Theorem 1.3.4 *If* $f = \{f(n)\} \in \mathcal{RV}$ *is a strictly decreasing sequence, such that* $\lim_{n\to\infty} f(n) = 0$, *then for each* $\gamma \in \mathbb{R}$

$$\lim_{n\to\infty} f(n)^{-\gamma} \sum_{k=n}^{\infty} f(k)^{\gamma-1}\Big(-\Delta f(k)\Big) = \frac{1}{\gamma}. \tag{1.3.1}$$

If $g = \{g(n)\} \in \mathcal{RV}$ is a strictly increasing sequence such that $\lim_{n\to\infty} g(n) = \infty$, then

$$\lim_{n\to\infty} g(n)^{-\gamma}\sum_{k=1}^{n-1} g(k)^{\gamma-1}\Delta g(k) = \frac{1}{\gamma}. \tag{1.3.2}$$

The next inequality, which directly follows by Bernoulli's inequality, will be used in the proof of next theorem.

Lemma 1.3.1 *For all $n \in \mathbb{N}$ and $\alpha < -1$ the following inequality holds*

$$\frac{n^{\alpha+1} - (n-1)^{\alpha+1}}{\alpha+1} \le n^{\alpha} \le \frac{(n+1)^{\alpha+1} - n^{\alpha+1}}{\alpha+1}.$$

The following theorem can be seen as *the discrete analog of the Karamata's integration theorem* and plays a central role in the proving this thesis's main results. We prove some parts (which are missed) in [62]. Also, some parts of this theorem's proof can be found in [11] and [114].

Theorem 1.3.5 *Let $l = \{l(n)\} \in \mathcal{SV}$.*

(i) *If* $\alpha > -1$, *then* $\displaystyle\lim_{n\to\infty}\frac{1}{n^{\alpha+1}l(n)}\sum_{k=1}^{n} k^{\alpha} l(k) = \frac{1}{1+\alpha}$;

(ii) *If* $\alpha < -1$, *then* $\displaystyle\lim_{n\to\infty}\frac{1}{n^{\alpha+1}l(n)}\sum_{k=n}^{\infty} k^{\alpha} l(k) = -\frac{1}{1+\alpha}$;

(iii) *If* $\displaystyle\sum_{k=1}^{\infty}\frac{l(k)}{k} < \infty$, *then* $\displaystyle S_\star(n) = \sum_{k=n}^{\infty}\frac{l(k)}{k}$, $S_\star \in \mathcal{SV}$ *and* $\displaystyle\lim_{n\to\infty}\frac{S_\star(n)}{l(n)} = \infty$;

(iv) *If* $\displaystyle\sum_{k=1}^{\infty}\frac{l(k)}{k} = \infty$, *then* $\displaystyle S^\star(n) = \sum_{k=1}^{n}\frac{l(k)}{k}$, $S^\star \in \mathcal{SV}$ *and* $\displaystyle\lim_{n\to\infty}\frac{S^\star(n)}{l(n)} = \infty$.

Proof. (i) See [11, Theorem 6].

(ii) Assume that $l \in \mathcal{SV}$ and let $\alpha < -1$. Choose $\nu > 0$ such that $\alpha < -\nu - 1$. Then, by Lemma 1.3.1

$$\begin{aligned}\sum_{k=n}^{\infty} k^{\alpha} l(k) &= \sum_{k=n}^{\infty} k^{\alpha+\nu}k^{-\nu}l_k \le \sup_{k\ge n}\Big(k^{-\nu}l(k)\Big)\sum_{k=n}^{\infty}k^{\alpha+\nu}\\ &\le \sup_{k\ge n}\Big(k^{-\nu}l(k)\Big)\sum_{k=n}^{\infty}\frac{(k+1)^{\alpha+\nu+1}-k^{\alpha+\nu+1}}{\alpha+\nu+1} = \sup_{k\ge n}\Big(k^{-\nu}l(k)\Big)\left(-\frac{n^{1+\alpha+\nu}}{1+\alpha+\nu}\right)\end{aligned}$$

implying that

$$\frac{1}{n^{\alpha+1}l(n)}\sum_{k=n}^{\infty}k^{\alpha}l(k)\le-\frac{1}{1+\alpha+\nu}\cdot\frac{\sup_{k\ge n}\big(k^{-\nu}l(k)\big)}{n^{-\nu}l(n)}.$$

Then, Theorem 1.3.3-(v) (for $\rho=0$) yields

$$\limsup_{n\to\infty}\frac{1}{n^{\alpha+1}l(n)}\sum_{k=n}^{\infty}k^{\alpha}l(k)\le-\frac{1}{1+\alpha+\nu}\,. \tag{1.3.3}$$

Moreover, if we choose $\sigma<0$ such that $\alpha<-\sigma-1$, we get by using Lemma 1.3.1

$$\sum_{k=n}^{\infty}k^{\alpha}l(k)=\sum_{k=n}^{\infty}k^{\alpha+\sigma}k^{-\sigma}l(k)\ge\inf_{k\ge n}\big(k^{-\sigma}l(k)\big)\sum_{k=n}^{\infty}k^{\alpha+\sigma}\ge\inf_{k\ge n}\big(k^{-\sigma}l(k)\big)\left(-\frac{(n-1)^{1+\alpha+\sigma}}{1+\alpha+\sigma}\right),$$

or

$$\frac{1}{n^{\alpha+1}l(n)}\sum_{k=n}^{\infty}k^{\alpha}l(k)\ge-\frac{1}{1+\alpha+\sigma}\cdot\left(1-\frac{1}{n}\right)^{1+\alpha+\sigma}\frac{\inf_{k\ge n}\big(k^{-\sigma}l(k)\big)}{n^{-\sigma}l(n)}.$$

Thus, by Theorem 1.3.3-(v) (for $\rho=0$)

$$\liminf_{n\to\infty}\frac{1}{n^{\alpha+1}l(n)}\sum_{k=n}^{\infty}k^{\alpha}l(k)\ge-\frac{1}{1+\alpha+\sigma}\,. \tag{1.3.4}$$

Finally, from (1.3.3) and (1.3.4), since $\nu>0$ and $\sigma<0$ can be chosen arbitrarily close to zero

$$\lim_{n\to\infty}\frac{1}{n^{\alpha+1}l(n)}\sum_{k=n}^{\infty}k^{\alpha}l(k)=-\frac{1}{1+\alpha}\,.$$

(iii) From Theorem 1.3.2, for $l\in\mathcal{SV}$ we have that the function $l([x])$, $x\ge1$ is slowly varying. Also,

$$S_{\star}(n)=\sum_{k=n}^{\infty}\frac{l(k)}{k}=\int_n^{\infty}\frac{l([x])}{[x]}dx=\int_n^{\infty}\frac{l([x])\cdot\frac{x}{[x]}}{x}dx, \tag{1.3.5}$$

for every $n\in\mathbb{N}$. Because the sum $\sum_{k=1}^{\infty}\frac{l(k)}{k}$ is convergent, the integral $\int_1^{\infty}\frac{L(x)}{x}dx$ is convergent too, where $L(x)=l([x])\cdot\frac{x}{[x]}$, $x\ge1$. Also, L is slowly varying function, because $L(x)\sim l([x])$, $x\to\infty$, which follows from fact that $x\sim[x]$, when $x\to\infty$. Now, from Karamata's integration theorem (Theorem 1.2.1) function Y defined as $Y(x)=\int_x^{\infty}\frac{L(t)}{t}dt$, $x\ge1$ is slowly varying, so the restriction $y=\{y(n)\}$ of function Y on the set of naturals numbers, where $y(n)=\int_n^{\infty}\frac{L(t)}{t}dt$, $n\in\mathbb{N}$, is slowly varying sequence. From (1.3.5), $S_{\star}$ is slowly varying.

Further, from Karamata's integration theorem follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{l(n)}\cdot\sum_{k=n}^{\infty}\frac{l(k)}{k} &= \lim_{x\to\infty}\frac{1}{l([x])}\cdot\int_{[x]}^{\infty}\frac{l([t])}{[t]}dt = \lim_{x\to\infty}\frac{1}{l([x])\cdot\frac{x}{[x]}}\cdot\int_{[x]}^{\infty}\frac{l([t])\cdot\frac{t}{[t]}}{t}dt \\
&= \lim_{x\to\infty}\frac{1}{L(x)}\cdot\int_{x}^{\infty}\frac{L(t)}{t}dt = \infty.
\end{aligned}$$

(*iv*) Similarly as in (*iii*). □

Remark 1.3.1 It is easy to see, in view of Theorem 1.3.3-(*iii*) and Theorem 1.3.5-(*i*), that for $l \in \mathcal{SV}$, if $\alpha > -1$, we have

$$\sum_{k=1}^{n-1} k^{\alpha} l(k) \sim \frac{(n-1)^{\alpha+1} l(n-1)}{\alpha+1} \sim \frac{n^{\alpha+1} l(n)}{\alpha+1} \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty,$$

and since $\lim_{n\to\infty}\sum_{k=1}^{n-1} k^{\alpha} l(k) = \infty$, we also get

$$\sum_{k=n_0}^{n} k^{\alpha} l(k) \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty.$$

If $\lim_{n\to\infty}\sum_{k=1}^{n} k^{-1} l(k) = \infty$, we have

$$\sum_{k=n_0}^{n} k^{-1} l(k) \sim \sum_{k=1}^{n} k^{-1} l(k), \qquad n \to \infty.$$

Chapter 2

Fourth order nonlinear differential equations

2.1 Introduction

In this chapter we are going to study the equation

$$(E_1) \qquad \Big(p(t)|x''(t)|^{\alpha-1}x''(t)\Big)'' + q(t)|x(t)|^{\beta-1}x(t) = 0, \quad t \ge a > 0,$$

where

(i) α and β are positive constants such that $\alpha > \beta$,

(ii) $p, q : [a, \infty) \to (0, \infty)$ are continuous functions and p satisfies

$$(C) \qquad \int_a^\infty \frac{t^{1+(1/\alpha)}}{p(t)^{1/\alpha}}\, dt < \infty.$$

In the existing literature (see [57,60,86,108,109,137,138]) the existence and asymptotic behavior of eventually positive solutions as well as oscillation criteria for the equation (E_1), was discussed depending on the convergence or divergence of the following integrals

$$I_1 = \int_a^\infty \frac{t}{p(t)^{1/\alpha}}\, dt, \qquad I_2 = \int_a^\infty \left(\frac{t}{p(t)}\right)^{1/\alpha} dt\,.$$

Through classification of eventually positive solutions, it has been established that there are classes of so-called primitive solutions satisfying $x(t) \sim \phi_i(t),\ t \to \infty$, for precisely defined functions $\phi_i(t)$, $i \in \{1,2,3,4\}$ (commonly in some integral form depending on the coefficient of the equation (E_1)). Existence of primitive solutions has been fully characterized by necessary and sufficient conditions. The other main objective in these papers was to establish necessary and/or sufficient conditions for oscillation of all solutions of (E_1). However, the existence of so-called intermediate solutions (satisfying e.g. $\phi_1(t) \prec x(t) \prec \phi_2(t)$, $t \to \infty$) was not considered in any of the mentioned papers, until recently in [77, 102]. In fact, the existence and asymptotic representations of intermediate solutions of (E_1), under assumptions $I_1 = \infty$, $I_2 = \infty$, was studied in the framework of regular variation by Kusano, Manojlović, Milošević in [77] and Milošević, Manojlović in [102]. Recently, intermediate solutions of (E_1) was considered under assumptions $I_1 < \infty, I_2 = \infty$, in [32].

The oscillatory and asymptotic behavior of solutions of (E_1) under the condition (C) was already considered in [78, 82]. Kusano and Tanigawa in [82] performed a complete classification of eventually positive solutions and established necessary and sufficient conditions for the existence of four types of primitive solutions. Unlike primitive solutions, establishing necessary and sufficient conditions for the existence of the intermediate solutions seems to be much more difficult task. Thus, only sufficient conditions for the existence of these solutions was obtained in [82]. Afterwards, sharp oscillation criteria (establishing necessary and sufficient conditions for oscillation of all solutions) was obtained by Kusano, Manojlović and Tanigawa in [78].

In this chapter, motivated by papers [77, 78, 82, 102], considering open problem of obtaining necessary and sufficient conditions for (E_1) to possess two types of intermediate solutions, our task is to solve this problem and moreover to determine precisely asymptotic behavior at infinity of these two types of intermediate solutions. Since this problem is very difficult for the equation (E_1) with general continuous coefficients p and q, we solve the problem in the framework of regular variation, that is, limiting ourselves to the case where p and q are regularly varying functions and placing attention on regularly varying solutions. In fact, to make clear the dependence of asymptotic behavior of intermediate solutions on the condition (C), we use theory of generalized regularly varying functions (or generalized Karamata functions). Thereafter, we show that the problem of getting necessary and sufficient conditions for the existence of intermediate solutions which are regularly varying in the sense of Karamata, can be embedded in the framework of generalized regularly varying functions. The obtained results, combined with existing results on the existence of primitive solutions of (E_1) (Theorems 2.2.1-2.2.4), enable us to finally present the full structure of the set of regularly varying solutions for the equation (E_1) with regularly varying coefficients.

Sections 2.3, 2.4, 2.5 are based on the original results published in [132].

2.2 Classification of positive solutions

Definition 2.2.1 *Function $x : [T, \infty) \to \mathbb{R}, T \geq a$ is a solution of (E_1) if and only if it is twice continuously differentiable together with $p|x''|^{\alpha-1}x''$ on $[T, \infty)$ and satisfies the equation (E_1) at every point in $[T, \infty)$.*

A solution x of (E_1) is said to be *nonoscillatory* if there exists $T \geq a$ such that $x(t) \neq 0$ for all $t \geq T$ and *oscillatory* otherwise. It is clear that, if x is a solution of (E_1), then so does $-x$, and so in studying nonoscillatory solutions of (E_1) it suffices to restrict our attention to its (eventually) positive solutions. The equation (E_1) is called *sub-half-linear* if $\beta < \alpha$ and *super-half-linear* if $\beta > \alpha$.

Kusano and Tanigawa in [82] made a detailed classification of all positive solutions of the equation (E_1) under the condition (C) and established conditions for the existence of such solutions. It was proved that the following four types of combination of the signs of x', x'' and $\left(p|x''|^{\alpha-1}x''\right)'$ are possible for an eventually positive solution x of (E_1):

$$(p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, \quad x''(t) > 0, \quad x'(t) > 0 \quad \text{for all large } t, \tag{2.2.1}$$

$$(p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, \quad x''(t) > 0, \quad x'(t) < 0 \quad \text{for all large } t, \tag{2.2.2}$$

$$(p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, \quad x''(t) < 0, \quad x'(t) > 0 \quad \text{for all large } t, \tag{2.2.3}$$

$$(p(t)|x''(t)|^{\alpha-1}x''(t))' < 0, \quad x''(t) < 0, \quad x'(t) > 0 \quad \text{for all large } t. \tag{2.2.4}$$

In order to describe all positive solutions of (E_1), a special role is played by the four functions

$$\varphi_1(t) = \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}}\,ds, \quad \varphi_2(t) = \int_t^\infty (s-t)\left(\frac{s}{p(s)}\right)^{1/\alpha} ds, \quad \psi_1(t) = 1, \quad \psi_2(t) = t,$$

which are particular solutions of the unperturbed differential equation

$$(p(t)|x''(t)|^{\alpha-1}x''(t))'' = 0.$$

Note that functions φ_i and ψ_i, $i = 1, 2$ defined above satisfy the dominance relation

$$\varphi_1(t) \prec \varphi_2(t) \prec \psi_1(t) \prec \psi_2(t), \quad t \to \infty.$$

As a result of further analysis of the four types of solutions mentioned above, Kusano and Tanigawa in [82] have shown that the following six types are possible for the asymptotic behavior of positive solutions of (E_1):

(P1) $x(t) \sim c_1\varphi_1(t)$, as $t \to \infty$,

(P2) $x(t) \sim c_2\varphi_2(t)$ as $t \to \infty$, (P3) $x(t) \sim c_3$ as $t \to \infty$,

(P4) $x(t) \sim c_4 t$ as $t \to \infty$,

(I1) $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ as $t \to \infty$,

(I2) $1 \prec x(t) \prec t$ as $t \to \infty$,

where $c_i > 0, i = 1, 2, 3, 4$ are constants. Positive solutions of (E_1) having the asymptotic behavior (P1)–(P4) are collectively called *primitive positive solutions* of the equation (E_1), while the solutions having the asymptotic behavior (I1) and (I2) are referred to as *intermediate solutions* of the equation (E_1).

The interrelation between the types (2.2.1)-(2.2.4) of the derivatives of solutions and the types (P1)–(P4), (I1) and (I2) of the asymptotic behavior of solutions is as follows:

(i) All solutions of a type (2.2.1) have the asymptotic behavior of type (P1);

(ii) A solution of type (2.2.2) has the asymptotic behavior of one of the types (P1), (P2), (P3) and (I1);

(iii) A solution of type (2.2.3) has the asymptotic behavior of one of the types (P3) and (P4);

(iv) A solution of type (2.2.4) has the asymptotic behavior of one of the types (P3), (P4) and (I2).

The existence of four types of primitive solutions has been completely characterized for both sub-half-linear and super-half-linear case of (E_1) with continuous coefficients p and q as the following theorems proved in [82] show.

Theorem 2.2.1 *Let $p, q \in C[a, \infty)$. The equation (E_1) has a positive solution x satisfying* (P3) *if and only if*

$$\mathcal{J}_1 = \int_a^\infty t \left(\frac{1}{p(t)} \int_a^t (t-s) q(s)\, ds \right)^{1/\alpha} dt < \infty. \tag{2.2.5}$$

Theorem 2.2.2 *Let $p, q \in C[a, \infty)$. The equation (E_1) has a positive solution x satisfying* (P4) *if and only if*

$$\mathcal{J}_2 = \int_a^\infty \left(\frac{1}{p(t)} \int_a^t (t-s) s^\beta q(s)\, ds \right)^{1/\alpha} dt < \infty. \tag{2.2.6}$$

Theorem 2.2.3 *Let $p, q \in C[a,\infty)$. The equation (E_1) has a positive solution x satisfying* (P1) *if and only if*

$$\mathcal{J}_3 = \int_a^\infty tq(t)\varphi_1(t)^\beta \, dt < \infty. \tag{2.2.7}$$

Theorem 2.2.4 *Let $p, q \in C[a,\infty)$. The equation (E_1) has a positive solution x satisfying* (P2) *if and only if*

$$\mathcal{J}_4 = \int_a^\infty q(t)\varphi_2(t)^\beta \, dt < \infty. \tag{2.2.8}$$

Next two theorems, proved in [82], give sufficient conditions for the existence of intermediate solutions.

Theorem 2.2.5 *If* (2.2.8) *holds and if*

$$\mathcal{J}_3 = \int_a^\infty tq(t)\varphi_1(t)^\beta \, dt = \infty,$$

then the equation (E_1) has a intermediate type positive solution satisfying $(I1)$.

Theorem 2.2.6 *If* (2.2.6) *holds and*

$$\mathcal{J}_1 = \int_a^\infty t\left(\frac{1}{p(t)}\int_a^t (t-s)q(s)\,ds\right)^{1/\alpha} dt = \infty,$$

then (E_1) has a intermediate type positive solution satisfying $(I2)$.

Further, sharp conditions for the oscillation of all solutions of (E_1) in both cases (sub-half-linear and super-half-linear) have been obtained in [78].

Theorem 2.2.7 *Let $\beta < 1 \le \alpha$. All solutions of (E_1) are oscillatory if and only if*

$$\mathcal{J}_2 = \infty\,.$$

Theorem 2.2.8 *Let $\alpha \le 1 < \beta$. All solutions of (E_1) are oscillatory if and only if*

$$\mathcal{J}_3 = \infty\,.$$

2.3 Asymptotic behavior of intermediate generalized regularly varying solutions

In what follows it is always assumed that functions p and q are generalized regularly varying of index η and σ with respect to R, with R defined with

$$R(t) = \left(\int_t^\infty \frac{s^{1+\frac{1}{\alpha}}}{p(s)^{1/\alpha}}\, ds \right)^{-1}, \tag{2.3.1}$$

and expressed as

$$p(t) = R(t)^\eta l_p(t),\ l_p \in \mathcal{SV}_R \quad \text{and} \quad q(t) = R(t)^\sigma l_q(t),\ l_q \in \mathcal{SV}_R\,. \tag{2.3.2}$$

From (2.3.1) and (2.3.2) we have that

$$t^{1+\frac{1}{\alpha}} = R'(t)R(t)^{\frac{\eta}{\alpha}-2} l_p(t)^{1/\alpha}. \tag{2.3.3}$$

Integrating (2.3.3) from a to t we have

$$\frac{t^{2+\frac{1}{\alpha}}}{2+\frac{1}{\alpha}} = \int_a^t R'(s)R(s)^{\frac{\eta}{\alpha}-2} l_p(s)^{1/\alpha} ds, \quad t \to \infty, \tag{2.3.4}$$

implying that $\frac{\eta}{\alpha} \geq 1$. In what follows, we limit ourselves to the case where $\eta > \alpha$ excludes other possibilities because of computational difficulty. Applying the generalized Karamata integration theorem (Theorem 1.2.2) at the right-hand side of (2.3.4) we obtain

$$t \sim \left(\frac{\eta-\alpha}{2\alpha+1} \right)^{-\frac{\alpha}{2\alpha+1}} R(t)^{\frac{\eta-\alpha}{2\alpha+1}} l_p(t)^{\frac{1}{2\alpha+1}}, \quad t \to \infty. \tag{2.3.5}$$

From (2.3.3) and (2.3.5) we can express $R'(t)$ as follows

$$R'(t) \sim \left(\frac{\eta-\alpha}{2\alpha+1} \right)^{-\frac{\alpha+1}{2\alpha+1}} R(t)^{\frac{3\alpha+1-\eta}{2\alpha+1}} l_p(t)^{-\frac{1}{2\alpha+1}}, \quad t \to \infty\,, \tag{2.3.6}$$

which can be rewritten in the form

$$1 \sim \left(\frac{\eta-\alpha}{2\alpha+1} \right)^{\frac{\alpha+1}{2\alpha+1}} R'(t)R(t)^{m_2(\alpha,\eta)-1} l_p(t)^{\frac{1}{2\alpha+1}}, \quad t \to \infty. \tag{2.3.7}$$

The next lemma, following directly from the generalized Karamata integration theorem using (2.3.7), will be frequently used in our later discussions. To that end and to further simplifying formulation of our main results, we introduce the notation:

$$m_1(\alpha,\eta) = \frac{-2\alpha^2-\eta}{\alpha(2\alpha+1)}, \quad m_2(\alpha,\eta) = \frac{\eta-\alpha}{2\alpha+1}. \tag{2.3.8}$$

It is clear that $m_1(\alpha,\eta) < -1 < 0 < m_2(\alpha,\eta)$ and

$$\text{(2.3.9)} \qquad \text{(i)} \quad m_1(\alpha,\eta) = 2m_2(\alpha,\eta) - \frac{\eta}{\alpha}; \quad \text{(ii)} \quad \frac{m_2(\alpha,\eta)-\eta}{\alpha} = -2m_2(\alpha,\eta) - 1\,.$$

In our main results constants $m_i(\alpha,\eta)$, $i = 1,2$, will be abbreviated as m_i, $i = 1,2$, respectively.

Lemma 2.3.1 *Let $f(t) = R(t)^{\mu}L_f(t)$, $L_f \in \mathcal{SV}_R$. Then:*

(i) If $\mu > -m_2(\alpha,\eta)$,

$$\int_a^t f(s)\,ds \sim \frac{m_2(\alpha,\eta)^{\frac{\alpha+1}{2\alpha+1}}}{\mu + m_2(\alpha,\eta)}\, R(t)^{\mu+m_2(\alpha,\eta)}L_f(t)l_p(t)^{\frac{1}{2\alpha+1}}, \quad t\to\infty;$$

(ii) If $\mu < -m_2(\alpha,\eta)$,

$$\int_t^\infty f(s)\,ds \sim \frac{m_2(\alpha,\eta)^{\frac{\alpha+1}{2\alpha+1}}}{-(\mu + m_2(\alpha,\eta))} R(t)^{\mu+m_2(\alpha,\eta)}L_f(t)l_p(t)^{\frac{1}{2\alpha+1}}, \quad t\to\infty;$$

(iii) If $\mu = -m_2(\alpha,\eta)$, then functions

$$\int_a^t f(s)\,ds = \int_a^t R(s)^{-m_2(\alpha,\eta)}L_f(s)\,ds,$$
$$\int_t^\infty f(s)\,ds = \int_t^\infty R(s)^{-m_2(\alpha,\eta)}L_f(s)\,ds$$

are slowly varying with respect to R.

To make an in-depth analysis of intermediate solutions of type (I1) and (I2) of (E_1) we need a fair knowledge of the structure of the functions ψ_1,ψ_2,φ_1 and φ_2 regarded as generalized regularly varying functions with respect to R. It is clear that $\psi_1 \in \mathcal{SV}_R$ and from (2.3.5) we see that $\psi_2 \in \mathcal{RV}_R(m_2(\alpha,\eta))$. Using (2.3.2) and applying Lemma 2.3.1 twice, we obtain

$$\text{(2.3.10)} \qquad \begin{aligned} \varphi_1(t) &= \int_t^\infty\int_s^\infty R(r)^{-\eta/\alpha}l_p(r)^{-1/\alpha}\,dr\,ds \\ &\sim \frac{m_2(\alpha,\eta)^{\frac{2(\alpha+1)}{2\alpha+1}}}{m_1(\alpha,\eta)(m_1(\alpha,\eta)-m_2(\alpha,\eta))}R(t)^{m_1(\alpha,\eta)}\, l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}}, \quad t\to\infty, \end{aligned}$$

which shows that $\varphi_1 \in \mathcal{RV}_R\,(m_1(\alpha,\eta))$. Further, by (2.3.2) and (2.3.5), in view of (2.3.9)-(ii), another two applications of Lemma 2.3.1 yield

$$\text{(2.3.11)} \qquad \begin{aligned} \varphi_2(t) &\sim m_2(\alpha,\eta)^{-\frac{1}{2\alpha+1}}\int_t^\infty\int_s^\infty R(r)^{-2m_2(\alpha,\eta)-1}\, l_p(r)^{-\frac{2}{2\alpha+1}}\,dr\,ds \\ &\sim \frac{m_2(\alpha,\eta)}{m_2(\alpha,\eta)+1}R(t)^{-1}, \quad t\to\infty, \end{aligned}$$

implying $\varphi_2 \in \mathcal{RV}_R(-1)$.

2.3.1 Intermediate solutions of type (I1)

The first subsection is devoted to the study of the existence and asymptotic behavior of generalized regularly varying solutions with respect to R of type (I1) with p and q satisfying (2.3.2). Expressing such a solution x of (E_1) in the form

$$x(t) = R(t)^{\rho}\, l_x(t), \quad l_x \in \mathcal{SV}_R, \tag{2.3.12}$$

since $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$, $t \to \infty$, the regularity index ρ of x must satisfy

$$m_1(\alpha, \eta) \leq \rho \leq -1.$$

If $\rho = m_1(\alpha,\eta)$, then x is a member of $\mathcal{RV}_R(m_1(\alpha,\eta))$, while if $\rho = -1$, then since $x(t)/\varphi_2(t) = c \cdot l_x(t) \to 0$, $t \to \infty$, where c is a real constant, x is a member of $ntr-\mathcal{RV}_R(-1)$. Thus the set of all generalized regularly varying solutions of type (I1) will be divided into the three disjoint classes

$$\begin{gathered} \mathcal{RV}_R(m_1(\alpha,\eta)) \quad \text{or} \\ \mathcal{RV}_R(\rho) \ \text{ with } \rho \in (m_1(\alpha,\eta)\,,\ -1) \quad \text{or} \quad ntr-\mathcal{RV}_R\,(-1)\,. \end{gathered} \tag{2.3.13}$$

Our aim is to establish necessary and sufficient conditions for each of the above classes to have a member and furthermore to show that the asymptotic behavior of all members of each class is governed by a unique explicit formula describing the decay order at infinity accurately.

Let x be a solution of (E_1) on $[t_0,\infty)$ such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ as $t \to \infty$. Since

$$\lim_{t\to\infty}\Big(p(t)(x''(t))^{\alpha}\Big)' = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} p(t)(x''(t))^{\alpha} = \infty, \tag{2.3.14}$$

integrating (E_1) first on $[t,\infty)$, and then on $[t_0,t]$ and finally twice on $[t,\infty)$, we obtain

$$x(t) = \int_t^{\infty} \frac{s-t}{p(s)^{1/\alpha}} \left(\xi_2 + \int_{t_0}^{s}\int_r^{\infty} q(u)x(u)^{\beta}\, du\, dr\right)^{1/\alpha} ds, \quad t \geq t_0, \tag{2.3.15}$$

where $\xi_2 = p(t_0)x''(t_0)^{\alpha}$.

To prove the existence of intermediate solutions of type (I1) it is sufficient to prove the existence of a positive solution of the integral equation (2.3.15) for some constants $t_0 \geq a$ and $\xi_2 > 0$, which is most commonly achieved by the application of Schauder-Tychonoff fixed point theorem. Denoting by $\mathcal{G}x(t)$ the right-hand side of (2.3.15), to find a fixed point of $\mathcal{G}$ it is crucial to choose a closed convex subset $\mathcal{X} \subset C[t_0,\infty)$ on which $\mathcal{G}$ is a self-map. Since our primary goal is not only proving the existence of generalized $\mathcal{RV}$ intermediate solutions, but establishing a precise asymptotic formula for such solutions, a choice of a subset $\mathcal{X}$ must be made appropriately. It will be shown that such a choice of $\mathcal{X}$ is possible by solving the integral asymptotic relation

$$x(t) \sim \int_t^{\infty} \frac{s-t}{p(s)^{1/\alpha}} \left(\int_b^{s}\int_r^{\infty} q(u)x(u)^{\beta}\, du\, dr\right)^{1/\alpha} ds, \quad t \to \infty, \tag{2.3.16}$$

for some $b \geq t_0$, which can be considered as an approximation (at infinity) of (2.3.15) in the sense that it is satisfied by all possible solutions of type (I1) of (E_1). Theory of regular variation will in fact ensure the solvability of (2.3.16) in the framework of generalized Karamata functions.

Main results for the intermediate solutions of type $(I1)$ are listed below and completely characterize the membership of each of the three classes of solutions given in (2.3.13).

Theorem 2.3.1 *Let $p \in \mathcal{RV}_R(\eta), q \in \mathcal{RV}_R(\sigma)$. The equation (E_1) has intermediate solutions $x \in \mathcal{RV}_R(m_1)$ satisfying* (I1) *if and only if*

$$\sigma = -\beta m_1 - 2m_2 \quad \text{and} \quad \int_a^\infty t q(t) \varphi_1(t)^\beta \, dt = \infty. \tag{2.3.17}$$

The asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim X_1(t) = \varphi_1(t) \left(\frac{\alpha - \beta}{\alpha} \int_a^t s q(s) \varphi_1(s)^\beta \, ds \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.18}$$

Theorem 2.3.2 *Let $p \in \mathcal{RV}_R(\eta), q \in \mathcal{RV}_R(\sigma)$. The equation (E_1) has intermediate solutions $x \in \mathcal{RV}_R(\rho)$ with $\rho \in (m_1, -1)$ if and only if*

$$-\beta m_1 - 2m_2 < \sigma < \beta - m_2, \tag{2.3.19}$$

in which case ρ is given by

$$\rho = \frac{\sigma + m_2 - \alpha}{\alpha - \beta} \tag{2.3.20}$$

and the asymptotic behavior of any such solution x is given by the unique formula

$$x(t) \sim X_2(t) = \left(\left(\frac{m_2^{\frac{(\alpha+1)^2}{2\alpha+1}}}{\alpha} \right)^2 \frac{p(t)^{\frac{1}{2\alpha+1}} q(t) R(t)^{-2\frac{\alpha(\alpha+1)}{2\alpha+1}}}{(m_1 - \rho)(\rho + 1)(\rho(\rho - m_2))^\alpha} \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.21}$$

Theorem 2.3.3 *Let $p \in \mathcal{RV}_R(\eta)$, $q \in \mathcal{RV}_R(\sigma)$. The equation (E_1) has intermediate solutions $x \in ntr - \mathcal{RV}_R(-1)$ satisfying* (I1) *if and only if*

$$\sigma = \beta - m_2 \quad \text{and} \quad \int_a^\infty q(t) \varphi_2(t)^\beta \, dt < \infty. \tag{2.3.22}$$

The asymptotic behavior of any such solution x is given by the unique formula

$$x(t) \sim X_3(t) = \varphi_2(t) \left(\frac{\alpha - \beta}{\alpha} \int_t^\infty q(s) \, \varphi_2(s)^\beta \, ds \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.23}$$

As preparatory steps toward the proofs of main results, we show that functions $X_i, i = 1,2,3$ defined by (2.3.18), (2.3.21) and (2.3.23) are generalized $\mathcal{RV}$-functions satisfying the asymptotic relation (2.3.16).

Lemma 2.3.2 *Suppose that* (2.3.17) *holds. The function* X_1 *given by* (2.3.18) *satisfies the asymptotic relation* (2.3.16) *for any* $b \geq a$ *and* $X_1 \in \mathcal{RV}_R(m_1)$.

Proof. From (2.3.2), (2.3.5) and (2.3.10), we have

$$tq(t)\varphi_1(t)^\beta \sim \frac{m_2^{\frac{2\beta(\alpha+1)-\alpha}{2\alpha+1}}}{(m_1(m_1-m_2))^\beta} R(t)^{\sigma+\beta m_1+m_2} l_p(t)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(t), \quad t\to\infty,$$

and applying (iii) of Lemma 2.3.1, in view of (2.3.17), we obtain

$$\int_a^t sq(s)\varphi_1(s)^\beta\, ds \sim \frac{m_2^{\frac{2\beta(\alpha+1)-\alpha}{2\alpha+1}}}{(m_1(m_1-m_2))^\beta} \int_a^t R(s)^{-m_2} l_p(s)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(s)\, ds \in \mathcal{SV}_R, \tag{2.3.24}$$

as $t \to \infty$, which together with (2.3.18) gives

$$X_1(t) \sim \varphi_1(t)\left(\frac{m_2^{\frac{2\beta(\alpha+1)-\alpha}{2\alpha+1}}}{(m_1(m_1-m_2))^\beta}\frac{\alpha-\beta}{\alpha}J_1(t)\right)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty,$$

where

$$J_1(t) = \int_a^t R(s)^{-m_2} l_p(s)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(s)\, ds. \tag{2.3.25}$$

Thus, since $J_1 \in \mathcal{SV}_R$, we conclude that $X_1 \in \mathcal{RV}_R(m_1(\alpha,\eta))$ and rewrite the previous relation, using (2.3.10), as

$$X_1(t) \sim R(t)^{m_1} l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} \left(\left(\frac{m_2}{m_1(m_1-m_2)}\right)^\alpha \frac{\alpha-\beta}{\alpha} J_1(t)\right)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty. \tag{2.3.26}$$

To prove that (2.3.16) is satisfied by X_1, we first integrate $q(t)X_1(t)^\beta$ on $[t,\infty)$, applying Lemma 2.3.1 and using (2.3.17), we have

$$\int_t^\infty q(s)\, X_1(s)^\beta\, ds \sim m_2^{-\frac{\alpha}{2\alpha+1}} \left(\left(\frac{m_2}{m_1(m_1-m_2)}\right)^\alpha \frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} R(t)^{-m_2} l_p(t)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(t) J_1(t)^{\frac{\beta}{\alpha-\beta}},$$

as $t \to \infty$. Integrating the above relation on $[b,t]$, for any $b \geq a$, we obtain

$$\begin{aligned}\int_b^t \int_s^\infty q(r)\, X_1(r)^\beta \, dr\, ds &\sim m_2^{-\frac{\alpha}{2\alpha+1}} \left(\left(\frac{m_2}{m_1(m_1-m_2)} \right)^\alpha \frac{\alpha-\beta}{\alpha} \right)^{\frac{\beta}{\alpha-\beta}} \\ &\quad \times \int_b^t R(s)^{-m_2} l_p(s)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(s) J_1(s)^{\frac{\beta}{\alpha-\beta}}\, ds \\ &= m_2^{-\frac{\alpha}{2\alpha+1}} \left(\left(\frac{m_2}{m_1(m_1-m_2)} \right)^\alpha \frac{\alpha-\beta}{\alpha} \right)^{\frac{\beta}{\alpha-\beta}} \int_b^t J_1(s)^{\frac{\beta}{\alpha-\beta}}\, dJ_1(s) \\ &= m_2^{-\frac{\alpha}{2\alpha+1}} \left(\left(\frac{m_2}{m_1(m_1-m_2)} \right)^\beta \frac{\alpha-\beta}{\alpha} \right)^{\frac{\alpha}{\alpha-\beta}} J_1(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty.\end{aligned}$$

Applying Lemma 2.3.1 and using (2.3.9)-(i), we obtain

$$\begin{aligned}&\int_t^\infty \int_s^\infty \left(\frac{1}{p(r)} \int_a^r \int_u^\infty q(\omega) X_1(\omega)^\beta \, d\omega du \right)^{1/\alpha} dr\, ds \\ &\sim \left(\left(\frac{m_2}{m_1(m_1-m_2)} \right)^\beta \frac{\alpha-\beta}{\alpha} \right)^{\frac{1}{\alpha-\beta}} \frac{m_2}{m_1(m_1-m_2)} R(t)^{m_1} l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} J_1(t)^{\frac{1}{\alpha-\beta}},\end{aligned}$$

as $t \to \infty$, which due to (2.3.26) proves that X_1 satisfies the desired asymptotic relation (2.3.16) for any $b \geq a$. □

Lemma 2.3.3 *Suppose that* (2.3.19) *holds and let ρ be defined by* (2.3.20). *The function X_2 given by* (2.3.21) *satisfies the asymptotic relation* (2.3.16) *for any $b \geq a$ and belongs to $\mathcal{RV}_R(\rho)$.*

Proof. Using (2.3.8), (2.3.9)-(i) and (2.3.20) we obtain

$$\sigma + \rho\beta + m_2 = \alpha(\rho+1), \quad \sigma + \rho\beta + 2m_2 = \alpha(\rho - m_1) = \rho - 2m_2 + \frac{\eta}{\alpha}. \tag{2.3.27}$$

The function X_2 given by (2.3.21) can be expressed in the form

$$X_2(t) \sim (\lambda\alpha^2)^{-\frac{1}{\alpha-\beta}} m_2^{\frac{2(\alpha+1)^2}{(2\alpha+1)(\alpha-\beta)}} R(t)^\rho \left(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t) \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty, \tag{2.3.28}$$

where

$$\lambda = (\rho(\rho - m_2))^\alpha (m_1 - \rho)(\rho+1).$$

Thus, $X_2 \in \mathcal{RV}_R(\rho)$. Using (2.3.27) and (2.3.28), applying Lemma 2.3.1 twice, we find

$$\int_t^\infty q(s)\, X_2(s)^\beta \, ds \sim -\frac{m_2^{\frac{(\alpha+1)(2\alpha\beta+\alpha+\beta)}{(2\alpha+1)(\alpha-\beta)}}}{(\lambda\alpha^2)^{\frac{\beta}{\alpha-\beta}} (\sigma + \rho\beta + m_2)} R(t)^{\sigma+\rho\beta+m_2} \left(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t) \right)^{\frac{\alpha}{\alpha-\beta}},$$

and

$$\int_b^t \int_s^\infty q(r)\, X_2(r)^\beta \, dr\, ds$$

$$\sim \frac{m_2^{\frac{2\alpha(\alpha+1)(\beta+1)}{(2\alpha+1)(\alpha-\beta)}}}{\left(\lambda\alpha^2\right)^{\frac{\beta}{\alpha-\beta}}(-(\sigma+\rho\beta+m_2))(\sigma+\rho\beta+2m_2)} R(t)^{\sigma+\rho\beta+2m_2}\Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\Big)^{\frac{1}{\alpha-\beta}}$$

$$= \frac{m_2^{\frac{2\alpha(\alpha+1)(\beta+1)}{(2\alpha+1)(\alpha-\beta)}}}{\left(\lambda\alpha^2\right)^{\frac{\beta}{\alpha-\beta}}\alpha^2(\rho+1)(m_1-\rho)} R(t)^{\alpha(\rho-2m_2+\frac{\eta}{\alpha})}\Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\Big)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty,$$

for any $b \geq a$. Therefore,

$$\left(\frac{1}{p(t)}\int_b^t\int_r^\infty q(u)\, X_2(u)^\beta\, du\, dr\right)^{1/\alpha}$$

$$\sim \frac{m_2^{\frac{2(\alpha+1)(\beta+1)}{(\alpha-\beta)(2\alpha+1)}}}{(\lambda\alpha^2)^{\frac{\beta}{\alpha(\alpha-\beta)}}\,(\alpha^2(m_1-\rho)(\rho+1))^{1/\alpha}} R(t)^{\rho-2m_2}\left(l_p(t)^{\frac{2\beta-2\alpha+1}{2\alpha+1}} l_q(t)\right)^{\frac{1}{\alpha-\beta}},$$

and the integration of the previous relation over $[t,\infty)$ twice, with the application of Lemma 2.3.1, gives

$$\int_t^\infty\int_s^\infty \left(\frac{1}{p(r)}\int_b^r\int_u^\infty q(\omega)\, X_2(\omega)^\beta\, d\omega du\right)^{1/\alpha} dr\, ds$$

$$\sim \frac{m_2^{\frac{2(\alpha+1)^2}{(\alpha-\beta)(2\alpha+1)}}}{\left(\lambda\alpha^2\right)^{\frac{\beta}{\alpha-\beta}}\rho(\rho-m_2)(\alpha^2(m_1-\rho)(\rho+1))^{1/\alpha}} R(t)^\rho\left(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t)\right)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty.$$

This, due to (2.3.28), completes the proof of Lemma 2.3.3. □

Lemma 2.3.4 *Suppose that* (2.3.22) *holds. Then the function* X_3 *given by* (2.3.23) *satisfies the asymptotic relation* (2.3.16) *for any* $b \geq a$ *and* $X_3 \in ntr - \mathcal{RV}_R(-1)$.

Proof. Using (2.3.2), (2.3.11), (2.3.22) and applying (iii) of Lemma 2.3.1, we obtain

$$\int_t^\infty q(s)\,\varphi_2(s)^\beta\, ds \sim \left(\frac{m_2}{m_2+1}\right)^\beta J_3(t), \quad t\to\infty, \tag{2.3.29}$$

where

$$J_3(t) = \int_t^\infty R(s)^{-m_2} l_q(s)\, ds, \quad J_3 \in \mathcal{SV}_R. \tag{2.3.30}$$

Thus, expression (2.3.23) is of the form

$$X_3(t) \sim \left(\frac{m_2}{m_2+1}\right)^{\frac{\alpha}{\alpha-\beta}} R(t)^{-1} \left(\frac{\alpha-\beta}{\alpha} J_3(t)\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.31}$$

From (2.3.30) and (2.3.31), using (2.3.14), we see that $X_3 \in ntr - \mathcal{RV}_R(-1)$. Next, we integrate $q(t)\, X_3(t)^\beta$ on $[t, \infty)$ and using (2.3.22), we obtain

$$\begin{aligned}
\int_t^\infty q(s)\, X_3(s)^\beta\, ds &\sim \left(\frac{m_2}{m_2+1}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_t^\infty R(s)^{-m_2} l_q(s)\, J_3(s)^{\frac{\beta}{\alpha-\beta}} ds \\
&= \left(\frac{m_2}{m_2+1}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_t^\infty J_3(s)^{\frac{\beta}{\alpha-\beta}} (-dJ_3(s)) \\
&= \left(\frac{m_2}{m_2+1}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} J_3(t)^{\frac{\alpha}{\alpha-\beta}} \in \mathcal{SV}_R, \quad t \to \infty.
\end{aligned}$$

Further, integrating previous relation on $[b, t]$ for any fixed $b \geq a$, by Lemma 2.3.1, we have

$$\int_b^t \int_s^\infty q(r) X_3(r)^\beta\, dr\, ds \sim \left(\frac{m_2}{m_2+1}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} m_2^{-\frac{\alpha}{2\alpha+1}} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} J_3(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty.$$

As a result of the application of Lemma 2.3.1, with the help of (2.3.9)-(ii), we obtain

$$\begin{aligned}
&\int_t^\infty \left(\frac{1}{p(s)} \int_b^s \int_r^\infty q(u) X_3(u)^\beta\, du\, dr\right)^{1/\alpha} ds \\
&\quad \sim \left(\frac{m_2}{m_2+1}\right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} \frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_2+1} R(t)^{-m_2-1} l_p(t)^{-\frac{\alpha}{\alpha(2\alpha+1)}} J_3(t)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^\infty \int_s^\infty \left(\frac{1}{p(r)} \int_b^r \int_u^\infty q(\omega) X_3(\omega)^\beta\, d\omega du\right)^{1/\alpha} dr\, ds \\
&\quad \sim \left(\frac{m_2}{m_2+1}\right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} \frac{m_2}{m_2+1} R(t)^{-1} J_3(t)^{\frac{1}{\alpha-\beta}} \sim X_3(t), \quad t \to \infty,
\end{aligned}$$

which in view of (2.3.31), completes the proof of Lemma 2.3.4. □

Now, we are ready to prove main results.

Proof of the "only if" part of Theorems 2.3.1, 2.3.2 and 2.3.3: Suppose that (E_1) has a type (I1) intermediate solution $x \in \mathcal{RV}_R(\rho)$ on $[t_0, \infty)$. Clearly, $\rho \in [m_1, -1]$. Using (2.3.2) and (2.3.12), we obtain integrating (E_1) on $[t, \infty)$

$$(p(t)(x''(t))^\alpha)' = \int_t^\infty q(s)x(s)^\beta\, ds = \int_t^\infty R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta\, ds. \tag{2.3.32}$$

Noting that the last integral is convergent, we conclude that $\sigma + \beta\rho + m_2 \leq 0$ and distinguish two cases:

$$(1) \quad \sigma + \beta\rho + m_2 = 0 \quad \text{and} \quad (2) \quad \sigma + \beta\rho + m_2 < 0.$$

Assume that (1) holds. By Lemma 2.3.1-(iii) function S_3 defined with

$$S_3(t) = \int_t^\infty R(s)^{-m_2} l_q(s) l_x(s)^\beta\, ds, \tag{2.3.33}$$

is slowly varying with respect to R, and according to (2.3.14) and (2.3.32) follows that $\lim_{t\to\infty} S_3(t) = 0$. Integration of (2.3.32) on $[t_0, t]$ shows that

$$p(t)(x''(t))^\alpha \sim m_2^{-\frac{\alpha}{2\alpha+1}} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} S_3(t), \quad t \to \infty, \tag{2.3.34}$$

which is rewritten using (2.3.9)-(ii) as

$$x''(t) \sim m_2^{-\frac{1}{2\alpha+1}} R(t)^{-2m_2-1} l_p(t)^{-\frac{2}{2\alpha+1}} S_3(t)^{1/\alpha}, \quad t \to \infty.$$

Integrability of x'' on $[t, \infty)$, and $-m_2 - 1 < 0$, allows us to integrate the previous relation on $[t, \infty)$, implying

$$-x'(t) \sim \frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_2+1} R(t)^{-m_2-1} l_p(t)^{-\frac{1}{2\alpha+1}} S_3(t)^{1/\alpha}, \quad t \to \infty,$$

which we may integrate once more on $[t, \infty)$, to obtain

$$x(t) \sim \frac{m_2}{m_2+1} R(t)^{-1} S_3(t)^{1/\alpha}, \quad t \to \infty. \tag{2.3.35}$$

Since S_3 tends to zero, this shows that $x \in ntr - \mathcal{RV}_R(-1)$. Thus, $\rho = -1$ and from (1) $\sigma = -\beta - m_2$.

Assume next that (2) holds. From (2.3.32) we find that

$$(p(t)(x''(t))^\alpha)' \sim -\frac{m_2^{\frac{\alpha+1}{2\alpha+1}}}{\sigma+\beta\rho+m_2} R(t)^{\sigma+\beta\rho+m_2} l_p(t)^{\frac{1}{2\alpha+1}} l_q(t) l_x(t)^\beta, \quad t \to \infty,$$

which by integration on $[t_0, t]$ implies

$$p(t)(x''(t))^\alpha \sim -\frac{m_2^{\frac{\alpha+1}{2\alpha+1}}}{\sigma+\beta\rho+m_2}\int_{t_0}^{t} R(s)^{\sigma+\beta\rho+m_2} l_p(s)^{\frac{1}{2\alpha+1}} l_q(s) l_x(s)^\beta ds, \tag{2.3.36}$$

as $t \to \infty$. In view of (2.3.14), the integral on right-hand side is divergent, so $\sigma + \beta\rho + 2m_2 \geq 0$. We distinguish the cases:

$$\text{(2.a)} \quad \sigma+\beta\rho+2m_2 = 0 \quad \text{and} \quad \text{(2.b)} \quad \sigma+\beta\rho+2m_2 > 0.$$

Assume that (2.a) holds. Denote by

$$S_1(t) = \int_{t_0}^{t} R(s)^{-m_2} l_p(s)^{\frac{1}{2\alpha+1}} l_q(s) l_x(s)^\beta ds\,. \tag{2.3.37}$$

Then $S_1 \in \mathcal{SV}_R$ and using (2.3.2) we rewrite (2.3.36) as

$$x''(t) \sim m_2^{-\frac{1}{2\alpha+1}} R(t)^{-\eta/\alpha} l_p(t)^{-1/\alpha} S_1(t)^{1/\alpha}, \quad t \to \infty. \tag{2.3.38}$$

Because of integrability of x'' on $[t, \infty]$ and the fact that $-\frac{\eta}{\alpha} + m_2 = m_1 - m_2 < 0$, via Lemma 2.3.1, we conclude by integration of (2.3.38) on $[t, \infty]$ that

$$-x'(t) \sim -\frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_1 - m_2} R(t)^{m_1-m_2}\, l_p(t)^{-\frac{\alpha+1}{\alpha(2\alpha+1)}} S_1(t)^{1/\alpha}, \quad t \to \infty,$$

which because of integrability of x' on $[t, \infty)$ and $m_1 < 0$, we may integrate once more on $[t, \infty)$ to get

$$x(t) \sim \frac{m_2}{m_1(m_1-m_2)} R(t)^{m_1} l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} S_1(t)^{1/\alpha}, \quad t \to \infty. \tag{2.3.39}$$

The last relation implies that $x \in \mathcal{RV}_R(m_1)$. Therefore, $\rho = m_1$ and from (2.a) $\sigma = -\beta m_1 - m_2$.

Assume that (2.*b*) holds. From (2.3.36), the application of Lemma 2.3.1 gives

$$p(t)(x''(t))^\alpha \sim -\frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}}}{(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2)} R(t)^{\sigma+\beta\rho+2m_2} l_p(t)^{\frac{2}{2\alpha+1}} l_q(t) l_x(t)^\beta,$$

as $t \to \infty$, which yields

$$x''(t) \sim \frac{m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}} R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} l_p(t)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha},\ t \to \infty.$$

Integrability of x'' on $[t,\infty]$ allows us to integrate the previous relation on $[t,\infty)$, implying

$$
\begin{aligned}
-x'(t) \sim\ & \frac{m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}} \\
& \times \int_t^\infty R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} l_p(s)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \quad t\to\infty,
\end{aligned} \tag{2.3.40}
$$

where $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2 \le 0$, because of the convergence of the last integral. We distinguish two cases:

$$
\text{(2.b.1)}\quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2=0 \quad \text{and} \quad \text{(2.b.2)}\quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2<0.
$$

The case (2.b.1) is impossible because the left-hand side of (2.3.40) is integrable on $[t_0,\infty)$, while the right-hand side is not, because it is slowly varying with respect to R.

Assume now that (2.b.2) holds. Then, the application of Lemma (2.3.1) in (2.3.40) and integration of resulting relation on $[t,\infty)$ leads to

$$
\begin{aligned}
x(t) \sim -\ & \frac{m_2^{\frac{(\alpha+1)(\alpha+2)}{\alpha(2\alpha+1)}}}{(-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2)} \\
& \times \int_t^\infty R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \quad t\to\infty,
\end{aligned} \tag{2.3.41}
$$

which brings us to the observation of two possible cases:

$$
\text{(2.b.2.1)}\quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2=0 \ \text{ and } \ \text{(2.b.2.2)}\quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2<0.
$$

In the case (2.b.2.1) the integral on the right-hand side of the relation (2.3.41) is slowly varying with respect to R by Lemma 2.3.1 and so $x \in \mathcal{SV}_R$. This is impossible because $\rho \in [m_1,-1]$.

In the case (2.b.2.2) the application of Lemma 2.3.1 gives

$$
\begin{aligned}
x(t) \sim\ & m_2^{\frac{2(\alpha+1)^2}{\alpha(2\alpha+1)}} \Bigg((-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha} \\
& \times \left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2\right)\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2\right)\Bigg)^{-1} \\
& \times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2} l_p(t)^{\frac{1}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha}, \quad t\to\infty,
\end{aligned} \tag{2.3.42}
$$

implying that $x \in \mathcal{RV}_R(\rho)$, where ρ satisfies

$$\rho = \frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + 2m_2. \tag{2.3.43}$$

The last relation implies that the index of regularity of x is given by (2.3.20).

Suppose that x is a type (I1) solution of (E_1) belonging to $\mathcal{RV}_R(m_1)$. From the above observations this is possible only when (2.a) holds, in which case (2.3.39) is satisfied. Thus, $\rho = m_1$ and $\sigma = -m_1\beta - 2m_2$. Using $x(t) = R(t)^{m_1} l_x(t)$, (2.3.39) can be expressed as

$$l_x(t) \sim K_1 l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} S_1(t)^{1/\alpha}, \quad t \to \infty, \ \text{ where } K_1 = \frac{m_2}{m_1(m_1 - m_2)}, \tag{2.3.44}$$

and S_1 is defined by (2.3.37). Then (2.3.44) is transformed into the differential asymptotic relation for S_1:

$$S_1(t)^{-\frac{\beta}{\alpha}} \, S_1'(t) \sim K_1^{\beta} R(t)^{-m_2} l_p(t)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(t), \quad t \to \infty. \tag{2.3.45}$$

From (2.3.10) and (2.3.39), since $\lim_{t\to\infty} x(t)/\varphi_1(t) = \infty$, we have $\lim_{t\to\infty} S_1(t) = \infty$. Therefore, integrating (2.3.45) on $[t_0, t]$, in view of the notation (2.3.25) and the fact $J_1 \in \mathcal{SV}_R$, we find that the second condition in (2.3.17) is satisfied and

$$S_1(t)^{1/\alpha} \sim \left(\frac{\alpha - \beta}{\alpha} K_1^{\beta} J_1(t) \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,$$

implying with (2.3.44) that

$$x(t) \sim R(t)^{m_1} l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} \left(\frac{\alpha - \beta}{\alpha} K_1^{\alpha} J_1(t) \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.46}$$

In the proof of Lemma 2.3.2, using (2.3.2), (2.3.5) and (2.3.10), we have obtained an expression (2.3.26) for X_1 given by (2.3.18). Thus, (2.3.46) in fact proves that $x(t) \sim X_1(t)$, $t \to \infty$, completing the "only if" part of the proof of Theorem 2.3.1.

Next, suppose that x is a solution of (E_1) belonging to $\mathcal{RV}_R(\rho)$, $\rho \in (m_1, -1)$. This is possible only when (2.b.2.2) holds, in which case x satisfies the asymptotic relation (2.3.42). Therefore, ρ satisfies (2.3.43) which justifies (2.3.20). An elementary calculation shows that

$$m_1 < \rho < -1 \implies -\beta m_1 - 2m_2 < \sigma < \beta - m_2,$$

which determines the range (2.3.19) for σ. In view of (2.3.27) and (2.3.43), we conclude from (2.3.42) that x enjoys the asymptotic behavior $x(t) \sim X_2(t)$, $t \to \infty$, where X_2 is given by (2.3.21). This proves the "only if" part of the Theorem 2.3.2.

Finally, suppose that x is a type (I1) intermediate solution of (E_1) belonging to $ntr-\mathcal{RV}_R(-1)$. Then, the case (1) is the only possibility for x, which means that $\sigma = \beta - m_2$ and (2.3.35) is satisfied by x, with S_3 defined by (2.3.33). Using $x(t) = R(t)^{-1} l_x(t)$, (2.3.35) can be expressed as

$$\text{(2.3.47)} \qquad l_x(t) \sim K_3\, S_3(t)^{1/\alpha},\ t \to \infty, \quad \text{where } K_3 = \frac{m_2}{m_2+1},$$

implying the differential asymptotic relation

$$\text{(2.3.48)} \qquad -S_3(t)^{-\frac{\beta}{\alpha}}\, S_3'(t) \sim K_3^{\beta} R(t)^{-m_2} l_q(t), \quad t \to \infty.$$

From (2.3.11) and (2.3.35), since $\lim_{t\to\infty} x(t)/\varphi_2(t) = 0$, we have $\lim_{t\to\infty} S_3(t) = 0$, implying that the left-hand side of (2.3.48) is integrable over $[t_0,\infty)$. This, in view of (2.3.30), implies the second condition in (2.3.22). Integrating (2.3.48) on $[t,\infty)$ with the use of (2.3.47), yields

$$x(t) \sim R(t)^{-1}\Big(\frac{\alpha-\beta}{\alpha} K_3^{\alpha} J_3(t)\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,$$

which due to the expression (2.3.31) gives $x(t) \sim X_3(t)$ as $t \to \infty$. This proves the "only" if part of Theorem 2.3.3.

Proof of the part "if" of Theorems 2.3.1, 2.3.2 and 2.3.3: Suppose that (2.3.17) or (2.3.19) or (2.3.22) holds. From Lemmas 2.3.2, 2.3.3 and 2.3.4 it is known that X_i, $i = 1,2,3$, defined by (2.3.18), (2.3.21) and (2.3.23) satisfy the asymptotic relation (2.3.16) for any $b \geq a$. We perform the simultaneous proof for X_i, $i = 1,2,3$ so the subscripts $i = 1,2,3$ will be deleted in the rest of the proof. Let us denote

$$\text{(2.3.49)} \qquad I(t,a,\xi_2) = \int_t^{\infty} \frac{s-t}{p(s)^{1/\alpha}} \left(\xi_2 + \int_a^s \int_r^{\infty} q(u)\, X(u)^{\beta}\, du\, dr\right)^{1/\alpha} ds, \quad t \geq a$$

where ξ_2 is an arbitrary fixed positive constant. It is clear that

$$\text{(2.3.50)} \qquad I(t,a,\xi_2) \sim \int_t^{\infty} \frac{s-t}{p(s)^{1/\alpha}} \left(\int_a^s \int_r^{\infty} q(u)\, X(u)^{\beta}\, du\, dr\right)^{1/\alpha} ds, \quad t \to \infty.$$

Therefore, by (2.3.16) there exist $T_1 \geq T_0 \geq a$ such that

$$\text{(2.3.51)} \qquad I(t,T_0,\xi_2) \leq 2X(t) \quad \text{for} \quad t \geq T_0$$

and

$$\text{(2.3.52)} \qquad \frac{X(t)}{2} \leq I(t,T_0,\xi_2), \quad \text{for} \quad t \geq T_1.$$

Let such T_0 and T_1 be fixed and choose positive constants c, P such that

$$c \le \frac{\varphi_1(t)}{X(t)} \le P, \quad T_0 \le t \le T_1. \tag{2.3.53}$$

Constants c and P exist because a continuous function is bounded on every compact set. Further, choose positive constants m and M such that

$$m \le \min\{c\,{\xi_2}^{1/\alpha},\, 2^{\frac{\alpha}{\beta-\alpha}}\}, \quad M^{1-\frac{\beta}{\alpha}} \ge 2. \tag{2.3.54}$$

Define the integral operator

$$\mathcal{G}x(t) = \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} \left(\xi_2 + \int_{T_0}^s \int_r^\infty q(u)\,x(u)^\beta\,du\,dr \right)^{1/\alpha} ds, \quad t \ge T_0, \tag{2.3.55}$$

and let it act on the set

$$\mathcal{X} = \{x \in C[T_0,\infty) : mX(t) \le x(t) \le M\,X(t),\ t \ge T_0\}. \tag{2.3.56}$$

It is clear that $\mathcal{X}$ is a closed, convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$.

It can be shown that $\mathcal{G}$ is a continuous self-map on $\mathcal{X}$ and that the set $\mathcal{G}(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$.
(i) $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$: Let $x \in \mathcal{X}$. Using (2.3.51), (2.3.54) and (2.3.56) we obtain

$$\begin{aligned}
\mathcal{G}x(t) &\le M^{\beta/\alpha} \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} \left(\frac{\xi_2}{M^\beta} + \int_{T_0}^s \int_r^\infty q(u)\,X(u)^\beta\,du\,dr \right)^{1/\alpha} ds \\
&\le M^{\beta/\alpha} \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} \left(\xi_2 + \int_{T_0}^s \int_r^\infty q(u)\,X(u)^\beta\,du\,dr \right)^{1/\alpha} ds \\
&\le 2M^{\beta/\alpha}\,X(t) \le MX(t), \quad t \ge T_0.
\end{aligned}$$

On the other hand, using (2.3.52), for $t \ge T_1$

$$\begin{aligned}
\mathcal{G}x(t) &\ge \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} \left(\xi_2 + m^\beta \int_{T_0}^s \int_r^\infty q(u)\,X(u)^\beta\,du\,dr \right)^{1/\alpha} ds \\
&= m^{\beta/\alpha} \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} \left(\frac{\xi_2}{m^\beta} + \int_{T_0}^s \int_r^\infty q(u)\,X(u)^\beta\,du\,dr \right)^{1/\alpha} ds \\
&\ge m^{\beta/\alpha} \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} \left(\xi_2 + \int_{T_0}^s \int_r^\infty q(u)\,X(u)^\beta\,du\,dr \right)^{1/\alpha} ds \\
&\ge m^{\beta/\alpha}\,\frac{X(t)}{2} \ge mX(t),
\end{aligned}$$

and using (2.3.53) for $t \in [T_0, T_1]$ we have

$$\mathcal{G}x(t) \geq {\xi_2}^{1/\alpha} \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} = {\xi_2}^{1/\alpha}\, \varphi_1(t) \geq {\xi_2}^{1/\alpha}\, c\, X(t) \geq mX(t).$$

This shows that $\mathcal{G}x \in \mathcal{X}$; that is, $\mathcal{G}$ maps $\mathcal{X}$ into itself.
(ii) $\mathcal{G}(\mathcal{X})$ *is relatively compact:* The inclusion $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$ ensures that $\mathcal{G}(\mathcal{X})$ is locally uniformly bounded on $[T_0, T_2]$, for any $T_2 > T_0$. From (2.3.55), we have

$$(\mathcal{G}x)'(t) = -\int_t^\infty \frac{1}{p(r)^{1/\alpha}} \left(\xi_2 + \int_{T_0}^s \int_r^\infty q(u)\, x(u)^\beta\, du\, dr \right)^{1/\alpha} ds, \quad t \in [T_0, T_2].$$

From the inequality

$$-M^{\beta/\alpha} \int_t^\infty \frac{1}{p(r)^{1/\alpha}} \left(\xi_2 + \int_{T_0}^s \int_r^\infty q(u)\, X(u)^\beta\, du\, dr \right)^{1/\alpha} ds \leq (\mathcal{G}x)'(t) \leq 0,$$

where $t \in [T_0, T_2]$, holding for all $x \in \mathcal{X}$, it follows that $\mathcal{G}(\mathcal{X})$ is locally equicontinuous on $[T_0, T_2] \subset [T_0, \infty)$. Then, the relative compactness of $\mathcal{G}(\mathcal{X})$ follows from the Arzela-Ascoli lemma.
(iii) $\mathcal{G}$ *is continuous on* $\mathcal{X}$: Let $\{x_n\}$ be a sequence in $\mathcal{X}$ converging to $x \in \mathcal{X}$ uniformly on any compact subinterval of $[T_0, \infty)$. Let $T_2 > T_0$ be any fixed real number. From (2.3.55) we have

$$|\mathcal{G}x_n(t) - \mathcal{G}x(t)| \leq \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} G_n(s)\, ds, \quad t \in [T_0, T_2],$$

where

$$G_n(t) = \left| \left(\xi_2 + \int_{T_0}^t \int_r^\infty q(s)\, x_n(s)^\beta\, ds dr \right)^{1/\alpha} - \left(\xi_2 + \int_{T_0}^t \int_r^\infty q(s)\, x(s)^\beta\, ds dr \right)^{1/\alpha} \right|.$$

Using the inequality $|x^\lambda - y^\lambda| \leq |x-y|^\lambda$, $x, y \in \mathbb{R}^+$ holding for $\lambda \in (0,1)$, we see that if $\alpha \geq 1$, then

$$G_n(t) \leq \left(\int_{T_0}^t \int_r^\infty q(s) |x_n(s)^\beta - x(s)^\beta| ds dr \right)^{1/\alpha}.$$

On the other hand, using the mean value theorem, if $\alpha < 1$ we obtain

$$G_n(t) \leq \frac{1}{\alpha} \left(\xi_2 + M^\beta \int_{T_0}^t \int_r^\infty q(s) X(s)^\beta ds dr \right)^{\frac{\alpha-1}{\alpha}} \int_{T_0}^t \int_r^\infty q(s) |x_n(s)^\beta - x(s)^\beta| ds dr.$$

Thus, using that $q(t)\big|x_n(t)^\beta - x(t)^\beta| \to 0$ as $n \to \infty$ at each point $t \in [T_0, \infty)$ and $q(t)\big|x_n(t)^\beta - x(t)^\beta| \le 2M^\beta q(t)X(t)^\beta$ for $t \ge T_0$, while $q(t)X(t)^\beta$ is integrable on $[T_0, \infty)$, the uniform convergence $G_n(t) \rightrightarrows 0$, $n \to \infty$ on $[T_0, \infty)$ follows by the application of the Lebesgue dominated convergence theorem. We conclude that $\mathcal{G}x_n(t) \to \mathcal{G}x(t)$ uniformly on any compact subinterval of $[T_0, \infty)$ as $n \to \infty$, which proves the continuity of $\mathcal{G}$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x \in \mathcal{X}$ of $\mathcal{G}$, which satisfies the integral equation

$$x(t) = \int_t^\infty \frac{s-t}{p(s)} \left(\xi_2 + \int_{T_0}^s \int_r^\infty q(u)\, x(u)^\beta \, du\, dr \right)^{1/\alpha} ds, \quad t \ge T_0.$$

Differentiating the above expression four times shows that x is a solution of (E_1) on $[T_0, \infty)$, which due to (2.3.56) is an intermediate solution of type (I1). Therefore, the proof of our main results will be completed with the verification that the intermediate solution of (E_1) constructed above is actually regularly varying function with respect to R. Consider the function $I(t, T_0, \xi_2)$ for $t \ge T_0$ defined by (2.3.49), and put

$$l = \liminf_{t\to\infty} \frac{x(t)}{I(t, T_0, \xi_2)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{I(t, T_0, \xi_2)}.$$

By Lemmas 2.3.2, 2.3.3 and 2.3.4 and (2.3.50), we have $X(t) \sim I(t, T_0, \xi_2)$, $t \to \infty$. Since, $x \in \mathcal{X}$, it is clear that $0 < l \le L < \infty$. Applying Theorem 1.1.7 four times, we obtain

$$\begin{aligned}
L &\le \limsup_{t\to\infty} \frac{x'(t)}{I'(t, T_0, \xi_2)} = \limsup_{t\to\infty} \frac{\int_t^\infty \left(\frac{1}{p(s)} \left(\xi_2 + \int_{T_0}^s \int_r^\infty q(u)\, x(u)^\beta \, du\, dr \right)\right)^{1/\alpha} ds}{\int_t^\infty \left(\frac{1}{p(s)} \int_{T_0}^s \int_r^\infty q(u)\, X(u)^\beta \, du\, dr \right)^{1/\alpha} ds} \\
&\le \limsup_{t\to\infty} \left(\frac{\xi_2 + \int_t^\infty (s-t) q(s) x(s)^\beta \, ds}{\int_t^\infty (s-t) q(s) X(s)^\beta \, ds} \right)^{1/\alpha} \le \left(\limsup_{t\to\infty} \frac{\int_t^\infty q(s) x(s)^\beta \, ds}{\int_t^\infty q(s) X(s)^\beta \, ds} \right)^{1/\alpha} \\
&\le \left(\limsup_{t\to\infty} \frac{q(t)x(t)^\beta}{q(t)X(t)^\beta} \right)^{1/\alpha} = \left(\limsup_{t\to\infty} \frac{x(t)}{X(t)} \right)^{\beta/\alpha} = \left(\limsup_{t\to\infty} \frac{x(t)}{I(t, T_0, \xi_2)} \right)^{\beta/\alpha} = L^{\beta/\alpha},
\end{aligned}$$

where we have used $X(t) \sim I(t, T_0, \xi_2)$, $t \to \infty$, in the last step. Since $\beta/\alpha < 1$, the inequality $L \le L^{\beta/\alpha}$ implies that $L \le 1$. Similarly, it may be verified that $l \ge 1$, from which it follows that $L = l = 1$, that is,

$$\lim_{t\to\infty} \frac{x(t)}{I(t, T_0, \xi_2)} = 1 \implies x(t) \sim I(t, T_0, \xi_2) \sim X(t), \ t \to \infty.$$

Therefore it is concluded that if $p \in \mathcal{RV}_R(\eta)$ and $q \in \mathcal{RV}_R(\sigma)$, then the type-(I1) solution x under consideration is a member of $\mathcal{RV}_R(\rho)$, where

$$\rho = m_1 \quad \text{or} \quad \rho = \frac{\sigma + m_2 - \alpha}{\alpha - \beta} \in (m_1, -1) \quad \text{or} \quad \rho = -1,$$

according to whether the pair (η,σ) satisfies (2.3.17), (2.3.19) or (2.3.22), respectively. Needless to say, any such solution $x \in \mathcal{RV}_R(\rho)$ enjoys one and the same asymptotic behavior (2.3.18), (2.3.21) or (2.3.23), respectively. This completes the "if" parts of Theorems 2.3.1, 2.3.2 and 2.3.3.

2.3.2 Intermediate solutions of type (I2)

Let us turn our attention to the study of intermediate solutions of type (I2) of the equation (E_1); that is, those solutions x such that $1 \prec x(t) \prec t$ as $t \to \infty$. As in the preceding subsection the use is made of the expressions (2.3.2) and (2.3.12) for the coefficients p, q and solutions x. Since $\psi_1 \in \mathcal{SV}_R$ and $\psi_2 \in \mathcal{RV}_R(m_2(\alpha,\eta))$ (cf. (2.3.7) and (2.3.5)), the regularity index ρ of x must satisfy $0 \le \rho \le m_2(\alpha,\eta)$. If $\rho = 0$, then since $x(t) = l_x(t) \to \infty$, $t \to \infty$, x is a member of $ntr - \mathcal{SV}_R$, while if $\rho = m_2(\alpha,\eta)$, then x is a member of $\mathcal{RV}_R(m_2(\alpha,\eta))$. If $0 < \rho < m_2(\alpha,\eta)$, then x belongs to $\mathcal{RV}_R(\rho)$ and clearly satisfies $x(t) \to \infty$ and $x(t)/R(t)^{m_2(\alpha,\eta)} \to 0$ as $t \to \infty$. Therefore, the totality of type (I2) intermediate solutions of (E_1) is divided into the following three classes

$$ntr - \mathcal{SV}_R, \quad \mathcal{RV}_R(\rho),\ \rho \in (0, m_2(\alpha,\eta)), \quad \mathcal{RV}_R(m_2(\alpha,\eta))$$

and our purpose is to show that, for each of the above classes, necessary and sufficient conditions for the membership can be established and that the asymptotic behavior at infinity of all members of each class is determined precisely by a unique explicit formula.

Let x be a type (I2) intermediate solution of (E_1) defined on $[t_0,\infty)$. It is known that

$$\begin{gathered}\lim_{t\to\infty} x'(t) = 0,\\ \lim_{t\to\infty}(p(t)|x''(t)|^{\alpha-1}x''(t))' = \lim_{t\to\infty} p(t)|x''(t)|^{\alpha-1}x''(t) = \lim_{t\to\infty} x(t) = \infty\,.\end{gathered} \tag{2.3.57}$$

Integrating (E_1) twice on $[t_0,t]$, then on $[t_0,\infty)$ and finally on $[t_0,t]$, we obtain, for $t \ge t_0 \ge a$,

$$x(t) = c_0 + \int_{t_0}^{t}\int_{s}^{\infty} \frac{1}{p(r)^{1/\alpha}}\left(c_2 + c_3(r-t_0) + \int_{t_0}^{r}(r-u)q(u)x(u)^{\beta}\,du\right)^{1/\alpha} dr\,ds, \tag{2.3.58}$$

where $c_0 = x(t_0)$, $c_2 = p(t_0)(-x''(t_0))^{\alpha}$, and $c_3 = (p(t_0)(-x''(t_0))^{\alpha})'$. From (2.3.58) we easily see that x satisfies the integral asymptotic relation

$$x(t) \sim \int_{b}^{t}\int_{s}^{\infty}\left(\frac{1}{p(r)}\int_{b}^{r}(r-u)q(u)x(u)^{\beta}\,du\right)^{1/\alpha} dr\,ds, \quad t\to\infty, \tag{2.3.59}$$

for some $b \ge a$, which will play a central role in constructing generalized $\mathcal{RV}$-intermediate solutions of type (I2).

Theorems below represent main results for intermediate solutions of type (I2).

Theorem 2.3.4 *Let $p \in \mathcal{RV}_R(\eta), q \in \mathcal{RV}_R(\sigma)$. Then* (E_1) *has intermediate solutions $x \in ntr - \mathcal{SV}_R$ satisfying* (I2) *if and only if*

$$\sigma = \alpha - m_2 \quad \textit{and} \quad \int_a^\infty t\left(\frac{1}{p(t)}\int_a^t (t-s)\,q(s)\,ds\right)^{1/\alpha} dt = \infty. \tag{2.3.60}$$

The asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim Y_1(t) = \left(\frac{\alpha-\beta}{\alpha}\int_a^t s\left(\frac{1}{p(s)}\int_a^s (s-r)q(r)\,dr\right)^{1/\alpha} ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.61}$$

Theorem 2.3.5 *Let $p \in \mathcal{RV}_R(\eta), q \in \mathcal{RV}_R(\sigma)$. Then* (E_1) *has intermediate solutions $x \in \mathcal{RV}_R(\rho)$ with $\rho \in (0, m_2)$ if and only if*

$$\alpha - m_2 < \sigma < \eta - (\alpha+\beta+2)m_2 \tag{2.3.62}$$

in which case ρ is given by (2.3.20) *and the asymptotic behavior of any such solution x is governed by the unique formula*

$$x(t) \sim Y_2(t) = \left(\left(\frac{m_2^{\frac{(\alpha+1)^2}{2\alpha+1}}}{\alpha}\right)^2 \frac{p(t)^{\frac{1}{2\alpha+1}}q(t)R(t)^{-2\frac{\alpha(\alpha+1)}{2\alpha+1}}}{\left(\rho^\alpha(m_2-\rho)\right)^\alpha\,(\rho-m_1)\,(\rho+1)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.63}$$

Theorem 2.3.6 *Let $p \in \mathcal{RV}_R(\eta), q \in \mathcal{RV}_R(\sigma)$. Then* (E_1) *has intermediate solutions $x \in \mathcal{RV}_R\,(m_2)$ satisfying* (I2) *if and only if*

$$\sigma = \eta - (\alpha+\beta+2)m_2 \quad \text{and} \quad \int_a^\infty \left(\frac{1}{p(t)}\int_a^t (t-s)\,s^\beta\,q(s)\,ds\right)^{1/\alpha} dt < \infty. \tag{2.3.64}$$

The asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim Y_3(t) = t\left(\frac{\alpha-\beta}{\alpha}\int_t^\infty \left(\frac{1}{p(s)}\int_a^s (s-r)r^\beta q(r)\,dr\right)^{1/\alpha} ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.65}$$

In order to facilitate the proofs of the main results, we prove the following lemmas.

Lemma 2.3.5 *Suppose that* (2.3.60) *holds. Then the function Y_1 given by* (2.3.61) *satisfies the asymptotic relation* (2.3.59) *for any $b \geq a$ and belongs to $\mathcal{SV}_R$.*

Proof. First, we give an expression for Y_1 in terms of R, l_p and l_q. Applying Lemma 2.3.1 twice, we have

$$\int_a^t \int_a^s q(u)\,du\,ds = \int_a^t \int_a^s R(u)^{\alpha-m_2} l_q(u)\,du\,ds \sim \frac{m_2^{2\frac{\alpha+1}{2\alpha+1}}}{\alpha(\alpha+m_2)} R(t)^{\alpha+m_2} l_p(t)^{\frac{2}{2\alpha+1}} l_q(t), \quad t\to\infty.$$

Using (2.3.2), (2.3.5) and (2.3.9)-(ii), we have as $t\to\infty$

$$t\left(\frac{1}{p(t)}\int_a^t (t-s)q(s)\,ds\right)^{1/\alpha} \sim \frac{m_2^{\frac{2\alpha+2-\alpha^2}{\alpha(2\alpha+1)}}}{(\alpha(\alpha+m_2))^{1/\alpha}} R(t)^{-m_2} l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha}. \tag{2.3.66}$$

Integrating the above on $[b,t]$ for any $b\ge a$, we show that

$$Y_1(t) \sim W_1^{\frac{1}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha} Q_1(t)\right)^{\frac{\alpha}{\alpha-\beta}} \tag{2.3.67}$$

where

$$Q_1(t) = \int_b^t R(s)^{-m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha}\,ds, \quad Q_1\in\mathcal{SV}_R, \quad W_1 = \frac{m_2^{\frac{2\alpha+2-\alpha^2}{2\alpha+1}}}{\alpha(\alpha+m_2)}. \tag{2.3.68}$$

From (2.3.67), we conclude that $Y_1\in\mathcal{SV}_R$.

To verify the asymptotic relation (2.3.59) for Y_1, we integrate $q(t)Y_1(t)^\beta$ twice on $[b,t]$ and use $Y_1\in ntr-\mathcal{SV}_R$ to obtain

$$\int_b^t\int_b^s q(r)Y_1(r)^\beta\,dr\,ds \sim \frac{m_2^{2\frac{\alpha+1}{2\alpha+1}}}{(\sigma+m_2)(\sigma+2m_2)} R(t)^{\sigma+2m_2} l_p(t)^{\frac{2}{2\alpha+1}} l_q(t) Y_1(t)^\beta, \ \ t\to\infty,$$

, which together with (2.3.67), by assumption (2.3.60) and (2.3.9)-(ii), yields

$$\begin{aligned}&\left(\frac{1}{p(t)}\int_b^t (t-s)q(s)Y_1(s)^\beta ds\right)^{1/\alpha}\\ &\sim \left(\frac{m_2^{\frac{2\alpha+2-\alpha\beta}{2\alpha+1}}}{\alpha(\alpha+m_2)}\right)^{\frac{1}{\alpha-\beta}} R(t)^{-2m_2} l_p(t)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha}\left(\frac{\alpha-\beta}{\alpha}Q_1(t)\right)^{\frac{\beta}{\alpha-\beta}},\end{aligned} \tag{2.3.69}$$

as $t\to\infty$. Integration of (2.3.69) on $[t,\infty)$ gives

$$\begin{aligned}&\int_t^\infty \left(\frac{1}{p(r)}\int_b^r (r-u)q(u)Y_1(u)^\beta\,du\right)^{1/\alpha} dr\\ &\sim \left(\frac{m_2^{\frac{2\alpha+2-\alpha\beta}{2\alpha+1}}}{\alpha(\alpha+m_2)}\right)^{\frac{1}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} m_2^{-\frac{\alpha}{2\alpha+1}} R(t)^{-m_2} l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} Q_1(t)^{\frac{\beta}{\alpha-\beta}},\end{aligned}$$

as $t \to \infty$. Previous relation, by integration on $[b, t]$, implies

$$\begin{aligned}
&\int_b^t \int_s^\infty \left(\frac{1}{p(r)} \int_b^r (r-u)q(u)Y_1(u)^\beta \, du\right)^{1/\alpha} dr\, ds \\
&\sim W_1^{\frac{1}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_b^t R(s)^{-m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} Q_1(s)^{\frac{\beta}{\alpha-\beta}} \, ds \\
&\sim W_1^{\frac{1}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \int_b^t Q_1(s)^{\frac{\beta}{\alpha-\beta}} \, dQ_1(s) = W_1^{\frac{1}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} Q_1(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty,
\end{aligned}$$

establishing, in view of (2.3.67), that Y_1 satisfies the asymptotic relation (2.3.59). □

Lemma 2.3.6 *Suppose that* (2.3.62) *holds and let ρ be defined by* (2.3.20)*. Then, the function Y_2 given by* (2.3.63) *satisfies the asymptotic relation* (2.3.59) *for any $b \geq a$ and belongs to $\mathcal{RV}_R(\rho)$.*

Proof. Using (2.3.2) and (2.3.8), since $\frac{\eta-2\alpha(\alpha+1)}{2\alpha+1} = m_2 - \alpha$, we can express Y_2 in the form

$$Y_2(t) \sim W_2 R(t)^\rho \left(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t)\right)^{\frac{1}{\alpha-\beta}}, \tag{2.3.70}$$

where

$$C = m_2^{\frac{(\alpha+1)^2}{2\alpha+1}}, \quad \nu = (\rho(m_2-\rho))^\alpha (\rho - m_1)(\rho+1), \quad W_2 = \left(\frac{C^2}{\alpha^2 \nu}\right)^{\frac{1}{\alpha-\beta}}. \tag{2.3.71}$$

Therefore, $Y_2 \in \mathcal{RV}_R(\rho)$. Next, we prove that Y_2 satisfies the asymptotic relation (2.3.59) and to that end, we first integrate $q(t)Y_2(t)^\beta$ twice on $[b, t]$ for some $b \geq a$, with the application of Lemma 2.3.1 and due to (2.3.9), (2.3.27), we get

$$\begin{aligned}
&\int_b^t \int_b^s q(r) Y_2(r)^\beta \, dr\, ds \\
&\sim W_2^\beta \int_b^t \int_b^s R(r)^{\sigma+\rho\beta} \left(l_p(t)^{\frac{\beta}{2\alpha+1}} l_q(t)^\alpha\right)^{\frac{1}{\alpha-\beta}} drds \\
&\sim \frac{W_2^\beta}{(\sigma+\rho\beta+m_2)(\sigma+\rho\beta+2m_2)} m_2^{2\frac{\alpha+1}{2\alpha+1}} R(t)^{\sigma+\rho\beta+2m_2} \left(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\right)^{\frac{1}{\alpha-\beta}} \\
&= \frac{W_2^\beta}{\alpha^2(\rho+1)(\rho-m_1)} m_2^{2\frac{\alpha+1}{2\alpha+1}} R(t)^{\alpha(\rho-m_1)} \left(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\right)^{\frac{1}{\alpha-\beta}} \\
&= \frac{W_2^\beta}{\alpha^2(\rho+1)(\rho-m_1)} m_2^{2\frac{\alpha+1}{2\alpha+1}} R(t)^{\alpha(\rho-2m_2-\frac{\eta}{\alpha})} \left(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,
\end{aligned}$$

implying further that

$$\begin{aligned}
&\int_b^t \int_s^\infty \left(\frac{1}{p(r)} \int_b^r (r-u)q(u)Y_2(u)^\beta \, du\right)^{1/\alpha} dr\, ds \\
&\sim \left(\frac{W_2^\beta}{\alpha^2(\rho+1)(\rho-m_1)} m_2^{2\frac{\alpha+1}{2\alpha+1}}\right)^{1/\alpha} \int_b^t \int_s^\infty R(r)^{\rho-2m_2} \left(l_p(r)^{\frac{2\beta-2\alpha+1}{2\alpha+1}} l_q(r)\right)^{\frac{1}{\alpha-\beta}} dr\, ds \\
&\sim \frac{W_2^{\beta/\alpha} m_2^{2\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}}{(\alpha^2(\rho+1)(\rho-m_1))^{1/\alpha}(m_2-\rho)\rho} R(t)^\rho \left(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t)\right)^{\frac{1}{\alpha(\alpha-\beta)}} \\
&= W_2^{\beta/\alpha} \left(\frac{C^2}{\nu\alpha^2}\right)^{1/\alpha} R(t)^\rho \left(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t)\right)^{\frac{1}{\alpha(\alpha-\beta)}}, \quad t \to \infty.
\end{aligned}$$

By (2.3.70) and (2.3.71) proves that Y_2 satisfies the asymptotic relation (2.3.59).

Lemma 2.3.7 *Suppose that* (2.3.64) *holds. Then the function* Y_3 *given by* (2.3.65) *satisfies the asymptotic relation* (2.3.59) *for any* $b \geq a$ *and belongs to* $\mathcal{RV}_R(m_2)$. □

Proof. According to (2.3.5) and (2.3.64), the application of the Lemma 2.3.1, gives

$$\begin{aligned}
\int_b^t \int_b^s r^\beta q(r)\, dr\, ds &\sim m_2^{-\frac{\alpha\beta}{2\alpha+1}} \int_b^t \int_b^s R(r)^{\eta-(\alpha+2)m_2} l_p(s)^{\frac{\beta}{2\alpha+1}} l_q(s) ds \\
&\sim \frac{m_2^{\frac{2(\alpha+1)-\alpha\beta}{2\alpha+1}}}{(\eta-(\alpha+1)m_2)(\eta-\alpha m_2)} R(t)^{\eta-\alpha m_2} l_p(t)^{\frac{\beta+2}{2\alpha+1}} l_q(t),
\end{aligned}$$

as $t \to \infty$. Since by (2.3.9)-(ii) we have that

$$\eta - (\alpha+1)m_2 = \alpha(m_2+1), \tag{2.3.72}$$

from the last relation, we conclude that

$$\begin{aligned}
&\int_t^\infty \left(\frac{1}{p(s)} \int_b^s (s-r) r^\beta q(r) dr\right)^{1/\alpha} ds \\
&\sim \left(\frac{m_2^{\frac{2(\alpha+1)-\alpha\beta}{2\alpha+1}}}{\alpha(m_2+1)(\alpha+\alpha m_2+m_2)}\right)^{1/\alpha} \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{\beta-2\alpha+1}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} ds,
\end{aligned} \tag{2.3.73}$$

as $t \to \infty$. We denote by

$$Q_3(t) = \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{\beta-2\alpha+1}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} ds, \quad Q_3 \in \mathcal{SV}_R \tag{2.3.74}$$

and combining (2.3.73) with (2.3.65) and (2.3.5), we obtain the following asymptotic representation for Y_3 in terms of R, l_p and l_q:

$$Y_3(t) \sim W_3^{\frac{1}{\alpha-\beta}} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} \left(\frac{\alpha-\beta}{\alpha} Q_3(t)\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty, \tag{2.3.75}$$

where

$$W_3 = \frac{m_2^{\frac{-\alpha^2+2\alpha+2}{2\alpha+1}}}{\alpha(m_2+1)(\alpha m_2 + m_2 + \alpha)}. \tag{2.3.76}$$

From (2.3.75) we conclude that $Y_3 \in \mathcal{RV}_R(m_2)$ and compute with the help of Lemma 2.3.1,

$$\begin{aligned}
&\int_b^t \int_b^s q(r) Y_3(r)^\beta \, dr\, ds \\
&\sim \left(\frac{\alpha-\beta}{\alpha} Q_3(t)\right)^{\frac{\alpha\beta}{\alpha-\beta}} W_3^{\frac{\beta}{\alpha-\beta}} \frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}} R(t)^{\sigma+m_2\beta+2m_2}}{(\sigma+m_2\beta+2m_2)(\sigma+m_2\beta+m_2)} l_p(t)^{\frac{\beta+2}{2\alpha+1}} l_q(t),
\end{aligned}$$

as $t \to \infty$. Next, using (2.3.64) and (2.3.72) we obtain

$$\begin{aligned}
&\int_t^\infty \left(\frac{1}{p(s)} \int_b^s (s-r) q(r) Y_3(r)^\beta \, dr\right)^{1/\alpha} ds \\
&\sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}} \\
&\quad \times \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{\beta-2\alpha+1}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} Q_3(s)^{\frac{\beta}{\alpha-\beta}} ds \\
&\sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha\,(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}} \int_t^\infty Q_3(s)^{\frac{\beta}{\alpha-\beta}} d(-Q_3(s)) \\
&= \left(\frac{\alpha-\beta}{\alpha} Q_3(t)\right)^{\frac{\alpha}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha\,(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}}, \quad t \to \infty.
\end{aligned}$$

Noting that the last expression in the previous relation is slowly varying with respect to R, the integration of this relation over $[b,t]$ leads to

$$
\begin{aligned}
&\int_b^t \int_s^\infty \left(\frac{1}{p(r)} \int_b^r (r-u)q(u)Y_3(u)^\beta\,du \right)^{1/\alpha} dr\,ds \\
&\sim \left(\frac{\alpha-\beta}{\alpha} Q_3(t) \right)^{\frac{\alpha}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{(\alpha+2)(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}} \frac{R(t)^{m_2}}{m_2} l_p(t)^{\frac{1}{2\alpha+1}} \\
&= \left(\frac{\alpha-\beta}{\alpha} Q_3(t) \right)^{\frac{\alpha}{\alpha-\beta}} W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} W_3^{1/\alpha} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}}, \quad t\to\infty.
\end{aligned}
$$

In view of (2.3.75) this proves that the desired integral asymptotic relation (2.3.59) is satisfied by Y_3. $\square$

Using previous results we can prove the main results of this subsection.

Proof of the "only if" part of Theorems 2.3.4, 2.3.5 and 2.3.6: Suppose that (E_1) has a type (I2) intermediate solution $x \in \mathcal{RV}_R(\rho)$, $\rho \in [0, m_2]$, defined on $[t_0,\infty)$. We begin by integrating (E_1) on $[t_0,t]$. Using (2.3.2), (2.3.12), we have

$$
(p(t)(-x''(t))^\alpha)' \sim \int_{t_0}^t q(s)x(s)^\beta ds = \int_{t_0}^t R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta\,ds, t\to\infty \tag{2.3.77}
$$

and conclude by (2.3.57) that $\sigma+\beta\rho+m_2 \ge 0$. Thus, we distinguish two cases:

$$
(1)\quad \sigma+\beta\rho+m_2 = 0 \quad \text{and} \quad (2)\quad \sigma+\beta\rho+m_2 > 0.
$$

Let the case (1) holds, so that

$$
H_4(t) = \int_{t_0}^t R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta\,ds = \int_{t_0}^t R(s)^{-m_2} l_q(s) l_x(s)^\beta\,ds, \tag{2.3.78}
$$

and $H_4 \in \mathcal{SV}_R$. The integration of (2.3.77) on $[t_0,t]$ with (2.3.9)-(ii) yields

$$
\begin{aligned}
-x''(t) &\sim m_2^{-\frac{1}{2\alpha+1}} R(t)^{\frac{m_2-\eta}{\alpha}} l_p(t)^{-\frac{2}{2\alpha+1}} H_4(t)^{\frac{1}{\alpha}} \\
&= m_2^{-\frac{1}{2\alpha+1}} R(t)^{-2m_2-1} l_p(t)^{-\frac{2}{2\alpha+1}} H_4(t)^{1/\alpha}, \quad t\to\infty,
\end{aligned}
$$

Since $-m_2-1<0$, we may integrate previous relation on $[t,\infty)$ and obtain via Lemma 2.3.1 that

$$
x'(t) \sim \frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_2+1} R(t)^{-m_2-1} H_4(t)^{1/\alpha}, \quad t\to\infty.
$$

The right-hand side in the last relation is integrable on $[t,\infty)$, because $-m_2-1<-m_2$, but on the other hand, in view of (2.3.57), the left-hand side of the last relation is not integrable on $[t,\infty)$, so we conclude that this case is impossible.

Let the case (2) holds. Then, from (2.3.77) it follows that

$$(p(t)(-x''(t))^{\alpha})' \sim \frac{m_2^{\frac{\alpha+1}{2\alpha+1}}}{\sigma+\beta\rho+m_2} R(t)^{\sigma+\beta\rho+m_2} l_p(t)^{\frac{1}{2\alpha+1}} l_q(t) l_x(t)^{\beta}.$$

Since $\sigma+\beta\rho+2m_2>0$, the integration of the previous relation on $[t_0,t]$ gives

$$-x''(t) \sim \left(\frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}}}{(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2)}\right)^{1/\alpha} R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} l_p(t)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha}, \quad t\to\infty,$$

implying in view of (2.3.57), by integration on $[t,\infty)$,

(2.3.79)
$$\begin{aligned} x'(t) \sim & \left(\frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}}}{(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2)}\right)^{1/\alpha} \\ & \times \int_t^{\infty} R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} \left(l_p(s)^{\frac{1-2\alpha}{2\alpha+1}} l_q(s) l_x(s)^{\beta}\right)^{1/\alpha} ds, \quad t\to\infty. \end{aligned}$$

Thus, we further consider the following two possible cases:

$$\text{(2.a)} \quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2=0 \quad \text{and} \quad \text{(2.b)} \quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2<0.$$

Suppose that (2.a) holds, and let

(2.3.80)
$$H_3(t)=\int_t^{\infty} R(s)^{-m_2} l_p(s)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds.$$

Using (2.3.72) and (2.3.9)-(ii), since we have $\sigma+\rho\beta+m_2=\alpha(m_2+1)$, the integration of (2.3.79) on $[t_0,t]$ implies

(2.3.81)
$$x(t) \sim \left(\frac{m_2^{\frac{-\alpha^2+2(\alpha+1)}{2\alpha+1}}}{\alpha(m_2+1)(\alpha(m_2+1)+m_2)}\right)^{1/\alpha} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} H_3(t), \quad t\to\infty.$$

Since $H_3\in\mathcal{SV}_R$, we conclude that $x\in\mathcal{RV}_R(m_2)$.

Suppose that (2.b) holds. The application of Lemma 2.3.1 in (2.3.79) implies

(2.3.82)
$$\begin{aligned} x'(t) \sim & -\frac{m_2^{\frac{(\alpha+1)(\alpha+2)}{\alpha(2\alpha+1)}}}{((\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2\right)} \\ & \times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2} l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha}, \quad t\to\infty. \end{aligned}$$

Integrating (2.3.82) on $[t_0, t]$, using (2.3.57), we obtain

$$x(t) \sim \frac{m_2^{\frac{(\alpha+1)(\alpha+2)}{\alpha(2\alpha+1)}}}{((\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}\left(-\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2\right)\right)} \times \int_{t_0}^{t} R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \quad t \to \infty. \tag{2.3.83}$$

Thus, since $x(t) \to \infty$ as $t \to \infty$, from the previous relation we conclude that two possibilities may hold:

$$\text{(2.b.1)} \quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2 = 0 \ \text{ or } \ \text{(2.b.2)} \quad \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2 > 0.$$

In the case (2.b.1), using (2.3.9)-(ii), we obtain $\sigma + \beta\rho + m_2 = \alpha$. The application of Lemma 2.3.1 in (2.3.83) leads us to

$$x(t) \sim \left(\frac{m_2^{\frac{-\alpha^2+2(\alpha+1)}{2\alpha+1}}}{\alpha(\alpha+m_2)}\right)^{1/\alpha} H_1(t), \quad t \to \infty, \tag{2.3.84}$$

where

$$H_1(t) = \int_{t_0}^{t} R(s)^{-m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \quad H_1 \in \mathcal{SV}_R\,. \tag{2.3.85}$$

Thus, since $x(t) \to \infty$ as $t \to \infty$, $x \in ntr - \mathcal{SV}_R$.

The application of Lemma 2.3.1 in (2.3.83) in the case (2.b.2) gives

$$x(t) \sim m_2^{\frac{2(\alpha+1)^2}{\alpha(2\alpha+1)}} \Big(((\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha} \times \left(-\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2\right)\right)\left(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2\right)\Big)^{-1} \times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2} l_p(t)^{\frac{1}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha} ds, \quad t \to \infty. \tag{2.3.86}$$

This implies that $x \in \mathcal{RV}(\rho)$ where ρ satisfies

$$\rho = \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2 \;\Leftrightarrow\; \rho = \frac{\sigma+m_2-\alpha}{\alpha-\beta}, \tag{2.3.87}$$

verifying that the regularity index ρ is given by (2.3.20).

Now, let x be a type (I2) intermediate solution of (E_1) belonging to $ntr - \mathcal{SV}_R$. Then, from the above observations, it is clear that only the case (2.b.1) is admissible, in which case $\sigma = \alpha - m_2$, and (2.3.84) is satisfied by x. Using $x(t) = l_x(t)$, from (2.3.84) we have

$$l_x(t) \sim W_1^{1/\alpha} H_1(t), \quad t \to \infty, \tag{2.3.88}$$

where W_1 is given by (2.3.68) and H_1 is defined by (2.3.85). Then, (2.3.88) is transformed into the following differential asymptotic relation for H_1,

$$H_1(t)^{-\frac{\beta}{\alpha}} H_1'(t) \sim W_1^{\beta/\alpha} R(t)^{-m_2} l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha}, \quad t \to \infty. \tag{2.3.89}$$

From (2.3.57), since $\lim_{t\to\infty} x(t) = \infty$, we have $\lim_{t\to\infty} H_1(t)^{\frac{\alpha-\beta}{\alpha}} = \infty$. Integrating (2.3.89) on $[t_0, t]$, in view of the relation (2.3.66) and the notation (2.3.68), we find

$$\frac{\alpha}{\alpha - \beta} H_1(t)^{\frac{\alpha-\beta}{\alpha}} \sim W_1^{\beta/\alpha} Q_1(t) \sim W_1^{\beta/\alpha} \int_{t_0}^{t} R(s)^{-m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} ds, \quad t \to \infty.$$

This implies that the second condition in (2.3.60) is satisfied and with (2.3.88) implies

$$x(t) \sim W_1^{\frac{1}{\alpha-\beta}} \left(\frac{\alpha - \beta}{\alpha} Q_1(t) \right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty. \tag{2.3.90}$$

Note that in Lemma 2.3.5 we have obtained the expression (2.3.67) for Y_1 given by (2.3.61). Therefore, (2.3.90) in fact proves that $x(t) \sim Y_1(t)$, $t \to \infty$, completing the "only if" part of Theorem 2.3.4.

Remark 2.3.1 *From the previous observation, we see that Y_1 is not only $\mathcal{SV}_R$ but $ntr - \mathcal{SV}_R$.*

Next, let x be a type (I2) intermediate solution of (E_1) belonging to $\mathcal{RV}_R(\rho)$ for some $\rho \in (0, m_2)$. Clearly, only the case (2.b.2) can hold and hence x satisfies the asymptotic relation (2.3.86) and ρ is given by (2.3.20). An elementary computation shows that

$$0 < \rho < m_2 \;\Rightarrow\; \alpha - m_2 < \sigma < \alpha + m_2(\alpha - \beta - 1),$$

showing that the range of σ is given by (2.3.62). In view of (2.3.27) and (2.3.87), we conclude from (2.3.86) that x enjoys the asymptotic behavior $x(t) \sim Y_2(t)$, $t \to \infty$, where Y_2 is given by (2.3.63). This proves the "only if" part of the Theorem 2.3.5.

Finally, let x is a type (I2) intermediate solution of (E_1) belonging to $\mathcal{RV}_R(m_2)$. Since only the case (2.a) is possible for x, it satisfies (2.3.81), where H_3 is defined by (2.3.80), implying $\rho = m_2$ and $\sigma = \alpha + m_2(\alpha - \beta - 1)$. Using $x(t) = R(t)^{m_2} l_x(t)$, (2.3.81) can be expressed as

$$l_x(t) \sim W_3^{1/\alpha} l_p(t)^{\frac{1}{2\alpha+1}} H_3(t), \; t \to \infty, \tag{2.3.91}$$

where W_3 is defined by (2.3.76), implying the differential asymptotic relation

$$\text{(2.3.92)} \qquad -H_3(t)^{-\frac{\beta}{\alpha}} H_3'(t) \sim W_3^{\frac{\beta}{\alpha^2}} R(t)^{-m_2} l_p(t)^{\frac{\beta+1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha}, \quad t \to \infty.$$

From (2.3.5) and (2.3.81), since $\lim_{t\to\infty} \psi_2(t)^{-1}x(t) = 0$, where $\psi_2(t) = t$, we have that $\lim_{t\to\infty} H_3(t)^{\frac{\alpha-\beta}{\alpha}} = 0$, implying that the left-hand side od (2.3.92) is integrable over $[t,\infty)$. In view of (2.3.73) and (2.3.74), integration of (2.3.92) implies

$$\text{(2.3.93)} \qquad \frac{\alpha}{\alpha-\beta} H_3(t)^{\frac{\alpha-\beta}{\alpha}} \sim W_3^{\frac{\beta}{\alpha^2}} \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{\beta+1-2\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} ds = W_3^{\frac{\beta}{\alpha^2}} Q_3(t), \quad t \to \infty.$$

Therefore, the second condition in (2.3.64) is satisfied. Combining (2.3.93) with (2.3.91), using the expression (2.3.75), we find that

$$x(t) \sim W_3^{\frac{1}{\alpha-\beta}} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} \left(\frac{\alpha}{\alpha-\beta} Q_3(t) \right)^{\frac{\alpha}{\alpha-\beta}} \sim Y_3(t), \quad t \to \infty.$$

Thus the "only if" part of the Theorem 2.3.6 has been proved.

Proof of the "if" part of Theorem 2.3.4, 2.3.5 and 2.3.6: Suppose that (2.3.60) or (2.3.62) or (2.3.64) holds. From Lemmas 2.3.5, 2.3.6 and 2.3.7 it is known that Y_i, $i = 1,2,3$, defined by (2.3.61), (2.3.63) and (2.3.65) satisfy the asymptotic relation (2.3.59). We perform the simultaneous proof for Y_i, $i = 1,2,3$ so the subscripts $i = 1,2,3$ will be deleted in the rest of the proof. By (2.3.59), there exists $T_0 > a$ such that

$$\int_{T_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_{T_0}^r (r-u) q(u) Y(u)^\beta \, du \right)^{1/\alpha} dr\, ds \le 2Y(t), \;\; t \ge T_0.$$

Let such a T_0 be fixed. We may assume that Y is increasing on $[T_0,\infty)$. Since (2.3.59) holds with $b = T_0$, there exists $T_1 > T_0$ such that

$$\int_{T_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_{T_0}^r (r-u) q(u) Y(u)^\beta \, du \right)^{1/\alpha} dr\, ds \ge \frac{Y(t)}{2}, \quad t \ge T_1.$$

Choose positive constants k and K such that

$$k^{1-\frac{\beta}{\alpha}} \le \frac{1}{2}, \quad K^{1-\frac{\beta}{\alpha}} \ge 4, \quad 2kY(T_1) \le KY(T_0).$$

Considering the integral operator

$$\mathcal{H}y(t) = y_0 + \int_{T_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_{T_0}^r (r-u) q(u)\, y(u)^\beta \, du \right)^{1/\alpha} dr\, ds, \quad t \ge T_0,$$

where y_0 is a constant such that $kY(T_1) \le y_0 \le \dfrac{K}{2}Y(T_0)$, we may verify that $\mathcal{H}$ is a continuous self-map on the set

$$\mathcal{Y} = \{y \in C[T_0,\infty) : kY(t) \le y(t) \le KY(t),\ t \ge T_0\},$$

and that $\mathcal{H}$ sends $\mathcal{Y}$ into a relatively compact subset of $C[T_0,\infty)$. Thus, $\mathcal{H}$ has a fixed point $y \in \mathcal{Y}$, which generates a solution of the equation (E_1) of the type (I2) satisfying

$$0 < \liminf_{t\to\infty} \frac{y(t)}{Y(t)} \le \limsup_{t\to\infty} \frac{y(t)}{Y(t)} < \infty.$$

Denoting

$$L(t) = \int_a^t \int_s^\infty \Big(\frac{1}{p(r)} \int_a^r (r-u)q(u)Y(u)^\beta\, du\Big)^{1/\alpha} dr\, ds$$

and using $Y(t) \sim L(t)$ as $t \to \infty$ we obtain

$$0 < \liminf_{t\to\infty} \frac{y(t)}{L(t)} \le \limsup_{t\to\infty} \frac{y(t)}{L(t)} < \infty.$$

Then, proceeding exactly as in the proof of the "if" part of Theorems 2.3.1–2.3.3, with the application of Theorem 1.1.7, we conclude that $y(t) \sim L(t) \sim Y(t), t \to \infty$. Therefore, y is a generalized regularly varying solution with respect to R with requested regularity index and with the asymptotic representation (2.3.61), (2.3.63), (2.3.65) depending on if $q \in \mathcal{RV}_R(\sigma)$ satisfies, respectively, (2.3.60) or (2.3.62) or (2.3.64). Thus, the "if part" of Theorems 2.3.4, 2.3.5 and 2.3.6 has been proved.

2.4 Asymptotic behavior of intermediate regularly varying solutions

The final section is concerned with the equation (E_1) whose coefficients p and q are regularly varying functions (in the sense of Karamata). It is natural to expect that such an equation may possess regularly varying intermediate solutions. Our purpose here is to show that the problem of getting necessary and sufficient conditions for the existence of intermediate solutions, which are regularly varying in the sense of Karamata, can be embedded in the framework of generalized regularly varying functions, so that the results of the preceding section provide full information about the existence and the precise asymptotic behavior of intermediate regularly varying solutions of (E_1).

We assume that p and q are regularly varying functions of indices η and σ, respectively, i.e.,

$$p(t) = t^\eta l_p(t), \quad q(t) = t^\sigma l_q(t), \quad l_p, l_q \in \mathcal{SV}, \tag{2.4.1}$$

and seek regularly varying solutions x of (E_1) expressed in the from

$$x(t) = t^{\rho} l_x(t), \quad l_x \in \mathcal{SV}. \tag{2.4.2}$$

Note first that the condition (C) holds only if we assume that $\eta \geq 1 + 2\alpha$. Since R is defined by (2.3.1), due to (2.4.1), it takes the form

$$R(t) = \left(\int_t^{\infty} s^{\frac{\alpha+1-\eta}{\alpha}} l_p(s)^{-1/\alpha} ds \right)^{-1}.$$

It is easy to see that

$$R \in \mathcal{SV} \quad \text{if} \quad \eta = 2\alpha + 1 \quad \text{and} \quad R \in \mathcal{RV}\left(\frac{\eta - 1 - 2\alpha}{\alpha} \right) \quad \text{if} \quad \eta > 2\alpha + 1. \tag{2.4.3}$$

An important remark is that the possibility $\eta = 2\alpha + 1$ should be excluded. If this equality holds, then R is slowly varying by (2.4.3), and this fact prevents p from being a generalized regularly varying function with respect to R. In fact, if $p \in \mathcal{RV}_R(\eta^*)$ for some η^*, then there exists $f \in \mathcal{RV}(\eta^*)$ such that $p(t) = f(R(t))$, which implies that $p \in \mathcal{SV}$. But this contradicts the hypothesis that $p \in \mathcal{RV}(\eta) = \mathcal{RV}(2\alpha + 1)$. Thus, the case $\eta = 2\alpha + 1$ is impossible, and so η must be restricted to

$$\eta > 1 + 2\alpha, \tag{2.4.4}$$

in which case R satisfies

$$R(t) \sim \frac{\eta - 2\alpha - 1}{\alpha} t^{\frac{\eta - 2\alpha - 1}{\alpha}} l_p(t)^{1/\alpha}, \quad t \to \infty. \tag{2.4.5}$$

Since R is monotone increasing, its inverse function $R^{-1}(t)$ is a regularly varying of index $\alpha/(\eta - 2\alpha - 1)$. Therefore, any regularly varying function of index λ is considered as a generalized regularly varying function with respect to R which regularity index is $\alpha\lambda/(\eta - 2\alpha - 1)$, and conversely, any generalized regularly varying function with respect to R of index λ^* is regarded as a regularly varying function in the sense of Karamata of index $\lambda = \lambda^*(\eta - 2\alpha - 1)/\alpha$. It follows from (2.4.1) and (2.4.2) that

$$p \in \mathcal{RV}_R\left(\frac{\alpha\,\eta}{\eta - 2\alpha - 1} \right), \quad q \in \mathcal{RV}_R\left(\frac{\alpha\,\sigma}{\eta - 2\alpha - 1} \right), \quad x \in \mathcal{RV}_R\left(\frac{\alpha\,\rho}{\eta - 2\alpha - 1} \right).$$

Put

$$\eta^* = \frac{\alpha\eta}{\eta - 2\alpha - 1}, \quad \sigma^* = \frac{\alpha\sigma}{\eta - 2\alpha - 1}, \quad \rho^* = \frac{\alpha\rho}{\eta - 2\alpha - 1}.$$

Note that (2.4.4) implies $\eta > \alpha$ because $\alpha > 0$ and that the two constants given by (2.3.8) are reduced to

$$m_1(\alpha, \eta^*) = \frac{2\alpha - \eta}{\eta - 2\alpha - 1}, \quad m_2(\alpha, \eta^*) = \frac{\alpha}{\eta - 2\alpha - 1}.$$

It turns out therefore that any type (I1) intermediate regularly varying solution of (E_1) is a member of one of the next three classes

$$\mathcal{RV}\left(\frac{2\alpha-\eta}{\alpha}\right), \quad \mathcal{RV}(\rho),\ \rho \in \left(\frac{2\alpha-\eta}{\alpha}, \frac{1+2\alpha-\eta}{\alpha}\right), \quad ntr-\mathcal{RV}\left(\frac{1+2\alpha-\eta}{\alpha}\right),$$

while any type (I2) intermediate regularly varying solution belongs to one of the three classes

$$ntr-\mathcal{SV}, \quad \mathcal{RV}(\rho),\ \rho \in (0,1), \quad \mathcal{RV}(1).$$

Based on the above observations we are able to apply our main results in Section 3, establishing necessary and sufficient conditions for the existence of intermediate regularly varying solutions of (E_1) and determining the asymptotic behavior of all such solutions explicitly.

First, we state the results on the type (I1) intermediate solutions that can be derived as corollaries of Theorems 2.3.1, 2.3.2 and 2.3.3.

Theorem 2.4.1 *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. The equation (E_1) possess intermediate solutions belonging to $\mathcal{RV}\left(\frac{2\alpha-\eta}{\alpha}\right)$ if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2 \quad \text{and} \quad \int_a^\infty tq(t)\varphi_1(t)^\beta\,dt = \infty.$$

Any such solution x enjoys one and the same asymptotic behavior $x(t) \sim X_1(t)$ as $t \to \infty$, where X_1 is given by (2.3.18)*.*

Theorem 2.4.2 *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. The equation (E_1) possess intermediate regularly varying solutions of index ρ with $\rho \in \left(\frac{2\alpha-\eta}{\alpha}, \frac{1+2\alpha-\eta}{\alpha}\right)$ if and only if*

$$\frac{\beta}{\alpha}\eta - 2\beta - 2 < \sigma < \frac{\beta}{\alpha}(\eta - 1) - 2\beta - 1,$$

in which case ρ is given by

$$\rho = \frac{2\alpha - \eta + \sigma + 2}{\alpha - \beta} \tag{2.4.6}$$

and any such solution x enjoys one and the same asymptotic behavior

$$x(t) \sim \left(\frac{t^2\,p(t)^{-1}\,q(t)}{(\rho(\rho-1))^\alpha\ (2\alpha-\eta)\ (\rho\alpha+\eta-1-2\alpha)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 2.4.3 *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. The equation (E_1) possess intermediate solutions belonging to $ntr-\mathcal{RV}\left(\frac{1+2\alpha-\eta}{\alpha}\right)$ if and only if*

$$\sigma = \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1 \quad \text{and} \quad \int_a^\infty q(t)\,\varphi_2(t)^\beta\,dt < \infty.$$

Any such solution x enjoys one and the same asymptotic behavior $x(t) \sim X_3(t)$ as $t \to \infty$, where X_3 is given by (2.3.23).

To prove Theorem 2.4.1 and 2.4.3 we need only to check that

$$\sigma^* = -m_1(\alpha,\eta^*)\beta - 2m_2(\alpha,\eta^*) \;\Leftrightarrow\; \sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2,$$

$$\sigma^* = \beta - m_2(\alpha,\eta^*) \;\Leftrightarrow\; \sigma = \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1,$$

and to prove Theorem 2.4.2 it suffices to note that

$$\rho^* = \frac{\sigma^* + m_2(\alpha,\eta^*) - \alpha}{\alpha-\beta} \;\Leftrightarrow\; \rho = \frac{2\alpha+\sigma-\eta+2}{\alpha-\beta},$$

and to combine the relation (2.4.5) with the equality

$$\begin{aligned}&\alpha^2 m_2(\alpha,\eta^*)^{-\frac{2(\alpha+1)^2}{2\alpha+1}}\Big[(m_1(\alpha,\eta^*)-\rho^*)(\rho^*+1)\Big((\rho^*-m_2(\alpha,\eta^*)\rho^*\Big)^\alpha\Big]\\ &= (2\alpha-\eta)(\rho\alpha+\eta-1-2\alpha)(\rho(\rho-1))^\alpha.\end{aligned}$$

Similarly, we are able to gain a thorough knowledge of type-(I2) intermediate regularly varying solutions of (E_1) from Theorems 2.3.4, 2.3.5 and 2.3.6.

Theorem 2.4.4 *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. The equation (E_1) possess intermediate nontrivial slowly varying solutions if and only if*

$$\sigma = \eta - 2\alpha - 2 \quad \text{and} \quad \int_a^\infty t\left(\frac{1}{p(t)}\int_a^t (t-s)\,q(s)\,ds\right)^{1/\alpha} dt = \infty.$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_1(t)$, $t \to \infty$, where Y_1 is given by (2.3.61).

Theorem 2.4.5 *Assume that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. The equation (E_1) possess intermediate regularly varying solutions of index ρ with $\rho \in (0,1)$ if and only if*

$$\eta - 2\alpha - 2 < \sigma < \eta - \alpha - \beta - 2,$$

in which case ρ is given by (2.4.6) *and the asymptotic behavior of any such solution x is governed by the unique formula*

$$x(t) \sim \left(\frac{t^2\,p(t)^{-1}\,q(t)}{(\rho(1-\rho))^\alpha\;(\eta-2\alpha)\;(\rho\alpha+\eta-1-2\alpha)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 2.4.6 *Assume that $p(t) \in \mathcal{RV}(\eta)$ and $q(t) \in \mathcal{RV}(\sigma)$. The equation (E_1) possess intermediate regularly varying solutions of index 1 if and only if*

$$\sigma = \eta - \alpha - \beta - 2 \quad \textit{and} \quad \int_a^{\infty} \left(\frac{1}{p(t)} \int_a^t (t-s) s^{\beta} q(s) ds \right)^{1/\alpha} dt < \infty.$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_3(t)$, $t \to \infty$, where Y_3 is given by (2.3.65).

2.5 Complete structure of the class of regularly varying solutions

Theorems 2.4.1–2.4.6 combined with Theorems 2.2.1–2.2.4 enable us to describe in full details the structure of $\mathcal{RV}$-solutions of the equation (E_1) with $\mathcal{RV}$-coefficients. Let $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$. Denote with $\mathcal{R}$ the set of all regularly varying solutions of (E_1) and define the subsets

$$\mathcal{R}(\rho) = \mathcal{R} \cap \mathcal{RV}(\rho), \quad tr - \mathcal{R}(\rho) = \mathcal{R} \cap tr - \mathcal{RV}(\rho), \quad ntr - \mathcal{R}(\rho) = \mathcal{R} \cap ntr - \mathcal{RV}(\rho).$$

(i) If $\sigma < \frac{\beta}{\alpha}\eta - 2\beta - 2$, or $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ and $\mathcal{J}_3 < \infty$, then

$$\mathcal{R} = tr - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup tr - \mathcal{R}\left(\frac{1 + 2\alpha - \eta}{\alpha}\right) \cup tr - \mathcal{R}(0) \cup tr - \mathcal{R}(1).$$

(ii) If $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ and $\mathcal{J}_3 = \infty$, then

$$\mathcal{R} = \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup tr - \mathcal{R}\left(\frac{1 + 2\alpha - \eta}{\alpha}\right) \cup tr - \mathcal{R}(0) \cup tr - \mathcal{R}(1).$$

(iii) If $\sigma \in \left(\frac{\beta}{\alpha}\eta - 2\beta - 2, \frac{\beta}{\alpha}(\eta - 1) - 2\beta - 1\right)$, then

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma + 2\alpha + 2 - \eta}{\alpha - \beta}\right) \cup tr - \mathcal{R}\left(\frac{1 + 2\alpha - \eta}{\alpha}\right) \cup tr - \mathcal{R}(0) \cup tr - \mathcal{R}(1).$$

(iv) If $\sigma = \frac{\beta}{\alpha}(\eta - 1) - 2\beta - 1$ and $\mathcal{J}_4 < \infty$, then

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}\left(\frac{1 + 2\alpha - \eta}{\alpha}\right) \cup ntr - \mathcal{R}\left(\frac{1 + 2\alpha - \eta}{\alpha}\right) \cup tr - \mathcal{R}(0) \cup tr - \mathcal{R}(1).$$

(v) If $\sigma = \frac{\beta}{\alpha}(\eta - 1) - 2\beta - 1$ and $\mathcal{J}_4 = \infty$, or $\sigma \in \left(\frac{\beta}{\alpha}(\eta - 1) - 2\beta - 1, \eta - 2\alpha - 2\right)$, or $\sigma = \eta - 2\alpha - 2$ and $\mathcal{J}_1 < \infty$, then

$$\mathcal{R} = tr - \mathcal{R}(0) \cup tr - \mathcal{R}(1).$$

(vi) If $\sigma = \eta - 2\alpha - 2$ and $\mathcal{J}_1 = \infty$, then

$$\mathcal{R} = ntr - \mathcal{R}(0) \cup tr - \mathcal{R}(1).$$

(vii) If $\sigma \in (\eta - 2\alpha - 2, \eta - \alpha - \beta - 2)$, then

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma + 2\alpha + 2 - \eta}{\alpha - \beta}\right) \cup tr - \mathcal{R}(1).$$

(viii) If $\sigma = \eta - \alpha - \beta - 2$ and $\mathcal{J}_2 < \infty$, then

$$\mathcal{R} = tr - \mathcal{R}(1) \cup \mathcal{R}(1).$$

(ix) If $\sigma = \eta - \alpha - \beta - 2$ and $\mathcal{J}_2 = \infty$, or $\sigma > \eta - \alpha - \beta - 2$, then $\mathcal{R} = \emptyset$.

Chapter 3

Second order Emden-Fowler type difference equation

3.1 Introduction

In this chapter we are considering the nonlinear difference equation of the second order

$$(E_2) \qquad \Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) = q(n)|x(n+1)|^{\beta-1}x(n+1), \quad n \geq 1,$$

in the sublinear case i.e. for $\alpha > \beta > 0$, $p = \{p(n)\}$, $q = \{q(n)\}$ are positive real sequences and Δ is forward difference operator defined as $\Delta x(n) = x(n+1) - x(n)$. In the case $0 < \alpha < \beta$ equation is referred as superlinear, while if $\alpha = \beta > 0$ then equation (E_2) is called half-linear.

The equation (E_2) is one of the most frequently studied nonlinear difference equations in the existing literature. Considering system of two first-order difference equations which leads to the equation (E_2), Agarwal, Li and Pang in [4] have shown that all solutions (except trivial) are nonoscillatory. Usually, the equation (E_2) is considered depending on the convergence or divergence of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{p(n)^{1/\alpha}},$$

depending on which we have differently classification of positive solutions according to their asymptotic behavior in infinity. The classification of solutions and the existence of nonoscillatory solutions were studied in [4, 14–16, 23, 24, 123, 124, 131, 136] and references therein. These results are summarized in Section 3.2. Other interesting contributions can be found also in monographs [1,2]. It well known that the problem of determining the conditions for the existence of extremal solutions (strongly decreasing and strongly increasing solutions) is in most cases very difficult and mostly, only necessary or sufficient conditions are known. Namely, in the existing literature there are almost no results for the existence of strongly decreasing solutions of sublinear equation (E_2), while for the existence of strongly increasing solutions of sublinear equation (E_2) only sufficient conditions are known. On the other hand, asymptotic formulas are not known for any of the mentioned two types of solutions. Because of that, assuming that coefficients of the equation (E_2) are regularly varying sequences, in sections 3.3 and 3.4, both necessary and sufficient conditions for the existence of extremal solutions will be presented and their precise asymptotic formulas will be given. In Section 3.5 the complete structure of a set of regularly varying solutions will be presented, while in Section 3.6 the main results will be illustrated with examples.

The obtained results represent a continuation of the application of the discrete theory of regular variation in the asymptotic analysis of difference equations, initiated by Matucci and Rehak in [96]. Except in the mentioned paper, theory of regularly varying sequences has been further developed and applied in the asymptotic analysis of second-order linear and half-linear difference equations in succeeding papers [97–100, 114, 115] of Matucci and Rehak. However, theory of regularly varying sequences has not been applied in the asymptotic analysis of nonlinear difference equations except by Agarwal and Manojlović in [7].

Presented results are original results published in the papers [62, 63].

3.2 Classification of positive solutions of (E_2)

By a solution of (E_2) we mean a not trivial real sequence $x = \{x(n)\}$ satisfying (E_2). A solution x of the equation (E_2) is called oscillatory if for every $M \in \mathbb{N}$ there exist $m, n \in \mathbb{N}, M \leq m < n$ such that $x_m x_n < 0$, otherwise, it is called nonoscillatory. In other words, a solution x is called nonoscillatory if it is eventually positive or eventually negative. It is known that every solution of (E_2) is nonoscillatory. If $x = \{x(n)\}$ is a solution of (3.1.1), then clearly $-x = \{-x(n)\}$ is also a solution. Thus, in studying nonoscillatory solutions of (E_2), for the sake of simplicity, we restrict ourselves to solutions which are eventually positive. Any such solution $\{x(n)\}$ is eventually strongly monotone and belongs to one of the two classes listed below (see [14, Lemma 1]):

$$\begin{aligned} \mathbb{M}^+ &= \{x \text{ solution of } (E_2) \,|\, \exists\, n_0 \geq 1 \ : \ x(n) > 0, \ \ \Delta x(n) > 0, \ \ \text{for } n \geq n_0\}, \\ \mathbb{M}^- &= \{x \text{ solution of } (E_2) \,|\, x(n) > 0, \ \ \Delta x(n) < 0, \ \ \text{for } n \geq 1\}. \end{aligned}$$

It is well-known that the differential equation

$$\Big(p(t)|x'|^{\alpha-1}\, x'\Big) = q(t)|x|^{\beta-1}\, x, \quad \alpha, \beta > 0, \tag{3.2.1}$$

where p, q are continuous positive functions on $[a, \infty)$, may have a nontrivial solution x, with the property that there exists $T_x < \infty$, such that $x(t) \equiv 0$ on $[T_x, \infty)$. Such a solution is said to be an *extinct singular solution* or *singular solution of the first kind*. On the contrary, such solutions of the difference equation (E_2) do not exist. Also, *singular solutions of the second kind*, i.e. solutions that are not extendable to infinity, do not exist in the discrete case. One more difference between differential and difference equations is that for the differential equation (3.2.1) classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be empty (see for example [66]), while for the difference equation (E_2), this case cannot occur (see [2, Theorem 5.3.3] and [14]).

For any solution x of (E_2) denote by $x^{[1]} = \left\{x^{[1]}(n)\right\}$ its quasi-difference $x^{[1]}(n) = p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)$. Thus, the classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be a-priori divided into the following subclasses:

$$\begin{aligned}
\mathbb{M}^+_{\infty,\infty} &= \{x \in \mathbb{M}^+ : \lim_n x(n) = \infty,\ \lim_n x^{[1]}(n) = \infty, \}\,,\\
\mathbb{M}^+_{\infty,l} &= \{x \in \mathbb{M}^+ : \lim_n x(n) = \infty,\ \lim_n x^{[1]}(n) = l,\ 0 < l < \infty\}\,,\\
\mathbb{M}^+_{k,\infty} &= \{x \in \mathbb{M}^+ : \lim_n x(n) = k,\ 0 < k < \infty,\ \lim_n x^{[1]}(n) = \infty\}\,,\\
\mathbb{M}^+_{k,l} &= \{x \in \mathbb{M}^+ : \lim_n x(n) = k,\ 0 < k < \infty,\ \lim_n x^{[1]}(n) = l,\ 0 < l < \infty\}\,,\\
\mathbb{M}^-_{k,l} &= \{x \in \mathbb{M}^- : \lim_n x(n) = k,\ 0 < k < \infty,\ \lim_n x^{[1]}(n) = -l,\ 0 < l < \infty\}\,,\\
\mathbb{M}^-_{0,l} &= \{x \in \mathbb{M}^- : \lim_n x(n) = 0,\ \lim_n x^{[1]}(n) = -l,\ 0 < l < \infty\}\\
\mathbb{M}^-_{k,0} &= \{x \in \mathbb{M}^- : \lim_n x(n) = k,\ 0 < k < \infty,\ \lim_n x^{[1]}(n) = 0\}\,,\\
\mathbb{M}^-_{0,0} &= \{x \in \mathbb{M}^- : \lim_n x(n) = 0,\ \lim_n x^{[1]}(n) = 0\}\,.
\end{aligned}$$

A solution $x \in \mathbb{M}^+_{\infty,\infty}$ is said to be *strongly increasing* and a solution $x \in \mathbb{M}^-_{0,0}$ is said to be *strongly decreasing* or *strongly decaying*. For solutions which tend to some constant we use $\mathbb{M}^-_B = \mathbb{M}^-_{k,0} \cup \mathbb{M}^-_{k,l}$, $\mathbb{M}^+_B = \mathbb{M}^+_{k,\infty} \cup \mathbb{M}^+_{k,l}$ and for decreasing solutions which tend to zero we use $\mathbb{M}^-_0 = \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_{0,0}$. Solutions in $\mathbb{M}^-_B$ and $\mathbb{M}^+_B$ are called *asymptotically constant solutions*, while solutions in $\mathbb{M}^-_0$ are called *decaying solutions*.

Depending on whether $S = \infty$ or $S < \infty$ some of the above classes may be empty.

(i) If $S = \infty$ then

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \cup \mathbb{M}^+_{\infty,l} \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_{k,0} \cup \mathbb{M}^-_{0,0}, \quad \text{i.e.} \quad \mathbb{M}^+_B = \emptyset\,,\ \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_{k,l} = \emptyset.$$

(ii) If $S < \infty$ then

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \cup \mathbb{M}^+_B \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_0 \cup \mathbb{M}^-_B, \quad \text{i.e.} \quad \mathbb{M}^+_{\infty,l} = \emptyset.$$

The asymptotic behavior of solutions in $\mathbb{M}^-$ depends on series:

$$I_1 = \sum_{n=1}^{\infty}\left(\frac{1}{p(n)}\sum_{k=n}^{\infty} q(k)\right)^{\frac{1}{\alpha}}, \quad J_1 = \sum_{n=1}^{\infty} q(n)\left(\sum_{k=n}^{\infty}\frac{1}{p(k+1)^{1/\alpha}}\right)^{\beta}$$

and the one of solutions in $\mathbb{M}^+$ depends on series

$$I_2 = \sum_{n=2}^{\infty} \left(\frac{1}{p(n)} \sum_{k=1}^{n-1} q(k) \right)^{\frac{1}{\alpha}}, \quad J_2 = \sum_{n=2}^{\infty} q(n) \left(\sum_{k=1}^{n-1} \frac{1}{p(k+1)^{1/\alpha}} \right)^{\beta}.$$

Concerning the existence of solutions in the classes $\mathbb{M}_B^+, \mathbb{M}_B^-, \mathbb{M}_{\infty,l}^+$ and $\mathbb{M}_{0,l}^-$, the following holds.

Theorem 3.2.1 (i) ([14, Theorem 2 and Theorem 5-(a)]) *Equation* (E_2) *has solutions in* $\mathbb{M}_B^-$ *if and only if* $I_1 < \infty$.

(ii) ([16, Theorem 2.2 and Theorem 3.1] and [83, Theorem 9]) *Equation* (E_2) *has solutions in* $\mathbb{M}_{0,l}^-$ *if and only if* $J_1 < \infty$.

Theorem 3.2.2 (i) ([15, Proposition 1 and Theorem 2]) *Equation* (E_2) *has solutions in* $\mathbb{M}_B^+$ *if and only if* $I_2 < \infty$.

(ii) ([15, Theorem 5] and [4, Theorem 2]) *Equation* (E_2) *has solutions in* $\mathbb{M}_{\infty,l}^+$ *if and only if* $J_2 < \infty$.

In both Theorems 3.2.1 and 3.2.2 statements (i) are valid for all positive α and β, while statements (ii) are valid only in the sublinear case.

Theorem 3.2.3 ([16, Corollary 3.3]) *For equation* (E_2) *the following hold:*

(a) *If* $I_1 = \infty$ *and* $J_1 = \infty$ *then* $\mathbb{M}^- = \mathbb{M}_{0,0}^- \neq \emptyset$;

(b) *If* $I_1 = \infty$ *and* $J_1 < \infty$ *then* $\mathbb{M}_l^- = \emptyset$, $\mathbb{M}_{0,l}^- \neq \emptyset$;

(c) *If* $I_1 < \infty$ *and* $J_1 = \infty$ *then* $\mathbb{M}_l^- \neq \emptyset$, $\mathbb{M}_{0,l}^- = \emptyset$;

(d) *If* $I_1 < \infty$ *and* $J_1 < \infty$ *then* $\mathbb{M}_l^- \neq \emptyset$, $\mathbb{M}_{0,l}^- \neq \emptyset$.

In addition, for $\alpha \leq \beta$, *we have* $\mathbb{M}_{0,0}^- = \emptyset$ *in claims* (b) $-$ (d).

Theorem 3.2.4 ([15, Theorem 7]) *For equation* (E_2) *the following hold:*

(a) *If* $I_2 = \infty$ *and* $J_2 = \infty$ *then* $\mathbb{M}^+ = \mathbb{M}_{\infty,\infty}^+ \neq \emptyset$;

(b) *If* $I_2 = \infty$ *and* $J_2 < \infty$ *then* $\mathbb{M}_l^+ = \emptyset$, $\mathbb{M}^+ = \mathbb{M}_{\infty,l}^+ \neq \emptyset$;

(c) *If* $I_2 < \infty$ *then* $\mathbb{M}_l^+ \neq \emptyset$, $\mathbb{M}_{\infty,l}^+ = \emptyset$.

In addition, for $\alpha \geq \beta$, *we have* $\mathbb{M}_{\infty,\infty}^+ = \emptyset$ *in claims* (b), (c).

As we see from Theorem 3.2.3, it is an open problem whether it is true that $\mathbb{M}^-_{0,0} \neq \emptyset$ in cases $(b)-(d)$ if $\alpha > \beta$. The existence of strongly decreasing solutions in the continuous case, that is for the differential equation (3.2.1), can be proved as in [121] with the help of fixed point theory by proving that the operator

$$(\mathcal{F}x)(t) = \int_t^\infty \left(\frac{1}{p(s)} \int_s^\infty q(r)x(r)^\beta \, dr \right)^{\frac{1}{\alpha}} ds$$

has a nonzero fixed point. To this end the operator $\mathcal{F}$ acts on the set

$$\Omega = \{x \in C[t_0, \infty] : z(t) \leq x(t) \leq z(t_0),\ t \geq t_0\},$$

where z is a singular solution of the first kind of (3.2.1). The second approach, due to [104], is to construct the sequence $\{x(n)\}$ of asymptotically constant solutions of the differential equation (3.2.1), having the limit function x, and it gives rise to a positive strongly decreasing solution of (3.2.1). This approach, however, requires lower bound for such a sequence of solutions, which is again given by a singular solution of the first kind of (3.2.1). Clearly, due to the nonexistence of singular solutions in the discrete case, neither of these two approaches work. Therefore, in Section 3.3 we will give the necessary and sufficient conditions that the equation (E_2) has solutions in class $\mathbb{M}^-_{0,0}$, but limiting ourself to the case when the coefficients of the observed equation are regularly varying sequences. In addition, we will be able to give a precise asymptotic representation of these solutions.

Regarding the existence of strongly increasing solutions, there are only partial results given in Theorem 3.2.4. Therefore, continuing in this direction, in section 3.4 the necessary and sufficient conditions for the existence of a regularly varying solution of the equation (E_2), whose coefficients are regularly varying sequences, will be given, and precise asymptotic formulas of strongly increasing solutions will be obtained.

3.3 Strongly decreasing solutions

In what follows we assume that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and use expressions

(3.3.1) $$p(n) = n^\eta \xi(n) \quad q(n) = n^\sigma \omega(n), \quad \xi = \{\xi(n)\},\ \omega = \{\omega(n)\} \in \mathcal{SV},$$

considering strongly decreasing $\mathcal{RV}$−solutions expressed as

(3.3.2) $$x(n) = n^\rho l(n), \quad l = \{l(n)\} \in \mathcal{SV}.$$

Moreover, we assume that $\eta \neq \alpha$ and distinguish two mutually exclusive cases:

(3.3.3) $$\begin{aligned} &(i) \quad \eta < \alpha \quad \text{implying that} \quad S = \infty; \\ &(ii) \quad \eta > \alpha \quad \text{implying that} \quad S < \infty. \end{aligned}$$

Case (i): It is clear that for any strongly decreasing solution of (E_2) it holds that

$x(n) \le c$, for large n. Thus, we have that the index of regularity ρ of strongly decreasing $\mathcal{RV}-$solution x must satisfy $\rho \le 0$. If $\rho = 0$ then $l(n) = x(n) \to 0$, so x is a member of $ntr-\mathcal{SV}$.

Case (ii): Using (3.3.1) and Theorem 1.3.5 we have

$$\pi(n) = \sum_{k=n}^{\infty} \frac{1}{p(k)^{1/\alpha}} = \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \xi(k)^{-\frac{1}{\alpha}} \sim \frac{\alpha}{\eta-\alpha} n^{\frac{\alpha-\eta}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} = \frac{\alpha}{\eta-\alpha} \cdot \frac{n}{p(n)^{1/\alpha}}, \tag{3.3.4}$$

as $n \to \infty$, so that $\{\pi(n)\} \in \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. For any strongly decreasing solution x of (E_2), by application of Lemma 1.1.9, we have that

$$\lim_{n\to\infty} \frac{x(n)}{\pi(n)} = \lim_{n\to\infty} \frac{\Delta x(n)}{-p(n)^{-\frac{1}{\alpha}}} = \lim_{n\to\infty} \left(x^{[1]}(n)\right)^{\frac{1}{\alpha}} = 0\,,$$

implying that the index of regularity ρ of strongly decreasing solutions must satisfy $\rho \le \frac{\alpha-\eta}{\alpha}$.

If $\eta < \alpha$, the totality of strongly decreasing $\mathcal{RV}-$solutions will be divided into the following two classes

$$ntr-\mathcal{SV} \quad \text{or} \quad \mathcal{RV}(\rho) \ \text{ with } \ \rho < 0,$$

while, if $\eta > \alpha$, the totality of strongly decreasing $\mathcal{RV}-$solutions of (E_2) will be divided into the following two subclasses:

$$\mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right) \quad \text{or} \quad \mathcal{RV}(\rho) \ \text{ with } \ \rho < \frac{\alpha-\eta}{\alpha}.$$

Our purpose is to show that all solutions in each of these four subclasses of strongly decreasing $\mathcal{RV}-$solutions of (E_2) enjoy one and the same asymptotic behavior as $n \to \infty$, whereby the regularity index of such a solution is uniquely determined by α, β and the regularity indices η, σ of coefficients p, q. Moreover, necessary and sufficient conditions for the existence of solutions, belonging to these four subclasses of strongly decreasing $\mathcal{RV}-$solutions will be established.

3.3.1 Existence of strongly decreasing solutions

Sufficient conditions for the existence of a strongly decreasing solution of the differential equation (3.2.1) are given by the following theorem:

Theorem 3.3.1 (i) ([104, Theorem 3.2], [22, Proposition 2-(e)]) *Let* $\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty$, $a \geq 0$. *If*

$$\int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty q(s)ds\right)^{\frac{1}{\alpha}} dt < \infty,$$

then equation (3.2.1) *has a strongly decreasing solution.*

(ii) ([121, Theorem 4.3]) *Let* $\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty,\ a \geq 0$. *If*

$$\int_a^\infty q(t)\left(\int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}}\right)^\beta dt < \infty,$$

then equation (3.2.1) *has a strongly decreasing solution.*

According to these continuous results, it is expected that in the discrete case, the existence of strongly decreasing solution is characterized by the assumption $I_1 < \infty$ if $S = \infty$ and by the assumption $J_1 < \infty$ if $S < \infty$. In fact, we prove

Theorem 3.3.2 *Suppose that* $p \in \mathcal{RV}(\eta)$ *and* $q \in \mathcal{RV}(\sigma)$.
(i) Let $\eta < \alpha$. *If* $I_1 < \infty$, *then* $\mathbb{M}^-_{0,0} \neq \emptyset$.
(ii) Let $\eta > \alpha$. *If* $J_1 < \infty$, *then* $\mathbb{M}^-_{0,0} \neq \emptyset$.

First of all, let us notice that if $\eta < \alpha$, then $\sigma < -1$ is a necessary condition for $I_1 < \infty$. Then, using discrete Karamata theorem, (3.3.1) and (3.3.4), we have

$$\left(\frac{1}{p(k)}\sum_{j=k}^\infty q(j)\right)^{\frac{1}{\alpha}} \sim \frac{1}{(-(\sigma+1))^{\frac{1}{\alpha}}}\left(\frac{k^{\sigma+1-\eta}\omega(k)}{\xi(k)}\right)^{\frac{1}{\alpha}}, \quad k \to \infty\,.$$

On the other hand, if $\eta > \alpha$ application of discrete Karamata theorem gives

$$q(k)\left(\sum_{j=k}^\infty \frac{1}{p(j)^{1/\alpha}}\right)^\beta \sim \left(\frac{\alpha}{\eta-\alpha}\right)^\beta k^{\sigma+\beta-\frac{\beta}{\alpha}\eta}\,\frac{\omega(k)}{\xi(k)^{\beta/\alpha}},\ \ k \to \infty.$$

Consequently,

(i) for $\eta < \alpha$, $I_1 < \infty$ if and only if

(3.3.5) $$\sigma < \eta - \alpha - 1$$

or

(3.3.6) $$\sigma = \eta - \alpha - 1 \quad \text{and} \quad \sum_{k=1}^\infty k^{-1}\left(\frac{\omega(k)}{\xi(k)}\right)^{\frac{1}{\alpha}} < \infty\,;$$

(ii) for $\eta > \alpha$, $J_1 < \infty$ if and only if

$$\sigma < \frac{\beta\eta}{\alpha} - \beta - 1 \tag{3.3.7}$$

or

$$\sigma = \frac{\beta\eta}{\alpha} - \beta - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1} \frac{\omega(k)}{\xi(k)^{\beta/\alpha}} < \infty\,. \tag{3.3.8}$$

Taking into account the previous consideration, Theorem 3.3.2 will be proved by considering the above four cases.

Theorem 3.3.3 *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$.*
(i) Let $\eta < \alpha$. If (3.3.5) *holds, then equation (E_2) possesses a solution $x \in \mathbb{M}^{-}_{0,0}$.*
(ii) Let $\eta > \alpha$. If (3.3.7) *holds, then equation (E_2) possesses a solution $x \in \mathbb{M}^{-}_{0,0}$.*

Proof. Suppose either $\eta < \alpha$ and (3.3.5) holds or $\eta > \alpha$ and (3.3.7) holds. Denote

$$X(n) = \left[\frac{n^{\alpha+1} p(n)^{-1} q(n)}{(-\rho)^{\alpha}(\alpha - \eta - \rho\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \qquad n \geq 1, \tag{3.3.9}$$

and $\lambda = (-\rho)^{\alpha}(\alpha - \eta - \rho\alpha)$, where ρ is given by

$$\rho = \frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}. \tag{3.3.10}$$

Clearly, $X = \{X(n)\} \in \mathcal{RV}(\rho)$ and it may be expressed in the form

$$X(n) = \lambda^{-\frac{1}{\alpha-\beta}} n^{\rho} \left(\frac{\omega(n)}{\xi(n)}\right)^{\frac{1}{\alpha-\beta}}. \tag{3.3.11}$$

Notice that (3.3.5) and (3.3.10) imply that $\rho < 0$, while (3.3.7) and (3.3.10) imply that $\rho < \frac{\alpha-\eta}{\alpha}$, so that by Theorem 1.3.3-(v),(vii), $X(n) \to 0$ as $n \to \infty$ and $\{X(n)\}$ is eventually decreasing, in both cases (*i*) and (*ii*).

Let us first prove that the sequence X satisfies the asymptotic relation

$$\sum_{k=n}^{\infty} \left(\frac{1}{p(k)} \sum_{j=k}^{\infty} q(j) X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} \sim X(n), \qquad n \to \infty. \tag{3.3.12}$$

Using (3.3.1), by application of Theorem 1.3.3-(iii) and Theorem 1.3.5-(ii), we get

$$
\begin{aligned}
\sum_{k=n}^{\infty} q(k)X(k+1)^{\beta} &\sim \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{\sigma+\rho\beta}\xi(k)^{-\frac{\beta}{\alpha-\beta}}\omega(k)^{\frac{\alpha}{\alpha-\beta}} \\
&= \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{\alpha(\rho-1)+\eta-1}\xi(k)^{-\frac{\beta}{\alpha-\beta}}\omega(k)^{\frac{\alpha}{\alpha-\beta}} \\
&\sim \lambda^{-\frac{\beta}{\alpha-\beta}} \frac{n^{\alpha(\rho-1)+\eta}\xi(n)^{-\frac{\beta}{\alpha-\beta}}\omega(n)^{\frac{\alpha}{\alpha-\beta}}}{-(\alpha(\rho-1)+\eta)}, \quad n \to \infty.
\end{aligned} \tag{3.3.13}
$$

Notice that $\alpha(\rho-1)+\eta < 0$ in both cases (i) and (ii). From (3.3.13), applying Theorem 1.3.5-(ii), we obtain the desired asymptotic relation for X:

$$
\begin{aligned}
\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty} q(j)X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} &\sim \left(\frac{\lambda^{-\frac{\beta}{\alpha-\beta}}}{\alpha(1-\rho)-\eta}\right)^{\frac{1}{\alpha}} \sum_{k=n}^{\infty} k^{\rho-1}\left(\frac{\omega(k)}{\xi(k)}\right)^{\frac{1}{\alpha-\beta}} \\
&\sim \left(\frac{\lambda^{-\frac{\beta}{\alpha-\beta}}}{\alpha(1-\rho)-\eta}\right)^{\frac{1}{\alpha}} \frac{n^{\rho}}{-\rho}\left(\frac{\omega(n)}{\xi(n)}\right)^{\frac{1}{\alpha-\beta}} \\
&= \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \cdot \lambda^{-\frac{1}{\alpha}}\, n^{\rho}\left(\frac{\omega(n)}{\xi(n)}\right)^{\frac{1}{\alpha-\beta}} = X(n),
\end{aligned}
$$

as $n \to \infty$. Thus, there exists $n_0 > 1$ such that

$$
\begin{gathered}
X(n+1) \le X(n) \quad \text{and} \\
\frac{1}{2}X(n) \le \sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty} q(j)X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} \le 2X(n), \quad \text{for } n \ge n_0\,.
\end{gathered} \tag{3.3.14}
$$

Let such n_0 be fixed. We choose constants $\kappa \in (0,1)$ and $K > 1$ such that

$$
\kappa^{1-\frac{\beta}{\alpha}} \le \frac{1}{2} \qquad \text{and} \quad K^{1-\frac{\beta}{\alpha}} \ge 2\,. \tag{3.3.15}
$$

Consider the space Υ_{n_0} of all real sequences $x = \{x(n)\}_{n=n_0}^{\infty}$ such that $\{x(n)/X(n)\}$ is bounded. Then, Υ_{n_0} is a Banach space, endowed with the norm

$$
\|x\| = \sup_{n\ge n_0} \frac{x(n)}{X(n)}\,.
$$

Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\le$: for $x, y \in \Upsilon_{n_0}$, $x \le y$ means $x(n) \le y(n)$ for all $n \ge n_0$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ by

$$
\mathcal{X} = \{x \in \Upsilon_{n_0} : \kappa X(n) \le x(n) \le KX(n),\ n \ge n_0\,\}. \tag{3.3.16}
$$

For any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. Next, define the operator $\mathcal{F} : \mathcal{X} \to \Upsilon_{n_0}$ by

$$(\mathcal{F}x)(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p(k)} \sum_{j=k}^{\infty} q(j)x(j+1)^{\beta} \right)^{\frac{1}{\alpha}}, \quad n \geq n_0, \tag{3.3.17}$$

and show that $\mathcal{F}$ has a fixed point by using Theorem 1.1.1. Namely, the operator $\mathcal{F}$ has the following properties:

(i) *Operator $\mathcal{F}$ maps $\mathcal{X}$ into itself*: Let $x \in \mathcal{X}$. Using (3.3.14), (3.3.15), (3.3.16) and (3.3.17), we get

$$(\mathcal{F}x)(n) \leq K^{\frac{\beta}{\alpha}} \sum_{k=n}^{\infty} \left(\frac{1}{p(k)} \sum_{j=k}^{\infty} q(j)X(j+1)^{\beta} \right)^{\frac{1}{\alpha}} \leq 2K^{\frac{\beta}{\alpha}} X(n) \leq K\,X(n), \quad n \geq n_0\,.$$

$$(\mathcal{F}x)(n) \geq \kappa^{\frac{\beta}{\alpha}} \sum_{k=n}^{\infty} \left(\frac{1}{p(k)} \sum_{j=k}^{\infty} q(j)X(j+1)^{\beta} \right)^{\frac{1}{\alpha}} \geq \kappa^{\frac{\beta}{\alpha}} \frac{X(n)}{2} \geq \kappa X(n), \quad n \geq n_0\,.$$

This shows that $(\mathcal{F}x)_n \in \mathcal{X}$, for all $n \geq n_0$, that is, $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$.

(ii) *Operator $\mathcal{F}$ is increasing*, i.e. for any $x, y \in \mathcal{X}$, $x \leq y$ implies $\mathcal{F}x \leq \mathcal{F}y$.

Thus all the hypotheses of Theorem 1.1.1 are fulfilled implying the existence of a fixed point $x \in \mathcal{X}$ of $\mathcal{F}$, satisfying

$$x(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p(k)} \sum_{j=k}^{\infty} q(j)x(j+1)^{\beta} \right)^{\frac{1}{\alpha}}, \qquad n \geq n_0\,. \tag{3.3.18}$$

It is clear in view of (3.3.16) and the fact that $X(n) \to 0$, $n \to \infty$, that x is a positive solution of (E_2) which satisfies $x(n) \to 0$, $n \to \infty$. Moreover, due to (3.3.11), (3.3.14) and (3.3.16), we have

$$p(n)(-\Delta x(n))^{\alpha} \leq K^{\beta} \sum_{k=n}^{\infty} q(k)X(k+1)^{\beta} \leq m \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} f(k), \tag{3.3.19}$$

where

$$f(k) = \left(\frac{\omega(k)^{\alpha}}{\xi(k)^{\beta}} \right)^{\frac{1}{\alpha-\beta}}, \quad f = \{f(k)\} \in \mathcal{SV} \quad \text{and} \quad m = K^{\beta} \lambda^{-\frac{\beta}{\alpha-\beta}}.$$

Since, $\eta < \alpha$ and (3.3.5) as well as $\eta > \alpha$ and (3.3.7) imply that $\sigma + \rho\beta < -1$, from (3.3.19) we conclude that $x^{[1]}(n) \to 0$, $n \to \infty$, that is $x \in \mathbb{M}^{-}_{0,0}$. $\square$

Theorem 3.3.4 *Suppose that* $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ *and* $q \in \mathcal{RV}(\sigma)$. *If* (3.3.6) *holds, then there exists* $x \in \mathbb{M}^-_{0,0}$.

Proof. Suppose (3.3.6) holds. Define sequences $T = \{T(n)\}$ and $G = \{G(n)\}$ by

$$G(n) = \sum_{k=n}^{\infty} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}}, \quad T(n) = \left(\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty} q(j)\right)^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-\beta}}, \tag{3.3.20}$$

for $n \geq 1$. Since the first condition from (3.3.6) implies $\sigma < -1$, application of Theorem 1.3.5 gives

$$\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty} q(j)\right)^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}}\sum_{k=n}^{\infty} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}}, \quad n \to \infty,$$

so that

$$T(n) \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} G(n)^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty$$

Clearly, $G \in ntr-\mathcal{SV}$ and $T \in ntr-\mathcal{SV}$. Applying Theorem 1.3.5-(ii) and using the first condition from (3.3.6) we get

$$\sum_{k=n}^{\infty} q(k)T(k+1)^{\beta} \sim \frac{1}{(\alpha-\eta)^{\frac{\alpha}{\alpha-\beta}}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} n^{\eta-\alpha}\omega(n)G(n)^{\frac{\alpha\beta}{\alpha-\beta}},\ n \to \infty.$$

Thus, by Theorem 1.3.4, the previous relation gives

$$\begin{aligned}
\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty} q(j)T(j+1)^{\beta}\right)^{\frac{1}{\alpha}} &\sim \left(\frac{(\alpha-\beta)^{\beta}}{\alpha^{\beta}(\alpha-\eta)}\right)^{\frac{1}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{-1}\left(\frac{\omega(k)}{\xi(k)}\right)^{\frac{1}{\alpha}} G(k)^{\frac{\beta}{\alpha-\beta}} \\
&\sim \left(\frac{(\alpha-\beta)^{\beta}}{\alpha^{\beta}(\alpha-\eta)}\right)^{\frac{1}{\alpha-\beta}} \sum_{k=n}^{\infty} (-\Delta G(k))\cdot G(k)^{\frac{\beta}{\alpha-\beta}} \\
&\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} G(n)^{\frac{\alpha}{\alpha-\beta}} \sim T(n),
\end{aligned}$$

as $n \to \infty$. Consequently, we conclude that T satisfies the asymptotic relation (3.3.12).

The rest of the proof is the same as the proof of Theorem 3.3.3 where $X(n)$ is replaced with $T(n)$. Then, a solution x of the equation (E_2) satisfying $\kappa\, T(n) \leq x(n) \leq K\, T(n)$, for large n, is obtained by the application of Knaster-Tarski fixed point theorem (Theorem 1.1.1) and belongs to the class $\mathbb{M}^-_{0,0}$. □

Theorem 3.3.5 *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. If* (3.3.8) *holds, then there exists $x \in \mathbb{M}^-_{0,0}$.*

Proof. Suppose (3.3.8) holds. Using (3.3.1) and the assumption (3.3.8), we have that

$$\begin{aligned}\sum_{k=1}^{\infty} q(k)\left(\sum_{j=k}^{\infty}\frac{1}{p(k)^{1/\alpha}}\right)^{\beta} &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta}\sum_{k=1}^{\infty} k^{\beta}q(k)p(k)^{-\frac{\beta}{\alpha}}\\ &= \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta}\sum_{k=1}^{\infty} k^{-1}\omega(k)\xi(k)^{-\frac{\beta}{\alpha}},\ \ n\to\infty\,.\end{aligned}$$

Define sequences $Y = \{Y(n)\}$ and $W = \{W(n)\}$ by

$$\begin{aligned} W(n) &= \sum_{k=n}^{\infty} k^{-1}\omega(k)\xi(k)^{-\frac{\beta}{\alpha}},\\ Y(n) &= \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} n\,p(n)^{-\frac{1}{\alpha}}W(n)^{\frac{1}{\alpha-\beta}},\ \ n\geq 1\,.\end{aligned}\tag{3.3.21}$$

Note that $W \in \mathcal{SV}$ and since $n\,p(n)^{-\frac{1}{\alpha}} = n^{\frac{\alpha-\eta}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}$, we see that $Y \in \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Thus, the application of Theorem 1.3.4 gives

$$\begin{aligned}\sum_{k=n}^{\infty} q(k)Y(k+1)^{\beta} &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\sum_{k=n}^{\infty} k^{-1}\omega(k)\xi(k)^{-\frac{\beta}{\alpha}}W(k)^{\frac{\beta}{\alpha-\beta}}\\ &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\sum_{k=n}^{\infty}(-\Delta W(k))\cdot W(k)^{\frac{\beta}{\alpha-\beta}}\\ &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} W(n)^{\frac{\alpha}{\alpha-\beta}},\quad n\to\infty,\end{aligned}$$

which yields with the help of Theorem 1.3.5-(ii)

$$\begin{aligned}\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty} q(j)Y(j+1)^{\beta}\right)^{\frac{1}{\alpha}} &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}}\sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}}\xi(k)^{-\frac{1}{\alpha}}W(k)^{\frac{1}{\alpha-\beta}}\\ &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} n^{\frac{\alpha-\eta}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}W(n)^{\frac{1}{\alpha-\beta}} = Y(n),\ \ n\to\infty.\end{aligned}$$

Therefore, $Y = \{Y(n)\}$ satisfies the asymptotic relation (3.3.12). Then, proceeding exactly as in the proof of Theorem 3.3.3, replacing $X(n)$ with $Y(n)$, a solution x satisfying

$\kappa\, Y(n) \le x(n) \le K\, Y(n)$, for large n, is obtained by the application of Theorem 1.1.1, and belongs to a class $\mathbb{M}^-_{0,0}$. □

Proof of Theorem 3.3.2:
(i) Follows from Theorem 3.3.3-(i) and Theorem 3.3.4.
(ii) Follows from Theorem 3.3.3-(ii) and Theorem 3.3.5. □

3.3.2 Asymptotic representation of strongly decreasing $\mathcal{RV}$-solutions

To simplify the "only if" part of the proof of main results we prove the next two lemmas.

Lemma 3.3.1 *Let $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. For any $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le 0$, only one of the following two statements holds:*
(i) $\rho = 0$ *and*

$$x(n) \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}} \sum_{k=n}^{\infty} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}} l(k)^{\frac{\beta}{\alpha}}, \quad n \to \infty\,. \tag{3.3.22}$$

Then, it is $\sigma = \eta - \alpha - 1 < -1$.
(ii) ρ *is given by* (3.3.10) *and*

$$x(n) \sim \left[\frac{n^{\alpha+1}p(n)^{-1}q(n)}{(-\rho)^{\alpha}(\alpha-\eta-\rho\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{3.3.23}$$

Then, it is $\sigma < \eta - \alpha - 1$.

Proof. Suppose that (E_2) has a solution $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le 0$, satisfying $x(n) > 0$, $\Delta x(n) < 0$ for $n \ge n_0 + 1 \ge 2$ and expressed with (3.3.2). Summing (E_2) for $k \ge n \ge n_0$, we get

$$p(n)(-\Delta x(n))^{\alpha} = \sum_{k=n}^{\infty} q(k)x(k+1)^{\beta}\,,$$

which yields, using (3.3.1) and (3.3.2)

$$p(n)(-\Delta x(n))^{\alpha} \sim \sum_{k=n}^{\infty} q(k)x(k)^{\beta} = \sum_{k=n}^{\infty} k^{\sigma+\rho\beta}\omega(k)l(k)^{\beta}, \quad n \to \infty. \tag{3.3.24}$$

The fact that $x^{[1]}(n) = p(n)(-\Delta x(n))^{\alpha} \to 0$ as $n \to \infty$ implies

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} k^{\sigma+\rho\beta}\omega(k)l(k)^{\beta} = 0,$$

so it must be $\sigma + \rho\beta \leq -1$. We are first considering the case $\sigma + \rho\beta = -1$. Then,

$$p(n)(-\Delta x(n))^{\alpha} \sim \sum_{k=n}^{\infty} k^{-1}\omega(k)l(k)^{\beta} = \Omega(n), \quad n \to \infty, \tag{3.3.25}$$

where $\Omega = \{\Omega(n)\} \in \mathcal{SV}$ and $\Omega(n) \to 0$, $n \to \infty$. Consequently

$$-\Delta x(n) \sim \left(\frac{\Omega(n)}{p(n)}\right)^{\frac{1}{\alpha}} = n^{-\frac{\eta}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}\Omega(n)^{\frac{1}{\alpha}}, \quad n \to \infty.$$

Since $\lim_{n\to\infty} x(n) = 0$, summing previous relation from n to ∞, we get

$$x(n) \sim \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}}\left(\frac{\Omega(k)}{\xi(k)}\right)^{\frac{1}{\alpha}}, \quad n \to \infty, \tag{3.3.26}$$

implying that $1 - \frac{\eta}{\alpha} \leq 0$ i.e. $\eta \geq \alpha$ which is a contradiction, so this case is impossible.

Therefore, $\sigma + \rho\beta < -1$. An application of Theorem 1.3.5-(ii) in (3.3.24) gives

$$-\Delta x(n) = \left(\frac{1}{p(n)}\sum_{k=n}^{\infty} q(k)x(k+1)^{\beta}\right)^{\frac{1}{\alpha}} \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}}\omega(n)^{\frac{1}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}l(n)^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}}, \tag{3.3.27}$$

as $n \to \infty$. Because $x(n) \to 0$, $n \to \infty$, summing (3.3.27) from n to ∞ we get

$$x(n) \sim \sum_{k=n}^{\infty} \frac{k^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}}\omega(k)^{\frac{1}{\alpha}}\xi(k)^{-\frac{1}{\alpha}}l(k)^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}}, \quad n \to \infty. \tag{3.3.28}$$

From the last relation we conclude that it must be $(\sigma + \rho\beta + 1 - \eta)/\alpha \leq -1$, so we distinguish two possibilities:

$$\text{(a)} \ \frac{\sigma+\rho\beta+1-\eta}{\alpha} = -1, \qquad \text{(b)} \ \frac{\sigma+\rho\beta+1-\eta}{\alpha} < -1\,. \tag{3.3.29}$$

If (a) holds, then $\sigma + \rho\beta + 1 = \eta - \alpha$. From (3.3.28), we get that (3.3.22) holds, and according to Theorem 1.3.5-(iii), $x \in \mathcal{SV}$. Thus, $\rho = 0$ and (a) implies that $\sigma = \eta - \alpha - 1$. On the other hand, if (b) holds, from (3.3.28), by Theorem 1.3.5-(ii), we obtain

$$x(n) \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1}\omega(n)^{\frac{1}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}l(n)^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}\left(-\frac{\sigma+\rho\beta+1-\eta}{\alpha}-1\right)}, \quad n \to \infty. \tag{3.3.30}$$

Thus it must be

$$\rho = \frac{\sigma+\rho\beta+1-\eta}{\alpha} + 1, \tag{3.3.31}$$

implying that the regularity index of x is given by (3.3.10). Combined this with the assumption $\rho < 0$, we get that $\sigma < \eta - \alpha - 1$. Moreover, using (3.3.10) i.e. (3.3.31), we obtain

$$(-(\sigma + \rho\beta + 1))^{\frac{1}{\alpha}}\left(-\frac{\sigma + \rho\beta + 1 - \eta}{\alpha} - 1\right) = \left((\alpha - \eta - \rho\alpha)(-\rho)^{\alpha}\right)^{\frac{1}{\alpha}}, \tag{3.3.32}$$

and

$$n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1}\xi(n)^{-\frac{1}{\alpha}}\omega(n)^{\frac{1}{\alpha}}l(n)^{\frac{\beta}{\alpha}} = \left(n^{\alpha+1}p(n)^{-1}q(n)\right)^{\frac{1}{\alpha}} x(n)^{\frac{\beta}{\alpha}}. \tag{3.3.33}$$

Then, from (3.3.30) we obtain that the asymptotic representation of x is given by (3.3.23). □

Lemma 3.3.2 *Let $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. For any $x \in \mathbb{M}^{-}_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le \frac{\alpha-\eta}{\alpha}$ only one of the following two statements holds:*
(i) $\rho = \dfrac{\alpha - \eta}{\alpha}$ *and*

$$x(n) \sim \frac{\alpha}{\eta - \alpha} n^{\frac{\alpha-\eta}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} \left(\sum_{k=n}^{\infty} k^{-1}\omega(k)l(k)^{\beta}\right)^{\frac{1}{\alpha}}, \ n \to \infty; \tag{3.3.34}$$

Then, it is $\sigma = \beta\dfrac{\eta - \alpha}{\alpha} - 1$.
(ii) ρ *is given by* (3.3.10) *and* (3.3.23) *holds. Then, it is* $\sigma < \beta\dfrac{\eta - \alpha}{\alpha} - 1$.

Proof. Suppose that (E_2) has a solution $x \in \mathbb{M}^{-}_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le \frac{\alpha-\eta}{\alpha}$, satisfying $x(n) > 0$, $\Delta x(n) < 0$ for $n \ge n_0 + 1 \ge 2$ and expressed with (3.3.2). Using (3.3.1) and (3.3.2) we have (3.3.24). As in the proof of previous lemma, the fact that $x^{[1]}(n) = p(n)(\Delta x(n))^{\alpha} \to 0$ as $n \to \infty$ implies that $\sigma + \rho\beta \le -1$.

If $\sigma + \rho\beta = -1$, then as in the proof of previous lemma we get (3.3.26), where $\Omega(n)$ is given in (3.3.25). Using that $\eta > \alpha$, application of Theorem (1.3.5)-(ii) in (3.3.26) gives us (3.3.34). Thus, $\rho = \frac{\alpha-\eta}{\alpha}$, implying that $\sigma = \beta\frac{\eta-\alpha}{\alpha} - 1$.

Next, we are considering the case $\sigma + \rho\beta < -1$. An application of Theorem 1.3.5-(ii) in (3.3.24) give us (3.3.28) implying, as previously, two possibilities (a) or (b) in (3.3.29). However, the case (a) is not possible, because $\sigma + \rho\beta < -1$ implies

$$-1 = \frac{\sigma + \rho\beta + 1 - \eta}{\alpha} < -\frac{\eta}{\alpha},$$

which is a contradiction with $\eta > \alpha$. Thus, only (b) in (3.3.29) can be valid and so from (3.3.28), as previously, we obtain that ρ is given by (3.3.10) and x satisfies (3.3.23). Since, $\rho < \frac{\alpha-\eta}{\alpha}$ from (3.3.10) we conclude that $\sigma < \frac{\beta\eta}{\alpha} - \beta - 1$. □

Now, we are in a position to prove the main results.

Theorem 3.3.6 *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$.*

(i) *Let $\eta < \alpha$. Equation (E_2) possesses regularly varying solutions x of index $\rho < 0$ if and only if* (3.3.5) *holds.*

(ii) *Let $\eta > \alpha$. Equation (E_2) possesses regularly varying solutions x of index $\rho < \frac{\alpha-\eta}{\alpha}$ if and only if* (3.3.7).

In both cases ρ is given by (3.3.10) *and the asymptotic behavior of any such solution x is governed by the unique formula* (3.3.23).

Proof. **The "only if" part:** Suppose that $\eta < \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho < 0$. According to Theorem 1.3.3-(v) and (vi), $x \in \mathbb{M}^-$ and $\lim_{n\to\infty} x(n) = 0$. It is easy to prove (see [14, Lemma 3]) that if $S = \infty$, then for any solution in the class $\mathbb{M}^-$, it holds $\lim_{n\to\infty} x^{[1]}(n) = 0$. Thus, $x \in \mathbb{M}^-_{0,0}$. Then, it is clear that only the case (ii) of Lemma 3.3.1 is admissible for x. Thus, the regularity index of x is given by (3.3.10) and σ satisfies (3.3.5).

Suppose that $\eta > \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho < \frac{\alpha-\eta}{\alpha}$. Since $\rho < 0$ as previously we conclude that $x \in \mathbb{M}^-_0$. Therewith, in view of (3.3.4), by Theorem 1.3.3-(vi) we get

$$\lim_{n\to\infty} \frac{x(n)}{\pi(n)} = \frac{\eta-\alpha}{\alpha} \lim_{n\to\infty} n^{\varrho-\frac{\alpha-\eta}{\alpha}} l(n)\xi(n)^{\frac{1}{\alpha}} = 0,$$

implying that $x \in \mathbb{M}^-_{0,0}$. It is clear that only the case (ii) of Lemma 3.3.2 is admissible for x, implying that the regularity index of x is given by (3.3.10) and that (3.3.7) holds.

From Lemmas 3.3.1 and 3.3.2 we obtain that the asymptotic representation of regularly varying solution x of index ρ is given by (3.3.23) in each of the two cases (i) and (ii).

The "if" part: We perform the simultaneous proof for both of the cases (i) and (ii). From Theorem 3.3.3 follows the existence of a solution $x \in \mathbb{M}^-_{0,0}$. It remains to prove that x satisfying (3.3.16) or (3.3.18) is a regularly varying sequence of index ρ. From (3.3.16) we have

$$0 < \liminf_{n\to\infty} \frac{x(n)}{X(n)} \le \limsup_{n\to\infty} \frac{x(n)}{X(n)} < \infty,$$

where $X(n)$ is given by (3.3.9). Application of Lemma 1.1.8, using (3.3.12) and (3.3.18), yields

$$\begin{aligned} L &= \limsup_{n\to\infty} \frac{x(n)}{X(n)} \le \limsup_{n\to\infty} \frac{\Delta x(n)}{\Delta X(n)} = \limsup_{n\to\infty} \frac{-\left(\frac{1}{p(k)}\sum_{k=n}^{\infty} q(k)x(k+1)^{\beta}\right)^{1/\alpha}}{-\left(\frac{1}{p(k)}\sum_{k=n}^{\infty} q(k)X(k+1)^{\beta}\right)^{1/\alpha}} \\ &\le \left(\limsup_{n\to\infty} \frac{\sum_{k=n}^{\infty} q(k)x(k+1)^{\beta}}{\sum_{k=n}^{\infty} q(k)X(k+1)^{\beta}}\right)^{1/\alpha} \le \left(\limsup_{n\to\infty} \frac{-q(n)x(n+1)^{\beta}}{-q(n)X(n+1)^{\beta}}\right)^{1/\alpha} \\ &\le \left(\limsup_{n\to\infty} \frac{x(n+1)}{X(n+1)}\right)^{\beta/\alpha} = L^{\frac{\beta}{\alpha}}. \end{aligned}$$

Since $\beta < \alpha$, from above we conclude that $0 < L \leq 1$. Similarly, we can see that $l = \liminf_{n\to\infty} x(n)/X(n)$ satisfies $1 \leq l < \infty$. Therefore, we obtain that $l = L = 1$, which means that $x(n) \sim X(n)$, $n \to \infty$ and ensures that x is a regularly varying solution of (E_2) with requested regularity index and the asymptotic representation given by (3.3.23). □

Theorem 3.3.7 *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. There exists $x \in \mathbb{M}^-_{0,0} \cap ntr - \mathcal{SV}$ if and only if* (3.3.6) *holds. All such solutions of (E) enjoy the precise asymptotic formula*

$$x(n) \sim \left[\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty}q(j)\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty. \tag{3.3.35}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^-_{0,0} \cap ntr - \mathcal{SV}$. Then, clearly, only the statement (i) of Lemma 3.3.1 could hold. Therefore, $\rho = 0$, $\sigma = \eta - \alpha - 1$ and x satisfies (3.3.22). Then, since $\sigma < -1$, application of Theorem 1.3.5 gives

$$\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty}q(j)\right)^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}}\sum_{k=n}^{\infty}k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}}, \quad n \to \infty, \tag{3.3.36}$$

where we used that $\sigma + 1 = \alpha - \eta$. Denote

$$z(n) = \sum_{k=n}^{\infty}k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}}l(k)^{\frac{\beta}{\alpha}}. \tag{3.3.37}$$

From Theorem 1.3.5-(iii) clearly $z = \{z(n)\} \in \mathcal{SV}$ and (3.3.22) becomes

$$x(n) = l(n) \sim \frac{z(n)}{(\alpha-\eta)^{\frac{1}{\alpha}}}, \quad n \to \infty. \tag{3.3.38}$$

From (3.3.37) and (3.3.38) we obtain the asymptotic relation

$$z(n)^{-\frac{\beta}{\alpha}}(-\Delta z(n)) \sim \frac{n^{-1}\xi(n)^{-\frac{1}{\alpha}}\omega(n)^{\frac{1}{\alpha}}}{(\alpha-\eta)^{\frac{\beta}{\alpha^2}}}, \quad n \to \infty. \tag{3.3.39}$$

By (3.3.38), we have that $z(n) \to 0$, $n \to \infty$ and clearly $\{z(n)\}$ is strictly decreasing. Summing (3.3.39) from n to ∞, using Theorem 1.3.4 and (3.3.36), we obtain

$$\begin{aligned}\frac{\alpha}{\alpha-\beta}z(n)^{1-\frac{\beta}{\alpha}} &\sim \frac{1}{(\alpha-\eta)^{\frac{\beta}{\alpha^2}}}\sum_{k=n}^{\infty}k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}} \\ &\sim \frac{1}{(\alpha-\eta)^{\frac{\beta-\alpha}{\alpha^2}}}\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty}q(j)\right)^{\frac{1}{\alpha}}, \ n \to \infty.\end{aligned} \tag{3.3.40}$$

Because $1 - \frac{\beta}{\alpha} > 0$, $z(n)^{1-\frac{\beta}{\alpha}} \to 0$, $n \to \infty$, so (3.3.40) yields that the second condition in (3.3.6) is satisfied as well as that the asymptotic expression for x is

$$\begin{aligned} x(n) &\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \sum_{k=n}^{\infty} k^{-1} \xi(k)^{-\frac{1}{\alpha}} \omega(k)^{\frac{1}{\alpha}} \right)^{\frac{\alpha}{\alpha-\beta}} \\ &\sim \left(\frac{\alpha-\beta}{\alpha} \sum_{k=n}^{\infty} \left(\frac{1}{p(k)} \sum_{j=k}^{\infty} q(j) \right)^{\frac{1}{\alpha}} \right)^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty. \end{aligned}$$

This completes the "only if" part of the proof of Theorem 3.3.7.

The "if" part: From Theorem 3.3.4 we have the existence of a solution $x \in \mathbb{M}^{-}_{0,0}$. In the same way as in the proof of Theorem 3.3.6, replacing $X(n)$ with $T(n)$ given by (3.3.20) and with the application of Lemma 1.1.8 we obtain that $x(n) \sim T(n)$, $n \to \infty$, implying that such a solution is slowly varying and enjoys the precise asymptotic behavior (3.3.35). □

Theorem 3.3.8 *Suppose that* $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ *and* $q \in \mathcal{RV}(\sigma)$. *There exists* $x \in \mathbb{M}^{-}_{0,0} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$ *if and only if* (3.3.8) *holds. All such solutions of (E) enjoy the precise asymptotic behaviour*

$$x(n) \sim \left(\alpha^{\alpha-1} \frac{\alpha-\beta}{(\eta-\alpha)^{\alpha}} \right)^{\frac{1}{\alpha-\beta}} n\, p(n)^{-\frac{1}{\alpha}} \left[\sum_{k=n}^{\infty} k^{\beta} q(k) p(k)^{-\frac{\beta}{\alpha}} \right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{3.3.41}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^{-}_{0,0} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Then, clearly only the statement (i) of Lemma 3.3.2 could hold. Therefore, $\rho = \frac{\alpha-\eta}{\alpha}$, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and x satisfies (3.3.34). From (3.3.2) and (3.3.34) we get

$$l(n) \sim \frac{\alpha}{\eta-\alpha} \xi(n)^{-\frac{1}{\alpha}} \Omega(n)^{\frac{1}{\alpha}}, \ n \to \infty, \tag{3.3.42}$$

where $\Omega(n)$ is given in (3.3.25). From (3.3.25), we conclude that $\Omega \in \mathcal{SV}$, $\Omega(n) \to 0$ as $n \to \infty$ and $\{\Omega(n)\}$ is strictly decreasing. We transform (3.3.42) into the asymptotic relation for Ω

$$\begin{aligned} \Omega(n)^{-\frac{\beta}{\alpha}} \Delta\Omega(n) &\sim -\left(\frac{\alpha}{\eta-\alpha} \right)^{\beta} n^{-1} \omega(n) \xi(n)^{-\frac{\beta}{\alpha}} \\ &= -\left(\frac{\alpha}{\eta-\alpha} \right)^{\beta} n^{\beta} q(n) p(n)^{-\frac{\beta}{\alpha}}, \quad n \to \infty. \end{aligned} \tag{3.3.43}$$

Summing (3.3.43) from n to ∞ and using Theorem 1.3.4 we obtain

$$\frac{\alpha}{\alpha-\beta}\Omega(n)^{1-\frac{\beta}{\alpha}} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta}\sum_{k=n}^{\infty}k^{\beta}q(k)p(k)^{-\frac{\beta}{\alpha}}, \quad n\to\infty. \tag{3.3.44}$$

Because $\Omega(n)^{1-\frac{\beta}{\alpha}} \to 0$ as $n\to\infty$, (3.3.44) yields that the second condition in (3.3.8) is satisfied. The asymptotic expression (3.3.34) for x becomes

$$\begin{aligned} x(n) &\sim \frac{\alpha}{\eta-\alpha}n^{\frac{\alpha-\eta}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}\Omega(n)^{\frac{1}{\alpha}} \\ &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} n\,p(n)^{-\frac{1}{\alpha}}\left[\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty}k^{\beta}q(k)p(k)^{-\frac{\beta}{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n\to\infty. \end{aligned}$$

This completes the "only if" part of the proof of Theorem 3.3.8.

The "if" part: From Theorem 3.3.5 we obtain the existence of a solution $x \in \mathbb{M}^{-}_{0,0}$, while application of Lemma 1.1.8 as in the proof of Theorem 3.3.6, with $Y(n)$ instead of $X(n)$, where $Y(n)$ is given by (3.3.21), proves that $x(n) \sim Y(n)$, $n\to\infty$, so that such a solution is in fact a $\mathcal{RV}$−solution of index $\frac{\alpha-\eta}{\alpha}$, with the precise asymptotic behavior given by (3.3.41). □

Summarizing the results given in this section, we see that the existence of strongly decreasing $\mathcal{RV}$−solutions for the equation (E_2) with $\mathcal{RV}$ coefficients is fully characterized by the assumption $I<\infty$ if $S=\infty$ and by the assumption $J<\infty$ if $S<\infty$. In fact, the following corollary holds.

Corollary 3.3.1 *Suppose that $p\in\mathcal{RV}(\eta)$, $\eta\neq\alpha$ and $q\in\mathcal{RV}(\sigma)$.*

(i) Let $S=\infty$. Equation (E_2) has strongly decreasing $\mathcal{RV}$-solutions if and only if $I_1<\infty$.

(ii) Let $S<\infty$. Equation (E_2) has strongly decreasing $\mathcal{RV}$-solutions if and only if $J_1<\infty$.

Moreover, if $S=\infty$, then $J=\infty$ so by Theorem 3.2.1 $\mathbb{M}^{-}_{0,l}=\emptyset$. Otherwise, if $S<\infty$, denoting the series $Q=\sum_{k=1}^{\infty}q_k$, we have two cases:

(a) If $Q=\infty$, then $I_1=\infty$, so by Theorem 3.2.1 we have $\mathbb{M}^{-}=\mathbb{M}^{-}_{0}$ i.e. $\mathbb{M}^{-}_{B}=\emptyset$.

(b) If $Q<\infty$, then $I_1<\infty$, so by Theorem 3.2.1 we have $\mathbb{M}^{-}=\mathbb{M}^{-}_{0}\cup\mathbb{M}^{-}_{B}$.

3.4 Strongly increasing solutions

As in the previous section, we assume that (3.3.1) and (3.3.2) hold, so again, we have two cases given in (3.3.3).

Case (*i*): Using (3.3.1) and Theorem 1.3.5 we have

$$\Pi(n) = \sum_{k=1}^{n-1} \frac{1}{p(k)^{1/\alpha}} = \sum_{k=1}^{n-1} k^{-\frac{\eta}{\alpha}} \xi(k)^{-\frac{1}{\alpha}} \sim \frac{\alpha}{\alpha-\eta} n^{\frac{\alpha-\eta}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} = \frac{\alpha}{\alpha-\eta} \cdot \frac{n}{p(n)^{1/\alpha}}, \quad n \to \infty, \tag{3.4.1}$$

so that $\{\Pi(n)\} \in \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. For any strongly increasing solution x of equation (E_1) we have that

$$\lim_{n\to\infty} \frac{x(n)}{\Pi(n)} = \infty\,,$$

implying that the index of regularity ρ of strongly increasing solutions must satisfy $\rho \geq \frac{\alpha-\eta}{\alpha}$. If $\rho = \frac{\alpha-\eta}{\alpha}$ then x is a member of $\mathcal{RV}(\frac{\alpha-\eta}{\alpha})$ and if $\rho > \frac{\alpha-\eta}{\alpha}$ then $x \in \mathcal{RV}(\rho)$ and clearly satisfies $x(n)/n^{\frac{\alpha-\eta}{\alpha}} \to \infty$ when $n \to \infty$. Therefore, if $\eta < \alpha$, the totality of strongly increasing $\mathcal{RV}$−solutions of (E_1) will be divided into the following two subclasses:

$$\mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right) \quad \text{or} \quad \mathcal{RV}(\rho) \;\; \text{with} \;\; \rho > \frac{\alpha-\eta}{\alpha}. \tag{3.4.2}$$

Case (*ii*): It is clear that for any strongly increasing solution of (E_1) it holds that $x(n) \geq c$, for large n. Thus, we have that the index of regularity ρ of strongly increasing $\mathcal{RV}$−solution x must satisfy $\rho \geq 0$. If $\rho = 0$ then $l(n) = x(n) \to \infty$ so x is a member of $ntr-\mathcal{SV}$. Therefore, the totality of strongly increasing $\mathcal{RV}$−solutions in case $\eta > \alpha$ will be divided into the following two classes

$$ntr-\mathcal{SV} \quad \text{or} \quad \mathcal{RV}(\rho) \;\; \text{with} \;\; \rho > 0. \tag{3.4.3}$$

For equation (E_2) with arbitrary coefficients, there are only sufficient conditions for the existence of strongly increasing solutions (see Theorem 3.2.3). We further wish to establish the necessary and sufficient conditions for the existence of the solutions which are regularly varying and belong to one of the subclasses given in (3.4.2) and (3.4.3). Moreover, we will show that these solutions of the corresponding subclass have the same asymptotic behavior when $n \to \infty$, where the regularity index of these solutions is uniquely determined by α and β and indices of regularity η and σ, of the coefficients p and q.

3.4.1 Existence and asymptotic representation of strongly increasing $\mathcal{RV}$−solutions

Before giving some results, let us notice that if $\eta > \alpha$ then $\sigma > -1$ is a necessary condition for $I_2 = \infty$. Then, using discrete Karamata theorem and (3.3.1), we have

$$\left(\frac{1}{p(k)}\sum_{j=1}^{k-1} q(j)\right)^{\frac{1}{\alpha}} \sim \frac{1}{(\sigma+1)^{\frac{1}{\alpha}}}\left(\frac{k^{\sigma+1-\eta}\omega(k)}{\xi(k)}\right)^{\frac{1}{\alpha}}, \quad k \to \infty.$$

On the other hand, if $\eta < \alpha$ application of discrete Karamata theorem with the help of (3.4.1) gives

$$q(k)\left(\sum_{j=1}^{k-1}\frac{1}{p(j)^{1/\alpha}}\right)^{\beta} \sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\beta} q(k)\left(\frac{k}{p(k)^{1/\alpha}}\right)^{\beta} = \left(\frac{\alpha}{\alpha-\eta}\right)^{\beta} k^{\sigma+\beta-\frac{\beta}{\alpha}\eta}\,\frac{\omega(k)}{\xi(k)^{\beta/\alpha}}, \quad \text{as} \quad k \to \infty.$$

Consequently

(i) for $\eta < \alpha$, $J_2 = \infty$ if and only if

$$\sigma > \frac{\beta}{\alpha}\eta - \beta - 1 \tag{3.4.4}$$

or

$$\sigma = \frac{\beta}{\alpha}\eta - \beta - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1}\frac{\omega(k)}{\xi(k)^{\frac{\beta}{\alpha}}} = \infty\,; \tag{3.4.5}$$

(ii) for $\eta > \alpha$, $I_2 = \infty$ if and only if

$$\sigma > \eta - \alpha - 1 \tag{3.4.6}$$

or

$$\sigma = \eta - \alpha - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1}\left(\frac{\omega(k)}{\xi(k)}\right)^{\frac{1}{\alpha}} = \infty\,; \tag{3.4.7}$$

To simplify the "only if" part of the proof of main results we prove the next two lemmas.

Lemma 3.4.1 *Let $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. For any $x \in \mathbb{M}^{+}_{\infty,\infty} \cap \mathcal{RV}(\rho)$ with $\rho \geq \frac{\alpha-\eta}{\alpha}$ only one of the following two statements holds:*

(i) $\rho = 1 - \frac{\eta}{\alpha}$ *and*

$$x(n) \sim \frac{\alpha}{\alpha - \eta} n^{\frac{\alpha-\eta}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} \left(\sum_{k=1}^{n-1} k^{-1} \omega(k) l(k)^{\beta} \right)^{\frac{1}{\alpha}}, \; n \to \infty. \tag{3.4.8}$$

Then, it is $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$.
(ii) ρ *is given by* (3.3.10) *and*

$$x(n) \sim \left[\frac{n^{\alpha+1} p(n)^{-1} q(n)}{\rho^{\alpha}(\rho\alpha - \alpha + \eta)} \right]^{\frac{1}{\alpha-\beta}} \quad \text{as} \quad n \to \infty. \tag{3.4.9}$$

Then, it is $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$.

Proof. Suppose that (E_1) has a solution $x \in \mathbb{M}^{+}_{\infty,\infty} \cap \mathcal{RV}(\rho)$ with $\rho \geq \frac{\alpha-\eta}{\alpha}$, satisfying $x(n) > 0$, $\Delta x(n) > 0$ for $n \geq n_0 + 1 \geq 2$ and expressed with (3.3.2). Summing (E_1), for $n \geq n_0 + 1$ we get

$$p(n)(\Delta x(n))^{\alpha} = p(n_0)(\Delta x(n_0))^{\alpha} + \sum_{k=n_0}^{n-1} q(k) x(k+1)^{\beta}$$

which yields, using (3.3.1) and (3.3.2), that

$$p(n)(\Delta x(n))^{\alpha} \sim \sum_{k=n_0}^{n-1} q(k) x(k)^{\beta} = \sum_{k=n_0}^{n-1} k^{\sigma+\rho\beta} \omega(k) l(k)^{\beta}, \quad n \to \infty. \tag{3.4.10}$$

The fact that $x^{[1]}(n) = p(n)(\Delta x(n))^{\alpha} \to \infty$ when $n \to \infty$ implies

$$\lim_{n\to\infty} \sum_{k=n_0}^{n-1} k^{\sigma+\rho\beta} \omega(k) l(k)^{\beta} = \infty,$$

so it must be $\sigma + \rho\beta \geq -1$. We first consider the case $\sigma + \rho\beta = -1$. Then,

$$p(n)(\Delta x(n))^{\alpha} \sim \sum_{k=n_0}^{n-1} k^{-1} \omega(k) l(k)^{\beta} \sim \sum_{k=1}^{n-1} k^{-1} \omega(k) l(k)^{\beta} = H(n), \tag{3.4.11}$$

as $n \to \infty$, where $H = \{H(n)\} \in \mathcal{SV}$ and $H(n) \to \infty$, $n \to \infty$. Consequently

$$\Delta x(n) \sim \left(\frac{H(n)}{p(n)} \right)^{\frac{1}{\alpha}} = n^{-\frac{\eta}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} H(n)^{\frac{1}{\alpha}}, \quad n \to \infty. \tag{3.4.12}$$

Summing (3.4.12) from $n_0 + 1$ to n, using that $\eta < \alpha$, with application of Theorem 1.3.5-(i) and Remark 1.3.1, we get (3.4.8). Therefore, $\rho = 1 - \frac{\eta}{\alpha}$. From $\sigma + \rho\beta = -1$ it follows that $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$.

Next we consider the case $\sigma + \rho\beta > -1$. An application of Theorem 1.3.5-(i) in (3.4.10) implies

$$\Delta x(n) = \left(\frac{1}{p(n)} \sum_{k=n_0}^{n-1} q(k) x(k+1)^{\beta} \right)^{\frac{1}{\alpha}} \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}} \omega(n)^{\frac{1}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} l(n)^{\frac{\beta}{\alpha}}}{(\sigma + \rho\beta + 1)^{\frac{1}{\alpha}}}, \tag{3.4.13}$$

as $n \to \infty$. Since, $\sigma + \rho\beta > -1$ and $\eta < \alpha$ imply $(\sigma + \rho\beta + 1 - \eta)/\alpha > -1$, summing (3.4.13) from $n_0 + 1$ to $n - 1$ with application of Theorem 1.3.5-(i) gives

$$x(n) \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1} \omega(n)^{\frac{1}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} l(n)^{\frac{\beta}{\alpha}}}{(\sigma + \rho\beta + 1)^{\frac{1}{\alpha}} \left(\frac{\sigma+\rho\beta+1-\eta}{\alpha} + 1 \right)}, \quad n \to \infty. \tag{3.4.14}$$

Thus, it holds that $\rho = \frac{\sigma+\rho\beta+1-\eta}{\alpha} + 1$ implying that regularity index of x is given with

$$\rho = \frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}$$

which is also given by (3.3.10). Combined this with $\sigma + \rho\beta > -1$, we get $\sigma + \rho\beta > -1$. Moreover, using (3.3.10) we obtain

$$(\sigma + \rho\beta + 1)^{\frac{1}{\alpha}} \left(\frac{\sigma + \rho\beta + 1 - \eta}{\alpha} + 1 \right) = \left(\rho^{\alpha} \left(\rho\alpha + \eta - \alpha \right) \right)^{\frac{1}{\alpha}}$$

and

$$n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1} \omega(n)^{\frac{1}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} l(n)^{\frac{\beta}{\alpha}} = \left(n^{\alpha+1} p(n)^{-1} q(n) \right)^{\frac{1}{\alpha}} x(n)^{\frac{\beta}{\alpha}}.$$

Then, from (3.4.14) we obtain that the asymptotic representation of x is given by (3.4.9). □

Lemma 3.4.2 *Let $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. For any $x \in \mathbb{M}^{+}_{\infty,\infty} \cap \mathcal{RV}(\rho)$ with $\rho \geq 0$ only one of the following two statements holds:*

(i) $\rho = 0$ *and*

$$x(n) \sim \frac{1}{(\sigma + \rho\beta + 1)^{\frac{1}{\alpha}}} \sum_{k=1}^{n-1} k^{-1} \xi(k)^{-\frac{1}{\alpha}} \omega(k)^{\frac{1}{\alpha}} l(k)^{\frac{\beta}{\alpha}} \; n \to \infty. \tag{3.4.15}$$

Then, it is $\sigma = \eta - \alpha - 1$.

(ii) ρ *is given by* (3.3.10) *and* (3.4.9) *holds. Then,* $\sigma > \eta - \alpha - 1$.

Proof. Suppose that (E_1) has a solution $x \in \mathbb{M}^+_{\infty,\infty} \cap \mathcal{RV}(\rho)$ with $\rho \geq 0$, satisfying $x(n) > 0$ and $\Delta x(n) > 0$ for $n \geq n_0 + 1 \geq 2$ and expressed with (3.3.2). Using (3.3.1) and (3.3.2) we have (3.4.10). As in the proof of previous lemma, the fact that $x^{[1]}(n) = p(n)(\Delta x(n))^\alpha \to \infty$ when $n \to \infty$ implies that $\sigma + \rho\beta \geq -1$, although the case $\sigma + \rho\beta = -1$ is now impossible. Indeed, from (3.4.12), since $\lim_{n\to\infty} x(n) = \infty$, we obtain

$$\lim_{n\to\infty} \sum_{k=n_0}^{n-1} n^{-\frac{\eta}{\alpha}} \xi(n)^{-\frac{1}{\alpha}} H(n)^{\frac{1}{\alpha}} = \infty,$$

but this is the contradiction with our assumption $\eta > \alpha$.

Therefore, we have that $\sigma + \rho\beta > -1$ and obtain (3.4.13) with an application of Theorem 1.3.5-(i) in (3.4.10). Because $x(n) \to \infty$, $n \to \infty$, we conclude from (3.4.13) that it must be $(\sigma + \rho\beta + 1 - \eta)/\alpha \geq -1$, so we distinguish two possibilities:

$$\text{(a)}\ \ \frac{\sigma + \rho\beta + 1 - \eta}{\alpha} = -1, \qquad \text{(b)}\ \ \frac{\sigma + \rho\beta + 1 - \eta}{\alpha} > -1\,.$$

If (a) holds, summing (3.4.13) from $n_0 + 1$ to $n - 1$, we obtain (3.4.15) which shows that $x \in \mathcal{SV}$ according to Theorem 1.3.5-(iv). On the other hand, if (b) holds, as in the proof of Lemma 3.4.1 we show that (3.4.9) holds and that $\sigma > \eta - \alpha - 1$ using that $\rho > 1 - \frac{\eta}{\alpha}$. □

We are now in a position to give necessary and sufficient conditions for the existence of regularly varying solutions from the class $\mathbb{M}^+_{\infty,\infty}$ and to give precise asymptotic representation of all these solutions.

Theorem 3.4.1 *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$.*

(i) *Let $\eta < \alpha$. Equation (E_1) possesses a regularly varying solution x of index $\rho > \frac{\alpha-\eta}{\alpha}$ if and only if* (3.4.4) *holds.*

(ii) *Let $\eta > \alpha$. Equation (E_1) possesses a regularly varying solution x of index $\rho > 0$ if and only if* (3.4.6) *holds.*

In both cases ρ is given by (3.3.10) *and the asymptotic behavior of any such solution x is governed by the unique formula* (3.4.9).

Proof. **The "only if" part:** Suppose that $\eta < \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho > \frac{\alpha-\eta}{\alpha}$. According to Theorem 1.3.3-(v) and (vi), since $\rho > 0$, we conclude that $x \in \mathbb{M}^+$ and $\lim_{n\to\infty} x(n) = \infty$. How

$$\lim_{n\to\infty} \frac{x(n)}{\Pi(n)} = \lim_{n\to\infty} \frac{n^\rho l(n)}{\frac{\alpha}{\alpha-\eta} \cdot \frac{n}{n^{\frac{\eta}{\alpha}} \xi(n)^{\frac{1}{\alpha}}}} = \frac{\alpha - \eta}{\alpha} \lim_{n\to\infty} n^{\rho - \frac{\alpha-\eta}{\alpha}}\, l(n) \xi(n)^{\frac{1}{\alpha}} = \infty$$

it follows that $x \in \mathbb{M}^+_{\infty,\infty}$. Then, it is clear that only the case (ii) of Lemma 3.4.1 is admissible for x. Thus the regularity index of x is given by (3.3.10) and σ satisfies (3.4.4).

Suppose that $\eta > \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho > 0$. As previous, $x \in \mathbb{M}^+$ and $\lim_{n\to\infty} x(n) = \infty$. It is easy to show that it must be $\lim_{n\to\infty} x^{[1]}(n) = \infty$. Then, it is clear that only case (ii) of Lemma 3.4.2 is admissible for x, implying that the regularity index of x is given by (3.3.10) and (3.4.6) holds. From Lemmas 3.4.1 and 3.4.2 we obtain that the asymptotic representation of regularly varying solution x of index ρ is given by (3.4.9) in each of two cases (i) and (ii).

The "if" part: Suppose either $\eta < \alpha$ and (3.4.4) holds or $\eta > \alpha$ and (3.4.6) holds. Denote

$$X(n) = \left[\frac{n^{\alpha+1}p(n)^{-1}q(n)}{\rho^{\alpha}(\rho\alpha - \alpha + \eta)}\right]^{\frac{1}{\alpha-\beta}}, \qquad n \geq 1,$$

and $\lambda = \rho^{\alpha}(\rho\alpha - \alpha + \eta)$, where ρ is given by (3.3.10). Clearly, $X = \{X(n)\} \in \mathcal{RV}(\rho)$ and it may be expressed in the form

$$X(n) = \lambda^{-\frac{1}{\alpha-\beta}} n^{\rho} \xi(n)^{-\frac{1}{\alpha-\beta}} \omega(n)^{\frac{1}{\alpha-\beta}} . \tag{3.4.16}$$

Notice that (3.4.4) and (3.3.10) imply that $\rho > \frac{\alpha-\eta}{\alpha}$, while (3.4.6) and (3.3.10) imply that $\rho > 0$. Therefore, by Theorem 1.3.3 $\lim_{n\to\infty} X(n) = \infty$.

Let us first prove that sequence X satisfies the asymptotic relation

$$\sum_{k=2}^{n-1}\left(\frac{1}{p(k)}\sum_{j=1}^{k-1} q(j)X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} \sim X(n), \qquad n \to \infty. \tag{3.4.17}$$

Using (3.3.1), by application of Theorem 1.3.5-(i) and Theorem 1.3.3-(iv) we get

$$\begin{aligned}
\sum_{k=1}^{n-1} q(k)X(k+1)^{\beta} &= \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=1}^{n-1} k^{\sigma+\rho\beta}\xi(k)^{-\frac{\beta}{\alpha-\beta}}\omega(k)^{\frac{\alpha}{\alpha-\beta}} \\
&= \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=1}^{n-1} k^{\alpha(\rho-1)+\eta-1}\xi(k)^{-\frac{\beta}{\alpha-\beta}}\omega(k)^{\frac{\alpha}{\alpha-\beta}} \\
&\sim \lambda^{-\frac{\beta}{\alpha-\beta}} \frac{n^{\alpha(\rho-1)+\eta}\xi(n)^{-\frac{\beta}{\alpha-\beta}}\omega(n)^{\frac{\alpha}{\alpha-\beta}}}{\alpha(\rho-1)+\eta}, \quad n \to \infty.
\end{aligned} \tag{3.4.18}$$

From (3.4.18), applying Theorem 1.3.5-(i), we obtain the desired asymptotic relation for X:

$$\begin{aligned}
\sum_{k=2}^{n-1}\left(\frac{1}{p(k)}\sum_{j=1}^{k-1} q(j)X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} &\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} (\alpha(\rho-1)+\eta)^{-\frac{1}{\alpha}} \sum_{k=2}^{n-1} k^{\rho-1}\xi(k)^{-\frac{1}{\alpha-\beta}}\omega(k)^{\frac{1}{\alpha-\beta}} \\
&\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} (\alpha(\rho-1)+\eta)^{-\frac{1}{\alpha}} \frac{n^{\rho}\xi(n)^{-\frac{1}{\alpha-\beta}}\omega(n)^{\frac{1}{\alpha-\beta}}}{\rho} \\
&= \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \cdot \lambda^{-\frac{1}{\alpha}} n^{\rho}\xi(n)^{-\frac{1}{\alpha-\beta}}\omega(n)^{\frac{1}{\alpha-\beta}} = X(n), \qquad n \to \infty .
\end{aligned}$$

Thus, there exists $n_0 > 1$ such that

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p(k)}\sum_{j=n_0-1}^{k-1} q(j)X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} \le 2X(n), \quad n \ge n_0\,. \tag{3.4.19}$$

Let such n_0 be fixed. We may assume that $X(n)$ is increasing for $n \ge n_0$ (see Theorem 1.3.3−(vii)). Since from (3.4.17) we have

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p(k)}\sum_{j=n_0-1}^{k-1} q(j)X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} \sim X(n), \quad n \to \infty,$$

there exists $n_1 > n_0$ such that

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p(k)}\sum_{j=n_0-1}^{k-1} q(j)X(j+1)^{\beta}\right)^{\frac{1}{\alpha}} \ge \frac{X(n)}{2}, \quad n \ge n_1. \tag{3.4.20}$$

Let such n_1 be fixed. We choose constants $\kappa \in (0,1)$ and $K > 1$ such that

$$\kappa^{1-\frac{\beta}{\alpha}} \le \frac{1}{2}, \qquad K^{1-\frac{\beta}{\alpha}} \ge 4 \quad \text{and} \quad K \ge 2\kappa\frac{X(n_1)}{X(n_0)}\,. \tag{3.4.21}$$

Consider the space Υ_{n_0} of all real sequences $x = \{x(n)\}_{n=n_0}^{\infty}$ such that $\{\frac{x(n)}{X(n)}\}$ is bounded. Then, Υ_{n_0} is a Banach space endowed with the norm

$$\|x\| = \sup_{n\ge n_0}\frac{x(n)}{X(n)}\,,$$

Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\le$: for $x, y \in \Upsilon_{n_0}$, $x \le y$ means $x(n) \le y(n)$ for all $n \ge n_0$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ like

$$\mathcal{X} = \{x \in \Upsilon_{n_0} : \kappa X(n) \le x(n) \le KX(n),\, n \ge n_0\,\}. \tag{3.4.22}$$

For any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. We will define the operator $\mathcal{F} : \mathcal{X} \to \Upsilon_{n_0}$ by

$$\Big(\mathcal{F}x\Big)(n) = x_0 + \sum_{k=n_0}^{n-1}\left(\frac{1}{p(k)}\sum_{j=n_0-1}^{k-1} q(j)x(j+1)^{\beta}\right)^{\frac{1}{\alpha}}, \quad n \ge n_0, \tag{3.4.23}$$

where $x_0 > 0$ satisfies

$$\kappa X(n_1) \le x_0 \le \frac{K}{2}X(n_0), \tag{3.4.24}$$

and show that $\mathcal{F}$ has a fixed point by using Theorem 1.1.1. Namely, the operator $\mathcal{F}$ has the following properties:

(i) *Operator $\mathcal{F}$ maps $\mathcal{X}$ into itself*: Let $x \in \mathcal{X}$. Using (3.4.19), (3.4.21), (3.4.22), (3.4.23) and (3.4.24), we get

$$\begin{aligned}(\mathcal{F}x)(n) &\leq x_0 + K^{\frac{\beta}{\alpha}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p(k)} \sum_{j=n_0-1}^{k-1} q(j) X(j+1)^{\beta} \right)^{\frac{1}{\alpha}} \\ &\leq \frac{K}{2} X(n_0) + 2K^{\frac{\beta}{\alpha}} X(n) \leq \frac{K}{2} X(n) + \frac{K}{2} X(n) = K\,X(n), \quad n \geq n_0 .\end{aligned}$$

On the other hand, using (3.4.20), (3.4.21), (3.4.22), (3.4.23) and (3.4.24), we have

$$(\mathcal{F}x)(n) \geq \kappa^{\frac{\beta}{\alpha}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p(k)} \sum_{j=n_0-1}^{k-1} q(j) X(j+1)^{\beta} \right)^{\frac{1}{\alpha}} \geq \kappa^{\frac{\beta}{\alpha}} \frac{X(n)}{2} \geq \kappa X(n), \quad n \geq n_1$$

and

$$(\mathcal{F}x)(n) \geq x_0 \geq \kappa X(n_1) \geq \kappa X(n), \quad n_0 \leq n \leq n_1 .$$

This shows that $(\mathcal{F}x)(n) \in \mathcal{X}$, for all $n \geq n_0$, that is, $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$.

(ii) *Operator $\mathcal{F}$ is increasing*, i.e. for any $x, y \in \mathcal{X}$, $x \leq y$ implies $\mathcal{F}x \leq \mathcal{F}y$.

Thus all the hypotheses of Theorem 1.1.1 are fulfilled implying the existence of a fixed point $x \in \mathcal{X}$ of $\mathcal{F}$, satisfying

$$x(n) = x_0 + \sum_{k=n_0}^{n-1} \left(\frac{1}{p(k)} \sum_{j=n_0-1}^{k-1} q(j) x(j+1)^{\beta} \right)^{\frac{1}{\alpha}}, \qquad n \geq n_0 . \tag{3.4.25}$$

It is clear in view of (3.4.22) and the fact that $X(n) \to \infty$, when $n \to \infty$, that x is a positive solution of (E_1) which satisfies $x(n) \to \infty$, $n \to \infty$. Moreover, due to (3.4.16) and (3.4.22)

$$p(n)(\Delta x(n))^{\alpha} \geq \kappa^{\beta} \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=n_0-1}^{n-1} k^{\sigma+\rho\beta} (\omega(k)/\xi(k))^{\frac{1}{\alpha-\beta}} . \tag{3.4.26}$$

Since, $\eta < \alpha$ and (3.4.4) as well as $\eta > \alpha$ and (3.4.6) imply that $\sigma + \rho\beta > -1$, from (3.4.26) we conclude that $x^{[1]}(n) \to \infty$, $n \to \infty$, that is $x \in \mathbb{M}^{+}_{\infty,\infty}$.

It remains to prove that x satisfying (3.4.22) is a regularly varying sequence of index ρ. From (3.4.22) we have

$$0 < \liminf_{n\to\infty} \frac{x(n)}{X(n)} \leq \limsup_{n\to\infty} \frac{x(n)}{X(n)} < \infty\,. \tag{3.4.27}$$

Application of Lemma 1.1.8, using (3.4.17) and (3.4.25), yields

$$\begin{aligned}
L &= \limsup_{n\to\infty} \frac{x(n)}{X(n)} \leq \limsup_{n\to\infty} \frac{\Delta x(n)}{\Delta X(n)} = \limsup_{n\to\infty} \frac{\left(\frac{1}{p(k)}\sum_{k=1}^{n-1} q(k)x(k+1)^{\beta}\right)^{1/\alpha}}{\left(\frac{1}{p(k)}\sum_{k=1}^{n-1} q(k)X(k+1)^{\beta}\right)^{1/\alpha}} \\
&\leq \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^{n-1} q(k)x(k+1)^{\beta}}{\sum_{k=1}^{n-1} q(k)X(k+1)^{\beta}}\right)^{1/\alpha} \leq \left(\limsup_{n\to\infty} \frac{q(n)x(n+1)^{\beta}}{q(n)X(n+1)^{\beta}}\right)^{1/\alpha} \\
&\leq \left(\limsup_{n\to\infty} \frac{x(n+1)}{X(n+1)}\right)^{\beta/\alpha} = L^{\frac{\beta}{\alpha}}.
\end{aligned}$$

Since $\beta < \alpha$, from the above we conclude that

$$0 < L \leq 1\,. \tag{3.4.28}$$

Similarly, we can see that $l = \liminf_{n\to\infty} x(n)/X(n)$ satisfies

$$1 \leq l < \infty\,. \tag{3.4.29}$$

From (3.4.28) and (3.4.29) we obtain that $l = L = 1$, which means that $x(n) \sim X(n)$, $n \to \infty$ and ensures that x is a regularly varying solution of (E_1) with requested regularity index and the asymptotic representation given by (3.4.9). □

Theorem 3.4.2 *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. There exists $x \in \mathbb{M}^{+}_{\infty,\infty} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$ if and only if* (3.4.5) *holds. All such solutions of (E_1) have the precise asymptotic property*

$$x(n) \sim \left(\alpha^{\alpha-1}\frac{\alpha-\beta}{(\alpha-\eta)^{\alpha}}\right)^{\frac{1}{\alpha-\beta}} \frac{n}{p(n)^{1/\alpha}} \left[\sum_{k=1}^{n-1} k^{\beta} q(k) p(k)^{-\frac{\beta}{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{3.4.30}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^{+}_{\infty,\infty} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Then, clearly only the statement (i) of Lemma 3.4.1 could hold and (3.4.11) is satisfied.Therefore, $\rho = \frac{\alpha-\eta}{\alpha}$ and $\sigma + \rho\beta = -1$ implies $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and x satisfies (3.4.8). From (3.3.2) and (3.4.8) we get

$$l(n) \sim \frac{\alpha}{\alpha-\eta}\xi(n)^{-\frac{1}{\alpha}} H(n),\ n \to \infty, \tag{3.4.31}$$

where $H(n)$ is given by (3.4.11). Using (3.4.11) we transform (3.4.31) into the asymptotic relation for H:

$$\begin{aligned} H(n)^{-\frac{\beta}{\alpha}}\Delta H(n) &\sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\beta} n^{-1}\omega(n)\xi(n)^{-\frac{\beta}{\alpha}} \\ &= \left(\frac{\alpha}{\alpha-\eta}\right)^{\beta} n^{\beta}q(n)p(n)^{-\frac{\beta}{\alpha}}, \quad n\to\infty. \end{aligned} \tag{3.4.32}$$

Summing (3.4.32) for $n \geq 2$ and using Theorem 1.3.4 we obtain

$$\frac{\alpha}{\alpha-\beta}H(n)^{1-\frac{\beta}{\alpha}} \sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\beta}\sum_{k=1}^{n-1}k^{\beta}q(k)p(k)^{-\frac{\beta}{\alpha}}, \quad n\to\infty. \tag{3.4.33}$$

Because $1-\frac{\beta}{\alpha}>0$, $H(n)^{1-\frac{\beta}{\alpha}}\to\infty$, $n\to\infty$, (3.4.33) yields that the second condition in (3.4.5) is satisfied. The asymptotic expression (3.4.8) for x becomes

$$x(n) \sim \frac{\alpha}{\alpha-\eta}n^{\frac{\alpha-\eta}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}H(n)^{\frac{1}{\alpha}} \sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\frac{\alpha}{\alpha-\beta}}\frac{n}{p(n)^{1/\alpha}}\left[\frac{\alpha-\beta}{\alpha}\sum_{k=1}^{n-1}k^{\beta}q(k)p(k)^{-\frac{\beta}{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n\to\infty.$$

This completes the "only if" part of the proof of Theorem 3.4.2.

The "if" part: Suppose (3.4.5) holds. Define sequences $Y=\{Y(n)\}$ and $W=\{W(n)\}$ with

$$W(n)=\sum_{k=1}^{n-1}k^{-1}\omega(k)\xi(k)^{-\frac{\beta}{\alpha}}, \quad Y(n)=\left(\frac{\alpha}{\alpha-\eta}\right)^{\frac{\alpha}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}}\frac{n}{p(n)^{1/\alpha}}W(n)^{\frac{1}{\alpha-\beta}}, \ n\geq 2\,.$$

Note that $W\in ntr-\mathcal{SV}$ and since $n/p(n)^{1/\alpha}=n^{\frac{\alpha-\eta}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}$, we see that $Y\in\mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Thus, the application of Theorem 1.3.4 gives

$$\begin{aligned} \sum_{k=1}^{n-1}q(k)Y(k+1)^{\beta} &= \left(\frac{\alpha}{\alpha-\eta}\right)^{\frac{\alpha\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\sum_{k=1}^{n-1}k^{-1}\omega(k)\xi(k)^{-\frac{\beta}{\alpha}}W(k+1)^{\frac{\beta}{\alpha-\beta}} \\ &\sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\frac{\alpha\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\sum_{k=1}^{n-1}\Delta W(k)\cdot W(k+1)^{\frac{\beta}{\alpha-\beta}} \\ &\sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\frac{\alpha\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}}W(n)^{\frac{\alpha}{\alpha-\beta}}, \ n\to\infty, \end{aligned}$$

which yields with the help of Theorem 1.3.5-(i)

$$\sum_{k=2}^{n-1}\left(\frac{1}{p(k)}\sum_{j=1}^{k-1}q(j)Y(j+1)^{\beta}\right)^{\frac{1}{\alpha}} \sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\frac{\beta}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}}\sum_{k=2}^{n-1}k^{-\frac{\eta}{\alpha}}\xi(k)^{-\frac{1}{\alpha}}W(k)^{\frac{1}{\alpha-\beta}}$$
$$\sim \left(\frac{\alpha}{\alpha-\eta}\right)^{\frac{\alpha}{\alpha-\beta}}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} n^{\frac{\alpha-\eta}{\alpha}}\xi(n)^{-\frac{1}{\alpha}}W(n)^{\frac{1}{\alpha-\beta}} = Y(n), \;\; n\to\infty.$$

Therefore, $Y = \{Y(n)\}$ satisfies the asymptotic relation (3.4.17). Then, proceeding exactly as in the proof of Theorem 3.4.1, replacing $X(n)$ with $Y(n)$, solution x satisfying $\kappa\, Y(n) \le x(n) \le K\, Y(n)$, for large n, is obtained by the application of the Knaster-Tarski fixed point theorem (Theorem 1.1.1), while application of Lemma 1.1.8 proves that $x(n) \sim Y(n)$, $n \to \infty$, showing that such solution is in fact $\mathcal{RV}-$solution of index $\frac{\alpha-\eta}{\alpha}$, with the precise asymptotic behavior given by (3.4.30). □

Theorem 3.4.3 *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. There exists $x \in \mathbb{M}^{+}_{\infty,\infty} \cap ntr - \mathcal{SV}$ if and only if* (3.4.7) *holds. All such solutions of* (E_1) *have the precise asymptotic property*

$$x(n) \sim \left[\frac{\alpha-\beta}{\alpha}\sum_{k=2}^{n-1}\left(\frac{1}{p(k)}\sum_{j=1}^{k-1}q(j)\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha}{\alpha-\beta}}, \quad n\to\infty. \tag{3.4.34}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^{+}_{\infty,\infty} \cap ntr - \mathcal{SV}$. Then, clearly only the statement (i) of Lemma 3.4.2 could hold. Therefore, $\rho = 0$, $\sigma = \eta - \alpha - 1$ and x satisfies (3.4.15). Then, since $\sigma > -1$, application of Theorem 1.3.5 gives

$$\sum_{k=2}^{n-1}\left(\frac{1}{p(k)}\sum_{j=1}^{k-1}q(j)\right)^{\frac{1}{\alpha}} \sim \frac{1}{(\eta-\alpha)^{\frac{1}{\alpha}}}\sum_{k=1}^{n-1}k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}}, \quad n\to\infty. \tag{3.4.35}$$

Denote $z(n) = \sum_{k=n_0}^{n-1} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}}l(k)^{\frac{\beta}{\alpha}}$. Then (3.4.15) becomes

$$x(n) = l(n) \sim \frac{z(n)}{(\eta-\alpha)^{\frac{1}{\alpha}}}, \;\; n\to\infty, \tag{3.4.36}$$

so that $z(n) \to \infty$, $n \to \infty$. By Theorem 1.3.5-(iv) clearly $z = \{z(n)\} \in ntr - \mathcal{SV}$. From (3.4.36) we obtain the asymptotic relation:

$$z(n)^{-\frac{\beta}{\alpha}}\Delta z(n) \sim \frac{n^{-1}\xi(n)^{-\frac{1}{\alpha}}\omega(n)^{\frac{1}{\alpha}}}{(\eta-\alpha)^{\frac{\beta}{\alpha^2}}}, \quad n\to\infty, \tag{3.4.37}$$

where we used that $\sigma + \rho\beta + 1 = \eta - \alpha$. Summing (3.4.37) for $n \geq 2$, using Theorem 1.3.4 and (3.4.35), we obtain

$$\begin{aligned}\frac{\alpha}{\alpha-\beta} z(n)^{1-\frac{\beta}{\alpha}} &\sim \frac{1}{(\eta-\alpha)^{\frac{\beta}{\alpha^2}}} \sum_{k=1}^{n-1} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}} \\ &\sim \frac{1}{(\eta-\alpha)^{\frac{\beta-\alpha}{\alpha^2}}} \sum_{k=2}^{n-1} \left(\frac{1}{p(k)} \sum_{j=1}^{k-1} q(j) \right)^{\frac{1}{\alpha}}, \ n \to \infty.\end{aligned} \tag{3.4.38}$$

Because $1 - \frac{\beta}{\alpha} > 0$, $z(n)^{1-\frac{\beta}{\alpha}} \to \infty$, $n \to \infty$, so (3.4.38) yields that the second condition in (3.4.7) is satisfied as well as that the asymptotic expression for x is

$$x(n) \sim \frac{1}{(\eta-\alpha)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \sum_{k=2}^{n-1} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}} \right)^{\frac{\alpha}{\alpha-\beta}} \sim \left(\frac{\alpha-\beta}{\alpha} \sum_{k=2}^{n-1} \left(\frac{1}{p(k)} \sum_{j=1}^{k-1} q(j) \right)^{\frac{1}{\alpha}} \right)^{\frac{\alpha}{\alpha-\beta}}, \ n \to \infty.$$

This completes the "only if" part of the proof of Theorem 3.4.3.

The "if" part: Suppose (3.4.7) holds. Define sequences $T = \{T(n)\}$ and $G = \{G(n)\}$ with

$$G(n) = \sum_{k=2}^{n-1} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}}, \quad T(n) = \left(\frac{\alpha-\beta}{\alpha} \sum_{k=2}^{n-1} \left(\frac{1}{p(k)} \sum_{j=1}^{k-1} q(j) \right)^{\frac{1}{\alpha}} \right)^{\frac{\alpha}{\alpha-\beta}}, \quad n \geq 3\,.$$

Because (3.4.7) implies $\sigma > -1$, application of Theorem 1.3.5 gives (3.4.35), so that

$$T(n) \sim \frac{1}{(\eta-\alpha)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \right)^{\frac{\alpha}{\alpha-\beta}} G(n)^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty$$

Clearly, $G \in ntr - \mathcal{SV}$ and $T \in ntr - \mathcal{SV}$. Applying Theorem 1.3.5-(i) and using the first condition from (3.4.7) we get

$$\sum_{k=1}^{n-1} q(k)T(k+1)^{\beta} \sim \frac{1}{(\eta-\alpha)^{\frac{\alpha}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \right)^{\frac{\alpha\beta}{\alpha-\beta}} n^{\eta-\alpha}\omega(n)G(n)^{\frac{\alpha\beta}{\alpha-\beta}}, \ n \to \infty.$$

As a result of the application of Theorem 1.3.4 to the previous relation, we obtain

$$\begin{aligned}\sum_{k=2}^{n-1} \left(\frac{1}{p(k)} \sum_{j=1}^{k-1} q(j)T(j+1)^{\beta} \right)^{\frac{1}{\alpha}} &\sim \frac{1}{(\eta-\alpha)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=2}^{n-1} k^{-1}\xi(k)^{-\frac{1}{\alpha}}\omega(k)^{\frac{1}{\alpha}} G(k)^{\frac{\beta}{\alpha-\beta}} \\ &\sim \frac{1}{(\eta-\alpha)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=2}^{n-1} \Delta G(k) \cdot G(k)^{\frac{\beta}{\alpha-\beta}} \\ &\sim \frac{1}{(\eta-\alpha)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \right)^{\frac{\alpha}{\alpha-\beta}} G(n)^{\frac{\alpha}{\alpha-\beta}} \sim T(n), \quad n \to \infty.\end{aligned}$$

Consequently, we conclude that T satisfies the asymptotic relation (3.4.17).

The rest of the proof is the same as the proof of Theorem 3.4.1 where $X(n)$ is replaced with $T(n)$. Then, solution x of equation (E_1) satisfying $\kappa\, T(n) \leq x(n) \leq K\, T(n)$, for large n, is obtained by the application of the Knaster-Tarski fixed point theorem (Theorem 1.1.1), while application of Lemma 1.1.8 proves that $x(n) \sim T(n)$, $n \to \infty$, showing that such solution is a nontrivial, slowly varying, and has the precise asymptotic behavior (3.4.34). □

According to the results given in this section, we can say that the existence of strongly increasing $\mathcal{RV}-$solutions is fully characterized by the assumption $J_2 = \infty$ if $S = \infty$ and by the assumption $I_2 = \infty$ if $S < \infty$. In fact, the following corollary holds.

Corollary 3.4.1 *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta \neq \alpha$ and $q \in \mathcal{RV}(\sigma)$.*

(i) Let $S = \infty$. Equation (E_1) has strongly increasing $\mathcal{RV}-$solutions if and only if $J_2 = \infty$.

(ii) Let $S < \infty$. Equation (E_1) has strongly increasing $\mathcal{RV}-$solutions if and only if $I_2 = \infty$.

3.5 Complete structure of the class of $\mathcal{RV}$-solutions

In order to describe a set of regularly varying solutions, we will use the following symbols:

* $\mathcal{R}$ denote the set of all regularly varying solutions,
* $\mathcal{R}^-$ denote the set of all decreasing regularly varying solutions,
* $\mathcal{R}^+$ denote the set of all increasing regularly varying solutions,
* $\mathcal{R}_0^- = \mathcal{R} \cap \mathbb{M}_0^-$.
* $\mathcal{R}_{0,0}^- = \mathcal{R} \cap \mathbb{M}_{0,0}^-$.

Using conclusions from Theorem 3.2.1 and Corollary 3.3.1, we get the next two corollaries:

Corollary 3.5.1 *Suppose that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $S = \infty$. Then,*

$$\mathcal{R}^- = ntr - \mathcal{SV} \cup \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}_B^-$$

if and only if $I_1 < \infty$.

Corollary 3.5.2 *Suppose that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $S < \infty$. Then,*

(i) If $\sigma < -1$ or $\sigma = -1$ and $Q < \infty$, then

$$\mathcal{R}^- = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}_{0,l}^- \cup \mathbb{M}_B^-.$$

(ii) If $\sigma = -1$ *and* $Q = \infty$ *or* $-1 < \sigma < \frac{\beta\eta}{\alpha} - \beta - 1$, *then*

$$\mathcal{R}^- = \mathcal{R}_0^- = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}_{0,l}^-.$$

(iii) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ *and* $J < \infty$, *then*

$$\mathcal{R}^- = \mathcal{R}_0^- = \mathcal{RV}\left(\frac{\alpha - \eta}{\alpha}\right) \cup \mathbb{M}_{0,l}^-.$$

(iv) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ *and* $J = \infty$ *or* $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, *then*

$$\mathcal{R}^- = \emptyset.$$

Next two corollaries follows from Theorem 3.2.2 and Corollary 3.4.1.

Corollary 3.5.3 *Suppose that* $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ *and* $S = \infty$. *Then,*

(i) $\sigma < \frac{\beta\eta}{\alpha} - \beta - 1$ *or* $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ *and* $J_2 < \infty$, *then*

$$\mathcal{R}^+ = \mathbb{M}_{\infty,l}^+.$$

(ii) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ *and* $J_2 = \infty$, *then*

$$\mathcal{R}^+ = \mathcal{RV}\left(\frac{\alpha - \eta}{\alpha}\right).$$

(iii) If $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, *then*

$$\mathcal{R}^+ = \left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right).$$

Corollary 3.5.4 *Suppose that* $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ *and* $S < \infty$. *Then,*

(i) $\sigma < \eta - \alpha - 1$ *or* $\sigma = \eta - \alpha - 1$ *and* $I_2 < \infty$, *then*

$$\mathcal{R}^+ = \mathbb{M}_B^+.$$

(ii) If $\sigma = \eta - \alpha - 1$ *and* $I_2 = \infty$, *then*

$$\mathcal{R}^+ = ntr - \mathcal{SV}.$$

(iii) If $\sigma > \eta - \alpha - 1$, then

$$\mathcal{R}^{+} = \left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right).$$

Previous corollaries enable us to describe in full details the simple and clear structure of $\mathcal{RV}-$solutions of equation (E_2) with $\mathcal{RV}-$coefficients.

Corollary 3.5.5 *Let $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $\eta < \alpha$.*

(i) If $\sigma < \eta - \alpha - 1$ then

$$\mathcal{R} = ntr - \mathcal{SV} \cup \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}_B^{-} \cup \mathbb{M}_{\infty,l}^{+}.$$

(ii) If $\sigma = \eta - \alpha - 1$ and $I_1 < \infty$, then

$$\mathcal{R} = ntr - \mathcal{SV} \cup \mathbb{M}_B^{-} \cup \mathbb{M}_{\infty,l}^{+}.$$

(iii) If $\sigma = \eta - \alpha - 1$ and $I_1 = \infty$, or $\eta - \alpha - 1 < \sigma < \frac{\beta\eta}{\alpha} - \beta - 1$, or $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J_2 < \infty$ then

$$\mathcal{R} = \mathbb{M}_{\infty,l}^{+}.$$

(iv) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J_2 = \infty$, then

$$\mathcal{R} = \mathcal{R}^{+} = \mathcal{RV}\left(\frac{\alpha - \eta}{\alpha}\right).$$

(v) If $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, then

$$\mathcal{R} = \mathcal{R}^{+} = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right).$$

Corollary 3.5.6 *Let $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $\eta > \alpha$.*

(i) If $\sigma < -1$, or $\sigma = -1$ and $\sum_{n=1}^{\infty} q(n) < \infty$ then

$$\mathcal{R} = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}_{0,l}^{-} \cup \mathbb{M}_B^{-} \cup \mathbb{M}_B^{+}.$$

(ii) If $\sigma = -1$ and $\sum_{n=1}^{\infty} q(n) = \infty$, or $-1 < \sigma < \frac{\beta\eta}{\alpha} - \beta - 1$ then

$$\mathcal{R} = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}_{0,l}^{-} \cup \mathbb{M}_B^{+}.$$

(iii) *If* $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ *and* $J_1 < \infty$ *then*

$$\mathcal{R} = \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right) \cup \mathbb{M}^-_{0,l} \cup \mathbb{M}^+_B.$$

(iv) *If* $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ *and* $J_1 = \infty$, *or* $\frac{\beta\eta}{\alpha} - \beta - 1 < \sigma < \eta - \alpha - 1$, *or* $\sigma = \eta - \alpha - 1$ *and* $I_2 < \infty$ *then*

$$\mathcal{R} = \mathbb{M}^+_B.$$

(v) *If* $\sigma = \eta - \alpha - 1$ *and* $I_2 = \infty$ *then*

$$\mathcal{R} = \mathcal{R}^+ = ntr - \mathcal{SV}.$$

(vi) *If* $\sigma > \eta - \alpha - 1$ *then*

$$\mathcal{R} = \mathcal{R}^+ = \mathcal{RV}\left(\frac{\sigma+\alpha+1-\eta}{\alpha-\beta}\right).$$

We emphasize that in previous corollaries $tr - \mathcal{SV} = \mathbb{M}^-_B \cup \mathbb{M}^+_B$, $\mathbb{M}^-_{0,l} \subset \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$ and $\mathbb{M}^+_{\infty,l} \subset \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$.

Remark 3.5.1 One important thing that we can conclude from previous corollaries is that in same cases (see Corollary 3.5.5 (*iii*) − (*v*) and Corollary 3.5.6 (*iv*) − (*vi*)) decreasing $\mathcal{RV}$-solutions do not exist. Why is this important? At the beginning of this chapter, we have said that classes $\mathbb{M}^+$ and $\mathbb{M}^-$ are nonempty. Therefore, nonexistence of $\mathcal{RV}$-solutions implies that there are solutions which are not regularly varying sequences although coefficients are regularly varying sequences. This is also one more difference between differential and difference equations, because it has been shown that all solutions of the differential equation (3.2.1) are regularly varying under the assumption that p, q are regularly varying functions (see Matucci, Rehak [100], Rehak [113], Kusano, Manojlović, Marić [68–70, 85]) in the case $p(t) \equiv 1$.

3.6 Examples

In the following two examples we illustrate our main results in Section 3.3.

Example 3.6.1 Consider difference equation

$$\Delta\left(\frac{n^\eta}{\log n}(\Delta x(n))^3\right) = \frac{n^{\eta-7}\varphi(n)}{\log^5 n}\sqrt{x(n+1)^3}, \quad n \geq 1, \tag{3.6.1}$$

where $\varphi(n)$ is positive real-value sequence such that $\lim_{n\to\infty}\varphi(n) = \delta$ and $\eta \neq 3$. In this equation, $\alpha = 3$, $\beta = \dfrac{3}{2}$, $\{p(n)\} \in \mathcal{RV}(\eta)$ and $\{q(n)\} \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 7$.

(i) Suppose that $\eta < 3$. In this case

$$\sigma = \eta - 7 < \eta - 4 = \eta - \alpha - 1,$$

so in view of Theorem 3.3.6-(i) and Theorem 3.2.1 this equation has strongly decreasing $\mathcal{RV}$-solutions of index $\rho < 0$. There is also a decreasing solution from $tr - \mathcal{SV}$ and a increasing solution from $\mathbb{M}^+_{\infty,l}$. More precisely, by Theorem 3.3.6-(i) the equation (3.6.1) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$. That solution has asymptotic behavior

$$x(n) \sim \left(\frac{\delta}{8(9-\eta)}\right)^{\frac{2}{3}} n^{-2} (\log n)^{-\frac{8}{3}}, \quad n \to \infty. \tag{3.6.2}$$

If

$$\varphi(n) = \frac{n^7 (n+1)^3}{(\log n)^4 (\log(n+1))^5} \left((\log n)^9 \psi(n) - (\log(n+1))^9 \left(\frac{n+1}{n}\right)^{\eta} \psi(n+1) \right), \tag{3.6.3}$$

where

$$\psi(n) = \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \left(\frac{\log n}{\log(n+1)} \right)^{\frac{8}{3}} \right)^3 \tag{3.6.4}$$

then $\delta = 8(9-\eta)$ and considered equation has an exact solution $n^{-2} (\log n)^{-\frac{8}{3}}$.

(ii) For $\eta \in (3,9)$ we have that $\eta > \alpha$ and $\sigma = \eta - 7 < \frac{\eta-5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$ so in view of Theorem 3.3.6-(*ii*) the equation (3.6.1) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$ and satisfies (3.6.2). Also, if $\varphi(n)$ is given by (3.6.3) then $n^{-2} (\log n)^{-\frac{8}{3}}$ is an exact solution of the equation (3.6.1). In case when $\eta \in (3,6]$, this equation posses solutions which are not strongly decreasing and belong to classes $\mathbb{M}^-_{0,l}$, $\mathbb{M}^-_B$ and $\mathbb{M}^+_B$, while for $\eta \in (6,9)$, there are solutions in classes $\mathbb{M}^-_{0,l}$, $\mathbb{M}^+_B$.

(iii) Let $\eta = 9$. Then, $\sigma = 2 = \dfrac{\beta\eta}{\alpha} - \beta - 1$, $q(n) \sim \delta\, n^2 (\log n)^{-5}$ and $J_1 < \infty$. By Theorem 3.3.8 the equation (3.6.1) has a solution $x \in \mathcal{RV}(1 - \frac{\eta}{\alpha}) = \mathcal{RV}(-2)$ and any such solution x has an asymptotic representation

$$x(n) \sim \left(\frac{\delta}{56}\right)^{\frac{2}{3}} (n \,\log n)^{-2}, \quad n \to \infty.$$

If

$$\varphi(n) = \frac{(n+1)^3 (\log(n+1))^3 (\log n)^5}{n^2} (\chi(n) - \chi(n+1)),$$

where

$$\chi(n) = \frac{n^3}{(\log n)^7} \left(1 - \left(\frac{n \log n}{(n+1)\log(n+1)} \right)^2 \right)^3,$$

then $\lim_{n\to\infty}\varphi(n) = 56$ and $x(n) = n^{-2}(\log n)^{-2}$ is an exact solution of the equation (3.6.1). As in the previous case, the equation here has also solutions in classes $\mathbb{M}^-_{0,l}$ and $\mathbb{M}^+_B$.

(iv) If $\eta > 9$ then $\sigma = \eta - 7 > \frac{\eta-5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$ so $J_1 = \infty$. Therefore, by Corollary 3.3.1-(*ii*) equation (3.6.1) does not have decreasing regularly varying solutions, but have solution which belong to class $\mathbb{M}^+_B$.

Note that this equation does not have strongly increasing solutions. □

Example 3.6.2 Consider a difference equation

$$\Delta\left(-n^{\eta}\sqrt{\log n}\,(\Delta x(n))^2\right) = \frac{n^{\eta-3}\,\varphi(n)}{(\log n)^{19/6}}\sqrt[3]{x(n+1)}, \quad n \ge 1, \tag{3.6.5}$$

where $\varphi(n)$ is a positive real-value sequence such that $\lim_{n\to\infty}\varphi(n) = \delta$ and $\eta \neq 2$. Here, $p(n) = n^{\eta}\sqrt{\log n}$, and $q(n) = n^{\eta-3}\,\varphi(n)\,(\log n)^{-19/6}$, so $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 3 = \eta - \alpha - 1$.

Let $\eta < 2 = \alpha$. Using that

$$\sum_{k=n}^{\infty}\left(\frac{1}{p(k)}\sum_{j=k}^{\infty}q(j)\right)^{\frac{1}{\alpha}} \sim \sum_{k=n}^{\infty}\sqrt{\frac{\varphi(n)}{2-\eta}}\,\frac{1}{k\,(\log k)^{11/6}} < \infty, \quad n \to \infty,$$

by Theorem 3.3.7, the equation (3.6.5) has a nontrivial slowly varying solution and any such solution x has an asymptotic representation

$$x(n) \sim \left(\frac{\delta}{2-\eta}\right)^{\frac{3}{5}} \cdot (\log n)^{-1}, \quad n \to \infty.$$

If

$$\varphi(n) = n^3\left(\frac{\log n}{\log(n+1)}\right)^{\frac{19}{6}}\left[\frac{(\log\frac{n+1}{n})^2}{(\log n)^{\frac{3}{2}}(\log(n+1))^{\frac{1}{2}}} - \left(\frac{n+1}{n}\right)^{\eta}\left(\frac{\log\frac{n+2}{n+1}}{\log(n+2)}\right)^2\right],$$

then $\delta = 2 - \eta$ and considered equation has an exact solution $x(n) = (\log n)^{-1}$, $x \in ntr - \mathcal{SV}$. In this case, classes $\mathbb{M}^+_{\infty,l}$ and $\mathbb{M}^-_B$ are also nonempty.

Notice that in the case $\eta > 2 = \alpha$, since $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, using Corollary 3.3.1-(*ii*) , we conclude that $\mathcal{R}^- = \emptyset$. How $I_2 < \infty$, $\mathcal{R} = \mathbb{M}^+_B$.

Note that this equations doesn't have strongly increasing solutions. □

In the next two examples we illustrate our main results in Section 3.4.

Example 3.6.3 Consider the difference equation

$$\Delta\left(n^{\eta}\sqrt{\log n}\,(\Delta x(n))^3\right) = n^{\eta+1}\varphi(n)\sqrt[12]{\log n}\,\sqrt{x(n+1)}, \quad n \ge 1, \tag{3.6.6}$$

where $\varphi(n)$ is a positive real-value sequence such that $\lim_{n\to\infty}\varphi(n) = \delta$ and $\eta \neq 3$. In this equation, $\alpha = 3$, $\beta = \frac{1}{2}$, $\{p(n)\} \in \mathcal{RV}(\eta)$ and $\{q(n)\} \in \mathcal{RV}(\sigma)$, where $\sigma = \eta + 1$ and $\frac{\beta\eta}{\alpha} - \beta - 1 = \frac{\eta - 9}{6}$.

(i) Suppose that $\eta < -3$. In this case $\eta < \alpha$ and $\sigma < \frac{\beta\eta}{\alpha} - \beta - 1$, so in view of Theorem 3.4.1-(i) and Theorem 3.4.2, this equation does not have strongly increasing $\mathcal{RV}$-solutions. However, since $J_2 < \infty$, the equation (3.6.6) has solutions in $\mathbb{M}^{+}_{\infty,l}$. Also, this equation does not have strongly decreasing $\mathcal{RV}$-solutions.

(ii) Let $\eta = -3$. Then, $\sigma = -2 = \frac{\beta\eta}{\alpha} - \beta - 1$, $q(n) \sim \delta\, n^{-2} \sqrt[12]{\log n}$ and

$$\sum_{k=1}^{n} k^{\beta} q(k) p_k^{\frac{-\beta}{\alpha}} = \delta \sum_{k=1}^{n} \frac{1}{k} \to \infty, \quad n \to \infty.$$

By Theorem 3.4.2 there exist a solution $x \in \mathcal{RV}(2)$ of the equation (3.6.6) and any such solution x has the asymptotic behavior

$$x(n) \sim \frac{n^2}{\sqrt[6]{\log n}} \left(\frac{5\delta}{48}\right)^{\frac{2}{5}} \left(\sum_{k=1}^{n} \frac{1}{k}\right)^{\frac{2}{5}} \sim \left(\frac{5\delta}{48}\right)^{\frac{2}{5}} n^2 (\log n)^{\frac{7}{30}}, \quad n \to \infty.$$

In the last asymptotic relation we used that $\sum_{k=1}^{n} \frac{1}{k} \sim \log n$, $n \to \infty$, which follows from

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = \gamma,$$

where γ is Euler - Mascheroni constant (also called Euler's constant).

If

$$\varphi(n) = \frac{n^2}{(n+1)^4} \left(\frac{(\log(n+1))^{\frac{23}{5}}}{\log n}\right)^{\frac{1}{12}} \cdot \psi(n+1) - \frac{1}{n(n+1)} \left(\frac{(\log n)^5}{(\log(n+1))^{\frac{7}{5}}}\right)^{\frac{1}{12}} \cdot \psi(n),$$

where $\psi(n) = \left(\Delta\left(n^2 (\log n)^{7/30}\right)\right)^3$, then $\lim_{n\to\infty}\varphi(n) = 48/5$ and $x(n) = n^2(\log n)^{7/30}$ is an exact solution of the equation (3.6.6).

(iii) For $\eta \in (-3, 3)$ we have that $\eta < \alpha$ and $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$ so by Theorem 3.4.1-(i) equation (3.6.6) has $\mathcal{RV}(2)$-solution x satisfying

$$x(n) \sim \left(\frac{\delta}{8(\eta+3)}\right)^{\frac{2}{5}} \frac{n^2}{\sqrt[6]{\log n}}, \quad n \to \infty. \tag{3.6.7}$$

If in particular

$$\varphi(n)=\left(\frac{\log(n+1)}{\log n}\right)^{1/12}\cdot\frac{\chi(n+1)-\chi(n)}{n^{1+\eta}(n+1)}, \tag{3.6.8}$$

where

$$\chi(n)=n^{\eta}\left((n+1)^2\left(\frac{\log n}{\log(n+1)}\right)^{1/6}-n^2\right)^3,$$

then $\delta=8(\eta+3)$ and the considered equation has an exact solution $x(n)=n^2(\log n)^{-1/6}$.

(iv) If $\eta>3=\alpha$, then $\sigma=\eta+1>\eta-4=\eta-\alpha-1$. Therefore, by Theorem 3.4.1-(*ii*) the equation (3.6.6) has $\mathcal{RV}(2)$-solution x satisfying (3.6.7) and if $\varphi(n)$ is given by (3.6.8) then $x(n)=n^2(\log n)^{-1/6}$ is an exact solution of (3.6.6).

In the summary, if $\mathcal{R}_+$ denotes the set of all strongly increasing regularly varying solutions,

$$\mathcal{R}_+=\begin{cases}\emptyset, & \eta\in(-\infty,-3)\\ ntr-\mathcal{RV}(2), & \eta=-3\\ \mathcal{RV}(2), & \eta\in(-3,3)\cup(3,\infty)\end{cases}$$

Note that this equation does not have any decreasing $\mathcal{RV}$-solution. □

Example 3.6.4 Consider the difference equation

$$\Delta\left(n^{\eta}\sqrt{\log n}\,(\Delta x(n))^3\right)=n^{\eta-4}\,\varphi(n)\sqrt{x(n+1)\log n},\quad n\geq 1, \tag{3.6.9}$$

where $\varphi(n)$ is a positive real-value sequence such that $\lim_{n\to\infty}\varphi(n)=\delta$ and $\eta\neq 3$. Here, $\alpha=3$, $\beta=1/2$, $p(n)=n^{\eta}\sqrt{\log n}$, and $q(n)=n^{\eta-4}\sqrt{\log n}$, so $p\in\mathcal{RV}(\eta)$ and $q\in\mathcal{RV}(\sigma)$, where $\sigma=\eta-4=\eta-\alpha-1$.

In the case $\eta<3=\alpha$, since $\sigma<\dfrac{\beta\eta}{\alpha}-\beta-1$, implying that $J_2<\infty$, using Theorem 3.2.2-(*ii*) and Theorem 3.4.1-(*i*), we conclude that there exist solutions in $\mathbb{M}^+_{\infty,l}$, while the set of all strongly increasing regularly varying solutions $\mathcal{R}_+$ is empty. Since $I_1=\infty$, the set of decreasing $\mathcal{RV}$-solutions is also empty. Let $\eta>3=\alpha$. Using that

$$\sum_{k=2}^{n}\left(\frac{1}{p(k)}\sum_{j=1}^{k-1}q(j)\right)^{\frac{1}{\alpha}}\sim\frac{1}{\sqrt[3]{\eta-3}}\sum_{k=2}^{n}\frac{1}{k}\to\infty,\quad n\to\infty,$$

by Theorem 3.4.3 there exists nontrivial slowly varying solution of the equation (3.6.9) and any such solution x has asymptotic representation

$$x(n)\sim\left(\frac{125\delta}{216(\eta-3)}\right)^{\frac{2}{5}}\cdot(\log n)^{\frac{6}{5}},\quad n\to\infty.$$

In fact, due to Theorem 3.4.1-(*ii*), we conclude that $\mathcal{R}_+ = ntr - \mathcal{SV}$.

If

$$\varphi(n) = n^4 \left[\left(\frac{n+1}{n}\right)^{\eta} \frac{(\log(n+2))^{\frac{18}{5}}}{(\log n)^{\frac{1}{2}}(\log(n+1))^{\frac{1}{10}}} \cdot \nu(n+1) - (\log(n+1))^3 \cdot \nu(n) \right],$$

where

$$\nu(n) = \left(1 - \left(\frac{\log n}{\log(n+1)}\right)^{6/5}\right)^3,$$

then $\delta = 216(\eta - 3)/125$ and the considered equation has an exact solution $x(n) = (\log n)^{6/5}$, $x \in ntr - \mathcal{SV}$.

As in previous case there are no decreasing solutions. □

Chapter 4

Cyclic systems of second-order difference equations

4.1 Introduction

To further generalize results established in the previous chapter, systems of nonlinear difference equations that will be studied in this chapter are the following cyclic systems:

$$(SE+) \qquad \Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) + q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1) = 0,$$

and

$$(SE-) \qquad \Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) - q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1) = 0,$$

where $i = \overline{1,N}$, $x_{N+1} = x_1$, $n \in \mathbb{N}$, and following conditions hold:

(*a*) α_i and β_i, $i = \overline{1,N}$ are positive constants such that

$$\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N > \beta_1\beta_2 \cdot \ldots \cdot \beta_N;$$

(*b*) $p_i = \{p_i(n)\}$, $q_i = \{q_i(n)\}$ are positive real sequences;

(*c*) All p_i, $i = \overline{1,N}$ simultaneously satisfy either

$$(I) \qquad S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} = \infty,$$

or

$$(II) \qquad S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} < \infty.$$

Systems $(SE+)$ and $(SE-)$ are called sublinear if the condition (a) holds. In the case when opposite inequality holds, we say that the systems are superlinear. If $\alpha_1\alpha_2\cdot\ldots\cdot\alpha_N = \beta_1\beta_2\cdot\ldots\cdot\beta_N$, then systems are called half-linear.

Existence of positive solutions and oscillation of discrete nonlinear systems are widely studied in the literature (see, e.g. [4,55,56,83,84,94,95,110–112] and references therein). However, in the existing literature, there are no results concerning asymptotic analysis of solutions of a cyclic system of difference equations of second order. In this regard, the first task will be to classify solutions based on their behavior at infinity. The second task is to determine the necessary and sufficient conditions for the existence of these solutions. Determining precise asymptotic formulas is the third and the most difficult task. For so-called primitive solutions, asymptotic behavior is already known. However, for intermediate solutions of systems $(SE+)$, that is, extreme solutions (strongly increasing and strongly decreasing) of systems $(SE-)$, it is not easy to find appropriate asymptotic formulas. Therefore, as in the previous chapter, we will limit ourselves to examination of systems whose coefficients $p_i = \{p_i(n)\}$, $q_i = \{q_i(n)\}$ are regularly varying sequences and focus our attention on the existence and asymptotic behavior of regularly varying solutions.

The obtained results can be considered as a discrete analogue of the results in a continuous case (see [50,51,54]). Also, the obtained results can be applied to systems of N equations of the first order in the case when N is even.

One-dimensional system is in fact well-known sublinear second order Emden-Fowler type difference equation

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) \pm q(n)|x(n+1)|^{\beta-1}x(n+1) = 0, \tag{4.1.1}$$

which has been studied a lot in the literature (see e.g. [7,14–16,19–21,23,24,123,124,131, 136]). Also, two-dimensional system can be easily reduced to fourth-order nonlinear difference equations

$$\Delta^2\left(p(n)\left|\Delta^2 x(n)\right|^{\alpha-1}\Delta^2 x(n)\right) + q(n)x(n+2)^{\beta} = 0, \tag{4.1.2}$$

and

$$\Delta\left(a(n)\left(\Delta\, b(n)\,(\Delta\, c(n)\,(\Delta x(n))^{\gamma})^{\beta}\right)^{\alpha}\right) + d(n)x_1(n+2)^{\lambda} = 0\,. \tag{4.1.3}$$

The oscillatory and asymptotic properties of solutions of the equation (4.1.2) have been investigated by various authors [5,6,87,125,126,128–131], while more general equation (4.1.3) has been considered in [3,6,27–29]. Therefore, our main results can be seen as an extension of the quoted existence results for solutions of equations (4.1.1), (4.1.2), (4.1.3) as well as an improvement of the quoted result in the sense of giving exact asymptotic representation formula of solutions of these second and fourth order difference equation.

The whole chapter is based on the original results, among which results presented in Section 4.3 was published in [61].

4.2 Preliminaries

By a solution of $(SE+)$ (respectively $(SE-)$) we mean a vector sequence

$$\mathbf{x} = (x_1, x_2, \ldots, x_N) = \Big(\{x_1(n)\}, \{x_2(n)\}, \ldots, \{x_N(n)\}\Big) \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R},$$

where ${}^{\mathbb{N}}\mathbb{R} = \{f \mid f : \mathbb{N} \to \mathbb{R}\}$, whose components $x_i = \{x_i(n)\}$, $i = \overline{1,N}$ satisfy $(SE+)$ (respectively $(SE-)$) for $n \in \mathbb{N}$. In what following, we will observe sequences for sufficiently large n, i.e. $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Therefore, we introduced notation ${}^{\mathbb{N}_{n_0}}\mathbb{R} = \{f \mid f : \mathbb{N}_{n_0} \to \mathbb{R}\}$, where $\mathbb{N}_{n_0} = \{n \in \mathbb{N} \;\big|\; n \geq n_0\}$.

We are interested in nonoscillatory positive solutions, i.e. solutions whose all components are eventually positive. Similar, if all components of solution $\mathbf{x}$ are increasing (decreasing), we say that $\mathbf{x}$ is increasing (decreasing) solution. For every component of any solution $\mathbf{x}$ of $(SE+)$ or $(SE-)$, let we denote by $x_i^{[1]} = \left\{x_i^{[1]}(n)\right\}$ its quasi-difference $x_i^{[1]}(n) = p_i(n)|\Delta x_i(n)|^{\alpha-1}\Delta x_i(n)$, $i = \overline{1,N}$.

Since we will consider systems which have regularly varying coefficients, we assume that $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$ and represent them with

$$p_i(n) = n^{\lambda_i} l_i(n), \quad q_i(n) = n^{\mu_i} m_i(n), \quad l_i, m_i \in \mathcal{SV}, \quad i = \overline{1,N}. \tag{4.2.1}$$

We will search the regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ of the observed systems in the form

$$x_i(n) = n^{\rho_i} \xi_i(n), \quad \xi_i \in \mathcal{SV}, \quad i = \overline{1,N}. \tag{4.2.2}$$

We also assume that all sequences p_i, $i = \overline{1,N}$ satisfy either (I) or (II). Condition (I), resp. (II), holds if and only if

$$\lambda_i < \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i \;\text{ and } \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} = \infty, \tag{4.2.3}$$

resp.

$$\lambda_i > \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i \;\text{ and } \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} < \infty. \tag{4.2.4}$$

Sequences $P_i = \{P_i(n)\}$ given by

$$P_i(n) = \sum_{k=1}^{n-1} p_i(k)^{-\frac{1}{\alpha_i}}, \quad i = \overline{1,N} \tag{4.2.5}$$

and $\pi_i = \{\pi_i(n)\}$ given by

$$\pi_i(n) = \sum_{k=n}^{\infty} p_i(k)^{-\frac{1}{\alpha_i}}, \quad i = \overline{1,N} \tag{4.2.6}$$

play a very important role in a classification of solutions of considered systems, as well as, in their asymptotic analysis. Therefore, for P_i and π_i we will use the following asymptotic equivalences which are obtained with the help of Theorem 1.3.5.

In the case when $\lambda_i < \alpha_i$, $i = \overline{1,N}$ we have

$$P_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}, \tag{4.2.7}$$

and if $\lambda_i = \alpha_i$, $i = \overline{1,N}$ then

$$P_i(n) \sim \sum_{n=1}^{n-1} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}}. \tag{4.2.8}$$

If the case (II) holds, then for sequence π_i we have

$$\pi_i(n) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}. \tag{4.2.9}$$

or

$$\pi_i(n) \sim \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}}, \tag{4.2.10}$$

depending whether $\lambda_i > \alpha_i$, $i = \overline{1,N}$ or $\lambda_i = \alpha_i$, $i = \overline{1,N}$ respectively.

Also, to simplify notation we denote $A_N = \alpha_1 \alpha_2 \cdot \ldots \cdot \alpha_N$, $B_N = \beta_1 \beta_2 \cdot \ldots \cdot \beta_N$ and use matrix

$$M = \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \cdots & \cdots & \frac{\beta_1\beta_2\cdot\ldots\cdot\beta_{N-1}}{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{N-1}} \\ & 1 & \frac{\beta_2}{\alpha_2} & \frac{\beta_2\beta_3}{\alpha_2\alpha_3} & \cdots & \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_{N-1}}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{N-1}} \\ & & 1 & \frac{\beta_3}{\alpha_3} & \cdots & \frac{\beta_3\cdot\ldots\cdot\beta_{N-1}}{\alpha_3\cdot\ldots\cdot\alpha_{N-1}} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & \frac{\beta_{N-1}}{\alpha_{N-1}} \\ & & & & & 1 \end{pmatrix}, \tag{4.2.11}$$

whose elements will be denoted by $M = (M_{ij})$, where the lower triangular elements are omitted for the economy of notation. In fact, the i–th row of (M_{ij}) is obtained by shifting the vector

$$\left(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \ldots, \frac{\beta_i\beta_{i+1}\cdot\ldots\beta_{i+(N-2)}}{\alpha_i\alpha_{i+1}\cdot\ldots\alpha_{i+(N-2)}}\right), \quad \alpha_{N+j} = \alpha_j,\ \beta_{N+j} = \beta_j,\ j = \overline{1,N-2}$$

$(i-1)-$times to the right cyclically, so that the lower triangular elements $M_{ij}, i > j$, satisfy the relation

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N}, \quad i > j, \quad i = \overline{2,N}.$$

It is easy to see that for the elements of matrix M hold

$$\text{(4.2.12)} \qquad M_{i+1,i}\frac{\beta_i}{\alpha_i} = \frac{B_N}{A_N}, \quad M_{i+1,j}\frac{\beta_i}{\alpha_i} = M_{ij}, \quad \text{for} \quad j \neq i,$$

where $M_{N+1,j} = M_{1,j}$, $j = \overline{1,N}$.

Throughout the text, $n \geq n_0$ means that n is sufficiently large so that n_0 need not to be the same at each occurrence.

4.3 The system $(SE+)$

4.3.1 Classification of positive solutions

In order to fully describe a set of positive solutions, we first classify positive solutions according to their behavior at infinity. It can easily be seen that all components of eventually positive solution $\mathbf{x}$ of the system $(SE+)$ satisfy

$$\text{(4.3.1)} \qquad c_i \leq x_i(n) \leq C_i \cdot P_i(n), \ \text{for large } n, \quad i = \overline{1,N} \quad \text{if} \quad (I) \quad \text{holds}$$

or

$$\text{(4.3.2)} \qquad k_i\pi_i(n) \leq x_i(n) \leq K_i, \ \text{for large } n, \quad i = \overline{1,N} \quad \text{if} \quad (II) \quad \text{holds}$$

where P_i and π_i for $i = \overline{1,N}$, are given by (4.2.5) and (4.2.6) respectively, and c_i, C_i, k_i and K_i are positive real constants.

Indeed, if (I) holds, then is easy to see that x_i, $i = \overline{1,N}$ are eventually increasing, i.e.

$$\text{(4.3.3)} \qquad x_i(n) > 0, \quad \Delta x_i(n) > 0, \quad \text{for } n \geq n_0, \quad i = \overline{1,N}.$$

This implies left inequality in (4.3.1). Also, $x_i^{[1]}$, $i = \overline{1,N}$ are eventually decreasing, so there exist positive constants b_i such that $p_i(n)\Delta x_i(n)^{\alpha_i} \leq b_i$, $n \geq n_0$ and $i = \overline{1,N}$, which by summation implies that

$$x_i(n) \leq x_i(n_0) + b_i^{\frac{1}{\alpha_i}} \sum_{k=n_0}^{n-1} \frac{1}{p_i(k)^{1/\alpha_i}} \leq x_i(n_0) + b_i^{\frac{1}{\alpha_i}} P_i(n), \quad i = \overline{1,N}.$$

Since for all $i = \overline{1,N}$, P_i are increasing and $P_i(n) \to \infty$, $n \to \infty$, we get the right side inequality in (4.3.1).

Similarly, if (II) holds, then x_i, for $i = \overline{1,N}$ are eventually decreasing, i.e.

$$x_i(n) > 0, \quad \Delta x_i(n) < 0, \quad \text{for } n \geq n_0, \quad i = \overline{1,N}, \tag{4.3.4}$$

so right inequality in (4.3.2) holds. Therefore, $\lim_{n\to\infty} x_i(n) = x_i(\infty) < \infty$. On the other hand, since $x_i^{[1]}(n) = -p_i(n)\,(-\Delta x_i(n))^{\alpha_i} \leq 0$ and $x_i^{[1]}$, $i = \overline{1,N}$ are decreasing, it follows that $-p_i(n)\,(-\Delta x_i(n))^{\alpha_i} \leq -h_i$, $h_i \in \mathbb{R}^+$, $n \geq n_0$, $i = \overline{1,N}$, implying that

$$x_i(n) \geq x_i(\infty) + h_i^{\frac{1}{\alpha_i}} \pi_i(n), \quad n \geq n_0.$$

Using that π_i, $i = \overline{1,N}$ are decreasing and tend to zero, we get the left side inequality in (4.3.2).

In the case (I) for each component x_i of solution $\mathbf{x}$ only one of next three possibilities holds:

$(S1+)$ $\lim_{n\to\infty} \dfrac{x_i(n)}{P_i(n)} = const > 0$, i.e. $x_i(n) \sim \kappa_i P_i(n)$, $n \to \infty$, $\kappa_i > 0$,

$(IM1)$ $\lim_{n\to\infty} x_i(n) = \infty$, $\lim_{n\to\infty} \dfrac{x_i(n)}{P_i(n)} = 0$,

$(AC+)$ $\lim_{n\to\infty} x_i(n) = const > 0$, i.e. $x_i(n) \sim \kappa_i$, $n \to \infty$, $\kappa_i > 0$.

In the case (II) for each component x_i of solution $\mathbf{x}$ only one of next three possibilities holds:

$(AC+)$ $\lim_{n\to\infty} x_i(n) = const > 0$, i.e. $x_i(n) \sim \kappa_i$, $n \to \infty$, $\kappa_i > 0$,

$(IM2)$ $\lim_{n\to\infty} x_i(n) = 0$, $\lim_{n\to\infty} \dfrac{x_i(n)}{\pi_i(n)} = \infty$,

$(S2+)$ $\lim_{n\to\infty} \dfrac{x_i(n)}{\pi_i(n)} = const > 0$, i.e. $x_i(n) \sim \kappa_i \pi_i(n)$, $n \to \infty$, $\kappa_i > 0$.

Note that we consider only solutions whose all components are the same type. Solutions of the type $(S1+)$, $(S2+)$ and $(AC+)$ are called primitive and the existence of such solutions will be discussed in the next subsection. Solutions of type $(IM1)$ and $(IM2)$ are called *intermediate solutions*. The existence, as well as the asymptotic formulas of these solutions will be studied in details in Subsection 4.3.3.

4.3.2 Existence of primitive solutions

This section is dedicated to solutions of types $(S1+)$, $(S2+)$, and $(AC+)$. Problem of the existence of this types of solutions can be solved without the assumption that coefficients of the system $(SE+)$ are regularly varying sequences.

Theorem 4.3.1 *Let* (I) *holds. The system* $(SE+)$ *has a solution* $\mathbf{x}$ *whose each component satisfies* $(AC1+)$ *if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\right)^{\frac{1}{\alpha_i}}<\infty,\quad i=\overline{1,N}. \tag{4.3.5}$$

Proof. Let $\mathbf{x}=(x_1,x_2,\dots,x_N)$ be a solution of $(SE+)$ whose each component satisfies $\lim_{n\to\infty}x_i(n)=c_i$. Then, there exist positive constants k_i and n_0, such that $k_i\le x_i(n)$, $n\ge n_0$. We claim that $\lim_{n\to\infty}x_i^{[1]}(n)=0$, for all i. Indeed, since $x_i^{[1]}$, $i=\overline{1,N}$ are positive and decreasing, it follows that $\lim_{n\to\infty}x_i^{[1]}(n)=\omega_i\ge 0$. If $\omega_i>0$ for arbitrary fixed i, then there exist $m_0\ge n_0$ such that $x_i^{[1]}(n)\ge\omega_i$, for $n\ge m_0$. In that case, we obtain that

$$x_i(n)\ge x_i(m_0)+\omega_i^{\frac{1}{\alpha_i}}\sum_{k=m_0}^{n-1}\frac{1}{p_i(k)^{1/\alpha_i}},\quad n\ge m_0\,,$$

implying that $\lim_{n\to\infty}x_i(n)=\infty$, which is a contradiction. Thus, $\lim_{n\to\infty}x_i^{[1]}(n)=0$ and from $(SE+)$ we have that

$$\Delta x_i(n)=\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\,x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}},\quad i=\overline{1,N}.$$

Summing previous equality from n_0 to ∞ we obtain

$$c_i-x_i(n_0)=\sum_{n=n_0}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\,x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\ge k_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{n=n_0}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\right)^{\frac{1}{\alpha_i}},\quad i=\overline{1,N}$$

which implies that the condition (4.3.5) is satisfied.

Assume now that (4.3.5) holds. Then, there exist $n_0\ge 1$ such that

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\right)^{\frac{1}{\alpha_i}}<1,\quad i=\overline{1,N}. \tag{4.3.6}$$

Denote with $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x}=(x_1,x_2,\dots,x_N)$, such that $x_i=\{x_i(n)\}\in{}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i=\overline{1,N}$ are bounded, and endowed with the topology of the norm

$$\|\mathbf{x}\|=\max_{1\le i\le N}\left\{\sup_{n\ge n_0}x_i(n)\right\}. \tag{4.3.7}$$

Clearly, $\mathcal{L}_{n_0}$ is a Banach space. Set

$$\Omega_1 = \left\{ \mathbf{x} \in \mathcal{L}_{n_0} \,\middle|\, \frac{c_i}{2} \le x_i(n) \le c_i, \quad n \ge n_0, \quad i = \overline{1,N} \right\}, \tag{4.3.8}$$

where c_i, $i = \overline{1,N}$ are positive constants such that

$$c_i \ge 2c_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad c_{N+1} = c_1. \tag{4.3.9}$$

An example of such choices is

$$c_i = 2^{\frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij}}, \quad i = \overline{1,N}, \tag{4.3.10}$$

where M_{ij} are elements of the matrix M given by (4.2.11).

It is easy to see that Ω_1 is bounded, closed and convex subset of $\mathcal{L}_{n_0}$. Define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = c_i - \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i} \right)^{\frac{1}{\alpha\, i}}, \quad n \ge n_0, \quad i = \overline{1,N}, \tag{4.3.11}$$

and define the mapping $\Theta : \Omega_1 \to \mathcal{L}_{n_0}$ by

$$\Theta(x_1, x_2, \ldots, x_N) = \Big(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \ldots, \mathcal{F}_N x_{N+1}\Big), \tag{4.3.12}$$

where $x_{N+1} = x_1$. We will show that Θ has a fixed point by using Schauder - Tychonoff fixed point theorem. Namely, the operator Θ has the following properties:

(i) Θ *maps* Ω_1 *into itself*: Let $\mathbf{x} \in \Omega_1$. Then, using (4.3.6), (4.3.8), (4.3.9) and (4.3.11), we see that

$$c_i \ge \mathcal{F}_i x_{i+1}(n) \ge c_i - c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{n=n_0}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) \right)^{\frac{1}{\alpha_i}} \ge \frac{c_i}{2}, \quad i = \overline{1,N}, \quad n \ge n_0.$$

(ii) Θ *is continuous*: Let $\varepsilon_i > 0, i = \overline{1,N}$ and $\{\mathbf{x}^{(m)}\}_{m\in\mathbb{N}} = \left\{(x_1^{(m)}, x_2^{(m)}, \ldots,\ x_N^{(m)})\right\}_{m\in\mathbb{N}}$, be a sequence in Ω_1 which converges to $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ as $m \to \infty$. Since, Ω_1 is closed, $\mathbf{x} \in \Omega_1$. The rest of the proof does not depend on i, so let $i \in \{1, 2, \ldots, N\}$ be arbitrary fixed. For every $n \ge n_0$, we have

$$\left|\mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n)\right| \le \sum_{k=n}^{\infty} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}} \left| \left(\sum_{s=k}^{\infty} q_i(s) x_{i+1}^{(m)}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} - \left(\sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \right|.$$

By (4.3.5), $\sum_{n=n_0}^{\infty} q_i(n)$ is convergent, implying that $\sum_{n=n_0}^{\infty} q_i(n)\, x_{i+1}^{(m)}(n+1)^{\beta_i}$ is totally convergent, because $q_i(n)\, x_{i+1}^{(m)}(n+1)^{\beta_i} \le c_{i+1}^{\beta_i} q_i(n)$, for every $n \ge n_0, m \in \mathbb{N}$. Then, by a discrete analogue of Lebesgue dominated convergence theorem (Theorem 1.1.4), it holds for every $k \ge n_0$

$$\lim_{m\to\infty} \left| \left(\sum_{s=k}^{\infty} q_i(s) x_{i+1}^{(m)}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} - \left(\sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \right| = 0,$$

which shows that

$$\lim_{m\to\infty} \sup_{n \ge n_0} \left| \mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n) \right| = 0.$$

Therefore, $\|\Theta \mathbf{x}^{(m)} - \Theta \mathbf{x}\| \to 0$ as $m \to \infty$, i.e. Θ is continuous.

(iii) $\Theta(\Omega_1)$ *is relatively compact*: To show this, by Theorem 1.1.6, it is sufficient to show that $\Theta(\Omega_1)$ is uniformly Cauchy in the topology of $\mathcal{L}_{n_0}$. For $\mathbf{x} \in \Omega_1$ and $m > n \ge n_0$ we have

$$\begin{aligned} |\mathcal{F}_i x_{i+1}(m) - \mathcal{F}_i x_{i+1}(n)| &= \left| \sum_{k=n}^{m-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \right| \\ &\le \sum_{k=n}^{m-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}} \left| \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right|^{\frac{1}{\alpha_i}} \le c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{m-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) \right)^{\frac{1}{\alpha_i}}. \end{aligned}$$

According to (4.3.5) it follows that $\Theta(\Omega_1)$ is uniformly Cauchy. Therefore, all the hypotheses of Schauder - Tychonoff fixed point theorem are fulfilled implying the existence of a fixed point $\mathbf{x} \in \Omega_1$ of Θ, which satisfies

$$x_i(n) = c_i - \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}.$$

It is clear that $\mathbf{x}$ is a positive solution of $(SE+)$ whose all components tend to constants.
□

Theorem 4.3.2 *Let* (I) *holds. The system* $(SE+)$ *has a solution* $\mathbf{x}$ *whose each component satisfies* $(S1+)$ *if and only if*

$$\sum_{n=1}^{\infty} q_i(n) \left(\sum_{k=1}^{n} \frac{1}{p_{i+1}(n)^{1/\alpha_{i+1}}} \right)^{\beta_i} < \infty, \quad i = \overline{1,N}. \tag{4.3.13}$$

Proof. Suppose that $\mathbf{x} = (x_1, x_2, \dots, x_N)$ is a solution of $(SE+)$ whose each component satisfies $(S1+)$. Then, $\lim_{n\to\infty} x_i(n) = \infty$ and $\lim_{n\to\infty} x_i^{[1]}(n) = d_i > 0$, $i = \overline{1,N}$. As $x_i^{[1]}$ are decreasing and tend to d_i, there exists n_0 such that $x_i^{[1]}(n) \geq d_i$, $n \geq n_0$, $i = \overline{1,N}$. By summation of $x_i^{[1]}(n)/p_i(n) = (\Delta x_i(n))^{\alpha_i}$ from n_0 to $n-1$, we get

$$x_i(n) = x_i(n_0) + \sum_{k=n_0}^{n-1} \left(\frac{x_i(k)^{[1]}}{p_i(k)}\right)^{\frac{1}{\alpha_i}} \geq d_i^{\frac{1}{\alpha_i}} \sum_{k=n_0}^{n-1} \frac{1}{p_i(k)^{1/\alpha_i}}, \quad n \geq n_0\,.$$

Then, from $(SE+)$ we obtain for $n \geq n_0$

$$x_i^{[1]}(n) - d_i = \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \geq d_{i+1}^{\frac{\beta_i}{\alpha_{i+1}}} \sum_{k=n}^{\infty} q_i(k) \left(\sum_{k=n_0}^{n-1} \frac{1}{p_{i+1}(k)^{1/\alpha_{i+1}}}\right)^{\beta_i}, \quad i = \overline{1,N}\,.$$

Letting $n \to \infty$, we get that the condition (4.3.13) holds.

On the other hand, if (4.3.13) holds, then there exists n_0 such that

$$\sum_{k=n_0}^{\infty} q_i(k) \left(\sum_{s=n_0}^{k} \frac{1}{p_{i+1}(k)^{1/\alpha_{i+1}}}\right)^{\beta_i} < 2^{\alpha_i - \beta_i}\,(2^{\alpha_i} - 1)\,, \quad i = \overline{1,N}. \tag{4.3.14}$$

Denote with $\mathcal{X}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/P_i^{n_0}(n)\}$, $i = \overline{1,N}$ are bounded, where

$$P_i^{n_0}(n) = \sum_{k=n_0}^{n-1} \frac{1}{p_i(k)^{1/\alpha_i}}, \quad i = \overline{1,N}\,. \tag{4.3.15}$$

The space $\mathcal{X}_{n_0}$ endowed with the norm

$$\|\mathbf{x}\| = \max_{1\leq i\leq N} \left\{\sup_{n\geq n_0} \frac{x_i(n)}{P_i^{n_0}(n)}\right\} \tag{4.3.16}$$

is a Banach space. Set

$$\Omega_2 = \left\{\, \mathbf{x} \in \mathcal{X}_{n_0} \;\middle|\; c_i P_i^{n_0}(n) \leq x_i(n) \leq 2c_i\, P_i^{n_0}(n), \quad n \geq n_0, \quad i = \overline{1,N} \right\}, \tag{4.3.17}$$

where c_i, $i = \overline{1,N}$ are positive constants which satisfy (4.3.9). It is easy to see that Ω_2 is bounded, closed and convex subset of $\mathcal{X}_{n_0}$. Define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \left(c_i^{\alpha_i} + \sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i}\right)\right)^{1/\alpha_i}, \quad n \geq n_0, \quad i = \overline{1,N},$$

and define the mapping $\Theta : \Omega_2 \to \mathcal{L}_{n_0}$ by (4.3.12). The mapping defined like this satisfies all conditions of Schauder-Tychonoff fixed point theorem. Indeed, because of (4.3.14) and (4.3.17), Θ maps Ω_2 into itself. Using discrete Lebesgue's dominated convergence theorem, it can be shown that Θ is continuous and that $\Theta(\Omega_2)$ is uniformly Cauchy. Applying Schauder-Tychonoff fixed point theorem, there exists $\mathbf{x} \in \Omega_2$ such that $\mathbf{x} = \Theta\mathbf{x}$. Then, it is easy to verify that $\mathbf{x}$ is a solution of $(SE+)$ satisfying $(S1+)$. □

Theorem 4.3.3 *Let* (II) *holds. The system* $(SE+)$ *has a solution* $\mathbf{x}$ *whose each component satisfies* $(AC+)$ *if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=1}^{n-1} q_i(k)\right)^{\frac{1}{\alpha_i}} < \infty, \quad i = \overline{1,N}. \tag{4.3.18}$$

Proof. Suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is a solution of $(SE+)$ such that $\lim_{n\to\infty} x_i(n) = c_i$, $i = \overline{1,N}$. Since all x_i are eventually decreasing, there exists n_0 such that $x_i(n) \ge c_i$, $n \ge n_0$. Then, for $i = \overline{1,N}$ we have

$$x_i(n) - c_i \ge \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_0}^{k-1} q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \ge c_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_0}^{k-1} q_i(s)\right)^{\frac{1}{\alpha_i}}, \quad n \ge n_0\,.$$

Letting $n \to \infty$, we obtain (4.3.18).

Conversely, suppose that (4.3.18) holds. Then, there exists n_0 such that

$$\sum_{k=n_0}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_0}^{k-1} q_i(s)\right)^{\frac{1}{\alpha_i}} < 2^{1-\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}.$$

Denote with $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, such that $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$ are bounded. The space $\mathcal{L}_{n_0}$ equipped with the norm (4.3.7) is Banach space. Further, $\mathcal{L}_{n_0}$ is partially ordered, with the usual pointwise ordering $\le$: for $\mathbf{x}, \mathbf{y} \in \mathcal{L}_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Define

$$\Omega_3 = \left\{ \mathbf{x} \in \mathcal{L}_{n_0} \,\middle|\, c_i \le x_i(n) \le 2c_i, \quad n \ge n_0, \quad i = \overline{1,N} \right\},$$

where c_i, $i = \overline{1,N}$ are positive constants satisfying (4.3.9).

For any subset B of Ω_3, it is obvious that $\sup B \in \Omega_3$ and $\inf B \in \Omega_3$. Let us further define operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = c_i + \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_0}^{k-1} q_i(s)x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha\, i}}, \quad n \ge n_0, \quad i = \overline{1,N},$$

and define the mapping $\Theta : \Omega_3 \to \mathcal{L}_{n_0}$ by (4.3.12). The mapping Θ satisfies the assumptions of Theorem 1.1.1. Indeed, Θ maps Ω_3 into itself, because if $\mathbf{x} \in \Omega_3$, then

$$\mathcal{F}_i x_{i+1}(n) \geq c_i, \quad i = \overline{1, N}$$

and

$$\begin{aligned}\mathcal{F}_i x_{i+1}(n) &\leq c_i + (2c_{i+1})^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) \right)^{\frac{1}{\alpha\, i}} \\ &\leq c_i + (2c_{i+1})^{\frac{\beta_i}{\alpha_i}} 2^{1-\frac{\beta_i}{\alpha_i}} \leq c_i + c_i = 2c_i, \quad n \geq n_0 \quad i = \overline{1, N}.\end{aligned}$$

Clearly, Θ is nondecreasing, i.e. for $\mathbf{x} \leq \mathbf{y}$ follows that $\Theta\mathbf{x} \leq \Theta\mathbf{y}$. By Theorem 1.1.1, mapping Θ has a fixed point $\mathbf{x} \in \Omega_3$, i.e. $\Theta\mathbf{x} = \mathbf{x}$, implying that $\mathbf{x}$ is a solution of $(SE+)$. It is easy to see that every component of the vector $\mathbf{x}$ tends to some constant. □

Theorem 4.3.4 *Let* (II) *holds. The system* $(SE+)$ *has a solution* $\mathbf{x}$ *whose each component satisfies* $(S2+)$ *if and only if*

$$\sum_{n=1}^{\infty} q_i(n) \left(\sum_{k=n}^{\infty} \frac{1}{p_{i+1}(n)^{1/\alpha_{i+1}}} \right)^{\beta_i} < \infty, \quad i = \overline{1, N}. \tag{4.3.19}$$

Proof. Suppose that $\mathbf{x} = (x_1, x_2, \dots, x_N)$ is a solution of $(SE+)$ whose each component satisfy $\lim_{n\to\infty} x_i(n)/\pi_i(n) = c_i$. Since the proof is the same for all components of vector $\mathbf{x}$, let $i \in \{1, 2, \dots, N\}$ be arbitrary fixed. Then, there exist positive constants δ_i, γ_i and n_0 such that

$$\delta_i \leq \frac{x_i(n)}{\pi_i(n)} \leq \gamma_i, \quad n \geq n_0. \tag{4.3.20}$$

Since $x_i^{[1]}$ is decreasing it follows

$$-p_i(n) \left(-\Delta x_i(n)\right)^{\alpha_i} \leq -p_i(m) \left(-\Delta x_i(m)\right)^{\alpha_i}, \quad n \geq m,$$

i.e.

$$-\Delta x_i(n) \geq \frac{p_i(m)^{\frac{1}{\alpha_i}} \left(-\Delta x_i(m)\right)}{p_i(n)^{\frac{1}{\alpha_i}}}, \quad n \geq m. \tag{4.3.21}$$

Summing (4.3.21) from m to $k-1$, we get

$$x_i(m) \geq x_i(m) - x_i(k) \geq p_i(m)^{\frac{1}{\alpha_i}} \left(-\Delta x_i(m)\right) \sum_{n=m}^{k-1} \frac{1}{p_i(n)^{\frac{1}{\alpha_i}}}.$$

Letting $k \to \infty$, we obtain $x_i(m) \geq p_i(m)^{\frac{1}{\alpha_i}} (-\Delta x_i(m))\, \pi_i(m)$. Then, using the last inequality and (4.3.20), $(SE+)$ gives

$$\sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \leq p_i(n)\,(-\Delta x_i(n))^{\alpha_i} \leq \left(\frac{x_i(m)}{\pi_i(m)}\right)^{\alpha_i} \leq \gamma_i^{\alpha_i}, \quad n \geq n_0 + 1\,.$$

Letting $n \to \infty$, we get that the condition (4.3.19) is satisfied.

Conversely, let (4.3.19) holds. Then, there exist $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} q_i(n) \left(\sum_{k=n}^{\infty} \frac{1}{p_{i+1}(n)^{1/\alpha_{i+1}}}\right)^{\beta_i} < 2^{\alpha_i - \beta_i}\,(2^{\alpha_i} - 1)\,, \quad i = \overline{1,N}.$$

Let $\mathcal{W}_{n_0}$ be the space of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$ such that $\{x_i(n)/\pi_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, $\mathcal{W}_{n_0}$ is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1 \leq i \leq N} \left\{ \sup_{n \geq n_0} \frac{x_i(n)}{\pi_i(n)} \right\}. \tag{4.3.22}$$

Set

$$\Omega_4 = \left\{ \mathbf{x} \in \mathcal{W}_{n_0} \,\middle|\, d_i \pi_i(n) \leq x_i(n) \leq 2 d_i\, \pi_i(n), \quad n \geq n_0, \quad i = \overline{1,N} \right\},$$

where d_i, $i = \overline{1,N}$ are positive constants such that

$$d_i \geq 2 d_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad d_{N+1} = d_1.$$

An example of such choice is

$$d_i = 2^{\frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij}}, \quad i = \overline{1,N}\,,$$

where M_{ij} are elements of the matrix M given by (4.2.11). It is easy to see that Ω_4 is bounded, closed and convex subset of $\mathcal{W}_{n_0}$. Define the operators $\mathcal{G}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{G}_i x(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left((2d_i)^{\alpha_i} - \sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha\, i}}, \quad n \geq n_0, \quad i = \overline{1,N},$$

and define the mapping $\Theta : \Omega_1 \to \mathcal{W}_{n_0}$ as (4.3.12). By means of similar reasoning used in the proof of Theorem 4.3.1, it can be verified that the mapping Θ satisfies all conditions of Schauder-Tychonoff fixed point theorem. Therefore, there exists $\mathbf{x} \in \Omega_4$ such that $\mathbf{x} = \Theta \mathbf{x}$. It is easy to verify that $\mathbf{x}$ is a solution of $(SE+)$ satisfying $(S2+)$. □

4.3.3 Asymptotic behavior of intermediate regularly varying solutions

In what follows we assume that coefficients p_i, q_i, $i = \overline{1,N}$ are regularly varying sequences expressed by (4.2.1). Intermediate solutions of the system $(SE+)$ are solutions of a system of equations

$$x_i(n) = c_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N},$$

if (I) holds, and

$$x_i(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N}$$

if (II) holds, for some constants $n_0 \geq 1$ and $c_i, h_i > 0$, $i = \overline{1,N}$. It follows therefore that intermediate solution of $(SE+)$ satisfies the following systems of asymptotic relations

$$x_i(n) \sim \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \quad i = \overline{1,N}, \tag{4.3.23}$$

or

$$x_i(n) \sim \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \quad i = \overline{1,N} \tag{4.3.24}$$

in cases (I) or (II), respectively.

In what follows we do not consider cases when $\lambda_i = \alpha_i$ for one or all i (these cases lead to $\rho_i = 0$), because of computational difficulty. Therefore, we have requirements of positivity or negativity for regularity indices of solutions.

The following theorem gives us necessary and sufficient condition for the existence of regularly varying solution $\mathbf{x}$ of a positive index $(\rho_1, \rho_2, \ldots, \rho_N)$ of the system of asymptotic relations (4.3.23).

Theorem 4.3.5 *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$ and suppose that $\lambda_i < \alpha_i$, $i = \overline{1,N}$. The system of asymptotic relations* (4.3.23) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i \in \left(0, \frac{\alpha_i - \lambda_i}{\alpha_i}\right)$, $i = \overline{1,N}$ if and only if*

$$0 < \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < \frac{\alpha_i - \lambda_i}{\alpha_i} \left(1 - \frac{B_N}{A_N}\right), \quad i = \overline{1,N}, \tag{4.3.25}$$

in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = \overline{1,N} \tag{4.3.26}$$

and the asymptotic behavior of any such solution is governed by the unique formula

$$x_i(n) \sim \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N-B_N}}, \quad n \to \infty, \quad i = \overline{1,N}. \tag{4.3.27}$$

where

$$D_j = (\alpha_j - \lambda_j - \alpha_j \rho_j)^{\frac{1}{\alpha_j}} \rho_j, \quad j = \overline{1,N}. \tag{4.3.28}$$

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $0 < \rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, be a solution of (4.3.23) whose components are expressed in the form (4.2.2). Then, by Theorem 1.3.3 - (v), follows that $x_i(n) \to \infty$, $n \to \infty$, $i = \overline{1,N}$. From Theorem 1.3.3 - (vii) we have that components of $\mathbf{x}$ satisfy (4.3.3). Since indices of regularity of x_i/P_i, $i = \overline{1,N}$ are less then zero, Theorem 1.3.3 - (v) implies that $\lim_{n\to\infty} x_i(n)/P_i(n) = \lim_{n\to\infty} x^{[1]}(n) = 0$. Thus, using (4.2.1) and (4.2.2), we get

$$x_i^{[1]}(n) = \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n}^{\infty} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \;\; n \geq n_0, \;\; i = \overline{1,N}. \tag{4.3.29}$$

The convergence of the above sums implies that $\mu_i + \beta_i \rho_{i+1} \leq -1$, $i = \overline{1,N}$. If for some i equality holds, then

$$\Delta x_i(n) = \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} \left(\sum_{k=n}^{\infty} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i} \right)^{\frac{1}{\alpha_i}}. \tag{4.3.30}$$

Summing (4.3.30) from n_0 to $n-1$ we find that

$$x_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} \left(\sum_{k=n}^{\infty} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad n \to \infty,$$

implying that $\rho_i = \frac{\alpha_i - \lambda_i}{\alpha_i}$, which is a contradiction. It follows that $\mu_i + \beta_i \rho_{i+1} < -1$ for $i = \overline{1,N}$. Application of Theorem 1.3.5 to (4.3.29) gives for $i = \overline{1,N}$

$$\Delta x_i(n) = \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i \rho_{i+1} + 1))^{\frac{1}{\alpha_i}}}, \tag{4.3.31}$$

when $n \to \infty$. Because $x_i(n) \to \infty$, $n \to \infty$, we conclude from (4.3.31) that it must be $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \geq -1$, $i = \overline{1,N}$. Here, also, the equality should be ruled out. If the equality holds for some i, then summing (4.3.31) from n_0 to $n-1$ we have

$$x_i(n) \sim \left(\frac{1}{\alpha_i - \lambda_i}\right) \sum_{k=n_0}^{n-1} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}}, \quad n \to \infty,$$

implying that $x_i \in \mathcal{SV}$, which is impossible. Therefore, $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i > -1$, $i = \overline{1,N}$. Summing (4.3.31) from n_0 to $n-1$ and applying Theorem 1.3.5, we conclude that

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i\rho_{i+1} + 1))^{\frac{1}{\alpha_i}} \left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} + 1\right)}, \quad n \to \infty, \quad i = \overline{1,N}. \tag{4.3.32}$$

From the previous relation, we see that

$$\rho_i = \frac{-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} + 1, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1\,, \tag{4.3.33}$$

which is equivalent to a linear cyclic system of equations

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1. \tag{4.3.34}$$

The matrix of system (4.3.34)

$$A = A\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \dots, \frac{\beta_N}{\alpha_N}\right) = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \dots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\frac{\beta_{N-1}}{\alpha_{N-1}} \\ -\frac{\beta_N}{\alpha_N} & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{4.3.35}$$

is a nonsingular because according to condition (a),

$$\det(A) = 1 - \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N} > 0\,. \tag{4.3.36}$$

Thus, the matrix A is invertible

$$A^{-1} = \frac{A_N}{A_N - B_N} M\,, \tag{4.3.37}$$

where matrix M is given by (4.2.11) and the system (4.3.34) has the unique solution ρ_i, $i = \overline{1,N}$ given explicitly by (4.3.26). From (4.3.26) we can see that all ρ_i satisfy $0 < \rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if and only if (4.3.25) holds.

Using (4.2.1) and (4.2.2) we can transform (4.3.32) in the following form

$$x_i(n) \sim \frac{n^{\frac{\alpha_i+1}{\alpha_i}} p_i(n)^{-\frac{1}{\alpha_i}} q_i(n)^{\frac{1}{\alpha_i}} x_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n \to \infty, \tag{4.3.38}$$

where D_i, $i = \overline{1,N}$ are given by (4.3.28). Without difficulty, we can obtain explicit formula (4.3.27) for each x_i from the cyclic system of asymptotic relations (4.3.38). The relation (4.3.27) can be rewritten in the following form

$$x_i(n) \sim n^{\rho_i} \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \quad n \to \infty, \quad i = \overline{1,N}. \tag{4.3.39}$$

implying that the regularity index of x_i is exactly ρ_i.

Suppose now that (4.3.25) holds and define ρ_i with (4.3.26), D_i with (4.3.28). Denote

$$X_i(n) = \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \quad i = \overline{1,N} \text{ and } X_i = \{X_i(n)\}. \tag{4.3.40}$$

Clearly, $X_i \in \mathcal{RV}(\rho_i)$, $i = \overline{1,N}$ and X_i satisfy the system of asymptotic relations (4.3.23), i.e.

$$\sum_{k=n_1}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{4.3.41}$$

where $X_{N+1} = X_1$. Indeed, $X_i(n)$ can be expressed as

$$X_i(n) = n^{\rho_i} \chi_i(n), \quad \chi_i(n) = \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \tag{4.3.42}$$

and using Theorem 1.3.5, we obtain

$$\sum_{k=n_1}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \tag{4.3.43}$$

as $n \to \infty$. Since (4.2.12) holds for the elements of matrix M, relation (4.3.43) can be transformed as

$$\frac{l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}}}{D_i} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}} = \frac{l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}}}{D_i} \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{i+1,j}\frac{\beta_i}{\alpha_i}} \right]^{\frac{A_N}{A_N - B_N}}$$

(4.3.44)

$$= \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}} = \chi_i(n), \quad i = \overline{1,N}, \quad \chi_{N+1} = \chi_N ,$$

so from (4.3.43), we obtain that X_i, $i = \overline{1,N}$ satisfy (4.3.41). □

Assuming that (II) holds, we are in a position to find necessary and sufficient condition that the system of asymptotic relations (4.3.24) possesses a regularly varying solution $\mathbf{x}$ of negative index $(\rho_1, \rho_2, \dots, \rho_N)$.

Theorem 4.3.6 *Let* $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$ *and suppose that* $\lambda_i > \alpha_i$, $i = \overline{1,N}$. *The system of asymptotic relations* (4.3.24) *has a regularly varying solution* $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ *with* $\rho_i \in \left(\frac{\alpha_i - \lambda_i}{\alpha_i}, 0\right)$, $i = \overline{1,N}$ *if and only if*

$$\frac{\alpha_i - \lambda_i}{\alpha_i}\left(1 - \frac{B_N}{A_N}\right) < \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < 0 \tag{4.3.45}$$

in which case ρ_i *are given by* (4.3.26) *and the asymptotic behavior of any such solution is governed by the unique formula* (4.3.27) *where*

$$D_j = (\lambda_j - \alpha_j + \alpha_j \rho_j)^{\frac{1}{\alpha_j}} (-\rho_j), \quad j = \overline{1,N}. \tag{4.3.46}$$

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\frac{\alpha_i - \lambda_i}{\alpha_i} < \rho_i < 0$, $i = \overline{1,N}$, be a solution of (4.3.24), whose components are given by (4.2.2). From Theorem 1.3.3 - (vii) we see that all components of $\mathbf{x}$ satisfy (4.3.4). Using (4.2.1) and (4.2.2), we obtain

$$-x_i^{[1]}(n) \sim \sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n_0}^{n-1} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \quad n \geq n_0, \ i = \overline{1,N}, \tag{4.3.47}$$

as $n \to \infty$. Since indices of regularity of x_i/π_i, $i = \overline{1,N}$ are greater then zero, it follows, by Theorem 1.3.3 - (v), that $\lim_{n\to\infty} x_i(n)/\pi_i(n) = \infty$, implying that $\lim_{n\to\infty} x^{[1]}(n) = -\infty$.

Therefore, from (4.3.47) we have that $\mu_i + \beta_i\rho_{i+1} \geq -1$ for all i. If the equality holds for some i, then noting that

$$-\Delta x_i(n) \sim \left(\frac{1}{p_i(n)}\sum_{k=n_0}^{n-1} q_i(k)x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}\left(\sum_{k=n_0}^{n-1} k^{-1}m_i(k)\xi_{i+1}(k)^{\beta_i}\right), \quad n \to \infty,$$

and summing this from n to ∞, with the help of Theorem 1.3.5 we get

$$x_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i-\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}\left(\sum_{k=n_0}^{n-1} k^{-1}m_i(k)\xi_{i+1}(k)^{\beta_i}\right), \quad n \to \infty.$$

The previous relation implies that $x_i \in \mathcal{RV}\left(\frac{\alpha_i-\lambda_i}{\alpha_i}\right)$ which is impossible, due to $\rho_i > \frac{\alpha_i-\lambda_i}{\alpha_i}$. Therefore, $\mu_i + \beta_i\rho_{i+1} > -1$ for all i. Applying Theorem 1.3.5 to (4.3.47), we get

$$\text{(4.3.48)} \quad -\Delta x_i(n) \sim \left(\frac{1}{p_i(n)}\sum_{k=n_0}^{n-1} q_i(k)x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i+\beta_i\rho_{i+1}+1)^{\frac{1}{\alpha_i}}},$$

as $n \to \infty$. Since $x_i(n) \to 0$, $n \to \infty$, we see that $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \leq -1$ for all i. All inequalities should be strict, because if the equality holds for some i, then summing (4.3.48) from n to ∞, we get

$$x_i(n) \sim (\lambda_i - \alpha_i)^{-\frac{1}{\alpha_i}} \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}}, \quad n \to \infty,$$

i.e. $x_i \in \mathcal{SV}$, which is also impossible, due to assumption $\rho_i < 0$. Therefore, we see that $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i < -1$ for all i, in which case summing (4.3.48) from n to ∞, using Theorem 1.3.5, we get

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i+\beta_i\rho_{i+1}+1)^{\frac{1}{\alpha_i}}\left[-\left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1\right)\right]}, \quad n \to \infty, \quad i = \overline{1,N}.$$

This implies that (4.3.33) holds, which is equivalent to the linear algebraic system (4.3.34). Proceeding exactly like in the proof of the previous theorem we get that system (4.3.24) has a regularly varying solution of indices $\rho_i \in \left(\frac{\alpha_i-\lambda_i}{\alpha_i}, 0\right)$ if and only if (4.3.45) is fulfilled.

Now assume that (4.3.45) holds. Define $\rho_i \in \left(\frac{\alpha_i-\lambda_i}{\alpha_i}, 0\right)$ by (4.3.26) and consider $X_i \in \mathcal{RV}(\rho_i)$ defined by (4.3.40), with D_j given by (4.3.46). It can be verified that X_i, $i = \overline{1,N}$ satisfy the system of asymptotic relations

$$\text{(4.3.49)} \qquad \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N},$$

for any $n_1 \in \mathbb{N}$, where $X_{N+1} = X_1$. In fact, we can use the expression (4.3.42) for X_i, where D_j, $j = \overline{1,N}$ are given by (4.3.46). Then, we obtain the asymptotic relation

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i} l_i(n)^{-\frac{1}{\alpha_j}} m_i(n)^{\frac{1}{\alpha_j}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad i = \overline{1,N}, \tag{4.3.50}$$

as $n \to \infty$, with $\chi_{N+1} = \chi_N$. As in proof of the previous theorem, with the help of (4.2.12), it can be verified that

$$\frac{l_i(n)^{-\frac{1}{\alpha_j}} m_i(n)^{\frac{1}{\alpha_j}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i} = \chi_i(n)$$

and the desired relation (4.3.49) immediately follows from (4.3.50). This completes the proof of the theorem. □

We are now in a position to state and prove our main results on the existence and the precise asymptotic behavior of regularly varying intermediate solutions of system $(SE+)$ with regularly varying coefficients p_i and q_i. Use is made of the notation and properties of the matrices (4.2.11), (4.3.35) and (4.3.37).

Theorem 4.3.7 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that $\lambda_i < \alpha_i$ for all $i = \overline{1,N}$. System $(SE+)$ possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i \in \left(0, \frac{\alpha_i - \lambda_i}{\alpha_i}\right)$, $i = \overline{1,N}$, if and only if* (4.3.25) *holds, in which case ρ_i are given by* (4.3.26) *and the asymptotic behavior of any such solution $\mathbf{x}$ is governed by the unique formula* (4.3.27)*, whereby* (4.3.28) *holds.*

Theorem 4.3.8 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that $\lambda_i > \alpha_i$ for all $i = \overline{1,N}$. System $(SE+)$ possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i \in \left(\frac{\alpha_i - \lambda_i}{\alpha_i}, 0\right)$, $i = \overline{1,N}$, if and only if* (4.3.45) *holds, in which case ρ_i are given by* (4.3.26) *and the asymptotic behavior of any such solution $\mathbf{x}$ is governed by the unique formula* (4.3.27) *and D_j, $j = \overline{1,N}$, are given by* (4.3.46).

We remark that the "only if" parts of these theorems follow immediately from the corresponding parts of Theorem 4.3.5 and Theorem 4.3.6 because any $\mathcal{RV}$−solution $\mathbf{x}$ of $(SE+)$ with the indicated property satisfies $(IM1)$ or $(IM2)$ and accordingly the system of asymptotic relations (4.3.23) and (4.3.24).

Proof of the "if" part of Theorem 4.3.7: Let $X_i = \{X_i(n)\} \in \mathcal{RV}(\rho_i)$ denote sequences defined by (4.3.40), where D_j for $j = \overline{1,N}$ are given by (4.3.28). It is known that

$$\sum_{k=1}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{4.3.51}$$

implying that there exists $n_0 > 1$ such that

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \le 2X_i(n), \quad n \ge n_0, \quad i = \overline{1,N}. \tag{4.3.52}$$

Without loss of generality, we may assume that each X_i is increasing for $n \ge n_0$, because it is known that any regularly varying sequence of a positive index is asymptotically equivalent to an increasing $\mathcal{RV}$ sequence of the same index. Since (4.3.51) holds for n_0 it is possible to choose $n_1 > n_0 + 1$ so large that

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \ge \frac{1}{2}X_i(n), \quad n \ge n_1, \quad i = \overline{1,N}. \tag{4.3.53}$$

Let we choose positive constants c_i and C_i so that

$$c_i \le \frac{1}{2}c_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad C_i \ge 4C_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad c_{N+1} = c_1, \quad C_{N+1} = C_1. \tag{4.3.54}$$

An example of such choices is

$$c_i = \left(\frac{1}{2}\right)^{\frac{A_N}{A_N-B_N}\sum_{j=1}^{N} M_{ij}}, \quad C_i = 4^{\frac{A_N}{A_N-B_N}\sum_{j=1}^{N} M_{ij}} \tag{4.3.55}$$

for $i = \overline{1,N}$. Clearly $c_i \le 1 \le C_i$. The constants c_i and C_i can be chosen so that $C_i/c_i \ge 2X_i(n_1)/X_i(n_0)$, that is

$$2c_iX_i(n_1) \le C_iX_i(n_0), \quad i = \overline{1,N}, \tag{4.3.56}$$

because these constants are independent of X_i and the choice of n_0 and n_1.

Consider the space Υ_{n_0} of vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/X_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, Υ_{n_0} is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N}\left\{\sup_{n\ge n_0}\frac{x_i(n)}{X_i(n)}\right\}. \tag{4.3.57}$$

Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\le$: For $\mathbf{x}, \mathbf{y} \in \Upsilon_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ with

$$\mathcal{X} = \left\{\, \mathbf{x} \in \Upsilon_{n_0} \;\middle|\; c_iX_i(n) \le x_i(n) \le C_iX_i(n), \quad n \ge n_0, \quad i = \overline{1,N}\right\}. \tag{4.3.58}$$

It is easy to see that for any $\mathbf{x} \in \mathcal{X}$ the norm of $\mathbf{x}$ is finite. Also, for any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. We will define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = b_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i} \right)^{\frac{1}{\alpha i}}, \quad n \geq n_0, \quad i = \overline{1,N}, \tag{4.3.59}$$

where b_i are positive constants such that

$$c_i X_i(n_1) \leq b_i \leq \frac{1}{2} C_i X_i(n_0), \quad i = \overline{1,N}, \tag{4.3.60}$$

and define the mapping $\Phi : \mathcal{X} \to \Upsilon_{n_0}$ by

$$\Phi(x_1, x_2, \dots, x_N) = \Big(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \dots, \mathcal{F}_N x_{N+1} \Big), \quad x_{N+1} = x_1. \tag{4.3.61}$$

We will show that Φ has a fixed point by using Theorem 1.1.1. Namely, the operator Φ has the following properties:

(i) Φ *maps* $\mathcal{X}$ *into itself*: Let $\mathbf{x} \in \mathcal{X}$. Then, using (4.3.52)-(4.3.61), we see that

$$\begin{aligned}
\mathcal{F}_i x_{i+1}(n) &\leq \frac{1}{2} C_i X_i(n_0) + C_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \\
&\leq \frac{1}{2} C_i X_i(n_0) + 2 C_{i+1}^{\frac{\beta_i}{\alpha_i}} X_i(n) \leq \frac{1}{2} C_i X_i(n) + \frac{1}{2} C_i X_i(n) = C_i X_i(n)
\end{aligned}$$

for $n \geq n_0$ and

$$\mathcal{F}_i x_{i+1}(n) \geq b_i \geq c_i X_i(n_1) \geq c_i X_i(n), \quad \text{for} \quad n_0 \leq n \leq n_1,$$

$$\mathcal{F}_i x_{i+1}(n) \geq c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \geq \frac{1}{2} c_{i+1}^{\frac{\beta_i}{\alpha_i}} X_i(n) \geq c_i X_i(n), \quad n \geq n_1.$$

This shows that $\Phi \mathbf{x} \in \mathcal{X}$, that is, Φ is a self-map on $\mathcal{X}$.

(ii) Φ *is increasing,* i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} \leq \mathbf{y}$ implies $\Phi \mathbf{x} \leq \Phi \mathbf{y}$.

Thus all the hypotheses of Theorem 1.1.1 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{X}$ of Φ, which satisfies

$$x_i(n) = \mathcal{F}_i x_{i+1}(n) = b_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha i}}, \quad n \geq n_0,$$

for every $i = \overline{1, N}$. This shows that $\mathbf{x} \in \mathcal{X}$ is a solution of system $(SE+)$. It is clear, in view of (4.3.58) and $\lim_{n\to\infty} X_i(n) = \infty$, that we have $\lim_{n\to\infty} x_i(n) = \infty$. Also,

$$p_i(n)\,(\Delta x_i(n))^{\alpha_i} = \sum_{k=n}^{\infty} q_i(k)x_{i+1}(k+1)^{\beta_i} \le c_i^{\beta_i} \sum_{k=n}^{\infty} q_i(k)X_{i+1}(k+1)^{\beta_i}, \quad i = \overline{1, N}.$$

Letting $n \to \infty$ in the last relation, we get that $x_i^{[1]}(n) \to 0$, as $n \to \infty$. Therefore, $\mathbf{x}$ is an intermediate solution.

It remains to verify that $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$. We define

$$u_i(n) = \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \; i = \overline{1, N},$$

and put

$$r_i = \liminf_{n\to\infty} \frac{x_i(n)}{u_i(n)}, \quad R_i = \limsup_{n\to\infty} \frac{x_i(n)}{u_i(n)}.$$

Since $c_i X_i(n) \le x_i(n) \le C_i X_i(n)$, $n \ge n_0$, $i = \overline{1, N}$ and

$$u_i(n) \sim X_i(n), \quad n \to \infty, \quad i = \overline{1, N}, \tag{4.3.62}$$

it follows that $0 < r_i \le R_i < \infty$, $i = \overline{1, N}$. Using Theorem 1.1.8 we obtain

$$\begin{aligned} r_i &\ge \liminf_{n\to\infty} \frac{\Delta x_i(n)}{\Delta u_i(n)} = \liminf_{n\to\infty} \frac{\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k)x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}{\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k)X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}} \\ &= \liminf_{n\to\infty} \left(\frac{\sum_{k=n}^{\infty} q_i(k)x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k)X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \left(\liminf_{n\to\infty} \frac{\sum_{k=n}^{\infty} q_i(k)x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k)X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} \\ &\ge \left(\liminf_{n\to\infty} \frac{q_i(n)x_{i+1}(n+1)^{\beta_i}}{q_i(n)X_{i+1}(n+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \liminf_{n\to\infty} \left(\frac{x_{i+1}(n+1)}{X_{i+1}(n+1)} \right)^{\frac{\beta_i}{\alpha_i}} = r_{i+1}^{\frac{\beta_i}{\alpha_i}}, \end{aligned}$$

where (4.3.62) has been used in the last step. Thus, r_i satisfy the cyclic system of inequalities

$$r_i \ge r_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1, N}, \quad r_{N+1} = r_1. \tag{4.3.63}$$

Likewise, by taking the upper limits instead of the lower limits, we are led to the cyclic inequalities

$$R_i \le R_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1, N}, \quad R_{N+1} = R_1. \tag{4.3.64}$$

From (4.3.63) and (4.3.64) we easily see that

$$r_i \geq r_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}}, \quad R_i \leq R_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}}, \tag{4.3.65}$$

whence, because of the hypothesis $\beta_1\beta_2\ldots\beta_N/\alpha_1\alpha_2\ldots\alpha_N < 1$, we find that $r_i \geq 1$ and $R_i \leq 1$, $i = \overline{1,N}$. It follows therefore that $r_i = R_i = 1$ i.e. $\lim_{n\to\infty} x_i(n)/u_i(n) = 1$ for $i = \overline{1,N}$, which combined with (4.3.62) implies that $x_i(n) \sim u_i(n) \sim X_i(n)$ as $n \to \infty$. Therefore, each x_i is a regularly varying sequence of index ρ_i. Thus the proof of the "if" part of Theorem 4.3.7. is completed. □

Proof of the "if" part of Theorem 4.3.8: Suppose that (4.3.45) holds. Define ρ_i and D_j by (4.3.26) and (4.3.46), respectively, and consider the regularly varying sequences $Y_i = \{Y_i(n)\}$ of indices ρ_i defined by

$$Y_i(n) = n^{\rho_i}\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N - B_N}}, \quad i = \overline{1,N}.$$

Since Y_i, $i = \overline{1,N}$ satisfy the asymptotic relations

$$Y_i(n) \sim \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=1}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + \sum_{s=1}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}},$$

as $n \to \infty$, where $h_i > 0$ are arbitrary fixed real constants for $i = \overline{1,N}$, one can choose $n_1 > n_0 + 1 > 2$ so that

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \leq 2Y_i(n), \quad n \geq n_0, \quad i = \overline{1,N} \tag{4.3.66}$$

and

$$\frac{1}{2}Y_i(n) \leq \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}}, \quad n \geq n_1, \quad i = \overline{1,N}. \tag{4.3.67}$$

Let us choose the positive constants ω_i and W_i which satisfy the cyclic system of inequalities

$$\omega_i \leq \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad W_i \geq 2\, W_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad \omega_{N+1} = \omega_1, \quad W_{N+1} = W_1, \tag{4.3.68}$$

and $\omega_i \le h_i^{\frac{1}{\alpha_i}} \cdot \min_{n_0\le k\le n_1}\left\{\frac{\pi_i(k)}{Y_i(k)}\right\}$. An example of such choices is

$$\omega_i = \min\left\{c_i, \gamma_i h_i^{\frac{1}{\alpha_i}}\right\}, \quad W_i = 2^{\frac{A_N}{A_N-B_N}\sum_{j=1}^N M_{ij}}, \quad i=\overline{1,N}$$

where

$$\gamma_i = \min\left\{\min_{n_0\le k\le n_1}\left\{\frac{\pi_i(k)}{Y_i(k)}\right\}, h_i^{-\frac{1}{\alpha_i}}c_i\right\} \text{ and } c_i = \left(\frac{1}{2}\right)^{\frac{A_N}{A_N-B_N}\sum_{j=1}^N M_{ij}}.$$

It is easy to see that for such constants $\omega_i \le 1 \le W_i$.

Consider the space Υ_{n_0} of vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i=\overline{1,N}$ such that $\{x_i(n)/Y_i(n)\}$, $i=\overline{1,N}$ are bounded.Then, Υ_{n_0} is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N}\left\{\sup_{n\ge n_0}\frac{x_i(n)}{Y_i(n)}\right\}.$$

Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\le$: For $\mathbf{x},\mathbf{y}\in\Upsilon_{n_0}$, $\mathbf{x}\le\mathbf{y}$ means $x_i(n)\le y_i(n)$ for all $n\ge n_0$ and $i=\overline{1,N}$. Define the subset $\mathcal{Y}\subset\Upsilon_{n_0}$ like

$$\mathcal{Y} = \left\{ \mathbf{x}\in\Upsilon_{n_0} \;\middle|\; \omega_i Y_i(n) \le x_i(n) \le W_i Y_i(n), \quad n\ge n_0, \quad i=\overline{1,N}\right\}. \tag{4.3.69}$$

For any subset $B\subset\mathcal{Y}$, it is obvious that $\inf B\in\mathcal{Y}$ and $\sup B\in\mathcal{Y}$. We consider the mapping $\Psi:\mathcal{Y}\to\Upsilon_{n_0}$ defined by

$$\Psi(x_1, x_2, \dots, x_N) = \left(\mathcal{G}_1 x_2, \mathcal{G}_2 x_3, \dots, \mathcal{G}_N x_{N+1}\right), \;\; x_{N+1} = x_1, \tag{4.3.70}$$

where $\mathcal{G}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ and

$$\mathcal{G}_i x(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + \sum_{s=n_0}^{k-1} q_i(s) x(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}}, \quad n\ge n_0, \quad i=\overline{1,N}. \tag{4.3.71}$$

We will show that Ψ has a fixed point by using Theorem 1.1.1. Namely, the operator Ψ has the following properties:

(i) Ψ *maps* $\mathcal{Y}$ *into itself*: Let $\mathbf{x}\in\mathcal{Y}$. Then, using (4.3.66)-(4.3.71), we see that

$$\begin{aligned}
\mathcal{G}_i x_{i+1}(n) &\le \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + W_{i+1}^{\beta_i}\sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \\
&= \sum_{k=n}^{\infty}\left(\frac{W_{i+1}^{\beta_i}}{p_i(k)}\left(\frac{h_i}{W_{i+1}^{\beta_i}} + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \\
&\le W_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \le \frac{1}{2}W_i\, 2Y_i(n) = W_i Y_i(n)
\end{aligned}$$

for $n \geq n_0$ and

$$\mathcal{G}_i x_{i+1}(n) \geq \sum_{k=n}^{\infty} \left(\frac{h_i}{p_i(k)} \right)^{\frac{1}{\alpha_i}} = h_i^{\frac{1}{\alpha_i}} \pi_i(n) \geq h_i^{\frac{1}{\alpha_i}} \gamma_i Y_i(n) \geq \omega_i Y_i(n) \quad \text{for} \quad n_0 \leq n \leq n_1,$$

$$\mathcal{G}_i x_{i+1}(n) \geq \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \omega_{i+1}^{\beta_i} \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}}$$

$$= \omega_{i+1}^{\beta_i} \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(\frac{h_i}{\omega_{i+1}^{\beta_i}} + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}}$$

$$\geq \omega_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \geq 2\omega_i \frac{1}{2} Y_i(n) = \omega_i Y_i(n), \;\; n \geq n_1.$$

This shows that $\Psi \mathbf{x} \in \mathcal{Y}$, that is, Ψ is a self-map on $\mathcal{Y}$.

(ii) Ψ *is increasing*, i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{Y}$, $\mathbf{x} \leq \mathbf{y}$ implies $\Psi \mathbf{x} \leq \Psi \mathbf{y}$.

Thus all the hypotheses of Theorem 1.1.1 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{Y}$ of Ψ, which satisfies

$$x_i(n) = \mathcal{G}_i x_{i+1}(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}}, \quad n \geq n_0.$$

This shows that $\mathbf{x} \in \mathcal{Y}$ is a solution of the system ($SE+$) and it is easy to see that it is an intermediate solution. In the essentially same way as in the proof of the previous theorem, we obtain that $\mathbf{x}$ is a regularly varying solution of ($SE+$) of the index $(\rho_1, \rho_2, \ldots, \rho_N)$. □

Application. Main results of this section can be applied to the well-known second order difference equation of Emden-Fowler type (4.1.1) to provide new results on the existence and the asymptotic behavior of its intermediate solutions. As an immediate consequence of Theorem 4.3.7 and Theorem 4.3.8, we have the following two results for the equation (4.1.1).

Theorem 4.3.9 *Let $\{p(n)\} \in \mathcal{RV}(\lambda)$ and $\{q(n)\} \in \mathcal{RV}(\mu)$. Suppose that $\lambda < \alpha$. Equation* (4.1.1) *possesses a regularly varying solution x of index $\rho \in \left(0, \frac{\alpha-\lambda}{\alpha}\right)$ if and only if*

$$\lambda - \alpha - 1 < \mu < \lambda \frac{\beta}{\alpha} - \beta - 1$$

in which case ρ is given by

$$\rho = \frac{\alpha - \lambda + \mu + 1}{\alpha - \beta}, \tag{4.3.72}$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x(n) \sim \left[\frac{n^{\alpha+1} p(n)^{-1} q(n)}{(\alpha - \lambda - \alpha\rho)\rho^{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

Theorem 4.3.10 *Let $\{p(n)\} \in \mathcal{RV}(\lambda)$ and $\{q(n)\} \in \mathcal{RV}(\mu)$. Suppose that $\lambda > \alpha$. Equation* (4.1.1) *possesses a regularly varying solution x of index $\rho \in \left(\frac{\alpha-\lambda}{\alpha}, 0\right)$ if and only if*

$$\lambda \frac{\beta}{\alpha} - \beta - 1 < \mu < \lambda - \alpha - 1$$

in which case ρ is given by (4.3.72) *and the asymptotic behavior of any such solution x is governed by the unique formula*

$$x(n) \sim \left[\frac{n^{\alpha+1} p(n)^{-1} q(n)}{(\lambda - \alpha + \alpha\rho)(-\rho)^{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

A special case of the equation (4.1.1) with $p(n) = 1, n \in \mathbb{N}$ and with the coefficient q belonging to the set of normalized $\mathcal{RV}$ sequences was considered in [7]. Here, we obtained necessary and sufficient conditions for the existence of $\mathcal{RV}$ solutions and their precise asymptotic behavior, for the equation (4.1.1) with coefficients p, q belonging to a larger set of $\mathcal{RV}$ sequences. Thus, Theorem 4.3.9 and Theorem 4.3.10 greatly improve results in [7].

4.4 The system $(SE-)$

Let us now turn our attention to the $(SE-)$ system. In the case when $N = 1$, we get a second-order difference equation usually called Thomas-Fermy type. Note that the mentioned equation has been discussed in detail in the previous chapter.

4.4.1 Classification of positive solutions

As we have mentioned earlier, the first task is to classify the positive solutions. In this regard, assuming that x_i, $i = \overline{1, N}$, are eventually positive, from $(SE-)$ we have that $x_i^{[1]} = \left\{x_i^{[1]}(n)\right\}$, $i = \overline{1, N}$ are eventually increasing.

(a) Let we first consider a case when all x_i are eventually increasing, i.e.

$$x_i(n) > 0, \quad \Delta x_i(n) > 0, \quad \text{for } n \geq n_0, \quad i = \overline{1,N}. \tag{4.4.1}$$

Then, there are two possibilities for $x_i^{[1]}$:

$$\lim_{n\to\infty} x_i^{[1]}(n) = c_i > 0 \quad \text{or} \quad \lim_{n\to\infty} x_i^{[1]}(n) = \infty.$$

If (I) holds, since $x_i^{[1]}$, $i = \overline{1,N}$ are increasing, then there exist m_0 such that $p_i(n)\Delta x_i(n)^{\alpha_i} \geq x_i^{[1]}(m_0), n \geq m_0$, implying that

$$x_i(n) \geq x_i(m_0) + x_i^{[1]}(m_0)^{\frac{1}{\alpha_i}} \sum_{k=m_0}^{n-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}, \quad i = \overline{1,N}.$$

Letting $n \to \infty$, we get that $\lim_{n\to\infty} x_i(n) = \infty$. Note that in the case when $\lim_{n\to\infty} x_i^{[1]}(n) = c_i > 0$, we get

$$\lim_{n\to\infty} \frac{x_i(n)}{P_i(n)} = c_i^{1/\alpha_i} \quad \text{i.e.} \quad x_i(n) \sim c_i^{1/\alpha_i} P_i(n), \quad n \to \infty,\ i = \overline{1,N}.$$

If (II) holds, then in the case when $\lim_{n\to\infty} x_i^{[1]}(n) = c_i > 0$ we have that there exists $m_0 \in \mathbb{N}$ such that $x_i^{[1]}(n) \leq c_i, n \geq m_0$ for $i = \overline{1,N}$. Therefore, it follows that

$$x_i(n) \leq x_i(m_0) + c_i^{\frac{1}{\alpha_i}} \sum_{k=m_0}^{n-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}, \quad i = \overline{1,N}.$$

From the last inequality, letting $n \to \infty$, we get that $\lim_{n\to\infty} x_i(n) = k_i$, for some positive constants $k_i, i = \overline{1,N}$.

(b) Next, consider $x_i, i = \overline{1,N}$ are eventually decreasing, i.e.

$$x_i(n) > 0, \quad \Delta x_i(n) < 0, \quad \text{for } n \geq n_0, \quad i = \overline{1,N}. \tag{4.4.2}$$

Then, for $x_i^{[1]}$, $i = \overline{1,N}$, one of the following two equalities holds:

$$\lim_{n\to\infty} x_i^{[1]}(n) = -l_i, l_i > 0 \quad \text{or} \quad \lim_{n\to\infty} x_i^{[1]}(n) = 0.$$

If (I) holds, then $\lim_{n\to\infty} x_i^{[1]}(n) = 0$. Indeed, if $\lim_{n\to\infty} x_i^{[1]}(n) = -l_i, l_i > 0$, then $-p_i(n)\,(-\Delta x_i(n))^{\alpha_i} \leq l_i$, $n \geq n_0$ i.e. $x_i(n) \leq x_i(n_0) - l_i^{1/\alpha_i} \sum_{k=n_0}^{n-1} p_i(k)^{-1/\alpha_i}$. As the right-hand side tends to $-\infty$ contradicts positivity of x_i, we have the desired conclusion.

If (II) holds, the case $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = 0$ leads to $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$ and the case $\lim_{n\to\infty} x_i(n) = 0, \lim_{n\to\infty} x_i^{[1]}(n) = const. < 0$ leads to $\lim_{n\to\infty} x_i(n)/\pi_i(n) = const. > 0$

Therefore, if (I) holds, each component x_i of solution $\mathbf{x}$ is either increasing and satisfies:

(SI) $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = \infty$,

$(S1-)$ $\lim_{n\to\infty} \dfrac{x_i(n)}{P_i(n)} = const. > 0$, i.e. $x_i(n) \sim \kappa_i P_i(n)$, $n \to \infty$, $\kappa_i > 0$,

or is decreasing and satisfies:

(SD) $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = 0$,

$(AC-)$ $\lim_{n\to\infty} x_i(n) = const. > 0$, i.e. $x_i(n) \sim \kappa_i$, $n \to \infty$, $\kappa_i > 0$.

On the other hand, if (II) holds each component x_i of solution $\mathbf{x}$ is either increasing and satisfies

(SI) $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = \infty$,

$(AC-)$ $\lim_{n\to\infty} x_i(n) = const. > 0$, i.e. $x_i(n) \sim \kappa_i$, $n \to \infty$, $\kappa_i > 0$,

or is decreasing and satisfies

(SD) $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = 0$,

$(S2-)$ $\lim_{n\to\infty} \dfrac{x_i(n)}{\pi_i(n)} = const. > 0$, i.e. $x_i(n) \sim \kappa_i \pi_i(n)$, $n \to \infty$, $\kappa_i > 0$,

$(AC-)$ $\lim_{n\to\infty} x_i(n) = const. > 0$, i.e. $x_i(n) \sim \kappa_i$, $n \to \infty$, $\kappa_i > 0$,

Solutions of type $(S1-)$, $(S2-)$ and $(AC-)$ are called primitive solutions, while solutions (SI) and (SD) are said to be extreme. We emphasize that if (I) holds (SI) is equivalent to $\lim_{n\to\infty} x_i(n)/P_i(n) = \infty$, while if (II) holds (SD) is equivalent to $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$.

Necessary and sufficient conditions for the existence of primitive solutions will be established in the next subsection. On the other hand, existence and precise asymptotic formulas of extreme solutions are not easy to determine in the general case. Therefore, in Subsections 4.4.3 and 4.4.4, we will assume that the coefficients of the system are regularly varying sequences and thus, by finding regularly varying solutions, solve the problem. Subsection 4.4.3 is dedicated to strongly increasing solutions, while the conditions of the existence and asymptotic behavior of strongly decreasing solutions will be discussed in Subsection 4.4.4.

4.4.2 Existence of primitive solutions

The asymptotic behavior of primitive solutions, as we already said, is evident from the classification itself. The following theorems prove similar to those in Subsection 4.3.2, determining the necessary and sufficient conditions for the existence of these solutions, using Fixed point theory and without the assumption that coefficients are regularly varying sequences.

Theorem 4.4.1 *Let* (I) *holds. The system* $(SE-)$ *has a solution* $\mathbf{x}$ *whose each component satisfies* $(S1-)$ *if and only if*

$$\sum_{n=1}^{\infty} q_i(n)\left(\sum_{k=1}^{n}\frac{1}{p_{i+1}(k)^{1/\alpha_{i+1}}}\right)^{\beta_i}<\infty,\quad i=\overline{1,N}. \tag{4.4.3}$$

Proof. Let $\mathbf{x}=(x_1,x_2,\dots,x_N)$ be a solution whose each component satisfies $(S1-)$. Clearly, $\mathbf{x}$ satisfies (4.4.1). Then, there exist $n_0\in\mathbb{N}$ and $m_i>0$ such that $x_i^{[1]}(n)\le m_i$ for $n\ge n_0$, $i=\overline{1,N}$. Since $x_i^{[1]}$ are positive and increasing, we have for $n\ge n_0$

$$x_i(n)=x_i(n_0)+\sum_{k=n_0}^{n-1}\frac{\left(x_i^{[1]}(k)\right)^{1/\alpha_i}}{p_i(k)^{1/\alpha_i}}\ge k_i^{1/\alpha_i}\sum_{k=n_0}^{n-1}\frac{1}{p_i(k)^{1/\alpha_i}},\ \ k_i=x_i^{[1]}(n_0),\quad i=\overline{1,N}. \tag{4.4.4}$$

Using (4.4.4), summing $(SE-)$ from n_0 to $n-1$, we get for $n\ge n_0$

$$\begin{aligned} m_i\ge x_i^{[1]}(n)&=x_i^{[1]}(n_0)+\sum_{k=n_0}^{n-1}q_i(k)x_{i+1}(k+1)^{\beta_i}\\ &\ge x_i^{[1]}(n_0)+k_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n_0}^{n-1}q_i(k)\left(\sum_{s=n_0}^{k}\frac{1}{p_{i+1}(s)^{1/\alpha_{i+1}}}\right)^{\beta_i},\quad i=\overline{1,N}.\end{aligned}$$

Letting $n\to\infty$, follows that (4.4.3) holds.

Conversely, suppose that (4.4.3) holds. Then, there exists $n_0\in\mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} q_i(n)\left(\sum_{k=n_0}^{n}\frac{1}{p_{i+1}(k)^{1/\alpha_{i+1}}}\right)^{\beta_i}<2^{\alpha_i}-1,\quad i=\overline{1,N}.$$

Denote with $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x}=(x_1,x_2,\dots,x_N)$,$x_i\in{}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i=\overline{1,N}$, such that $\{x_i(n)/P_i^{n_0}(n)\}$, $i=\overline{1,N}$ are bounded, where $P_i^{n_0}$ are given by (4.3.15). Then, $\mathcal{L}_{n_0}$ is a Banach space endowed with the norm (4.3.16). Set

$$\Lambda_1=\left\{\ \mathbf{x}\in\mathcal{L}_{n_0}\ \middle|\ \frac{c_i}{2}P_i^{n_0}(n)\le x_i(n)\le c_iP_i^{n_0}(n),\quad n\ge n_0,\quad i=\overline{1,N}\right\},$$

where c_i, $i=\overline{1,N}$ are positive constants which satisfy (4.3.9).

Define operators $\mathcal{F}_i:{}^{\mathbb{N}_{n_0}}\mathbb{R}\to{}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_ix(n)=\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\left(c_i^{\alpha_i}-\sum_{s=k}^{\infty}q_i(s)x(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha i}},\quad n\ge n_0,\quad i=\overline{1,N},$$

and define the mapping $\Theta : \Lambda_1 \to \mathcal{L}_{n_0}$ by

$$\Theta(x_1, x_2, \dots, x_N) = \Big(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \dots, \mathcal{F}_N x_{N+1}\Big), \quad x_{N+1} = x_1. \tag{4.4.5}$$

Then, it is a matter of the routine procedure to show that Θ satisfies all the hypothesis of Schauder-Tychonoff fixed point theorem. Therefore, Θ has a fixed element $\mathbf{x}$ in the set Λ_1. It is easy to see that $\mathbf{x}$ is a solution of the system $(SE-)$ and that its components satisfy $(S1-)$. □

Theorem 4.4.2 *Let* (II) *holds. The system* $(SE-)$ *has an eventually increasing solution* $\mathbf{x}$ *whose each component satisfies* $(AC-)$ *if and only if*

$$\sum_{n=2}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=1}^{n-1} q_i(k) \right)^{\frac{1}{\alpha_i}} < \infty, \quad i = \overline{1, N}. \tag{4.4.6}$$

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_N)$ be a solution of $(SE-)$ whose each component satisfy (4.4.1) and $\lim_{n\to\infty} x_i(n) = c_i$. Then, there exist $n_0 \in \mathbb{N}$ and $m_i > 0$ such that $m_i \le x_i(n) \le c_i$, $n \ge n_0$, $i = \overline{1, N}$. Summing the equations of $(SE-)$ first from n_0 to $n-1$, and then from n_0 to ∞, we get for $i = \overline{1, N}$

$$c_i - x_i(n_0) = \sum_{k=n_0}^{\infty} \left(\frac{1}{p_i(k)} \left(x_i^{[1]}(n_0) + \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \ge m_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) \right)^{\frac{1}{\alpha_i}},$$

implying that (4.4.6) holds.

Conversely, suppose that (4.4.6) holds. Then, there exists $n_0 \ge 1$ such that

$$\sum_{k=n_0}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) \right)^{\frac{1}{\alpha_i}} < 1, \quad i = \overline{1, N}.$$

Denote with $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)$, such that $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1, N}$ are bounded. Then, $\mathcal{L}_{n_0}$ is a Banach space endowed with the norm (4.3.7). Set

$$\Lambda_2 = \left\{ \mathbf{x} \in \mathcal{L}_{n_0} \,\middle|\, \frac{c_i}{2} \le x_i(n) \le c_i, \quad n \ge n_0, \quad i = \overline{1, N} \right\},$$

where c_i, $i = \overline{1, N}$ are positive constants which satisfy (4.3.9).

Define operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = \frac{c_i}{2} + \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) x(s+1)^{\beta_i} \right)^{1/\alpha_i}, \quad n \ge n_0, \quad i = \overline{1, N},$$

and define the mapping $\Theta : \Lambda_2 \to \mathcal{L}_{n_0}$ by (4.4.5). By the SchauderTychonoff theorem Θ has a fixed element $\mathbf{x}$ in the set Λ_2. Further, it is easy to see that $\mathbf{x}$ is a solution of the system $(SE-)$ and that its components satisfy $(AC-)$. □

Theorem 4.4.3 *Let* (I) *holds. The system* $(SE-)$ *has a solution* $\mathbf{x}$ *whose each component satisfies* $(AC-)$ *if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\right)^{\frac{1}{\alpha_i}}<\infty,\quad i=\overline{1,N}. \tag{4.4.7}$$

Proof. Let $\mathbf{x}=(x_1,x_2,\dots,x_N)$ be a solution of $(SE-)$ whose each component satisfies $\lim_{n\to\infty}x_i(n)=c_i$. As previously shown $\lim_{n\to\infty}x_i^{[1]}(n)=0$, $i=\overline{1,N}$. From classification it is clear that $\mathbf{x}$ is a decreasing solution, so there exist $n_0\in\mathbb{N}$ such that $x_i(n)\geq c_i$, $n\geq n_0$, $i=\overline{1,N}$. Summing equations of $(SE-)$ twice from n to ∞, we obtain

$$x_i(n)=c_i+\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\geq c_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)\right)^{\frac{1}{\alpha_i}},\quad n\geq n_0\,,$$

for $i=\overline{1,N}$. Letting $n\to\infty$, we get that the condition (4.4.7) is satisfied.

Conversely, suppose that (4.4.7) holds. Then, there exists $n_0\in\mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\right)^{\frac{1}{\alpha_i}}<2^{\frac{\alpha_i}{\beta_i}}-1,\quad i=\overline{1,N}.$$

Denote with $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x}=(x_1,x_2,\dots,x_N)$, such that $x_i=\{x_i(n)\}\in{}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i=\overline{1,N}$ are bounded. Then, $\mathcal{L}_{n_0}$ is a Banach space endowed with the norm (4.3.7). Set

$$\Lambda_3=\left\{\ \mathbf{x}\in\mathcal{L}_{n_0}\ \middle|\ c_i\leq x_i(n)\leq 2c_i,\quad n\geq n_0,\quad i=\overline{1,N}\right\},$$

where c_i, $i=\overline{1,N}$ are positive constants which satisfy (4.3.9).

Define operators $\mathcal{F}_i:{}^{\mathbb{N}_{n_0}}\mathbb{R}\to{}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_ix(n)=c_i+\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}},\quad n\geq n_0,\quad i=\overline{1,N},$$

and the mapping $\Theta:\Lambda_3\to\mathcal{L}_{n_0}$ by (4.4.5). Then, the existence of a desired solution of $(SE-)$ in the set Λ_3 follows from the SchauderTychonoff fixed point theorem applied to the mapping Θ. □

Theorem 4.4.4 *Let* (II) *holds. The system* $(SE-)$ *has a solution* $\mathbf{x}$ *whose each component satisfies* $(S2-)$ *if and only if*

$$\sum_{n=1}^{\infty}q_i(n)\left(\sum_{k=n+1}^{\infty}\frac{1}{p_{i+1}(k)^{1/\alpha_{i+1}}}\right)^{\beta_i}<\infty,\quad i=\overline{1,N}. \tag{4.4.8}$$

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_N)$ be a solution whose each component satisfies $(S2-)$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = const. < 0$, $i = \overline{1,N}$. Since $x_i^{[1]}$, $i = \overline{1,N}$ are eventually negative and increasing, there exist $n_0 \in \mathbb{N}$ and $c_i > 0$ such that $x_i^{[1]}(n) \le c_i$ for $n \ge n_0$, $i = \overline{1,N}$. Then, for $n \ge n_0$, we have

$$x_i(n) = \sum_{k=n}^{\infty} \frac{\left(-x_i^{[1]}(k)\right)^{1/\alpha_i}}{p_i(k)^{1/\alpha_i}} \ge (-c_i)^{\frac{1}{\alpha_i}} \pi_i(n), \quad i = \overline{1,N}.$$

Summing equations of the system $(SE-)$ twice from n to ∞ we get for $n \ge n_0$,

$$\begin{aligned} x_i(n) &= \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(c_i + \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\ &\ge \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(c_i + (-c_{i+1})^{\frac{\beta_i}{\alpha_i}} \sum_{s=k}^{\infty} q_i(s) \pi_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N}. \end{aligned} \tag{4.4.9}$$

Letting that $n \to \infty$ in (4.4.9), follows that (4.4.8) holds.

Conversely, suppose that (4.4.8) holds. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} q_i(n) \left(\sum_{k=n+1}^{\infty} \frac{1}{p_{i+1}(k)^{1/\alpha_{i+1}}} \right)^{\beta_i} < 2^{\alpha_i - \beta_i} \left(2^{\alpha_i} - 1\right), \quad i = \overline{1,N}.$$

Denote with $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/\pi_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, $\mathcal{L}_{n_0}$ is a Banach space endowed with the norm (4.3.22) Set

$$\Lambda_4 = \left\{ \ \mathbf{x} \in \mathcal{L}_{n_0} \ \middle| c_i \pi_i(n) \le x_i(n) \le 2c_i \pi_i(n), \quad n \ge n_0, \quad i = \overline{1,N} \right\},$$

where c_i, $i = \overline{1,N}$ are positive constants which satisfy (4.3.9).

Define operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(c_i^{\alpha_i} + \sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha i}}, \quad n \ge n_0, \quad i = \overline{1,N},$$

and define the mapping $\Theta : \Lambda_4 \to \mathcal{L}_{n_0}$ by (4.4.5).

Then, by Knaster-Tarski fixed point theorem (Theorem 1.1.1), Θ has a fixed element $\mathbf{x}$ in the set Λ_4. It is easy to see that $\mathbf{x}$ is a solution of the system $(SE-)$ and that its components satisfy $(S2-)$. □

The following theorem can be proven in the essentially same way as Theorem 4.4.3.

Theorem 4.4.5 *Let* (II) *holds. The system* $(SE-)$ *has a decreasing solution* $\mathbf{x}$ *whose each component satisfies* $(AC-)$ *if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)\right)^{\frac{1}{\alpha_i}}<\infty,\quad i=\overline{1,N}. \tag{4.4.10}$$

4.4.3 Asymptotic behavior of strongly increasing regularly varying solutions

Strongly increasing solutions of $(SE-)$ are solutions of the system

$$x_i(n)=a_i+\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(n)}\left(b_i+\sum_{s=n_0}^{k-1}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}},\quad i=\overline{1,N}$$

for some constants $n_0\geq 1$ and $a_i,b_i>0$. Note that components of strongly increasing solution of $(SE-)$ is required to satisfy $\sum_{n=n_0}^{\infty}q_i(n)x_{i+1}(n+1)=\infty$, $i=\overline{1,N}$. It follows that strongly increasing solution of $(SE-)$ satisfies the following system of asymptotic relations

$$x_i(n)\sim\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(n)}\left(\sum_{s=n_0}^{k-1}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}},\quad n\to\infty\quad i=\overline{1,N}. \tag{4.4.11}$$

The previous system of asymptotic relations can be considered as an approximation of system $(SE-)$. To find strongly increasing solution of the system $(SE-)$, i.e. solution of (4.4.11), we will restrict our attention on the case when coefficients p_i and q_i are regularly varying sequences. Therefore, we use expressions (4.2.1) for p_i and q_i, and for the components x_i of a solution $\mathbf{x}$ of $(SE-)$ we use expression (4.2.2).

Because of computational difficulty, we restrict ourselves to consider only solutions with a positive index of regularity. As we have seen in Section 4.2, the case (I) is equivalent to (4.2.3) and the case (II) to (4.2.4). Therefore, if the case (I) is satisfied, then we will distinguish two cases:

$$\lambda_i<\alpha_i,\quad i=\overline{1,N}\quad\text{or}\quad\lambda_i=\alpha_i\quad i=\overline{1,N}$$

which imply that P_i are given by (4.2.7) or (4.2.8), respectively. In the case (II), when $\lambda_i>\alpha_i$, for sequences $\pi_i=\{\pi_i(n)\}$ we have that (4.2.9) holds.

The following Theorem provides complete information about the existence and asymptotic behavior of regularly varying solution of a positive index $(\rho_1,\rho_2,\ldots,\rho_N)$ for the system of asymptotic relations (4.4.11).

Theorem 4.4.6 *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that $\lambda_i < \alpha_i$, $i = \overline{1,N}$. The system of asymptotic relations* (4.4.11) *has a regularly varying solution* $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ *with $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right) > 0, \quad i = \overline{1,N} \tag{4.4.12}$$

holds, where $\alpha_{N+1} = \alpha_1, \lambda_{N+1} = \lambda_1$, in which case ρ_i are uniquely determined by (4.3.26) *and the asymptotic behavior of any such solution is governed by the unique formulas* (4.3.27) *where*

$$D_j = \left(\lambda_j - \alpha_j + \alpha_j \rho_j\right)^{\frac{1}{\alpha_j}} \rho_j, \quad j = \overline{1,N}. \tag{4.4.13}$$

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$ be a solution of (4.4.11). Then, all x_i satisfy (4.4.1) by Theorem 1.3.3 - (*vii*). Using (4.2.1) and (4.2.2), we obtain

$$x_i^{[1]}(n) \sim \sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n_0}^{n-1} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \;\; n \geq n_0, \;\; i = \overline{1,N}. \tag{4.4.14}$$

Since indices of regularity of x_i/P_i, $i = \overline{1,N}$ are greater then zero, from Theorem 1.3.3 - (v), we have that $\lim_{n\to\infty} x_i(n)/P_i(n) = \infty$ and $\lim_{n\to\infty} x_i(n) = \infty$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = \infty$. Therefore, both sums in (4.4.14) are divergent, implying that $\mu_i + \beta_i \rho_{i+1} \geq -1$, $i = \overline{1,N}$. If for some i equality holds, then

$$\Delta x_i(n) \sim \left(\frac{1}{p_i(n)} \sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} K_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{4.4.15}$$

where

$$K_i(n) = \sum_{k=n_0}^{n-1} k^{-1} m_i(k) \xi_{i+1}(k+1)^{\beta_i}, \quad K_i \in \mathcal{SV}.$$

Since $\lambda_i < \alpha_i$, from (4.4.11) and Theorem 1.3.5 we find that

$$x_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} K_i(n)^{\frac{1}{\alpha_i}}, n \to \infty.$$

This implies that $\rho_i = \frac{\alpha_i - \lambda_i}{\alpha_i}$, which is a contradiction. Therefore, $\mu_i + \beta_i \rho_{i+1} > -1$ for $i = \overline{1,N}$. Application of Theorem 1.3.5 to (4.4.14) gives

$$\Delta x_i(n) \sim \left(\frac{1}{p_i(n)} \sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i + \beta_i \rho_{i+1} + 1)^{\frac{1}{\alpha_i}}}, \tag{4.4.16}$$

as $n \to \infty$, which yields

$$x_i(n) \sim \sum_{k=n_0}^{n-1} \frac{k^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i+\beta_i\rho_{i+1}+1)^{\frac{1}{\alpha_i}}}, \tag{4.4.17}$$

$i = \overline{1,N}$. Since $x_i \to \infty$, $n \to \infty$, from (4.4.17) we conclude that $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \geq -1$, $i = \overline{1,N}$. All inequalities should be strict because the equality for some i would imply that $0 < \mu_i + \beta_i\rho_{i+1} + 1 = \lambda_i - \alpha_i < 0$, which is impossible. Therefore, $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i > -1$, $i = \overline{1,N}$. Applying Theorem 1.3.5, from (4.4.17) we get

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i+\beta_i\rho_{i+1}+1)^{\frac{1}{\alpha_i}} \left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1\right)}, \quad n \to \infty, \quad i = \overline{1,N}. \tag{4.4.18}$$

From previous relation we see that ρ_i, $i = \overline{1,N}$, satisfy (4.3.33) i.e. ρ_i, $i = \overline{1,N}$, will be determined as a solution of the system (4.3.34). Thus, ρ_i, $i = \overline{1,N}$ are given explicitly by (4.3.26). Let we denote $d_i = \rho_i - \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$. Then, the system (4.3.34) becomes

$$d_i - \frac{\beta_i}{\alpha_i} d_{i+1} = \frac{\mu_i + 1}{\alpha_i} + \frac{\beta_i(\alpha_{i+1} - \lambda_{i+1})}{\alpha_i\alpha_{i+1}}, \quad i = \overline{1,N}, \quad d_{N+1} = d_1. \tag{4.4.19}$$

Matrix of the system (4.4.19) is given by (4.3.35). Since A is nonsingular according to (4.3.36), the system (4.4.19) has a unique solution $d_i, i = \overline{1,N}$, where

$$d_i = \sum_{j=1}^{N} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1}) - \lambda_{j+1}}{\alpha_j\alpha_{j+1}} \right), \quad i = \overline{1,N}. \tag{4.4.20}$$

Using that $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$ if and only if $d_i > 0$, we conclude that the condition (4.4.12) is satisfied.

Using (4.2.1) and (4.2.2) we can transform (4.4.18) in the form (4.3.38) where $D_i, i = \overline{1,N}$ are given by (4.4.13). It easy to obtain formulas (4.3.27) for each x_i from the cyclic system of asymptotic relations (4.3.38), which can be rewritten in the form (4.3.39), implying that the regularity index of x_i is exactly ρ_i.

Suppose now that (4.4.12) holds, define ρ_i and D_i with (4.3.26) and (4.4.13), respectively, and let $X_i, i = \overline{1,N}$ be sequences defined with (4.3.40). Clearly, $X_i \in \mathcal{RV}(\rho_i)$, $i = \overline{1,N}$ and X_i satisfy the system of asymptotic relations

$$\sum_{k=n_1}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_1}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \quad X_{N+1} = X_1, \tag{4.4.21}$$

for arbitrary $n_1 \in \mathbb{N}$. Indeed, X_i can be expressed as (4.3.42) and using Theorem 1.3.5, we obtain

$$\sum_{k=n_1}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=n_1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n\to\infty. \tag{4.4.22}$$

Relation (4.4.22) can be transformed as in (4.3.44), implying that X_i, $i = \overline{1,N}$ satisfy (4.4.21). □

Theorem 4.4.7 *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose $\lambda_i = \alpha_i$ and $S_i = \infty$, $i = \overline{1,N}$. The system of asymptotic relations* (4.4.11) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i > 0$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij}\frac{\mu_j+1}{\alpha_j} > 0, \quad i = \overline{1,N} \tag{4.4.23}$$

in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N}\sum_{j=1}^{N} M_{ij}\frac{\mu_j+1}{\alpha_j}, \quad i = \overline{1,N} \tag{4.4.24}$$

and the asymptotic behavior of any such solution is governed by the unique formulas (4.3.27) *with $D_j = \left(\alpha_j \rho_j^{\alpha_j+1}\right)^{1/\alpha_j}$, $j = \overline{1,N}$.*

Proof. Suppose that the system (4.4.11) has a solution $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$, with $\rho_i > 0, i = \overline{1,N}$. Using (4.2.1) and (4.2.2), we obtain (4.4.14). For all $i = \overline{1,N}$, indices of regularity of x_i/P_i are $\rho_i > 0$, due to fact that $P_i \in \mathcal{SV}$. Thus, from Theorem 1.3.3 - (v), we have that $\lim_{n\to\infty} x_i(n)/P_i(n) = \infty$ and $\lim_{n\to\infty} x_i(n) = \infty$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = \infty$. Therefore, both sums in (4.4.14) are divergent, so it must be $\mu_i + \beta_i\rho_{i+1} \geq -1$. If equality holds for some i, then from (4.4.15) it follows that

$$x_i(n) \sim \sum_{k=n_0}^{n-1} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} K_i(k)^{\frac{1}{\alpha_i}}, \quad n\to\infty, \quad i = \overline{1,N},$$

implying that $x_i \in \mathcal{SV}$, which is a contradiction. Therefore, $\mu_i + \beta_i\rho_{i+1} > -1$ for $i = \overline{1,N}$. Application of Theorem 1.3.5 to (4.4.14) gives (4.4.16) and since $x_i(n) \to \infty$, $n\to\infty$, it must be $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \geq -1, i = \overline{1,N}$. If equality holds for any i, then $\mu_i + \beta_i\rho_{i+1} = -1$, which is impossible. Thus, $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i > -1, i = \overline{1,N}$. Summing (4.4.16) from n_0 to $n-1$ and using Theorem 1.3.5, we get (4.4.18). Using assumption $\lambda_i = \alpha_i$, $i = \overline{1,N}$, from (4.4.18) we obtain the following cyclic system by the unknown ρ_i,

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\mu_i + 1}{\alpha_i}, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1. \tag{4.4.25}$$

Matrix of the system (4.4.25) is given by (4.3.35), and therefore, the system has a unique solution $\rho_i, i = \overline{1,N}$ given by (4.4.24). All ρ_i are positive if and only if (4.4.23) holds. Proceeding exactly as in the proof of the previous theorem, we get (4.3.27), where constants D_j are reduced to $D_j = \left(\alpha_j \rho_j^{\alpha_j+1}\right)^{1/\alpha_j}, j = \overline{1,N}$. □

The solution of the problem of determining the necessary and sufficient conditions for the system of asymptotic relations (4.4.11) to have a regularly varying solution **x** of the positive regularity index $(\rho_1, \rho_2, \ldots, \rho_N)$ in the case (II) is given by the following theorem.

Theorem 4.4.8 *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$ and suppose that $\lambda_i > \alpha_i, i = \overline{1,N}$. The system of asymptotic relations* (4.4.11) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i > 0, i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} > 0 \tag{4.4.26}$$

in which case ρ_i are given by (4.3.26) *and the asymptotic behavior of any such solution is governed by the unique formula* (4.3.27) *with $D_j, j = \overline{1,N}$ given by* (4.4.13).

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i > 0$ be a solution of (4.4.11). Clearly, all components of the solution **x** satisfies (4.4.1). From Theorem 1.3.3 - (v) we have that $\lim_{n\to\infty} x_i(n) = \infty, i = \overline{1,N}$. Since $x_i^{[1]}, i = \overline{1,N}$ are positive and increasing it follows that

$$\text{(a)} \quad \lim_{n\to\infty} x_i^{[1]}(n) = c_i > 0 \quad \text{or} \quad \text{(b)} \quad \lim_{n\to\infty} x_i^{[1]}(n) = \infty.$$

Case (a) implies that $\lim_{n\to\infty} x_i(n) = const.$, that is $x_i \in \mathcal{SV}$, which is impossible. Thus, for $x_i^{[1]}, i = \overline{1,N}$ we have that (b) holds. As in the previous theorem we can obtain (4.4.14) and conclude that $\mu_i + \beta_i \rho_{i+1} \geq -1, i = \overline{1,N}$. If the equality holds for some i, then summing (4.4.15) from n_0 to $n-1$ and using Theorem 1.3.5 we get

$$x_i(n) \sim \sum_{k=n_0}^{n-1} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} K_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty.$$

Because $x_i(n) \to \infty, n \to \infty$, it must be $-\frac{\lambda_i}{\alpha_i} \geq -1$, i.e. $\lambda_i \leq \alpha_i$, which is impossible. Therefore, $\mu_i + \beta_i \rho_{i+1} > -1$ for all i. Proceeding exactly as in the proof of Theorem 4.4.6

we conclude that ρ_i, $i = \overline{1,N}$ are given by (4.3.26). It is obvious that $\rho_i > 0$ if and only if (4.4.26) is fulfilled. Like in the proof of the previous theorem, we get that the system (4.4.11) has a regularly varying solution satisfying (4.3.27).

Proof of the "if" part of the theorem is the same as of Theorem 4.4.6. □

From the statements of Theorem 4.4.7 and Theorem 4.4.8, we have the following result for the existence of regularly varying solution of the system of asymptotic relations (4.4.11) with positive index of regularity.

Theorem 4.4.9 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that for each of λ_i, α_i, $i = \overline{1,N}$, some of the following conditions is satisfied:*

$$\text{either} \quad \lambda_i > \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i,\ S_i = \infty.$$

The system of asymptotic relations (4.4.11) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i > 0$, $i = \overline{1,N}$ if and only if* (4.4.26) *holds, in which case ρ_i are given by* (4.3.26) *and the asymptotic behavior of any such solution is governed by the unique formula* (4.3.27)*, with D_j, $j = \overline{1,N}$ given by* (4.4.13).

Focusing on strongly increasing solutions of the system $(SE-)$ with a regularly varying coefficients p_i and q_i, we can formulate and prove the following statements.

Theorem 4.4.10 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that $\lambda_i < \alpha_i$ for all $i = \overline{1,N}$. The system $(SE-)$ possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$, if and only if* (4.4.12) *holds, in which case ρ_i are given by* (4.3.26) *and the asymptotic behavior of any such solution $\mathbf{x}$ is governed by the unique formula* (4.3.27)*, with D_j, $j = \overline{1,N}$ given by* (4.4.13).

Theorem 4.4.11 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that for each of λ_i, α_i, $i = \overline{1,N}$, some of the following conditions is satisfied:*

$$\text{either} \quad \lambda_i > \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i,\ S_i = \infty.$$

The system $(SE-)$ possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i > 0$, $i = \overline{1,N}$, if and only if (4.4.26) *holds, in which case ρ_i are given by* (4.3.26) *and the asymptotic behavior of any such solution $\mathbf{x}$ is governed by the unique formula* (4.3.27)*, with D_j, $j = \overline{1,N}$ given by* (4.4.13).

We remark that the "only if" parts of these theorems follow immediately from the corresponding parts of Theorem 4.4.6 and Theorem 4.4.8 because any solution $\mathbf{x}$ of $(SE-)$ with the indicated property satisfies the system of asymptotic relation (4.4.11).

Proof of the "if" part of Theorem 4.4.10: Suppose (4.4.12) is satisfied. Let we define the sequences $X_i = \{X_i(n)\} \in \mathcal{RV}(\rho_i)$ by (4.3.40), where D_j for $j = \overline{1,N}$ are given by (4.4.13). It is known that

$$\sum_{k=1}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=1}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{4.4.27}$$

implying that there exists $n_0 > 1$ such that

$$\sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \leq 2X_i(n), \quad n \geq n_0, \quad i = \overline{1,N}. \tag{4.4.28}$$

Without loss of generality, we may assume that each X_i is eventually increasing. Since (4.4.27) holds for n_0, it is possible to choose $n_1 > n_0 + 1$ so large that

$$\sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \geq \frac{1}{2} X_i(n), \quad n \geq n_1, \quad i = \overline{1,N}. \tag{4.4.29}$$

Let we choose positive constants c_i and C_i so that

$$c_i \leq \frac{1}{2} c_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad C_i \geq 4C_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad c_{N+1} = c_1, \quad C_{N+1} = C_1. \tag{4.4.30}$$

An example of such choices is given in (4.3.55). Constants c_i and C_i can be chosen so that

$$2c_i X_i(n_1) \leq C_i X_i(n_0), \quad i = \overline{1,N}, \tag{4.4.31}$$

because these constants are independent of X_i as well as of the choice of n_0 and n_1.

Consider the space Υ_{n_0} of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/X_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, Υ_{n_0} is a Banach space endowed with the norm (4.3.57). Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\leq$: For $\mathbf{x}, \mathbf{y} \in \Upsilon_{n_0}$, $\mathbf{x} \leq \mathbf{y}$ means $x_i(n) \leq y_i(n)$ for all $n \geq n_0$ and $i = \overline{1,N}$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ with

$$\mathcal{X} = \left\{ \mathbf{x} \in \Upsilon_{n_0} \,\middle|\, c_i X_i(n) \leq x_i(n) \leq C_i X_i(n), \quad n \geq n_0, \quad i = \overline{1,N} \right\}. \tag{4.4.32}$$

It is easy to see that for any $\mathbf{x} \in \mathcal{X}$, the norm of $\mathbf{x}$ is finite and that for any subset $B \subset \mathcal{X}$, $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. Define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = b_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) x(s+1)^{\beta_i} \right)^{\frac{1}{\alpha i}}, \quad n \geq n_0, \quad i = \overline{1,N}, \tag{4.4.33}$$

where b_i are positive constants such that

$$c_i X_i(n_1) \le b_i \le \frac{1}{2} C_i X_i(n_0), \quad i = \overline{1,N}, \tag{4.4.34}$$

and define the mapping $\Phi : \mathcal{X} \to \Upsilon_{n_0}$ by

$$\Phi(x_1, x_2, \dots, x_N) = \Big(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \dots, \mathcal{F}_N x_{N+1}\Big), \;\; x_{N+1} = x_1. \tag{4.4.35}$$

We will show that Φ has a fixed point by using Theorem 1.1.1. Namely, the operator Φ has the following properties:

(i) Φ *maps* $\mathcal{X}$ *into itself*: Let $\mathbf{x} \in \mathcal{X}$. Then, using (4.4.28)-(4.4.35), we see that

$$\begin{aligned}
\mathcal{F}_i x_{i+1}(n) &\le \frac{1}{2} C_i X_i(n_0) + C_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \\
&\le \frac{1}{2} C_i X_i(n_0) + 2 C_{i+1}^{\frac{\beta_i}{\alpha_i}} X_i(n) \le \frac{1}{2} C_i X_i(n) + \frac{1}{2} C_i X_i(n) = C_i X_i(n)
\end{aligned}$$

for $n \ge n_0$ and

$$\mathcal{F}_i x_{i+1}(n) \ge b_i \ge c_i X_i(n_1) \ge c_i X_i(n), \quad \text{for} \quad n_0 \le n \le n_1,$$

$$\mathcal{F}_i x_{i+1}(n) \ge c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \ge \frac{1}{2} c_{i+1}^{\frac{\beta_i}{\alpha_i}} X_i(n) \ge c_i X_i(n), \quad n \ge n_1.$$

This shows that $\Phi \mathbf{x} \in \mathcal{X}$, that is, Φ is a self-map on $\mathcal{X}$.

(ii) Φ *is increasing,* i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} \le \mathbf{y}$ implies $\Phi \mathbf{x} \le \Phi \mathbf{y}$.

Thus all the hypotheses of Theorem 1.1.1 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{X}$ of Φ, which satisfies

$$x_i(n) = \mathcal{F}_i x_{i+1}(n) = b_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha i}}, \quad n \ge n_0,$$

for $i = \overline{1,N}$. This shows that $\mathbf{x} \in \mathcal{X}$ is a solution of system $(SE-)$ and it is easy to see that it is a strongly increasing solution.

It remains to verify that $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$. We define

$$u_i(n) = \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \;\; i = \overline{1,N},$$

and put

$$r_i = \liminf_{n\to\infty} \frac{x_i(n)}{u_i(n)}, \quad R_i = \limsup_{n\to\infty} \frac{x_i(n)}{u_i(n)}.$$

Using (4.4.32) and

$$u_i(n) \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{4.4.36}$$

it follows that $0 < r_i \leq R_i < \infty$, $i = \overline{1,N}$. Using Theorem 1.1.8 we obtain

$$\begin{aligned}
r_i &\geq \liminf_{n\to\infty} \frac{\Delta x_i(n)}{\Delta u_i(n)} = \liminf_{n\to\infty} \frac{\left(\frac{1}{p_i(n)} \sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}{\left(\frac{1}{p_i(n)} \sum_{k=n_0}^{n-1} q_i(k) X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}} \\
&= \liminf_{n\to\infty} \left(\frac{\sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n_0}^{n-1} q_i(k) X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \left(\liminf_{n\to\infty} \frac{\sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n_0}^{n-1} q_i(k) X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} \\
&\geq \left(\liminf_{n\to\infty} \frac{q_i(n) x_{i+1}(n+1)^{\beta_i}}{q_i(n) X_{i+1}(n+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \liminf_{n\to\infty} \left(\frac{x_{i+1}(n+1)}{X_{i+1}(n+1)} \right)^{\frac{\beta_i}{\alpha_i}} = r_{i+1}^{\frac{\beta_i}{\alpha_i}}
\end{aligned}$$

where (4.4.36) has been used in the last step. Thus, r_i satisfy the cyclic system of inequalities (4.3.63). If we take the upper limits instead of the lower limits, we are led to the cyclic system of inequalities (4.3.64).

From (4.3.63) and (4.3.64) we easily see that (4.3.65) holds and using the hypothesis $\beta_1\beta_2\ldots\beta_N/\alpha_1\alpha_2\ldots\alpha_N < 1$, we find that $r_i \geq 1$ and $R_i \leq 1$, $i = \overline{1,N}$. It follows therefore that $r_i = R_i = 1$ i.e. $\lim_{n\to\infty} x_i(n)/u_i(n) = 1$ for $i = \overline{1,N}$. Combined this with (4.4.36) implies that $x_i(n) \sim u_i(n) \sim X_i(n)$ as $n \to \infty$, which shows that each x_i is a regularly varying sequence of index ρ_i. Thus the proof of the "if" part of Theorem 4.4.10 is completed. □

The "if" part of the Theorem 4.4.11 can be proved in the essentially same way as the "if" part of the previous theorem.

Application. Obtained results can be applied to the well-known second order difference equation of Thomas-Fermy type (4.1.1) which has been studied in Chapter 3. As a direct consequence of Theorem 4.4.10 and Theorem 4.4.11, we have Theorem 3.4.1. However, in the previous chapter, due to the calculation difficulty , the case when the regularity index of the coefficient p of the equation (4.1.1) is equal to α, has not been considered. For that reason, as a consequence of Theorem 4.4.11, we obtain a new result for the equation (4.1.1).

Theorem 4.4.12 *Let $p \in \mathcal{RV}(\lambda)$ and $q \in \mathcal{RV}(\mu)$. Suppose that*

$$\lambda = \alpha \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{p(n)^{\alpha}} = \infty.$$

The equation (4.1.1) *possesses a regularly varying solution of index $\rho > 0$ if and only if $\mu > -1$, in which case ρ is given by*

$$\rho = \frac{\mu + 1}{\alpha - \beta}, \tag{4.4.37}$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x(n) \sim \left[\frac{n^{\alpha+1} p(n)^{-1} q(n)}{\alpha \rho^{\alpha+1}} \right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

4.4.4 Asymptotic behavior of strongly decreasing regularly varying solutions

In this subsection the aim is the same as earlier, to investigate the asymptotic behavior of solutions of type (SD) and to find conditions under which those solutions exist. Every strongly decreasing solutions of ($SE-$) is a solutions of the system

$$x_i(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{j=k}^{\infty} q_i(j) x_{i+1}(j+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N}, \quad n \geq n_0, \tag{4.4.38}$$

for some $n_0 \geq 1$. To obtain an asymptotic formula of solution $\mathbf{x}$ of the system ($SE-$), an essential role is played by the following system od asymptotic relations

$$x_i(n) \sim \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{j=k}^{\infty} q_i(j) x_{i+1}(j+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N}, \quad n \to \infty, \tag{4.4.39}$$

which can observe as an approximation of the system (4.4.38). As in the previous sections, the use is made of the expression (4.2.1) for the coefficients p_i and q_i and (4.2.2) for the components x_i of the solution $\mathbf{x}$ of the system ($SE-$). Also, in the proof of the main theorems, we use matrix M given by (4.2.11).

As we saw, cases (I) and (II) are equivalent to (4.2.3) and (4.2.4), respectively. Since we consider only solutions with a negative index of regularity (slowly varying solutions we leave for further research because computational difficulty), cases (I) and (II) would be observed as follows:

- If the case (I) is satisfied, i.e. when $\lambda_i < \alpha_i$, for the sequences $P_i = \{P_i(n)\}$ we have that (4.2.7) holds.
- In the case (II) we will distinguish two cases:

$$\lambda_i > \alpha_i, \quad i = \overline{1,N} \quad \text{and} \quad \lambda_i = \alpha_i \quad i = \overline{1,N}$$

which imply (4.2.9) and (4.2.10), respectively.

Assuming that the condition (I) is satisfied, the following theorem will give the necessary and sufficient conditions for a system of asymptotic relations (4.4.39) to have a regularly varying solution $\mathbf{x}$ of a negative index $(\rho_1, \rho_2, \dots, \rho_N)$, as well as its asymptotic formula.

Theorem 4.4.13 *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that $\lambda_i < \alpha_i$, $i = \overline{1,N}$. The system of asymptotic relations* (4.4.39) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i < 0$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < 0, \quad i = \overline{1,N} \tag{4.4.40}$$

holds, in which case ρ_i are uniquely determined by (4.3.26) *and the asymptotic behavior of any such solution is governed by the unique formulas* (4.3.27) *where*

$$D_j = \left(\alpha_j - \lambda_j - \alpha_j \rho_j\right)^{\frac{1}{\alpha_j}} (-\rho_j), \quad j = \overline{1,N}. \tag{4.4.41}$$

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with all $\rho_i < 0$ be a solution of (4.4.39). Then, by Theorem 1.3.3 - (vii), follows that each x_i satisfies (4.4.2) and Theorem 1.3.3 - (v) yields that $x_i(n) \to 0, n \to \infty$. Also, since (I) holds, as shown in a classification of positive solution in Subsection 1.4.1. we have that $x_i^{[1]}(n) \to 0$, as $n \to \infty$. Using (4.2.1) and (4.2.2), we obtain

$$-x_i^{[1]}(n) \sim \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n}^{\infty} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \;\; n \geq n_0, \;\; i = \overline{1,N}. \tag{4.4.42}$$

As the left-hand side of (4.4.42) tends to zero as $n \to \infty$, it must be $\mu_i + \beta_i \rho_{i+1} \leq -1$, $i = \overline{1,N}$. If for some i equality holds, then

$$-\Delta x_i(n) \sim \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} H_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{4.4.43}$$

where $H_i(n) = \sum_{k=n}^{\infty} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i}$. Summing (4.4.43) from n to ∞, we obtain

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty.$$

Using that $x_i(n) \to 0, n \to \infty$, we conclude that $\lambda_i \geq \alpha_i, i = \overline{1,N}$. This is a contradiction with our assumption. Therefore, it follows that $\mu_i + \beta_i \rho_{i+1} < -1$ for $i = \overline{1,N}$. Application of Theorem 1.3.5 to (4.4.42) gives for $i = \overline{1,N}$

$$\text{(4.4.44)} \quad -\Delta x_i(n) \sim \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i \rho_{i+1} + 1))^{\frac{1}{\alpha_i}}},$$

as $n \to \infty$. Summing (4.4.44) from n to ∞, we obtain

$$\text{(4.4.45)} \qquad x_i(n) \sim \sum_{k=n}^{\infty} \frac{k^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i \rho_{i+1} + 1))^{\frac{1}{\alpha_i}}}, \ \ n \to \infty,$$

implying that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \leq -1$, $i = \overline{1,N}$. All inequalities should be strict because if the equality holds for some i then (4.4.45) implies that $\rho_i = 0$, which is contradiction. Therefore, $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i < -1$, $i = \overline{1,N}$. Applying Theorem 1.3.5 we conclude that

$$\text{(4.4.46)} \qquad x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{-\left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} + 1\right)(-(\mu_i + \beta_i \rho_{i+1} + 1))^{\frac{1}{\alpha_i}}}, \quad n \to \infty, \quad i = \overline{1,N}.$$

From the previous relation, we get the system (4.3.34), whose unique solution is explicitly given by (4.3.26). It is obvious that $\rho_i < 0$, $i = \overline{1,N}$ if and only if (4.4.40) holds.

Using (4.2.1) and (4.2.2), we can transform (4.4.46) in the form (4.3.38), where D_i are given by (4.4.41). Without difficulty, we can obtain explicit formula (4.3.27) for each x_i from the cyclic system of asymptotic relations (4.3.38). Relation (4.3.38) can be rewritten as (4.3.39), implying that the regularity index of x_i is exactly ρ_i.

Suppose now that (4.4.40) holds. Define ρ_i with (4.3.26) and sequences X_i, $i = \overline{1,N}$, by (4.3.40), where D_j, $j = \overline{1,N}$ are given by (4.4.41). All sequences X_i are regularly varying of index ρ_i and satisfy the system of asymptotic relations (4.4.39), i.e.

$$\text{(4.4.47)} \qquad \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N},$$

for any $n > n_0$, where $X_{N+1} = X_1$. Indeed, X_i can be expressed with (4.3.42). Using Theorem 1.3.5, we obtain

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n \to \infty, \tag{4.4.48}$$

With the help of (4.2.12), relation (4.4.48) can be transformed in the same way as in (4.3.44). Therefore, from (4.4.48) we obtain that X_i, $i = \overline{1,N}$ satisfy (4.4.47). □

Assuming that the condition (II) is satisfied, the next two theorems gives the necessary and sufficient conditions for the existence of regularly varying solution of the system (4.4.39).

Theorem 4.4.14 *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$. Suppose that $\lambda_i > \alpha_i$, $i = \overline{1,N}$. The system of asymptotic relations* (4.4.39) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij}\left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}}\right) < 0, \quad i = \overline{1,N} \tag{4.4.49}$$

holds, where $\alpha_{N+1} = \alpha_1, \lambda_{N+1} = \lambda_1$, in which case ρ_i are uniquely determined with (4.3.26) *and the asymptotic behavior of any such solution is governed by the unique formulas* (4.3.27) *with D_j, $j = \overline{1,N}$ given by* (4.4.41).

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with all $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$ be a solution of (4.4.39). Since ρ_i, $i = \overline{1,N}$ are negative, from Theorem 1.3.3 - (v) we have that $x_i(n) \to 0$, $n \to \infty$. As indices of regularity of x_i/π_i, $i = \overline{1,N}$ are less then zero, from Theorem 1.3.3 - (v) we have that $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = 0$. Using (4.2.1) and (4.2.2) we obtain (4.4.42). The left-hand side of (4.4.42) tends to zero as $n \to \infty$, implying that $\mu_i + \beta_i\rho_{i+1} \leq -1$, $i = \overline{1,N}$. If the equality holds for some i, then summing (4.4.43) from n to ∞ and using Theorem 1.3.5, we get

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}} \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} H_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty,$$

where $H_i(n) = \sum_{k=n}^{\infty} k^{-1} m_i(k)\xi_{i+1}(k)^{\beta_i}$, $i = \overline{1,N}$. From the previous relation follows that $x_i \in \mathcal{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i}\right)$, contradicting the hypothesis $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$. Therefore, $\mu_i + \beta_i\rho_{i+1} < -1$ for all i. Proceeding exactly as in the proof of the previous theorem we get that (4.4.44) holds and conclude that $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \leq -1$ for all i. All inequalities should be strict, because if the equality holds for some i, then $\lambda_i - \alpha_i = \mu_i + \beta_i\rho_{i+1} + 1 < 0$

which is impossible due to assumption $\lambda_i > \alpha_i$, $i = \overline{1,N}$. Thus, (4.4.46) holds, which yields that $\rho_i, i = \overline{1,N}$ is a solutions of the system (4.3.34) and so given by (4.3.26). To verify the condition (4.4.49), let we denote $d_i = \rho_i + \lambda_i/\alpha_i - 1$, $i = \overline{1,N}$. Then, the linear system of equations (4.3.34) is transformed into the system (4.4.19) whose solution is given by (4.4.20). Thus, $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$ if and only if $d_i < 0$ if and only if (4.4.49) holds.

The "if" part of the theorem, as well as the explicit formulas for x_i, can be obtained as in the previous theorem. □

Theorem 4.4.15 *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$. Suppose $\lambda_i = \alpha_i$ and $S_i < \infty$, $i = \overline{1,N}$. The system of asymptotic relations* (4.4.39) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i < 0$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\mu_j + 1}{\alpha_j} < 0, \quad i = \overline{1,N} \tag{4.4.50}$$

in which case ρ_i are uniquely determined by (4.4.24) *and the asymptotic behavior of any such solution is governed by the unique formulas* (4.3.27) *with $D_j = \Big(\alpha_j(-\rho_j)^{\alpha_j+1}\Big)^{1/\alpha_j}$, $j = \overline{1,N}$.*

Proof. Suppose that the system (4.4.39) has a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$, $\rho_i < 0$, $i = \overline{1,N}$, $\mathbf{x} = (x_1, x_2, \dots, x_N)$. Note that $\pi_i \in \mathcal{SV}$ and so indices of regularity of x_i/π_i are $\rho_i < 0, i = \overline{1,N}$. Therefore, by Theorem 1.3.3 - (v), we have that $\lim_{n\to\infty} x_i(n) = 0$ and $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$ implying that $\lim_{n\to\infty} x_i^{[1]}(n) = 0$. From (4.4.42), since the left-hand side tends to zero, we conclude that it must be $\mu_i + \beta_i\rho_{i+1} \le -1$. If equality holds for some i then from (4.4.43), it follows that

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \quad i = \overline{1,N},$$

i.e. $x_i \in \mathcal{SV}$, which is a contradiction. Therefore, $\mu_i + \beta_i\rho_{i+1} < -1$ for $i = \overline{1,N}$. Application of Theorem 1.3.5 to (4.4.42) gives (4.4.44) and since $x_i(n) \to 0$, $n \to \infty$ it must be $(\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \le -1, i = \overline{1,N}$. If equality holds for any i, then $\mu_i + \beta_i\rho_{i+1} = -1$ which is impossible. Thus, $(\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i < -1, i = \overline{1,N}$. Summing (4.4.44) from n to ∞ and using Theorem 1.3.5 yields (4.4.46). Using assumption $\lambda_i = \alpha_i$, $i = \overline{1,N}$ we get the cyclic system (4.4.25). As verified in the proof of Theorem 4.4.7 solution of system (4.4.25) is $\rho_i, i = \overline{1,N}$ given by (4.4.24). All ρ_i are negative if and only if (4.4.50) holds. Proceeding exactly as in the proof of the previous theorem we get (4.3.27), where constants D_j are reduced to $D_j = \Big(\alpha_j/(-\rho_i)^{\alpha_j+1}\Big)^{1/\alpha_j}$, $j = \overline{1,N}$. The "if" part is the same as in the previous theorem. □

Note that Theorem 4.4.13 and Theorem 4.4.15 can be unified into the following statement.

Theorem 4.4.16 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that for each of λ_i, α_i, $i = \overline{1,N}$, some of the following conditions is satisfied:*

$$\text{either} \quad \lambda_i < \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i,\ S_i < \infty.$$

The system (4.4.39) *has a regularly varying solution* $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ *with* $\rho_i < 0$, $i = \overline{1,N}$ *if and only if* (4.4.40) *holds, in which case* ρ_i *are given by* (4.3.26) *and the asymptotic behavior of any such solution is governed by the unique formula* (4.3.27) *with* D_j, $j = \overline{1,N}$ *given by* (4.4.41).

The main results of the existence and asymptotic behavior of strongly decreasing solutions of the system $(SE-)$ will be given and shown in the following theorems.

Theorem 4.4.17 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that for each of λ_i, α_i, $i = \overline{1,N}$, some of the following conditions is satisfied:*

$$\text{either} \quad \lambda_i < \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i,\ S_i < \infty.$$

The system $(SE-)$ possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ *with* $\rho_i < 0$, $i = \overline{1,N}$, *if and only if* (4.4.40) *holds, in which case* ρ_i *are given by* (4.3.26) *and the asymptotic behavior of any such solution* $\mathbf{x}$ *is governed by the unique formula* (4.3.27) *with* D_j, $j = \overline{1,N}$ *given by* (4.4.41).

Theorem 4.4.18 *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that $\lambda_i > \alpha_i$ for all $i = \overline{1,N}$. System $(SE-)$ possesses a solution* $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ *with* $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$, *if and only if* (4.4.49) *holds, in which case* ρ_i *are given by* (4.3.26) *and the asymptotic behavior of any such solution* $\mathbf{x}$ *is governed by the unique formula* (4.3.27) *with* D_j, $j = \overline{1,N}$ *given by* (4.4.41).

We remark that the "only if" parts of these theorems follow immediately from the corresponding parts of Theorem 4.4.13 and Theorem 4.4.14 because any solution $\mathbf{x}$ of $(SE-)$ with the indicated property satisfies the asymptotic relation (4.4.39).

Proof of the "if" part of Theorem 4.4.17: Using (4.2.1) let we define the sequences $X_i = \{X_i(n)\} \in \mathcal{RV}(\rho_i)$ as in (4.3.40), where D_j for $j = \overline{1,N}$ are given by (4.4.41). It is known that

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N},$$

from which it follows that there exists $n_0 > 1$ such that for $n > n_0$ holds

$$\frac{1}{2} X_i(n) \leq \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \leq 2X_i(n), \quad i = \overline{1,N}. \tag{4.4.51}$$

Let we choose positive constants ω_i and W_i so that

$$\omega_i \le \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad W_i \ge 2W_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad \omega_{N+1} = \omega_1, \quad W_{N+1} = W_1. \tag{4.4.52}$$

An example of such choices is

$$\omega_i = \left(\frac{1}{2}\right)^{\frac{A_N}{A_N - B_N}\sum_{j=1}^N M_{ij}}, \quad W_i = 2^{\frac{A_N}{A_N - B_N}\sum_{j=1}^N M_{ij}} \tag{4.4.53}$$

for $i = \overline{1,N}$. Clearly $\omega_i \le 1 \le W_i$.

Consider the space Υ_{n_0} of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/X_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, Υ_{n_0} is a Banach space endowed with the norm (4.3.57). Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\le$: For $\mathbf{x}, \mathbf{y} \in \Upsilon_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ with

$$\mathcal{X} = \left\{ \mathbf{x} \in \Upsilon_{n_0} \;\middle|\; \omega_i X_i(n) \le x_i(n) \le W_i X_i(n), \quad n \ge n_0, \quad i = \overline{1,N} \right\}. \tag{4.4.54}$$

It is easy to see that for any $\mathbf{x} \in \mathcal{X}$, the norm of $\mathbf{x}$ is finite. Also, for any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. We will define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \quad n \ge n_0, \quad i = \overline{1,N}, \tag{4.4.55}$$

and define the mapping $\Phi : \mathcal{X} \to \Upsilon_{n_0}$ by

$$\Phi(x_1, x_2, \ldots, x_N) = \Big(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \ldots, \mathcal{F}_N x_{N+1}\Big), \quad x_{N+1} = x_1. \tag{4.4.56}$$

We will show that Φ has a fixed point by using Theorem 1.1.1. Namely, the operator Φ has the following properties:

(i) Φ *maps* $\mathcal{X}$ *into itself*: Let $\mathbf{x} \in \mathcal{X}$. Then, using (4.4.51)-(4.4.56), we see that

$$\mathcal{F}_i x_{i+1}(n) \le W_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \le 2W_{i+1}^{\frac{\beta_i}{\alpha_i}} X_i(n) \le W_i X_i(n), \;\; n \ge n_0\,,$$

$$\mathcal{F}_i x_{i+1}(n) \ge \omega_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \ge \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}} X_i(n) \ge \omega_i X_i(n), \quad n \ge n_0.$$

This shows that $\Phi\mathbf{x} \in \mathcal{X}$, that is, Φ is a self-map on $\mathcal{X}$.

(ii) Φ *is increasing,* i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} \leq \mathbf{y}$ implies $\Phi\mathbf{x} \leq \Phi\mathbf{y}$.

Thus all the hypotheses of Theorem 1.1.1 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{X}$ of Φ, which satisfies

$$x_i(n) = \mathcal{F}_i x_{i+1}(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad n \geq n_0, \quad i = \overline{1, N}.$$

This shows that $\mathbf{x} \in \mathcal{X}$ is a solution of system $(SE-)$ and it is easy to see that it is an strongly decreasing solution. That $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ can be verified in the same way as in the proof of Theorem 4.4.10 (or Theorem 4.3.7), by the application of Theorem 1.1.8. □

Theorem 4.4.18 can be proved in the essentially same way.

Application. As mentioned, one-dimensional system is the equation (4.1.1), which has been studied in the previous chapter. Then Theorems 4.4.17 and 4.4.18 are reduced to Theorem 3.3.6. Moreover, we get here new result in the case when the regularity index of the coefficient p is equal to α, which has not been considered in the previous chapter.

Theorem 4.4.19 *Let $\{p(n)\} \in \mathcal{RV}(\lambda)$ and $\{q(n)\} \in \mathcal{RV}(\mu)$. Suppose that*

$$\lambda = \alpha \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{p(n)^{\alpha}} < \infty.$$

The equation (4.1.1) *possesses a regularly varying solution of index $\rho < 0$ if and only if $\mu < -1$, in which case ρ is given by* (4.4.37) *and the asymptotic behavior of any such solution x is governed by the unique formula*

$$x(n) \sim \left[\frac{n^{\alpha+1} p(n)^{-1} q(n)}{\alpha(-\rho)^{\alpha+1}} \right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

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Biography

Aleksandra Kapešić was born on February 5, 1988 in Niš, Serbia. She finished elementary school ”Sveti Sava” and high school ”Svetozar Markovic” (special department for talented mathematicians) as a holder of Vuk's diploma. During that period, she was a participant in numerous national competitions in mathematics, physics and chemistry. In the school year 2007/ 2008, she enrolled in the study of mathematics at the Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš. She received her bachelor's degree in 2010 and her master's degree in 2012, both with an average grade of 10/10. Master thesis title was ”Numerical solution of ordinary differential equations”. In the same year, she enrolled in PhD studies at the same faculty, where she also passed all exams with an average grade of 10/10.

Since 2013, Aleksandra has been working at the Faculty of Sciences and Mathematics in Niš, Department of Mathematics, first as a researcher-trainee, and a year later as an teaching assistant. The courses she is engaged in are Introduction to Differential Equations, Elementary Mathematics, Mathematical Analysis 1 and 2 (Department of Computer Science) in undergraduate studies, and Numerical Methods in Differential Equations and Methods of Teaching Mathematics on graduate level. As a researcher, she was a participant in project Functional analysis, stochastic analysis and applications, No. OI-174007 supported by the Ministry of Education, Science and Technological Development of Republic of Serbia, from 2013. to 2019. Since 2015, she has also been working at high school ”Svetozar Markovic”, at the department for talented students of physics. As an author and co-author, she has published four papers in international mathematical journals with IF.

List of author's publications:

- A. B. Trajković, J. V. Manojlović, *Asymptotic behavior of intermediate solutions of fourth-order nonlinear differential equations with regularly varying coefficients*, Electronic Journal of Differential Equations, Vol. 2016, No. 129, (2016), 1–32. **M21**
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