UNIVERSITY OF BELGRADE

FACULTY OF PHYSICS

Dragoljub Gočanin

FIELD THEORY IN $SO(2,3)_\star$ MODEL OF NONCOMMUTATIVE GRAVITY

Doctoral Dissertation

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TEORIJA POLJA U $SO(2,3)_\star$ MODELU NEKOMUTATIVNE GRAVITACIJE

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In Belgrade, September 2019
Dragoljub Gočanin

Abstract

Arguably the greatest challenge of contemporary theoretical physics is to understand the profound interplay between Quantum Mechanics (QM) and the General Theory of Relativity (GR). To solve the conundrum of "Quantum Gravity" (QG) one has to transcend some deeply rooted assumptions on which we are accustomed, in particular, at very short length scales we might have to abandon the notion of a *continuous* space-time and the associated mathematical construct of a smooth manifold that describes it. Field theory on *noncommutative* (NC) space-time is one distinguished approach to QG, and the one that will be advocated in this thesis. NC field theory is based on the method of *quantization by deformation*, originally developed for the purpose of establishing phase-space quantum mechanics. One speaks of a deformation of an object/structure whenever there is a family of similar objects/structures of which the "distortion" from the original, undeformed one can be somehow parametrized. In physics, this so-called *deformation parameter* is usually related to some fundamental constant of nature that measures the deviation from the classical (i.e. undeformed) theory. To deform classical space-time, one introduces an abstract algebra of NC coordinates, denoted by $\hat{x}^\mu$, that satisfy some non-trivial commutation relations. The simplest case of noncommutativity is the so-called *canonical (or θ-constant) noncommutativity*, $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \sim \Lambda_{NC}^2$, where $\theta^{\mu\nu}$ are components of a *constant* antisymmetric matrix, and Λ_{NC} is a hypothetical length scale at which NC effects become relevant. Instead of deforming abstract algebra of coordinates, one can introduce space-time noncommutativity in the form of NC products of functions (fields) on *commutative* space-time. These products are called *star products* ($\star$-products). In particular, *canonical noncommutativity* is effected by the *Moyal $\star$-product*.

During the previous studies of the theory of NC gravity, it was found that NC corrections to GR can be obtained by canonical deformation of anti-de Sitter (AdS) gauge field theory. Starting with an action of the MacDowell-Mansouri type, invariant under $SO(2,3)$ gauge transformations, one obtains the Einstein-Hilbert action with cosmological constant term, after choosing a certain gauge. NC deformation is based on the Seiberg-Witten approach to NC gauge field theory, and the first non-vanishing NC correction is quadratic in $\theta^{\mu\nu}$. This model also predicts a non-trivial NC deformation of Minkowski space and offers an explanation for the apparent breaking of diffeomorphism invariance in the NC theory. Namely, the structure of the NC-deformed Minkowski metric suggests that, by assuming canonical noncommutativity, we implicitly choose a preferred frame of reference - the Fermi inertial frame.

Building on these results, we proceeded by introducing matter fields within the $SO(2,3)$ framework, in particular, we considered Dirac spinor field, $U(1)$ gauge field and non-Abelian Yang-Mills gauge field. It turns out that inclusion of matter fields produces non-vanishing *linear* NC correction involving various new matter-gravity couplings that arise due to space-time noncommutativity. This feature is a significant improvement in comparison to the pure NC gravity model, and *a priori* unexpected result. Moreover, some NC terms pertain even in Minkowski space, effecting NC deformation of the Dirac equation. Predictions of this model of NC Electrodynamics include the *NC birefringence effect* (helicity-dependent energy levels of an electron in NC space-time) and *NC-deformed Landau levels* of an electron in background magnetic field.

Finally, we upgraded the model of pure NC gravity to include Supersymmetry (SUSY). It is well-known that one can define a consistent theory of extended $N=2$ AdS_4 Supergravity (SUGRA). This model of SUGRA involves a pair of Majorana vector-spinor fields that can be mixed to form a pair of Dirac spinors (charged gravitini) coupled to $U(1)$ gauge field. Besides local $SO(1,3)\times U(1)$ gauge symmetry, the action is also invariant under complex local SUSY. We present a geometric action that involves two "inhomogeneous" parts: an orthosymplectic $OSp(4|2)$ gauge-invariant action of the MacDowell-Mansouri type that has *vanishing* first order NC correction, and a supplementary action invariant under purely bosonic $SO(2,3)\times U(1)\sim Sp(4)\times SO(2)$ sector of $OSp(4|2)$, that needs to be added for consistency. This additional action provides a non-trivial *linear* NC correction that is calculated explicitly. Also, a recurring theme of the thesis will be the relation between the canonical NC deformation and Wigner-Inönü group contraction.

Key words: deformation quantization, Moyal product, NC gravity, Seiberg-Witten map, AdS gauge theory, Orthosymplectic SUGRA
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Rezime

Jedan od najvećih izazova savremene teorijske fizike jeste usaglašavanje Opšte teorije relativnosti (OTR) i Kvantne mehanike. Da bismo razrešili problem "kvantne gravitacije" neophodno je da prevaziđemo neke duboko ukorenjene pretpostavke na kojima se zasnivaju sve naše dosadašnje teorije. Jedna od njih je i pretpostavka da je struktura prostor-vremena *kontinualna na svim skalama* i da shodno tome odgovara matematičkom konceptu glatke mnogostrukosti. Teorija polja na nekomutativnom (NK) prostor-vremenu je jedan dobro definisani pristup problemu kvantne gravitacije, i taj pristup će biti zastupljen u ovoj disertaciji. NK teorija polja počiva na metodu deformacione kvantizacije, originalno razvijenom radi zasnivanja kvantne mehanike u faznom prostoru. O deformaciji nekog objekta/strukture govorimo onda kada postoji familija srodnih objekata/struktura kod koje se odstupanje od nedeformisanog originala može na određeni način paramatrizovati. U fizici se ovaj takozvani *parametar deformacije* javlja u vidu neke fundamentalne konstante prirode i predstavlja meru odstupanje od "klasične" (tj. nedeformisane) teorije. Da bismo deformisali klasično prostor-vreme, uvodimo apstraktnu algebru nekomutativnih koordinata, u oznaci $\hat{x}^\mu$, koje zadovoljavaju neke netrivijalne komutacione relacije. Najjednostavniji primer je takozvana *kanonska* (ili θ-konstantna) nekomutativnost, $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \sim \Lambda_{NC}^2$, gde su $\theta^{\mu\nu}$ komponente konstantne antisimetrične matrice, a Λ_{NC} hipotetička skala dužine na kojoj efekti nekomutativnosti postaju značajni. Umesto deformisanja apstraktne algebre koordinata, nekomutativnost možemo uvesti u vidu nekomutativnih proizvoda funkcija (polja) običnih komutativnih koordinata. Ovi proizvodi se nazivaju star-proizvodi ($\star$-proizvodi). Konkretno, kanonskoj nekomutativnosti odgovara Mojalov $\star$-proizvod.

Tokom prethodnih istraživanja teorije NK gravitacije, ustanovljeno je da se nekomutativna verzija OTR može dobiti kanonskom deformacijom anti-de Siter (AdS) gradijentne teorije gravitacije. Predloženo klasično dejstvo Jang-Milsovog tipa, invarijantno na lokalne $SO(2,3)$ transformacije, se pri određenom kalibracionom uslovu svodi na standardno Ajnštajn-Hilbertovo dejstvo sa kosmološkom konstantom. NK deformacija je sprovodena sledeći Sajberg-Vitenov pristup NK teoriji gradijentnih polja, i ispostavlja se da je prva nenulta NK korekcija kvadratna po $\theta^{\mu\nu}$. Ovaj model takođe predviđa netrivijalnu deformaciju prostora Minkovskog i pruža objašnjenje porekla narušenja opšte kovarijantnosti koje je prisutno u NK teoriji. Naime, struktura NK-deformisane metrike Minkovskog ukazuje na to da, uvodeći kanonsku nekomutativnost, mi implicitno prelazimo u određeni referentni sistem - onaj koji odgovara Fermijevim inercijalnim koordinatama duž geodezika.

Na osnovu pomenutih rezultata, u ovoj tezi je unapređen $SO(2,3)$ model čiste NK gravitacije uvođenjem polja materije, i to: Dirakovog spinorskog polja, $U(1)$ i Jang-Milsovog gradijentnog polja. Ispostavlja se da materija proizvodi nenultu NK korekciju *prvog reda* u vidu novih tipova interakcije sa gravitacijom, a usled nekomutativnosti prostor-vremena. Ovo je značajan i neočekivan napredak u odnosu na čistu NK gravitaciju. Štaviše, neki od novih interakcionih članova opstaju čak i u prostoru Minkovskog i uzrokuju deformaciju Dirakove jednačine. Neka od predviđanja ovog modela NK elektrodinamike su *efekat NK dvojnog prelamanja* (tj. zavisnost energetskih nivoa elektrona od njihovog heliciteta) i *NK-deformisani Landauovi nivoi* elektrona u pozadinskom magnetnom polju.

Konačno, izvršeno je i uopštenje modela čiste NK gravitacije koje uključuje supersimetriju. Poznato je da je moguće definisati konzistentnu teoriju $N=2$ AdS_4 supergravitacije (SUGRA). Model sadrži par Majorana vektor-spinora koji obrazuju par Dirakovih spinora (naelektrisana gravitina) kuplovanih sa $U(1)$ gradijentnim poljem. Pored lokalne $SO(1,3)\times U(1)$ simetrije, dejstvo je invarijantno i na kompleksnu lokalnu supersimetriju. Predstavićemo geometrijsko dejstvo koje sadrži dva "nehomogena" dela: ortosimplektičko dejstvo Jang-Milsovog tipa invarijantno na lokalne $OSp(4|2)$ transformacije, koje nema NK korekciju prvog reda, i dopunsko dejstvo invarijanto na čisto bozonski $SO(2,3)\times U(1) \sim Sp(4)\times SO(2)$ sektor $OSp(4|2)$ grupe, koje se mora dodati radi konsistentnosti sa klasičnom teorijom. Ovo dopunsko dejstvo poseduje netrivijalnu *linearnu* NK korekciju. Pitanje koje će se iznova postavljati je pitanje odnos kanonske NK deformacije i Vigner-Inonuove kontrakcije.

Ključne reči: kvantizacija deformacijom, Mojalov proizvod, NK gravitacija, Sajberg-Vitenovo preslikavanje, AdS gradijentna teorija, ortosimplektička SUGRA
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1 Introduction

Our Universe is a Cosmos. This is the fundamental assumption on which we base our scientific enterprise. In our intellectual struggle, a desire to understand the universal laws of the physical world and to grasp their ultimate meaning is often subdued by our own limitations and pragmatism - eventually, we want to be able to calculate something useful, and produce a working model for some restricted class of phenomena that are accessible to our current mathematical and technological resources. As a rule, the relation between profoundness of a certain theory and its capacity to produce concrete results is inversely proportional - underlying theories, considered to be more fundamental, tend to be less operational. It would be a challenging task, for example, to derive the laws of molecular dynamics starting from the Lagrangian of the Standard Model (SM) of particle physics with all its intricacies, although, in principle, this is possible. Therefore, it is an extraordinarily significant fact, and by no means obvious, that the complex hierarchical organization of the physical world and the manner in which its layers are intertwined between each other make it possible for us to construct well-defined *effective descriptions* of phenomena, reliable only for some restricted set of values of the relevant parameters. Every effective theory is characterized by its domain of applicability (scale), the degrees of freedom associated with that scale and symmetries of their dynamical laws. Pushing an effective theory beyond its area of applicability is typically marked by the appearance of singularities in its mathematical structure, the prototypical examples being the SM and the General Theory of Relativity (GR). A lack of constraint on infinitely small/large quantities of any kind signifies that a theoretical model is incomplete, in which case it should be replaced by a wider framework able to tame the infinities. Therefore, it seems reasonable to expect that our progress towards a hypothetical final "*theory of everything*" is going to proceed in steps, through a sequence of effective descriptions of increasing generality and ever-growing domain of validity. The transition from a particular effective theory to a more fundamental one (that contains the former as a limiting case) can be formalized through the notion of *deformation*. One speaks of a deformation of an object/structure whenever there is a family of similar objects/structures of which the "distortion" from the original, undeformed one, can be somehow parametrized. In physics, this so-called *deformation parameter* usually appears as some fundamental constant of nature that measures the deviation from the classical (meaning undeformed) theory. We can articulate this more precisely by the following (very abstract) definition.

Definition 1.1. *Let X be an object in a certain category $\mathcal{C}$. A deformation of X is a family of objects $X_\epsilon \in Obj(\mathcal{C})$ parametrized by ϵ, such that $X_{\epsilon_0} = X$ for some ϵ_0.*

An effective theory may be regarded as the *leading order term* in a perturbative expansion of a more general, deformed theory, in powers of a certain deformation parameter. In that respect, we consider the Special Theory of Relativity (STR) to be a deformation of the Newtonian mechanics, the deformation parameter being v/c; when $v/c \to 0$ (low-speed limit) Newtonian mechanics is restored. Another way to look at this relation is from the aspect of symmetry. Deformation parameter often plays the role of a *contraction parameter* in the Wigner-Inönü (WI) Lie algebra contraction procedure [1, 2]. As an illustration, consider the homogeneous Lorentz algebra $\mathfrak{so}(1,3)$. It has a total of six generators M_{ab} (rotations of Minkowski space $\mathcal{M}_4$), satisfying the following commutation relations,

$$[M_{ab}, M_{cd}] = i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \ . \tag{1.1}$$

After separating generators M_{ab} into three 3-rotation generators $J_i = i\varepsilon_{ijk}M_{jk}$ and three boost generators $K_i = M_{i0}$, we can recast (1.1) into a more explicit form,

$$\begin{aligned} [J_i, J_j] &= i\varepsilon_{ijk}J_k \ , \\ [J_i, K_j] &= i\varepsilon_{ijk}K_k \ , \\ [K_i, K_j] &= -i\varepsilon_{ijk}K_k \ . \end{aligned} \tag{1.2}$$

Now we use the speed of light c as a contraction (deformation) parameter and define $\widetilde{K}_i := K_i/c$. In the limit $v/c \to 0$ we obtain homogeneous Galilean Lie algebra,

$$\begin{aligned} [J_i, J_j] &= i\varepsilon_{ijk}J_k \ , \\ [J_i, \widetilde{K}_j] &= i\varepsilon_{ijk}\widetilde{K}_k \ , \\ [\widetilde{K}_i, \widetilde{K}_j] &= 0 \ . \end{aligned} \tag{1.3}$$

Another case that will be important for us, later on, is the WI contraction of anti-de Sitter (AdS) $\mathfrak{so}(2,3)$ algebra into Poincaré algebra.

In this general context, *quantum theories are recognized as deformations of the corresponding classical ones*, deformation parameter being the Planck constant $\hbar$. This is the content of the *principle of correspondence*, one of the most important guiding principles in physics, due to N. Bohr [3]. It poses a general constraint on every new-developed theory of physics - besides giving us a refined conceptual framework to deal with some class of phenomena, it must also be consistent with the corresponding less accurate theory that precedes it and reduce to that theory in a certain limit. This paradigm is inherited by the modern physics (especially high-energy physics), one of its guises being the concept of *quantization by deformation*.

1.1 Deformation quantization - quantum mechanics

That quantization (of a classical system) can be realized as a *deformation* has already been anticipated by P. Dirac in the early twenties [4, 5]. He was the first to note the resemblance between the *Poisson bracket* (Lie bracket for the algebra of functions on phase space) and *quantum commutator*, and suggested that it might be possible to define an associative, but *non-commutative* product of functions on phase space that could encapsulate the non-commutative character of quantum mechanics. The Poisson bracket would then be identified with the leading order term in the $\hbar$-expansion of a certain, more general, "quantum bracket" on phase space (today know as the Moyal bracket). In this expansion, higher-order terms (those containing $\hbar$) would be responsible for quantum effects and in the limit $\hbar \to 0$ classical structure would be restored, in accord with the principle of correspondence. This observation was the first incentive for the theory of star-products ($\star$-products) - associative, but non-commutative deformations of ordinary commutative point-wise products of functions on classical phase space - that lies at the heart of the method of deformation quantization.

Another important source of inspiration for the theory of $\star$-products came from the work of H. Weyl, J. von Neumann and E. Wigner. In [6] Weyl defined a certain formal map - *the Weyl transform* - that takes a function on phase space and assigns to it an operator on Hilbert space (the so-called associated Weyl-operator). This construction relates the theory of $\star$-products and Weyl quantization procedure based on the symmetric ordering scheme. Eventually, it became clear that Weyl transform is not an intrinsically special quantization prescription and that deformation quantization provides a more general framework. In 1931, von Neumann utilized the Weyl transform as an equivalent representation of the Heisenberg algebra [7]. He also worked out an analogue of Hilbert space operator multiplication in phase space and thus effectively discovered the rule governing the noncommutative product of the corresponding phase space functions — an early version of the $\star$-product. Nevertheless, von Neumann ignored his own discovery concerning the $\star$-product and just proceeded to postulate the standard correspondence rules between classical and quantum mechanics [8].

Wigner, on the other hand, was searching for an alternative formulation of Quantum Mechanics (QM), an operator-free formulation that could be defined *directly on classical phase space*. He developed a theory of *quasi-probability distributions* (*Wigner functions*) [9] to calculate quantum corrections to classical statistical mechanics. A central object in his approach is the *Wigner transform*, a map that takes an operator on a Hilbert space and assigns to it a function on phase space. For a self-adjoint operator this function is real. In particular, to a general quantum state (statistical operator) Wigner transform assigns a quasi-probability distribution that can be used to calculate statistical averages of any classical observable while accounting for the quantum effects. As it turns out, Wigner transform is an *inverse* of Weyl transform and the whole construction is therefore named *the Wigner-Weyl (WW) correspondence*. It is proved that WW correspondence is a one-to-one map between phase space functions and quantum operators.

In his 1946 thesis [10], H. L. Groenewold explored the consistency of the von Neumann's general quantization prescription. As a tool, he utilized a fully developed formulation of the WW correspondence, regarded as a formal invertible transform. The essence of this correspondence is the $\star$-product (today known as the Moyal-Weyl-Groenewold product, or just the Moyal product, for short). This realization helped Groenewold to prove that it is not possible to find a fully consistent quantization procedure (in the von Neumann sense), which means that it is not possible to promote classical Poisson bracket of *any* two functions onto their quantum commutator. This result is known as the *Groenewold's no-go theorem* [11] and it was one of the main reasons to look for another method of quantization. Groenewold's observation, and the counterexamples that he found, have been generalized and codified to what is now known as the Groenewold – Van Hove theorem [12]. Groenewold also realized that the Wigner transform of a quantum commutators gives a generalization of the Poisson bracket (the Moyal bracket), which contains Poisson bracket as its classical limit.

At the same time, J. H. Moyal was developing essentially the same theory but from a different point of view [13], one that is more related to statistical mechanics. He focused on all expectation values of quantum operator monomials, $\hat{q}^n\hat{p}^m$, symmetrized by Weyl ordering. Moyal realized that these expectation values could be generated out of a classical-valued characteristic function on phase space, which he later recognized as the Wigner transform of a statistical operator. He observed that many familiar operations of standard QM could be apparently bypassed and verified that the uncertainty principle is incorporated in this structure. Less systematically than Groenewold, Moyal also obtained the quantum evolution of the Wigner function by deforming the classical Poisson bracket into the Moyal bracket, thus establishing a more comprehensive notion of the "$\hbar \to 0$" limit based on the asymptotic $\hbar$-expansion — as opposed to the less intuitive method of taking the limit of large occupation numbers or computing expectation values in coherent quantum states.

Before the advent of deformation quantization techniques, there seemed to be no organic connection between classical and quantum systems and the workings of the correspondence principle were somewhat obscure. *Functorial quantization* [14] is the most general framework for defining quantization that assumes that classical and quantum systems are, as far as their mathematical structure is concerned, *categorically different*. To define a quantization procedure, one has to construct a *covariant functor* (an arrow preserving functor) that assigns to each classical system (phase space) a quantum system (Hilbert space). Classical systems are properly described by symplectic category S. Its objects are symplectic manifolds (M, ω) and its maps are symplectomorphism (canonical transformations). On the other hand, quantum systems are described by unitary category $\mathcal{U}$. Its objects are Hilbert spaces $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and its maps are unitary transformations. It seems that a reasonable definition of a quantization of a classical system would, therefore, be a covariant functor $\mathcal{F} : \mathcal{S} \to \mathcal{U}$ satisfying the following conditions:

1. To every symplectic manifold $(\mathcal{M}, \omega)$ is associated a Hilbert space $\mathcal{F}(\mathcal{M}, \omega) = (\mathcal{F}[\mathcal{M}], \mathcal{F}[\omega])$. In this Hilbert space, $\mathcal{F}[\omega] : \mathcal{F}[\mathcal{M}] \times \mathcal{F}[\mathcal{M}] \to \mathbb{C}$ is the inner product.

2. To every symplectomorphism $\varphi : (\mathcal{M}, \omega_{\mathcal{M}}) \to (\mathcal{N}, \omega_{\mathcal{N}})$ is associated a unitary transformation $\mathcal{F}[\varphi] : \mathcal{F}(\mathcal{M}, \omega_{\mathcal{M}}) \to \mathcal{F}(\mathcal{N}, \omega_{\mathcal{N}})$.

The question is, however, whether it is possible to find a functor consistent with the actual theory of quantum mechanics, that is, with Schrödinger representation. As it turns out, there can be no such functor that could provide a complete, physically sensible quantization. This is a strong motivation to consider an alternative approach to the problem of quantization, in general.

In 1978, F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer published a milestone papers that set the course of the modern theory of deformation quantization [15, 16]. Their goal was to endow classical phase space with noncommutative structure by *deforming the commutative algebra of functions on phase space*. The ordinary commutative product is replaced by a suitable noncommutative $\star$-product (which $\star$-product is to be applied, depends on the details of the deformation procedure), thus yielding a deformed algebra capable of capturing the noncommutative character of quantum mechanics. For example, the Moyal $\star$-product is associated with the constant phase space deformation, using $\hbar$ as a deformation parameter. Since the mapping of brackets (Poisson bracket to Moyal bracket) need only be satisfied asymptotically, up to $\mathcal{O}(\hbar^2)$, so as to find classical mechanics in the limit where $\hbar \to 0$, the inconsistencies found by Groenewold are resolved. Although we cannot claim that deformation quantization amounts to a definite solution to the problem of quantization, it does provide the most transparent formulation of the correspondence principle. The method of deformation quantization became fully appreciated after M. Kontsevich's proof of his famous *Formality conjecture* [17]. A corollary of this conjecture is that every classical system (properly described by a Poisson manifold) can be (almost) uniquely quantized by deformation [18], in accord with the general principle of correspondence and the original Dirac's intuition. For a more detailed account on the history of the subject see, for example, [19, 20].

1.2 Deformation quantization - quantum gravity

Besides its role in establishing a completely new interpretation of QM, "deformation philosophy" found its most attractive application in the study of the fundamental structure of space-time itself. To resolve the conundrum of "Quantum Gravity", we must be prepared to go beyond the usual assumptions on which we are accustomed, in particular, at very short length scales (very high energies) we might have to abandon the notion of a *continuous* space-time and the associated mathematical construct of a smooth manifold that describes it [21, 22]. There is a famous heuristic argument that supports this attitude. By combining (perhaps naively) the basic principles of QM and GR - the uncertainty principle and the geometric character of gravity, respectively - seems to imply that there exists a natural "defence mechanism" preventing us from observing the structure of the physical world beyond the Planck scale ($l_P \sim 10^{-35}m$). Namely, Heisenberg's uncertainty relations $\Delta q \Delta p \geq \hbar/2$ imply that in order to probe smaller length scales, we need to provide larger amounts of energy/momentum. This fact, however, brings us in conflict with GR, assuming the continuity of space-time at all scales [23]. According to GR, a sufficient amount of energy density creates a black hole with a Schwarzchild radius $R_S = 2MG_N/c^2$ proportional to the energy density. Therefore, with a further increase in energy, the size of the black hole increases proportionally, thus preventing us from accessing the region within. There is an uncertainty relation between R_S and the radial coordinate $\Delta R_s \Delta r \geq l_p^2$ that predicts the appearance of virtual black holes and wormholes (quantum foam) at the Planck scale [24, 25]. It follows from the Heisenberg relation $\Delta(Mc)\Delta r \geq \hbar/2$ that gets saturated at the Planck scale $(m_p c) l_P = \hbar/2$ ($m_P \sim 10^{-8}kg$ is the Planck mass). It seems that beyond the Planck scale, space-time loses its empirical meaning (as we know it), and since we do believe that there is physics beyond the Planck wall, this contradiction has to be resolved.

Over the course of the 20^{th} century, several well-established and quite distinct approaches to quantum gravity appeared. In spite of being different in so many respects, all of them, however, recognize that the crucial problem lies in the notion of *continuity of space-time* and *the principle of locality* that goes with it. To transcend this deeply rooted assumption, the proponents of String Theory [26] suggest that the fundamental building blocks of nature are not point-like elementary particles interacting at a single space-time point, but tiny vibrating string (or, more generally, branes) that, due to their extension, interact non-locally. Each vibrational mode of the string corresponds to a different particle (infinitely many of them) including graviton - a quantum of the gravitational field. The most attractive features of this theory are its unification power and a plethora of exotic mathematical structures such as Supersymmetry (SUSY) and extra (compactified) spatial dimensions. In String Theory, space-time is an emergent phenomenon grounded in the dynamics of the fundamental strings. The theory lacks ability to reproduce the known low-energy phenomenology (except gravity). Then there is Supergravity (SUGRA), in its simple and extended versions. SUSY greatly improves the renormalizability of non-gravitational gauge field theories through loop cancellation and offers a natural solution to the hierarchy problem. To include gravity, SUSY is promoted into gauge symmetry [27–29], the corresponding gauge field being the gravitino spin-3/2 field. It is a well-developed theory with great unification capacity, but it is not complete. Others claim that space-time itself has a discrete, granular structure and define the "quantum of volume"; this is the theory of Loop Quantum Gravity, and it most faithfully preserves the deep notion of background independence that lies at the heart of GR [30]. Then, there is an algebraic approach of Noncommutative Geometry where one abandons the geometrical notion of a point and instead defines a deformed space-time in terms of its C^*-algebra of functions [31]. Other approaches include Causal Set Theory [32], Quantum Measure Theory [33], Consistent Histories [34], Euclidean path integral approach [35], Asymptotic safety program [36], some recent developments such as gauge-gravity duality (AdS/CFT correspondence) and $EPR = ER$ conjecture [37, 38].

The concept of "space-time noncommutativity" appeared already in the 1930s when W. Heisenberg suggested that the problem of UV-divergences in Quantum Field Theory (QFT) could perhaps be solved by postulating non-vanishing commutation relations between coordinate operators, analogous to the canonical commutation relations between coordinates and their canonically-conjugated momenta [39]. The first model of NC geometry came from Snyder [40] who showed that one could have Lorentz symmetry in deformed space-time. However, this line of development was overshadowed by the success of the renormalization program of Schwinger, Feynman and Tomonaga. Nevertheless, noncommutativity started to appear unexpectedly in various contexts. After realizing the connection between String Theory and Noncommutative geometry, the latter once again became an interesting topic in high energy physics. Namely, the theory of open strings in a constant Kalb-Ramond B-field implies that the endpoint coordinates of a string attached to a D-brane do not commute [41]. This implies that a QFT on NC space-time can be interpreted as a low energy limit of the theory of open strings. There are more elementary examples coming from classical mechanics. If we take a single particle of mass m and charge q, constrained to move in the xy-plane, and apply homogeneous magnetic field in the z-direction, the Poisson bracket $\{x, y\}$ is not equal to zero in the limit of the strong magnetic field B. Making a transition to quantum mechanics, we obtain $[\hat{x}, \hat{y}] \sim 1/B$. The geometrical notion of a point lacks meaning in NC spaces and it makes no sense to introduce ordinary c-number coordinates. An important mathematical result of Gelfand and Naimark [42, 43] is that every unital commutative C^*-algebra over $\mathbb{C}$ is isomorphic to the one of $\mathbb{C}$-valued continuous functions on compact Hausdorff topological space. It implies that we do not have to regard space as a set of points in order to describe its properties; instead, we can use the commutative algebra of functions defined on that space. Analogously, for an NC space we use the corresponding NC C^*-algebra of functions.

There are many ways in which space-time noncommutativity can be introduced. One distinguished approach, and the one that will be advocated in this thesis, is *Noncommutative (NC) Field Theory* - field theory on *noncommutative space-time* - based on the method of $\star$-product NC deformation [22, 44]. This way of "quantizing" space-time is essentially different from the standard QFT quantization procedure for matter fields. Loosely speaking, the idea is that different space-time dimensions (usual $3+1$) are mutually "incompatible", in a sense that there exist a lower bound for the product of uncertainties $\Delta x^\mu \Delta x^\nu$ for a pair of different coordinates. Deformation quantization formalizes this notion of "pointlessness" by introducing an abstract algebra of NC coordinates as a deformation of the classical space-time structure described by ordinary commuting coordinates. These NC coordinates, denoted by $\hat{x}^\mu$, satisfy some non-trivial commutation relations, and so, it is no longer the case that $[\hat{x}^\mu, \hat{x}^\nu] = 0$. Other ways of introducing space-time noncommutativity include spectral triplets [45], NC vierbein formalism [46], matrix models [47]. These approaches are not entirely independent of each other. For example, the NC algebra of Schwartz functions, defined by Moyal $\star$-product, is actually a NC spectral triplet [48]. Spectral triplet composed of a Hilbert space, algebra of operators defined on that space and a Dirac operator form a basis of the Connes' noncommutative geometry.

The simplest case of noncommutativity is the so called *canonical (or θ-constant) noncommutativity*,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \sim \Lambda_{NC}^2 \ , \tag{1.4}$$

where $\theta^{\mu\nu}$ are components of a *constant* antisymmetric matrix and Λ_{NC} is the yet unknown length scale at which NC effects become relevant. Deformation parameter is a fundamental constant, like the Planck length or the speed of light. Other important choices include Lie algebra-like deformation and κ-deformation. Instead of deforming abstract algebra of coordinates, noncommutativity (deformation) of space-time can be encoded in the form of NC products of functions (fields) on classical space-time. These products are called *star products* ($\star$-products). In particular, to establish *canonical noncommutativity*, we use the Moyal $\star$-product, as a deformation of the ordinary commutative product,

$$\begin{aligned}(\hat{f} \star \hat{g})(x) &= e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \mathcal{O}(\theta^2) \ .\end{aligned} \tag{1.5}$$

The first term in the expansion of the exponential is the ordinary point-wise multiplication of the fields and the higher-order terms are non-classical NC corrections.

In Section 2, we will introduce Moyal $\star$-product more formally, both in QM and NC gauge field theory. After that, in Section 3, we study the Seiberg-Witten method of constructing NC gauge field theory that will be used throughout the thesis. Anti-de Sitter gauge field theory, its relation to GR, and its canonical NC deformation are discussed in Section 4. There we present, in some detail, the $SO(2,3)_\star$ model of NC gravity. In the remaining sections, we introduce matter fields (Dirac spinor field, $U(1)$ gauge field and non-Abelian Yang-Mills field) coupled to NC gravity (Sections $5-7$, respectively). These constitute the main part of the thesis. Finally, in Section 8, we upgrade the $SO(2,3)_\star$ model of pure NC gravity to include SUSY, and consider canonical NC deformation of $N=2$ AdS SUGRA in $D=4$, based on the orthosymplectic gauge supergroup $OSp(4|2)$. We conclude by proposing some new directions of research.

2 Moyal-Weyl-Groenewold $\star$-product

The uncertainty principle of QM compels us to abandon the concept of *phase space*, as a space of dynamical states of classical systems, and to promote generalized coordinates and their canonically-conjugated momenta into mutually incompatible self-adjoint operators acting on a Hilbert space. In the spirit of deformation philosophy, quantization procedure can be understood as an act of imposing a noncommmutative geometry structure on classical phase space. This deformed "quantum phase space" is the prime example of a noncommuttive space ever to be studied. Therefore, to begin with, we will introduce the Moyal-Weyl-Groenewold (MWG) $\star$-product in the context of phase space quantum mechanics, where it was originally founded. By studying the method of deformation quantization of classical structures we will become familiar with the theory of $\star$-products and prepare the ground for the next section, where we define the abstract concept of NC space-time on which NC field theory is based.

2.1 $\star$-product in quantum mechanics

The proper framework for studying classical mechanics is *symplectic geometry*. This is the setting in which Kontsevich proved his *formality conjecture* [17]. Consider a real vector space V of dimension m and let $\Omega : V^2 \to \mathbb{R}$ be a bilinear, skew-symmetric map. If for all $w \in V$, $\Omega(u, w) = 0$ implies $u = 0$, the map Ω is called *symplectic* or *non-degenerate*; (V, Ω) is called *symplectic vector space.* The property of non-degeneracy implies that V must be even-dimensional.

Definition 2.1. *(Symplectic Manifold). A pair $(\mathcal{M}, \omega)$ of smooth manifold $\mathcal{M}$ and 2-form field $\omega_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is called symplectic manifold if $d\omega = 0$ (if this is the case the form is said to be closed) and if ω_p is symplectic for all $p \in M$. A trivial example is $\mathcal{M} = \mathbb{R}^{2n}$ with $\omega = \sum_{i=1}^{n} \mathrm{d}\, x^i \wedge \mathrm{d}\, x^i$.*

The algebra of smooth functions $C^\infty(\mathcal{M})$ on the symplectic manifold $(\mathcal{M}, \omega)$ can be endowed with a Poisson bracket, which turns it into a *Poisson manifold* $(\mathcal{M}, C^\infty(\mathcal{M}), \{\cdot, \cdot\})$. Non-degeneracy of the symplectic 2-form ω implies that there is a unique vector field X_f assigned to each function $f \in C^\infty(\mathcal{M})$ such that $\omega(X_f, \cdot) = \mathrm{d}f$. The Poisson bracket $\{\cdot, \cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is defined by

$$\{f, g\} = \omega(X_f, X_g) \; ; \tag{2.6}$$

it is bilinear, skew-symmetric and satisfies the Jacobi identity and the Leibniz rule.

Therefore, all classical systems that can be described in terms of symplectic manifolds carry a natural Poisson structure. The Poisson algebra $(\mathcal{C}^\infty(\mathcal{M}), \{\cdot,\cdot\})$ of smooth functions on $\mathcal{M}$ is the algebraic structure that one should deform in order to quanatize the classical system. Let $\mathcal{Q}$ be a configuration space of some classical system with n degrees of freedom. The corresponding phase space has a natural symplectic manifold structure. It is defined as a pair $(T^*\mathcal{Q}, \omega)$ consisting of the cotangent bundle of $\mathcal{Q}$ and the canonical symplectic 2-form $\omega = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$. Poisson bracket is the familiar bracket from the Hamilton's canonical formalism,

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) . \tag{2.7}$$

For simplicity, we will consider only the configuration space $\mathcal{Q} = \mathbb{R}$ (one degree of freedom, no constraints). In that case, phase space manifold $\mathcal{M}$ can be identified with $\mathbb{R}^2$. The dynamical state of a classical system is completely determined by a pair $(q, p) \in \mathbb{R}^2$. Classical observables, like angular momentum and hamiltonian, can now be seen as smooth functions of (q, p). In general, classical observables f are elements of the *commutative* C^**-algebra* (with ordinary point-wise multiplication) of smooth functions on phase space, $C^\infty(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{R} \mid f \text{ is smooth}\}$. This algebra carries a complete information on the underlying phase space (Gelfand-Naimark theorem). In this setting, a systematic way to quantize a classical theory was introduced by H. Weyl [6], and was later called the *Weyl quantization*. He introduced a formal mapping - *Weyl transform* - that associates a quantum operator $\widehat{W}[f]$ to every phase space observable $f \in C^\infty(\mathcal{M})$. The procedure relies heavily on the invertible character of the Fourier transform on a certain class of well-behaving functions. The inverse of the Weyl transform is the already mentioned Wigner transform and the whole one-one correspondence is therefore known as the *Weyl-Wigner (WW) correspondence.*

Definition 2.2. *The WW correspondence consists of the following steps:*

1. *Let $a, b \in \mathbb{R}$; define the Fourier transform of $f \in L^2(\mathbb{R}^2)$ as*

$$\widetilde{f}(a, b) = \int\int dq dp \ \exp[-i(ap + bq)] f(q, p) \ . \tag{2.8}$$

2. *Perform a formal substitution $p \to \hat{p}$, $q \to \hat{q}$ and define*

$$\widehat{W}[f](\hat{q}, \hat{p}) = \int\int \frac{da}{2\pi} \frac{db}{2\pi} \exp[i(a\hat{p} + b\hat{q})] \widetilde{f}(a, b) \ , \tag{2.9}$$

   which is known as the associated Weyl-operator.

For mathematical simplicity, the procedure is only defined for functions $f \in L^2(\mathbb{R}^2)$. Since this space is a Hilbert space, the integration theory tells us that the Fourier transform and its inverse are well-defined. The reason why this definition is not yet satisfactory can be seen from the fact that it cannot handle even the harmonic oscillator hamiltonian $H(q,p) \sim (q^2+p^2)$; nevertheless, the procedure can be extended to all physical relevant functions, such as polynomials [10]. The essence of the $\star$-product approach to QM is that it captures the noncommutative character of QM directly on phase phase. For a given pair of Weyl-operators associated to a pair of phase space functions, we want to find a phase space function that corresponds to the composition of the two Weyl operators. This will be the noncommutative MWG $\star$-product of the two functions that we started with.

Let $f, g \in C^\infty(\mathcal{M})$. The goal is to find a function $h \in C^\infty(\mathcal{M})$ such that $\widehat{W}[f]\widehat{W}[g] = \widehat{W}[h]$ (on the left hand side, we have ordinary operator composition). Moreover, we want to obtain h as a function of f and g. By definition,

$$\widehat{W}[f]\widehat{W}[g] = \int\int \frac{\mathrm{d}a}{2\pi}\frac{\mathrm{d}b}{2\pi} e^{i(a\hat{p}+b\hat{q})}\widetilde{f}(a,b) \int\int \frac{\mathrm{d}a'}{2\pi}\frac{\mathrm{d}b'}{2\pi} e^{i(a'\hat{p}+b'\hat{q})}\widetilde{g}(a',b') \ . \tag{2.10}$$

Mixing the integrals, the exponents can be put together by using a variant of the Baker-Campbell-Hausdorff (BCH) formula. It states that for a pair of linear operators A and B that both commute with their commutator $[A,B]$, the following relation holds,

$$\exp(A)\exp(B) = \exp(A+B)\exp([A,B]/2) \ . \tag{2.11}$$

Applying this result yields

$$e^{i(a\hat{p}+b\hat{q})}e^{i(a'\hat{p}+b'\hat{q})} = \exp[i((a+a')\hat{p}+(b+b')\hat{q})]\exp[i\hbar(ab'-ba')/2] \ . \tag{2.12}$$

Performing the shifts $a \to a-a'$, $b \to b-b'$ gives us the requested expression,

$$\widehat{W}[f]\widehat{W}[g] = \int\int \frac{\mathrm{d}a}{2\pi}\frac{\mathrm{d}b}{2\pi} e^{i(a\hat{p}+b\hat{q})}(\widetilde{f \star_\hbar g})(a,b) = \widehat{W}[f \star_\hbar g] \ , \tag{2.13}$$

where we introduced

$$(\widetilde{f \star_\hbar g})(a,b) = \int\int \frac{\mathrm{d}a'}{2\pi}\frac{\mathrm{d}b'}{2\pi} \widetilde{f}(a-a',b-b')e^{i\hbar[(a-a')b'-(b-b')a']/2}\widetilde{g}(a',b') \ . \tag{2.14}$$

The MWG $\star$-product is defined as the inverse Fourier transform of $\widetilde{f \star_\hbar g}$,

$$(f \star_\hbar g)(q,p) := \mathcal{F}^{-1}[\widetilde{f \star_\hbar g}] = \int\int \frac{\mathrm{d}a}{2\pi}\frac{\mathrm{d}b}{2\pi} e^{i(a\hat{p}+b\hat{q})}(\widetilde{f \star_\hbar g})(a,b) \ . \tag{2.15}$$

Note that the star product $f \star_\hbar g$ depends on the classical variables (q, p) and that it defines a smooth function on phase space; it is therefore a product on the algebra of functions on phase space $(C^\infty(M), \star_\hbar)$ and it can be easily verified that it is *noncommutative*. To obtain the standard form of the MWG product, as found in the papers of Moyal and Groenewold, it suffices to perform the substitutions $a, a' \to -i\partial/\partial p$, $b, b' \to -i\partial/\partial q$ under the inverse Fourier transform and to remark that derivatives of smooth functions commute; therefore, we have

$$\begin{aligned}(f \star_\hbar g)(q,p) &= f(q,p) \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] g(q,p) \\ &= (f \cdot g)(q,p) + \sum_{n=1}^{\infty} \left(\frac{i\hbar}{2}\right)^n \mathcal{C}_n[f,g](q,p) \ , \end{aligned} \tag{2.16}$$

where "$\cdot$" stands for the commutative point-wise product, and we introduced

$$\mathcal{C}_n[f,g](q,p) = \frac{1}{n!} f(q,p) \left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)^n g(q,p) \ . \tag{2.17}$$

The *Moyal bracket* is defined by

$$\{f, g\}_{\star_\hbar} = \frac{1}{i\hbar}[f \overset{\star_\hbar}{,} g] = \frac{1}{i\hbar}(f \star_\hbar g - g \star_\hbar f) = \{f, g\} + \mathcal{O}(\hbar) \ , \tag{2.18}$$

where $\{\cdot, \cdot\}$ stands for the standard Poisson bracket on $C^\infty(\mathcal{M})$. Therefore, we have $f \star_\hbar g \to f \cdot g$ and $\{f, g\}_{\star\hbar} \to \{f, g\}$ as $\hbar \to 0$. Note that this is exactly what Dirac had anticipated. Moreover, one can easily check that MWG $\star$-product correctly reproduces the canonical QM commutation relations between between a conjugate pair of coordinate and momentum (Heisenberg algebra),

$$x \star_\hbar p - p \star_\hbar x = i\hbar \ . \tag{2.19}$$

Also, the evolution equation of some classical observable f in deformed classical mechanic involves the Moyal bracket,

$$\frac{df}{dt} = \frac{1}{i\hbar}(f \star_\hbar H - H \star_\hbar f) = \{f, H\} + \mathcal{O}(\hbar) \ . \tag{2.20}$$

Since $\widehat{W}$ is a linear operator, it follows that $[\widehat{W}[f], \widehat{W}[g]] = \widehat{W}([f \overset{\star_\hbar}{,} g])$. This amounts to a homomorphism of Lie algebras; the Lie bracket on the left is the commutator on the space of quantum operators, and on the right we have the Moyal commutator on $C^\infty(\mathcal{M})$.

Therefore, we may conclude that by introducing associative, but noncommutative MWG $\star$-product on the space $C^\infty(\mathbb{R}^2)$ of smooth functions on $\mathbb{R}^2$, we indeed implement a deformation of the classical Poisson algebra $(C^\infty(\mathbb{R}^2), \{\cdot,\cdot\})$, the deformation parameter being $\hbar$, and promote the classical phase space into a noncommutative space. Weyl transform defines a homomorphism between the $\star$-deformed Lie algebra of functions on phase space $\mathbb{R}^2$ and the Lie algebra of operators on the Hilbert space $L^2(\mathbb{R}^2)$ of square-integrable functions. In general, a quantum system is completely determined by its C^*-algebra of linear operators on the Hilbert space. Since Moyal commutator inherits the properties of a quantum commutator (it satisfies the Heisenberg algebra), according to the Stone-von Neumann theorem [7], the NC-deformed algebra $(C^\infty(\mathbb{R}^2), \{\cdot,\cdot\}_{\star_\hbar})$ amounts to an equivalent alternative representation of quantum mechanics, directly on phase space.

However, it is not quite clear whether WW correspondence (and therefore the MWG $\star$-product) provides the *physical* quantization procedure, since it relates classical systems involving commuting observables to quantum systems involving operators that do not commute in general, and in the latter case, different choices of ordering yield different quantum operators. As a simple example, consider the function $f(q,p) = q \cdot p = p \cdot q = f(p,q)$. Its quantum counterpart is not defined unambiguously, namely $f(\hat{q},\hat{p}) = \hat{q}\hat{p} = \hat{p}\hat{q} + i\hbar \neq \hat{p}\hat{q} = f(\hat{p},\hat{q})$. The two natural choices of ordering, standard (or naive) and symmetric (or Weyl), lead to two different $\star$-products. The two are related by a bijective linear map. We will show that the symmetric ordering corresponds to the MWG $\star$-product.

Definition 2.3. *(Naive quantization). The linear operator $Q_N : \mathbb{C}[q,p] \to Diff(\mathbb{R})$ is defined by*

$$1 \to Q_N(1) = \mathbb{I} \ , \qquad q \to Q_N(q) = \hat{q} \ , \tag{2.21}$$

$$p \to Q_N(p) = \hat{p} \ , \qquad q^n \cdot p^m \to \hat{q}^n \hat{p}^m \ , \tag{2.22}$$

where $\mathbb{C}[q,p]$ is the ring of complex polynomials of two variables and $Diff(\mathbb{R})$ is the space of differential operators with polynomial coefficients in the space $\mathcal{C}^\infty(\mathbb{R})$, i.e. an element $D \in Diff(\mathbb{R})$ takes the form $D = \sum_{k=0}^N f_k \partial^k/\partial q^k$ where $f_0, ..., f_N \in \mathbb{C}[q]$.

First, note that Q_N is a well-defined map since it is defined on a basis of the ring $\mathbb{C}[q,p]$; it is extended to the entire ring by $\mathbb{C}$-bilinearity. Furthermore, note that Q_N is *bijection* since its inverse is evidently well-defined. By applying this procedure to the aforementioned example $f(q,p) = q \cdot p$ would yield $Q_N(p \cdot q) = Q_N(q \cdot p) = \hat{q}\hat{p} = \hat{p}\hat{q} + i\hbar \neq Q_N(p)Q_N(q)$. This means that Q_N is not a homomorphism of (associative) algebras $\mathbb{C}[q,p]$ and $Diff(\mathbb{R})$. However, classically, when $\hbar \to 0$, this map turns into a homomorphism. Using the naive quantization procedure, it is possible to construct an associative NC product on $\mathbb{C}[q,p]$ that *almost satisfies* the correspondence principle. The idea is to pullback the NC multiplication in the space of differential operators $Diff(\mathbb{R})$ to the ring $\mathbb{C}[q,p]$ using the bijective map Q_N, and define a $\star$-product that is compatible with the naive quantization procedure; we call this product $\star_N$.

Theorem 2.1. *Let $f, g \in \mathbb{C}[q,p]$ and $\phi \in \mathcal{C}^\infty(\mathbb{R})$. Then,*

$$f \star_N g = Q_N^{-1}(Q_N(f)Q_N(g)) = \sum_{m=0}^{\infty} \frac{(\hbar/i)^m}{m!}\frac{\partial^m f}{\partial p^m}\frac{\partial^m g}{\partial q^m} = \exp((\hbar/i)\partial_p \otimes \partial_q)(f \otimes g) \tag{2.23}$$

defines an associative NC product on the ring $\mathbb{C}[q,p]$.

Since Q_N is a bijective linear map, the associativity of the product in $Diff(\mathbb{R})$ directly carries over to $\star_N$. The former is generally noncommutative and so the same holds for the latter. In the classical limit we get

$$f \star_N g = f \cdot g - i\hbar \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + \mathcal{O}(\hbar^2) \ , \tag{2.24}$$

which almost satisfies the correspondence principle. Note also that Q_N *is* a homomorphism of algebras with respect to the $\star_N$-product: for all $f, g \in \mathbb{C}[q,p]$, $Q_N(f \star_N g) = Q_N(f)Q_N(g)$. This means that, although the correspondence principle is not exactly satisfied, Q_N does define a complete quantization by deformation.

To obtain a physical deformation quantization, one that satisfies the correspondence principle, we introduce an operator Q_S symmetric under the exchange $\hat{q} \leftrightarrow \hat{p}$.

Definition 2.4. *(Symmetric quantization) The linear operator $Q_S : \mathbb{C}[q,p] \to Diff(\mathbb{R})$ is defined on the basis of monomials of the ring $\mathbb{C}[q,p]$ by*

$$1 \to Q_S(1) = \mathbb{I} \ , \qquad q \to Q_S(q) = \hat{q} \ , \tag{2.25}$$

$$p \to Q_S(p) = \hat{p} \ , \qquad q^n \cdot p^m \to Q_S(q^n \cdot p^m) = S_{n,m}(\hat{q},\hat{p}) \ . \tag{2.26}$$

Function $S_{n,m}(\hat{q},\hat{p})$ is a polynomial in $\hat{q}$, $\hat{p}$, symmetric under the exchange $\hat{q} \leftrightarrow \hat{p}$ and it is given by

$$S_{n,m}(\hat{q},\hat{p}) = \left(\frac{\partial^{n+m}}{\partial x^n \partial y^m} e^{x\hat{q}+y\hat{p}}\right)|_{x,y=0} \ . \tag{2.27}$$

The quantization operator Q_S is extended to the whole ring by $\mathbb{C}$-bilinearity. Again, by invoking the BCH formula and noting that $\hat{q}$ and $\hat{p}$ commute with their commutator, we can obtain the connection between Q_S and Q_N,

$$\begin{aligned} Q_S(e^{xq+yp}) &= e^{x\hat{q}+y\hat{p}} = \exp[\hbar xy/2i]e^{x\hat{q}}e^{y\hat{p}} = \exp[\hbar xy/2i]Q_N(e^{xq}e^{yp}) \\ &= \exp[\hbar xy/2i]Q_N(e^{xq+yp}) = Q_N(\exp[\hbar xy/2i]e^{xq+yp}) \ . \end{aligned} \tag{2.28}$$

So there is a bijective map $N : \mathbb{C}[q,p] \to \mathbb{C}[q,p]$ defined by

$$Nf(q,p) = \exp[(\hbar/2i)\partial^2/\partial q\partial p]f(q,p) \ , \tag{2.29}$$

linking the two orderings by $Q_N(Nf) = Q_S(f)$ for all $f \in \mathbb{C}[q,p]$. Note that N's inverse is given by $N^{-1} = \exp[-(\hbar/2i)\partial^2/\partial q\partial p]$ and so $Q_N(f) = Q_S(N^{-1}f)$ for all $f \in \mathcal{C}[q,p]$. Since it was shown that $Q_N(Nf) = Q_S(f)$ for all $f \in \mathbb{C}[q,p]$, the isomoprphism of the two $\star$-products follows directly from Theorem 2.1. The associativity of the product in $Diff(\mathbb{R})$ carries over to $\star_S$, since $Q_S = Q_N \circ N$ is a bijection. Hence, $\star_S$ is also an NC product on $\mathbb{C}[q,p]$.

Theorem 2.2. *Let $f,g \in \mathbb{C}[q,p]$ and $\phi \in \mathcal{C}^\infty(\mathbb{R})$. Then,*

$$f \star_S g = Q_S^{-1}(Q_S(f)Q_S(g)) = \sum_{m=0}^{\infty} \frac{(\hbar/i)^m}{m!} \sum_{k=0}^{\infty} \binom{m}{k} (-1)^{m-k} \frac{\partial^m f}{\partial q^k \partial p^{m-k}} \frac{\partial^m g}{\partial q^{m-k}\partial p^k} \tag{2.30}$$

defines an associative NC product on the ring $\mathbb{C}[q,p]$. Moreover this product is isomorphic to the product $\star_N$ of naive quantization via the linear bijective map N:

$$N(f \star_N g) = (Nf) \star_S (Ng) \ , \quad \textit{for all } \ f,g \in \mathbb{C}[q,p]. \tag{2.31}$$

Operator Q_S is a homomorphism of algebras with respect to $\star_S$: for all $f,g \in \mathbb{C}[q,p]$ we have $Q_S(f \star_S g) = Q_S(f)Q_S(g)$. This time, however, we have the following asymptotic relation:

$$f \star_S g = f \cdot g + \frac{i\hbar}{2}\{f,g\} + \mathcal{O}(\hbar^2) \ . \tag{2.32}$$

Therefore, operator Q_S defines a deformation quantization on the ring of polynomials $\mathbb{C}[q,p]$ in the sense that to every function in the ring it associates a differential operator that acts on the space $\mathcal{C}^\infty(\mathbb{R})$ and $\star_S$-product captures the NC character of these differential operators transferring it back to $\mathbb{C}[q,p]$. Moreover, $\star_S$-product satisfies the correspondence principle.

It turns out that $\star_S$-product induced by the symmetric quantization is exactly the same as the MWG $\star_\hbar$-product induced by the WW-correspondence. By definition, for any $f, g \in \mathbb{C}[q,p]$,

$$\begin{aligned}(f \star_S g)(q,p) &= \sum_{m=0}^{\infty} \frac{(i\hbar/2)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\partial^m f(q,p)}{\partial q^k \partial p^{m-k}} \frac{\partial^m g(q,p)}{\partial q^{m-k} \partial p^k} \\ &= \sum_{m=0}^{\infty} \frac{(i\hbar/2)^m}{m!} \sum_{k=0}^{m} \binom{n}{k} f(q,p) \left(\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} \right)^m g(q,p) \\ &= f(q,p) \exp \left[\frac{i\hbar}{2} \left(\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} \right) \right] g(q,p) \\ &= (f \star_\hbar g)(q,p) \ . \end{aligned} \tag{2.33}$$

Therefore, for all $f, g \in \mathbb{C}[q,p]$, we have $f \star_S g = f \star_\hbar g$. This means that although the MWG product is only defined on $L^2(\mathbb{R}^2)$, apparently its definition also works for functions in the ring $\mathbb{C}[q,p]$. Since symmetric quantization is well-defined, we see now that the Weyl transform of polynomial functions, such as the harmonic oscillator hamiltonian, is also well-defined. Moreover, the symmetric quantization is in a 1-1 correspondence with the naive quantization which is a bijective quantization.

2.2 Noncommutative space-time

Classical phase space is an abstract *space of states* and not an actual coordinate space $\mathbb{R}^3$. However, classical space-time is also described by a smooth manifold and a commutative algebra of fields defined on it. Therefore, we should be able to apply the $\star$-product formalism to obtain the deformation of continuous space-time. This approach leads us to an algebraic definition of a "quantum", noncommutative space-time. *The following exposition relies heavily on* [22].

To deform the algebraic structure of continuous space-time, we first consider a unital, associative, freely generated algebra of *formal polynomials* over the field of complex numbers, i.e. $\mathbb{C}[x^1, ..., x^N]$. It is a noncommutative analogue of a polynomial ring, and in return, a polynomial ring may be regarded as a commutative free algebra. The basis of $\mathbb{C}[x^1, ..., x^N]$ consists of all finite formal products of N elements $x^1, ..., x^N$ ("coordinates") including the unit element 1 which is of zero order. The product of two basis elements is naturally defined by concatenation,

$$(x^{i_1} x^{i_2} ... x^{i_p})(x^{j_1} x^{j_2} ... x^{j_q}) = x^{i_1} x^{i_2} ... x^{i_p} x^{j_1} x^{j_2} ... x^{j_q} \ . \tag{2.34}$$

There are equivalence classes in $\mathbb{C}[x^1, ..., x^N]$ defined by the relation $\mathcal{R}_x : [x^\mu, x^\nu] = 0$ that generates a two-sided ideal in $\mathbb{C}[x^1, ..., x^N]$. The quotient

$$\mathcal{P}_x = \frac{\mathbb{C}[x^1, ..., x^N]}{I_{\mathcal{R}}} \tag{2.35}$$

is the algebra of polynomials in N *commuting* elements. This algebra can be extended by introducing a dimensionless parameter h and considering the algebra $\mathbb{C}[x^1, ..., x^N][[h]]$ of *formal power series* in h with coefficients in $\mathbb{C}[x^1, ..., x^N]$. The h-extension of $\mathcal{P}_x$ is the quotient

$$\mathcal{A}_x = \frac{\mathbb{C}[x^1, ..., x^N][[h]]}{I_{\mathcal{R}}} \ . \tag{2.36}$$

This is the algebra of commuting coordinates of a classical space-time that we want to deform by introducing NC coordinates $\hat{x}^\mu$. The deformation is imposed on the relation $\mathcal{R}_x$ by making it non-trivial. In general,

$$\hat{\mathcal{R}}_x : [\hat{x}^\mu, \hat{x}^\nu] - ihC^{\mu\nu}(\hat{x}^\nu) = 0 \ , \tag{2.37}$$

where $C^{\mu\nu}(\hat{x}) \in \mathbb{C}[\hat{x}^1, ..., \hat{x}^N][[h]]$. For $h = 0$ we consistently obtain the original algebra of commuting coordinates $\mathcal{A}_x$. The deformed relations define a two sided ideal $I_{\hat{\mathcal{R}}}$ in $\mathbb{C}[\hat{x}^1, ..., \hat{x}^N][[h]]$ spanned by the elements of the form

$$(\hat{x}...\hat{x}) \left([\hat{x}^\mu, x^\nu] - ihC^{\mu\nu}(\hat{x}^\nu)\right) (\hat{x}...\hat{x}) \ , \tag{2.38}$$

where $(\hat{x}...\hat{x})$ stands for an arbitrary product of NC coordinates from $\mathbb{C}[\hat{x}^1, ..., \hat{x}^N]$. Finally, the quotient

$$\hat{\mathcal{A}}_{\hat{x}} = \frac{\mathbb{C}[\hat{x}^1, ..., \hat{x}^N][[h]]}{I_{\hat{\mathcal{R}}}} \tag{2.39}$$

is the algebra of NC coordinates $\hat{x}$ - a deformation of the original commutative structure $\mathcal{A}_x$.

There are several important examples of such algebras that we should mention. First, we have *constant deformation*, with $\hat{x}$-independent $C^{\mu\nu}$. This is the analog of the Heisenberg phase space algebra, and it is therefore called *canonical* (θ-constant) deformation:

$$[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu} \ , \tag{2.40}$$

with $C^{\mu\nu}(\hat{x}) \equiv \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R}$. Then there is Lie algebra type of deformation, with deformation functions $C^{\mu\nu}(\hat{x})$ linear in $\hat{x}$,

$$[\hat{x}^\mu, \hat{x}^\nu] = ihf_\rho^{\mu\nu}\hat{x}^\rho \ . \tag{2.41}$$

In this case, the algebra $\hat{\mathcal{A}}_{\hat{x}}$ of NC coordinates is the universal enveloping algebra of the Lie algebra defined by (2.41). A particular example would be

$$[\hat{x}^\mu, \hat{x}^\nu] = i(a^\mu\hat{x}^\nu - a^\nu\hat{x}^\mu) \ , \tag{2.42}$$

with real parameters a^μ. In the basis where $a^i = 0$ for $i \neq N$ and $a^N = 1/\kappa$ we can identify this algebra with the κ-deformation algebra. To be consistent with the reality property $(x^\mu)^* = (x^\mu)$ we demand a conjugation for $\hat{x}$ as well as $(\hat{x}^\mu)^* = \hat{x}^\mu$ and $(\hat{x}^\mu\hat{x}^\nu)^* = (\hat{x}^\nu)^*(\hat{x}^\mu)^*$. This implies $(C^{\mu\nu})^* = -C^{\nu\mu} = C^{\mu\nu}$.

The vector space of $\mathcal{A}_x$ can be decomposed into finite dimensional subspaces V_r spanned by the monomials of degree r. A basis in V_r is given by the monomials $x^{i_1}...x^{i_r}$ with $i_1 \leq ... \leq i_r$. Consider the vector space $F_r = \oplus_{s=0}^r V_s$ spanned by all monomials up to degree r. We require that vector space $\hat{F}_r$ in $\hat{\mathcal{A}}_{\hat{x}}$ of all NC polynomials up to degree r to have *the same dimension* as F_r. We also require the ordered monomials up to degree r, that is $\hat{x}^{i_1}...\hat{x}^{i_s}$ with $i_1 \leq ... \leq i_s$ and $0 \leq s \leq r$, to constitute a basis in $\hat{F}_r$. This is the Poincaré-Birkhoff-Witt (PBW) property of the NC algebra $\hat{\mathcal{A}}_{\hat{x}}$. For canonical deformation and Lie algebra deformation the PBW property holds (PBW theorem). If algebra $\hat{\mathcal{A}}_{\hat{x}}$ has PBW property, the set of all monomials ordered with respect to a given fixed ordering, forms a basis. A natural choice, but not the only one, is the symmetric ordering that gives fully symmetrized monomials. The linear span of the basis elements of degree r defines a vector space $\hat{V}_r$. By construction, this space has the same dimension as the vector space V_r of polynomials of degree r in N commuting coordinates. We can extend the vector space isomorphism $\hat{V}_r \sim V_r$ to an algebra isomorphism $\hat{\mathcal{A}}_{\hat{x}} \sim \mathcal{A}_x^\star$ since their vector spaces coincide. The $\star$-product in $\mathcal{A}_x^\star$ is defined so that the algebras $\hat{\mathcal{A}}_{\hat{x}}$ and $\mathcal{A}_x^\star$ are *isomorphic*. By the vector space isomorphism, we map polynomials,

$$p(x) \longleftrightarrow \hat{p}(\hat{x}) \ . \tag{2.43}$$

A pair of polynomials $\hat{p}_1(\hat{x})$ and $\hat{p}_2(\hat{x})$ are multiplied as

$$\hat{p}_1(\hat{x}) \cdot \hat{p}_2(\hat{x}) = \widehat{p_1 p_2}(\hat{x}) \ . \tag{2.44}$$

By (2.43) we map this polynomial back to a polynomial in $\mathcal{A}^{\star}_{x}$,

$$\widehat{p_1 p_2}(\hat{x}) \to (p_1 \star p_2)(x) \ . \tag{2.45}$$

This defines the NC $\star$-product of two polynomial functions.

Different deformations correspond to different $\star$-products. For θ-constant deformation, the associated $\star$-product is the Moyal $\star$-product given by

$$(p_1 \star p_2)(x) = \mu\left(e^{\frac{i}{2}h\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} p_1 \otimes p_2\right)(x) \ , \tag{2.46}$$

with point-wise multiplication map,

$$\mu(f \otimes g)(x) = f(x) \cdot g(x) \ . \tag{2.47}$$

Moyal product can be extended to C^∞ functions, remaining bilinear and associative. The power series in h will not converge in general (for an arbitrary C^∞ function) and we should regard it as a formal power series. In general,

$$\begin{aligned} \mu_\star(f \otimes g) &= f \star g = \mu\Big(e^{\frac{i}{2}h\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} f \otimes g\Big) \\ &= \sum_{n=0}^{\infty}\left(\frac{ih}{2}\right)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}...\theta^{\rho_n\sigma_n}(\partial_{\rho_1}...\partial_{\rho_n} f)(\partial_{\sigma_1}...\partial_{\sigma_n} g) \ . \end{aligned} \tag{2.48}$$

Expansion in h yields

$$f \star g = f \cdot g + \mathcal{O}(h) \ , \tag{2.49}$$

and

$$f \star g - g \star f = \frac{ih}{2}\theta^{\rho\sigma}\left(\partial_\rho f \partial_\sigma g - \partial_\rho g \partial_\sigma f\right) + \mathcal{O}(h^2) \ . \tag{2.50}$$

This equation defines a Poisson structure known as the Moyal bracket,

$$\{f, g\}_\star = \frac{i}{2}\theta^{\rho\sigma}\left(\partial_\rho f \partial_\sigma g - \partial_\rho g \partial_\sigma f\right) \ , \tag{2.51}$$

that consistently reduces to zero when $\theta^{\rho\sigma} \to 0$.

We will present an explicit derivation of the Moyal $\star$-product that is associated with the canonical deformation. For that matter, consider the algebra $\mathcal{P}_x$ of polynomial functions in N commuting coordinates $x^1, ..., x^N$. Any function can be expanded in the monomial basis

$$f(x) = \sum_i C_{\mu_1 \dots \mu_i} x^{\mu_1} \dots x^{\mu_i} = C + C_\mu x^\mu + C_{\mu\nu} x^\mu x^\nu + \dots \tag{2.52}$$

and it is uniquely determined by the expansion coefficients $C_{\mu_1 \dots \mu_i}$ that are completely symmetric in their indices. Now consider the algebra $\hat{P}_{\hat{x}}$ of polynomial functions of N noncommmuting coordinates $\hat{x}^1, \dots, \hat{x}^N$ that satisfies PBW condition. PBW property enables us to define a basis of ordered monomials. There are many possible orderings. The most often used ones are the symmetric and the normal ordering. If we choose the symmetric ordering, the basis in the NC algebra is given by

$$\begin{aligned} :1: &= 1 \ , \\ :\hat{x}^\mu: &= \hat{x}^\mu \ , \\ :\hat{x}^\mu \hat{x}^\nu: &= \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu) \ , \\ &\dots \ . \end{aligned} \tag{2.53}$$

An arbitrary element of $\hat{\mathcal{P}}_{\hat{x}}$ can be expanded in this basis,

$$\hat{f}(\hat{x}) = \sum_i C_{\mu_1 \dots \mu_i} : \hat{x}^{\mu 1} \dots \hat{x}^{\mu_i} := C + C_\mu : \hat{x}^\mu : + C_{\mu\nu} : \hat{x}^\mu \hat{x}^\nu : + \dots \ , \tag{2.54}$$

and it is fully characterized by the completely symmetric coefficients $C_{\mu_1 \dots \mu_i}$.

We will now apply the procedure analogues to the Weyl quantization. We define an isomorphism $\mathcal{W}$ between vector spaces $\mathcal{P}_x$ and $\hat{\mathcal{P}}_{\hat{x}}$ by mapping the basis of $\mathcal{P}_x$ into the basis of $\hat{\mathcal{P}}_{\hat{x}}$ according to the chosen ordering prescription,

$$\mathcal{W} : f(x) \longrightarrow \hat{f}(\hat{x}) \ . \tag{2.55}$$

In the case of symmetric ordering,

$$\hat{f}(\hat{x}) = \mathcal{W}[f] = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N k \ \widetilde{f}(k) e^{ik_\mu x^\mu} \ , \tag{2.56}$$

where $\widetilde{f}(k)$ is the usual Fourier transform of $f(x)$,

$$\widetilde{f}(k) = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N x \ f(x) e^{-ik_\mu x^\mu} \ . \tag{2.57}$$

For an arbitrary monomial

$$\begin{aligned}
\mathcal{W}[x^{\mu_1}...x^{\mu_j}] &= (i)^j \int \mathrm{d}^N k \left(\partial_{k_{\mu_1}}...\partial_{k_{\mu_j}} \delta^{(N)}(k)\right) e^{ik_\rho x^\rho} \\
&= (i)^j \int \mathrm{d}^N k\ \delta^{(N)}(k) \left(\partial_{k_{\mu_1}}...\partial_{k_{\mu_j}} e^{ik_\rho x^\rho}\right) \\
&= (i)^{2j}(-1)^j \frac{1}{j!} \sum_{\sigma\in S_j} (\hat{x}^{\sigma(\mu_1)}...\hat{x}^{\sigma(\mu_j)}) \\
&=: \hat{x}^{\mu_1}...\hat{x}^{\mu_j} : \ .
\end{aligned} \tag{2.58}$$

In general, the $\star$-product is defined by

$$\mathcal{W}[f\star g] = \mathcal{W}[f]\cdot\mathcal{W}[g] = \hat{f}(\hat{x})\cdot\hat{g}(\hat{x}) \ . \tag{2.59}$$

For the case of θ-deformation, we have the Moyal $\star$-product. We start with

$$\begin{aligned}
\mathcal{W}[f]\cdot\mathcal{W}[g] &= \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N k\ \widetilde{f}(k) e^{ik_\rho\hat{x}^\rho} \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N p\ \widetilde{f}(p) e^{ip_\sigma\hat{x}^\sigma} \\
&= \frac{1}{(2\pi)^N} \int \mathrm{d}^N k \int \mathrm{d}^N p\ \widetilde{f}(k)\widetilde{f}(p) e^{ik_\rho\hat{x}^\rho} e^{ip_\sigma\hat{x}^\sigma} \ .
\end{aligned} \tag{2.60}$$

Since the exponents do not commute, we must again use the BCH formula

$$\exp(A)\exp(B) = \exp(A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+...) \ , \tag{2.61}$$

where A and B are two noncommuting operators. In the case of θ-deformation the BCH formula terminates since terms with more than two commutators vanish,

$$\begin{aligned}
\mathcal{W}[f]\cdot\mathcal{W}[g] &= \frac{1}{(2\pi)^N} \int \mathrm{d}^N k \int \mathrm{d}^N p\ \widetilde{f}(k)\widetilde{g}(p) e^{i(k+p)_\rho\hat{x}^\rho - \frac{i}{2}\theta^{\kappa\rho}k_\rho p_\sigma} \\
&= \frac{1}{(2\pi)^N} \int \mathrm{d}^N k \int \mathrm{d}^N q\ \widetilde{f}(k)\widetilde{g}(q-k) e^{iq_\rho\hat{x}^\rho - \frac{i}{2}\theta^{\kappa\rho}k_\rho(q-k)_\sigma} \ ,
\end{aligned} \tag{2.62}$$

where we introduced $q = k + p$. Comparing this expression with

$$\mathcal{W}(f\star g) = \frac{1}{(2\pi)^N} \int \mathrm{d}^N q (\widetilde{f\star g})(q) e^{iq_\rho\hat{x}^\rho} \ , \tag{2.63}$$

we conclude

$$(\widetilde{f\star g})(q) = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N k\ \widetilde{f}(k)\widetilde{g}(q-k) e^{-\frac{i}{2}\theta^{\rho\sigma}k_\rho(q-k)_\sigma} \ . \tag{2.64}$$

Finally, we take the inverse Fourier transform and obtain

$$f \star g = \frac{1}{(2\pi)^N} \int \mathrm{d}^N p \int \mathrm{d}^N k \; \widetilde{f}(k) e^{-ik_\sigma x^\sigma} e^{-\frac{i}{2} k_\rho \theta^{\rho\sigma} p_\sigma} \widetilde{g}(p) e^{-ip_\sigma x^\sigma} \; . \tag{2.65}$$

To evaluate the integral, we expand in powers of $\theta^{\rho\sigma}$,

$$\begin{aligned} f \star g &= fg + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g) - \frac{1}{8}\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}(\partial_{\rho_1}\partial_{\sigma_1} f)(\partial_{\rho_2}\partial_{\sigma_2} g) + \dots \\ &= \mu\{e^{\frac{i}{2}h\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\} \; , \end{aligned} \tag{2.66}$$

with commutative point-wise multiplication $\mu\{f \otimes g\} = f \cdot g \equiv fg$.

2.3 Noncommutative calculus

To completely develop a NC field theory, we need to have some tools at our disposal. First we introduce NC derivative $\hat{\partial}_\rho$ as a map of NC algebra $\hat{\mathcal{A}}_{\hat{x}}$ to itself. It is a deformation of the ordinary partial derivative, and we assume that

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta) \; . \tag{2.67}$$

Here, $f^\mu_\rho(\hat{\partial}, \theta)$ is an operator on $\hat{\mathcal{A}}_{\hat{x}}$ that does not depend on $\hat{x}$. Also, NC derivatives should commute $[\hat{\partial}_\rho, \hat{\partial}_\sigma] = 0$.

Since $\hat{\partial} : \hat{\mathcal{A}}_{\hat{x}} \to \hat{\mathcal{A}}_{\hat{x}}$ the relation (2.67) must be consistent with θ-deformation,

$$\hat{\partial}_\rho([\hat{x}_\mu, \hat{x}_\nu] - i\theta^{\mu\nu}) - ([\hat{x}_\mu, \hat{x}_\nu] - i\theta^{\mu\nu})\hat{\partial}_\rho = 0 \; , \tag{2.68}$$

that is, interchanging NC derivative and NC coordinates does not produce a new commutation relations between coordinates. From this follows that $f^\mu_\rho(\hat{\partial}, \theta) = 0$, and hence

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho \; . \tag{2.69}$$

This relation is a peculiarity of the θ-constant deformation; it does not hold in general. We can determine the properties of $\hat{\partial}_\mu$ from the following diagram

$$\begin{tikzcd} \hat{f}(\hat{x}) \arrow[r, "PBW"] \arrow[d, "\hat{\partial}_\mu"'] & f(x) \arrow[d, "\partial^\star_\mu"] \\ (\hat{\partial}_\mu \hat{f})(\hat{x}) \arrow[r, "PBW"] & (\partial^\star_\mu f)(x) \end{tikzcd} \tag{2.70}$$

The PBW property allows us to map $\hat{f}(\hat{x})$ and $(\hat{\partial}_\mu \hat{f})(\hat{x})$ to $\mathcal{A}^\star_x$. Then, by comparing the images $f(x)$ and $(\partial^\star_\mu f)(x)$, we can deduce the form of $\partial^\star_\mu$. In the case of θ-constant deformation, the representation of $\hat{\partial}_\mu$ on $\mathcal{A}^\star_x$ is given by

$$\hat{\partial}_\mu \to \partial^\star_\mu = \partial_\mu \ , \tag{2.71}$$

meaning that it does not get deformed. The derivative $\partial^\star_\mu$ satisfies the *undeformed* Leibniz rule,

$$\big(\partial^\star_\mu (f \star g)\big) = \partial_\mu (f \star g) = (\partial^\star_\mu f) \star g + f \star (\partial^\star_\mu g) = (\partial_\mu f) \star g + f \star (\partial_\mu g) \ . \tag{2.72}$$

To be able to formulate actions, we have to introduce integrals on NC space-time. One can readily check the *cyclicity* of ordinary integral over classical space-time (with suitable boundary conditions),

$$\int \mathrm{d}^4x \ f \star g = \int \mathrm{d}^4x \ g \star f = \int \mathrm{d}^4x \ f \cdot g \ . \tag{2.73}$$

and this holds in any number of dimensions. From (2.73) follows

$$\int \mathrm{d}^4x \ (f_1 \star ... \star f_k) = \int \mathrm{d}^4x \ (f_k \star f_1 \star ... \star f_{k-1}) \ , \tag{2.74}$$

that is, cyclic permutations under integral are allowed. This is important for establishing variational principle that gives us equations of motion. We can use the Leibniz rule for the functional variation and cyclicity to eliminate one $\star$-product and extract the result. For example,

$$\begin{aligned} \frac{\delta}{\delta g(y)} \int \mathrm{d}^4x \ f \star g \star h &= \int \mathrm{d}^4x \ f \star \left(\frac{\delta}{\delta g(y)} g \right) \star h \\ &= \int \mathrm{d}^4x \ f \star \delta^{(4)}(y-x) \star h \\ &= \int \mathrm{d}^4x \ \delta^{(4)}(y-x) \star (h \star f) \\ &= \int \mathrm{d}^4x \ \delta^{(4)}(y-x)(h \star f) = (h \star f)(y) \ . \end{aligned} \tag{2.75}$$

Here, $\delta^{(4)}(y-x)$ is the ordinary four-dimensional Dirac delta-function.

3 Seiberg-Witten gauge field theory

To begin with, we will briefly summarize the basic elements of *classical* (undeformed) gauge field theories (gauge field theories on classical space-time). Let hermitian generators T_A $(A = 1, 2, ..., N)$ of some non-Abelian, N-parametric gauge group $\mathcal{G}$, satisfy the following Lie algebra commutation relations

$$[T_A, T_B] = if_{AB}{}^C T_C \ , \tag{3.1}$$

with totally antisymmetric structure constants $f_{AB}{}^C$. Variation of matter field ψ (we assume that matter fields belong to the fundamental representation of the gauge group) under infinitesimal gauge transformation is given by

$$\delta_\alpha \psi = i\alpha\psi = i\alpha^A(x)T_A\psi \ , \tag{3.2}$$

where infinitesimal gauge parameter $\alpha(x) = \alpha^A(x)T_A$ belongs to the Lie algebra of the gauge group and depends on space-time coordinates. These transformations close in the algebra,

$$[\delta_\alpha, \delta_\beta] = \delta_{-i[\alpha,\beta]} \ . \tag{3.3}$$

Covariant derivative acts on ψ as

$$D_\mu\psi = \partial_\mu\psi - iv_\mu\psi \ , \tag{3.4}$$

where $v_\mu(x) = v_\mu{}^A(x)T_A$ is a Lie algebra-valued gauge potential. By definition, the covariant derivative of ψ transforms covariantly,

$$\delta_\alpha D_\mu\psi = i\alpha D_\mu\psi \ , \tag{3.5}$$

and from this condition follows the inhomogeneous transformation law of the gauge potential,

$$\delta_\alpha v_\mu = \partial_\mu\alpha + i[\alpha, v_\mu] \ . \tag{3.6}$$

Gauge field strength,

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu] \ , \tag{3.7}$$

is also Lie algebra-valued, $F_{\mu\nu} = F_{\mu\nu}{}^A T_A$, and it transforms in the adjoint representation of the gauge group,

$$\delta_\alpha F_{\mu\nu} = i[\alpha, F_{\mu\nu}] \ . \tag{3.8}$$

We can rewrite the gauge field strength in terms of covariant derivative in a curvature-like fashion,

$$F_{\mu\nu} = i[D_\mu, D_\nu] \ . \tag{3.9}$$

When acting on an adjoint field, such as $F_{\mu\nu}$, covariant derivative reads

$$D_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu} - i[v_\alpha, F_{\mu\nu}] \ . \tag{3.10}$$

In the Seiberg-Witten (SW) approach to NC gauge field theories [22, 41, 49, 50] the basic structure of classical gauge field theories is kept, but instead of ordinary fields and ordinary commutative multiplication, one introduces NC fields and Moyal $\star$-product, respectively. We will mainly follow the exposition given in [22]. Variation of NC matter field $\hat{\psi}$ (NC fields are denoted by a "hat" symbol) under infinitesimal, NC-deformed gauge transformation is, by definition,

$$\delta^\star_\Lambda \hat{\psi} = i\hat{\Lambda} \star \hat{\psi} \ . \tag{3.11}$$

where $\hat{\Lambda} = \hat{\Lambda}(x)$ is an NC gauge parameter. Acting on a $\star$-product of two NC fields, NC variation satisfies the Leibniz rule:

$$\delta^\star_\Lambda(\hat{\phi} \star \hat{\psi}) = (\delta^\star_\Lambda \hat{\phi}) \star \hat{\psi} + \hat{\phi} \star (\delta^\star_\Lambda \hat{\psi}) \ . \tag{3.12}$$

To establish closure, for a given pair of NC gauge parameters $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$, we would like to find a third one, $\hat{\Lambda}_3$, such that

$$[\delta^\star_1 \stackrel{\star}{,} \delta^\star_2] = \delta^\star_3 \ . \tag{3.13}$$

There is however a difficulty, in general, concerning the closure axiom for NC gauge transformations. Namely, if NC gauge parameter $\hat{\Lambda}$ is supposed to be Lie algebra-valued, $\hat{\Lambda}(x) = \hat{\Lambda}^A(x) T_A$, then, for some NC field $\hat{\psi}$ from the fundamental representation (the following argument holds in any representation), we have

$$\begin{aligned} [\delta^\star_1 \stackrel{\star}{,} \delta^\star_2]\hat{\psi} &= (\hat{\Lambda}_1 \star \hat{\Lambda}_2 - \hat{\Lambda}_2 \star \hat{\Lambda}_1) \star \hat{\psi} \\ &= \frac{1}{2}\Big([\hat{\Lambda}^A_1 \stackrel{\star}{,} \hat{\Lambda}^B_2]\{T_A, T_B\} + \{\hat{\Lambda}^A_1 \stackrel{\star}{,} \hat{\Lambda}^B_2\}[T_A, T_B]\Big) \star \hat{\psi} = i\hat{\Lambda}_3 \star \hat{\psi} \ . \end{aligned} \tag{3.14}$$

We see that the NC closure rule

$$[\delta^\star_{\hat{\Lambda}_1} \stackrel{\star}{,} \delta^\star_{\hat{\Lambda}_2}] = \delta^\star_{-i[\hat{\Lambda}_1 \stackrel{\star}{,} \hat{\Lambda}_2]} \ , \tag{3.15}$$

consistently generalizes its commutative counterpart (3.3). However, (3.14) implies that $\star$-commutator of two NC gauge transformations does not generally close in the Lie algebra, because anti-commutator $\{T_A, T_B\}$ is not, in general, an element of the algebra, except for $U(N)$ gauge group; only in this particular case one can study non-expended (in orders of the ;deformation parameter) NC gauge theories, as in [41]. Such actions look the same as actions describing the corresponding undeformed theories, except that, instead of the usual point-wise field multiplication, one has the Moyal product. Quntization of non-expended theories leads to the phenomena of UV/IR mixing [51, 52]. But this is not enough if one wants to study the Standard Model. Therefore, we will employ the *enveloping algebra approach* [49, 50].

3.1 Universal enveloping algebra approach

The universal enveloping algebra (UEA) is the largest unital associative algebra in which we can embed a given Lie algebra, so that the abstract bracket operation of the Lie algebra is now the *commutator* in the associative algebra.

Definition 3.1. *(Universal enveloping algebra). For a given Lie algebra $\mathfrak{g}$ of dimension n, over a field $\mathbb{K}$, with generators g_1,..., g_n satisfying $[g_i, g_j] = c_i{}^k{}_j g_k$, one can define a freely generated tensor algebra $\mathcal{T}(\mathfrak{g}) = \oplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} = \mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus ...$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is obtained by taking a quotient with respect to the relation $a \otimes b - b \otimes a = [a, b]$ for all a and b in the embedding of $\mathfrak{g}$ in $\mathcal{T}(\mathfrak{g})$, that is, $\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I}$, where $\mathcal{I}$ is the two-sided ideal generated by the elements of the form $a \otimes b - b \otimes a - [a, b] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \subset \mathcal{T}(\mathfrak{g})$; note that $[a, b] = c_i{}^k{}_j a^i b^j g_k$.*

In general, elements of UEA are linear combinations of (tensor) products of the generators of the original Lie algebra in all possible orders. Using the defining relations of UEA, we can always re-arrange those products in a particular manner, for example, elements of the form (we omit the tensor product) $g_1^{k_1} g_2^{k_2} ... g_n^{k_n}$ with $k_i \in \mathbb{Z}_+ \cup \{0\}$, span UEA. Basis of UEA is always *infinite dimensional.* In our case, assuming *symmetric ordering*, a basis (omitting "1") is provided by

$$
\begin{aligned}
:T_A: &= T_A \ , \\
:T_A T_B: &= \frac{1}{2}(T_A T_B + T_B T_A) \ , \\
&\dots , \\
:T_{A_1} ... T_{A_n}: &= \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma(A_1)} ... T_{\sigma(A_n)} \ ,
\end{aligned} \tag{3.16}
$$

where S_n is the set of permutations of n objects; $|S_n| = n!$.

Universal enveloping algebra of a gauge group is "large enough" to ensure that the closure property for NC gauge transformations holds, provided that NC gauge parameter $\hat{\Lambda}$ is UEA-valued,

$$\begin{aligned}\hat{\Lambda}(x) &= \sum_{n=1}^{\infty}\sum_{\textbf{basis}} \hat{\Lambda}^{n,A_1...A_n} : T_{A_1}...T_{A_n} : \\ &= \hat{\Lambda}^{1,A} : T_A : + \hat{\Lambda}^{2,AB}(x) : T_A T_B : + ... \; . \end{aligned} \tag{3.17}$$

In this case, $\star$-commutator of two NC gauge transformations closes in the enveloping algebra. NC covariant derivative in the fundamental representation is defined by

$$D_\mu \hat{\psi} = \partial_\mu \hat{\psi} - i\hat{V}_\mu \star \hat{\psi} \; , \tag{3.18}$$

where $\hat{V}_\mu$ stands for NC gauge field. NC field strength is defined by analogy with the classical case

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu - i[\hat{V}_\mu \overset{\star}{,} \hat{V}_\nu] \; . \tag{3.19}$$

The covariant derivative of $\hat{\psi}$ transforms covariantly,

$$\delta^\star_\Lambda D_\mu \hat{\psi} = i\hat{\Lambda} \star D_\mu \hat{\psi} \; , \tag{3.20}$$

implying the inhomogeneous transformation law for the NC gauge field,

$$\delta^\star_\Lambda \hat{V}_\mu = \partial_\mu \hat{\Lambda} + i[\hat{\Lambda} \overset{\star}{,} \hat{V}_\mu] \; . \tag{3.21}$$

From this follows that NC gauge field must also be UEA-valued, and it can be represented in its basis,

$$\begin{aligned}\hat{V}_\mu &= \sum_{n=1}^{\infty}\sum_{\text{basis}} \hat{V}_\mu^{nA_1...A_n} : T_{A_1}...T_{A_n} : \\ &= \hat{V}_\mu^{1,A} T_A + \frac{1}{2!}\hat{V}_\mu^{2,AB}\{T_A, T_B\} \\ &+ \frac{1}{3!}\hat{V}_\mu^{3,ABC}(T_A\{T_B, T_C\} + T_B\{T_C, T_A\} + T_C\{T_A, T_B\}) + ... \; . \end{aligned} \tag{3.22}$$

Components $\hat{V}_\mu^{1,A}$, $\hat{V}_\mu^{2,AB}$, $\hat{V}_\mu^{3,ABC}$,... are *new independent fields* in the theory, and since UEA has an infinite basis, it seems that by invoking it we actually introduced an infinite number of new degrees of freedom in the NC theory. This unwanted feature of UEA-valued gauge field is tamed by the Seiberg-Witten map [41, 53].

3.2 Seiberg-Witten map

As we saw, the main difficulty with UAE-valued gauge field theory is that it seems to compel us to introduce an infinite number of new un-physical degrees of freedom, making this approach unrealistic. Fortunately, however, it is demonstrated in [54], by studying the cohomology of UEA gauge theory, that all components of the UEA-valued gauge field can be obtained from the Lie algebra-valued gauge field $V^1_\mu = V^{1,A}_\mu T_A$. Therefore, these new components do not represent new degrees of freedom; *NC gauge field theory possess the same number of degrees of freedom as the corresponding gauge field theory on classical space-time*. This structural feature of UEA gauge theory and the fact that NC fields have to reduce to their classical counterparts when $\theta^{\alpha\beta} \to 0$, as dictated by the principle of correspondence, are the basis of the SW construction, originally constructed in [41].

This map provides a way to represent NC fields as perturbation series in powers of the deformation parameter $\theta^{\alpha\beta}$, with coefficients built out of the fields from the corresponding classical theory. This means that one can define an NC gauge field theory in terms of its classical counterpart. The expansion can be defined by *demanding* that NC gauge transformations are *induced* by the corresponding undeformed gauge transformations. This, in turn, implies that NC gauge parameter and NC gauge field have the following structure

$$\hat{\Lambda} = \hat{\Lambda}(\alpha, \partial\alpha, ...; v_\mu, \partial v_\mu, ...) \ , \tag{3.23}$$

$$\hat{V}_\mu = \hat{V}_\mu(v_\mu, \partial v_\mu, ...) \ , \tag{3.24}$$

where dots stand for higher-order derivatives (we will omit the derivatives to simplify the notation). Thus we can relate NC gauge transformation $\delta^\star_{\hat{\Lambda}} \equiv \delta^\star_{\hat{\Lambda}_\alpha} \equiv \delta^\star_\alpha$ with the undeformed gauge transformation δ_α by

$$\delta^\star_\alpha \hat{\Lambda}(\alpha, v_\mu) = \hat{\Lambda}(\alpha, v_\mu + \delta_\alpha v_\mu) - \hat{\Lambda}(\alpha, v_\mu) \ , \tag{3.25}$$

$$\delta^\star_\alpha \hat{V}_\mu(v_\mu) = \hat{V}_\mu(v_\mu + \delta_\alpha v_\mu) - \hat{V}_\mu(v_\mu) \ , \tag{3.26}$$

with classical variation of the classical gauge field, $\delta_\alpha v_\mu = \partial_\mu \alpha + i[\alpha, v_\mu]$.

Inserting $\hat{\Lambda}_\alpha \equiv \hat{\Lambda}(\alpha, v_\mu)$ into (3.14) yields

$$\hat{\Lambda}_\alpha \star \hat{\Lambda}_\beta - \hat{\Lambda}_\beta \star \hat{\Lambda}_\alpha + i(\delta^\star_\alpha \hat{\Lambda}_\beta - \delta^\star_\beta \hat{\Lambda}_\alpha) = \delta^\star_{-i[\alpha,\beta]} \ . \tag{3.27}$$

In order to solve this equation perturbatively, one has to expand $\hat{\Lambda}_\alpha$ in powers of the deformation parameter $\theta^{\alpha\beta}$ as

$$\hat{\Lambda}_\alpha = \hat{\Lambda}^{(0)}_\alpha + \hat{\Lambda}^{(1)}_\alpha + \hat{\Lambda}^{(2)}_\alpha + \dots \ , \tag{3.28}$$

with $\hat{\Lambda}^{(n)}_\alpha \sim \theta^n$. At zeroth order, $\hat{\Lambda}_\alpha$ reduce to its undeformed counterparts, $\hat{\Lambda}^{(0)}_\alpha = \alpha$. Note that this expansion is not the same as the basis expansion (3.17).

The first order NC correction $\hat{\Lambda}^{(1)}_\alpha \sim \theta$ satisfies the inhomogeneous equation

$$\delta^\star_\alpha \hat{\Lambda}^{(1)}_\beta - \delta^\star_\beta \hat{\Lambda}^{(1)}_\alpha - i[\alpha, \Lambda^{(1)}_\beta] - i[\hat{\Lambda}^{(1)}_\alpha, \beta] - \hat{\Lambda}^{(1)}_{-i[\alpha,\beta]} = -\frac{1}{2}\theta^{\mu\nu}\{\partial_\mu\alpha, \partial_\nu\beta\} \ . \tag{3.29}$$

Up to the first order, the solution is given by

$$\hat{\Lambda}_\alpha = \alpha - \frac{1}{4}\theta^{\mu\nu}\{v_\mu, \partial_\nu\alpha\} + \mathcal{O}(\theta^2) \ . \tag{3.30}$$

This solution is not unique, since one can add to it solutions of the homogeneous equation

$$\delta^\star_\alpha \hat{\Lambda}^{(1)}_\beta - \delta^\star_\beta \hat{\Lambda}^{(1)}_\alpha - i[\alpha, \hat{\Lambda}^{(1)}_\beta] - i[\hat{\Lambda}^{(1)}_\alpha, \beta] - \hat{\Lambda}^{(1)}_{-i[\alpha,\beta]} = 0 \ . \tag{3.31}$$

We will not discuss this aspect here. Detailed analyses of non uniqueness of the SW map can be found in [55].

In this way one can obtain SW expansions for NC gauge parameter, NC gauge potential, NC field strength and NC matter field; they are given by

$$\hat{\Lambda}_\alpha = \alpha - \frac{1}{4}\theta^{\alpha\beta}\{v_\alpha, \partial_\beta\alpha\} + \mathcal{O}(\theta^2) \ , \tag{3.32}$$

$$\hat{V}_\mu = v_\mu - \frac{1}{4}\theta^{\alpha\beta}\{v_\alpha, \partial_\beta v_\mu + F_{\beta\mu}\} + \mathcal{O}(\theta^2) \ , \tag{3.33}$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{1}{4}\theta^{\alpha\beta}\{v_\alpha, (\partial_\beta + D_\beta)F_{\mu\nu}\} + \frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu}, F_{\beta\nu}\} + \mathcal{O}(\theta^2) \ , \tag{3.34}$$

$$\hat{\psi} = \psi - \frac{1}{4}\theta^{\alpha\beta}v_\alpha(\partial_\beta + D_\beta)\psi + \mathcal{O}(\theta^2) \ . \tag{3.35}$$

It is clear that, at the leading order, all NC fields consistently reduce to their classical counterparts, in accord with the principle of correspondence. We can use SW map to expend NC-deformed actions and analyze them perturbatively. There are no new fields in this expansion and the leading order action will always be the original classical action. By the virtue of SW map, expanded actions are endowed with the original *undeformed* gauge symmetry of the classical action, order-by-order in $\theta^{\alpha\beta}$. This method was used for constructing the NC Standard Model, as in [56].

4 $SO(2,3)_\star$ model of NC gravity

In this section, we introduce NC gravity as an $SO(2,3)_\star$ gauge theory on canonically deformed space-time. We will go through some main results obtained in [57–60], where the theory was founded, without getting into details of the calculation. The main emphasis will be on the structure of the theory and the method of its construction. The intention is to set a general framework for dealing with matter fields and supergravity, later on.

Instating the $SO(2,3)_\star$ model of NC gravity involves several steps. To begin with, one introduces classical (undeformed) action invariant under $SO(2,3)$ gauge transformations. This action consists of three parts: the first one is a Mac-Dowell Mansouri type of action, quadratic in $SO(2,3)$ field strength and the other two parts are suitable $SO(2,3)$ generalizations of the Einstein-Hilbert action and the cosmological constant term. To relate the AdS gauge theory to gravity (GR), a gauge fixing condition that reduces the original $SO(2,3)$ gauge symmetry down to $SO(1,3)$ needs to be imposed. For this purpose, a constrained auxiliary field is employed, in the manner of Stelle and West. After the symmetry breaking, the AdS action consistently reduces to (*Einstein-Hilbert*) + (*Cosmological constant*) + (*topological Gauss-Bonnet term*).

Canonical NC deformation of classical space-time is performed by introducing the Moyal $\star$-product that replaces ordinary commutative field multiplication, thus yielding an NC action invariant under deformed $SO(2,3)_\star$ gauge transformations. At this stage, a direct symmetry braking would not provide the desired result, since it would not render an $SO(1,3)_\star$ invariant action. The way to proceed is to follow the SW approach to NC gauge field theory and expand the $SO(2,3)_\star$ gauge-invariant NC action in powers of the deformation parameter $\theta^{\mu\nu}$. By the virtue of SW map, this expansion is invariant under classical $SO(2,3)$ gauge transformations, order-by-order in $\theta^{\mu\nu}$. After gauge fixing, the obtained NC corrections of all orders will necessarily possess $SO(1,3)$ gauge symmetry. The second-order NC correction is calculated explicitly (the first-order correction vanishes), and the low-energy approximation of the theory is studied, including the equations of motion. In particular, it is demonstrated that $SO(2,3)_\star$ model implies a non-trivial deformation of Minkowski space and reveals that noncommutativity can be regarded as a source of curvature and torsion. Furthermore, the structure of the NC-deformed Minkowski metric suggests that the lack of diffeomorphism invariance in the NC theory can be understood as a consequence of the fact that constant noncommutativity implies working in a preferred coordinate system - the Fermi inertial frame.

4.1 AdS gauge theory of gravity

Before presenting classical AdS gauge-invariant action and its NC deformation, let us accentuate the main point of the subject - that GR *with the cosmological constant* can be formulated as a Yang-Mills-like theory of AdS gauge group $SO(2,3)$. By now, a vast body of literature concerning the relation of GR to Yang-Mills gauge theories has been accumulated. Since the original papers of Utiyama [61], Kibble [62] and Sciama [63], there has been considerable interest in this subject and, instead of giving a full historical account, we refer to the several available reviews [64–66]. The main result of these efforts has been to establish a connection between GR, expressed in the first-order formalism, and the Poincaré gauge theory (PGT), with the spin-connection representing the gauge field for the local Lorentz rotations, and the vierbein field being considered as the gauge field for translations in space-time [67, 68]. However, the analogy with Yang-Mills gauge theories is not complete because of the specific treatment of translations. Nevertheless, it is possible to formulate gauge theory of gravity in a way that treats the whole Poincaré group in a more unified way, and naturally includes the cosmological constant. The approach is based on the AdS gauge group $SO(2,3)$. A significant incentive for studying gauge-theoretic formulations of gravity came with the development of SUGRA (extended SUGRA theories combine space-time and internal symmetries). Pure SUGRA is related to a gauge theory of the Poincaré supergroup, or in the generalization of SUGRA to include cosmological constant, to a gauge theory of the orthosymplectic $OSp(4|1)$ supergroup. We put our attention on the AdS group $SO(2,3)$, which is locally isomorphic to the symplectic group $Sp(4)$ (bosonic sector of $OSp(4|1)$). For our purposes, we could equally well use the de Sitter group $SO(1,4)$; the choice of $SO(3,2)$ is made to retain the connection to SUGRA. We will mainly follow the course set in [69–74].

AdS_4 is a maximally symmetric space with Lorentzian signature $(+---)$ and constant negative curvature; it can be represented as a hyperboloid embedded in a five-dimensional flat ambient space with signature $(+---+)$. In AdS gauge theory, we start with an action invariant under $SO(2,3)$ gauge transformations. In order to relate AdS gauge theory to GR, one has to reduce the original $SO(2,3)$ gauge symmetry to $SO(1,3)$. For this purpose, in [71] Stelle and West introduced a nondynamical $SO(2,3)$ five-vector field ϕ^A with dimensions of length. The auxiliary field is constrained to take values in the AdS_4 submanifold of the five-dimensional flat internal space with metric η_{AB}, a copy of which is associated to each point of the space-time manifold. The constraint $\eta_{AB}\phi^A\phi^B = l^2$ defines the AdS_4 embedding equation, where l is related to the cosmological constant by $\Lambda = -3/l^2$.

Lie group $SO(2,3)$ is the isometry group of AdS_4. AdS algebra $\mathfrak{so}(2,3)$ is spanned by ten generators $M_{AB} = -M_{BA}$ $(A, B = 0, 1, 2, 3, 5)$ satisfying $\mathfrak{so}(2,3)$ commutation relations

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}) \ . \tag{4.1}$$

By splitting the set of generators into six AdS rotation generators M_{ab} $(a, b = 0, 1, 2, 3)$ and four AdS translation generators M_{a5}, we can recast the AdS algebra relations (8.1) in a more explicit form,

$$\begin{aligned}
[M_{a5}, M_{b5}] &= -iM_{ab} \ , \\
[M_{ab}, M_{c5}] &= i(\eta_{bc}M_{a5} - \eta_{ac}M_{b5}) \ , \\
[M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \ .
\end{aligned} \tag{4.2}$$

By introducing rescaled generators $P_a := l^{-1}M_{a5}$, (4.2) can be transformed into

$$\begin{aligned}
[P_a, P_b] &= -il^{-2}M_{ab} \ , \\
[M_{ab}, P_c] &= i(\eta_{bc}P_a - \eta_{ac}P_b) \ , \\
[M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \ .
\end{aligned} \tag{4.3}$$

In the limit $l \to \infty$ the AdS algebra reduces to Poincaré algebra, in particular, we have $[P_a, P_b] = 0$ with all other commutators left unchanged. This is a famous example of the Wigner-Inönü contraction. In this sense, AdS_4 can be regarded as a deformation of $\mathcal{M}_4$. A realization of AdS algebra is provided by 5D gamma-matrices Γ^A satisfying Clifford algebra $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$; the generators are given by $M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B]$. One choice of 5D gamma matrices is $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, where γ_a are the usual 4D gamma-matrices. In this particular representation, $SO(2,3)$ generators are $M_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab}$ and $M_{5a} = \frac{1}{2}\gamma_a$. The AdS group $SO(2,3)$ acts on matter fields in the tangent space as a gauge group of internal symmetries. AdS gauge field splits into two components ${\omega_\mu}^{ab}$ and ${\omega_\mu}^{a5}$,

$$\omega_\mu = \frac{1}{2}{\omega_\mu}^{AB}M_{AB} = \frac{1}{4}{\omega_\mu}^{ab}\sigma_{ab} - \frac{1}{2}{\omega_\mu}^{a5}\gamma_a \ , \tag{4.4}$$

Its variation under infinitesimal gauge transformation is given by

$$\delta_\epsilon\omega_\mu = \partial_\mu\epsilon + i[\epsilon, \omega_\mu] \ , \tag{4.5}$$

for some $\mathfrak{so}(2,3)$ algebra-valued gauge parameter $\epsilon = \frac{1}{2}\epsilon^{AB}(x)M_{AB}$.

More explicitly, in terms of components,

$$
\begin{aligned}
\delta_\epsilon \omega_\mu{}^{AB} &= \partial_\mu \epsilon^{AB} - \epsilon^A{}_C \omega_\mu{}^{CB} + \epsilon^B{}_C \omega_\mu{}^{CA} \ , \\
\delta_\epsilon \omega_\mu{}^{ab} &= \partial_\mu \epsilon^{ab} - \epsilon^a{}_c \omega_\mu{}^{cb} + \epsilon^b{}_c \omega_\mu{}^{ca} - \epsilon^a{}_5 \omega_\mu{}^{5b} + \epsilon^b{}_5 \omega_\mu{}^{5a} \ , \\
\delta_\epsilon \omega_\mu{}^{a5} &= \partial_\mu \epsilon^{a5} - \epsilon^a{}_c \omega_\mu{}^{c5} + \epsilon^5{}_c \omega_\mu{}^{ca} \ .
\end{aligned} \tag{4.6}
$$

AdS field strength is defined in the usual way,

$$
F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2} F_{\mu\nu}{}^{AB} M_{AB} \ , \tag{4.7}
$$

and just like the gauge field, its components split into $F_{\mu\nu}{}^{ab}$ and $F_{\mu\nu}{}^{a5}$, yielding

$$
F_{\mu\nu} = \frac{1}{4}\Big(R_{\mu\nu}{}^{ab} - (\omega_\mu{}^{a5}\omega_\nu{}^{b5} - \omega_\nu{}^{a5}\omega_\mu{}^{b5}) \Big) \sigma_{ab} - \frac{1}{2} F_{\mu\nu}{}^{a5}\gamma_a \ , \tag{4.8}
$$

where

$$
R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^a{}_c \, \omega_\nu{}^{cb} - \omega_\mu{}^b{}_c \, \omega_\nu{}^{ca} \ , \tag{4.9}
$$

$$
F_{\mu\nu}{}^{a5} = D^L_\mu \omega_\nu{}^{a5} - D^L_\nu \omega_\mu{}^{a5} \ . \tag{4.10}
$$

Note that D^L_μ stands for the Lorentz $SO(1,3)$ covariant derivative.

Under local AdS transformations, field strength transforms in the adjoint representation of $SO(2,3)$ gauge group,

$$
\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}] \ , \tag{4.11}
$$

or, more explicitly,

$$
\begin{aligned}
\delta_\epsilon F_{\mu\nu}{}^{ab} &= -\epsilon^{ac} F_{\mu\nu c}{}^b + \epsilon^{bc} F_{\mu\nu c}{}^a - \epsilon^{a5} F_{\mu\nu 5}{}^b + \epsilon^{b5} F_{\mu\nu 5}{}^a \ , \\
\delta_\epsilon F_{\mu\nu}{}^{a5} &= -\epsilon^{ac} F_{\mu\nu c}{}^5 + \epsilon^{5c} F_{\mu\nu c}{}^a \ .
\end{aligned} \tag{4.12}
$$

Equations (4.4), (4.6), (4.8) and (4.12) suggest that after setting $\epsilon^{a5} = 0$ (by doing this we restrict the group of gauge transformations to $SO(1,3)$) we may identify $\omega_\mu{}^{ab}$ component of the AdS gauge field with the Lorentz $SO(1,3)$ spin-connection of PGT, $\omega_\mu{}^{a5}$ with the (rescaled) vierbein e^a_μ/l, field strength component $R_{\mu\nu}{}^{ab}$ with the curvature tensor, and $F_{\mu\nu}{}^{a5}$ with (rescaled) torsion $T_{\mu\nu}{}^a/l$. It has been demonstrated in the 70s that one could indeed make such identification and relate AdS gauge theory with GR. One approach was proposed by MacDowell and Mansouri [69]. They start from an $SO(2,3)$ gauge invariant theory but make an additional assumption

- that all fields in the theory transform covariantly under infinitesimal diffeomorphisms. The action is written in a way which breaks the $SO(2,3)$ gauge symmetry down to $SO(1,3)$, and it is invariant under infinitesimal diffeomorphisms. One can then identify $\omega_\mu{}^{a5}$ with the vierbein and obtain GR after going to the second-order formalism[1]. A similar approach was discussed by Towsend in [70].

A more elegant way of relating AdS gauge theory with GR was introduced by Stelle and West [71]. They also start from an AdS gauge theory, but they *spontaneously* break the $SO(2,3)$ gauge symmetry down to $SO(1,3)$. Their start with an $SO(2,3)$ gauge-invariant action, and introduce an auxiliary field ϕ in order to perform the symmetry breaking. In a particular gauge, their action reduces to the MacDowell-Mansouri action which is invariant under the $SO(1,3)$ gauge transformations, and again $\omega_\mu{}^{a5}$ can be interpreted as the vierbeine. In that way, the diffeomorphism invariance follows from the spontaneous symmetry breaking and does not have to be introduced by hand at the very beginning.

The auxiliary field $\phi = \phi^A\Gamma_A$ is a space-time scalar and internal space 5-vector transforming in the adjoint representation of $SO(2,3)$, that is $\delta_\epsilon\phi = i[\epsilon,\phi]$. It has dimensions of length and it is constrained by $\phi^2 = \eta_{AB}\phi^A\phi^B = l^2$. Using this auxiliary field one can write the following $SO(2,3)$ gauge invariant actions [72],

$$S_1 = \frac{il}{64\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\phi \ , \tag{4.13}$$

$$S_2 = \frac{1}{128\pi G_N l}\mathrm{Tr}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}D_\rho\phi D_\sigma\phi\phi + c.c. \ , \tag{4.14}$$

$$S_3 = -\frac{i}{128\pi G_N l}\mathrm{Tr}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi \ , \tag{4.15}$$

with $SO(2,3)$ covariant derivative in the adjoin representation,

$$D_\mu\phi = \partial_\mu\phi - i[\omega_\mu,\phi] \ . \tag{4.16}$$

The complete commutative model of AdS gauge field theory is defined by the sum of these three actions,

$$S = c_1S_1 + c_2S_2 + c_3S_3 \ , \tag{4.17}$$

where we introduced free parameters c_1, c_2 and c_3 that will be determined from some additional constraints. The action (4.17) is real and manifestly invariant under $SO(2,3)$ gauge group.

[1] This holds if there are no spinors in the theory. If the spinor fields appear, the torsion is nonzero, and the pure gravity part of the theory does not reduce to GR.

In the approach taken in [57–60] (the one that will be advocated in this thesis), there is no spontaneous symmetry breaking. Instead, gauge symmetry is broken *directly* from $SO(2,3)$ to $SO(1,3)$ by setting $\phi^a = 0$ and $\phi^5 = l$ (physical gauge), which is consistent with $\phi^2 = l^2$, yielding

$$\phi|_{\text{g.f.}} = l\gamma_5 \ . \tag{4.18}$$

The components of $D_\mu\phi$ then reduce to $(D_\mu\phi)^a_{\text{g.f.}} = e^a_\mu$ and $(D_\mu\phi)^5_{\text{g.f.}} = 0$. This is how we get the vierbein from the auxiliary field ϕ.

In the physical gauge, the classical action (4.17) becomes

$$\begin{aligned} S|_{\text{g.f.}} &= c_1 S_1|_{\text{g.f.}} + c_2 S_2|_{\text{g.f.}} + c_3 S_3|_{\text{g.f.}} \\ &= -\frac{1}{16\pi G_N}\int \mathrm{d}^4x \bigg(\frac{c_1 l^2}{16}\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{\ \ mn} R_{\rho\sigma}^{\ \ rs}\epsilon_{mnrs} \\ &\qquad\qquad + e\Big((c_1+c_2)R - \frac{6}{l^2}(c_1 + 2c_2 + 2c_3)\Big)\bigg). \end{aligned} \tag{4.19}$$

This is the GR action in the first order formalism. The vierbein e^a_μ and the spin-connection $\omega_\mu^{\ ab}$ are independent fields. Varying the action with respect to the spin-connection we obtain an equation that allows us to express the spin-connection in terms of the vierbein. Since there is no fermionic matter in the action this equation gives vanishing torsion. In that case, the first term in the action (quadratic in curvature) is the Gauss-Bonnet term. The second term is the Einstein-Hilbert action, while the last term is the cosmological constant. From the vierbein e^a_μ we can construct the metric tensor $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$ and $e = \sqrt{-g}$. In order to have the canonical normalization of the Einstein-Hilbert term, we impose the constraint $c_1 + c_2 = 1$. Gauss-Bonnet term is topological; it does not influence the equations of motion and we can safely omit it. Therefore, the original AdS action reduces to the Einstein Hilbert action with the cosmological constant

$$\Lambda = -3\frac{1 + c_2 + 2c_3}{l^2} \ . \tag{4.20}$$

Note that the cosmological constant Λ can be positive, negative or zero, regardless of the AdS symmetry of our model. Under WI contraction, it vanishes.

By this we conclude the first stage of constructing the theory of NC AdS gravity. Geometrical character of the classical AdS action (4.17) makes it suitable for NC deformation by the Seiberg-Witten method. The resulting NC action, invariant under NC-deformed AdS gauge transformations, is considered in the following section.

4.2 NC $SO(2,3)_\star$ gravity action

NC deformation of GR cannot be obtained in a straightforward manner, the main difficulty being the underlying diffeomorphism invariance of GR. A vast amount of literature concerning NC gravity has been accumulated over the years, offering a variety of different approaches to the subject. In [75–77] an NC deformation of pure Einstein gravity based on the SW construction is proposed. Then, there is twist approach including some NC solutions [78–81]. Lorentz symmetry in NC gauge field theories was studied in [82, 83]. In the case of emergent NC gravity, dynamical quantum geometry arises from NC gauge theory given by Yang-Mills matrix models [84, 85]. There are also fuzzy space gravity models [86, 87]. The SW map approach was related to NC gravity models via the Fedosov deformation quantization of endomorphism bundles [88, 89]. Other attempts to relate NC gravity models with some testable GR results like gravitational waves, cosmological solutions and Newtonian potential, can be found in [90–95]. The connection to SUGRA was established in [97, 98] and the extension of NC gauge theories to orthogonal and symplectic algebras was considered in [99, 100].

Having in mind that SW construction works very well for NC gauge theories and that we do know how to define a consistent classical AdS gauge theory of gravity, it seems reasonable to consider NC gravity as a SW gauge field theory of NC-deformed AdS gauge group $SO(2,3)_\star$. This theory was founded in [57–60]. However, one cannot simply impose a gauge fixing condition on the level of the non-extended NC action, because this will not yield an $SO(1,3)_\star$ invariant theory [57]. The main point is that NC deformation *does not commute* with the gauge fixing. Therefore, one first has to expand the NC action in powers of $\theta^{\mu\nu}$ using the UEA gauge field theory and the SW map. The *expanded* NC action is invariant under classical $SO(2,3)$ gauge transformations, order-by-order in $\theta^{\mu\nu}$, by the virtue of SW map. In this manner, the NC-deformed $SO(2,3)_\star$ gauge theory is related to NC gravity. Of course, one still has to impose the gauge fixing condition to obtain an $SO(1,3)$ gauge-invariant NC corrections. The first non-vanishing NC correction to GR action is of second-order and it was calculated explicitly [59]. This result is in accord with [101]. Due to the complexity of the NC gravity action, only the low-energy sector of the theory is studied. An important prediction of the $SO(2,3)_\star$ model is a non-trivial NC deformation of Minkowski space that leads to a new interpretation of noncommutativity as a source of curvature and torsion [59], and of diffeomorphism symmetry breaking in NC field theories [60]. Also, the model has the capacity to incorporate matter fields, and we will see later that inclusion of matter couplings produces a non-trivial linear NC deformation.

The NC generalization of the classical actions (4.13), (4.14) and (4.15) is obtained by promoting ordinary fields to their NC counterparts and commutative product between fields by the Moyal $\star$-product, yielding

$$S_1^\star = \frac{ilc_1}{64\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\star\hat{\phi}\ , \tag{4.21}$$

$$S_2^\star = \frac{c_2}{128\pi G_N l}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\hat{\phi}\star\hat{F}_{\mu\nu}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi} + c.c.\ , \tag{4.22}$$

$$S_3^\star = -\frac{ic_3}{128\pi G_N l}\mathrm{Tr}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi}\ . \tag{4.23}$$

The second action is not real, and we therefore have to add its complex conjugate by hand to impose the reality condition.

After the SW expansion and the gauge fixing (in that order), the second-order NC correction $S_{NC}^{(2)}$ was found explicitly. It is highly intricate and we will not write the full expression here. We refer to the original work [59]. The analysis of the exact action is very demanding, especially the resulting equations of motion, since it contains terms that are up to fourth power of curvature and up to second power of torsion. However, one can still analyze the model in different regimes of parameters. If we are interested in the low energy corrections, we should keep terms that have at most two derivatives on vierbeins. Therefore, we include only terms linear in curvature, and linear and quadratic in torsion. Additionally, we assume that the spin connection $\omega_\mu{}^{ab}$ and the first-order derivatives of vierbeins such are of the same order. The equations of motions are obtained by varying the action over vierbein and spin connection, independently. If we consider only the class of NC solutions with vanishing torsion $T_{\mu\nu}{}^a = 0$, in the *low energy limit*, equations of motion for the vierbein and the spin-connection are

$$\delta e^a_\mu :\ R_{\alpha\beta}{}^{cd}e^\alpha_a e^\beta_d e^\mu_c - \frac{1}{2}e^\mu_a R + \frac{3}{l^2}(1+c_2+2c_3)e^\mu_a = \tau_a{}^\mu = -\frac{8\pi G_N}{e}\frac{\delta S_{NC}^{(2)}}{\delta e^a_\mu}\ , \tag{4.24}$$

$$\delta\omega_\mu{}^{ab} :\ T_{ac}{}^c e^\mu_b - T_{bc}{}^c e^\mu_a - T_{ab}{}^\mu = S_{ab}{}^\mu = -\frac{16\pi G_N}{e}\frac{\delta S_{NC}^{(2)}}{\delta\omega_\mu{}^{ab}}\ . \tag{4.25}$$

Now we come to an important point. The effective energy-momentum tensor $\tau_a{}^\mu$ and the effective spin-tensor $S_{ab}{}^\mu$ in equations (4.24) and (4.25) depend on $\theta^{\mu\nu}$ (since they are obtained by varying NC correction $S_{NC}^{(2)}$ that is quadratic in $\theta^{\mu\nu}$) and we may conclude that *noncommutativity acts as a source of curvature and torsion*, that is, space-time can becomes curved as an effect of the noncommutative corrections. Also, a torsion-free solution could develop a non-zero torsion due to noncommutativity.

4.3 NC Minkowski space

To explore the consequences of space-time noncommutativity in some more detail, we consider the NC deformation of Minkowski space in the low-energy limit [60]. Minkowski space is a vacuum solution of Einstein field equations without the cosmological constant. Therefore, if we recall that the cosmological constant depends on the free parameters c_1, c_2 and c_3 as $\Lambda = -3(1+c_2+2c_3)/l^2$, we have to assume the constraint $1 + c_2 + 2c_3 = 0$ that eliminates the cosmological constant in the classical action. Regarding NC correction as a small perturbation around flat Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \Lambda_{NC}^{-4} h_{\mu\nu} \ , \tag{4.26}$$

where $h_{\mu\nu}$ is quadratic in $\theta^{\mu\nu} \sim \Lambda_{NC}^2$, field equations reduce to

$$\frac{1}{2}(\partial_\sigma \partial^\nu h^{\sigma\mu} + \partial_\sigma \partial^\mu h^{\sigma\nu} - \partial^\mu \partial^\nu h - \Box h^{\mu\nu}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h) = \Lambda_{NC}^4 \tau^{\mu\nu} \ , \tag{4.27}$$

with

$$\tau^{\mu\nu} = \frac{11}{4l^6}\left(2\eta_{\alpha\beta}\theta^{\alpha\mu}\theta^{\beta\nu} + \frac{1}{2} g_{\alpha\gamma} g_{\beta\delta} g^{\mu\nu} \theta^{\alpha\beta}\theta^{\gamma\delta}\right) \ .$$

The NC-deformed components of the metric tensor are given by

$$\begin{aligned} g^{00} &= 1 - \frac{11}{2l^6}\theta^{0m}\theta^{0n} x^m x^n - \frac{11}{8l^6}\theta^{\alpha\beta}\theta_{\alpha\beta} r^2 \ , \\ g^{0i} &= -\frac{11}{3l^6}\theta^{0m}\theta^{in} x^m x^n \ , \\ g^{ij} &= -\delta^{ij} - \frac{11}{6l^6}\theta^{im}\theta^{jn} x^m x^n + \frac{11}{24l^6}\delta^{ij}\theta^{\alpha\beta}\theta_{\alpha\beta} r^2 - \frac{11}{24l^6}\theta^{\alpha\beta}\theta_{\alpha\beta} x^i x^j \ . \end{aligned} \tag{4.28}$$

The Reimann tensor for this solution can be calculated easily, and the scalar curvature of the NC Minkowski space turns out to be $R = \frac{11}{l^6}\theta^2 = const.$ Under WI contraction it consistently vanishes. Thus, in the $SO(2,3)_\star$ model, there exists a non-trivial NC deformation of Minkowski space. A very interesting (and unexpected) conclusion emerges: having the components of the Riemann tensor, the components of the metric tensor can be represented as

$$\begin{aligned} g_{00} &= 1 - R_{0m0n} x^m x^n \ , \\ g_{0i} &= -\frac{2}{3} R_{0min} x^m x^n \ , \\ g_{ij} &= -\delta_{ij} - \frac{1}{3} R_{imjn} x^m x^n \ . \end{aligned} \tag{4.29}$$

This result suggests that the coordinates x^μ that we started with, are actually *Fermi normal coordinates*. These are the inertial coordinates of a local observer moving along a geodesic. The time coordinate x^0 is just the proper time of the observer, and space coordinates x^i are defined as affine parameters along the geodesics in the hypersurface orthogonal to the actual geodesic of the observer. Unlike Riemann normal coordinates which can be constructed in a small neighbourhood of a point, Fermi normal coordinates can be constructed in a small neighbourhood of a geodesic, that is, inside a small cylinder surrounding the geodesic [102–104]. Along the geodesic we have

$$g_{\mu\nu}|_{geod.} = \eta_{\mu\nu}, \quad \partial_\rho g_{\mu\nu}|_{geod.} = 0 \ . \tag{4.30}$$

The measurements performed by a local observer moving along a geodesic are described from a Fermi frame of reference, and this observer is the one that measures $\theta^{\mu\nu}$ to be constant. In any other reference frame (any other coordinate system) $\theta^{\mu\nu}$ will not be constant. The breaking of diffeomorphism symmetry due to canonical noncommutativity can now be understood as a consequence of working in a preferred frame of reference given by the Fermi normal coordinates.

In an arbitrary reference frame, the NC deformation is obtained by an appropriate coordinate transformation. Let y^α be an arbitrary coordinate system at a point P in a small neighborhood of the geodesic γ which defines our Fermi normal coordinates x^μ and $[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}$. The noncommutativity in y-coordinates is then given by

$$[y^\alpha \stackrel{\star}{,} y^\beta] = i\theta^{\mu\nu}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} - \frac{i}{24}\theta^{\mu\nu}\theta^{\rho\sigma}\theta^{\kappa\lambda}\frac{\partial^3 y^\alpha}{\partial x^\kappa\partial x^\rho\partial x^\mu}\frac{\partial^3 y^\beta}{\partial x^\lambda\partial x^\sigma\partial x^\nu} + \dots \ . \tag{4.31}$$

The $\star$-product is the Moyal $\star$-product and y^α are understood as functions of Fermi inertial coordinates x^μ.

Two NC gravity models with constant noncommutativity, one in x^μ coordinates, and the other in y^μ coordinates will not be equivalent. The result of [60] suggests that constant noncommutativity implies a preferred coordinate system. This choice breaks the diffeomorphism invariance of the NC theory. It is not clear whether the diffeomorphism invariance can be restored. To answer this question, we have to be able to rewrite the model in an arbitrary coordinate system. A step towards the resolution of this problem would be understanding better various solutions of the $SO(2,3)$ NC gravity model, such as the NC Schwarzschild solution and cosmological solutions. This remains to be done in the future.

5 Dirac field and NC gravity

The content of this section is originally presented in [105].

Dirac spinor field describes charged spin-1/2 fermions, such as an electron, or a quark. It transforms (although not necessarily) in the fundamental representation of a gauge group, and it is invariant under general coordinate transformations. To work with spinors in curved space-time, one has to use the first-order formalism. In this section we introduce Dirac field within the framework of AdS gauge theory of gravity, on the classical and noncommutative level. We start by presenting a classical (undeformed) action, invariant under $SO(2,3)$ gauge transformations, that coincides, after choosing a certain gauge, with the standard Dirac action in curved space-time with a universal mass-like term that vanishes under WI contraction. This mass-like term suggests (wrongly) that fermions have a mass equal to $2/l$ (l is the WI contraction parameter related to AdS radius). However, the correct interpretation would be that theory describes a massless electron in AdS_4 background geometry.

Due to its geometric character (before gauge fixing), the classical $SO(2,3)$ gauge-invariant action is straightforwardly deformed by introducing Moyal $\star$-product. The resulting NC action is invariant under NC-deformed group of $SO(2,3)_\star$ gauge transformations. We take the SW approach to NC gauge field theory and expand the NC action in powers of the deformation parameter $\theta^{\mu\nu}$. By construction, the expanded NC action is invariant under ordinary $SO(2,3)$ gauge transformations, order-by-order in $\theta^{\mu\nu}$. A significant consequence of having matter fields coupled to NC gravity is the non-vanishing *first order* NC correction (for pure NC gravity it is *quadratic*). This fact greatly simplifies the calculation and leads to some new phenomenological predictions. In particular, the linear NC correction pertains even in the flat space-time limit and produces NC deformation of the Dirac equation, Feynman propagator and dispersion relation of an electron. We arrive at an interesting conclusion concerning the relation between electron's energy and helicity, namely, the model predicts *NC birefringence effect* (analogues to the well-know effect in optics) for free electrons propagating in NC space-time. Therefore, NC-deformed space-time acts as a birefringent medium for electrons and causes a Zeeman-like splitting of their energy levels, even in the absence of an external magnetic field. Later on, we will see that if homogeneous background magnetic field is present, space-time noncommutativity modifies electron's Landau levels. We also note that some NC terms survive the WI contraction, and thus provide a possible way to explore the connection between the WI contraction and canonical NC deformation.

5.1 Dirac field in AdS framework

Let ψ be a Dirac spinor field transforming in the fundamental representation of $SO(2,3)$ gauge group. Its infinitesimal variation under the group action is

$$\delta_\epsilon \psi = i\epsilon\psi = \frac{i}{2}\epsilon^{AB} M_{AB}\psi \ , \tag{5.1}$$

where ϵ^{AB} are some antisymmetric gauge parameters of $SO(2,3)$. Therefore, we define the $SO(2,3)$ covariant derivative of a Dirac spinor as

$$D_\mu\psi = \partial_\mu\psi - \frac{i}{2}\omega_\mu{}^{AB} M_{AB}\psi \ , \tag{5.2}$$

and it can be dissolved in two parts

$$D_\mu\psi = D^L_\mu\psi + \frac{i}{2l}e^a_\mu\gamma_a\psi \ , \tag{5.3}$$

where D^L_μ, given by

$$D^L_\mu\psi = \partial_\mu\psi - \frac{i}{4}\omega_\mu{}^{ab}\sigma_{ab}\psi \ , \tag{5.4}$$

is the Lorentz $SO(1,3)$ covariant derivative. The vierbein term in (5.3) is AdS deformation that vanishes under WI contraction. As in the case of pure NC gravity (Section 4), we introduce a non-dynamical auxiliary field $\phi = \phi^A\Gamma_A$ that transforms in the adjoint representation AdS group, that is $\delta_\epsilon\phi = i[\epsilon, \phi]$.

Consider the spinor action (since it involves derivatives, we will call it "*kinetic*")

$$S_{\psi,kin} = \frac{i}{12}\int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi - D_\sigma\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\psi\Big] \ . \tag{5.5}$$

This action is manifestly invariant under $SO(2,3)$ gauge transformations, and it is hermitian up to the surface term that vanishes. To reduce $SO(2,3)$ gauge symmetry down to $SO(1,3)$ we choose the physical gauge and set $\phi^a = 0$ and $\phi^5 = l$. Consequently, we must set $D_\mu\phi^a|_{\text{g.f.}} = e^a_\mu$ and $D_\mu\phi^a|_{\text{g.f.}} = 0$, yielding

$$S_{\psi,kin}|_{\text{g.f.}} = \frac{i}{2}\int \mathrm{d}^4x \ e\left[\bar\psi\gamma^\sigma D^L_\sigma\psi - D^L_\sigma\bar\psi\gamma^\sigma\psi\right] - \frac{2}{l}\int \mathrm{d}^4x \ e \ \bar\psi\psi \ . \tag{5.6}$$

This is exactly the Dirac action in curved space-time for spinors of mass $2/l$. Now, spinors do not actually gain mass by gauge fixing. The correct interpretation is that cosmological mass-like term arises due to *AdS* background geometry. WI contraction eliminates this term.

There are five additional fermionic terms, invariant under $SO(2,3)$ gauge transformations, that can be used to supplement the original action and modify the cosmological mass-like term. They differ only in the position of the auxiliary field,

$$\begin{aligned}
&\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi \ , \ \bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi \ , \\
&\bar{\psi}D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi \ , \ \bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi \ , \\
&\bar{\psi}D_\mu\phi\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi \ . \qquad (5.7)
\end{aligned}$$

Using them, we can build only three independent hermitian mass-like actions (of the type $\bar{\psi}...\psi$) denoted by $S_{m,i}$ $(i = 1, 2, 3)$:

$$\begin{aligned}
S_{m,1} &= \frac{ic_1}{2}\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi + \bar{\psi}\phi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big] \ , \\
S_{m,2} &= \frac{ic_2}{2}\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi + \bar{\psi}D_\mu\phi\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big] \ , \\
S_{m,3} &= ic_3\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma} \ \ \bar{\psi}D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi \ . \qquad (5.8)
\end{aligned}$$

Free dimensionless parameters c_1, c_2 and c_3 are introduced for generality. After gauge fixing, sum of the three terms in (5.8), denoted by $S_{\psi,m}$, reduces to

$$S_{\psi,m}|_{\text{g.f.}} = \sum_{i=1}^{3} S_{m,i} = 24(c_2 - c_1 - c_3)\left(m - \frac{2}{l}\right)\int \mathrm{d}^4x \ e \ \bar{\psi}\psi \ . \qquad (5.9)$$

If we want to assume some particular value m for the mass parameter, the coefficients c_1, c_2, and c_3 must satisfy the constraint

$$c_2 - c_1 - c_3 = -\frac{1}{24} \ . \qquad (5.10)$$

Then (5.9) becomes

$$S_{\psi,m}|_{\text{g.f.}} = -\left(m - \frac{2}{l}\right)\int \mathrm{d}^4x \ e \ \bar{\psi}\psi \ . \qquad (5.11)$$

Terms in (5.6) and (5.11) that involve cosmological mass $2/l$ cancel each other out, and therefore, the total spinor action $S_\psi = S_{\psi,kin} + S_{\psi,m}$ comes down to

$$S_\psi|_{\text{g.f.}} = \frac{i}{2}\int \mathrm{d}^4x \ e \ \left[\bar{\psi}\gamma^\sigma D^L_\sigma\psi - D^L_\sigma\bar{\psi}\gamma^\sigma\psi\right] - m\int \mathrm{d}^4x \ e \ \bar{\psi}\psi \ . \qquad (5.12)$$

Thus, by imposing the gauge fixing condition, we reduced the original AdS gauge theory involving Dirac spinors, to the standard Dirac action in curved space-time.

5.2 NC Dirac action

To deform classical actions (5.5) and (5.8), we implement the Seiberg-Witten method of constructing NC gauge field theories out of the corresponding undeformed ones, presented in Section 3. First, we promote classical fields, ψ and ϕ, to their NC counterparts, $\hat{\psi}$ and $\hat{\phi}$, and introduce the Moyal $\star$-product (1.5) instead of commutative field multiplication. The resulting NC action is subsequently expanded in powers of $\theta^{\mu\nu}$ using the SW prescription. By construction, this expanded NC action is invariant under *undeformed* $SO(2,3)$ gauge transformations, order-by-order. In the case of pure NC gravity (without matter field), the lowest non-vanishing NC correction, in the physical gauge, is of second order in $\theta^{\mu\nu}$. This feature renders the theory computationally challenging. It is, therefore, a quite significant fact that having matter fields (Dirac spinors, in particular) coupled to NC gravity produces a non-vanishing linear NC correction in the physical gauge. We will present the calculation procedure for the linear NC correction to the kinetic term (5.5) and the three bilinear terms (5.8), separately.

NC coupling of spinors and gravity was previously treated by P. Aschieri and L. Castellani [106–108]. They choose to start with the classical action in curved space-time, thus having $SO(1,3)$ gauge symmetry from the beginning. NC deformation immediately produces $SO(1,3)_\star$ gauge-invariant action. In the case of massless Majorana spinors [106, 107], the first non-vanishing NC correction turns out to be quadratic in $\theta^{\mu\nu}$ (all odd-power corrections being equal to zero). Coupling of Dirac spinors and NC gravity is treated in [108] and the linear NC deformation is obtained, but the physical implications of this result have not been elaborated. The fact that AdS algebra reduces to Poincaré algebra under WI contraction, might be reflected on the relation of our NC AdS gauge theory with the NC theory of Aschieri and Castellani, based on deformed Lorentz group. For that matter, we point out that our theory implies that some parts of the linear NC correction to the Dirac action in curved space-time survive WI contraction and some residual NC effects are present even in flat space-time. This feature enables us to investigate potentially observable NC effects at the lowest possible order. It leads to an important physical prediction of the linearly deformed dispersion relation for electrons in NC Minkowski space, along with a Zeeman-like splitting of their undeformed energy levels. Also, the energy levels become helicity-dependent due to noncommutativity of the background space-time that behaves as a birefringent medium for the propagating electrons. Incidentally, that differences between the two models revealed themselves already in the case of pure NC gravity. Namely, as we saw in Section 4, the NC deformation of Minkowski space is obtained in the $SO(2,3)_\star$ model of NC gravity.

5.3 NC deformation of the kinetic spinor term

To deform the spinor action (5.5), we follow the general SW prescription elaborated in Section 3. As in general SW NC gauge field theory, we introduce NC spinor field $\hat{\psi}$ (from the fundamental representation), NC auxiliary field $\hat{\phi}$ (from the adjoint representation) and $SO(2,3)_\star$ gauge field $\hat{\omega}_\mu$. NC covariant derivative is defined as

$$D_\mu\hat{\psi} = \partial_\mu\hat{\psi} - i\hat{\omega}_\mu \star \hat{\psi} \ , \tag{5.13}$$

$$D_\mu\hat{\phi} = \partial_\mu\hat{\phi} - i[\hat{\omega}_\mu \stackrel{\star}{,} \hat{\phi}] \ . \tag{5.14}$$

The structure of NC covariant derivative, in both representations, is the same as in classical gauge field theory, the only difference being the use of the Moyal $\star$-product instead of ordinary point-wise multiplication. Under infinitesimal NC gauge transformations, $\hat{\psi}$, $\hat{\phi}$ and their covariant derivatives transform as

$$\begin{aligned} \delta^\star_\epsilon\hat{\psi} &= i\hat{\Lambda}_\epsilon \star \hat{\psi} \ , \quad \delta^\star_\epsilon D_\mu\hat{\psi} = i\hat{\Lambda}_\epsilon \star D_\mu\hat{\psi} \ , \\ \delta^\star_\epsilon\hat{\phi} &= i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\phi}] \ , \quad \delta^\star_\epsilon D_\mu\hat{\phi} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} D_\mu\hat{\phi}] \ . \end{aligned} \tag{5.15}$$

In these NC variation, $\hat{\Lambda}_\epsilon$ is an $SO(2,3)_\star$ gauge parameter that reduces to the corresponding classical $SO(2,3)$ gauge parameter $\epsilon = \frac{1}{2}\epsilon^{AB}M_{AB}$ when $\theta^{\alpha\beta} \to 0$, that is, $\hat{\Lambda}_\epsilon = \epsilon + \mathcal{O}(\theta)$. Likewise, we have $\hat{\omega}_\mu = \omega_\mu + \mathcal{O}(\theta)$ and $\hat{F}_{\mu\nu} = F_{\mu\nu} + \mathcal{O}(\theta)$.

The SW expansion of $\hat{\psi}$, $\hat{\phi}$ and their covariant derivatives, are given by

$$\hat{\psi} = \psi - \tfrac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)\psi + \mathcal{O}(\theta^2) \ , \tag{5.16}$$

$$\hat{\phi} = \phi - \tfrac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)\phi\} + \mathcal{O}(\theta^2) \ , \tag{5.17}$$

$$D_\mu\hat{\psi} = D_\mu\psi - \tfrac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)D_\mu\psi + \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\mu}D_\beta\psi + \mathcal{O}(\theta^2) \ , \tag{5.18}$$

$$D_\mu\hat{\phi} = D_\mu\phi - \tfrac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)D_\mu\phi\} + \tfrac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu}, D_\beta\phi\} + \mathcal{O}(\theta^2) \ . \tag{5.19}$$

Consider the NC version of the kinetic spinor action (5.5)

$$\begin{aligned} S^\star_{kin} = \frac{i}{12}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\ \Big[&\hat{\bar{\psi}} \star (D_\mu\hat{\phi}) \star (D_\nu\hat{\phi}) \star (D_\rho\hat{\phi}) \star (D_\sigma\hat{\psi}) \\ &-(D_\sigma\hat{\bar{\psi}}) \star (D_\mu\hat{\phi}) \star (D_\nu\hat{\phi}) \star (D_\rho\hat{\phi}) \star \hat{\psi}\Big] \ . \end{aligned} \tag{5.20}$$

It is obtained by a direct substitution of the ordinary commutative product with the Moyal $\star$-product. Using the NC variations (5.15) and cyclicity of the $\star$-product one can readily check that (5.20) is invariant under deformed $SO(2,3)_\star$ gauge transformations. Moreover, this action is real, up to the surface term that vanishes.

Now we expand this action up to first order in the deformation parameter $\theta^{\alpha\beta}$, using the SW map. Generally, for any pair of NC fields $\hat{A}$ and $\hat{B}$, the first order NC correction to their $\star$-product is given by

$$\left(\hat{A} \star \hat{B}\right)^{(1)} = \hat{A}^{(1)}B + A\hat{B}^{(1)} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha A\partial_\beta B \ . \tag{5.21}$$

If *both* of these two fields transform in the *adjoint representation*, the last formula assumes a more specific form

$$\begin{aligned}\left(\hat{A} \star \hat{B}\right)^{(1)} =& -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)AB\} + \frac{i}{2}\theta^{\alpha\beta}D_\alpha AD_\beta B \\ &+ cov(\hat{A}^{(1)})B + Acov(\hat{B}^{(1)}) \ ,\end{aligned} \tag{5.22}$$

where $cov(\hat{A}^{(1)})$ is the covariant part of $A's$ first order NC correction, and $cov(\hat{B}^{(1)})$, the covariant part of $B's$ first order NC correction. Applying the rule (5.22) twice, and using the expansion (5.19) for the covariant derivative of the adjoint field $\hat{\phi}$, we can obtain the first order NC correction to the product $D_\mu\hat{\phi} \star D_\nu\hat{\phi} \star D_\rho\hat{\phi}$,

$$\begin{aligned}\left(D_\mu\hat{\phi} \star D_\nu\hat{\phi} \star D_\rho\hat{\phi}\right)^{(1)} = \theta^{\alpha\beta}\Big[&-\frac{1}{4}\{\omega_\alpha, (\partial_\beta + D_\beta)(D_\mu\phi D_\nu\phi D_\rho\phi)\} \\ &+ \frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi) \\ &+ \frac{i}{2}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi \\ &+ \frac{1}{2}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi \\ &+ \frac{1}{2}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi \\ &+ \frac{1}{2}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}\Big] \ .\end{aligned} \tag{5.23}$$

The composite field $D_\mu\hat{\phi} \star D_\nu\hat{\phi} \star D_\rho\hat{\phi}$ also transforms in the adjoint representation of $SO(2,3)_\star$, being a product of fields that transform in the adjoint representation. Therefore, according to the rule (5.22), we could immediately say, without explicit calculation, what is the non-covariant part of the first order NC correction to $D_\mu\hat{\phi} \star D_\nu\hat{\phi} \star D_\rho\hat{\phi}$, that is, the first term in (5.23). It is non-covariant because of the manner in which it involves the gauge potential ω_α and the partial derivative ∂_β. The other terms appearing in (5.23) are manifestly covariant. The use of the rule (5.22) significantly simplifies the calculation. The non-covariant part of any NC field $\hat{A}$ that transforms in the adjoint representation has the same form, namely,

$$\text{non-}cov\left(\hat{A}^{(1)}\right) = -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)A\} \ . \tag{5.24}$$

If we have an NC field $\hat{A}$ that transforms in the adjoint representation, and an NC field $\hat{B}$ that transforms in the fundamental representation, the rule (5.21) again acquires a specific form,

$$\begin{aligned}\left(\hat{A}\star\hat{B}\right)^{(1)} = &-\frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta+D_\beta)(AB)+\frac{i}{2}\theta^{\alpha\beta}D_\alpha A D_\beta B\\ &+cov(\hat{A}^{(1)})B+Acov(\hat{B}^{(1)})\ .\end{aligned} \tag{5.25}$$

Similar relation can be found in [101]. The non-covariant part of any composite field that transforms in the fundamental representation has the same form as the second term in (5.16). This formula reviles the structure of NC corrections and greatly simplifies the calculation. Using the result (5.23) and the expansion (5.18) for the covariant derivative of a spinor field, can obtain first order NC correction to the noncommutative product $D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\rho\hat{\psi}$. Applying the rule (5.25), and setting $\hat{A}:=D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}$ and $\hat{B}:=D_\sigma\hat{\psi}$, yield

$$\begin{aligned}(D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\psi})^{(1)}=\theta^{\alpha\beta}\Big[&-\frac{1}{4}\omega_\alpha(\partial_\beta+D_\beta)(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi)\\ &+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi)\\ &+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi\\ &+\frac{i}{2}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi\\ &+\frac{1}{2}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi\\ &+\frac{1}{2}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}D_\rho\phi D_\sigma\psi\\ &+\frac{1}{2}D_\mu\phi D_\nu\phi\{F_{\alpha\rho},D_\beta\phi\}D_\sigma\psi\\ &-\frac{1}{2}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi\Big]\ .\end{aligned} \tag{5.26}$$

The composite field $D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\psi}$ transforms in the fundamental representation since it is a product of the field $D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}$ that transforms in the adjoint representation, and the field $D_\sigma\hat{\psi}$ that transforms in the fundamental representation of $SO(2,3)_\star$, and therefore, the first term in (5.26), the non-covariant one, has the same form as the corresponding non-covariant term in (5.16). Again, we could anticipate that from the general result (5.25). The remaining terms in (5.26) are manifestly covariant. Using the NC expansion of the Dirac adjoint field,

$$\hat{\bar{\psi}}=\bar{\psi}-\frac{1}{4}\theta^{\alpha\beta}\bar{\psi}(\overleftarrow{\partial}_\beta+\overleftarrow{D}_\beta)\omega_\alpha+\mathcal{O}(\theta^2)\ , \tag{5.27}$$

setting $\hat{A} := \hat{\bar{\psi}}$ and $\hat{B} := D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\psi}$, the general rule (5.21) gives us the first order NC correction to the $SO(2,3)_\star$ scalar $\hat{\bar{\psi}}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\psi}$:

$$
\begin{aligned}
(\hat{\bar{\psi}}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\psi})^{(1)} = \theta^{\alpha\beta}\Big[&-\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi \\
&+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi) \\
&+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi \\
&+\frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi \\
&+\frac{1}{2}\bar{\psi}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi \\
&+\frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi D_\sigma\psi \\
&+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\psi \\
&-\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi\Big]. \qquad (5.28)
\end{aligned}
$$

Finally, we can present the complete linear NC correction to the classical kinetic spinor action (5.5), that is, the $n=1$ term in the perturbative expansion of the full NC kinetic spinor action $S^\star_{\psi,kin} = \sum_{n=0}^{\infty} S^{(n)}_{\psi,kin}$, before gauge fixing,

$$
\begin{aligned}
S^{(1)}_{\psi,kin} = \frac{i}{12}\,\theta^{\alpha\beta}\int \mathrm{d}^4x\;\varepsilon^{\mu\nu\rho\sigma}\Bigg[&-\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi \\
&+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi \\
&+\frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi) \\
&+\frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi \\
&+\frac{1}{2}\bar{\psi}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi \\
&+\frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi D_\sigma\psi \\
&+\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\psi \\
&-\frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi\Bigg] + c.c. \qquad (5.29)
\end{aligned}
$$

By the virtue of SW map, all terms in the expansion of the NC action (5.29), which is invariant under NC-deformed $SO(2,3)_\star$ gauge transformations, are manifestly invariant under ordinary $SO(2,3)$ gauge transformation.

In order to break the $SO(2,3)$ gauge symmetry of the action (5.29) down to the local Lorentz $SO(1,3)$ symmetry, we choose the physical gauge and set $\phi^a = 0$ and $\phi^5 = l$ (the result for each term separately is given in Appendix A), yielding

$$
\begin{aligned}
S^{(1)}_{\psi,kin}|_{\text{g.f.}} = \theta^{\alpha\beta}\int \mathrm{d}^4x\, e\Big[& -\frac{1}{8}R_{\alpha\mu}{}^{ab}e^\mu_a(\bar\psi\gamma_b D^L_\beta\psi) - \frac{1}{16}R_{\alpha\beta}{}^{ab}e^\sigma_a(\bar\psi\gamma_b D^L_\sigma\psi) \\
& - \frac{i}{32}R_{\alpha\beta}{}^{ab}\varepsilon_{abc}{}^{d}e^\sigma_d(\bar\psi\gamma^c\gamma^5 D^L_\sigma\psi) - \frac{i}{24}R_{\alpha\mu}{}^{ab}\varepsilon_{abc}{}^{d}e^c_\beta(e^\mu_d e^\sigma_s - e^\mu_s e^\sigma_d)(\bar\psi\gamma^s\gamma^5 D^L_\sigma\psi) \\
& - \frac{i}{16}R_{\alpha\mu}{}^{bc}e^\mu_a\varepsilon^a{}_{bcm}(\bar\psi\gamma^m\gamma^5 D^L_\beta\psi) - \frac{i}{8l}T_{\alpha\beta}{}^{a}e^\sigma_a(\bar\psi D^L_\sigma\psi) + \frac{i}{8l}T_{\alpha\mu}{}^{a}e^\mu_a(\bar\psi D^L_\beta\psi) \\
& + \frac{1}{16l}T_{\alpha\beta}{}^{a}e^\mu_a(\bar\psi\sigma_\mu{}^\sigma D^L_\sigma\psi) + \frac{1}{8l}T_{\alpha\mu}{}^{a}e^\mu_b(\bar\psi\sigma_a{}^b D^L_\beta\psi) - \frac{1}{12l}T_{\alpha\mu}{}^{a}\varepsilon_{ab}{}^{cd}e^b_\beta e^\mu_c e^\sigma_d(\bar\psi\gamma^5 D^L_\sigma\psi) \\
& - \frac{1}{4}(D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\sigma_a e^\mu_b)(\bar\psi\gamma^b D^L_\beta D^L_\sigma\psi) - \frac{1}{4l}(\bar\psi\sigma_\alpha{}^\sigma D^L_\beta D^L_\sigma\psi) - \frac{i}{2l}(D^L_\alpha e^a_\mu)e^\mu_a(\bar\psi D^L_\beta\psi) \\
& + \frac{7i}{48l^2}\varepsilon_{abc}{}^{d}e^a_\alpha e^b_\beta e^\sigma_d(\bar\psi\gamma^c\gamma^5 D^L_\sigma\psi) - \frac{i}{8}\eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon^{cdrs}e^\mu_c e^\nu_d e^\sigma_s(\bar\psi\gamma_r\gamma_5 D^L_\sigma\psi) \\
& + \frac{i}{12}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon_b{}^{cds}e^\mu_c e^\nu_d e^\sigma_s(\bar\psi\gamma_a\gamma_5 D^L_\sigma\psi) - \frac{1}{12l}e^c_\alpha(D^L_\beta e^b_\nu)\varepsilon_{bc}{}^{ds}e^\nu_d e^\sigma_s(\bar\psi\gamma_5 D^L_\sigma\psi) \\
& - \frac{1}{8l}(D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\sigma_a e^\mu_b)e^c_\beta(\bar\psi\sigma^b{}_c D^L_\sigma\psi) - \frac{1}{8l}(D^L_\alpha e^a_\mu)e^\mu_b(\bar\psi\sigma_a{}^b D^L_\beta\psi) \\
& + \frac{1}{96l}R_{\alpha\beta}{}^{ab}(\bar\psi\sigma_{ab}\psi) - \frac{5}{48l}R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta(\bar\psi\sigma_{bc}\psi) - \frac{1}{16l}R_{\alpha\mu}{}^{ab}e_{\beta a}e^\mu_c(\bar\psi\sigma_b{}^c\psi) \\
& - \frac{3}{32l^2}T_{\alpha\beta}{}^{a}(\bar\psi\gamma_a\psi) - \frac{1}{16l^2}T_{\alpha\mu}{}^{a}e^\mu_a(\bar\psi\gamma_\beta\psi) + \frac{1}{16l^2}T_{\alpha\mu}{}^{a}e_{\beta a}(\bar\psi\gamma^\mu\psi) \\
& + \frac{1}{12l}\eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(\bar\psi\sigma^{\mu\nu}\psi) - \frac{1}{6l}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)(\bar\psi\sigma^c{}_b\psi) \\
& - \frac{3}{16l^2}(D^L_\alpha e^a_\mu)e^\mu_a(\bar\psi\gamma_\beta\psi) + \frac{1}{16l^2}(D^L_\alpha e^a_\mu)e_{\beta a}(\bar\psi\gamma^\mu\psi) - \frac{1}{3l^3}(\bar\psi\sigma_{\alpha\beta}\psi)\Big] + c.c.
\end{aligned}
\tag{5.30}
$$

After WI contraction ($l \to \infty$) many terms in (5.30) vanish, for example, all terms of the type $(\bar\psi ... \psi)$, and we are left with a severely reduced action,

$$
\begin{aligned}
S^{(1)}_{\psi,kin}|_{\text{g.f.}} \overset{\text{WI}}{=} \theta^{\alpha\beta}\int \mathrm{d}^4x\, e\Big[& -\frac{1}{8}R_{\alpha\mu}{}^{ab}e^\mu_a(\bar\psi\gamma_b D^L_\beta\psi) - \frac{1}{16}R_{\alpha\beta}{}^{ab}e^\sigma_a(\bar\psi\gamma_b D^L_\sigma\psi) \\
& - \frac{i}{32}R_{\alpha\beta}{}^{ab}\varepsilon_{abc}{}^{d}e^\sigma_d(\bar\psi\gamma^c\gamma_5 D^L_\sigma\psi) - \frac{i}{16}R_{\alpha\mu}{}^{bc}e^\mu_a\varepsilon^a{}_{bcm}(\bar\psi\gamma^m\gamma_5 D^L_\beta\psi) \\
& - \frac{i}{24}R_{\alpha\mu}{}^{ab}\varepsilon_{abc}{}^{d}e^c_\beta(e^\mu_d e^\sigma_s - e^\mu_s e^\sigma_d)(\bar\psi\gamma^s\gamma_5 D^L_\sigma\psi) \\
& - \frac{1}{4}(D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\sigma_a e^\mu_b)(\bar\psi\gamma^b D^L_\beta D^L_\sigma\psi) \\
& - \frac{i}{8}\eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon^{cdrs}e^\mu_c e^\nu_d e^\sigma_s(\bar\psi\gamma_r\gamma_5 D^L_\sigma\psi) \\
& + \frac{i}{12}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon_b{}^{cds}e^\mu_c e^\nu_d e^\sigma_s(\bar\psi\gamma_a\gamma_5 D^L_\sigma\psi)\Big] + c.c.
\end{aligned}
\tag{5.31}
$$

5.4 NC deformation of the mass-like spinor terms

We now have to deal with the bilinear mass-like spinor actions (5.8). Their NC deformation proceeds in the same manner as for the kinetic spinor term, and after some tedious calculation that involves sequential application of the rules (5.22) and (5.25), we obtain the following results.

NC-deformed mass-like spinor action before gauge fixing is

$$
\begin{aligned}
S^{\star}_{\psi,m} = \frac{i}{2l}\left(m-\frac{2}{l}\right)\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\Big[\, & c_1\hat{\bar{\psi}}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi}\star\hat{\psi} \\
& +c_2\hat{\bar{\psi}}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\psi} \\
& +c_3\hat{\bar{\psi}}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\psi}\Big] + c.c.
\end{aligned}
\tag{5.32}
$$

Again, by using the SW map, we can represent this action as an expansion in powers of the deformation parameter $\theta^{\alpha\beta}$, taking only linear NC correction into account. Below, we present the result of this operation for each of the three mass-like terms, separately. We denote them by $S^{(1)}_{m,i}$ $(i=1,2,3)$.

The first mass-like term:

$$
\begin{aligned}
S^{(1)}_{m,1} = \frac{ic_1}{2l}\left(m-\frac{2}{l}\right)\theta^{\alpha\beta}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}\Big[& +\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi)D_\beta \\
& -\frac{1}{4}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi \\
& +\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi)D_\beta\phi \\
& +\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi)D_\beta(D_\rho\phi D_\sigma\phi)\phi\psi \\
& +\frac{i}{2}D_\mu\phi D_\nu\phi(D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi)\phi \\
& +\frac{i}{2}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\phi\phi\psi \\
& +\frac{1}{2}\{F_{\alpha\mu},D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\phi\phi \\
& +\frac{1}{2}D_\mu\phi\{F_{\alpha\nu},D_\beta\phi\}D_\rho\phi D_\sigma\phi\phi\psi \\
& +\frac{1}{2}D_\mu\phi D_\nu\phi\{F_{\alpha\rho},D_\beta\phi\}D_\sigma\phi\phi \\
& +\frac{1}{2}D_\mu\phi D_\nu\phi D_\rho\phi\{F_{\alpha\sigma},D_\beta\phi\}\phi\Big]\psi\,.
\end{aligned}
\tag{5.33}
$$

The second mass-like term:

$$
\begin{aligned}
S^{(1)}_{m,2} = \frac{ic_2}{2l}\left(m-\frac{2}{l}\right)\,\theta^{\alpha\beta}\,\int \mathrm{d}^4x\; \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}\Bigg[&+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi)D_\beta \\
&-\frac{1}{4}F_{\alpha\beta}D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi \\
&+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi\phi)(D_\beta D_\sigma\phi) \\
&+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)D_\beta\phi D_\sigma\phi \\
&+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)\phi D_\sigma\phi \\
&+\frac{i}{2}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi\phi D_\sigma\phi \\
&+\frac{1}{2}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi\phi D_\sigma\phi \\
&+\frac{1}{2}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi\phi D_\sigma\phi \\
&+\frac{1}{2}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}\phi D_\sigma\phi \qquad (5.34) \\
&+\frac{1}{2}D_\mu\phi D_\nu\phi D_\rho\phi\phi\{F_{\alpha\sigma}, D_\beta\phi\}\Bigg]\psi\;.
\end{aligned}
$$

The third mass-like term:

$$
\begin{aligned}
S^{(1)}_{m,3} = \frac{ic_3}{2l}\left(m-\frac{2}{l}\right)\,\theta^{\alpha\beta}\,\int \mathrm{d}^4x\; \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}\Bigg[&+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi)D_\beta \\
&-\frac{1}{4}F_{\alpha\beta}D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi \\
&+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi\phi)D_\beta(D_\rho\phi D_\sigma\phi) \\
&+\frac{i}{2}D_\alpha(D_\mu\phi D_\nu\phi)D_\beta\phi D_\rho\phi D_\sigma\phi \\
&+\frac{i}{2}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)\phi D_\rho\phi D_\sigma\phi \\
&+\frac{i}{2}D_\mu\phi D_\nu\phi\phi(D_\alpha D_\rho\phi)(D_\beta D_\sigma\phi) \\
&+\frac{1}{2}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi\phi D_\rho\phi D_\sigma\phi \\
&+\frac{1}{2}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}\phi D_\rho\phi D_\sigma\phi \\
&+\frac{1}{2}D_\mu\phi D_\nu\phi\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\phi \qquad (5.35) \\
&+\frac{1}{2}D_\mu\phi D_\nu\phi\phi D_\rho\phi\{F_{\alpha\sigma}, D_\beta\phi\}\Bigg]\psi\;.
\end{aligned}
$$

Assuming that none of the three mass-like spinor terms in (5.32) is more preferable than the other, and taking into account the classical constraint (5.10), we choose to set $c_1 = -c_2 = c_3 = \frac{1}{72}$.

After imposing the gauge fixing condition, and defining $a(m, l) := m - 2/l$, the complete first order NC correction to the sum of the three classical mass-like terms (5.8) becomes

$$S^{(1)}_{\psi,m}|_{\text{g.f.}} = a(m,l)\,\theta^{\alpha\beta} \int \mathrm{d}^4x\; e\; \bar{\psi}\Big[-\frac{i}{4}(D^L_\alpha e^a_\mu) e^\mu_a \overleftrightarrow{D}^L_\beta + \frac{1}{12}\eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}$$
$$- \frac{1}{3}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu) e^\mu_a e^\nu_c \sigma^c{}_b - \frac{1}{18l}(D^L_\alpha e^a_\mu)e^\mu_a \gamma_\beta - \frac{1}{6} R_{\alpha\mu}{}^{ab} e^\mu_a e^c_\beta \sigma_{bc}$$
$$- \frac{1}{48} R_{\alpha\beta}{}^{ab}\sigma_{ab} - \frac{1}{36l} T_{\alpha\beta}{}^a \gamma_a - \frac{7}{36l} T_{\alpha\mu}{}^a e^\mu_a \gamma_\beta - \frac{1}{4l^2}\sigma_{\alpha\beta}\Big]\psi\ . \tag{5.36}$$

In Appendix A we present the result for each mass-like term separately.

After WI contraction we are left with

$$S^{(1)}_{\psi,m}|_{\text{g.f.}} \overset{\text{WI}}{=} \theta^{\alpha\beta} \int \mathrm{d}^4x\; e\; \bar{\psi}\Big[-\frac{im}{4}(D^L_\alpha e^a_\mu) e^\mu_a \overleftrightarrow{D}^L_\beta + \frac{m}{12}\eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}$$
$$- \frac{m}{3}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu) e^\mu_a e^\nu_c \sigma^c{}_b - \frac{m}{48} R_{\alpha\beta}{}^{ab}\sigma_{ab} - \frac{m}{6} R_{\alpha\mu}{}^{ab} e^\mu_a e^c_\beta \sigma_{bc}\Big]\psi\ . \tag{5.37}$$

The total first order NC correction in the physical gauge is the sum of the kinetic spinor term (5.30) and the mass-like spinor term (5.36),

$$S^{(1)}_{\psi}|_{\text{g.f.}} = S^{(1)}_{\psi,kin}|_{\text{g.f.}} + S^{(1)}_{\psi,m}|_{\text{g.f.}}\ . \tag{5.38}$$

The result (5.38) represents *the first order NC correction to the classical Dirac action in curved space-time*. The action exhibits couplings of Dirac spinors and gravity that arise due to space-time noncommutativity. The fact that some terms, those in (5.31) and (5.37), survive WI contraction is of special significance. Namely, it is not entirely clear whether WI contraction, in general, commutes with the canonical NC deformation. In this particular case, one would have to calculate *directly* the NC correction to the Dirac action in curved space-time (by deforming $SO(1,3)$ gauge symmetry) and compare the result with (5.38).

Also, having a non-trivial linear NC correction enables us to explore potentially observable NC effects at the lowest perturbative order. We will see in the next section that the first order NC correction pertains even in Minkowski space.

5.5 NC Dirac equation in Minkowski space

Up until now, we have not talked about the metric of space-time explicitly. We know from Section 4 that the $SO(2,3)_\star$ model of NC gravity predicts quadratic deformation of the Minkowski metric. Therefore, it is justified to consider the flat space-time limit of the NC spinor-gravity action (5.38). After setting $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(\theta^2)$, the NC action (5.38) becomes

$$S^{(1)}_{\psi,\text{flat}} = \theta^{\alpha\beta}\int \mathrm{d}^4x\, \Big[-\frac{1}{2l}(\bar\psi\sigma_\alpha{}^\sigma\partial_\beta\partial_\sigma\psi) + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}(\bar\psi\gamma_\rho\gamma_5\partial_\sigma\psi) - M^3(\bar\psi\sigma_{\alpha\beta}\psi)\Big] \ , \tag{5.39}$$

where we introduced $M^3 := \frac{m}{4l^2} + \frac{1}{6l^3}$. This action is the linear NC correction to the Dirac action in Minkowski space. Note that it vanishes under WI contraction. Noncommutativity appears in the form of new terms in the action. One of them is the mass-like term with the mass matrix $M^3\theta^{\alpha\beta}\sigma_{\alpha\beta}$. The total NC-deformed action in Minkowski space up to first order in $\theta^{\alpha\beta}$ is

$$\begin{aligned} S^\star_{\psi,\text{flat}} = S^{(0)}_{\psi,\text{flat}} + S^{(1)}_{\psi,\text{flat}} = \int \mathrm{d}^4x\ \bar\psi(i\gamma^\mu\partial_\mu - m)\psi + \theta^{\alpha\beta}\int \mathrm{d}^4x\, \Big[-\frac{1}{2l}(\bar\psi\sigma_\alpha{}^\sigma\partial_\beta\partial_\sigma\psi) \\ + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}(\bar\psi\gamma_\rho\gamma_5\partial_\sigma\psi) - M^3(\bar\psi\sigma_{\alpha\beta}\psi)\Big] \ . \end{aligned} \tag{5.40}$$

The existence of the first order NC correction to the Dirac action in Minkowski space is a non-trivial, and a priori unexpected, consequence of the NC AdS model. It is important to note that we are working with "free" electrons (they interact only with NC gravity). Therefore, if we were to deform classical Dirac action $S^{(0)}_{\psi,\text{flat}}$ by directly inserting the Moyal $\star$-product (minimal substitution), it follows from (2.73) that action would remain the same.

From (5.40) we derive the Feynman propagator (in momentum space),

$$\begin{aligned} iS_F(p) &= \int d^4x\ \langle\Omega|T\psi(x)\bar\psi(0)|\Omega\rangle e^{ipx} \\ &= \frac{i}{\not p - m + i\epsilon} + \frac{i}{\not p - m + i\epsilon}(i\theta^{\alpha\beta}D_{\alpha\beta})\frac{i}{\not p - m + i\epsilon} + \dots \ , \end{aligned} \tag{5.41}$$

with

$$D_{\alpha\beta} := \frac{1}{2l}\sigma_\alpha{}^\sigma p_\beta p_\sigma + \frac{7}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5 p_\sigma - M^3\sigma_{\alpha\beta} \ . \tag{5.42}$$

The Feynman propagator is modified due to space-time noncommutativity. Therefore, we may say that electrons effectively interact with the NC background itself, in a similar manner in which they interact with a background electromagnetic field.

By varying (5.40) with respect to $\bar{\psi}$ we derive the NC-deformed Dirac equation for ψ in Minkowski space,

$$\left[i\partial\!\!\!/ - m - \frac{1}{2l}\theta^{\alpha\beta}\sigma_\alpha{}^\sigma\partial_\beta\partial_\sigma + \frac{7i}{24l^2}\theta^{\alpha\beta}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\partial_\sigma - \theta^{\alpha\beta}M^3\sigma_{\alpha\beta}\right]\psi = 0\ . \tag{5.43}$$

To simplify further analysis, we will assume that we have only two spatial dimensions that are mutually incompatible, e.g. $[x^1, x^2] = i\theta^{12}$. Therefore we have $\theta^{12} = -\theta^{21} =: \theta \neq 0$, with all other components of $\theta^{\alpha\beta}$ equal to zero.

The equation (5.43) reduces to

$$\left[i\partial\!\!\!/ - m - \frac{\theta}{2l}(\sigma_1{}^\sigma\partial_2\partial_\sigma - \sigma_2{}^\sigma\partial_1\partial_\sigma) + \frac{7i\theta}{12l^2}(\gamma_0\gamma_5\partial_3 - \gamma_3\gamma_5\partial_0) - 2\theta M^3\sigma_{12}\right]\psi = 0\ , \tag{5.44}$$

and we choose $\varepsilon^{0123} = 1$.

Now we want to find an NC version of the dispersion relation for Dirac fermions. Since hamiltonian commutes with the total momentum operator, we can assume the plane wave ansatz $\psi(x) = u(\mathbf{p})e^{-ip\cdot x}$, where $u(\mathbf{p})$ stands for a yet undetermined spinor amplitude

$$u(\mathbf{p}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\ . \tag{5.45}$$

With this choice, equation (5.44) can be represented in the momentum space as

$$\left(\begin{pmatrix} E - m & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -E - m \end{pmatrix} + \theta\mathcal{M}\right)u(\mathbf{p}) = 0\ , \tag{5.46}$$

with matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{pmatrix} A & \frac{1}{2l}p_zp_- & -\frac{7}{12l^2}p_z & \frac{1}{2l}Ep_- \\ \frac{1}{2l}p_zp_+ & -A & -\frac{1}{2l}Ep_+ & -\frac{7}{12l^2}p_z \\ \frac{7}{12l^2}p_z & \frac{1}{2l}Ep_- & B & \frac{1}{2l}p_zp_- \\ -\frac{1}{2l}Ep_+ & \frac{7}{12l^2}p_z & \frac{1}{2l}p_zp_+ & -B \end{pmatrix}\ . \tag{5.47}$$

Quantities E and $\mathbf{p}$ denote energy and momentum of a particle, respectively, and the matrix elements A and B are given by

$$A := -\frac{1}{2l}(p_x^2 + p_y^2) + \frac{7E}{12l^2} - 2M^3 \ ,$$
$$B := -\frac{1}{2l}(p_x^2 + p_y^2) - \frac{7E}{12l^2} - 2M^3 \ , \tag{5.48}$$

We use the Dirac representation of $\gamma-$matrices.

Non trivial solutions of the homogeneous matrix equation (5.46) which, when written explicitly, states that (we use $p_\pm = p_x \pm ip_y$)

$$\begin{pmatrix} E - m + \theta A & \frac{\theta}{2l}p_z p_- & -p_z - \frac{7\theta}{12l^2}p_z & -p_- + \frac{\theta}{2l}Ep_- \\ \frac{\theta}{2l}p_z p_+ & E - m - \theta A & -p_+ - \frac{\theta}{2l}Ep_+ & p_z - \frac{7\theta}{12l^2}p_z \\ p_z + \frac{7\theta}{12l^2}p_z & p_- + \frac{\theta}{2l}Ep_- & -E - m + \theta B & \frac{\theta}{2l}p_z p_- \\ p_+ - \frac{\theta}{2l}Ep_+ & -p_z + \frac{7\theta}{12l^2}p_z & \frac{\theta}{2l}p_z p_+ & -E - m - \theta B \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0 \ , \tag{5.49}$$

exist, if and only if, the determinant of the matrix $\not{p} - m + \theta\mathcal{M}$ (which is the matrix appearing in (5.49)) equals zero. This condition will give us the dispersion relation. The determinant depends on energy which is also represented as a perturbative expansion in θ,

$$E = \sum_{n=0}^{+\infty} E^{(n)} \ , \quad \text{where} \quad E^{(n)} \sim \frac{\theta^n}{(length)^{2n+1}} \ . \tag{5.50}$$

If the determinant equals zero, it is equal to zero order-by-order in θ, and we can derive the momentum dependence of $E^{(1)}$ correction which is enough to see how non-commutativity modifies the dispersion relation. To get higher-order energy terms, we need higher-order perturbative corrections to the Dirac action.

First, we will consider an electron moving along z-direction, i.e. in the direction orthogonal to the NC xy-plane. The matrix equation (5.49) reduces to

$$\begin{pmatrix} E - m + \theta A(0) & 0 & -p_z - \frac{7\theta}{12l^2}p_z & 0 \\ 0 & E - m - \theta A(0) & 0 & p_z - \frac{7\theta}{12l^2}p_z \\ p_z + \frac{7\theta}{12l^2}p_z & 0 & -E - m + \theta B(0) & 0 \\ 0 & -p_z + \frac{7\theta}{12l^2}p_z & 0 & -E - m - \theta B(0) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0 \, , \tag{5.51}$$

where $A(0) = A(p_x = p_y = 0)$ and, likewise, $B(0) = B(p_x = p_y = 0)$.

Non trivial solution for spinor components a, b, c, and d exist if at least one of the following two conditions is satisfied:

$$\left[E - m \pm \left(\frac{7E}{12l^2} - 2M^3\right)\theta\right]\left[E + m \pm \left(\frac{7E}{12l^2} + 2M^3\right)\theta\right] = \left[p_z \pm \frac{7p_z}{12l^2}\theta\right]^2 . \tag{5.52}$$

Four different solutions for the energy (up to the first order in θ) are

$$\begin{aligned} E_{1,2} &= E_{\mathbf{p}} \mp \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2) , \\ E_{3,4} &= -E_{\mathbf{p}} \pm \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} + \mathcal{O}(\theta^2) , \end{aligned} \tag{5.53}$$

with $E_{\mathbf{p}} = \sqrt{m^2 + p_z^2}$. This is reminiscent of the quantum Zeeman effect. The deformation parameter θ plays the role of a constant background magnetic field that causes the splitting of atomic energy levels.

In the rest frame ($\mathbf{p} = 0$) the energies reduce to:

$$\begin{aligned} E_{1,2}(0) &= m \mp \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta + \mathcal{O}(\theta^2) , \\ E_{3,4}(0) &= -m \pm \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta + \mathcal{O}(\theta^2) . \end{aligned} \tag{5.54}$$

We see that the mass of an electron gets "renormalized" due to space-time noncommutativity and the NC correction is linear in the deformation parameter.

By solving the matrix equation (5.51) for each of the four energy functions in (5.53), we get four linearly independent spinor solutions of the NC Dirac equation (up to a normalization factor):

$$\psi_1 \sim \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1 + \left(\frac{m}{12l^2} - \frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \end{pmatrix} e^{-iE_1t + ip_zz} ,$$

$$\psi_2 \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1 - \left(\frac{m}{12l^2} - \frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \end{pmatrix} e^{-iE_2t - ip_zz} ,$$

$$\psi_3 \sim \begin{pmatrix} \frac{p_z}{E_{\mathbf{p}}+m}\left[1+\left(\frac{m}{12l^2}-\frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-iE_3t-ip_zz} \ ,$$

$$\psi_4 \sim \begin{pmatrix} 0 \\ \frac{p_z}{E_{\mathbf{p}}+m}\left[1-\left(\frac{m}{12l^2}-\frac{1}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}}\right] \\ 0 \\ 1 \end{pmatrix} e^{-iE_4t+ip_zz} \ . \tag{5.55}$$

Spinors ψ_1 and ψ_2 (ψ_3 and ψ_4) correspond to positive (negative) energy solutions of the NC Dirac equation. Note that, in the commutative case, the opposite helicity ($\pm\frac{1}{2}$) solutions have *the same* energy. However, in the NC theory, the solutions with opposite helicity have *different* energies. The noncommutativity of space, here taken to be constrained to xy-plane, causes the undeformed energy levels $\pm E_{\mathbf{p}}$ to split. The energy gap between the new levels is the same for $\pm E_{\mathbf{p}}$ and it equals

$$2\left[\frac{m^2}{12l^2}-\frac{m}{3l^3}\right]\frac{\theta}{E_{\mathbf{p}}} \ . \tag{5.56}$$

From dispersion relations (5.53) we can derive the (group) velocity of an electron. This velocity is defined by

$$\boldsymbol{v} \equiv \frac{\partial E}{\partial \mathbf{p}} \ . \tag{5.57}$$

For positive (negative) helicity solution ψ_1 (ψ_2) we get

$$\boldsymbol{v}_{1,2} = \frac{\mathbf{p}}{E_{\mathbf{p}}}\left[1 \pm \left(\frac{m^2}{12l^2}-\frac{m}{3l^3}\right)\frac{\theta}{E_{\mathbf{p}}^2}+\mathcal{O}(\theta^2)\right] \ . \tag{5.58}$$

These velocities can be represented as

$$\boldsymbol{v}_{1,2} = \frac{\mathbf{p}}{E_{1,2}}+\mathcal{O}(\theta^2) \ . \tag{5.59}$$

Therefore, we may conclude that group velocity of an electron moving in the z-direction *depends on its helicity.* This is analogues to the *birefringence effect*, i.e. an optical property of a material having a refractive index that depends on the polarization and propagation direction of light. NC background acts as a birefringent medium for electrons propagating in it.

The Dirac spinor ψ_1 can now be represented as

$$\psi_1 \sim \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_1+E_1(0)} \\ 0 \end{pmatrix} e^{-iE_1 t + ip_z z} \ , \tag{5.60}$$

and in the rest frame

$$\psi_1(0) \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-iE_1(0)t} \ , \tag{5.61}$$

where

$$E_1(0) = m - \left[\frac{m}{12l^2} - \frac{1}{3l^3} \right] \theta \ . \tag{5.62}$$

The boost along z-direction in spinor representation is given by

$$S(\varphi) = \cosh\left(\frac{\varphi}{2}\right) I - \sinh\left(\frac{\varphi}{2}\right) \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \ , \tag{5.63}$$

where $v = \tanh(\varphi)$. If we take $v = -v_1 = -\frac{p_z}{E_1}$ we can boost the rest frame solution $\psi_1(p_z = 0)$ into the solution $\psi_1(p_z)$,

$$S(-p_z)\psi_1(0) = \psi_1(p_z) \ , \tag{5.64}$$

with the boost matrix

$$S(-p_z) = \sqrt{\frac{E_1(p_z) + E_1(0)}{2E_1(0)}} I + \sqrt{\frac{E_1(p_z) - E_1(0)}{2E_1(0)}} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \ . \tag{5.65}$$

This tells us that constant noncommutativity in the xy-plane is compatible with a Lorentz boost along z-direction. Similar statement holds for the other solutions.

By the same procedure we get NC-deformed energy levels of an electron moving in the NC xy-plane, i.e. an electron with momentum $\mathbf{p} = (p_x, p_y, 0)$,

$$\begin{aligned} E_{1,4} &= \pm E_{\mathbf{p}} - \left[\frac{m}{12l^2} - \frac{1}{3l^3} \right] \theta \ , \\ E_{2,3} &= \pm E_{\mathbf{p}} + \left[\frac{m}{12l^2} - \frac{1}{3l^3} \right] \theta \ , \end{aligned} \tag{5.66}$$

with $E_{\mathbf{p}} = \sqrt{m^2 + p_x^2 + p_y^2}$. Note that, in this case, the NC corrections do not depend on the momentum of an electron, as opposed to the NC corrections of the energy levels of an electron moving along z-direction. Again, these energy levels exactly reduces to (5.54) when $\mathbf{p} = 0$.

The four independent Dirac spinors are

$$\psi_1 \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p_+}{E_{\mathbf{p}}+m}\left[1 + \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \end{pmatrix} e^{-iE_1 t + ip_x x + ip_y y} \, ,$$

$$\psi_2 \sim \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E_{\mathbf{p}}+m}\left[1 - \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \\ 0 \end{pmatrix} e^{-iE_2 t + ip_x x + ip_y y} \, ,$$

$$\psi_3 \sim \begin{pmatrix} 0 \\ \frac{p_+}{E_{\mathbf{p}}+m}\left[1 + \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \\ 1 \\ 0 \end{pmatrix} e^{-iE_3 t - ip_x x - ip_y y} \, ,$$

$$\psi_4 \sim \begin{pmatrix} \frac{p_-}{E_{\mathbf{p}}+m}\left[1 - \left(\frac{7}{12l^2} - \frac{m}{12l}\right)\theta\right] \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-iE_4 t - ip_x x - ip_y y} \, . \tag{5.67}$$

It turns out that these solutions cannot be obtained by boosting the corresponding rest frame solutions. This was to be expected since, as we have already mentioned, by choosing the canonical noncommutativity we have effectively fixed the coordinate system. In other words, we work in a preferred coordinate system in which only boosts along z-axis and rotations around z-axis are preserved. With this observation we conclude the analysis of the Dirac field in the $SO(2,3)_\star$ model of NC gravity.

6 NC Electrodynamics

The content of this section is originally presented in [109].

In the previous section, we have demonstrated that AdS gauge theory of gravity has the capacity to consistently incorporate Dirac spinors, both classically and noncommutativelly. We studied canonicaly deformed dispersion relation for *free electrons*, that is, for electrons that interact only with NC gravity and not among themselves. To establish a complete theory of NC Electrodynamics, we have to include $U(1)$ gauge field in the $SO(2,3)$ framework. In the first order formalism, fermions couple naturally to the gravitational field, however, to couple gauge fields to the gravitational field one normally requires the Hodge dual operation. The definition of the Hodge dual requires an explicit use of the metric tensor, which means working in the second order formalism. This becomes even more significant in the AdS gauge theory where the basic dynamical variable is the $SO(2,3)$ gauge field that splits into Lorentz $SO(1,3)$ spin-connection and vierbein only after imposing the physical gauge. In this section, we present a classical $SO(2,3)\times U(1)$ gauge-invariant action that reduces to the kinetic action for $U(1)$ gauge field in the physical gauge. NC correction is derived in the usual way and it is linear in $\theta^{\mu\nu}$. Special attention is placed on the residual NC effects after WI contraction and in Minkowski space. We discuss how noncommutativity modifies relativistic Landau levels of an electron, in a constant background magnetic field.

6.1 $U(1)$ gauge field in AdS framework

To include electromagnetic field in $SO(2,3)$ framework, we upgrade the original $SO(2,3)$ gauge group to $SO(2,3)\times U(1)$. The general gauge potential Ω_μ consists of two independent parts,

$$\Omega_\mu = \omega_\mu + A_\mu \; . \tag{6.1}$$

The first part is the $SO(2,3)$ gauge field ω_μ that splits into $SO(1,3)$ spin-connection and vierbein, and the second part, A_μ, is the $U(1)$ gauge field. To Ω_μ we associate field strength

$$\mathbb{F}_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - i[\Omega_\mu,\Omega_\nu] = F_{\mu\nu} + \mathcal{F}_{\mu\nu} \; , \tag{6.2}$$

comprised of $SO(2,3)$ field strength $F_{\mu\nu} = \frac{1}{2}F_{\mu\nu}{}^{AB}M_{AB}$ and $U(1)$ field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; . \tag{6.3}$$

Following the approach of [110], we define $SO(2,3)\times U(1)$ gauge-invariant action for $U(1)$ gauge field,

$$S_A = -\frac{1}{16l}\mathrm{Tr}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\left(f\mathbb{F}_{\mu\nu}\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi + \frac{i}{3!}ff\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi\right) + c.c. \tag{6.4}$$

where $\mathcal{D}_\mu$ stands for $SO(2,3)\times U(1)$ covariant derivative. The action involves an additional auxiliary field f which is a $U(1)$-neutral AdS algebra-valued 0-form, transforming in the adjoint representation of $SO(2,3)$, that is

$$f = \frac{1}{2}f^{AB}M_{AB}\ ,\quad \delta_\epsilon f = i[\epsilon, f]\ . \tag{6.5}$$

with $\epsilon = \frac{1}{2}\epsilon^{AB}M_{AB}$. Its role is to provide the canonical kinetic term for $U(1)$ gauge field in the absence of the Hodge dual operation that can not be defined without explicitly introduction of the metric tensor. This, however, is not possible, given that metric is not explicit in S_A.

Field ϕ is also a $U(1)$ scalar, and its covariant derivative reduces to the AdS part,

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - i[\Omega_\mu,\phi] = \partial_\mu\phi - i[\omega_\mu,\phi] = D_\mu\phi\ . \tag{6.6}$$

This simplification is not a peculiarity of the Abelian group $U(1)$; it also holds in a more general case of non-Abelian Yang-Mills theory, as we shall see later.

The action (6.4) can be recast in a more explicit form,

$$\begin{aligned} S_A = &-\frac{1}{16l}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\Bigg(\frac{1}{4}f^{AB}F_{\mu\nu}{}^{CD}(D_\rho\phi)^E(D_\sigma\phi)^F\phi^G\ \mathrm{Tr}(M_{AB}M_{CD}\Gamma_E\Gamma_F\Gamma_G)\\ &+\frac{i}{24}f^{AB}f^{CD}(D_\mu\phi)^E(D_\nu\phi)^F(D_\rho\phi)^G(D_\sigma\phi)^H\phi^R\ \mathrm{Tr}(M_{AB}M_{CD}\Gamma_E\Gamma_F\Gamma_G\Gamma_H\Gamma_R)\\ &+\frac{1}{2}f^{AB}\mathcal{F}_{\mu\nu}(D_\rho\phi)^E(D_\sigma\phi)^F\phi^G\ \mathrm{Tr}(M_{AB}\Gamma_E\Gamma_F\Gamma_G)\Bigg) + c.c. \end{aligned} \tag{6.7}$$

After calculating traces (see Appendix D) we obtain

$$\begin{aligned} S_A = &-\frac{i}{32l}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\Bigg(f^{AB}F_{\mu\nu}{}^{CD}(D_\rho\phi)^E(D_\sigma\phi)^F\phi^G(\eta_{FG}\varepsilon_{ABCDE} + 2\eta_{AD}\varepsilon_{BCEFG})\\ &-2if^{AB}\mathcal{F}_{\mu\nu}(D_\rho\phi)^E(D_\sigma\phi)^F\phi^G\varepsilon_{ABEFG}\\ &-\frac{i}{6}f^{AB}f_{AB}(D_\mu\phi)^E(D_\nu\phi)^F(D_\rho\phi)^G(D_\sigma\phi)^H\phi^R\varepsilon_{EFGHR}\Bigg) + c.c. \end{aligned} \tag{6.8}$$

The first term in (6.8) is *purely imaginary*, and since we imposed the reality condition on S_A by adding its complex conjugate, this term does not contribute. Therefore, after the gauge fixing, when $(D_\mu\phi)^a = e_\mu^a$ and $(D_\mu\phi)^5 = 0$, the action reduces to

$$\begin{aligned} S_A|_{\text{g.f.}} &= -\frac{1}{8}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big(f^{ab}\mathcal{F}_{\mu\nu}\varepsilon_{abef}e^\epsilon_\rho e^f_\sigma + \frac{1}{12}f^{AB}f_{AB}\varepsilon_{efgh}e^e_\mu e^f_\nu e^g_\rho e^h_\sigma\Big) \\ &= \frac{1}{2}\int \mathrm{d}^4x\ e\ \Big(f^{ab}e^\mu_a e^\nu_b\mathcal{F}_{\mu\nu} + \frac{1}{2}(f^{ab}f_{ab} + 2f^{a5}f_a{}^5)\Big)\ , \end{aligned} \tag{6.9}$$

with the vierbein determinant $e = \det(e^a_\mu) = \sqrt{-g}$.

Equations of motion for the components f_{ab} and f_{a5} of the auxiliary field f are

$$f_{a5} = 0\ , \quad f_{ab} = -e^\mu_a e^\nu_b\mathcal{F}_{\mu\nu}\ . \tag{6.10}$$

Inserting them back into the action (6.9), we can eliminate the auxiliary field f. This leaves us with the well-known action for pure $U(1)$ gauge field in curved space-time,

$$S_A|_{\text{g.f.}} = -\frac{1}{4}\int \mathrm{d}^4x\ e\ g^{\mu\rho}g^{\nu\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} = -\frac{1}{4}\int \mathrm{d}^4x\ e\ \mathcal{F}^2\ . \tag{6.11}$$

6.2 Interacting Dirac fermions

Dirac spinor field has already been treated in detail in Section 5. Here we will simply generalize those results to include $U(1)$ gauge field, the only difference being an additional A_μ term in the total covariant derivative. In this manner, we introduce interaction between Dirac spinors mediated by A_μ. Dirac spinor field ψ transforms in the fundamental representation of $SO(2,3)\times U(1)$ gauge group,

$$\delta_\epsilon\psi = i\epsilon\psi = \frac{i}{2}\epsilon^{AB}M_{AB}\psi + i\alpha\psi\ , \tag{6.12}$$

where ϵ^{AB} are infinitesimal antisymmetric gauge parameters of $SO(2,3)$, and α is an infinitesimal $U(1)$ gauge parameter. The covariant derivative of the full $SO(2,3)\times U(1)$ gauge group in the fundamental representation is given by

$$\begin{aligned} \mathcal{D}_\mu\psi &= \partial_\mu\psi - i\Omega_\mu\psi = \partial_\mu\psi - i(\omega_\mu + A_\mu)\psi \\ &= \mathcal{D}^L_\mu\psi + \frac{i}{2}e^a_\mu\gamma_a\psi\ , \end{aligned} \tag{6.13}$$

where we introduced $SO(1,3)\times U(1)$ covariant derivative $\mathcal{D}^L_\mu = D^L_\mu - iA_\mu$ and we set $q = -1$ for an electron.

The fermionic action consists of two parts: the kinetic action

$$S_{\psi,kin} = \frac{i}{12}\int \mathrm{d}^4x\ \epsilon^{\mu\nu\rho\sigma}\left(\bar{\psi}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\psi - \mathcal{D}_\sigma\bar{\psi}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\psi\right)\ , \tag{6.14}$$

and the three mass-like action terms

$$\begin{aligned} S_{\psi,m} =& \frac{i}{144l}\left(m - \frac{2}{l}\right)\int \mathrm{d}^4x\ \epsilon^{\mu\nu\rho\sigma}\Big(\bar{\psi}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi\psi \\ &- \bar{\psi}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\phi\mathcal{D}_\sigma\phi\psi + \bar{\psi}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\psi\Big) + c.c. \end{aligned} \tag{6.15}$$

After the symmetry breaking, the total spinor action reduces to

$$S_\psi|_{\text{g.f.}} = S_{\psi,kin}|_{\text{g.f.}} + S_{\psi,m}|_{\text{g.f.}} = \int \mathrm{d}^4x\ e\left(i\bar{\psi}\gamma^\sigma(D^L_\sigma - iA_\sigma)\psi - m\bar{\psi}\psi\right)\ . \tag{6.16}$$

This is the Dirac action for charged fermions with mass m in curved space-time. Together with the $U(1)$ kinetic term (6.11), it constitutes the total action for classical electrodynamics in curved space-time.

6.3 Canonical deformation of AdS Electrodynamics

To establish an NC field theory with $SO(2,3)_\star \times U(1)_\star$ gauge group, we need NC spinor field $\hat{\psi}$, NC gauge potential $\hat{\Omega}_\mu$ and NC adjoint field $\hat{\phi}$. The corresponding NC field strength is defined as

$$\hat{\mathbb{F}}_{\mu\nu} = \partial_\mu\hat{\Omega}_\nu - \partial_\nu\hat{\Omega}_\mu - i[\hat{\Omega}_\mu \stackrel{\star}{,} \hat{\Omega}_\nu]\ . \tag{6.17}$$

The covariant derivatives of $\hat{\psi}$ and $\hat{\phi}$ are

$$\mathcal{D}_\mu\hat{\psi} = \partial_\mu\hat{\psi} - i\hat{\Omega}_\mu \star \hat{\psi}\ , \tag{6.18}$$

$$\mathcal{D}_\mu\hat{\phi} = \partial_\mu\hat{\phi} - i[\hat{\Omega}_\mu \stackrel{\star}{,} \hat{\phi}]\ . \tag{6.19}$$

Fields $\hat{\psi}$ and $\hat{\phi}$, along with their covariant derivatives (6.18) and (6.19), transform in the fundamental and adjoint representation, respectively, under infinitesimal NC gauge transformations,

$$\begin{aligned} &\delta^\star_\epsilon\hat{\psi} = i\hat{\Lambda}_\epsilon \star \hat{\psi}\ , \quad \delta^\star_\epsilon\mathcal{D}_\mu\hat{\psi} = i\hat{\Lambda}_\epsilon \star \mathcal{D}_\mu\hat{\psi}\ , \\ &\delta^\star_\epsilon\hat{\phi} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\phi}]\ , \quad \delta^\star_\epsilon\mathcal{D}_\mu\hat{\phi} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \mathcal{D}_\mu\hat{\phi}]\ . \end{aligned} \tag{6.20}$$

The transformation laws for NC gauge potential and field strength are

$$\delta^\star_\epsilon \hat{\Omega}_\mu = \partial_\mu \hat{\Lambda}_\epsilon + i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\Omega}_\mu] \ , \tag{6.21}$$

$$\delta^\star_\epsilon \hat{\mathbb{F}}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\mathbb{F}}_{\mu\nu}] \ . \tag{6.22}$$

In these variations, $\hat{\Lambda}_\epsilon$ is an NC gauge parameter of the full $SO(2,3)_\star \times U(1)_\star$ gauge group, and $\hat{\Lambda}_\epsilon = \frac{1}{2}\epsilon^{AB}M_{AB} + \alpha$, as $\theta^{\alpha\beta} \to 0$.

The SW map enables us to express NC fields in terms of the corresponding commutative fields, without introducing new degrees of freedom in the theory:

$$\hat{\psi} = \psi - \frac{1}{4}\theta^{\alpha\beta}\Omega_\alpha(\partial_\beta + \mathcal{D}_\beta)\psi + \mathcal{O}(\theta^2) \ , \tag{6.23}$$

$$\hat{\phi} = \phi - \frac{1}{4}\theta^{\alpha\beta}\{\Omega_\alpha, (\partial_\beta + \mathcal{D}_\beta)\phi\} + \mathcal{O}(\theta^2) \ , \tag{6.24}$$

$$\hat{\Omega}_\mu = \Omega_\mu - \frac{1}{4}\theta^{\alpha\beta}\{\Omega_\alpha, \partial_\beta\Omega_\mu + \mathbb{F}_{\beta\mu}\} + \mathcal{O}(\theta^2) \ . \tag{6.25}$$

Using these expansions, we can derive similar ones for the field strength, and covariant derivatives of spinor and adjoint field. They are given by

$$\hat{\mathbb{F}}_{\mu\nu} = \mathbb{F}_{\mu\nu} - \frac{1}{4}\theta^{\alpha\beta}\{\Omega_\alpha, (\partial_\beta + \mathcal{D}_\beta)\mathbb{F}_{\mu\nu}\} + \frac{1}{2}\theta^{\alpha\beta}\{\mathbb{F}_{\alpha\mu}, \mathbb{F}_{\beta\nu}\} + \mathcal{O}(\theta^2) \ , \tag{6.26}$$

$$\mathcal{D}_\mu\hat{\psi} = \mathcal{D}_\mu\psi - \frac{1}{4}\theta^{\alpha\beta}\Omega_\alpha(\partial_\beta + \mathcal{D}_\beta)\mathcal{D}_\mu\psi + \frac{1}{2}\theta^{\alpha\beta}\mathbb{F}_{\alpha\mu}\mathcal{D}_\beta\psi + \mathcal{O}(\theta^2) \ , \tag{6.27}$$

$$\mathcal{D}_\mu\hat{\phi} = \mathcal{D}_\mu\phi - \frac{1}{4}\theta^{\alpha\beta}\{\Omega_\alpha, (\partial_\beta + \mathcal{D}_\beta)\mathcal{D}_\mu\phi\} + \frac{1}{2}\theta^{\alpha\beta}\{\mathbb{F}_{\alpha\mu}, \mathcal{D}_\beta\phi\} + \mathcal{O}(\theta^2) \ . \tag{6.28}$$

A *non-expanded* NC action with $SO(2,3)_\star \times U(1)_\star$ gauge symmetry is obtained directly from the classical action (6.4) by introducing Moyal $\star$-product,

$$\begin{aligned} S^\star_A = -\frac{1}{16l}\mathrm{Tr}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big(&\hat{f}\star\hat{\mathbb{F}}_{\mu\nu}\star\mathcal{D}_\rho\hat{\phi}\star\mathcal{D}_\sigma\hat{\phi}\star\hat{\phi} \\ &+\frac{i}{3!}\hat{f}\star\hat{f}\star\mathcal{D}_\mu\hat{\phi}\star\mathcal{D}_\nu\hat{\phi}\star\mathcal{D}_\rho\hat{\phi}\star\mathcal{D}_\sigma\hat{\phi}\star\hat{\phi}\Big) + c.c. \end{aligned} \tag{6.29}$$

This NC action involves an auxiliary NC field $\hat{f}$, classically defined in (6.5), that transforms in the adjoint representation of the $SO(2,3)_\star \times U(1)_\star$ gauge group

$$\delta^\star_\epsilon \hat{f} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{f}] \ . \tag{6.30}$$

The transformation laws (6.20), (6.22) and (6.30) ensure the invariance of action (6.29) under $SO(2,3)_\star \times U(1)_\star$ NC gauge transformations.

Again, we use the general rule for calculating linear NC correction to $\star$-product of a pair of adjoint NC fields

$$\begin{aligned}\left(\hat{A}\star\hat{B}\right)^{(1)} =& -\frac{1}{4}\theta^{\alpha\beta}\{\Omega_\alpha,(\partial_\beta+\mathcal{D}_\beta)AB\}+\frac{i}{2}\theta^{\alpha\beta}\mathcal{D}_\alpha A\mathcal{D}_\beta B\\ &+cov(\hat{A}^{(1)})B+Acov(\hat{B}^{(1)})\ ,\end{aligned}\tag{6.31}$$

where $cov(\hat{A}^{(1)})$ is the covariant part of $A's$ first order NC correction, and $cov(\hat{B}^{(1)})$, the covariant part of $B's$ first order NC correction. After some simplification, including a few partial integrations, the first order NC correction before gauge fixing can be expressed as

$$\begin{aligned}S_A^{(1)} =&S_{Af}^{(1)}+S_{Aff}^{(1)}\\ =&\frac{\theta^{\alpha\beta}}{32l}\,\mathrm{Tr}\int \mathrm{d}^4x\;\varepsilon^{\mu\nu\rho\sigma}\Bigg(\frac{1}{2}\{\mathbb{F}_{\alpha\beta},f\}\mathbb{F}_{\mu\nu}\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi+if\mathcal{D}_\beta\mathbb{F}_{\mu\nu}\mathcal{D}_\alpha(\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi)\\ &-f\{\mathbb{F}_{\alpha\mu},\mathbb{F}_{\beta\nu}\}\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi-if\mathbb{F}_{\mu\nu}\mathcal{D}_\alpha(\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi)\mathcal{D}_\beta\phi\\ &-if\mathbb{F}_{\mu\nu}(\mathcal{D}_\alpha\mathcal{D}_\rho\phi)(\mathcal{D}_\beta\mathcal{D}_\sigma\phi)\phi-f\mathbb{F}_{\mu\nu}\{\{\mathbb{F}_{\alpha\rho},\mathcal{D}_\beta\phi\},\mathcal{D}_\sigma\phi\}\phi\\ &+\frac{i}{3!}\Bigg(\frac{1}{2}\{\mathbb{F}_{\alpha\beta},f^2\}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi-f^2\{[\{\mathbb{F}_{\alpha\mu},\mathcal{D}_\beta\phi\},\mathcal{D}_\nu\phi],\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\}\phi\\ &-if^2\Big(\mathcal{D}_\alpha(\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi)\mathcal{D}_\beta\phi+\mathcal{D}_\alpha(\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi)(\mathcal{D}_\beta\mathcal{D}_\sigma\phi)\phi\\ &+\big(\mathcal{D}_\alpha(\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi)(\mathcal{D}_\beta\mathcal{D}_\rho\phi)+(\mathcal{D}_\alpha\mathcal{D}_\mu\phi)(\mathcal{D}_\beta\mathcal{D}_\nu\phi)\mathcal{D}_\rho\phi\big)\mathcal{D}_\sigma\phi\phi\Big)\Bigg)\Bigg)+c.c.\ ,\end{aligned}\tag{6.32}$$

where we distinguish the *linear* f*-part* and the *quadratic* f^2*-part*, and all terms are *manifestly* $SO(2,3)\times U(1)$ gauge-invariant, by the virtue of SW map. Note also that $\mathcal{D}_\mu\phi=D_\mu\phi$, since ϕ is not charged under $U(1)$.

After imposing the physical gauge condition, the six terms of $S_{Af}^{(1)}$ are a bit lengthy, are we present them in Appendix B. The $S_{Aff}^{(1)}$ part is much simpler and it is given by

$$S_{Aff}^{(1)}|_{\text{g.f.}}=\frac{\theta^{\alpha\beta}}{8}\int\mathrm{d}^4x\;e\;\mathcal{F}_{\alpha\beta}f^2=\frac{\theta^{\alpha\beta}}{8}\int\mathrm{d}^4x\;e\;\mathcal{F}_{\alpha\beta}(f^{ab}f_{ab}+2f^{a5}f_a{}^5)\ .\tag{6.33}$$

The gravitational part (that which involves quantities like curvature and torsion) of $S_{Aff}^{(1)}|_{\text{g.f.}}$ turns out to be *purely imaginary*, and therefore provides no contribution after adding its complex conjugate.

Now we have to evaluate the gauge-fixed NC action $S^\star_A|_{\text{g.f.}} = S^{(0)}_A|_{\text{g.f.}} + S^{(1)}_A|_{\text{g.f.}}$ on the EoM of the auxiliary field f, up to first order in $\theta^{\alpha\beta}$. The EoM up to order $\mathcal{O}(\theta)$ are obtained by varying $S^\star_A|_{\text{g.f.}}$ over f_{ab} and f_{a5} independently. The on-shell first order action, denoted by $S^{(1)}_{A,EoM}|_{\text{g.f.}}$, has two contributions: the first one comes from evaluating $S^{(1)}_A|_{\text{g.f.}}$ on the classical (zero order) EoM for f, which we have already calculated (6.10). The second contribution comes from evaluating $S^{(0)}_A|_{\text{g.f.}}$ on the first order EoM for f. It is straightforward, although tedious, to compute the first order EoM, but as it turns out, this is not necessary. It can be readily checked that, if we work only up to first order, the classical action (6.9) gets annihilated after inserting the first order EoM for f, whatever they might be. Thus, we only need to insert classical EoM for f (6.10) in the first order action $S^{(1)}_A|_{\text{g.f.}}$, thus yielding

$$S^{(1)}_{A,EoM}|_{\text{g.f.}} = \sum_{j=1}^{6} S^{(1)}_{A,EoMf.j}|_{\text{g.f.}} + S^{(1)}_{A,EoMff}|_{\text{g.f.}} \ , \tag{6.34}$$

$$\begin{aligned}
S^{(1)}_{A,EoMf.1}|_{\text{g.f.}} &= \frac{\theta^{\alpha\beta}}{64}\int \mathrm{d}^4x\ e\ \Big\{\mathcal{F}^{\mu\nu}R_{\mu\nu ab}\big(R_{\alpha\beta}{}^{ab} - \frac{2}{l^2}e^a_\alpha e^b_\beta\big)\\
&+ \mathcal{F}^{\rho\sigma}e^a_\rho e^b_\sigma\big(R_{\mu\nu ab}R_{\alpha\beta}{}^{cd}e^\mu_c e^\nu_d - \frac{2}{l^2}R_{\alpha\beta ab}\big) + 4\mathcal{F}^{\rho\mu}e^c_\rho\big(R_{\mu\nu ac}R_{\alpha\beta}{}^{ab}e^\nu_b - \frac{2}{l^2}R_{\mu\beta ac}e^a_\alpha\big)\\
&+ \mathcal{F}_{\lambda\tau}e^\lambda_a e^\tau_b R_{\alpha\beta}{}^{ab}\big(R_{\mu\nu}{}^{mn}\ e^\mu_m e^\nu_n - \frac{12}{l^2}\big) + \frac{2}{l^2}\mathcal{F}^{\mu\nu}T_{\alpha\beta}{}^{a}\big(T_{\mu\nu a} - 2T_{\rho\nu m}e^\rho_a e^m_\mu\big)\\
&- \frac{2}{l^2}\mathcal{F}_{\alpha\beta}\big(R_{\mu\nu}{}^{mn}e^\mu_m e^\nu_n - \frac{12}{l^2} - 4l^2\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big)\Big\} + c.c.
\end{aligned} \tag{6.35}$$

$$\begin{aligned}
S^{(1)}_{A,EoMf.2}|_{\text{g.f.}} &= \frac{\theta^{\alpha\beta}}{8}\int \mathrm{d}^4x\ e\ \Big\{ - (D^L_\beta R_{\mu\nu}{}^{mc})(D^L_\alpha e^r_\rho)e^\lambda_c\big(e^\nu_m(\mathcal{F}_\lambda{}^\mu e^\rho_r - \mathcal{F}_\lambda{}^\rho e^\mu_r) + \mathcal{F}_\lambda{}^\nu e^\mu_r e^\rho_m\big)\\
&+ \frac{1}{l^2}\mathcal{F}_\rho{}^\mu e^\rho_c\big(D^L_\beta T_{\alpha\mu}{}^c - e_{\beta b}R_{\alpha\mu}{}^{bc}\big) - \frac{4}{l^2}\mathcal{F}_\nu{}^\mu(D^L_\alpha e^r_\rho)(D^L_\beta e^m_\mu)e^\nu_m e^\rho_r\\
&- \frac{1}{l^2}(D^L_\alpha e^r_\rho)e^\rho_r\big(e^\nu_c\mathcal{F}_\beta{}^\mu T_{\mu\nu}{}^c - e^\nu_c\mathcal{F}_\nu{}^\mu T_{\mu\beta}{}^c\big) + \frac{1}{2l^2}T_{\alpha\beta}{}^r T_{\mu\nu}{}^c\mathcal{F}_\lambda{}^\nu e^\lambda_c e^\mu_r\\
&+ \frac{1}{l^2}(D^L_\alpha e^r_\rho)e^\nu_r\Big(T_{\mu\nu}{}^m\big(\mathcal{F}_\beta{}^\mu e^\rho_m - \mathcal{F}_\beta{}^\rho e^\mu_m\big) + T_{\beta\nu}{}^m\mathcal{F}_\mu{}^\rho e^\mu_m\Big)\\
&+ \frac{2}{l^2}\mathcal{F}_\lambda{}^\rho e^\lambda_c(D^L_\alpha e^r_\rho)(D^L_\beta e^c_\nu)e^\nu_r + \frac{1}{l^4}\mathcal{F}_{\alpha\beta}\Big\} + c.c.
\end{aligned} \tag{6.36}$$

$$\begin{aligned}
S^{(1)}_{A,EoMf.3}|_{\text{g.f.}} &= -\frac{\theta^{\alpha\beta}}{32}\int \mathrm{d}^4x\ e\ \Big\{\mathcal{F}^{\mu\nu}\Big(R_{\beta\nu am}\big(R_{\alpha\mu}{}^{am} - \frac{4}{l^2}e^a_\alpha e^m_\mu\big) + 8\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}\Big)\\
&+ \mathcal{F}_{\lambda\tau}R_{\alpha\mu}{}^{am}R_{\beta\nu}{}^{bn}\big(e^\mu_a e^\nu_m e^\lambda_b e^\tau_n + e^\lambda_a e^\tau_m e^\mu_b e^\nu_n + 2e^\lambda_n e^\tau_m(e^\mu_a e^\nu_b - e^\nu_a e^\mu_b)\big)\\
&+ \frac{2}{l^2}e^\lambda_n e^\tau_b\big(2\mathcal{F}_{\alpha\tau}R_{\beta\lambda}{}^{bn} - \mathcal{F}_{\lambda\tau}R_{\alpha\beta}{}^{bn}\big) + \frac{2}{l^2}\mathcal{F}^{\mu\nu}T_{\alpha\mu}{}^{a}T_{\beta\nu a}\Big\} + c.c.
\end{aligned} \tag{6.37}$$

$$
\begin{aligned}
&S^{(1)}_{A,EoMf.4}|_{\text{g.f.}} + S^{(1)}_{A,EoMf.5}|_{\text{g.f.}} = \frac{\theta^{\alpha\beta}}{32}\int \mathrm{d}^4x\; e\,\Big\{ \\
&+ R_{\mu\nu}{}^{ab}(D^L_\alpha e^m_\rho)(D^L_\beta e_{\sigma m})\Big(\mathcal{F}^{\mu\nu}e^\rho_a e^\sigma_b + \mathcal{F}^{\rho\sigma}e^\mu_a e^\nu_b - 4\mathcal{F}^{\mu\rho}e^\nu_a e^\sigma_b\Big) \\
&- \frac{1}{l^2}R_{\mu\nu}{}^{ab}\big(\mathcal{F}^{\mu\nu}e_{\alpha a}e_{\beta b} + \mathcal{F}_{\alpha\beta}e^\mu_a e^\nu_b + 4\mathcal{F}^\mu{}_\beta e^\nu_a e^\sigma_b\big) - \frac{1}{l^2}\mathcal{F}^{\rho\sigma}(D^L_\alpha e^m_\rho)(D^L_\beta e_{\sigma m}) \\
&+ \frac{2}{l^2}T_{\mu\nu}{}^c(D^L_\alpha e^d_\rho)\Big(\mathcal{F}^{\mu\nu}(2e^\rho_c e_{\beta d} - e^\rho_d e_{\beta c}) + 2\mathcal{F}_\beta{}^\mu(e^\rho_c e^\nu_d - e^\rho_d e^\nu_c) \\
&+ 2\mathcal{F}^{\rho\mu}(2e^\nu_c e_{\beta d} - e^\nu_d e_{\beta c}) + 2\mathcal{F}^\rho{}_\beta e^\mu_c e^\nu_d\Big) - \frac{2}{l^2}\mathcal{F}^{\rho\mu}T_{\mu\nu c}T_{\alpha\beta}{}^d e^c_\rho e^\nu_d \\
&- \frac{4}{l^2}T_{\beta\nu a}(D^L_\alpha e^d_\rho)e^a_\lambda\big(\mathcal{F}^{\lambda\nu}e^\rho_d - \mathcal{F}^{\lambda\rho}e^\nu_d\big) + \frac{4}{l^2}R_{\beta\nu}{}^c{}_a e^a_\lambda\big(\mathcal{F}^{\lambda\nu}e_{\alpha c} - \mathcal{F}^\lambda{}_\alpha e^\nu_c\big) - \frac{6}{l^4}\mathcal{F}_{\alpha\beta}\Big\} + c.c.
\end{aligned}
\tag{6.38}
$$

$$
\begin{aligned}
S^{(1)}_{A,EoMf.6}|_{\text{g.f.}} = &-\frac{\theta^{\alpha\beta}}{32}\int \mathrm{d}^4x\; e\,\Big\{\mathcal{F}^{\kappa\xi}e^a_\kappa e^b_\xi e^\mu_e e^\nu_f\big(R_{\alpha\beta}{}^{ef}R_{\mu\nu ab} - 2R_{\beta\nu}{}^{ef}R_{\alpha\mu ab}\big) \\
&+ 8\mathcal{F}^{\mu\nu}\big(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - 2\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}\big) + \frac{2}{l^2}\mathcal{F}^{\mu\nu}e^a_\mu e^b_\nu R_{\alpha\beta ab} - \frac{4}{l^2}\mathcal{F}_{\beta\nu}\epsilon^\nu_a e^\rho_b R_{\alpha\rho}{}^{ab} \\
&+ \frac{4}{l^2}\mathcal{F}^{\kappa\mu}e^a_\kappa e^\rho_e\big(T_{\rho\mu a}T_{\alpha\beta}{}^e + T_{\mu\beta a}T_{\alpha\rho}{}^e + T_{\beta\rho a}T_{\alpha\mu}{}^e\big) - \frac{8}{l^4}\mathcal{F}_{\alpha\beta}\Big\} + c.c.
\end{aligned}
\tag{6.39}
$$

And finally, the f^2-term,

$$
S^{(1)}_{A,EoMff}|_{\text{g.f.}} = \frac{\theta^{\alpha\beta}}{8}\int \mathrm{d}^4x\; e\;\mathcal{F}_{\alpha\beta}\mathcal{F}^2\;. \tag{6.40}
$$

Again we have non-trivial WI contraction. After performing contractions and some simplification, we obtain

$$
\begin{aligned}
S^{(1)}_A \overset{\text{WI}}{=} &\frac{\theta^{\alpha\beta}}{32}\int \mathrm{d}^4x\; e\,\Big\{\mathcal{F}^{\mu\nu}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} - R_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma} \\
&- 4\mathcal{F}^{\mu\rho}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\nu\sigma} + R_{\alpha\beta}{}^{\rho\sigma}\mathcal{F}_{\rho\sigma}R - 4\mathcal{F}^{\rho\sigma}R_{\alpha\sigma}R_{\beta\rho} \\
&- 4\mathcal{F}_{\rho\sigma}R_{\alpha\mu}{}^{\nu\rho}R_{\beta\nu}{}^{\mu\sigma} - 2\mathcal{F}^{\mu\nu}R_{\alpha\mu}{}^{\rho\sigma}R_{\beta\nu\rho\sigma} - 4\mathcal{F}_{\alpha\beta}\mathcal{F}^2 + 16\mathcal{F}^{\mu\nu}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu} \\
&- 8(D^L_\beta R_{\mu\nu}{}^{mc})(D^L_\alpha e^r_\rho)e^\lambda_c\Big(\mathcal{F}_\lambda{}^\mu e^\nu_m e^\rho_r - \mathcal{F}_\lambda{}^\rho e^\mu_r e^\nu_m + \mathcal{F}_\lambda{}^\nu e^\mu_r e^\rho_m\Big) \\
&+ 2R_{\mu\nu}{}^{ab}(D^L_\alpha e^m_\rho)(D^L_\beta e_{\sigma m})\Big(\mathcal{F}^{\mu\nu}e^\rho_a e^\sigma_b + \mathcal{F}^{\rho\sigma}e^\mu_a e^\nu_b - 4\mathcal{F}^{\mu\rho}e^\nu_a e^\sigma_b\Big)\Big\}\;.
\end{aligned}
\tag{6.41}
$$

If canonical deformation and WI contraction commute, we should obtain the same result for the NC action defined by minimal substitution $e\; g^{\mu\rho}g^{\nu\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} \to e \star g^{\mu\rho} \star g^{\nu\sigma} \star \mathcal{F}_{\mu\nu} \star \mathcal{F}_{\rho\sigma}$ under the integral. This remains to be seen.

6.4 NC theory of interacting Dirac fermions

The $SO(2,3)_\star$ model of *free* (not coupled to $U(1)$ gauge field) Dirac spinor field has already been treated in Section 5 and the first order NC correction has been calculated (5.38). To include $U(1)$-interaction, we generalize this result by making substitutions $D^L_\mu \to \mathcal{D}^L_\mu = D^L_\mu - iA_\mu$ and $F_{\mu\nu} \to \mathbb{F}_{\mu\nu} = F_{\mu\nu} + \mathcal{F}_{\mu\nu}$, yielding

$$\begin{aligned}
S^{(1)}_{\psi,m}|_{\text{g.f.}} &= \frac{\theta^{\alpha\beta}}{4}\left(m-\frac{2}{l}\right)\int \mathrm{d}^4x\; e\;\bar{\psi}\,\Big\{ -i(D^L_\alpha e^a_\mu)e^\mu_a\mathcal{D}^L_\beta + \frac{1}{6}\eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu} \\
&- \frac{1}{3}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b - \frac{1}{9l}(D^L_\alpha e^a_\mu)e^\mu_a\gamma_\beta - \frac{1}{24}R_{\alpha\beta}{}^{ab}\sigma_{ab} \\
&- \frac{1}{3}R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc} - \frac{1}{18l}T_{\alpha\beta}{}^a\gamma_a - \frac{7}{18l}T_{\alpha\mu}{}^a e^\mu_a\gamma_\beta - \frac{1}{2l^2}\sigma_{\alpha\beta} - \frac{3}{2}\mathcal{F}_{\alpha\beta}\Big\}\,\psi + c.c. \qquad (6.42)
\end{aligned}$$

$$\begin{aligned}
S^{(1)}_{\psi,kin}|_{\text{g.f.}} &= \frac{\theta^{\alpha\beta}}{8}\int \mathrm{d}^4x\; e\;\bar{\psi} \\
\Big\{ &- R_{\alpha\mu}{}^{ab}\Big(e^\mu_a\gamma_b + \frac{i}{2}e^\mu_c\varepsilon^c{}_{abd}\;\gamma^d\gamma^5\Big)\mathcal{D}^L_\beta + \frac{1}{2}R_{\alpha\beta}{}^{ab}\Big(e^\sigma_b\gamma_a - \frac{i}{2}\varepsilon_{abc}{}^d e^\sigma_d\;\gamma^c\gamma^5\Big)\mathcal{D}^L_\sigma \\
&- \frac{i}{3}R_{\alpha\mu}{}^{ab}\varepsilon_{abc}{}^d e^c_\beta(e^\mu_d e^\sigma_s - e^\mu_s e^\sigma_d)\;\gamma^s\gamma^5\mathcal{D}^L_\sigma + \frac{i}{l}T_{\alpha\mu}{}^a\Big(e^\mu_a - ie^\mu_b\sigma_a{}^b\Big)\mathcal{D}^L_\beta \\
&- \frac{i}{l}T_{\alpha\beta}{}^a\Big(e^\sigma_a + \frac{i}{2}e^\mu_a\sigma_\mu{}^\sigma\Big)\mathcal{D}^L_\sigma - \frac{2}{3l}T_{\alpha\mu}{}^a\varepsilon_{ab}{}^{cd}e^b_\beta e^\mu_c e^\sigma_d\gamma^5\mathcal{D}^L_\sigma + \frac{7i}{6l^2}\varepsilon_{abc}{}^d e^a_\alpha e^b_\beta e^\sigma_d\gamma^c\gamma^5\mathcal{D}^L_\sigma \\
&- 2\Big((D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\sigma_a e^\mu_b)\;\gamma^b + \frac{1}{l}\sigma_\alpha{}^\sigma\Big)\mathcal{D}^L_\beta\mathcal{D}^L_\sigma - \frac{2}{3l}e^c_\alpha(D^L_\beta e^b_\nu)\varepsilon_{bc}{}^{ds}e^\nu_d e^\sigma_s\gamma_5\mathcal{D}^L_\sigma \\
&+ i(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)e^\mu_c e^\nu_d e^\sigma_s\Big(\frac{2}{3}\varepsilon_b{}^{cds}\gamma_a - \eta_{ab}\varepsilon^{cdrs}\gamma_r\Big)\gamma_5\mathcal{D}^L_\sigma \\
&- \frac{1}{l}(D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\sigma_a e^\mu_b)e^c_\beta\sigma^b{}_c\mathcal{D}^L_\sigma - \frac{1}{l}(D^L_\alpha e^a_\mu)\Big(4ie^\mu_a + e^\mu_b\sigma_a{}^b\Big)\mathcal{D}^L_\beta \\
&+ \frac{1}{16l}R_{\alpha\beta}{}^{ab}\sigma_{ab} - \frac{1}{2l}R_{\alpha\mu}{}^{ab}\Big(\frac{5}{3}e^\mu_a e^c_\beta + e_{\beta a}e^{\mu c}\Big)\sigma_{bc} \\
&- \frac{3}{4l^2}T_{\alpha\beta}{}^a\gamma_a - \frac{1}{2l^2}T_{\alpha\mu}{}^a\Big(e^\mu_a\gamma_\beta - e_{\beta a}\gamma^\mu\Big) - \frac{8}{3l^3}\sigma_{\alpha\beta} \\
&+ \frac{2}{3l}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\Big(\eta_{ab}\sigma^{\mu\nu} - 2(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b\Big) - \frac{1}{2l^2}(D^L_\alpha e^a_\mu)\Big(3e^\mu_a\gamma_\beta - e_{\beta a}\gamma^\mu\Big) \\
&+ 3i\mathcal{F}_{\alpha\beta}e^\sigma_s\gamma^s\mathcal{D}^L_\sigma - 2i\mathcal{F}_{\alpha\mu}e^\mu_m\gamma^m\mathcal{D}^L_\beta - \frac{5}{l}\mathcal{F}_{\alpha\beta}\Big\}\,\psi + c.c. \qquad (6.43)
\end{aligned}$$

Putting the pieces together, we come to the final result: *the linear NC deformation of the classical $U(1)$ gauge field theory in curved space-time,*

$$S^\star|_{\text{g.f.}} = S^{(0)}|_{\text{g.f.}} + S^{(1)}_{\psi,kin}|_{\text{g.f.}} + S^{(1)}_{\psi,m}|_{\text{g.f.}} + S^{(1)}_{A,EoMf}|_{\text{g.f.}} + S^{(1)}_{A,EoMff}|_{\text{g.f.}}\;. \qquad (6.44)$$

Action $S^\star|_{\text{g.f.}}$, describing NC Electrodynamics, pertains even in Minkowski space.

6.5 NC Electrodynamics in Minkowski space

There are some residual effects of space-time noncommmutativity coming from the terms in $S^\star|_{\text{g.f.}}$ that survive the flat space-time limit. Therefore, from now on we work in Minkowski space.

The action for NC electrodynamics in Minkowski space, up to the first order in $\theta^{\alpha\beta}$, is given by

$$\begin{aligned}
S^\star_{flat} =&S^{(0)}_{flat} + S^{(1)}_{flat} = \int \mathrm{d}^4x\ \bar\psi(i\tilde{\not{\mathcal{D}}} - m)\psi - \frac{1}{4}\int \mathrm{d}^4x\ \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \\
&+ \theta^{\alpha\beta}\int \mathrm{d}^4x \left(\frac{1}{2}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}\mathcal{F}^{\mu\nu} - \frac{1}{8}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right) \\
&+ \theta^{\alpha\beta}\int \mathrm{d}^4x\ \bar\psi\Bigg(- \frac{1}{2l}\sigma_\alpha{}^\sigma\widetilde{\mathcal{D}}_\beta\widetilde{\mathcal{D}}_\sigma + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\widetilde{\mathcal{D}}_\sigma \\
&- \left(\frac{m}{4l^2} + \frac{1}{6l^3}\right)\sigma_{\alpha\beta} + \frac{3i}{4}\mathcal{F}_{\alpha\beta}\widetilde{\not{\mathcal{D}}} - \frac{i}{2}\mathcal{F}_{\alpha\mu}\gamma^\mu\widetilde{\mathcal{D}}_\beta - \left(\frac{3m}{4} - \frac{1}{4l}\right)\mathcal{F}_{\alpha\beta}\Bigg)\psi\ ,
\end{aligned} \tag{6.45}$$

where we introduced *flat* space-time covariant derivative $\widetilde{\mathcal{D}}_\mu := \partial_\mu - iA_\mu$. This action is obviously different from the other actions describing NC Electrodynamics already present in the literature [111–113]. The new terms stem from the residual interaction with NC gravity and they will trigger some non-trivial phenomena, such as deformed relativistic Landau levels of an electron. Also, we point out that not all of these terms vanish under WI contraction.

6.6 Deformed equations of motion

By varying (6.45) over A_ρ we obtain NC Maxwell equation with sources. Up to to first order, the equation is given by

$$\partial_\mu\mathcal{F}^{\mu\rho} - \frac{1}{4}\theta^{\alpha\beta}\mathcal{F}_{\alpha\beta}\partial_\mu\mathcal{F}^{\mu\rho} - \frac{1}{2}\theta^{\alpha\rho}\mathcal{F}_{\alpha\nu}\partial_\mu\mathcal{F}^{\mu\nu} + \theta^{\alpha\beta}\partial_\mu(\mathcal{F}_\alpha{}^\mu\mathcal{F}_\beta{}^\rho) \tag{6.46}$$

$$\begin{aligned}
&= -\bar\psi\gamma^\rho\psi - \frac{i}{2l}\theta^{\alpha\rho}\bar\psi\sigma_\alpha{}^\sigma\mathcal{D}_\sigma\psi - \frac{i}{2l}\theta^{\alpha\beta}\bar\psi\sigma_\alpha{}^\rho\mathcal{D}_\beta\psi + \frac{i}{2l}\theta^{\alpha\beta}\partial_\beta(\bar\psi\sigma_\alpha{}^\rho\psi) \\
&- \frac{7}{24l^2}\theta^{\alpha\beta}\epsilon_{\alpha\beta}{}^{\lambda\rho}\bar\psi\gamma_\lambda\gamma_5\psi - \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha(\bar\psi\gamma^\rho\mathcal{D}_\beta\psi) + \frac{i}{2}\theta^{\rho\beta}\partial_\mu(\bar\psi\gamma^\mu\mathcal{D}_\beta\psi) + \frac{1}{2l}\theta^{\alpha\rho}\partial_\alpha(\bar\psi\psi) \\
&\overset{\text{WI}}{=} -\bar\psi\gamma^\rho\psi - \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha(\bar\psi\gamma^\rho\mathcal{D}_\beta\psi) + \frac{i}{2}\theta^{\rho\beta}\partial_\mu(\bar\psi\gamma^\mu\mathcal{D}_\beta\psi)\ .
\end{aligned} \tag{6.47}$$

The analysis of this equation remains to be done.

By varying NC action (6.45) over $\bar{\psi}$ we obtain a deformed Dirac equation for an electron coupled to electromagnetic field:

$$\left(i\partial\!\!\!/ - m + A\!\!\!/ + \theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}\right)\psi = 0 \ , \tag{6.48}$$

where $\theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}$ is given by

$$\begin{aligned}\theta^{\alpha\beta}\mathcal{M}_{\alpha\beta} = \theta^{\alpha\beta}\Bigg\{ & -\frac{1}{2l}\sigma_\alpha{}^\sigma\widetilde{\mathcal{D}}_\beta\widetilde{\mathcal{D}}_\sigma + \frac{7i}{24l^2}\epsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\widetilde{\mathcal{D}}_\sigma - \left(\frac{m}{4l^2} + \frac{1}{6l^3}\right)\sigma_{\alpha\beta} \\ & + \frac{3i}{4}\mathcal{F}_{\alpha\beta}\widetilde{\mathcal{D}\!\!\!/} - \frac{i}{2}\mathcal{F}_{\alpha\mu}\gamma^\mu\widetilde{\mathcal{D}}_\beta - \left(\frac{3m}{4} - \frac{1}{4l}\right)\mathcal{F}_{\alpha\beta}\Bigg\} \ . \end{aligned} \tag{6.49}$$

From (6.49) we see that there are some new interaction terms in (6.48). For an electron, we set $q = -1$.

6.7 Electron in a background magnetic field

Using the NC-deformed Dirac equation (6.48) we can analyze the special case of an electron propagating in constant magnetic field $\mathbf{B} = B\mathbf{e}_z$. Accordingly, we choose $A_\mu = (0, By, 0, 0)$. An appropriate ansatz for (6.48) is

$$\psi = \begin{pmatrix} \varphi(y) \\ \chi(y) \end{pmatrix} e^{-iEt + ip_x x + ip_z z} \ . \tag{6.50}$$

Spinor components and energy function are all represented as perturbation series in powers of the deformation parameter,

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \mathcal{O}(\theta^2) \ , \tag{6.51}$$

$$\chi = \chi^{(0)} + \chi^{(1)} + \mathcal{O}(\theta^2) \ , \tag{6.52}$$

$$E = E^{(0)} + E^{(1)} + \mathcal{O}(\theta^2) \ . \tag{6.53}$$

Inserting the ansatz (6.50) in the Dirac equation (6.48) yields

$$\left[E\gamma^0 - p_x\gamma^1 + i\gamma^2\frac{d}{dy} - p_z\gamma^3 - m + By\gamma^1 + \theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}\right]\begin{pmatrix} \varphi(y) \\ \chi(y) \end{pmatrix} = 0 \ . \tag{6.54}$$

The zeroth order (undeformed) equation is therefore

$$\left[E^{(0)}\gamma^0 - p_x\gamma^1 + i\gamma^2\frac{d}{dy} - p_z\gamma^3 - m + By\gamma^1\right]\begin{pmatrix}\varphi^{(0)}\\ \chi^{(0)}\end{pmatrix} = 0 \ , \tag{6.55}$$

and at the first order we have

$$\begin{aligned}&\left[E^{(0)}\gamma^0 - p_x\gamma^1 + i\gamma^2\frac{d}{dy} - p_z\gamma^3 - m + By\gamma^1\right]\begin{pmatrix}\varphi^{(1)}\\ \chi^{(1)}\end{pmatrix}\\ &= -\left[E^{(1)}\gamma^0 + \theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}\right]\begin{pmatrix}\varphi^{(0)}\\ \chi^{(0)}\end{pmatrix} \ .\end{aligned} \tag{6.56}$$

Taking the adjoint of (6.55),

$$\bar{\psi}^{(0)}\left[E^{(0)}\gamma^0 - p_x\gamma^1 - i\gamma^2\frac{\overleftarrow{d}}{dy} - p_z\gamma^3 - m + By\gamma^1\right] = 0 \ , \tag{6.57}$$

multiplying (6.56) by $\bar{\psi}^{(0)}$ from the left and integrating over y, yields

$$E^{(1)}\int \mathrm{d}y\ \bar{\psi}^{(0)}\gamma^0\psi^{(0)} = -\theta^{\alpha\beta}\int \mathrm{d}y\ \bar{\psi}^{(0)}\mathcal{M}_{\alpha\beta}\psi^{(0)} \ .$$

Therefore, the first order NC correction to the energy of an electron is simply

$$E^{(1)} = \frac{-\theta^{\alpha\beta}\int \mathrm{d}y\ \bar{\psi}^{(0)}\mathcal{M}_{\alpha\beta}\psi^{(0)}}{\int \mathrm{d}y\ \bar{\psi}^{(0)}\gamma^0\psi^{(0)}}. \tag{6.58}$$

We us calculate explicitly the zeroth order solution $\psi^{(0)}$. From the unperturbed equation (6.55) we derive the equation for $\varphi^{(0)}$ spinor component. It is given by

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (p_x - By)^2 + (E^{(0)})^2 - p_z^2 - m^2 - B\sigma_3\right]\varphi^{(0)}(y) = 0 \ .$$

The unperturbed energy levels of an electron in constant magnetic field (*relativistic Landau levels*) are

$$E^{(0)}_{n,s} = \sqrt{p_z^2 + m^2 + (2n + s + 1)B} \ , \tag{6.59}$$

where $n = 0, 1, 2, \ldots$ is the principal quantum number, while $s = \pm 1$ is the eigenvalue of the Pauli matrix σ_3.

The complete undeformed Dirac spinor is

$$\psi_{n,s}^{(0)} = \begin{pmatrix} \varphi_{n,s}^{(0)} \\ \chi_{n,s}^{(0)} \end{pmatrix} e^{-iE_{n,s}^{(0)}t+ip_x x+ip_z z} , \tag{6.60}$$

with components

$$\varphi_{n,s}^{(0)} = \Phi_n \varphi_s , \tag{6.61}$$

$$\chi_{n,s}^{(0)} = \frac{1}{E_{n,s}^{(0)}+m}\left[-\sqrt{\frac{B}{2}}\left(\sqrt{n+1}\Phi_{n+1}\sigma_- - \sqrt{n}\Phi_{n-1}\sigma_+\right) + p_z\Phi_n\sigma_3\right]\varphi_s , \tag{6.62}$$

where $\sigma_\pm = \sigma_1 \pm i\sigma_2$, and φ_s is the eigenvector of σ_3 for eigenvalue $s = \pm 1$. Functions $\Phi_n(\xi)$ $(n = 0, 1, ...)$ are Hermitian functions defined by

$$\Phi_n(\xi) = \frac{1}{\sqrt{2^n n!\sqrt{\pi B}}} H_n\left(\frac{\xi}{\sqrt{B}}\right) e^{-\frac{\xi^2}{2B}} ,$$

where H_n are Hermitian polynomials and $\xi = By - p_x$.

Normalization of (6.60) gives us

$$\int dy\ \bar{\psi}_{n,s}^{(0)}\gamma^0\psi_{n,s}^{(0)} = \frac{2E_{n,s}^{(0)}}{B(E_{n,s}^{(0)}+m)} . \tag{6.63}$$

Using (6.58), we find

$$E_{n,s}^{(1)} = -\frac{\theta^{\alpha\beta}\int dy\ \bar{\psi}_{n,s}^{(0)}\mathcal{M}_{\alpha\beta}\psi_{n,s}^{(0)}}{\int dy\ \bar{\psi}_{n,s}^{(0)}\gamma^0\psi_{n,s}^{(0)}} . \tag{6.64}$$

In particular, for $\theta^{12} = -\theta^{21} = \theta \neq 0$ and all the other components $\theta^{\alpha\beta}$ equal to zero, we find

$$E_{n,s}^{(1)} = -\frac{\theta s}{E_{n,s}^{(0)}}\left(\frac{m^2}{12l^2} - \frac{m}{3l^3}\right)\left(1+\frac{B}{(E_{n,s}^{(0)}+m)}(2n+s+1)\right)+\frac{\theta B^2}{2E_{n,s}^{(0)}}(2n+s+1) . \tag{6.65}$$

In the absence of magnetic field we confirm the already established result,

$$E_{n,s}^{(1)} = -\frac{s\theta}{E_{n,s}^{(0)}}\left(\frac{m^2}{12l^2} - \frac{m}{3l^3}\right) .$$

The NC energy levels depend on $s = \pm 1$ and we see that constant noncommutative background causes Zeeman-like splitting of the undeformed energy levels. Therefore, it acts as a birefringent medium.

To obtain the non-relativistic limit of NC energy levels we expand the classical energy $E^{(0)}_{n,s}$ assuming that p_z^2, $B \ll m^2$:

$$
\begin{aligned}
E^{(0)}_{n,s} &= \sqrt{p_z^2 + m^2 + (2n+s+1)B} = m\left(1 + \frac{p_z^2 + (2n+s+1)B}{m^2}\right)^{1/2} \\
&= m\left(1 + \frac{p_z^2 + (2n+s+1)B}{2m^2} - \frac{(p_z^2 + (2n+s+1)B)^2}{8m^4}\right) .
\end{aligned}
$$

Expending (6.65), we obtain the first order NC correction to the energy levels of a non-relativistic electron:

$$
\begin{aligned}
E^{(1)}_{n,s} &= \left(\frac{\theta s}{3l^3} - \frac{\theta s m}{12 l^2}\right)\left(1 - \frac{p_z^2}{2m^2} + \frac{3p_z^4}{8m^4} + \frac{3p_z^2(2n+s+1)B}{8m^4}\right) \\
&\quad + \frac{\theta B^2}{2m}(2n+s+1)\left(1 - \frac{p_z^2 + (2n+s+1)B}{2m^2} + \frac{3(p_z + (2n+s+1)B)^2}{8m^4}\right) .
\end{aligned}
$$

If we take $p_z = 0$, the non-relativistic NC energy levels reduce to

$$
\begin{aligned}
&E_{n,s} = E^{(0)}_{n,s} + E^{(1)}_{n,s} + \mathcal{O}(\theta^2) \\
&= m + \frac{\theta s}{3l^3} - \frac{\theta s m}{12l^2} + \frac{2n+s+1}{2m}(B + \theta B^2) - \frac{(2n+s+1)^2}{8m^3}(B^2 + 2\theta B^3) + \mathcal{O}(\theta^2) \\
&= m - s\theta\left(\frac{m}{12l^2} - \frac{1}{3l^3}\right) + \frac{2n+s+1}{2m}B_{eff} - \frac{(2n+s+1)^2}{8m^3}B^2_{eff} + \mathcal{O}(\theta^2) , \quad (6.66)
\end{aligned}
$$

where we introduced $B_{eff} = (B + \theta B^2)$ as an *effective magnetic field*. As for the non-interacting electrons [105], the spin-dependent mass-shift is apparent. If we compare this expression with the one for the undeformed energy levels $E^{(0)}_{n,s}$, we see that the only effect of θ-constant noncommutaivity is to modify the mass of an electron and the value of the background magnetic field. This result is in accord with string theory. Namely, in [41] it is argued that the endpoint coordinate of an open string constrained to a D-brane in the presence of a constant Kalb-Ramond B-field satisfy constant noncommutativity algebra. The implication is that NC field theory can be interpreted as a low energy limit, i.e. an effective theory, of the theory of open strings.

Having the energy function (6.66), we can derive NC deformation of the induced magnetic moment of an electron in the nth Landau level, in the limit of weak magnetic field,

$$
\mu_{n,s} = -\frac{\partial E_{n,s}}{\partial B} = -\mu_B(2n+s+1)(1+\theta B) , \quad (6.67)
$$

where $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton. We recognize $-(2n+1)\mu_B$ as the *diamagnetic moment* of an electron and $-s\mu_B$ as the spin magnetic moment. The θB-term is another potentially observable phenomenological prediction. It is a linear NC correction to the induced dipole moment of an electron. As a next step, one could try to calculate the induced magnetization (induced magnetic moment per unit area) of a certain material. For that matter, we need a better understanding of the realization of noncommutaivity in many-body physics.

Canonical NC deformation of relativistic Landau levels has been considered in [114] and for other types of NC space-times in [115, 116]. It can be seen both from (6.65) and (6.66) that the NC correction to the (non)-relativistic Landau levels depends on the mass m, the principal quantum number n and the spin s. In particular, the NC correction to energy levels will be different for different levels. It would be interesting to explore how space-time noncommutativity modifies the degeneracy of Landau levels and we plan to address this problem in the future. It is well known that the physics of the Lowest Landau Level (LLL) is closely related to the physics of Quantum Hall Effect (QHE). Using the results of this section, we hope to obtain some constraints on the noncommutaivity parameter from condensed matter experiments.

Also, starting from (6.45), we can analyze renormalizability properties of our model. It was found that the so-called minimal NC Electrodynamics, a theory obtained by directly introducing the Moyal $\star$-product in the classical Dirac action in Minkowski space, is not a renormalisabile theory, because of the fermionic loop contributions [111, 112]. It would be interesting to see if additional NC terms that are present in the NC Eletrodynamics action (6.45) can improve this behaviour.

7 NC Yang-Mills theory

The content of this section is originally presented in [117].

The NC correction to the classical Yang-Mills action could be relevant for the physics of the early Universe. Namely, the standard cosmological model predicts that thermodynamic conditions in the early Universe were such that nuclear matter existed in a state of quark-gluon plasma (QGP), a distinct phase of nuclear matter that exists only at extremely high temperatures and densities, when low-energy composite states - hadrons - disintegrate into their fundamental constituents - quarks and gluons. On the other hand, it is generally expected that quantum gravity effects, such as space-time noncommutativity, become relevant near the cosmological singularity (just after the Big Bang). The existence of QGP, as a theoretical possibility, was realized in the mid-seventies after the development of the Quark Model of hadrons and the non-Abelian gauge field theory of strong interaction - Quantum Chromodynamics (QCD). The latter exhibits several remarkable features. At short distances, or large momenta q, the effective (renormalized) coupling constant $\alpha_s(q^2)$ decreases logarithmically, and quarks and gluons appear to be weakly coupled, the so-called *asymptotic freedom*. Since the interaction between quarks diminishes as they approach each other, at sufficiently high density, they are no longer confined inside hadrons, and become free. On the other hand, at large distances (or small momenta), the coupling becomes strong, resulting in the phenomena of *confinement* (quarks do not exist as isolated particles at low energies; they occur only as constituents of hadronic bound states - mesons and baryons). The commonly-adopted cosmological scenario of the subsequent cooling of the initially hot and dense Universe, assumes a series of phase transitions through various spontaneous symmetry-breakings related to non-Abelian gauge fields. Specifically, the hadronization of QGP is related to the spontaneous chiral symmetry breaking, described by a non-Abelian theory of strong interactions based on $SU_c(3)$ symmetry group, i.e. the QCD. The nature of this phase transition determines, to a great extent, the evolution of the early Universe.

That being said, our primary goal in this section is to generalize the previous results related to $U(1)$ gauge field and consistently incorporate non-Abelian Yang-Mills gauge field (describing gluons) into the $SO(2,3)_\star$ framework. Deformation of the classical action, invariant under $SO(2,3) \times SU(N)$ gauge group, will reveal the sort of couplings of quarks, gluons and gravity that arise due to space-time noncommutativity. This result can be regarded as the first step towards a full theoretical treatment of quarks and gluons in NC space-time.

7.1 Yang-Mills field in AdS framework

Introducing non-Abelian $SU(N)$ gauge field, $A_\mu = A_\mu^I T_I$, requires an upgrade of the original gauge group $SO(2,3)$ to $SO(2,3) \times SU(N)$. Generators T_I of $SU(N)$ group are hermitian and traceless, and they satisfy the (anti)commutation relations: $[T_I, T_J] = i f_{IJK} T_K$ and $\{T_I, T_J\} = d_{IJK} T_K$, with antisymmetric structure constants f_{IJK}, and totally symmetric symbols $d_{IJK} = Tr(\{T_I, T_J\} T_K)$. We choose the normalization $Tr(T_I T_J) = \delta_{IJ}$. $SU(N)$ group indices I, J,... run from 1 to $N^2 - 1$. The total gauge potential of $SO(2,3) \times SU(N)$ group is

$$\Omega_\mu = \frac{1}{2}\omega_\mu^{AB} M_{AB} \otimes \mathbb{I} + \mathbb{I} \otimes A_\mu^I T_I \ , \tag{7.1}$$

and the corresponding total field strength $\mathbb{F}_{\mu\nu}$ is the sum of the gravitational part $F_{\mu\nu}$ and the Yang-Mills part $\mathcal{F}_{\mu\nu}$,

$$\mathbb{F}_{\mu\nu} = \frac{1}{2} F_{\mu\nu}^{AB} M_{AB} \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{F}_{\mu\nu}^I T_I \ , \tag{7.2}$$

with the usual $\mathcal{F}_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + g f^{IJK} A^J A^K$, where g is the Yang-Mills coupling strength.

Action for Yang-Mills gauge field (a suitable generalization of the $U(1)$ action from the previous section),

$$S_A = -\frac{1}{16l}\ \mathrm{Tr} \int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma} \left(f \mathbb{F}_{\mu\nu} \mathcal{D}_\rho \phi \mathcal{D}_\sigma \phi \phi + \frac{i}{6} f^2 \mathcal{D}_\mu \phi \mathcal{D}_\nu \phi \mathcal{D}_\rho \phi \mathcal{D}_\sigma \phi \phi \right) + c.c. \tag{7.3}$$

is real and invariant under $SO(2,3) \times SU(N)$ gauge transformations. It involves an auxiliary field f defined by:

$$f = \frac{1}{2} f^{AB,I} M_{AB} \otimes T_I \ , \quad \delta_\epsilon f = i[\epsilon, f] \ , \tag{7.4}$$

where we have a gauge parameter ϵ that consists of the $SO(2,3)$ and the $SU(N)$ part,

$$\epsilon = \frac{1}{2}\epsilon^{AB} M_{AB} \otimes \mathbb{I} + \mathbb{I} \otimes \epsilon^I T_I \ . \tag{7.5}$$

Auxiliary field f transforms in the adjoint representation of $SO(2,3)$ and $SU(N)$ group. The role of this field is to produce the canonical kinetic term in curved space-time for the non Abelian $SU(N)$ gauge field in the absence the Hodge dual operator, which otherwise cannot be defined without an explicit use of the metric tensor. We still need to use an auxiliary field $\phi = \phi^A \Gamma_A$ to produce symmetry breaking. It is invariant under $SU(N)$ gauge transformations and its full covariant derivative is given by,

$$\mathcal{D}_\sigma \phi = \partial_\sigma \phi - ig[\Omega_\sigma, \phi] = \partial_\sigma \phi - i[\omega_\sigma, \phi] = D_\sigma \phi \ . \tag{7.6}$$

By setting $\phi^a = 0$ and $\phi^5 = l$ in (7.3) we break $SO(2,3) \times SU(N)$ gauge symmetry down to $SO(1,3) \times SU(N)$ and obtain

$$S_A|_{\text{g.f.}} = \frac{1}{2}\int \mathrm{d}^4x \ e \ f^{ab,I} \mathcal{F}^I_{\mu\nu} e^\mu_a e^\nu_b + \frac{1}{4}\int \mathrm{d}^4x \ e \ (f^{ab,I} f^I_{ab} + 2 f^{a5,I} f^I_{a5}) \ . \tag{7.7}$$

Varying this action independently in components f^I_{ab} and f^I_{a5} of the auxiliary field, we get their equations of motion,

$$f^I_{a5} = 0 \ , \quad f^I_{ab} = -e^\mu_a e^\nu_b \mathcal{F}^I_{\mu\nu} \ . \tag{7.8}$$

By evaluating (7.7) on these EoM we eliminate the auxiliary field, and obtain

$$S_{A,EoM}|_{\text{g.f.}} = -\frac{1}{4}\int \mathrm{d}^4x \ \sqrt{-g} \ g^{\mu\rho} g^{\nu\sigma} \mathcal{F}^I_{\mu\nu} \mathcal{F}^I_{\rho\sigma} \ , \tag{7.9}$$

which is exactly the canonical kinetic term for Yang-Mills gauge field in curved space-time.

In the context of $SU(N)$ Yang-Mills gauge theory, we need to introduce *a multiplet* of N Dirac spinors,

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \ , \tag{7.10}$$

that transforms in the fundamental representation of $SU(N)$ gauge group. Each component ψ_i transforms in the fundamental representation of $SO(2,3)$ gauge group, and so, infinitesimally, under a full gauge transformation,

$$(\delta_\epsilon \Psi)_i = i(\epsilon\Psi)_i = i\epsilon^I (T_I)_{ij} \psi_j = \frac{i}{2}\epsilon^I_{AB} (T_I)_{ij} M_{AB} \psi_i \ . \tag{7.11}$$

The i-th component of the total covariant can be decomposed in three parts, Lorentz, l-dependent AdS part, and $SU(N)$ part,

$$\begin{aligned}(\mathcal{D}_\sigma \Psi)_i &= \partial_\sigma \psi_i - ig\Omega_\sigma \psi_i = \partial_\sigma \psi_i - \frac{i}{2}\omega^{AB}_\sigma M_{AB} \psi_i - igA^I_\sigma (T_I)_{ij} \psi_j \\ &= D^L_\sigma \psi_i + \frac{i}{2l} e^a_\sigma \gamma_a \psi_i - igA^I_\sigma (T_I)_{ij} \psi_j \ . \end{aligned} \tag{7.12}$$

The undeformed fermionic action is

$$S_\Psi = \frac{i}{12}\int \mathrm{d}^4x\ \epsilon^{\mu\nu\rho\sigma}\left(\bar\Psi\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\Psi - \mathcal{D}_\sigma\bar\Psi\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\Psi\right), \tag{7.13}$$

and, after gauge fixing, it reduces to

$$S_\Psi|_{\text{g.f.}} = i\int \mathrm{d}^4x\ e\left(\bar\Psi e^\sigma_s\gamma^s D^L_\sigma\Psi - \frac{2}{l}\bar\Psi\Psi + g\bar\Psi e^\sigma_s\gamma^s A^I_\sigma T_I\Psi\right), \tag{7.14}$$

We can also include additional $SO(2,3)\times SU(N)$ invariant mass-like terms $S_{m,i}$ $(i=1,2,3)$ in the total action,

$$\begin{aligned}
S_{m,1} &= \frac{ic_1}{2l}\left(m-\frac{2}{l}\right)\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\ \bar\Psi\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\phi\Psi + c.c.\\
S_{m,2} &= \frac{ic_2}{2l}\left(m-\frac{2}{l}\right)\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\ \bar\Psi\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\mathcal{D}_\rho\phi\phi\mathcal{D}_\sigma\phi\Psi + c.c.\\
S_{m,3} &= \frac{ic_3}{l}\left(m-\frac{2}{l}\right)\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\ \bar\Psi\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi\phi\mathcal{D}_\rho\phi\mathcal{D}_\sigma\phi\Psi\ .
\end{aligned} \tag{7.15}$$

For the free dimensionless parameters c_1, c_2 and c_3 will again set $c_1 = -c_2 = c_3 = 1/72$. In that case, after the symmetry breaking, the sum of the three mass-like terms (7.15), denoted by S_m, reduces to

$$S_m|_{\text{g.f.}} = -\left(m-\frac{2}{l}\right)\int \mathrm{d}^4x\ e\ \bar\Psi\Psi\ . \tag{7.16}$$

Adding this to (7.14), the $2/l$ terms exactly cancel. In particular. We thus have a complete and consistent classical (undeformed) model of $SO(2,3)\times SU(N)$ gauge-invariant Yang-Mills theory.

7.2 NC-deformed Yang-Mills theory

The NC Yang-Mills action invariant under NC-deformed $SO(2,3)_\star \times SU(N)_\star$ gauge transformations is obtained by applying the procedure of $\star$-product deformation to the commutative actions S_Ψ, S_m, and S_A, given by Eqs. (7.3), (7.13) and (7.15), respectively. For example, pure Yang-Mills action S_A becomes

$$\begin{aligned}
S^\star_A = -\frac{1}{16l}\mathrm{Tr}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big\{&\hat f\star\hat{\mathbb{F}}_{\mu\nu}\star\mathcal{D}_\rho\hat\phi\star\mathcal{D}_\sigma\hat\phi\star\hat\phi\\
&+\frac{i}{6}\hat f\star\hat f\star\mathcal{D}_\mu\hat\phi\star\mathcal{D}_\nu\hat\phi\star\mathcal{D}_\rho\hat\phi\star\mathcal{D}_\sigma\hat\phi\star\hat\phi\Big\} + c.c.
\end{aligned} \tag{7.17}$$

We proceed by employing the SW map to perturbatively expanded the NC action in powers of $\theta^{\alpha\beta}$. SW map insures the invariance of this expansion under ordinary $SO(2,3)\times SU(N)$ gauge transformations, order-by-order.

After the symmetry breaking and elimination of the auxiliary field f by using its equations of motion - if we consider only terms linear in $\theta^{\alpha\beta}$, to eliminate the f-field, one only needs to insert the undeformed EoM (7.8) in the first order NC correction (as in the case of $U(1)$ gauge field), yielding

$$S_A^{(1)}|_{\text{g.f.}} = -\frac{\theta^{\alpha\beta}}{16}\int \mathrm{d}^4x\; e\; d_{IJK}\; g^{\mu\rho}g^{\nu\sigma}\left(\mathcal{F}^I_{\alpha\beta}\mathcal{F}^J_{\mu\nu}\mathcal{F}^K_{\rho\sigma} - 4\mathcal{F}^I_{\alpha\mu}\mathcal{F}^J_{\beta\nu}\mathcal{F}^K_{\rho\sigma}\right)\;. \tag{7.18}$$

This is the first order NC correction to the pure Yang-Mills action in curved space-time. It describes interaction of gauge-bosons with gravity that arises due to space-time noncommutativity. After the gauge fixing, the total linear NC correction to the spinor part of the Yang-Mills theory becomes

$$\begin{aligned}
\left(S_\Psi^{(1)}+S_m^{(1)}\right)|_{\text{g.f.}} = \theta^{\alpha\beta}\int \mathrm{d}^4x\; e\;\bar\Psi\Bigg[& -\frac{1}{8}R_{\alpha\mu}{}^{ab}e^\mu_a\gamma_b\mathcal{D}^L_\beta - \frac{i}{32}R_{\alpha\beta}{}^{ab}\varepsilon_{abc}{}^{d}e^\sigma_d\gamma^c\gamma^5\mathcal{D}^L_\sigma \\
&+\frac{1}{16}R_{\alpha\beta}{}^{ab}e^\sigma_b\gamma_a\mathcal{D}^L_\sigma - \frac{i}{16}R_{\alpha\mu}{}^{bc}e^\mu_a\varepsilon^a{}_{bcm}\gamma^m\gamma^5\mathcal{D}^L_\beta - \frac{i}{24}R_{\alpha\mu}{}^{ab}\varepsilon_{abc}{}^{d}e^c_\beta(e^\mu_d e^\sigma_s - e^\mu_s e^\sigma_d)\gamma^s\gamma^5\mathcal{D}^L_\sigma \\
&-\frac{i}{8l}T_{\alpha\beta}{}^{a}e^\sigma_a\mathcal{D}^L_\sigma + \frac{i}{8l}T_{\alpha\mu}{}^{a}e^\mu_a\mathcal{D}^L_\beta + \frac{1}{16l}T_{\alpha\beta}{}^{a}e^\mu_a\sigma_\mu{}^\sigma\mathcal{D}^L_\sigma + \frac{1}{8l}T_{\alpha\mu}{}^{a}e^\mu_b\sigma_a{}^b\mathcal{D}^L_\beta \\
&-\frac{1}{12l}T_{\alpha\mu}{}^{a}\varepsilon_{ab}{}^{cd}e^b_\beta e^\mu_c e^\sigma_d\gamma^5\mathcal{D}^L_\sigma + \frac{7i}{48l^2}\varepsilon_{abc}{}^{d}e^a_\alpha e^b_\beta e^\sigma_d\gamma^c\gamma^5\mathcal{D}^L_\sigma - \frac{1}{4l}\sigma_\alpha{}^\sigma\mathcal{D}^L_\beta\mathcal{D}_\sigma \\
&-\frac{1}{4}(D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\sigma_a e^\mu_b)\gamma^b\mathcal{D}^L_\beta\mathcal{D}^L_\sigma - \frac{i}{8}\eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon^{cdrs}e^\mu_c e^\nu_d e^\sigma_s\gamma_r\gamma_5\mathcal{D}^L_\sigma \\
&+\frac{i}{12}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon_b{}^{cds}e^\mu_c e^\nu_d e^\sigma_s\;\gamma_a\gamma_5\mathcal{D}^L_\sigma - \frac{1}{12l}e^c_\alpha(D^L_\beta e^b_\nu)\varepsilon_{bc}{}^{ds}e^\nu_d e^\sigma_s\gamma_5\mathcal{D}^L_\sigma \\
&-\frac{1}{8l}(D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\sigma_a e^\mu_b)e^c_\beta\sigma^b{}_c\mathcal{D}^L_\sigma - \frac{1}{8l}(D^L_\alpha e^a_\mu)e^\mu_b\sigma_a{}^b\mathcal{D}_\beta + \frac{3}{96l}R_{\alpha\beta}{}^{ab}\sigma_{ab} \\
&+\frac{1}{16l}R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc} - \frac{1}{16l}R_{\alpha\mu}{}^{ab}e_{\beta a}e^\mu_c\sigma_b{}^c - \frac{19}{288l^2}T_{\alpha\beta}{}^{a}\gamma_a + \frac{19}{144l^2}T_{\alpha\mu}{}^{a}e^\mu_a\gamma_\beta \\
&+\frac{1}{16l^2}T_{\alpha\mu}{}^{a}e_{\beta a}\gamma^\mu - \frac{19}{144l^2}(D^L_\alpha e^a_\mu)e^\mu_a\gamma_\beta + \frac{1}{16l^2}(D^L_\alpha e^a_\mu)e_{\beta a}\gamma^\mu - \frac{1}{12l^3}\sigma_{\alpha\beta} \\
&+\frac{3i}{8}\mathcal{F}_{\alpha\beta}e^\sigma_s\gamma^s\mathcal{D}^L_\sigma - \frac{i}{4}\mathcal{F}_{\alpha\sigma}e^\sigma_s\gamma^s\mathcal{D}^L_\beta + \frac{1}{8l}\mathcal{F}_{\alpha\beta}\Bigg]\Psi + c.c.
\end{aligned} \tag{7.19}$$

The $SO(1,3)\times SU(N)$ covariant derivative is $\mathcal{D}^L_\sigma\Psi = (D^L_\sigma - igA^I_\sigma T_I)\Psi$. This action describes interaction of fermions with gravity that emerge due to space-time noncommutativity. Evidently, some of them pertain even in flat space-time and this unexpected residual effect of space-time noncommutativity perhaps has some nontrivial consequences for the dynamics of quarks and gluons. The new terms, linear in $\theta^{\alpha\beta}$, modify the standard theory of strong interaction. Note that some of them survive the WI contraction.

From curved space-time NC actions (7.18) and (7.19) we can derive the NC-deformed action for $SU_c(3)$ Yang-Mills theory in Minkowski space,

$$
\begin{aligned}
S^{\star}_{flat} = \int \mathrm{d}^4x \Bigg\{ & i\bar{\Psi}^{(q)}\widetilde{\not{\mathcal{D}}}\Psi^{(q)} - \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}\mathcal{F}^I_{\mu\nu}\mathcal{F}^I_{\rho\sigma} \\
+\theta^{\alpha\beta}\Bigg[& -\frac{1}{2l}\bar{\Psi}^{(q)}\sigma_\alpha{}^\sigma\widetilde{\mathcal{D}}_\beta\widetilde{\mathcal{D}}_\sigma\Psi^{(q)} + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\bar{\Psi}^{(q)}\gamma_\rho\gamma_5\widetilde{\mathcal{D}}_\sigma\Psi^{(q)} + \frac{3i}{4}\bar{\Psi}^{(q)}\mathcal{F}_{\alpha\beta}\widetilde{\not{\mathcal{D}}}\Psi^{(q)} \\
& -\frac{i}{2}\bar{\Psi}^{(q)}\mathcal{F}_{\alpha\sigma}\gamma^\sigma\widetilde{\mathcal{D}}_\beta\Psi^{(q)} - \frac{1}{6l^3}\bar{\Psi}^{(q)}\sigma_{\alpha\beta}\Psi^{(q)} + \frac{1}{4l}\bar{\Psi}^{(q)}\mathcal{F}_{\alpha\beta}\Psi^{(q)}\Bigg] \\
& -\frac{\theta^{\alpha\beta}}{16}d_{IJK}\, g^{\mu\rho}g^{\nu\sigma}\left(\mathcal{F}^I_{\alpha\beta}\mathcal{F}^J_{\mu\nu}\mathcal{F}^K_{\rho\sigma} - 4\mathcal{F}^I_{\alpha\mu}\mathcal{F}^J_{\beta\nu}\mathcal{F}^K_{\rho\sigma}\right)\Bigg\}\,, \qquad (7.20)
\end{aligned}
$$

with $\Psi^{(q)}$ being the quark color triplet, and we imply the summation over all six flavours ($q = u, d, s, c, t, b$). The $SU_c(3)$ covariant derivative is given by

$$
(\widetilde{\mathcal{D}}_\sigma\Psi^{(q)})_i = \partial_\sigma\psi^{(q)}_i - \frac{i}{2}g_s A^I_\sigma(\lambda_I)_{ij}\psi^{(q)}_j\,, \qquad (7.21)
$$

with Gell-Mann matrices $\frac{1}{2}\lambda_I$ as generators of $SU_c(3)$ gauge group; index I goes from 1 to 8. We have a gluon field strength $\mathcal{F}^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g_s f^{IJK}A^J A^K$, with coupling g_s. From the experimental point of view, it is significant that the first non-vanishing NC correction is linear in $\theta^{\alpha\beta}$ since this could lead to some potentially observable effects. Following this approach, it is possible to progress towards generalizing the NC Standard Model. After quantization, the renormalizability of the model can be investigated, as in [118], especially the influence of noncommutativity on asymptotic freedom. This result sets a basis for further investigation of the effects of space-time noncommutativity on strong interaction and, by extension, the dynamics of the early Universe.

8 Canonical deformation of $N = 2$ AdS_4 SUGRA

The content of this section is originally presented in [119].

It is well known that one can define a consistent theory of extended $N = 2$ AdS SUGRA in $D = 4$. Besides the standard gravitational part (with a negative cosmological constant), this SUGRA model involves a $U(1)$ gauge field and a pair independent Majorana vector-spinors that can be mixed to form a pair of Dirac spinors (charged spin-3/2 gravitini). The action for $N = 2$ AdS_4 SUGRA is invariant under $SO(1,3) \times U(1)$ gauge transformations, and under local SUSY. We present a *geometric action* that involves two “inhomogeneous” parts: an orthosymplectic $OSp(4|2)$ gauge-invariant action of the MacDowell-Mansouri type, and a supplementary action invariant under the purely bosonic $SO(2,3) \times U(1) \sim Sp(4) \times SO(2)$ sector of $OSp(4|2)$, which needs to be added for consistency. This action reduces to $N = 2$ AdS_4 SUGRA after the gauge fixing. We show that $N = 2$ AdS_4 SUGRA has non-vanishing linear NC correction in the physical gauge, originating from the additional, purely bosonic term. For comparison, simple $N = 1$ Poinacaré SUGRA can be obtained in the same manner from an $OSp(4|1)$ gauge-invariant action (without introducing any additional terms). The first non-vanishing NC correction is quadratic in the deformation parameter $\theta^{\mu\nu}$, and therefore exceedingly difficult to calculate. Under WI contraction, $N = 2$ AdS_4 superalgebra reduces to $N = 2$ Poincaré superalgebra, and it is not at all clear whether this relation holds after canonical NC deformation. We present the linear NC correction to $N = 2$ AdS_4 SUGRA explicitly and discuss its low-energy limit and what remains of it after WI contraction.

To date, we still lack direct physical evidence of SUSY, at least in its simplest form. However, its beneficial influence on high-energy physics (it improves renormalizability in QFT and a provides a natural solution to the hierarchy problem), along with its mathematical consistency and unification power (especially the unification of gravity and the Standard Model within SUGRA, and an ultimate unification scheme such as Superstring theory), motivate us to seriously consider SUSY as a part of our description of nature. Since the original work of Freedman, van Nieuwenhuizen *et al.* [120, 121], and Deser and Zumino [122], the theory of supergravity has become a well-developed research field. SUGRA provides a unification of gravity with other fields by imposing the gauge principle on SUSY, the associated gauge field being the spin-3/2 gravitino field described by a Majorana vector-spinor. It was demonstrated in [123, 124] that one can define a consistent theory of $N = 2$ AdS_4 SUGRA with a complex, $U(1)$-charged gravitino. We propose a more geometric way of obtaining $N = 2$ AdS_4 SUGRA action and perform its NC deformation.

The following results amount to a supersymmetric extension of the theory of NC gravity based on the NC-deformed AdS gauge group $SO(2,3)$ developed in [57–60]. NC SUGRA can be established by gauging an appropriate supergroup [28, 29, 125–131] and performing canonical deformation. Since GR can be obtained by gauging AdS group $SO(2,3)$, orthosymplectic supergroup $OSp(4|1)$ appears as a natural choice for pure $N = 1$ Poincaré SUGRA. The bosonic sector of $\mathfrak{osp}(4|1)$ superalgebra - symplectic algebra $\mathfrak{sp}(4)$ - is isomorphic to AdS algebra $\mathfrak{so}(2,3)$ that reduces to Poincaré algebra under WI contraction [132]. The subject of NC SUGRA has been thoroughly treated in [96–98]. Classical action for $OSp(4|1)$ SUGRA presented in [98] is manifestly invariant under $OSp(4|1)$ gauge transformations, and we will use it as a starting point. However, obtaining explicit NC correction of this action is exceedingly difficult because the first non-vanishing NC correction is quadratic in $\theta^{\mu\nu}$. Taking a lesson from [105, 108, 109] that iby inlcuding Dirac spinors coupled to $U(1)$ gauge field (much simpler) linear NC correction emerges, we will make a transition to $OSp(4|2)$ SUGRA that involves a pair of Majorana spinors that can be mixed into a pair of gravitini charged under $U(1)$. We present an action that consists of two "inhomogeneous" geometric parts: an orthosymplectic, $OSp(4|2)$ gauge-invariant action of the MacDowell-Mansouri type and a supplementary action that is invariant under the purely bosonic $SO(2,3) \times U(1)$ sector of $OSp(4|2)$, that has to be included in order to obtain complete $N = 2$ AdS_4 SUGRA at the classical level; a non-trivial linear NC correction to $N = 2$ AdS_4 SUGRA comes from this additional bosonic term, after deformation.

We consider two classical SUGRA models based on the orthosymplectic $OSp(4|N)$ gauge group: the simple $N = 1$ AdS_4 SUGRA, describing pure supergravity with the negative cosmological constant, and the extended $N = 2$ AdS_4 SUGRA. We put our attentions on the latter ($N = 2$), since the former ($N = 1$) has been treated thoroughly in [98], including its NC deformation; we discuss it just for comparison. Significant differences of the two models in question have been manifested already at the level of their classical actions, and this reflects drastically on the structure of their NC corrections after deformation.

8.1 $OSp(4|2)$ SUGRA

Orthosymplectic group $OSp(4|2)$ has 19 generators, and they are of two sorts - bosonic and fermionic. Ten bosonic generators $\hat{M}_{AB} = -\hat{M}_{BA}$ $(A, B = 0, 1, 2, 3, 5)$ form a basis of AdS Lie algebra $\mathfrak{so}(2,3)$ (symmetry algebra of AdS_4),

$$[\hat{M}_{AB}, \hat{M}_{CD}] = i(\eta_{AD}\hat{M}_{BC} + \eta_{BC}\hat{M}_{AD} - \eta_{AC}\hat{M}_{BD} - \eta_{BD}\hat{M}_{AC}) \ , \tag{8.1}$$

where η_{AB} is flat 5D metric with signature $(+, -, -, -, +)$. By splitting this set of generators into six $\hat{M}_{ab}$ AdS rotation generators $(a, b = 0, 1, 2, 3)$ and four AdS translation generators $\hat{M}_{a5}$, we can recast $\mathfrak{so}(2,3)$ algebra in a more explicit form:

$$\begin{aligned}
[\hat{M}_{a5}, \hat{M}_{b5}] &= -i\hat{M}_{ab} \ , \\
[\hat{M}_{ab}, \hat{M}_{c5}] &= i(\eta_{bc}\hat{M}_{a5} - \eta_{ac}\hat{M}_{b5}) \ , \\
[\hat{M}_{ab}, \hat{M}_{cd}] &= i(\eta_{ad}\hat{M}_{bc} + \eta_{bc}\hat{M}_{ad} - \eta_{ac}\hat{M}_{bd} - \eta_{bd}\hat{M}_{ac}) \ .
\end{aligned} \tag{8.2}$$

If we introduce a new set of generators $(\hat{\mathcal{M}}_{ab}, \hat{\mathcal{P}}_a)$ defined by $\hat{\mathcal{M}}_{ab} := \hat{M}_{ab}$ and $\hat{\mathcal{P}}_a := l^{-1}\hat{M}_{a5} = \alpha\hat{M}_{a5}$, where l is a length scale related to AdS radius and $\alpha = l^{-1}$ (we will use both in the following formulae), the algebra (8.2) transforms into:

$$\begin{aligned}
[\hat{\mathcal{P}}_a, \hat{\mathcal{P}}_b] &= -i\alpha^2\hat{\mathcal{M}}_{ab} \ , \\
[\hat{\mathcal{M}}_{ab}, \hat{\mathcal{P}}_c] &= i(\eta_{bc}\hat{\mathcal{P}}_a - \eta_{ac}\hat{\mathcal{P}}_b) \ , \\
[\hat{\mathcal{M}}_{ab}, \hat{\mathcal{M}}_{cd}] &= i(\eta_{ad}\hat{\mathcal{M}}_{bc} + \eta_{bc}\hat{\mathcal{M}}_{ad} - \eta_{ac}\hat{\mathcal{M}}_{bd} - \eta_{bd}\hat{\mathcal{M}}_{ac}) \ .
\end{aligned} \tag{8.3}$$

In the limit $\alpha \to 0$ (or $l \to \infty$), AdS algebra reduces to Poincaré algebra; in particular, we obtain $[\hat{\mathcal{P}}_a, \hat{\mathcal{P}}_b] = 0$ with all other commutators left unchanged. This is a famous example of the Wigner-Inönü (WI) contraction, the contraction parameter being α (or l). This Lie-algebra contraction (or deformation) can be extended to AdS superalgebra, and we will be interested, later on, in its effect on the NC correction of $N = 2$ AdS_4 SUGRA.

A representation of the AdS sector of $\mathfrak{osp}(4|1)$ superalgebra is obtained by using 5D gamma matrices Γ_A satisfying Clifford algebra $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$; the AdS generators $\hat{M}_{AB}$ are represented by 6×6 *super-matrices*, which reduce to 4×4 matrices $M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B]$ in the AdS subspace, see Appendix C. One choice of Γ-matrices is $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, where γ_a are the usual 4D γ-matrices. In this representation, the components of M_{AB} are given by $M_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab}$ and $M_{a5} = -\frac{1}{2}\gamma_a$.

The ten AdS bosonic generators M_{AB} are accompanied by eight independent fermionic generators $\hat{Q}^I_\alpha$, with spinor index $\alpha = 1,2,3,4$ and $SO(2)$ index $I = 1,2$, comprising a pair of Majorana spinors, and one additional bosonic generator $\hat{T}$ related to $SO(2) \sim U(1)$ extension. Together, they satisfy $\mathfrak{osp}(4|2)$ superalgebra (for consistency, fermionic generators $\hat{Q}^I_\alpha$ have to transform as components of an AdS Majorana spinor),

$$
\begin{aligned}
[\hat{M}_{AB}, \hat{M}_{CD}] &= i(\eta_{AD}\hat{M}_{BC} + \eta_{BC}\hat{M}_{AD} - \eta_{AC}\hat{M}_{BD} - \eta_{BD}\hat{M}_{AC}) \ , \\
[\hat{M}_{AB}, \hat{Q}^I_\alpha] &= -(M_{AB})_\alpha{}^\beta \hat{Q}^I_\beta \ , \\
\{\hat{Q}^I_\alpha, \hat{Q}^J_\beta\} &= -2\delta^{IJ}(M^{AB}C^{-1})_{\alpha\beta}\hat{M}_{AB} - i\varepsilon^{IJ}C_{\alpha\beta}\hat{T} \ , \\
[\hat{T}, \hat{Q}^I_\alpha] &= -i\varepsilon^{IJ}\hat{Q}^I_\alpha \ ,
\end{aligned} \tag{8.4}
$$

with antisymmetric tensor ε^{IJ}, $\varepsilon^{12} = 1$. C^{-1} is the inverse of the charge-conjugation matrix (spinor metric) for which we use the following representation given in terms of Pauli matrices: $C = -\sigma^3 \otimes i\sigma^2$ and $C_{\alpha\beta} = -C_{\beta\alpha}$. Numerically, we have $C^{-1} = -C$, but the index structure of the two is different since $C_{\alpha\gamma}(C^{-1})^{\gamma\beta} = \delta^\beta_\alpha$.

More visually,

$$
C = \left(\begin{array}{c|c}
\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array} & 0_{2\times 2} \\ \hline
0_{2\times 2} & \begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}
\end{array}\right) \ . \tag{8.5}
$$

An explicit matrix representation of $\mathfrak{osp}(4|2)$ superalgebra is given in Appendix A.

By introducing a new set of generators $\{\hat{\mathcal{M}}_{ab} := \hat{M}_{ab}, \hat{\mathcal{P}}_a := \alpha\hat{M}_{a5}, \hat{\mathcal{Q}}^I_\alpha := \sqrt{\alpha}\hat{Q}^I_\alpha, \hat{\mathcal{T}} := \alpha\hat{T}\}$, we can recast the $\mathfrak{osp}(4|2)$ superalgebra (8.4) into the following form:

$$
\begin{aligned}
[\hat{\mathcal{P}}_a, \hat{\mathcal{P}}_b] &= -i\alpha^2\hat{\mathcal{M}}_{ab} \ , \\
[\hat{\mathcal{M}}_{ab}, \hat{\mathcal{P}}_c] &= i(\eta_{bc}\hat{\mathcal{P}}_a - \eta_{ac}\hat{\mathcal{P}}_b) \ , \\
[\hat{\mathcal{M}}_{ab}, \hat{\mathcal{M}}_{cd}] &= i(\eta_{ad}\hat{\mathcal{M}}_{bc} + \eta_{bc}\hat{\mathcal{M}}_{ad} - \eta_{ac}\hat{\mathcal{M}}_{bd} - \eta_{bd}\hat{\mathcal{M}}_{ac}) \ , \\
[\hat{\mathcal{P}}_a, \hat{\mathcal{Q}}^I_\alpha] &= -\alpha(M_{a5})_\alpha{}^\beta\hat{\mathcal{Q}}^I_\beta \ , \\
[\hat{\mathcal{M}}_{ab}, \hat{\mathcal{Q}}^I_\alpha] &= -(M_{ab})_\alpha{}^\beta\hat{\mathcal{Q}}^I_\beta \ , \\
[\hat{\mathcal{T}}, \hat{\mathcal{Q}}^I_\alpha] &= -i\varepsilon^{IJ}\hat{\mathcal{Q}}^I_\alpha \ , \\
\{\hat{\mathcal{Q}}^I_\alpha, \hat{\mathcal{Q}}^J_\beta\} &= -2\delta^{IJ}\alpha(M^{ab}C^{-1})_{\alpha\beta}\hat{\mathcal{M}}_{ab} - 2\delta^{IJ}(M^{a5}C^{-1})_{\alpha\beta}\hat{\mathcal{P}}_a - i\varepsilon^{IJ}C_{\alpha\beta}\hat{\mathcal{T}} \ .
\end{aligned} \tag{8.6}
$$

Under WI contraction $\alpha \to 0$, it reduces to $N = 2$ Poincaré superalgebra.

Orthosymplectic supergroup $OSp(2n|m)$ (the symplectic sector is always even-dimensional) consists of those super-matrices U that preserve the graded metric

$$G = \left(\begin{array}{c|c} \Sigma_{\alpha\beta} & 0_{2n\times m} \\ \hline 0_{m\times 2n} & \Delta_{ij} \end{array} \right) , \tag{8.7}$$

with some real $2n \times 2n$ matrix $\Sigma_{\alpha\beta} = -\Sigma_{\beta\alpha}$ and some real $m \times m$ matrix $\Delta_{ij} = \Delta_{ji}$. Considering only infinitesimal transformations $U = 1 + \epsilon M$, generated by some $\mathfrak{osp}(2n|m)$-valued supermatrix

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{8.8}$$

(bosonic blocks $A_{2n\times 2n}$ and $D_{m\times m}$ have ordinary *commuting* entries, and fermionic blocks $B_{2n\times m}$ and $C_{m\times 2n}$ have *Grassmann-valued* entries), the defining relation becomes

$$M^{ST}G + GM = 0 \ . \tag{8.9}$$

Super-transpose, super-hermitian adjoint and super-trace are defined by imposing the standard rules $(MN)^{ST} = N^{ST}M^{ST}$, $(MN)^{\dagger} = N^{\dagger}M^{\dagger}$ and $STr(MN) = STr(NM)$,

$$M^{ST} = \left(\begin{array}{c|c} A^T & C^T \\ \hline -B^T & D^T \end{array} \right) , \quad M^{\dagger} = \left(\begin{array}{c|c} A^{\dagger} & C^{\dagger} \\ \hline B^{\dagger} & D^{\dagger} \end{array} \right) , \quad \mathrm{STr}(M) = \mathrm{Tr}(A) - \mathrm{Tr}(D) \ . \tag{8.10}$$

Now, the key observation is that a pair of Majorana fields χ^I_μ (describing a pair of neutral spin-3/2 gravitini) constitute the fermionic sector of the $\mathfrak{osp}(4|2)$ connection super-matrix Ω_μ. We can expand this super-connection over the basis $\{\hat{M}_{ab}, \hat{M}_{a5}, \hat{Q}^I_\alpha, \hat{T}\}$ with the corresponding gauge fields $\{{\omega_\mu}^{ab}, {\omega_\mu}^{a5}, \bar{\chi}^I_\mu, A_\mu\}$, as

$$\Omega_\mu = \frac{1}{2}{\omega_\mu}^{ab}\hat{M}_{ab} + {\omega_\mu}^{a5}\hat{M}_{a5} + (\bar{\chi}^I_\mu)^\alpha \hat{Q}^I_\alpha + A_\mu \hat{T} = \left(\begin{array}{c|cc} \omega_\mu & \chi^1_\mu & \chi^2_\mu \\ \hline \bar{\chi}^1_\mu & 0 & \multicolumn{1}{|c}{iA_\mu} \\ \cline{2-3} \bar{\chi}^2_\mu & -iA_\mu & \multicolumn{1}{|c}{0} \end{array} \right) , \tag{8.11}$$

where we have $\mathfrak{so}(2,3)$ gauge field $\omega_\mu = \frac{1}{2}{\omega_\mu}^{AB}M_{AB} = \frac{1}{2}{\omega_\mu}^{ab}M_{ab} + {\omega_\mu}^{a5}M_{a5} = \frac{1}{4}{\omega_\mu}^{ab}\sigma_{ab} - \frac{1}{2}{\omega_\mu}^{a5}\gamma_a$, a pair of Majorana vector-spinors χ^I_μ with components $(\chi^I_\mu)_\alpha$, and their Dirac-adjoints $\bar{\chi}^I_\mu = -(\chi^I_\mu)^T C^{-1}$ with components $(\bar{\chi}^I_\mu)^\alpha = -(\chi^I_\mu)_\beta (C^{-1})^{\beta\alpha}$ $(\alpha = 1, 2, 3, 4)$.

Equivalently, we can expand Ω_μ over the rescaled basis $\{\hat{\mathcal{M}}_{ab}, \hat{\mathcal{P}}_a, \hat{\mathcal{Q}}^I_\alpha, \hat{\mathcal{T}}\}$, but with a different set of gauge fields $\{\omega_\mu{}^{ab}, e^a_\mu := \frac{1}{\alpha}\omega_\mu{}^{a5}, \bar{\psi}^I_\mu := \frac{1}{\sqrt{\alpha}}\bar{\chi}^I_\mu, \mathcal{A}_\mu := \frac{1}{\alpha}A_\mu\}$, as

$$\Omega_\mu = \frac{1}{2}\omega_\mu{}^{ab}\hat{\mathcal{M}}_{ab} + \tfrac{1}{\alpha}\omega_\mu{}^{a5}\hat{\mathcal{P}}_a + \tfrac{1}{\sqrt{\alpha}}\bar{\chi}^\alpha_\mu\hat{\mathcal{Q}}_\alpha = \left(\begin{array}{c|cc} \omega_\mu & \sqrt{\alpha}\psi^1_\mu & \sqrt{\alpha}\psi^2_\mu \\ \hline \sqrt{\alpha}\bar{\psi}^1_\mu & 0 & i\alpha\mathcal{A}_\mu \\ \sqrt{\alpha}\bar{\psi}^2_\mu & -i\alpha\mathcal{A}_\mu & 0 \end{array}\right) , \tag{8.12}$$

where we again have $\mathfrak{so}(2,3)$ gauge field $\omega_\mu = \frac{1}{4}\omega_\mu{}^{ab}\sigma_{ab} - \frac{\alpha}{2}e^a_\mu\gamma_a$, two independent Majorana spinors ψ^I_μ, and (dimensionless) $U(1)$ vector potential $\mathcal{A}_\mu$. We will use this particular representation because it makes WI contraction more transparent.

The two Majorana spinors, ψ^1_μ and ψ^2_μ, can be combined into an $SO(2)$ doublet,

$$\Psi_\mu = \begin{pmatrix} \psi^1_\mu \\ \psi^2_\mu \end{pmatrix} . \tag{8.13}$$

It can be readily confirmed that the gauge supermatrix (8.12) satisfies the defining relation for the elements of $\mathfrak{osp}(4|2)$ superalgebra (C is the charge-conjugation matrix (8.5)),

$$\Omega_\mu^{ST}\left(\begin{array}{c|cc} C & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) + \left(\begin{array}{c|cc} C & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\Omega_\mu = 0 . \tag{8.14}$$

By generalization, we introduce the $Osp(4|2)$ field strength $\mathbb{F}_{\mu\nu}$ associated with the super-connection Ω_μ,

$$\begin{aligned}\mathbb{F}_{\mu\nu} &= \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - i[\Omega_\mu, \Omega_\nu] \\ &= \left(\begin{array}{c|cc} \widetilde{F}_{\mu\nu} & \sqrt{\alpha}(\mathcal{D}_\mu\psi^1_\nu - \mathcal{D}_\nu\psi^1_\mu) & \sqrt{\alpha}(\mathcal{D}_\mu\psi^2_\nu - \mathcal{D}_\nu\psi^2_\mu) \\ \hline \sqrt{\alpha}(\mathcal{D}_\mu\bar{\psi}^1_\nu - \mathcal{D}_\nu\bar{\psi}^1_\mu) & 0 & i\alpha\widetilde{\mathcal{F}}_{\mu\nu} \\ \sqrt{\alpha}(\mathcal{D}_\mu\bar{\psi}^2_\nu - \mathcal{D}_\nu\bar{\psi}^2_\mu) & -i\alpha\widetilde{\mathcal{F}}_{\mu\nu} & 0 \end{array}\right) ,\end{aligned} \tag{8.15}$$

with extended AdS field strength $\widetilde{F}_{\mu\nu}$ (summation over $I = 1, 2$ is implied)

$$\widetilde{F}_{\mu\nu} = F_{\mu\nu} - i\alpha(\psi^I_\mu\bar{\psi}^I_\nu - \psi^I_\nu\bar{\psi}^I_\mu) = \frac{1}{4}\widetilde{R}_{\mu\nu}{}^{mn}\sigma_{mn} - \frac{\alpha}{2}\widetilde{T}_{\mu\nu}{}^m\gamma_m , \tag{8.16}$$

involving extended curvature tensor $\widetilde{R}_{\mu\nu}{}^{mn}$ and extended torsion $\widetilde{T}_{\mu\nu}{}^{m}$, given by

$$\widetilde{R}_{\mu\nu}{}^{mn} := R_{\mu\nu}{}^{mn} - \alpha^2(e_\mu^m e_\nu^n - e_\mu^n e_\nu^m) - i\alpha(\bar{\Psi}_\mu \sigma^{mn}\Psi_\nu) \ , \tag{8.17}$$

$$\widetilde{T}_{\mu\nu}{}^{m} := T_{\mu\nu}{}^{m} + i(\bar{\Psi}_\mu \gamma^m \Psi_\nu) \ . \tag{8.18}$$

Electromagnetic field strength is also modified by a bilinear current term $\mathcal{J}_{(e)}$,

$$\widetilde{\mathcal{F}}_{\mu\nu} := \mathcal{F}_{\mu\nu} - \mathcal{J}_{(e)\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \bar{\Psi}_\mu i\sigma^2 \Psi_\nu \ . \tag{8.19}$$

Note that Pauli matrix $i\sigma^2$ mixes the two Majorana components in $\mathcal{J}_{(e)}$.

In the fermionic sector of $\mathbb{F}_{\mu\nu}$, we introduced

$$\mathcal{D}_\mu \psi_\nu^1 := D_\mu \psi_\nu^1 + \alpha \mathcal{A}_\mu \psi_\nu^2 \ , \tag{8.20}$$

$$\mathcal{D}_\mu \psi_\nu^2 := D_\mu \psi_\nu^2 - \alpha \mathcal{A}_\mu \psi_\nu^1 \ , \tag{8.21}$$

where D_μ stands for $SO(2,3)$ covariant derivative. The fact that Majorana spinors ψ_μ^1 and ψ_ν^2 are not charged is reflected in the manner in which they couple to the gauge field $\mathcal{A}_\mu$. Using them, we can define two charged Dirac vector-spinors $\psi_\mu^\pm = \psi_\mu^1 \pm i\psi_\mu^2$, related to each other by C-conjugation, $\psi_\mu^- = \psi_\mu^{c+} = C\bar{\psi}_\mu^{+T}$, that do couple to $\mathcal{A}_\mu$ in the right way. Using the Pauli matrix $i\sigma^2$ we can unify (8.20) and (8.21)

$$\mathcal{D}_\mu \Psi_\nu = (D_\mu + \alpha \mathcal{A}_\mu i\sigma^2)\Psi_\nu = \left(D_\mu^L + \frac{i\alpha}{2}\gamma_\mu + \alpha \mathcal{A}_\mu i\sigma^2 \right) \Psi_\nu \ . \tag{8.22}$$

Now consider an action, similar to the one defined in (4.13) for pure gravity, but now appropriately generalized to be invariant under extended $OSp(4|2)$ gauge transformations,

$$S_{42} = \mathrm{STr} \int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma} \mathbb{F}_{\mu\nu}(a\mathbb{I}_{6\times 6} + b\Phi^2/l^2)\mathbb{F}_{\rho\sigma}\Phi \ . \tag{8.23}$$

The action is real and we introduced a pair of free parameters, a and b, that will by fixed later. The first part of (8.23) is quadratic in the gauge field strength and the second part (b-term) is necessary for having local SUSY after gauge fixing.

Generalized auxiliary field Φ is given by the following supermatrix (there are two Majorana spinors λ_1 and λ_2, and scalar fields π, m and σ, see also [98, 126–128]),

$$\Phi = \left(\begin{array}{c|cc} \frac{1}{4}\pi + i\phi^a\gamma_a\gamma_5 + \phi^5\gamma_5 & \lambda_1 & \lambda_2 \\ \hline -\bar{\lambda}_1 & \pi - \sigma & m \\ \cline{2-3} -\bar{\lambda}_2 & m & \sigma \end{array} \right) \ . \tag{8.24}$$

In the physical gauge, $\lambda_1 = \lambda_2 = \pi = \sigma = m = \phi^a = 0$ and $\phi^5 = l$, yielding

$$\Phi|_{\text{g.f.}} = \left(\begin{array}{c|cc} l\gamma_5 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{1}{|c}{0} \\ \cline{2-3} 0 & 0 & \multicolumn{1}{|c}{0} \end{array}\right) . \tag{8.25}$$

Field strength $\mathbb{F}_{\mu\nu}$ and the auxiliary field transform in the adjoint representation of $OSp(4|2)$, with infinitesimal variations

$$\delta_\epsilon \mathbb{F}_{\mu\nu} = i[\epsilon, \mathbb{F}_{\mu\nu}] \ , \quad \delta_\epsilon \Phi = i[\epsilon, \Phi] \ , \tag{8.26}$$

for some $\mathfrak{osp}(4|2)$-valued gauge parameter ϵ given by a supermatrix,

$$\epsilon = \left(\begin{array}{c|cc} \frac{1}{2}\epsilon^{AB} M_{AB} & \xi_1 & \xi_2 \\ \hline \bar{\xi}_1 & 0 & \multicolumn{1}{|c}{i\alpha} \\ \cline{2-3} \bar{\xi}_2 & -i\alpha & \multicolumn{1}{|c}{0} \end{array}\right) . \tag{8.27}$$

From (8.26), the invariance of the action (8.23) follows immediately.

After the gauge fixing, field Φ^2/l^2 that appears in the second term of (8.23) becomes a *projector* that reduces any $\mathfrak{osp}(4|2)$ supermatrix to its $\mathfrak{so}(2,3)$ sector, and the classical $OSp(4|2)$ gauge-invariant action (8.23) reduces to

$$S_{42}|_{\text{g.f.}} = \int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma} \left(\frac{(a+b)il}{4} \widetilde{R}_{\mu\nu}{}^{mn} \widetilde{R}_{\rho\sigma}{}^{rs} \varepsilon_{mnrs} - 4al(\mathcal{D}_\mu \bar{\Psi}_\nu \gamma_5 \mathcal{D}_\rho \Psi_\sigma) \right) . \tag{8.28}$$

The term that is quadratic in the Lorentz $SO(1,3)$ covariant derivative D^L_μ can be transformed by partial integration,

$$\int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma} (D^L_\mu \bar{\Psi}_\nu \gamma_5 D^L_\rho \Psi_\sigma) = \frac{1}{16} \int \mathrm{d}^4x \ \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{mn} (\bar{\Psi}_\rho \sigma^{rs} \Psi_\sigma) \varepsilon_{mnrs} \ , \tag{8.29}$$

where we invoked the commutator of two Lorentz covariant derivatives

$$i[D^L_\mu, D^L_\nu]\Psi_\sigma = \tfrac{1}{4} R_{\mu\nu}{}^{mn} \sigma_{mn} \Psi_\sigma \ . \tag{8.30}$$

A term of the same type appears in the first part of the action (8.28). These two contributions have to cancel each other in order to have SUSY, and this implies the constraint $b = -a/2$. Moreover, to obtain the correct normalization of the

Einstein-Hilbert term, we set $a = il/32\pi G_N = il/4\kappa^2$, yielding

$$S_{42}|_{\text{g.f.}} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x \left(e\Big(R(e,\omega) - 6\alpha^2\Big) + \frac{1}{16\alpha^2} R_{\mu\nu}{}^{mn} R_{\rho\sigma}{}^{rs} \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{mnrs} \right. \tag{8.31}$$
$$\left. + \varepsilon^{\mu\nu\rho\sigma}\Big(2\bar{\Psi}_\mu\gamma_5\gamma_\nu(D_\rho + \alpha\mathcal{A}_\rho i\sigma^2)\Psi_\sigma + i\mathcal{F}_{\mu\nu}(\bar{\Psi}_\rho\gamma_5 i\sigma^2\Psi_\sigma) - \frac{i}{2}(\bar{\Psi}_\mu i\sigma^2\Psi_\nu)(\bar{\Psi}_\rho\gamma_5 i\sigma^2\Psi_\sigma)\Big)\right) .$$

However, this is *not* the full $N = 2$ AdS_4 SUGRA action. The gravity part is correct (we can omit the topological Gauss-Bonnet term) and we also get the correct kinetic term for the gravitino doublet. There are also two bilinear source terms, electric and magnetic,

$$\mathcal{J}_{(e)\mu\nu} := \bar{\Psi}_\mu i\sigma^2\Psi_\nu \ , \quad \mathcal{J}^{\mu\nu}_{(m)} := \tfrac{i}{2e}\varepsilon^{\mu\nu\rho\sigma}(\bar{\Psi}_\rho\gamma_5 i\sigma^2\Psi_\sigma) \ . \tag{8.32}$$

But we are missing the contribution from the $SO(2)$ part of the bosonic sector, in particular, the kinetic term for $U(1)$ gauge field $\mathcal{A}_\mu$. The reason for this defect can be traced back to the specific form that the auxiliary field assumes in the physical gauge $\Phi|_{\text{g.f.}}$ (8.25); it completely annihilates the $SO(2)$ sector of any $\mathfrak{osp}(4|2)$ supermatrix. To restore the missing terms, we must introduce an additional action, supplementing (8.23). In [109], following the approach of [110], we defined a classical action invariant under $SO(2,3)\times U(1)$ gauge transformations ($\sim$bosonic sector of $OSp(4|2)$) that involves an *additional auxiliary field* $f = \frac{1}{2}f^{AB}M_{AB}$. Its role is to produce the canonical kinetic term for $U(1)$ gauge field in the absence of the Hodge dual operator (this is, of course, the crucial point, we are trying to construct a purely geometrical action that does not involve the metric tensor $g_{\mu\nu}$ explicitly). This auxiliary field f is a $U(1)$-neutral 0-form that takes values in $\mathfrak{so}(2,3)$ algebra, and it transforms in the adjoint representation of $SO(2,3)$.

The way to proceed is to employ this auxiliary field method to include the modified $U(1)$ field strength $\widetilde{\mathcal{F}}_{\mu\nu}$ defined in (8.19). However, there seems to be no way to construct an $OSp(4|2)$ gauge invariant action that is compatible with this procedure. Therefore, we will use an action, analogous to the one in [109], invariant under the purely bosonic $SO(2,3)\times U(1)$ sector of $OSp(4|2)$, involving the bosonic field strength $\widetilde{f}_{\mu\nu} := \widetilde{F}_{\mu\nu} + \kappa^{-1}\widetilde{\mathcal{F}}_{\mu\nu} = \widetilde{F}_{\mu\nu} + \kappa^{-1}(\mathcal{F}_{\mu\nu} - \mathcal{J}_{(e)\mu\nu})$ of $SO(2,3)\times U(1)$. The action is given by

$$S_A = \mathrm{Tr}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\left(c\, f\widetilde{f}_{\mu\nu}D_\rho\phi D_\sigma\phi\phi + d\, f^2 D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi \right) + c.c. \tag{8.33}$$

Note that, by doing this, we lose the complete $OSp(4|2)$ gauge invariance of the undeformed action before the symmetry breaking. Nevertheless, we will obtain the correct action for $N=2$ SdS_4 SUGRA in the physical gauge, and this is the only requirement that has to be satisfied in order to perform NC deformation.

After calculating traces (see Appendix B) we obtain

$$\begin{aligned} S_A = \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} \bigg(& \frac{ic}{2} f^{AB} F_{\mu\nu}{}^{CD} (D_\rho\phi)^E (D_\sigma\phi)^F \phi^G (\eta_{FG}\varepsilon_{ABCDE} + 2\eta_{AD}\varepsilon_{BCEFG}) \\ & + c\kappa^{-1} f^{AB} \widetilde{\mathcal{F}}_{\mu\nu} (D_\rho\phi)^E (D_\sigma\phi)^F \phi^G \varepsilon_{ABEFG} \\ & - \frac{id}{2} f^{AB} f_{AB} (D_\mu\phi)^E (D_\nu\phi)^F (D_\rho\phi)^G (D_\sigma\phi)^H \phi^R \varepsilon_{EFGHR} \bigg) + c.c. \end{aligned} \tag{8.34}$$

We conclude that parameter c must be real; otherwise, the second term, involving $\widetilde{\mathcal{F}}_{\mu\nu}$, would be purely imaginary and would not contribute (and this term is the one that we need to include). Therefore, assuming real c, the first term (involving gravitational quantities like curvature tensor and torsion) becomes purely imaginary and vanishes after adding its complex conjugate (c.c.). Also, d must be purely imaginary for the procedure to work.

Gauge fixing yields

$$S_A|_{\text{g.f.}} = \int \mathrm{d}^4x\, e \Big(-8lc\kappa^{-1} f^{ab} \widetilde{\mathcal{F}}_{\mu\nu} e^a_\mu e^b_\nu + 24ild\, f^{AB} f_{AB} \Big) . \tag{8.35}$$

By varying this gauge fixed action over f^{ab} and f^{a5} independently, we obtain algebraic equations of motion (EoM) for the components f_{ab} and f_{a5} of the auxiliary field f, respectively, and they are given by

$$f_{ab} = -\frac{ic}{6\kappa d} \widetilde{\mathcal{F}}_{\mu\nu} e^\mu_a e^\nu_b \ , \quad f_{a5} = 0 \ . \tag{8.36}$$

Inserting them back into the action (8.35), we obtain

$$S_A|_{\text{g.f.}} = \frac{2ilc^2}{3\kappa^2 d} \int \mathrm{d}^4x\, e\, \widetilde{\mathcal{F}}^2 \ . \tag{8.37}$$

To get the consistent normalization, we set the prefactor to $(8\kappa^2)^{-1}$, yielding another constraint $16ilc^2 = 3d$ for the parameters c and d. To make the connection with the results of [109], we take $c = 1/32l$ and $d = i/192l$, implying

$$f_{ab} = -\kappa^{-1} \widetilde{\mathcal{F}}_{\mu\nu} e^\mu_a e^\nu_b \ . \tag{8.38}$$

Therefore, after imposing the physical gauge, the original bosonic action (8.33), invariant under $SO(2,3)\times U(1)$ gauge transformations, reduces to the $SO(1,3)\times U(1)$ gauge-invariant action containing the canonical kinetic term for $U(1)$ gauge field $\mathcal{A}_\mu$ in curved space-time and two additional terms involving gravitino current $\mathcal{J}_{(e)\mu\nu}=\bar{\Psi}_\mu i\sigma^2\Psi_\nu$,

$$S_A|_{\text{g.f.}}=\frac{1}{4\kappa^2}\int \mathrm{d}^4x\; e\;\widetilde{\mathcal{F}}^2=\frac{1}{4\kappa^2}\int \mathrm{d}^4x\; e\;\left(\mathcal{F}^2-2\mathcal{F}\cdot\mathcal{J}_{(e)}+\mathcal{J}_{(e)}^2\right)\;. \tag{8.39}$$

This is exactly the piece that was missing in (8.31). With this result in hand, we have the complete classical $N=2$ AdS_4 SUGRA action [28, 29],

$$\begin{aligned}(S_{42}+S_A)|_{\text{g.f.}}=-\frac{1}{2\kappa^2}\int \mathrm{d}^4x\; e\Bigg(&R-6\alpha^2+2e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\Psi}_\mu\gamma_5\gamma_\nu(D_\rho+\alpha\mathcal{A}_\rho i\sigma^2)\Psi_\sigma\\&+2\mathcal{F}\cdot\mathcal{J}_{(m)}-\mathcal{J}_{(e)}\cdot\mathcal{J}_{(m)}-\frac{1}{4}\left(\mathcal{F}^2+\mathcal{J}_{(e)}^2-2\mathcal{F}\cdot\mathcal{J}_{(e)}\right)\Bigg)\;.\end{aligned} \tag{8.40}$$

The most important characteristics of this SUGRA model are the negative cosmological constant $\Lambda=-3\alpha^2=-3/l^2$ and the fact that $U(1)$ coupling strength is equal to the WI contraction parameter α. Under WI contraction ($\alpha\to 0$), the $N=2$ AdS_4 SUGRA action consistently reduces to the $N=2$ Poincaré SUGRA action.

In terms of charged Dirac vector-spinors $\psi^\pm_\mu=\psi^1_\mu\pm i\psi^2_\mu$ (actually, we can use only one of them since they are related to each other by C-conjugation) the action becomes

$$\begin{aligned}(S_{42}+S_A)|_{\text{g.f.}}=-\frac{1}{2\kappa^2}\int \mathrm{d}^4x\; e\Bigg(&R(e,\omega)-6\alpha^2+2e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}^+_\mu\gamma_5\gamma_\nu(D_\rho-i\alpha\mathcal{A}_\rho)\psi^+_\sigma\\&+2\mathcal{F}\cdot\mathcal{J}^+_{(m)}-\mathcal{J}^+_{(e)}\cdot\mathcal{J}^+_{(m)}-\frac{1}{4}\left(\mathcal{F}^2-2\mathcal{F}\cdot\mathcal{J}^+_{(e)}+(\mathcal{J}^+_{(e)})^2\right)\Bigg)\;,\end{aligned} \tag{8.41}$$

with $\mathcal{J}^+_{(e)}=\frac{1}{2i}(\bar{\psi}^+_\mu\psi^+_\nu-\bar{\psi}^+_\nu\psi^+_\mu)$ and $\mathcal{J}^+_{(m)}=\frac{1}{4e}(\bar{\psi}^+_\mu\gamma_5\psi^+_\nu-\bar{\psi}^+_\nu\gamma_5\psi^+_\mu)$.

For later purposes, we note that action (8.41) contains a mass-like term for the charged gravitino (we absorb the parameter κ^{-1} into ψ^+_μ to obtain the canonical dimensions),

$$i\alpha\int \mathrm{d}^4x\; e\;\bar{\psi}^+_\mu\sigma^{\mu\nu}\psi^+_\nu\;, \tag{8.42}$$

with mass-like parameter equal to the WI contraction parameter.

8.2 $OSp(4|1)$ SUGRA

The $OSp(4|1)$ supergroup has 14 generators: ten bosonic AdS generators $\hat{M}_{AB}$, and four fermionic generators $\hat{Q}_\alpha$ comprising a single Majorana spinor (describing a single neutral gravitino). The supermatrix for the $OSp(4|1)$ gauge field Ω_μ is

$$\Omega_\mu = \left(\begin{array}{c|c} \omega_\mu & \sqrt{\alpha}\psi_\mu \\ \hline \sqrt{\alpha}\bar{\psi}_\mu & 0 \end{array}\right) . \tag{8.43}$$

Consider the following action invariant under $OSp(4|1)$ gauge transformations [97]:

$$S_{41} = \frac{il}{32\pi G_N} \mathrm{STr} \int \mathrm{d}^4x \; \varepsilon^{\mu\nu\rho\sigma} \mathbb{F}_{\mu\nu} (\mathbb{I}_{5\times 5} - \tfrac{1}{2l^2}\Phi^2) \mathbb{F}_{\rho\sigma} \Phi \; . \tag{8.44}$$

The auxiliary field is

$$\Phi = \left(\begin{array}{c|c} \frac{1}{4}\pi + i\phi^a\gamma_a\gamma_5 + \phi^5\gamma_5 & \lambda \\ \hline -\bar{\lambda} & \pi \end{array}\right) \overset{\text{g.f.}}{=} \left(\begin{array}{c|c} l\gamma_5 & 0 \\ \hline 0 & 0 \end{array}\right) . \tag{8.45}$$

In the physical gauge, the $OSp(4|1)$ gauge-invariant action (8.44) *exactly* the reduces to $N = 1$ AdS_4 SUGRA action [28, 29, 98],

$$\begin{aligned} S_{41}|_{\text{g.f.}} &= -\frac{1}{2\kappa^2} \int \mathrm{d}^4x \; \Big(e\big(R(e,\omega) - 6\alpha^2\big) + 2\varepsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma) \Big) \qquad (8.46) \\ &= -\frac{1}{2\kappa^2} \int \mathrm{d}^4x \; e \Big(R(e,\omega) - 6\alpha^2 + 2e^{-1}\varepsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\gamma_5\gamma_\nu D^L_\rho\psi_\sigma) - 2i\alpha(\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu) \Big) . \end{aligned}$$

It contains the Einstein-Hilbert term with the negative cosmological constant $\Lambda = -3/l^2$, the Rarita-Schwinger kinetic term for the *neutral* gravitino, and a mass-like gravitino term that is needed to insure the invariance under local SUSY (the gravitino actually remains massless). Topological Gauss-Bonnet term is omitted. The cosmological constant and the mass-like term vanish under WI contraction, yielding minimal $N = 1$ Poincaré SUGRA. Note that we do not need additional action terms in (8.44) to obtain a consistent classical theory.

It is shown in [98] that linear (in $\theta^{\mu\nu}$) NC correction to (8.44) vanishes, and that one has to calculate the second order NC correction in order to see NC effects, which is exceedingly difficult. In the following section, we use the Seiberg-Witten approach to NC gauge field theories, to calculate linear NC correction to $N = 2$ AdS_4 SUGRA, and conclude that *it is not equal to zero*. The non-vanishing part comes from the additional bosonic action, S_A.

8.3 NC deformation of $N = 2$ AdS_4 SUGRA

Canonical deformation of the orthoymplectic action (8.23) is obtained by replacing ordinary commutative field multiplication with the Moyal $\star$-product, yielding an NC action invariant under NC-deformed $OSp(4|2)_\star$ gauge transformations,

$$S^\star_{42} = \frac{il}{32\pi G_N}\mathrm{STr}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\left(\hat{\mathbb{F}}_{\mu\nu}\star\hat{\mathbb{F}}_{\rho\sigma}\star\hat{\Phi} - \tfrac{1}{2l^2}\hat{\mathbb{F}}_{\mu\nu}\star\hat{\Phi}\star\hat{\Phi}\star\hat{\mathbb{F}}_{\rho\sigma}\star\hat{\Phi}\right)\ . \quad (8.47)$$

Likewise, we have a canonically deformed version of the bosonic action (8.33) with $c = 1/32l$ and $d = i/192l$,

$$S^\star_A = \frac{1}{32l}\mathrm{Tr}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big(\hat{f}\star\hat{\tilde{f}}_{\mu\nu}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi}$$
$$+\frac{i}{6}\hat{f}\star\hat{f}\star D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star D_\rho\hat{\phi}\star D_\sigma\hat{\phi}\star\hat{\phi}\Big) + c.c. \quad (8.48)$$

Field strength $\hat{\mathbb{F}}_{\mu\nu}$ appearing in (8.47) is defined in terms of $OSp(4,2)_\star$ gauge potential $\hat{\Omega}_\mu$ as

$$\hat{\mathbb{F}}_{\mu\nu} = \partial_\mu\hat{\Omega}_\nu - \partial_\nu\hat{\Omega}_\mu - i[\hat{\Omega}_\mu \overset{\star}{,} \hat{\Omega}_\nu]\ . \quad (8.49)$$

It transforms in the adjoint representation of $OSp(4,2)_\star$ supergroup as well as the NC auxiliary field $\hat{\Phi}$,

$$\delta^\star_\epsilon\hat{\mathbb{F}}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \overset{\star}{,} \hat{\mathbb{F}}_{\mu\nu}]\ ,\quad \delta^\star_\epsilon\hat{\Phi} = i[\hat{\Lambda}_\epsilon \overset{\star}{,} \hat{\Phi}]\ . \quad (8.50)$$

We proceed by expanding the $OSp(4|2)_*$ gauge-invariant NC action (8.47) in powers of the deformation parameter $\theta^{\mu\nu}$ via SW map. By construction, SW map ensures invariance of the expansion under ordinary $OSp(4|2)$ gauge transformations, order-by-order.

Now we present some relevant steps in the expansion procedure of the NC action (8.47). Our goal is to calculate and analyze linear NC correction to the classical action (8.23). According to the SW map, the first order NC corrections of the auxiliary field Φ and the $OSp(4|2)$ field strength $\mathbb{F}_{\mu\nu}$ are given by

$$\hat{\Phi}^{(1)} = -\frac{1}{4}\theta^{\rho\sigma}\{\Omega_\rho, (\partial_\sigma + \widehat{D}_\sigma)\Phi\}\ , \quad (8.51)$$

$$\hat{\mathbb{F}}^{(1)}_{\mu\nu} = -\frac{1}{4}\theta^{\rho\sigma}\{\Omega_\rho, (\partial_\sigma + \widehat{D}_\sigma)\mathbb{F}_{\mu\nu}\} + \frac{1}{2}\theta^{\rho\sigma}\{\mathbb{F}_{\rho\mu}, \mathbb{F}_{\sigma\nu}\}\ , \quad (8.52)$$

where $\widehat{D}_\mu$ stands for the $OSp(4|2)$ covariant derivative (associated to Ω_μ).

Successive application of this rule gives us the first order NC correction to the classical action (8.23):

$$
\begin{aligned}
S_{42}^{(1)} = \frac{il\theta^{\lambda\tau}}{32\pi G_N}\mathrm{STr}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\bigg(&-\frac{1}{4}\{\mathbb{F}_{\lambda\tau},\mathbb{F}_{\mu\nu}\mathbb{F}_{\rho\sigma}\}\Phi + \frac{i}{2}\widehat{D}_\lambda\mathbb{F}_{\mu\nu}\widehat{D}_\tau\mathbb{F}_{\rho\sigma}\Phi \\
&+\frac{1}{2}\{\mathbb{F}_{\lambda\mu},\mathbb{F}_{\tau\nu}\}\mathbb{F}_{\rho\sigma}\Phi + \frac{1}{2}\mathbb{F}_{\mu\nu}\{\mathbb{F}_{\lambda\rho},\mathbb{F}_{\tau\sigma}\}\Phi \\
-\frac{1}{2l^2}\Big(&-\frac{1}{4}\{\mathbb{F}_{\lambda\tau},\mathbb{F}_{\mu\nu}\Phi^2\}\mathbb{F}_{\rho\sigma}\Phi + \frac{i}{2}\widehat{D}_\lambda\mathbb{F}_{\mu\nu}\widehat{D}_\tau\Phi^2\mathbb{F}_{\rho\sigma}\Phi \\
&+\frac{1}{2}\{\mathbb{F}_{\lambda\mu},\mathbb{F}_{\tau\nu}\}\Phi^2\mathbb{F}_{\rho\sigma}\Phi + \frac{i}{4}\mathbb{F}_{\mu\nu}[\widehat{D}_\lambda\Phi,\widehat{D}_\tau\Phi]\mathbb{F}_{\rho\sigma}\Phi \\
&+\frac{i}{2}\mathbb{F}_{\mu\nu}\Phi^2\widehat{D}_\lambda\mathbb{F}_{\rho\sigma}\widehat{D}_\tau\Phi + \frac{1}{2}\mathbb{F}_{\mu\nu}\Phi^2\{\mathbb{F}_{\lambda\rho},\mathbb{F}_{\tau\sigma}\}\Phi\Big)\bigg) \ .
\end{aligned} \tag{8.53}
$$

This linear NC correction is real and invariant under $OSp(4,2)$ gauge transformations. However, a careful examination shows that after the gauge fixing it vanishes completely,

$$
S_{42}^{(1)}|_{g.f.} = 0 \ . \tag{8.54}
$$

But we still have the additional NC action $S_A^\star$ invariant under the purely bosonic NC-deformed $SO(2,3)_\star \times U(1)_\star$ gauge transformations. The only additional SW expansion we need is that of $\hat{f}$, namely

$$
\hat{f} = f - \frac{1}{4}\theta^{\rho\sigma}\{\Omega_\rho,(\partial_\sigma + D_\sigma)f\} + \mathcal{O}(\theta^2) \ . \tag{8.55}
$$

The first order NC correction to (8.48) before gauge fixing is given by

$$
\begin{aligned}
S_A^{(1)} =& S_{Af}^{(1)} + S_{Aff}^{(1)} \\
=& -\frac{\theta^{\lambda\tau}}{64l}\,\mathrm{Tr}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\bigg(- ifD_\lambda\widetilde{f}_{\mu\nu}D_\tau(D_\rho\phi D_\sigma\phi\phi) + \frac{1}{2}\{\widetilde{f}_{\lambda\tau},f\}\widetilde{f}_{\mu\nu}D_\rho\phi D_\sigma\phi\phi \\
&- f\{\widetilde{f}_{\lambda\mu},\widetilde{f}_{\tau\nu}\}D_\rho\phi D_\sigma\phi\phi - if\widetilde{f}_{\mu\nu}D_\lambda(D_\rho\phi D_\sigma\phi)D_\tau\phi \\
&- if\widetilde{f}_{\mu\nu}(D_\lambda D_\rho\phi)(D_\tau D_\sigma\phi)\phi - f\widetilde{f}_{\mu\nu}\{\{\widetilde{f}_{\lambda\rho},D_\tau\phi\},D_\sigma\phi\}\phi \\
&+ \frac{i}{3!}\bigg(\frac{1}{2}\{\widetilde{f}_{\lambda\tau},f^2\}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi - f^2\{[\{\widetilde{f}_{\lambda\mu},D_\tau\phi\},D_\nu\phi],D_\rho\phi D_\sigma\phi\}\phi \\
&- if^2\Big(D_\lambda(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi)D_\tau\phi + D_\lambda(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\tau D_\sigma\phi)\phi \\
&+ D_\lambda(D_\mu\phi D_\nu\phi)(D_\tau D_\rho\phi))D_\sigma\phi\phi + (D_\lambda D_\mu\phi)(D_\tau D_\nu\phi)D_\rho\phi D_\sigma\phi\phi\Big)\bigg)\bigg) + c.c.
\end{aligned}
$$

where we can distinguish the linear f-part and the quadratic f^2-part, and all terms are manifestly $SO(2,3)\times U(1)$ invariant by the virtue of SW map.

After calculating traces and evaluating the gauge-fixed action $S_A^{(1)}|_{\text{g.f.}}$ on the EoM of the components of the auxiliary field f (as it turns out, to obtain the first order NC correction, we only need to insert *zeroth order* (classical) EoM (8.36) in the gauge-fixed *first order* NC action $S_A^{(1)}|_{\text{g.f.}}$), we obtain

$$S^{(1)}_{A,EoM}|_{\text{g.f.}} = \sum_{j=1}^{6} S^{(1)}_{A,EoMf.j}|_{\text{g.f.}} + S^{(1)}_{A,EoMff}|_{\text{g.f.}} \ , \tag{8.56}$$

with the individual terms:

$$\begin{aligned}
S^{(1)}_{A,EoMf.1}|_{\text{g.f.}} &= -\frac{\theta^{\lambda\tau}}{64\kappa}\int \mathrm{d}^4x\; e \left\{ \widetilde{\mathcal{F}}^{\mu\nu} R_{\mu\nu ab}\left(R_{\lambda\tau}{}^{ab} - \frac{2}{l^2}e^a_\lambda e^b_\tau\right) \right.\\
&+ \widetilde{\mathcal{F}}^{\rho\sigma} e^a_\rho e^b_\sigma \left(R_{\mu\nu ab}R_{\lambda\tau}{}^{cd}e^\mu_c e^\nu_d - \frac{2}{l^2}R_{\lambda\tau ab}\right) + 4\widetilde{\mathcal{F}}^{\rho\mu}e^c_\rho\left(R_{\mu\nu ac}R_{\lambda\tau}{}^{ab}e^\nu_b + \frac{2}{l^2}R_{\mu\lambda ac}e^a_\tau\right)\\
&+ \widetilde{\mathcal{F}}_{\rho\sigma}e^\rho_a e^\sigma_b R_{\lambda\tau}{}^{ab}\left(R_{\mu\nu}{}^{mn}e^\mu_m e^\nu_n - \frac{12}{l^2}\right) + \frac{2}{l^2}\widetilde{\mathcal{F}}^{\mu\nu}T_{\lambda\tau}{}^a\left(T_{\mu\nu a} - 2T_{\rho\nu m}e^\rho_a e^m_\mu\right)\\
&\left. - \frac{2}{l^2}\widetilde{\mathcal{F}}_{\lambda\tau}\left(R_{\mu\nu}{}^{mn}e^\mu_m e^\nu_n - \frac{12}{l^2} - 4\frac{l^2}{\kappa^2}\widetilde{\mathcal{F}}^{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\right)\right\} ,
\end{aligned} \tag{8.57}$$

$$\begin{aligned}
S^{(1)}_{A,EoMf.2}|_{\text{g.f.}} &= -\frac{\theta^{\lambda\tau}}{8\kappa}\int \mathrm{d}^4x\; e \left\{ \vphantom{\frac{1}{1}} \right.\\
&+ (D^L_\lambda R_{\mu\nu}{}^{mc})(D^L_\tau e^r_\rho)e^\sigma_c\left(e^\nu_m(\widetilde{\mathcal{F}}_\sigma{}^\mu e^\rho_r - \widetilde{\mathcal{F}}_\sigma{}^\rho e^\mu_r) + \widetilde{\mathcal{F}}_\sigma{}^\nu e^\mu_r e^\rho_m\right)\\
&- \frac{1}{l^2}\widetilde{\mathcal{F}}_\rho{}^\mu e^\rho_c\left(D^L_\lambda T_{\tau\mu}{}^c - e_{\lambda b}R_{\tau\mu}{}^{bc}\right) - \frac{4}{l^2}\widetilde{\mathcal{F}}_\nu{}^\mu(D^L_\lambda e^r_\rho)(D^L_\tau e^m_\mu)e^\nu_m e^\rho_r\\
&- \frac{1}{l^2}(D^L_\lambda e^r_\rho)e^\rho_r\left(e^\nu_c\widetilde{\mathcal{F}}_\tau{}^\mu T_{\mu\nu}{}^c - e^\nu_c\widetilde{\mathcal{F}}_\nu{}^\mu T_{\mu\tau}{}^c\right) + \frac{1}{2l^2}T_{\lambda\tau}{}^r T_{\mu\nu}{}^c\widetilde{\mathcal{F}}_\sigma{}^\nu e^\sigma_c e^\mu_r\\
&+ \frac{1}{l^2}(D^L_\lambda e^r_\rho)e^\nu_r\left(T_{\mu\nu}{}^m(\widetilde{\mathcal{F}}_\tau{}^\mu e^\rho_m - \widetilde{\mathcal{F}}_\tau{}^\rho e^\mu_m) + T_{\tau\nu}{}^m\widetilde{\mathcal{F}}_\mu{}^\rho e^\mu_m\right)\\
&\left. + \frac{2}{l^2}\widetilde{\mathcal{F}}_\sigma{}^\rho e^\sigma_c(D^L_\lambda e^r_\rho)(D^L_\tau e^c_\nu)e^\nu_r + \frac{1}{l^4}\widetilde{\mathcal{F}}_{\lambda\tau}\right\} ,
\end{aligned} \tag{8.58}$$

$$\begin{aligned}
S^{(1)}_{A,EoMf.3}|_{\text{g.f.}} &= \frac{\theta^{\lambda\tau}}{32\kappa}\int \mathrm{d}^4x\; e \left\{ -\widetilde{\mathcal{F}}^{\mu\nu}R_{\lambda\nu am}\left(R_{\tau\mu}{}^{am} - \frac{4}{l^2}e^a_\tau e^m_\mu\right) \right.\\
&+ \widetilde{\mathcal{F}}_{\rho\sigma}R_{\lambda\mu}{}^{am}R_{\tau\nu}{}^{bn}\left(e^\mu_a e^\nu_m e^\rho_b e^\sigma_n + e^\rho_a e^\sigma_m e^\mu_b e^\nu_n + 2e^\rho_n e^\sigma_m(e^\mu_a e^\nu_b - e^\nu_a e^\mu_b)\right)\\
&\left. - \frac{2}{l^2}e^\rho_n e^\sigma_b\left(2\widetilde{\mathcal{F}}_{\lambda\rho}R_{\tau\sigma}{}^{bn} + \widetilde{\mathcal{F}}_{\rho\sigma}R_{\lambda\tau}{}^{bn}\right) + \frac{2}{l^2}\widetilde{\mathcal{F}}^{\mu\nu}T_{\lambda\mu}{}^a T_{\tau\nu a} + \frac{8}{\kappa^2}\widetilde{\mathcal{F}}^{\mu\nu}\widetilde{\mathcal{F}}_{\lambda\mu}\widetilde{\mathcal{F}}_{\tau\nu}\right\} ,
\end{aligned} \tag{8.59}$$

$$
\begin{aligned}
&\left(S^{(1)}_{A,EoMf.4}|_{\text{g.f.}} + S^{(1)}_{A,EoMf.5}|_{\text{g.f.}}\right) = -\frac{\theta^{\lambda\tau}}{32\kappa}\int \mathrm{d}^4x\, e\, \Bigg\{ \\
&+ R_{\mu\nu}{}^{ab}(D^L_\lambda e^m_\rho)(D^L_\tau e_{\sigma m})\Big(\widetilde{\mathcal{F}}^{\mu\nu}e^\rho_a e^\sigma_b + \widetilde{\mathcal{F}}^{\rho\sigma}e^\mu_a e^\nu_b - 4\widetilde{\mathcal{F}}^{\mu\rho}e^\nu_a e^\sigma_b\Big) \\
&- \frac{1}{l^2}R_{\mu\nu}{}^{ab}\big(\widetilde{\mathcal{F}}^{\mu\nu}e_{\lambda a}e_{\tau b} + \widetilde{\mathcal{F}}_{\lambda\tau}e^\mu_a e^\nu_b - 4\widetilde{\mathcal{F}}^\mu{}_\lambda e^\nu_a e_{\tau b}\big) - \frac{1}{l^2}\widetilde{\mathcal{F}}^{\mu\nu}\eta_{mn}(D^L_\lambda e^m_\mu)(D^L_\tau e^n_\nu) \\
&+ \frac{2}{l^2}T_{\mu\nu}{}^a(D^L_\lambda e^b_\rho)\Big(\widetilde{\mathcal{F}}^{\mu\nu}(2e^\rho_a e_{\tau b} - e^\rho_b e_{\tau a}) + 2\widetilde{\mathcal{F}}_\tau{}^\mu(e^\rho_a e^\nu_b - e^\rho_b e^\nu_a) \\
&+ 2\widetilde{\mathcal{F}}^{\rho\mu}(2e^\nu_a e_{\tau b} - e^\nu_b e_{\tau a}) + 2\widetilde{\mathcal{F}}^\rho{}_\tau e^\mu_a e^\nu_b\Big) - \frac{2}{l^2}\widetilde{\mathcal{F}}^{\rho\mu}T_{\mu\nu c}T_{\lambda\tau}{}^d e^c_\rho e^\nu_d \qquad (8.60) \\
&+ \frac{4}{l^2}T_{\lambda\nu a}(D^L_\tau e^d_\rho)e^a_\sigma\big(\widetilde{\mathcal{F}}^{\sigma\nu}e^\rho_d - \widetilde{\mathcal{F}}^{\sigma\rho}e^\nu_d\big) - \frac{4}{l^2}R_{\lambda\nu}{}^c{}_a e^a_\rho\big(\widetilde{\mathcal{F}}^{\rho\nu}e_{\tau c} - \widetilde{\mathcal{F}}^\rho{}_\tau e^\nu_c\big) - \frac{6}{l^4}\widetilde{\mathcal{F}}_{\lambda\tau}\Bigg\}\,,
\end{aligned}
$$

$$
\begin{aligned}
&S^{(1)}_{A,EoMf.6}|_{\text{g.f.}} = \frac{\theta^{\lambda\tau}}{32\kappa}\int \mathrm{d}^4x\, e\, \Bigg\{\widetilde{\mathcal{F}}^{\rho\sigma}e^a_\rho e^b_\sigma e^\mu_m e^\nu_n\big(R_{\lambda\tau}{}^{mn}R_{\mu\nu ab} - 2R_{\lambda\mu}{}^{mn}R_{\tau\nu ab}\big) \\
&+ \frac{8}{\kappa^2}\big(\widetilde{\mathcal{F}}_{\lambda\tau}\widetilde{\mathcal{F}}^2 - 2\widetilde{\mathcal{F}}^{\mu\nu}\widetilde{\mathcal{F}}_{\lambda\mu}\widetilde{\mathcal{F}}_{\tau\nu}\big) + \frac{2}{l^2}\widetilde{\mathcal{F}}^{\mu\nu}e^a_\mu e^b_\nu R_{\lambda\tau ab} + \frac{4}{l^2}\widetilde{\mathcal{F}}_{\lambda\nu}\epsilon^\nu_a e^\rho_b R_{\tau\rho}{}^{ab} \\
&- \frac{4}{l^2}\widetilde{\mathcal{F}}^{\mu\nu}e^a_\nu e^\rho_b\big(T_{\rho\mu a}T_{\lambda\tau}{}^b - T_{\mu\lambda a}T_{\tau\rho}{}^b - T_{\lambda\rho a}T_{\tau\mu}{}^b\big) - \frac{8}{l^4}\widetilde{\mathcal{F}}_{\lambda\tau}\Bigg\}\,, \qquad (8.61)
\end{aligned}
$$

and, finally, the f^2-term,

$$
S^{(1)}_{A,EoMff}|_{\text{g.f.}} = -\frac{\theta^{\lambda\tau}}{16\kappa^3}\int \mathrm{d}^4x\, e\, \widetilde{\mathcal{F}}_{\lambda\tau}\widetilde{\mathcal{F}}^2\,. \qquad (8.62)
$$

Action (8.56) represents the first order NC correction to $N = 2$ AdS_4 SUGRA. It involves various new couplings between $U(1)$ gauge field, gravity and gravitini fields that appear due to space-time noncommutativity. As it stands, this action seems too complicated to be analyzed in its entirety. However, we can restrict ourselves to some particular domain of parameters and work with an approximated NC action. In particular, we will derive a low-energy approximation of (8.56), by taking into account terms at most quadratic in the partial derivative. Therefore, we include only terms linear in curvature, and linear and quadratic in torsion. Additionally, we assume that spin connection $\omega_\mu{}^{ab}$ and the first order derivatives of the vierbeins are of the same order. Note also that the torsion constraint $\widetilde{T}_{\mu\nu}{}^a = 0$ (8.18) gives us $T_{\mu\nu}{}^a = -i\Psi_\mu\gamma^a\Psi_\nu$. These assumptions yield a very simple action,

$$
\begin{aligned}
S^{(1)}_{\text{low-energy}} &= -\frac{9\theta^{\mu\nu}}{16l^4\kappa}\int \mathrm{d}^4x\, e\, \widetilde{\mathcal{F}}_{\mu\nu} = -\frac{9\theta^{\mu\nu}}{16l^4\kappa}\int \mathrm{d}^4x\, e\, (\mathcal{F}_{\mu\nu} - \bar{\Psi}_\mu i\sigma^2\Psi_\nu) \\
&= -\frac{9}{8l^4\kappa}\int \mathrm{d}^4x\, e\, (\bar{\psi}^+_\mu i\Theta^{\mu\nu}\psi^+_\nu) + \text{surface term}\,. \qquad (8.63)
\end{aligned}
$$

This mass-like term for charged gravitino ψ_μ^+, minimally coupled to gravity, appears due to space-time noncommutativity and "renormalizes" the corresponding term (8.42) in the classical SUGRA action (8.41). If we again absorb κ^{-1} in ψ_μ^+ to obtain the canonical dimensions, the mass-like parameter is $\sim l_P\Lambda_{NC}^2/l^4$, and it vanishes under WI contraction.

After WI contraction, the action (8.56) reduces to

$$\begin{aligned}
S_A|_{\text{g.f.}} \overset{\text{WI}}{=} -\frac{\theta^{\lambda\tau}}{64\kappa}\int \mathrm{d}^4x\; e\,\Big\{ & \widetilde{\mathcal{F}}^{\mu\nu}R_{\mu\nu\rho\sigma}R_{\lambda\tau}{}^{\rho\sigma} - \widetilde{\mathcal{F}}^{\mu\nu}R_{\rho\sigma\mu\nu}R_{\lambda\tau}{}^{\rho\sigma} - 4\widetilde{\mathcal{F}}^{\mu\rho}R_{\mu\nu\rho\sigma}R_{\lambda\tau}{}^{\nu\sigma} \\
& - 2\widetilde{\mathcal{F}}^{\mu\nu}R_{\lambda\mu}{}^{\rho\sigma}R_{\tau\nu\rho\sigma} + 8\widetilde{\mathcal{F}}_{\rho\sigma}R_{\lambda\mu}{}^{\mu\rho}R_{\tau\nu}{}^{\nu\sigma} + \widetilde{\mathcal{F}}_{\mu\nu}R_{\lambda\tau}{}^{\mu\nu}R - \frac{4}{\kappa^2}\widetilde{\mathcal{F}}_{\lambda\tau}\widetilde{\mathcal{F}}^2 + \frac{16}{\kappa^2}\widetilde{\mathcal{F}}^{\mu\nu}\widetilde{\mathcal{F}}_{\lambda\mu}\widetilde{\mathcal{F}}_{\tau\nu} \\
& + 8(D^L_\lambda R_{\mu\nu}{}^{mc})(D^L_\tau e^r_\rho)e^\sigma_c\Big(\widetilde{\mathcal{F}}_\sigma{}^\mu e^\nu_m e^\rho_r - \widetilde{\mathcal{F}}_\sigma{}^\rho e^\mu_r e^\nu_m + \widetilde{\mathcal{F}}_\sigma{}^\nu e^\mu_r e^\rho_m\Big) \\
& + 2R_{\mu\nu}{}^{ab}\eta_{rs}(D^L_\lambda e^r_\rho)(D^L_\tau e^s_\sigma)\Big(\widetilde{\mathcal{F}}^{\mu\nu}e^\rho_a e^\sigma_b + \widetilde{\mathcal{F}}^{\rho\sigma}e^\mu_a e^\nu_b - 4\widetilde{\mathcal{F}}^{\mu\rho}e^\nu_a e^\sigma_b\Big)\Big\}\,.
\end{aligned} \tag{8.64}$$

At this point, we are confronted with an interesting question. The fact that $N=2$ AdS_4 superalgebra contracts to $N=2$ Poinacaré superalgebra when $l\to\infty$ is consistently reflected on the level of classical (undeformed) action (8.40); classical $N=2$ AdS_4 SUGRA reduces to classical $N=2$ Poincaré SUGRA under WI contraction. However, it is not *a priori* clear whether this relation holds *after NC deformation*, that is, whether NC deformation and WI contraction actually commute. For that matter, one would have to explicitly compute the NC correction to classical $N=2$ Poincaré SUGRA and compare it to the action (8.64).

9 Conclusion and Outlook

Let us conclude by giving a brief survey of the thesis, summarizing what is thus far accomplished by it and proposing some new directions of research. First of all, we should emphasize that the obtained results are to be regarded as an extension and upgrade of the substantial body of work that was previously established by many authors. In its present state, the content of the thesis certainly does not amount to a complete account on the subject and it opens a plethora of new questions that ought to be treated in the future. This is perhaps its greatest value.

There are several major themes in the thesis. In general, our goal was to define and study consistent NC deformations of some classical (i.e. undeformed) gauge field theories, including gravity. The AdS gauge theory, having $SO(2,3)$ connection as the only dynamical field in play, was our starting point. The advantage of the associated $SO(2,3)$ gauge-invariant action is that it does not *explicitly* involve quantities related to the underlying space-time manifold, such as metric, curvature or torsion. To relate this theory with GR, one has to break the original $SO(2,3)$ gauge symmetry down to the usual Lorentz $SO(1,3)$ gauge symmetry. For that we used a constrained auxiliary field (the method already advocated in the literature). The symmetry breaking is imposed directly, by fixing the components of the auxiliary field; this is another important general aspect of our approach. After the gauge fixing, the components of the $SO(2,3)$ gauge field are identified with the Lorentz spin-connection and the vierbein field, thus obtaining their proper unification. Canonical (θ-constant) NC deformation is performed along the lines of the Seiberg-Witten approach to NC gauge field theory. Classical actions are promoted into their NC counterparts by introducing Moyal $\star$-product instead of the commutative point-wise field multiplication. The resulting non-extended NC actions are subsequently expanded in powers of the deformation parameter $\theta^{\mu\nu}$ via SW map, up to the first order. By construction, expanded actions are endowed with the gauge symmetry of the corresponding classical actions, order-by-order. After imposing the gauge fixing condition (physical gauge), NC corrections emerge.

The $SO(2,3)_\star$ model of NC gravity has been previously established and well developed. It provided a basic framework for the research presented in this thesis. However, its most significant insight, concerning the origin of the apparent breaking of diffeomorphism invariance in canonically deformed theories, remains to be fully understood. This will certainly be one of the major research directions in the future. The $SO(2,3)_\star$ model of NC gravity exhibits quadratic (in the deformation parameter) NC correction to GR, which is notoriously difficult to analyze even in the low-energy regime. Moreover, it seems that this is a generic property of NC gravity. Inclusion of matter fields coupled to NC gravity improves the situation drastically; it provides a non-trivial linear NC correction to the classical action. We considered Dirac spinor field coupled to $U(1)$ gauge field and obtain a new model of NC Electrodynamics that can be analyzed both in curved and flat space-time. Some important predictions of this theory are *the NC birefringence effect* (energy levels of an electron become helicity-dependent due to space-time noncommutativity) and *NC-deformed relativistic Landau levels* of an electron in background magnetic field. Since they appear at the lowest perturbative order, these results could be used to constrain the yet unknown length scale Λ_{NC} at which NC effects become relevant. One could also proceed by calculating perturbative loop corrections and explore the renormalizability properties of this NC QED model. A generalization to NC Yang-Mills theory is straightforward and it may turn out to be relevant for the study of the early Universe and quark-gluon plasma.

To build a complete NC Standard Model within $SO(2,3)$ framework, one has to introduce scalar fields. It is fairly easy to construct an action for *real scalar field* φ,

$$S_\varphi = i\mathrm{Tr}\int \mathrm{d}^4x\; \varepsilon^{\mu\nu\rho\sigma}\left(fD_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\varphi + 6lf^2 D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\right) + c.c.$$

that reduces to the standard kinetic action after imposing the gauge condition,

$$S_\varphi\big|_{\mathrm{g.f.}} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\; g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi\ .$$

This action has quadratic NC correction after canonical deformation.

However, constructing an action for scalar electrodynamic that involves *complex scalar field* poses some severe difficulties. In this case, the total gauge group is $SO(2,3)\times U(1)$ and the auxiliary field f has to be $U(1)$-charged as well. Everything is consistent at the classical level, but the problem arises (and it seems to be a generic one) when we try to apply the Seiberg-Witten prescription to the auxiliary field f that transforms differently under $SO(2,3)$ versus $U(1)$. If one want to introduce the Higgs sector in the NC theory, this issue has to be resolved.

Regarding SUGRA, our primary goal was to obtain explicit NC correction to $N = 2$ AdS SUGRA in $D = 4$. We stared with an undeformed action (8.23) of the MacDowell-Mansouri type (already advocated in the literature), invariant under orthosymplectic $OSp(4|2)$ gauge transformations. However, this action alone is not enough to obtain $N = 2$ AdS_4 SUGRA after imposing the gauge fixing condition. In particular, one has to add a supplementary action (8.33) endowed with $SO(2,3) \times U(1)$ gauge symmetry (bosonic sector of $OSp(4|2)$) that provides the missing terms in the classical action obtained from (8.23) (e.g. the kinetic term for $U(1)$ gauge field). Therefore, we have the following schema:

$$\underbrace{\begin{array}{ccc} (OSp(4|2)\text{ invariant action}) & + & (SO(2,3)\times U(1)\text{ invariant action}) \\ \text{g.f.}\Big\downarrow & & \Big\downarrow\text{g.f.} \\ (SO(1,3)\times U(1)\text{ invariant action}) & + & (SO(1,3)\times U(1)\text{ invariant action}) \end{array}}_{\text{N=2 AdS SUGRA in D=4}}$$

This situation seems curious considering that a similar $OSp(4|1)$ gauge-invariant action (8.44) reduces to the complete $N = 1$ AdS_4 SUGRA action without the need of including any additional terms. We may conclude that the extended $N > 1$ AdS_4 SUGRA cannot be obtained simply by gauging the corresponding orthosymplectic group $OSp(4|N)$ and subsequently fixing the gauge. For $N > 2$ one is compelled to include an additional term, similar to the one for $N = 2$, that involves non-Abelian Yang-Mills gauge field.

For the $OSp(4|2)$ gauge-invariant part of the classical action, linear NC correction vanishes. This result was not expected. Namely, we have previously established that the canonical NC deformation of pure gravity, regarded as a gauge theory of $SO(2,3)_\star$ group, leads to the quadratic NC correction. However, after including matter fields, e.g. Dirac spinors coupled to $U(1)$ gauge field, linear NC correction appears. Since we can take a pair of Majorana vector-spinors of $OSp(4|2)$ SUGRA and form a pair of $U(1)$-charged Dirac vector-spinors, related to each other by C-conjugation, we expected to obtain a non-vanishing first order NC correction from the $OSp(4|2)$ action, as well. It is worth mentioning that the second order NC correction to $OSp(4|N)$ SUGRA has the same structure for every N. Analysis of the higher NC SUGRA corrections is another possible research problem.

The supplementary bosonic action does, however, provide a non-trivial linear NC correction that is calculated explicitly. It involves various new interaction terms that appear due to space-time noncommutativity. The classical action is constructed by applying the same auxiliary field method as for the AdS Electrodynamics. The full action is difficult to analyze, but we can restrict ourselves to the low-energy sector of the theory by taking into account only terms that are at most quadratic in partial derivatives. This leaves us with a single mass-like term for charged gravitino.

There are two additional terms with $OSp(4|2)$ gauge symmetry that we could take into account. We denote them by S' and S'' and they are given by

$$
\begin{aligned}
S' &= \frac{a'}{128\pi G_N l}\,\mathrm{STr}\int \mathrm{d}^4x\;\varepsilon^{\mu\nu\rho\sigma}\mathbb{F}_{\mu\nu}\widehat{D}_\rho\Phi\widehat{D}_\sigma\Phi\Phi + c.c.\ ,\\
S'' &= -\frac{ia''}{128\pi G_N l^3}\,\mathrm{STr}\int \mathrm{d}^4x\;\varepsilon^{\mu\nu\rho\sigma}\widehat{D}_\mu\Phi\widehat{D}_\nu\Phi\widehat{D}_\rho\Phi\widehat{D}_\sigma\Phi\Phi\ ,
\end{aligned}
$$

where we have two free dimensionless parameters a', a'' and $OSp(4|2)$ covariant derivative $\widehat{D}_\mu$. Their $SO(2,3)$ gauge-invariant counterparts have already been analyzed in the literature. After the gauge fixing, they modify the coefficients in the classical action but do not introduce new type of terms. In particular, they give us a freedom to eliminate the cosmological constant in the classical action. NC deformation of S' and S'' will change our final result, but their importance is not immediately evident. Analysis of these additional NC corrections remains to be done.

Finally, perhaps one of the most intriguing questions that arises out of these considerations concerns the compatibility of the Wigner-Inönü contraction and canonical NC deformation. WI contraction is a formal statement of the correspondence principle from the aspect of symmetry and it is not fully understood how NC deformation affects this operation. We have shown in this thesis that for AdS Electrodynamics and $N=2$ AdS_4 SUGRA WI contraction works well at the level of their classical actions. Also, after NC deformation, we obtained non-vanishing contracted NC actions. To determine whether the correspondence principle is consistent with the NC-deformed symmetry, we have to calculate NC corrections to classical electrodynamics in curved space-time and $N=2$ Poincaré SUGRA by the method of minimal substitution. It seems that our results suggest that the answer is negative, at least in Minkowski space.

A Spinor action - individual terms

The kinetic spinor action (5.29) contains eight terms *before* gauge fixing. Here we present them, in the order of appearance in (5.29), *after* the gauge fixing and what remains of them after the Wigner-Inönü (WI) contraction.

Terms from the kinetic spinor action after the gauge fixing:

$$
\begin{aligned}
S_1^{(1)} = &+ \frac{\theta^{\alpha\beta}}{16}\int \mathrm{d}^4x\; e\; R_{\alpha\beta}{}^{ab}e^\sigma_b(\bar\psi\gamma_a D^L_\sigma\psi) - \frac{i\theta^{\alpha\beta}}{32}\int \mathrm{d}^4x\; e\; R_{\alpha\beta}{}^{ab}\varepsilon_{abc}{}^{d}e^\sigma_d(\bar\psi\gamma^c\gamma^5 D^L_\sigma\psi)\\
&+ \frac{i\theta^{\alpha\beta}}{16l}\int \mathrm{d}^4x\; e\; T_{\alpha\beta}{}^{a}e^\sigma_a(\bar\psi D^L_\sigma\psi) + \frac{\theta^{\alpha\beta}}{16l}\int \mathrm{d}^4x\; e\; T_{\alpha\beta}{}^{a}e^\mu_a(\bar\psi\sigma_\mu{}^\sigma D^L_\sigma\psi)\\
&+ \frac{i\theta^{\alpha\beta}}{16l^2}\int \mathrm{d}^4x\; e\; \varepsilon_{abc}{}^{d}e^a_\alpha e^b_\beta e^\sigma_d(\bar\psi\gamma^c\gamma^5 D^L_\sigma\psi) - \frac{\theta^{\alpha\beta}}{8l^2}\int \mathrm{d}^4x\; e\; (\bar\psi\gamma_\alpha D^L_\beta\psi)\\
&+ \frac{\theta^{\alpha\beta}}{16l}\int \mathrm{d}^4x\; e\; R_{\alpha\beta}{}^{ab}(\bar\psi\sigma_{ab}\psi) - \frac{\theta^{\alpha\beta}}{8l^2}\int \mathrm{d}^4x\; e\; T_{\alpha\beta}{}^{a}(\bar\psi\gamma_a\psi)\\
&- \frac{\theta^{\alpha\beta}}{8l^3}\int \mathrm{d}^4x\; e\; (\bar\psi\sigma_{\alpha\beta}\psi)
\end{aligned}
\tag{A.1}
$$

$$
S_1^{(1)} \overset{\mathrm{WI}}{=} \frac{\theta^{\alpha\beta}}{16}\int \mathrm{d}^4x\; e\; \Big(R_{\alpha\beta}{}^{ab}e^\sigma_b(\bar\psi\gamma_a D^L_\sigma\psi) - \frac{i}{16}R_{\alpha\beta}{}^{ab}\varepsilon_{abc}{}^{d}e^\sigma_d(\bar\psi\gamma^c\gamma^5 D^L_\sigma\psi)\Big) \tag{A.2}
$$

$$
\begin{aligned}
S_2^{(1)} = &+ \frac{i\theta^{\alpha\beta}}{12}\int \mathrm{d}^4x\; e\; (D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon_b{}^{cds}e^\mu_c e^\nu_d e^\sigma_s(\bar\psi\gamma_a\gamma_5 D^L_\sigma\psi)\\
&- \frac{i\theta^{\alpha\beta}}{12}\int \mathrm{d}^4x\; e\; \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon^{rcds}e^\mu_c e^\nu_d e^\sigma_s(\bar\psi\gamma_r\gamma_5 D^L_\sigma\psi)\\
&- \frac{\theta^{\alpha\beta}}{12l}\int \mathrm{d}^4x\; e\; (D^L_\beta e^b_\nu)\varepsilon_b{}^{cds}e_{\alpha c}e^\nu_d e^\sigma_s(\bar\psi\gamma_5 D^L_\sigma\psi)\\
&- \frac{i\theta^{\alpha\beta}}{12l^2}\int \mathrm{d}^4x\; e\; \varepsilon_{ab}{}^{cd}e^a_\alpha e^b_\beta e^\sigma_d(\bar\psi\gamma_c\gamma_5 D^L_\sigma\psi)\\
&+ \frac{\theta^{\alpha\beta}}{24l}\int \mathrm{d}^4x\; e\; \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(\bar\psi\sigma^{\mu\nu}\psi)\\
&- \frac{i\theta^{\alpha\beta}}{24l}\int \mathrm{d}^4x\; e\; (D^L_\alpha e^a_\mu)e^\mu_b\varepsilon_a{}^{bcd}e_{\beta d}(\bar\psi\gamma_c\gamma_5\psi)\\
&- \frac{\theta^{\alpha\beta}}{24l}\int \mathrm{d}^4x\; e\; (D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)(\bar\psi\sigma^c{}_b\psi)\\
&+ \frac{\theta^{\alpha\beta}}{12l^3}\int \mathrm{d}^4x\; e\; (\bar\psi\sigma_{\alpha\beta}\psi)
\end{aligned}
\tag{A.3}
$$

$$
S_2^{(1)} \overset{\mathrm{WI}}{=} \frac{i\theta^{\alpha\beta}}{12}\int \mathrm{d}^4x\; e\; (D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)e^\mu_c e^\nu_d e^\sigma_s(\bar\psi\left(\varepsilon_b{}^{cds}\gamma_a - \eta_{ab}\varepsilon^{rcds}\gamma_r\right)\gamma_5 D^L_\sigma\psi) \tag{A.4}
$$

$$
\begin{aligned}
S_3^{(1)} =& -\frac{\theta^{\alpha\beta}}{4}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\mu_b e^\sigma_a)(\bar\psi\gamma^b D^L_\beta D^L_\sigma\psi) - \frac{\theta^{\alpha\beta}}{4l}\int \mathrm{d}^4x\ e\ (\bar\psi\sigma_\alpha{}^\sigma D^L_\beta D^L_\sigma\psi)\\
&- \frac{i\theta^{\alpha\beta}}{2l}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)e^\mu_a(\bar\psi D^L_\beta\psi) - \frac{\theta^{\alpha\beta}}{8l}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)e^\mu_b(\bar\psi\sigma_a{}^b D^L_\beta\psi)\\
&+ \frac{\theta^{\alpha\beta}}{2l^2}\int \mathrm{d}^4x\ e\ (\bar\psi\gamma_\alpha D^L_\beta\psi) + \frac{i\theta^{\alpha\beta}}{16l}\int \mathrm{d}^4x\ e\ T_{\alpha\beta}{}^a e^\sigma_a(\bar\psi D^L_\sigma\psi)\\
&- \frac{\theta^{\alpha\beta}}{8l}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\mu_b e^\sigma_a)e^c_\beta(\bar\psi\sigma^b{}_c D^L_\sigma\psi)\\
&+ \frac{i\theta^{\alpha\beta}}{8l^2}\int \mathrm{d}^4x\ e\ \varepsilon_{ab}{}^{cd}e^a_\alpha e^b_\beta e^\sigma_d(\bar\psi\gamma_c\gamma_5 D^L_\sigma\psi)\\
&- \frac{\theta^{\alpha\beta}}{32l^2}\int \mathrm{d}^4x\ e\ T_{\alpha\beta}{}^a(\bar\psi\gamma_a\psi) - \frac{3\theta^{\alpha\beta}}{16l^2}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)e^\mu_a(\bar\psi\gamma_\beta\psi)\\
&- \frac{\theta^{\alpha\beta}}{8l}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)(\bar\psi\sigma^c{}_b\psi)\\
&+ \frac{i\theta^{\alpha\beta}}{16l^2}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)e^\mu_b e^c_\beta\varepsilon_a{}^b{}_{cd}(\bar\psi\gamma^d\gamma_5\psi)\\
&+ \frac{\theta^{\alpha\beta}}{16l^2}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)e_{\beta a}(\bar\psi\gamma^\mu\psi) - \frac{\theta^{\alpha\beta}}{16l^3}\int \mathrm{d}^4x\ e\ (\bar\psi\sigma_{\alpha\beta}\psi)
\end{aligned}
\tag{A.5}
$$

$$
S_3^{(1)} \overset{\mathrm{WI}}{=} -\frac{\theta^{\alpha\beta}}{4}\int \mathrm{d}^4x\ e\ (D^L_\alpha e^a_\mu)(e^\mu_a e^\sigma_b - e^\mu_b e^\sigma_a)(\bar\psi\gamma^b D^L_\beta D^L_\sigma\psi) \tag{A.6}
$$

$$
\begin{aligned}
S_4^{(1)} =& -\frac{i\theta^{\alpha\beta}}{24}\int \mathrm{d}^4x\ e\ \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon^{cdrs}e^\mu_c e^\nu_d e^\sigma_d(\bar\psi\gamma_r\gamma_5 D^L_\sigma\psi)\\
&- \frac{i\theta^{\alpha\beta}}{24l^2}\int \mathrm{d}^4x\ e\ \varepsilon_{ab}{}^{cd}e^a_\alpha e^b_\beta e^\sigma_d(\bar\psi\gamma_c\gamma_5 D^L_\sigma\psi)\\
&+ \frac{\theta^{\alpha\beta}}{24l}\int \mathrm{d}^4x\ e\ \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(\bar\psi\sigma^{\mu\nu}\psi)\\
&+ \frac{\theta^{\alpha\beta}}{24l^3}\int \mathrm{d}^4x\ e\ (\bar\psi\sigma_{\alpha\beta}\psi)
\end{aligned}
\tag{A.7}
$$

$$
S_4^{(1)} \overset{\mathrm{WI}}{=} -\frac{i\theta^{\alpha\beta}}{24}\int \mathrm{d}^4x\ e\ \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\varepsilon^{cdrs}e^\mu_c e^\nu_d e^\sigma_d(\bar\psi\gamma_r\gamma_5 D^L_\sigma\psi) \tag{A.8}
$$

(Terms $S_5^{(1)}$, $S_6^{(1)}$ and $S_7^{(1)}$ are combined into a single term due to their similarity.)

$$\begin{aligned}
S_5^{(1)} + S_6^{(1)} + S_7^{(1)} = &- \frac{i\theta^{\alpha\beta}}{24} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} \varepsilon_{abc}{}^{d} e^c_\beta (e^\mu_d e^\sigma_s - e^\mu_s e^\sigma_d)(\bar{\psi}\gamma^s\gamma^5 D^L_\sigma \psi) \\
&- \frac{i\theta^{\alpha\beta}}{4l} \int \mathrm{d}^4x \; e \; T_{\alpha\beta}{}^{a} e^\sigma_a (\bar{\psi} D^L_\sigma \psi) + \frac{i\theta^{\alpha\beta}}{4l} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} e^\mu_a (\bar{\psi} D^L_\beta \psi) \\
&- \frac{\theta^{\alpha\beta}}{12l} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} \varepsilon_{ab}{}^{cd} e^b_\beta e^\mu_c e^\sigma_d (\bar{\psi}\gamma^5 D^L_\sigma \psi) \\
&+ \frac{i\theta^{\alpha\beta}}{12l^2} \int \mathrm{d}^4x \; e \; \varepsilon_{abc}{}^{d} e^a_\alpha e^b_\beta e^\sigma_d (\bar{\psi}\gamma^c\gamma^5 D^L_\sigma \psi) \\
&- \frac{\theta^{\alpha\beta}}{16l} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} \varepsilon_{abc}{}^{d} e^c_\beta e^\mu_d (\bar{\psi}\gamma^5 \psi) \\
&- \frac{\theta^{\alpha\beta}}{48l} \int \mathrm{d}^4x \; e \; R_{\alpha\beta}{}^{ab} (\bar{\psi}\sigma_{ab}\psi) - \frac{\theta^{\alpha\beta}}{24l} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} e^\mu_a e^c_\beta (\bar{\psi}\sigma_{bc}\psi) \\
&+ \frac{i\theta^{\alpha\beta}}{24l^2} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} e^b_\beta \varepsilon_{ab}{}^{cd} e^\mu_c (\bar{\psi}\gamma_d\gamma_5\psi) \\
&+ \frac{\theta^{\alpha\beta}}{8l^2} \int \mathrm{d}^4x \; e \; T_{\alpha\beta}{}^{a} (\bar{\psi}\gamma_a\psi) - \frac{\theta^{\alpha\beta}}{8l^2} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} e^\mu_a (\bar{\psi}\gamma_\beta\psi) \\
&- \frac{\theta^{\alpha\beta}}{12l^3} \int \mathrm{d}^4x \; e \; (\bar{\psi}\sigma_{\alpha\beta}\psi)
\end{aligned} \tag{A.9}$$

$$S_5^{(1)} + S_6^{(1)} + S_7^{(1)} \overset{\mathrm{WI}}{=} -\frac{i\theta^{\alpha\beta}}{24} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} \varepsilon_{abc}{}^{d} e^c_\beta (e^\mu_d e^\sigma_s - e^\mu_s e^\sigma_d)(\bar{\psi}\gamma^s\gamma^5 D^L_\sigma \psi) \tag{A.10}$$

$$\begin{aligned}
S_8^{(1)} = &- \frac{\theta^{\alpha\beta}}{8} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} e^\mu_a (\bar{\psi}\gamma_b D^L_\beta \psi) - \frac{i\theta^{\alpha\beta}}{16} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{bc} e^\mu_a \varepsilon^a{}_{bcm} (\bar{\psi}\gamma^m\gamma^5 D^L_\beta \psi) \\
&- \frac{i\theta^{\alpha\beta}}{8l} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} e^\mu_a (\bar{\psi} D^L_\beta \psi) + \frac{\theta^{\alpha\beta}}{8l} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} e^\mu_b (\bar{\psi}\sigma_a{}^b D^L_\beta \psi) \\
&- \frac{3\theta^{\alpha\beta}}{8l^2} \int \mathrm{d}^4x \; e \; (\bar{\psi}\gamma_\alpha D^L_\beta \psi) - \frac{\theta^{\alpha\beta}}{16l} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} e^\mu_a e^c_\beta (\bar{\psi}\sigma_{bc}\psi) \\
&- \frac{\theta^{\alpha\beta}}{32l} \int \mathrm{d}^4x \; e \; R_{\alpha\beta}{}^{ab} (\bar{\psi}\sigma_{ab}\psi) - \frac{\theta^{\alpha\beta}}{16l} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} e_{\beta a} e^\mu_c (\bar{\psi}\sigma_b{}^c\psi) \\
&- \frac{i\theta^{\alpha\beta}}{16l} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} e^\mu_a e_{\beta b} (\bar{\psi}\psi) + \frac{\theta^{\alpha\beta}}{32l} \int \mathrm{d}^4x \; e \; R_{\alpha\mu}{}^{ab} \varepsilon_{abc}{}^{d} e^c_\beta e^\mu_d (\bar{\psi}\gamma_5\psi) \\
&- \frac{\theta^{\alpha\beta}}{16l^2} \int \mathrm{d}^4x \; e \; T_{\alpha\beta}{}^{a} (\bar{\psi}\gamma_a\psi) + \frac{\theta^{\alpha\beta}}{16l^2} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} e^\mu_a (\bar{\psi}\gamma_\beta\psi) \\
&+ \frac{\theta^{\alpha\beta}}{16l^2} \int \mathrm{d}^4x \; e \; T_{\alpha\mu}{}^{a} e_{\beta a} (\bar{\psi}\gamma^\mu\psi) - \frac{3\theta^{\alpha\beta}}{16l^2} \int \mathrm{d}^4x \; e \; (\bar{\psi}\sigma_{\alpha\beta}\psi)
\end{aligned} \tag{A.11}$$

$$S_8^{(1)} \overset{\mathrm{WI}}{=} -\frac{\theta^{\alpha\beta}}{8} \int \mathrm{d}^4x \; e \; \bar{\psi} \left(R_{\alpha\mu}{}^{ab} e^\mu_a \gamma_b + \frac{i}{2} R_{\alpha\mu}{}^{bc} e^\mu_a \varepsilon^a{}_{bcm} \gamma^m \gamma^5 \right) D^L_\beta \psi \tag{A.12}$$

Now we consider separately each of the three bilinear mass-like spinor actions (5.33), (5.34) and (5.35), after the gauge fixing. WI contraction eliminates terms multiplied by the cosmological mass $2/l$ but some of the m-terms survive.

The first mass-like term after the gauge fixing:

$$\begin{aligned}S^{(1)}_{m,1} = c_1 a(m,l)\theta^{\alpha\beta}\int \mathrm{d}^4x\; e\; \bar{\psi}\Big(&-6i(D^L_\alpha e^a_\mu)e^\mu_a D^L_\beta + \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}\\ &-2(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b - \frac{3}{l}(D^L_\alpha e^a_\mu)e^\mu_a\gamma_\beta - \frac{1}{l^2}\sigma_{\alpha\beta}\\ &-\frac{1}{4}R_{\alpha\beta}{}^{ab}\sigma_{ab} - 2R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc} - \frac{3}{2l}T_{\alpha\beta}{}^a\gamma_a\Big)\psi\end{aligned} \tag{A.13}$$

$$\begin{aligned}S^{(1)}_{m,1} \overset{\mathrm{WI}}{=} c_1 m\theta^{\alpha\beta}\int \mathrm{d}^4x\; e\; \bar{\psi}\Big(&-6i(D^L_\alpha e^a_\mu)e^\mu_a D^L_\beta + \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}\\ &-2(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b - \frac{1}{4}R_{\alpha\beta}{}^{ab}\sigma_{ab} - 2R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc}\Big)\psi\end{aligned} \tag{A.14}$$

The second mass-like term after the gauge fixing:

$$\begin{aligned}S^{(1)}_{m,2} = c_2 a(m,l)\theta^{\alpha\beta}\int \mathrm{d}^4x\; e\; \bar{\psi}\Big(&6i(D^L_\alpha e^a_\mu)e^\mu_a D^L_\beta - \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}\\ &+2(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b + \frac{1}{4}R_{\alpha\beta}{}^{ab}\sigma_{ab} + 2R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc} + \frac{3}{l}T_{\alpha\mu}{}^a e^\mu_a\gamma_\beta\Big)\psi\end{aligned} \tag{A.15}$$

$$\begin{aligned}S^{(1)}_{m,2} \overset{\mathrm{WI}}{=} c_2 m\theta^{\alpha\beta}\int \mathrm{d}^4x\; e\; \bar{\psi}\Big(&6i(D^L_\alpha e^a_\mu)e^\mu_a D^L_\beta - \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}\\ &+2(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b + \frac{1}{4}R_{\alpha\beta}{}^{ab}\sigma_{ab} + 2R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc}\Big)\psi\end{aligned} \tag{A.16}$$

The third mass-like term after the gauge fixing:

$$\begin{aligned}S^{(1)}_{m,3} = c_3 a(m,l)\theta^{\alpha\beta}\int \mathrm{d}^4x\; e\; \bar{\psi}\Big(&-6i(D^L_\alpha e^a_\mu)e^\mu_a D^L_\beta + \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}\\ &-2(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b + \frac{1}{l}(D^L_\alpha e^a_\mu)e^\mu_a\gamma_\beta - \frac{1}{4}R_{\alpha\beta}{}^{ab}\sigma_{ab}\\ &-2R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc} + \frac{1}{2l}T_{\alpha\beta}{}^a\gamma_a - \frac{4}{l}T_{\alpha\mu}{}^a e^\mu_a\gamma_\beta + \frac{3}{l^2}\sigma_{\alpha\beta}\Big)\psi\end{aligned} \tag{A.17}$$

$$\begin{aligned}S^{(1)}_{m,3} \overset{\mathrm{WI}}{=} c_3 m\theta^{\alpha\beta}\int \mathrm{d}^4x\; e\; \bar{\psi}\Big(&-6i(D^L_\alpha e^a_\mu)e^\mu_a\, D^L_\beta + \eta_{ab}(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)\sigma^{\mu\nu}\\ &-2(D^L_\alpha e^a_\mu)(D^L_\beta e^b_\nu)(e^\mu_a e^\nu_c - e^\mu_c e^\nu_a)\sigma^c{}_b - \frac{1}{4}R_{\alpha\beta}{}^{ab}\sigma_{ab} - 2R_{\alpha\mu}{}^{ab}e^\mu_a e^c_\beta\sigma_{bc}\Big)\psi\end{aligned} \tag{A.18}$$

B Actions involving $U(1)$ gauge field

Here we present the six terms of $S^{(1)}_{Af} = \sum_{i=1}^{6} S^{(1)}_{Af.i}$ after the gauge fixing.

$$\begin{aligned} S^{(1)}_{Af.1} = -\frac{\theta^{\alpha\beta}}{16} \int \mathrm{d}^4x\; e\; \Big(&-\frac{1}{16}\epsilon_{abcd}\epsilon^{rspt} e^\mu_p e^\nu_t F_{\alpha\beta}{}^{ab} f^{cd} F_{\mu\nu rs} \\ &+ F_{\alpha\beta}{}^{ab} f^{c5}\big(F_{\mu\nu a5} e^\mu_b e^\nu_c + \frac{1}{2} F_{\mu\nu c5} e^\mu_a e^\nu_b\big) \\ &+ F_{\alpha\beta}{}^{a5} f^{cd}\big(F_{\mu\nu d5} e^\mu_a e^\nu_c + \frac{1}{2} F_{\mu\nu a5}\, e^\mu_c e^\nu_d\big) \\ &+ \frac{1}{4} F_{\mu\nu}{}^{mn} e^\mu_m e^\nu_n \big(F_{\alpha\beta}{}^{ab} f_{ab} + 2F_{\alpha\beta}{}^{a5} f_{a5}\big) + 2\mathcal{F}_{\alpha\beta} f^{cd} \mathcal{F}_{\mu\nu} e^\mu_c e^\nu_d \Big) + c.c. \end{aligned} \tag{B.19}$$

$$\begin{aligned} S^{(1)}_{Af.2} = -\frac{\theta^{\alpha\beta}}{8l} \int \mathrm{d}^4x\; e\; \Big(& e^\mu_d \big(f_c{}^d (D_\beta F_{\alpha\mu}{}^{c5}) + f_c{}^5 (D_\beta F_{\alpha\mu}{}^{dc})\big) \\ &- l(D^L_\alpha e^r_\rho)\big(f_c{}^d (D_\beta F_{\mu\nu}{}^{mc}) + f_5{}^d (D_\beta F_{\mu\nu}{}^{m5})\big)\big(e^\mu_d e^\nu_m e^\rho_r + e^\mu_r e^\nu_d e^\rho_m + e^\mu_m e^\nu_r e^\rho_d\big)\Big) + c.c. \end{aligned} \tag{B.20}$$

$$\begin{aligned} S^{(1)}_{Af.3} = \frac{\theta^{\alpha\beta}}{8} \int \mathrm{d}^4x\; e\; \Big(&-\frac{1}{16}\epsilon_{ambn}\epsilon^{cdpt} e^\mu_p e^\nu_t f_{cd} F_{\alpha\mu}{}^{am} F_{\beta\nu}{}^{bn} \\ &+ F_{\alpha\mu}{}^{am} F_{\beta\nu}{}^{b5}\big(f_{a5} e^\mu_m e^\nu_b + f_{b5} e^\mu_a e^\nu_m\big) + 2 f^{cd} \mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu} e^\mu_c e^\nu_d \\ &+ \frac{1}{4} e^\mu_c e^\nu_d \Big(f^{cd}\big(F_{\alpha\mu}{}^{am} F_{\beta\nu am} + 2F_{\alpha\mu}{}^{a5} F_{\beta\nu a5}\big) + 4 f_{m5} F_{\alpha\mu}{}^{dm} F_{\beta\nu}{}^{c5}\Big)\Big) + c.c. \end{aligned} \tag{B.21}$$

$$\begin{aligned} S^{(1)}_{Af.4} = -\frac{\theta^{\alpha\beta}}{32l} \int \mathrm{d}^4x\; e\; \epsilon^{\mu\nu\rho\sigma} \Big(& f^{a5} F_{\mu\nu}{}^{bc} (D^L_\alpha e^d_\rho) e^\delta_a e^\lambda_b e^\gamma_c \big(g_{\sigma\beta}\epsilon_{\delta\lambda\gamma\tau} e^\tau_d - \epsilon_{\delta\lambda\gamma\sigma} e_{\beta d}\big) \\ &+ f^{ab} F_{\mu\nu}{}^{c5} (D^L_\alpha e^d_\rho) e^\delta_a e^\lambda_b e^\gamma_c \big(g_{\sigma\beta}\epsilon_{\delta\lambda\gamma\tau} e^\tau_d - \epsilon_{\delta\lambda\gamma\sigma} e_{\beta d}\big) \\ &- 2(D^L_\alpha e^e_\rho)\epsilon_{\delta\lambda\sigma\beta} e^\delta_b e^\lambda_e \big(f^{a5} F_{\mu\nu}{}^b{}_a + f^{ab} F_{\mu\nu a5}\big) \\ &+ \frac{1}{2l}\epsilon_{\delta\lambda\gamma\tau} e^\delta_a e^\lambda_b e^\gamma_c e^\tau_d f^{ab} F_{\mu\nu}{}^{cd} g_{\alpha\rho} g_{\sigma\beta} + \frac{2}{l}\big(f^{ab} F_{\mu\nu}{}^c{}_a - f^{b5} F_{\mu\nu}{}^{c5}\big) g_{\alpha\rho}\epsilon_{\delta\lambda\sigma\beta} e^\delta_b e^\lambda_c \Big) + c.c. \end{aligned} \tag{B.22}$$

$$S^{(1)}_{Af.5} = \frac{\theta^{\alpha\beta}}{128} \int \mathrm{d}^4x\; e\; \epsilon^{\mu\nu\rho\sigma} \epsilon_{\delta\lambda\gamma\tau} e^\delta_a e^\lambda_b e^\gamma_c e^\tau_d f^{ab} F_{\mu\nu}{}^{cd} \Big((D^L_\alpha e^e_\rho)(D^L_\beta e_{\sigma e}) + \frac{1}{l^2} g_{\alpha\rho} g_{\sigma\beta}\Big) + c.c. \tag{B.23}$$

$$\begin{aligned} S^{(1)}_{Af.6} = -\frac{\theta^{\alpha\beta}}{64} \int \mathrm{d}^4x\; e\; \epsilon^{\mu\nu\rho\sigma} \Big(& \epsilon_{\delta\lambda\beta\sigma} e^\delta_e e^\lambda_f F_{\alpha\rho}{}^{ef} \big(f^{ab} F_{\mu\nu ab} + 2 f^{a5} F_{\mu\nu a5}\big) \\ &+ 4\epsilon_{\delta\lambda\beta\sigma} e^\delta_e e^\lambda_b \big(f^{ab} F_{\mu\nu a5} F_{\alpha\rho}{}^{e5} + f^{a5} F_{\mu\nu}{}^b{}_a F_{\alpha\rho}{}^{e5} + 2 f^{ab} \mathcal{F}_{\alpha\rho}\mathcal{F}_{\mu\nu}\big)\Big) + \text{c.c.} \end{aligned} \tag{B.24}$$

C Matrix representation of $\mathfrak{osp}(4|2)$ superalgebra

Here we present an explicit 6×6 matrix representation of $OSp(4|2)$ generators $\{\hat{M}_{AB}, \hat{Q}^I_\alpha, \hat{T}\}$ that span $\mathfrak{osp}(4|2)$ superalgebra.

Bosonic generators of $OSp(4|2)$:

$$\hat{M}_{AB} = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & M_{AB} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \hat{T} = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & 0_{4\times 4} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & -i & 0 \end{array}\right) \tag{C.1}$$

The imaginary unit in $\hat{T}$ is introduced for convenience.

Fermionic generators of $OSp(4|2)$:

$$(\hat{Q}^1)_1 = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & 0_{4\times 4} & & & 1 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (\hat{Q}^1)_2 = \left(\begin{array}{cccc|cc} & & & & -1 & 0 \\ & 0_{4\times 4} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{C.2}$$

$$(\hat{Q}^1)_3 = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & 0_{4\times 4} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & -1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (\hat{Q}^1)_4 = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & 0_{4\times 4} & & & 0 & 0 \\ & & & & 1 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{C.3}$$

The second set of fermionic generators $(\hat{Q}^2)_\alpha$ (for $I = 2$) is obtained from the first one simply by interchanging 5^{th} and 6^{th} column and 5^{th} and 6^{th} row. One can readily check that supermatrices (C.1), (C.2) and (C.3), along with the ones of the second set of fermionic generators, satisfy $\mathfrak{osp}(4|2)$ superalgebra (8.4).

D Majorana spinors and AdS identities

Some basic Fierz identities involving Majorana spinors ψ and χ:

$$\begin{aligned}
\bar{\psi}\chi &= \bar{\chi}\psi = (\bar{\psi}\chi)^\dagger \\
\bar{\psi}\gamma_5\chi &= \bar{\chi}\gamma_5\psi = -(\bar{\psi}\gamma_5\chi)^\dagger \\
\bar{\psi}\gamma_a\gamma_5\chi &= \bar{\chi}\gamma_a\gamma_5\psi = (\bar{\psi}\gamma_a\gamma_5\chi)^\dagger \\
\bar{\psi}\gamma_a\chi &= -\bar{\chi}\gamma_a\chi = -(\bar{\psi}\gamma_a\chi)^\dagger \\
\bar{\psi}\sigma_{ab}\chi &= -\bar{\chi}\sigma_{ab}\psi = -(\bar{\psi}\sigma_{ab}\chi)^\dagger
\end{aligned} \tag{D.1}$$

Also, we frequently use the following important identity in 4D. For any pair of Majorana spinors ψ and χ, we can expand $\psi\bar{\chi}$ in the Clifford algebra basis,

$$-4\psi\bar{\chi} = (\bar{\chi}\psi)\mathbb{I}_4 + (\bar{\chi}\gamma^a\psi)\gamma_a + (\bar{\chi}\gamma_5\psi)\gamma_5 + (\bar{\chi}\gamma^a\gamma_5\psi)\gamma_5\gamma_a + \frac{1}{2}(\bar{\chi}\sigma^{ab}\psi)\sigma_{ab} \tag{D.2}$$

Some AdS algebra relations:

$$\begin{aligned}
&[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}) \\
&\{M_{AB}, M_{CD}\} = \frac{i}{2}\epsilon_{ABCDE}\Gamma^E + \frac{1}{2}(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \\
&\{M_{AB}, \Gamma_C\} = i\epsilon_{ABCDE}M^{DE} \\
&[M_{AB}, \Gamma_C] = i(\eta_{BC}\Gamma_A - \eta_{AC}\Gamma_B) \\
&\Gamma_A^\dagger = -\gamma_0\Gamma_A\gamma_0\ , \quad M_{AB}^\dagger = \gamma_0 M_{AB}\gamma_0
\end{aligned} \tag{D.3}$$

Some useful identities involving γ-matrices and σ-matrices:

$$\begin{aligned}
&\gamma_a\gamma_b = \eta_{ab} - i\sigma_{ab} \\
&\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c - \eta_{ac}\gamma_b + \eta_{bc}\gamma_a + i\epsilon_{abcd}\gamma^d\gamma_5 \\
&\sigma_{ab}\gamma_c = i\eta_{bc}\gamma_a - i\eta_{ac}\gamma_b + \varepsilon_{abcd}\gamma_5\gamma^d \\
&\gamma_c\sigma_{ab} = i\eta_{ac}\gamma_b - i\eta_{bc}\gamma_a + \varepsilon_{abcd}\gamma_5\gamma^d \\
&\sigma_{ab}\gamma_5 = \tfrac{i}{2}\varepsilon_{abcd}\sigma^{cd} \\
&\sigma_{ab}\sigma_{cd} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + i\varepsilon_{abcd}\gamma_5 + i(\eta_{ad}\sigma_{bc} + \eta_{bc}\sigma_{ad} - \eta_{ac}\sigma_{bd} - \eta_{bd}\sigma_{ac}) \\
&\{\sigma_{ab}, \sigma_{cd}\} = 2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) + 2i\varepsilon_{abcd}\gamma_5, \\
&[\sigma_{ab}, \gamma_c] = 2i(\eta_{bc}\gamma_a - \eta_{ac}\gamma_b), \\
&\{\sigma_{ab}, \gamma_c\} = 2\varepsilon_{abcd}\gamma_5\gamma^d,
\end{aligned} \tag{D.4}$$

Identities with traces:

$$
\begin{aligned}
&\mathrm{Tr}(\Gamma_A \Gamma_B) = 4\eta_{AB} \\
&\mathrm{Tr}(\Gamma_A) = \mathrm{Tr}(\Gamma_A \Gamma_B \Gamma_C) = 0 \\
&\mathrm{Tr}(\Gamma_A \Gamma_B \Gamma_C \Gamma_D) = 4(\eta_{AB}\eta_{CD} - \eta_{AC}\eta_{BD} + \eta_{AD}\eta_{CB}) \\
&\mathrm{Tr}(\Gamma_A \Gamma_B \Gamma_C \Gamma_D \Gamma_E) = -4i\epsilon_{ABCDE} \\
&\mathrm{Tr}(M_{AB} M_{CD} \Gamma_E) = i\epsilon_{ABCDE} \\
&\mathrm{Tr}(M_{AB} M_{CD}) = -\eta_{AD}\eta_{CB} + \eta_{AC}\eta_{BD} \\
&\mathrm{Tr}(M_{AB} \Gamma_E \Gamma_F \Gamma_G) = 2\varepsilon_{ABEFG} \\
&\mathrm{Tr}(M_{AB} M_{CD} \Gamma_E \Gamma_F \Gamma_G) = i\varepsilon_{ABCDE}\eta_{FG} - i\varepsilon_{ABCDF}\eta_{EG} + i\varepsilon_{ABCDG}\eta_{EF} \\
&\qquad + i\varepsilon_{BCEFG}\eta_{AD} + i\varepsilon_{ADEFG}\eta_{BC} - i\varepsilon_{BDEFG}\eta_{AC} - i\varepsilon_{ACEFG}\eta_{BD}
\end{aligned} \tag{D.5}
$$

$$
\begin{aligned}
&\mathrm{Tr}(\gamma_a \gamma_b \gamma_c \gamma_d) = 4(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} + \eta_{cd}\eta_{bc}) \\
&\mathrm{Tr}(\sigma_{ab}\sigma_{cd}) = 4(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \\
&\mathrm{Tr}(\sigma_{ab}\sigma_{cd}\gamma_5) = 4i\varepsilon_{abcd} \\
&\mathrm{Tr}(\gamma_a \gamma_b \sigma_{cd}) = -4i(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \\
&\mathrm{Tr}(\gamma_a \gamma_b \sigma_{cd}\gamma_5) = 4\varepsilon_{abcd} \\
&\mathrm{Tr}(\sigma_{ab}\sigma_{cd}\sigma_{ef}) = 4i(\eta_{ad}(\eta_{be}\eta_{cf} - \eta_{bf}\eta_{ce}) + \eta_{bc}(\eta_{ae}\eta_{df} - \eta_{af}\eta_{de}) \\
&\qquad - \eta_{ac}(\eta_{be}\eta_{df} - \eta_{bf}\eta_{de}) - \eta_{bd}(\eta_{ae}\eta_{cf} - \eta_{af}\eta_{ce})) \\
&\mathrm{Tr}(\sigma_{ab}\sigma_{cd}\sigma_{ef}\gamma_5) = 4(\eta_{ac}\varepsilon_{bdef} + \eta_{bd}\varepsilon_{acef} - \eta_{ad}\varepsilon_{bcef} - \eta_{bc}\varepsilon_{adef})
\end{aligned} \tag{D.6}
$$

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Biografija

Dragoljub Gočanin je rođen 10. marta 1991. godine u Beogradu. Po završetku opšte Gimnazije u Vrnjačkoj Banji 2009. godine, započeo je osnovne akedemske studije na Fizičkom fakultetu Univerziteta u Beogradu, smer Teorijska i eksperimentalna fizika. Diplomirao je 2013. godine, sa prosečnom ocenom 9,84. Iste godine otpočinje Master studije na Fizičkom fakultetu Univerziteta u Beogradu i završava ih 2014. godine sa prosečnom ocenom 10, odbranivši tezu ,,*Funkcionalna renormalizaciona grupa u teoriji gradijentnih polja*", koja je sastavljena pod rukovodstvom prof. dr Voje Radovanovića. Od marta 2015. godine zaposlen je na Fizičkom fakultetu Univerziteta u Beogradu na projektu 171031 ,,Fizičke implikacije modifikovanog prostor-vremena" Ministarstva prosvete, nauke i tehnološkog razvija. U septembru 2018. godine je izabran za istraživača saradnika na Fizičkom fakultetu. Drži računske vežbe iz Teorijske mehanike, Osnova teorijske mehanike, Statističke fizike 1 i 2 i Kvantne statističke fizike. Doktorske studije na Fizičkom fakultetu Univerziteta u Beogradu otpočinje 2015. godine pod rukovodstvom prof. dr Voje Radovanovića.

Uža naučna oblast istraživanja Dragoljuba Gočanina je teorija gravitacije, klasična i kvantna teorija polja na nekomutativnim prostorima. Ova oblast se smatra jednom od značajnih oblasti istraživanja u fizici visokih energija.

Изјава о ауторству

Име и презиме аутора **Драгољуб Гочанин**

Број индекса **8006/2014**

Изјављујем

да је докторска дисертација под насловом

Теорија поља у $SO(2,3)_\star$ моделу некомутативне гравитације

- резултат сопственог истраживачког рада;
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Изјава о истоветности штампане и електронске верзије докторског рада

Име и презиме аутора **Драгољуб Гочанин**

Број индекса **8006/2014**

Студијски програм **Физика – квантна поља, честице и гравитација**

Наслов рада **Теорија поља у $SO(2,3)_\star$ моделу некомутативне гравитације**

Ментор **проф. др Воја Радовановић**

Изјављујем да је штампана верзија мог докторског рада истоветна електронској верзији коју сам предао/ла ради похрањена у **Дигиталном репозиторијуму Универзитета у Београду.**

Дозвољавам да се објаве моји лични подаци везани за добијање академског назива доктора наука, као што су име и презиме, година и место рођења и датум одбране рада.

Ови лични подаци могу се објавити на мрежним страницама дигиталне библиотеке, у електронском каталогу и у публикацијама Универзитета у Београду.

Потпис аутора

У Београду, **18.11.2019.**

Изјава о коришћењу

Овлашћујем Универзитетску библиотеку „Светозар Марковић“ да у Дигитални репозиторијум Универзитета у Београду унесе моју докторску дисертацију под насловом:

Теорија поља у SO(2,3)$_\star$ моделу некомутативне гравитације

која је моје ауторско дело.

Дисертацију са свим прилозима предао/ла сам у електронском формату погодном за трајно архивирање.

Моју докторску дисертацију похрањену у Дигиталном репозиторијуму Универзитета у Београду и доступну у отвореном приступу могу да користе сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons) за коју сам се одлучио/ла.

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