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Partial closure operators and applications in ordered set theory

Parcijalni operatori zatvaranja i primene u teoriji uređenih skupova

-Ph.D. thesis-

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CONTENTS

IZVOD

U ovoj tezi uopštavamo dobro poznate veze između operatora zatvaranja, sistema zatvaranja i potpunih mreža. Uvodimo posebnu vrstu parcijalnog operatora zatvaranja, koji nazivamo oštar parcijalni operator zatvaranja, i pokazujemo da svaki oštar parcijalni operator zatvaranja jedinstveno korespondira parcijalnom sistemu zatvaranja. Dalje uvodimo posebnu vrstu parcijalnog sistema zatvaranja, nazvan glavni parcijalni sistem zatvaranja, a zatim dokazujemo teoremu reprezentacije za posete u odnosu na uvedene parcijalne operatore zatvaranja i parcijalne sisteme zatvaranja.

Dalje, s obzirom na dobro poznatu vezu između matroida i geometrijskih mreža, a budući da se pojam matroida može na prirodan način uopštiti na parcijalne matroide (definišući ih preko parcijalnih operatora zatvaranja umesto preko operatora zatvaranja), definišemo geometrijske uređene skupove i pokazujemo da su povezani sa parcijalnim matroidima na isti način kao što su povezani i matroidi i geometrijske mreže. Osim toga, definišemo polumodularne uređene skupove i pokazujemo da su oni zaista uopštenje polumodularnih mreža i da ista veza postoji između polumodularnih i geometrijskih poseta kao što imamo između polumodularnih i geometrijskih mreža.

Konačno, konstatujemo da definisani pojmovi mogu biti primenjeni na implikacione sisteme, koji imaju veliku primenu u realnom svetu, posebno u analizi velikih podataka.

ABSTRACT

In this thesis we generalize the well-known connections between closure operators, closure systems and complete lattices. We introduce a special kind of a partial closure operator, named sharp partial closure operator, and show that each sharp partial closure operator uniquely corresponds to a partial closure system. We further introduce a special kind of a partial closure system, called principal partial closure system, and then prove the representation theorem for ordered sets with respect to the introduced partial closure operators and partial closure systems.

Further, motivated by a well-known connection between matroids and geometric lattices, given that the notion of matroids can be naturally generalized to partial matroids (by defining them with respect to a partial closure operator instead of with respect to a closure operator), we define geometric poset, and show that there is a same kind of connection between partial matroids and geometric posets as there is between matroids and geometric lattices. Furthermore, we then define semimodular poset, and show that it is indeed a generalization of semimodular lattices, and that there is a same kind of connection between semimodular and geometric posets as there is between semimodular and geometric lattices.

Finally, we note that the defined notions can be applied to implicational systems, that have many applications in real world, particularly in big data analysis.

PREFACE

This thesis belongs to the scope of the order theory and treats generalizations of closure operators and systems of sets connected with them. Historically speaking, this area has been studied since the middle of the 20th century. Along with introduction of particular types of orders, with study of lattices as ordered sets (R. Dedekind, R. Dilworth, G. Birkhoff, O. Ore, E. H. Moore and others), structures satisfying some given conditions and ordered by the set inclusion were also considered. Probably the most well-known of them are Moore's families or closure systems, which are collections of subsets of a given set closed under intersection. They are among the most important examples of complete lattices, which are getting more and more importance in modern algebra.

Closure systems correspond to special closure operators (defined on power set). As lattices and other ordered sets were becoming more and more significant, finding their place in different branches of mathematics and applied disciplines such as, for example, computer science, there emerged a need to study closure operators and closure systems satisfying some additional axioms, that is, having some special properties. A significant class of such structures are, for example, geometric lattices, which correspond to matroids as collections of sets.

Further investigation of structures ordered by the set inclusion considered structures closed not under intersection but under something else.

Such are, for example, centralized systems, which were studied in particular by M. Erné. In recent times, more attention is paid to complete ordered sets (CPOs), for which an interest emerged in the domain theory, information systems and in computer science in general (D. S. Scott, G. Markowski and others).

Closure operators and closure systems, as well as their connection with complete lattices, is a very popular research topic, which can be seen by numerous books and articles on this topic (some of which are [3, 8, 9, 18]). Collections of sets ordered by the set inclusion that are not lattices were gaining importance as they were arising in various branches of mathematics (order theory, combinatorial geometry, computer science). The corresponding (partial) closure operators were not introduced at the same time; their research has begun in more recent times. In [11, 12], there is a survey of closure systems on finite sets, their properties and properties of the corresponding lattices. In [8], there is analyzed a lattice of a particular kind of completion of a finite ordered set. Completion of ordered sets is also the subject of the paper [32]. A detailed research of closure systems and similar ordered sets has been conducted by M. Erné (for example, [21, 23]). In [28], there is studied a lattice of all Dedekind-MacNeille completions of ordered sets with fixed sup-irreducible elements. The number of closure systems on sets of a given cardinality is studied in [34], and in [35] the lattice of such systems is described.

In contrast to this, there are very little results on partial closure operators in the literature; practically all the existing results are implicit, that is, a byproduct of results on the corresponding collections of sets. In [48] a definition of partial closure operators is given that is a special case of the definition we use in this thesis (there they are defined only on lower sets in lattices; we hereby define them on arbitrary sets). This thesis can be considered as a continuation, or actually an extension of [53], which was the basis of it.

The work in this thesis is organized as follows.

In Chapter 1 we present necessary definitions and well known results that the research is built on. Section 1.5 here presents an original work.

Chapter 2 presents fully original work. The most of it is included in the paper [50]. Here we introduce a special type of partial closure operators, called sharp, which uniquely correspond to partial closure systems. Further, we introduce principal partial closure operators, motivated by systems of principal ideals on a poset. We further state and prove the representation theorem of posets with respect to sharp partial closure operators. In the last part, we analyze exact domains of partial closure operators and sharp partial closure operators, and give a necessary and sufficient condition for a collection of sets to be an exact domain.

In the first two sections of Chapter 3 we recall the definitions of geometric lattices and matroids, alternative definitions of them, and we give a connection between them. The other three sections of this chapter present an original work, mostly included in the paper [51]. In Section 3.3 we generalize the notion of geometric lattices to geometric posets and in Section 3.4 we generalize matroids to partial matroids. We show that geometric posets and partial matroids are connected in the same way as geometric lattices and matroids. In the last section of this chapter we introduce semimodularity for posets in such a way that the relationship between semimodular lattices and geometric lattices is fully preserved in these generalizations for posets.

Chapter 4 contains some applications of closure operators in implications systems. Section 4.2 here is an original work; using partial closure operators, we generalize unit implicational systems to partial unit implicational systems.

I would like to express my gratitude toward some persons without whom this thesis would not come into existence. First of all, I thank my parents for their unconditional support and for always being there for me. Further, I thank my supervisor, Prof. Branimir Šešelja, and the members of the Defend board: Prof. Andreja Tepavčević, Prof. Petar Marković, Prof. Miloš Kurilić and Prof. Jovanka Pantović. Thankfully to many useful comments of them, this thesis obtained the form in which it is now.

I am very grateful to my friends and colleagues, they helped me with going through the difficulties that came on the way to finishing this thesis, that I could not get through all by myself; the most of all, Bojan Bašić, who believed in me, encouraged me and who made sure I do not give up.

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CHAPTER

1 INTRODUCTION

In this chapter we introduce basic definitions and necessary theorems. Everything but Section 1.5, is well-known, and Section 1.5 presents original work.

1.1 Partially ordered sets

An **ordered set** is the pair $(P, \leqslant)$, where $\leqslant$ is a binary relation on a nonempty set P which is reflexive, antisymmetric and transitive. Another name for $(P, \leqslant)$ is a **partially ordered set**, or **poset** in short. We write only P instead of $(P, \leqslant)$ when the meaning is clear.

We say that x is **covered by** y (or y **covers** x) if and only if $x < y$ and $\neg(\exists z)(x < z < y)$; in that case we write $x \prec y$. If $x \prec y$ or $x = y$, then we write $x \preccurlyeq y$.

A subset T of P is called a **chain** (resp. an **antichain**) if every two different elements of T are comparable (resp. incomparable).

An element $a \in P$ is:

– the **least** (resp. the **greatest**), if for all $x \in P$ we have $a \leqslant x$ (resp. $x \leqslant a$); such elements are unique in a poset (if they exist) and usually are denoted by 0 and 1 respectively;

– **minimal** (resp. **maximal**), if for all $x \in P$, $x \leqslant a$ (resp. $a \leqslant x$) implies $x = a$.

For $p \in P$, we call the set

$$\downarrow p = \{x \in P \mid x \leqslant p\}$$

the **principal ideal** generated by the element p.

If P is a poset and $Q \subseteq P$, then the set of all lower bounds of the set Q, denoted by Q^d, is defined by

$$Q^d = \{a \in P \mid a \leqslant b, \text{ for all } b \in Q\},$$

and the set of all upper bounds of the set Q, denoted by Q^g, is defined by

$$Q^g = \{a \in P \mid b \leqslant a, \text{ for all } b \in Q\}.$$

A subset D of P is **directed** if every finite subset of D has an upper bound in D. A poset $(P, \leqslant)$ is called **complete** if it has the least element and if every directed subset D of P has the supremum. The complete partially ordered set is usually abbreviated by **CPO**.

An element a of a CPO P is called **compact** if for every directed subset D of P with $a \leqslant \bigvee D$, there exists $d \in D$ such that $a \leqslant d$.

Let $(P, \leqslant)$ be a poset and C the set of its compact elements. Then we say that P is **algebraic** if it is complete and for every $x \in P$ the set $\downarrow x \cap C$ is directed and $x = \bigvee(\downarrow x \cap C)$.

A poset $(L, \leqslant)$ in which every pair of elements has the supremum (infimum) is called a **upper semilattice** or **join-semilattice** (**lower semilattice** or **meet-semilattice**). A poset $(L, \leqslant)$ in which every pair of elements has the supremum and the infimum is called a **lattice**. A poset in which every subset has the supremum and the infimum is called a **complete lattice**.

It is well known that the so-called *Duality Principle* holds for posets, lattices and complete lattices, that is, if $(P, \leqslant)$ is a poset/lattice/complete lattice, then its dual $(P, \geqslant)$ is also a poset/lattice/complete lattice.

Remark 1.1. Notice that the Duality Principle does not hold for CPOs: a poset $(\{a, b, c\}, \leqslant)$, where $a \leqslant b$ and $a \leqslant c$, while b and c are uncomparable, is CPO, but its dual poset is not CPO, since it does not have the least element.

Another well known property of lattices is the following theorem.

Theorem 1.2. *A poset $(P, \leqslant)$ in which every subset has infimum (supremum) is a complete lattice.*

Proof. Since every subset has infimum, for $(P, \leqslant)$ to be a complete lattice, it remains to show that every subset has the supremum.

First we show that P has the greatest element. Since $\emptyset^d = P$ and since every subset of P has infimum, $\emptyset$ also has it, and therefore there exists the greatest element in P; denote it by 1, and $\bigwedge \emptyset = 1$. On the other hand, P has infimum, and hence there exists the least element; denote it by 0.

Now, let $A \subseteq P$. Then the set A^g contains 1, so it is not empty. Let us show that $\bigvee A = \bigwedge A^g$. Since $\bigwedge A^g$ exists, if we denote $a = \bigwedge A^g$, then, because each $x \in A$ is less than or equal to all the elements in A^g, we have $a \leqslant x$. On the other hand, if for some $p \in P$ we have $x \leqslant p$ for all $x \in A$, then $p \in A^g$, hence $a \leqslant p$, therefore we have $a = \bigvee A$. ■

An element a of a complete lattice L is **compact** if the following holds: whenever $a \leqslant \bigvee A$ for any subset A of L, it follows that there exists a finite subset B of A such that $a \leqslant \bigvee B$. Since a lattice is a special poset, it is natural to ask whether compact elements in a lattice have any connection with compact elements in a poset. In fact, it can be shown that if a CPO is a lattice, then compact elements in that poset are exactly the compact elements in that lattice.

A lattice is **compactly generated** if each of its elements is a supremum of compact elements. A lattice is **algebraic** if it is complete and compactly generated.

Theorem 1.3. *Let $(L, \leqslant)$ be a poset. Then every condition follows from the previous one:*

1. *L is an algebraic lattice,*
2. *L is a complete lattice,*
3. *L is a CPO.*

Theorem 1.4. *Let $(L, \leqslant)$ be a poset. Then every condition follows from the previous one:*

1. *L is an algebraic lattice,*
2. *L is an algebraic poset,*
3. *L is a CPO.*

Relations between these types of posets are illustrated in Figure 1. The rest of implications do not hold, which is shown by the following three examples.

Example 1. A poset $(\mathbb{N}, \leqslant)$ is not a CPO. A poset $([0,1], \leqslant)$ is a complete lattice, but it is not an algebraic poset since the only compact element is 0. The last example is $(\Sigma, \sqsubseteq)$, where Σ is the set of all words (finite and infinite) over the alphabet $\{0,1\}$ and $\sqsubseteq$ is a partial order on Σ defined by $u \sqsubseteq v$ if and only if u is a prefix of v. This is clearly not a complete lattice since it does not have the greatest element. On the other hand, $(\Sigma, \sqsubseteq)$ is an algebraic poset: there exists the least element (the empty word), a subset of Σ is directed if and only if it is a chain, and every chain has the supremum in Σ, and therefore $(\Sigma, \sqsubseteq)$ is CPO; finally, compact elements are finite words and this implies all the necessary conditions to for $(\Sigma, \sqsubseteq)$ to be an algebraic poset. $\square$

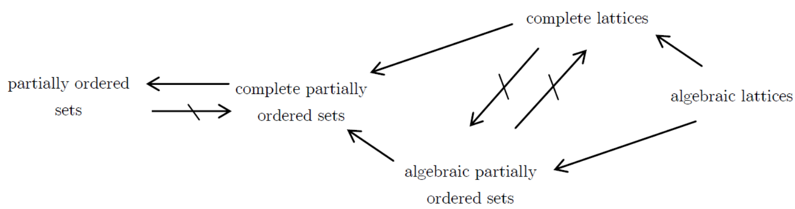

Figure 1.

Now we show a few statements that will be useful later.

Theorem 1.5 (**Iwamura's Lemma** [37])**.** *A poset is a CPO if and only if its every chain has the supremum.*

Proof. One direction is trivial, so let us assume that in a poset $(P, \leqslant)$ every chain has the supremum. First we show that P has the least element. Since every chain has the supremum, the empty chain has it too; denote $0 = \bigvee \emptyset$. Since $\emptyset^g = P$, thus the supremum 0 of $\emptyset$ is the least element of $\emptyset^g = P$.

Let now $C = \{c_\xi \mid \xi < \alpha\}$ be a directed subset of P and denote with A the set of all suprema of subsets of the set C, when supremum exists. Of course, A is directed and $C \subseteq A$. We shall also fix one well-ordering of the set A, which shall be needed later in the proof.

Using transfinite induction on $|B|$ we shall show: every subset B of the directed set A has an upper bound in A. In particular, we get that A has the supremum.

If B is finite, the statement holds. Assume that for all $B \subseteq A$ such that $|B| < \kappa$, we have that there exists an upper bound of B in the set A.

We need to show that the same holds for subsets of set A of cardinality κ. Let $B = \{b_\xi \mid \xi < \kappa\}$. We define the following sequences of sets:

$$B_0 := \{b_0\} \text{ and for every } \eta < \kappa,$$

$$B'_\eta := \bigcup_{\xi<\eta} B_\xi \text{ and } B_\eta := B'_\eta \cup \{b_\eta, g_{B'_\eta \cup \{b_\eta\}}\},$$

where g_X denotes the upper bound of the set X in the set A that is the smallest (with respect to the fixed well-ordering of the set A) among all such upper bounds. All these bounds indeed exist: by construction of these sets we have $|B_\xi| \leqslant |\xi|$, so $|B'_\eta \cup \{b_\eta\}| = |B'_\eta| = |B_\eta| \leqslant |\eta|^2 + 2 = |\eta|$ (there is actually another argument by transfinite induction involved here) and $\eta < \kappa$ (if η is finite, then the last argument does not hold, but nevertheless again $|B'_\eta \cup \{b_\eta\}| < \aleph_0 \leqslant \kappa$), and thus the inductive assumption holds, that is, every $B'_\eta \cup \{b_\eta\}$ has an upper bound in the set A.

Therefore, the collection $\{B_\eta \mid \eta < \kappa\}$ is a chain of directed sets. Every set in this chain has an upper bound which is in it, hence it has the supremum. These suprema $\{\bigvee B_\eta \mid \eta < \kappa\}$ also make chain of elements of the set A. By the assumption that every chain has the supremum, we get that $\bigvee_{\eta<\kappa} (\bigvee B_\eta)$ is an upper bound of the set B. This upper bound is in the set A. Indeed, all the elements of the set A can be written as supremum of elements from C, hence $\bigvee_{\eta<\kappa} (\bigvee B_\eta)$ can also be represented as supremum of elements from C, and therefore by definition of the set A this supremum is in A.

Therefore, the supremum of the set A exists, it is in A and it is the supremum of set C, which follows from the fact that every element of the set A is a supremum of some subset of C. This completes the proof. ■

Theorem 1.6. *If $(\mathcal{F}, \subseteq)$ is a collection of subsets of a set ordered by the set inclusion, then $\mathcal{F}$ is closed for unions of chains if and only if it is closed for unions of directed collections.*

Proof. Since every chain is a directed set, one direction of the statement holds.

For the other direction, let us assume that $\mathcal{G} = \{X_i \mid i = 1, 2, \dots\}$ is a directed subcollection of the collection $\mathcal{F}$ (we prove only the countable case, general case can be done by transfinite induction). We have to show that $\bigcup \mathcal{G} \in \mathcal{F}$.

First we define $\mathcal{H} = \{Y_i \mid i = 1, 2, \dots\}$ in the following way: $Y_1 := X_1$ and for all $i = 2, 3, \dots$ let Y_i be an upper bound of sets Y_{i-1} and X_i from the collection $\mathcal{G}$. Such an upper bound always exists, since $\mathcal{G}$ is a directed collection and $Y_i \subseteq Y_{i+1}$ for all $i = 1, 2, \dots$. Therefore, $\{Y_i \mid i = 1, 2, \dots\}$ is a chain in $\mathcal{G}$, so we have $\bigcup \mathcal{H} \in \mathcal{F}$, and because of the definition of $\mathcal{H}$ we have $\bigcup \mathcal{H} = \bigcup \mathcal{G}$; the proof is thus completed. ■

1.2 Closure systems

A **closure system** (also called **Moore's family**, after E. H. Moore, who introduced it in 1910) $\mathcal{F}$ on a nonempty set S is a collection of subsets of S that is closed under arbitrary set intersections. A closure system is **algebraic** (or **algebraic Moore's family**) if it is closed under unions of directed subcollections.

Theorem 1.7. *A closure system ordered by the set inclusion is a complete lattice.*

Proof. If a closure system $\mathcal{F}$ on A is ordered by the set inclusion, Theorem 1.2 gives that it is a complete lattice. Indeed: since infimum in a poset ordered this way is the set intersection, then every nonempty subfamily of $\mathcal{F}$ has the infimum, while infimum of the empty family is the whole set A. ■

A collection $\mathcal{F}$ of subsets of a nonempty set S is called a **partial closure system** on S (also known in the literature as a **centralized system** or **point closure system**; see, e. g., [23, 21]) if it fulfills the following conditions:

Ps_1: $\bigcup \mathcal{F} = S$,

Ps_2: for every $x \in S$ we have $\bigcap \{X \in \mathcal{F} \mid x \in X\} \in \mathcal{F}$.

We say that the set $\bigcap \{X \in \mathcal{F} \mid x \in X\}$ is a **centralized intersection** for $x \in S$.

Theorem 1.8. *Every partially ordered set $(P, \leqslant)$ is isomorphic to a partial closure system on P, ordered by the set inclusion.*

Proof. Let $(P, \leqslant)$ be a poset and let

$$\mathcal{F}(P) := \{\downarrow p \mid p \in P\}.$$

We shall show that $\mathcal{F}(P)$ is a partial closure system and, ordered by the set inclusion, it is isomorphic with P.

It is obvious that Ps_1 holds. In order to show that Ps_2 holds, we shall show that, for every $x \in P$, the intersection of all the principal ideals on P that contain x is exactly the principal ideal on P generated by x. Principal ideal $\downarrow x$ is in the collection $\{\downarrow p \in \mathcal{F}(P) \mid x \in \downarrow p\}$, since $x \in \downarrow x$; therefore $\bigcap\{\downarrow p \in \mathcal{F}(P) \mid x \in \downarrow p\} \subseteq \downarrow x$. On the other hand, since $x \in \downarrow p$ for every $\downarrow p$ from the observed collection, then all the elements $y \in P$ such that $y \leqslant x$ are also in $\downarrow p$ by the definition of principal ideal, and thus we have $\downarrow x \subseteq \bigcap\{\downarrow p \in \mathcal{F}(P) \mid x \in \downarrow p\}$. ■

Partial closure system $\mathcal{F}$ on a nonempty set S is **complete** if it fulfills:

Ps_3: Every chain in $\mathcal{F}$ has the supremum.

Theorem 1.9. *Every complete partial closure system is a CPO. Conversely, for every CPO there exists a complete partial closure system isomorphic with it.*

Proof. If $\mathcal{F}$ is a complete partial closure system, then by Theorem 1.5 it is a CPO, since it is a poset closed for suprema of chains.

On the other hand, assume that $(P, \leqslant)$ is a CPO and $\mathcal{F}(P) = \{\downarrow p \mid p \in P\}$. Obviously, $(P, \leqslant)$ is isomorphic with $(\mathcal{F}(P), \subseteq)$ (by Theorem 1.8). Let us show that every chain in $\mathcal{F}(P)$ has supremum. Let $\{\downarrow x_i \mid i \in I\}$ be a chain in $\mathcal{F}(P)$. Then $\{x_i \mid i \in I\}$ is a chain in P and its supremum is $\bigvee_{i \in I} x_i$, since

$$\bigvee\{\downarrow x_i \mid i \in I\} = \downarrow\left(\bigvee_{i \in I} x_i\right) \in \mathcal{F}(P).$$

Therefore, Ps_3 holds and the proof is complete. ■

Let $\mathcal{F}$ be a partial closure system on a set S which fulfills the next two conditions:

Ps_3': $\mathcal{F}$ is closed for unions of chains;

Ps_4': for every $X \in \mathcal{F}$, the family

$$\{Y \subseteq X \mid Y \text{ is compact element of } (\mathcal{F}, \subseteq)\}$$

is directed in $(\mathcal{F}, \subseteq)$.

Then $\mathcal{F}$ is called an **algebraic** partial closure system on S.

Theorem 1.10. *An algebraic partial closure system ordered by the set inclusion is an algebraic poset. Conversely, for every algebraic poset there exists an isomorphic algebraic partial closure system ordered by the set inclusion.*

Proof. Let $\mathcal{F}$ be an algebraic partial closure system on S. Then, by Theorem 1.9, $\mathcal{F}$ ordered by the set inclusion is a CPO. We shall show that this CPO is an algebraic poset, too. Let $\mathcal{K} \subseteq \mathcal{F}$ be the set of all compact elements of $\mathcal{F}$ and let $X \in \mathcal{F}$ be chosen arbitrarily. We need to prove that $\downarrow X \cap \mathcal{K}$ is a directed collection and that $X = \bigvee(\downarrow X \cap \mathcal{K})$, where $\downarrow X = \{Y \in \mathcal{F} \mid Y \subseteq X\}$. Note that $\downarrow X \cap \mathcal{K}$ is actually a collection from the condition Ps'_4, and from there it follows that this collection is directed.

There exists $\bigvee(\downarrow X \cap \mathcal{K})$. Indeed, since $\downarrow X \cap \mathcal{K}$ is directed and by Theorem 1.6 and condition Ps'_3 we have $\bigcup(\downarrow X \cap \mathcal{K}) \in \mathcal{F}$, we conclude that this union is also the supremum, and it belongs to the collection $\mathcal{F}$.

It is obvious that $\bigvee(\downarrow X \cap \mathcal{K}) \subseteq X$. Let us show that the other inclusion holds, too. It is sufficient to show that for every $x \in X$ there exists a compact subset Z_x of the set X that contains the element x. Let $x \in X$ and $Z_x = \bigcap\{Y \in \mathcal{F} \mid x \in Y\}$ (by Ps_2, this set is in $\mathcal{F}$). Obviously, $Z_x \subseteq X$, hence it will be sufficient to show that the set Z_x is compact. Let $\mathcal{D}$ be a directed subcollection of the collection $\mathcal{F}$ for which $Z_x \subseteq \bigvee \mathcal{D}$. By Ps'_3 and Theorem 1.6, we have $\bigvee \mathcal{D} = \bigcup \mathcal{D}$. Now from $x \in Z_x \subseteq \bigvee \mathcal{D} = \bigcup \mathcal{D}$ it follows that there exists $D \in \mathcal{D}$ such that $x \in D$. From here, by definition of the set Z_x, it follows that $Z_x \subseteq D \in \mathcal{D}$, which was to be proved.

Let now $(P, \leqslant)$ be an algebraic poset and let K be the set of its compact elements. We define $\mathcal{F} = \{\downarrow p \cap K \mid p \in P\}$ and let a mapping $f : P \to \mathcal{F}$ be defined by $f(p) = \downarrow p \cap K$. We shall prove that f is an isomorphism between $(P, \leqslant)$ and $(\mathcal{F}, \subseteq)$. First, f is obviously surjective. Second, if $p_1 \leqslant p_2$, it follows that $\downarrow p_1 \cap K \subseteq \downarrow p_2 \cap K$, and conversely, if $\downarrow p_1 \cap K \subseteq \downarrow p_2 \cap K$, then we have $\bigvee(\downarrow p_1 \cap K) \leqslant \bigvee(\downarrow p_2 \cap K)$, which together with the fact that P is compactly generated gives $p_1 \leqslant p_2$, therefore f is an isomorphism.

Finally, we shall prove that $(\mathcal{F}, \subseteq)$ is an algebraic partial closure system on the set K. The condition Ps_1 is trivial. Further, for every $x \in K$ we have

$$\bigcap\{X \in \mathcal{F} \mid x \in X\} = \bigcap\{\downarrow p \cap K \mid p \geqslant x\} = \downarrow x \cap K \in \mathcal{F},$$

so the condition Ps_2 is fulfilled. We now check Ps'_3. Let $\{x_i \cap K \mid i \in I\}$

be a chain in $\mathcal{F}$. We prove that

$$\bigcup\{\downarrow x_i \cap K \mid i \in I\} = \downarrow\left(\bigvee_{i\in I} x_i\right) \cap K.$$

First, let x be an element of the union on the left-hand side. Then for an $i \in I$ we have $x \leqslant x_i$ and $x \in K$, and it follows $x \in \downarrow\left(\bigvee_{i\in I} x_i\right) \cap K$. Therefore, one inclusion holds. Now, let $x \leqslant \bigvee_{i\in I} x_i$ and $x \in K$. Since $\{x_i \mid i \in I\}$ is a chain in P and therefore a directed subset, and since x is a compact element, it follows that $x \leqslant x_i$ for some $i \in I$. Therefore, x is in the union on the left-hand side. Hence, Ps'_3 holds too. At last, the condition Ps'_4 holds since $(P, \leqslant)$ is an algebraic poset and since $(P, \leqslant)$ and $(\mathcal{F}, \subseteq)$ are isomorphic. This completes the proof. ■

1.3 Closure operators

A **closure operator** on a nonempty set A is a unary operation $X \mapsto \overline{X}$ on the power set $P(\mathcal{A})$, which for all $X, Y \subseteq A$ satisfies the following conditions:

C_1: $X \subseteq \overline{X}$;

C_2: $X \subseteq Y$ implies $\overline{X} \subseteq \overline{Y}$;

C_3: $\overline{\overline{X}} = \overline{X}$.

If $X \subseteq A$ and $\overline{X} = X$, then X is a **closed** set and $\overline{X}$ is the **closure** of the set X. The family of closed sets $\mathcal{F}$ is the **range** of a closure operator.

Theorem 1.11. *The range of a closure operator on A is a closure system on the same set.*

Proof. Let $\mathcal{F}$ be the range of a given closure operator. By C_1 we have $A \in \mathcal{F}$, so it remains to show that $\mathcal{F}$ is closed for intersections of subcollections of sets. Let $\{X_i \mid i \in I\} \subseteq \mathcal{F}$. Then for all $i \in I$ we have $\bigcap\{X_i \mid i \in I\} \subseteq X_i$, so by C_2 it follows

$$\overline{\bigcap\{X_i \mid i \in I\}} \subseteq \overline{X_i} = X_i,$$

and hence we have

$$\overline{\bigcap\{X_i \mid i \in I\}} \subseteq \bigcap\{X_i \mid i \in I\}.$$

The other inclusion holds by C_1, therefore we have

$$\overline{\bigcap\{X_i \mid i \in I\}} = \bigcap\{X_i \mid i \in I\},$$

so the intersection is the closed set. ■

Theorem 1.12. *If $\mathcal{F}$ is a closure system on a set A, then the map $X \mapsto \overline{X}$ from $\mathcal{P}(A)$ to $\mathcal{P}(A)$, such that X is mapped to the intersection of all the elements of $\mathcal{F}$ that contain X, is a closure operator on the set A.*

Proof. Let $\mathcal{F}$ be a closure system and the map $\bar{\cdot} : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by:

$$\overline{X} = \bigcap\{Y \mid Y \in \mathcal{F} \text{ and } X \subseteq Y\}.$$

First, this function is well defined, because $X \subseteq A$, so A is in the collection on the right-hand side, hence the intersection exists.

Let $x \in X$. Then $x \in Y$ for all Y such that $X \subseteq Y$. Then also $x \in \bigcap\{Y \mid Y \in \mathcal{F} \text{ and } X \subseteq Y\}$, so we have the condition C_1.

Now we shall show that C_2 holds. Let $X_1 \subseteq X_2$. Then

$$\{Y \mid Y \in \mathcal{F} \text{ and } X_2 \subseteq Y\} \subseteq \{Y \mid Y \in \mathcal{F} \text{ and } X_1 \subseteq Y\},$$

which implies

$$\overline{X_1} = \bigcap\{Y \mid Y \in \mathcal{F} \text{ and } X_1 \subseteq Y\} \subseteq \bigcap\{Y \mid Y \in \mathcal{F} \text{ and } X_2 \subseteq Y\} = \overline{X_2}.$$

By C_1, we have $X \subseteq \overline{X}$, and by applying C_2 to this inclusion we obtain $\overline{X} \subseteq \overline{\overline{X}}$. Since $\overline{X} \in \mathcal{F}$, then $\overline{X} \in \{Y \mid Y \in \mathcal{F} \text{ and } \overline{X} \subseteq Y\}$, and since $\overline{\overline{X}} = \bigcap\{Y \mid Y \in \mathcal{F} \text{ and } \overline{X} \subseteq Y\}$, the reverse inclusion holds. Therefore, C_3 also holds and the proof is complete. ■

This correspondence among closure systems and corresponding operators is unique.

Theorem 1.13. *If $\mathcal{F}$ is the lattice of closed sets of some closure operator, then for each family $\{X_i \mid i \in I\} \subseteq \mathcal{F}$ we have*

$$\bigvee\{X_i \mid i \in I\} = \overline{\bigcup\{X_i \mid i \in I\}}.$$

Proof. Since for all $i \in I$ we have $X_i \subseteq \bigcup\{X_i \mid i \in I\}$, by C_2 we get

$$\overline{X_i} = X_i \subseteq \overline{\bigcup\{X_i \mid i \in I\}},$$

and also

$$\bigvee\{X_i \mid i \in I\} \subseteq \overline{\bigcup\{X_i \mid i \in I\}}.$$

Conversely, since

$$\bigcup\{X_i \mid i \in I\} \subseteq \bigvee\{X_i \mid i \in I\},$$

using C_2 we get

$$\overline{\bigcup\{X_i \mid i \in I\}} \subseteq \overline{\bigvee\{X_i \mid i \in I\}} = \bigvee\{X_i \mid i \in I\},$$

which completes the proof. ■

Theorem 1.14. *For every complete lattice L there exist a set and a closure operator on it such that L is isomorphic to the range of that closure operator.*

Proof. Let $(L, \leqslant)$ be a complete lattice. We define the following closure operator on the set L:

$$\overline{X} = \{x \in L \mid x \leqslant \bigvee X\},$$

where $X \subseteq L$.

Let us show that this is indeed a closure operator. It is obvious that C_1 and C_2 hold, so we prove that C_3 holds.

Let $a \in \overline{\overline{X}}$. Therefore, $a \in L$ and $a \leqslant \bigvee \overline{X} = \bigvee\{x \in L \mid x \leqslant \bigvee X\} = \bigvee X$, so we have $\overline{\overline{X}} \subseteq \overline{X}$, while the other inclusion holds by C_1.

Let now $(\mathcal{F}, \subseteq)$ be the lattice where $\mathcal{F}$ is the range of the defined closure operator and the map $i : L \to \mathcal{F}$ is defined by $i(x) := \downarrow x$. For every x, $i(x)$ is a closed set, and since x is the supremum of this set, we have that this map is well defined. By definition, i is surjective. By definition of a principal ideal we have

$$x \leqslant y \text{ if and only if } \downarrow x \subseteq \downarrow y,$$

so i is injective and both i and i^{-1} preserve the order. ■

A closure operator $X \mapsto \overline{X}$ on a set A is **algebraic** if it fulfills the following:

C_4: for all $X \subseteq A$ we have $\overline{X} = \bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}$.

Theorem 1.15. *Let $\mathcal{F}$ be a closure system on a set A. Then the family $\mathcal{F}$ is a range of an algebraic closure operator if and only if $\mathcal{F}$ is an algebraic closure system.*

Proof. Let $X \mapsto \overline{X}$ be an algebraic closure operator and let $\mathcal{F}$ be its range. Further, let $\mathcal{G} = \{Y_i \mid i \in I\}$ be a directed family of elements of $\mathcal{F}$, that is, for every finite subfamily of $\mathcal{G}$ there exists a set from $\mathcal{G}$ which contains (as a subset) every set from the considered subfamily. We shall prove that the union of all sets from $\mathcal{G}$ is a closed set; in other words, if we denote $G = \bigcup \mathcal{G}$, we shall prove $G = \overline{G}$. By C_4 we have

$$\overline{G} = \bigcup\{\overline{Y} \mid Y \subseteq G \text{ and } Y \text{ is finite}\}.$$

By C_1 we have $G \subseteq \overline{G}$. Now let $x \in \overline{G}$. Then for some finite subset U of G we have $x \in \overline{U}$. Since U is a finite subset of $G = \bigcup \mathcal{G}$, it follows that $U \subseteq \bigcup \mathcal{G}_1$ for some finite $\mathcal{G}_1 \subseteq \mathcal{G}$. By the fact that $\mathcal{G}$ is directed we have that there exists $H \in \mathcal{G}$ such that $\bigcup \mathcal{G}_1 \subseteq H$, that is, $U \subseteq H \in \mathcal{G}$. This implies $\overline{U} \subseteq \overline{H} = H$, and therefore $x \in \bigcup \mathcal{G} = G$, therefore $\overline{G} \subseteq G$ and $G = \overline{G}$. Hence, $\mathcal{F}$ is algebraic closure system.

For the other direction, let $\mathcal{F}$ be an algebraic closure system on the set A and $X \subseteq A$. We define $\overline{X} = \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}$. This is a closure operator by the proof of Theorem 1.12. We prove that this operator is algebraic. Let $x \in X$. Then

$$x \in \{x\} \subseteq \overline{\{x\}} \subseteq \bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\},$$

so we have

$$X \subseteq \bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}.$$

On the other hand, the set $\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}$ is directed since it contains an upper bound for every finite subfamily $\{\overline{Y_1}, \ldots, \overline{Y_n}\}$: this bound is $\overline{\bigcup_{i=1}^{n} Y_i}$, since $\bigcup_{i=1}^{n} Y_i$ is finite subset of X.

It follows from C_2 that

$$\bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\} \subseteq \overline{X}.$$

By the fact that $\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}$ is directed and that $\mathcal{F}$ is an algebraic closure system, we have that $\bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}$ is a closed set, and therefore

$$\overline{X} \subseteq \overline{\bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}} = \bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}.$$

Finally,

$$\overline{X} = \bigcup\{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\},$$

so the condition C_4 holds, therefore this operator is algebraic. ■

Theorem 1.16. *Let $\mathcal{F}$ be an algebraic closure system on a set A, $X \mapsto \overline{X}$ a closure operator on A such that $\mathcal{F}$ is its range, and $B \in \mathcal{F}$. Then B is compact in the lattice $(\mathcal{F}, \subseteq)$ if and only if $B = \overline{Y}$ for some finite subset Y of the set A.*

Proof. Let $B \subseteq A$ be a compact element of the lattice $(\mathcal{F}, \subseteq)$. Since $B \in \mathcal{F}$, and $\mathcal{F}$ is an algebraic closure system on A, we have that B is a closed subset of A. By (C_4) we have

$$B = \overline{B} = \bigcup\{\overline{Y} \mid Y \subseteq B \text{ and } Y \text{ is finite}\} = \overline{\bigcup\{\overline{Y} \mid Y \subseteq B \text{ and } Y \text{ is finite}\}}.$$

By Proposition 1.13 we have $B = \bigvee\{\overline{Y} \mid Y \subseteq B \text{ and } Y \text{ is finite}\}$, and since B is a compact element, it follows that there exists a natural number m such that $B = \overline{Y_1} \vee \overline{Y_2} \vee \ldots \overline{Y_m}$, where all Y_i are finite. If we denote $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_m$, then Y is a finite subset of A and we have $B = \overline{Y}$.

For the other direction, assume that for some $Y \subseteq A$ such that $|Y| = n$, $n \in \mathbb{N}$, we have $B = \overline{Y}$. We prove that B is a compact element in the lattice $(\mathcal{F}, \subseteq)$.

Let $B \subseteq \bigvee\{X_i \mid i \in I\}$. By Proposition 1.13 and the condition C_4, we have

$$\bigvee\{X_i \mid i \in I\} = \overline{\bigcup\{X_i \mid i \in I\}} = \bigcup\{\overline{Z} \mid Z \subseteq \bigcup X_i \text{ and } Z \text{ is finite}\}.$$

Therefore, for every $y_j \in Y$ there exists a finite set $Y_j \subseteq \bigcup\{X_i \mid i \in I\}$ such that $y_j \in \overline{Y_j}$. Since Y_j is finite, we have $Y_j \subseteq X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{k_j}}$ for finitely many elements of family $\{X_i \mid i \in I\}$. It follows $y_j \in \overline{X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{k_j}}}$, that is,

$$Y \subseteq \bigcup_{j=1}^{n} \overline{X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{k_j}}} \subseteq \overline{\bigcup_{j=1}^{n} X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{k_j}}}.$$

Hence, we have

$$B = \overline{Y} \subseteq \overline{\overline{\bigcup_{j=1}^{n} X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{k_j}}}} = \overline{\bigcup_{j=1}^{n} X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{k_j}}}$$

$$= \bigvee_{j=1}^{n} (X_{j_1} \vee X_{j_2} \vee \cdots \vee X_{j_{k_j}}),$$

so B is compact in $(\mathcal{F}, \subseteq)$. ■

Theorem 1.17. *A lattice $(\mathcal{F}, \subseteq)$, where $\mathcal{F}$ is the range of an algebraic closure operator on A, is algebraic.*

Proof. From Theorems 1.11 and 1.7 we know that $(\mathcal{F}, \subseteq)$ is a complete lattice. To show that it is algebraic, we need to prove that each of its elements is equal to supremum of some compact elements. Let $X \in \mathcal{F}$, that is, $X \subseteq A$ and $X = \overline{X}$. Then by C_4 we have

$$X = \bigcup \{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\},$$

and by Theorem 1.16 we see that X is union of some compact elements. Since X is closed, by Proposition 1.13 we have that it is also the supremum of these compact elements, which completes the proof. ■

Theorem 1.18. *Every algebraic lattice $(L, \leqslant)$ is isomorphic to the lattice $(\mathcal{F}, \subseteq)$, where $\mathcal{F}$ is the range of some algebraic closure operator.*

Proof. Let K be the set of all compact elements of an algebraic lattice L. We define the map from $\mathcal{P}(K)$ to $\mathcal{P}(K)$ such that $X \mapsto \overline{X}$, where

$$\overline{X} = \{k \in K \mid k \leqslant \bigvee X\}.$$

We prove that this map is an algebraic closure operator on the set K.

The condition C_1 holds, because if $x \in X \subseteq K$, then x is compact and $x \leqslant \bigvee X$, hence $x \in \overline{X}$.

Let $X \subseteq Y$. Then we have $\bigvee X \leqslant \bigvee Y$, which implies $\overline{X} \subseteq \overline{Y}$, so C_2 also holds.

Now we show $\overline{\overline{X}} \subseteq \overline{X}$. Let $x \in \overline{\overline{X}}$. Therefore, $x \in K$ and $x \leqslant \bigvee \overline{X}$. We have $\bigvee \overline{X} = \bigvee X$: the inequality ($\geqslant$) follows from C_1, and the inequality ($\leqslant$) follows from the fact that for all $k \in \overline{X}$ we have $k \leqslant \bigvee X$. Therefore, we have $x \leqslant \bigvee X$, and then $x \in \overline{X}$. Hence, C_3 holds.

Let now $x \in \overline{X}$. Then $x \leqslant \bigvee Y$ for some finite $Y \subseteq X$, since x is a compact element. Also we have $x \in \overline{Y}$, by definition of closure, so we have

$$\overline{X} \subseteq \bigcup \{\overline{Y} \mid Y \subseteq X \text{ and } Y \text{ is finite}\}.$$

The other inclusion is obvious, so it follows that this map is an algebraic closure operator.

Finally, we define a map φ from the lattice L into $\mathcal{P}(K)$ such that $\varphi : x \mapsto \{k \in K \mid k \leqslant x\}$. The image of x is a closed set, since $x = \bigvee\{k \in K \mid k \leqslant x\}$. Hence, this function maps L into the complete lattice of closed sets in K. If $X \subseteq K$ is a closed set, then $X = \overline{X} = \{k \in K \mid k \leqslant \bigvee X\}$, so $\varphi(\bigvee X) = X$ and hence the function φ is surjective. Finally, $x \leqslant y$ is equivalent to ${\downarrow}x \cap K \subseteq {\downarrow}y \cap K$, which is the same as $\varphi(x) \subseteq \varphi(y)$; therefore, the function φ is an isomorphism, so the proof is complete. ■

1.4 Partial closure operators

For a nonempty set S, let $C : \mathcal{P}(S) \to \mathcal{P}(S)$ be a partial mapping satisfying:

Pc_1: If $C(X)$ is defined, then $X \subseteq C(X)$.

Pc_2: If $C(X)$ and $C(Y)$ are defined, then $X \subseteq Y$ implies $C(X) \subseteq C(Y)$.

Pc_3: If $C(X)$ is defined, then $C(C(X))$ is also defined and $C(C(X)) = C(X)$.

Pc_4: $C(\{x\})$ is defined for every $x \in S$.

As defined in [53], a partial mapping C fulfilling properties Pc_1–Pc_4 is a **partial closure operator** on S. Note that partial closure operators are a generalization of closure operators.

As usual, if $X \subseteq S$ and $C(X) = X$, then we call X a **closed** set. The family of closed sets $\mathcal{F}_C$ is called the **range** of a partial closure operator C. The **exact domain** of a partial closure operator C on S is denoted by $Dom(C)$:

$$Dom(C) := \{X \mid X \subseteq S \text{ and } C(X) \text{ is defined}\}.$$

Let C be a partial closure operator on S. If $C(X)$ is defined, then it is straightforward to check that

$$C(X) = \bigcap\{Y \in \mathcal{F}_C \mid X \subseteq Y\} \tag{1.1}$$

(note that the same property also holds for closure operators).

Theorem 1.19. *The range of a partial closure operator on a set S is a partial closure system.*

Conversely, for every partial closure system $\mathcal{F}$ on S, there is a partial closure operator on S such that its range is $\mathcal{F}$.

Proof. Let C be a partial closure operator on a set S. The range $\mathcal{F}_C = \{X \mid X = C(X)\}$ is nonempty, since S is nonempty set, so by Pc_4 and Pc_3, we have $C(C(\{x\})) = C(\{x\})$.

The condition Ps_1 holds since every element of the set S belongs to a closed set, that is, $x \in \{x\} \subseteq C(\{x\}) = C(C(\{x\}))$, hence $\bigcup \mathcal{F}_C = S$.

Let $x \in S$ and $\mathcal{F}_x = \{X \in \mathcal{F}_C \mid x \in X\}$. If $X \in \mathcal{F}_x$, then we have $C(X) = X$ and $x \in X$. Therefore, $\{x\} \subseteq X$ and $C(\{x\}) \subseteq C(X) = X$, and thus we have $C(\{x\}) \subseteq \bigcap \mathcal{F}_x$. Since $C(\{x\}) \in \mathcal{F}_x$, it follows $C(\{x\}) = \bigcap \mathcal{F}_x$. For that reason we have $\bigcap \mathcal{F}_x \in \mathcal{F}$, hence Ps_2 holds, too.

Conversely, let $\mathcal{F}$ be a partial closure system on S. We define a partial closure operator $C : \mathcal{P}(S) \to \mathcal{P}(S)$ as follows:

$$C(X) := \bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\}$$

if the intersection on the right-hand side is in $\mathcal{F}$; otherwise $C(X)$ is not defined.

If, for some $X \subseteq S$, the closure $C(X)$ is defined, then it is easy to see that C has properties Pc_1–Pc_3. The property Pc_4 holds because, for $x \in S$, $C(\{x\})$ is defined by Ps_2. ■

Theorem 1.20. *Let $\mathcal{F}$ be a partial closure system and C a partial closure operator on S whose range is $\mathcal{F}$. Let*

$$\widehat{\mathcal{F}} := \{S\} \cup \{X \subseteq S \mid X = \bigcap \mathcal{G}, \text{ for } \mathcal{G} \subseteq \mathcal{F}\}.$$

Then $\widehat{\mathcal{F}}$ is a closure system and for the corresponding closure operator $\widehat{C}$ we have $\widehat{C}(X) = C(X)$ whenever $C(X)$ is defined.

Proof. The family $\widehat{\mathcal{F}}$ is a closure system since, by its definition, the set S belongs to it; further, it is closed for intersection, since

$$\bigcap \{\bigcap \mathcal{F}_i \mid i \in I\} = \bigcap (\bigcup \{\mathcal{F}_i \mid i \in I\}).$$

By Theorem 1.12, for $X \subseteq S$ we have

$$\widehat{C}(X) = \bigcap \{Y \in \widehat{\mathcal{F}} \mid X \subseteq Y\}.$$

Let $C(X)$ be defined. Since $\mathcal{F} \subseteq \widehat{\mathcal{F}}$, we have

$$\bigcap\{Y \in \widehat{\mathcal{F}} \mid X \subseteq Y\} \subseteq \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\},$$

hence $\widehat{C}(X) \subseteq C(X)$.

For the other direction, we need $C(X) \subseteq \widehat{C}(X)$, that is

$$C(X) \subseteq \bigcap\{Y \in \widehat{\mathcal{F}} \mid X \subseteq Y\}.$$

Let $Y \in \widehat{\mathcal{F}}$ be such that $X \subseteq Y$. We know that for all $Y \in \widehat{\mathcal{F}}$ there exists a subfamily $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ such that $Y = \bigcap \widetilde{\mathcal{F}}$. Therefore, for all $F \in \widetilde{\mathcal{F}}$ we have $X \subseteq Y \subseteq F$, hence $C(X) \subseteq C(F) = F$, which implies

$$C(X) \subseteq \bigcap \widetilde{\mathcal{F}} = Y.$$

This gives $\widehat{C}(X) = C(X)$, which completes the proof. ■

A partial closure operator C on a set S is **complete** if it satisfies the following condition:

Pc_5: if $\{X_i \mid i \in I\}$ is a chain and $C(X_i)$ is defined for all $i \in I$, then $C(\bigcup_{i \in I} X_i)$ is defined, too.

Theorem 1.21. *The range of a complete partial closure operator is a complete partial closure system.*

Conversely, a partial closure operator whose range is a complete partial closure system is complete.

Proof. Let C be a complete partial closure operator on S, and let $\{X_i \mid i \in I\}$ be a chain of sets from the range of the operator C. The range is the partial closure system by Theorem 1.19. By Pc_5, $C(\bigcup_{i \in I} X_i)$ exists and equals the supremum we need. Therefore, the range of C is a complete partial closure system.

Conversely, assume that $\mathcal{F}$ is a complete partial closure system on S and that C is a partial closure operator defined by

$$C(X) := \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}.$$

We will prove that Pc_5 holds for C.

Let $\{X_i \mid i \in I\}$ be a chain of subsets that have defined closure. By the condition Pc_2, $\{C(X_i) \mid i \in I\}$ is also a chain of sets. Further, by Ps_3 we have $\bigvee\{C(X_i) \mid i \in I\} \in \mathcal{F}$, therefore

$$C(\bigvee\{C(X_i) \mid i \in I\}) = \bigvee\{C(X_i) \mid i \in I\}.$$

Let $\widehat{C}$ be the map defined in Theorem 1.20. Then for $i \in I$, $X_i \subseteq \bigcup\{X_i \mid i \in I\}$ implies $\widehat{C}(X_i) \subseteq \widehat{C}(\bigcup\{X_i \mid i \in I\})$. By Theorem 1.20 we have

$$\bigvee\{C(X_i) \mid i \in I\} = \bigvee\{\widehat{C}(X_i) \mid i \in I\} \tag{1.2}$$

and also

$$\bigvee\{\widehat{C}(X_i) \mid i \in I\} \subseteq \widehat{C}(\bigcup\{X_i \mid i \in I\}). \tag{1.3}$$

We have

$$\widehat{C}(\bigcup\{X_i \mid i \in I\}) = \bigcap\{Y \in \widehat{\mathcal{F}} \mid \bigcup X_i \subseteq Y\}. \tag{1.4}$$

Since $X_i \subseteq C(X_i)$ for all $i \in I$, we have

$$\bigcup\{X_i \mid i \in I\} \subseteq \bigvee\{C(X_i) \mid i \in I\}.$$

By Ps_3, the supremum on the right-hand side is in $\mathcal{F}$, and $\mathcal{F} \subseteq \widehat{\mathcal{F}}$, which means that this is one of the sets Y in (1.4), and hence

$$\widehat{C}(\bigcup\{X_i \mid i \in I\}) \subseteq \bigvee\{C(X_i) \mid i \in I\}. \tag{1.5}$$

Therefore, by (1.2), (1.3) and (1.5) we obtain

$$\widehat{C}(\bigcup\{X_i \mid i \in I\}) = \bigvee\{C(X_i) \mid i \in I\}.$$

Since $\bigvee\{C(X_i) \mid i \in I\} \in \mathcal{F}$, it follows that $\widehat{C}(\bigcup\{X_i \mid i \in I\}) \in \mathcal{F}$, hence

$$\widehat{C}(\bigcup\{X_i \mid i \in I\}) = C(\bigcup\{X_i \mid i \in I\}).$$

Therefore, we have shown that $C(\bigcup\{X_i \mid i \in I\})$ is defined, that is, Pc_5 holds. ■

The next corollary follows directly from Theorem 1.9 and Theorem 1.21.

Corollary 1.22. *The range of a complete partial closure operator ordered by the set inclusion is a CPO, and vice versa, every CPO is isomorphic to the range of a complete partial closure operator.*

A complete partial closure operator C on a set S is **algebraic** if it fulfills:

Pc_6: if $C(X)$ is defined, then the set

$$\{C(Y) \mid Y \subseteq X,\ C(Y) \text{ is defined and } Y \text{ is finite}\} \tag{1.6}$$

is directed and

$$C(X) = \bigcup\{C(Y) \mid Y \subseteq X,\, C(Y) \text{ is defined and } Y \text{ is finite}\}. \tag{1.7}$$

Theorem 1.23. *If C is an algebraic partial closure operator, then its range is closed for unions of chains.*

Proof. Let C be an algebraic partial closure operator on a set S and let $\{X_i \mid i \in I\}$ be a chain of sets from the range of C. By Pc_5, $C(\bigcup X_i)$ exists, and if we denote $\bigcup\{X_i \mid i \in I\}$ by Z, then by Pc_6 we have

$$C(Z) = \bigcup\{C(Y) \mid Y \subseteq Z,\, C(Y) \text{ is defined and } Y \text{ is finite}\}.$$

We prove that $C(Z) = Z$. The inclusion $Z \subseteq C(Z)$ is trivial. On the other hand, let $x \in C(Z)$. Then for some finite subset Y of the set Z we have $x \in C(Y)$. Since Y is finite, it follows that $Y \subseteq \bigcup_{i\in J} X_i$, where $J \subseteq I$ is finite. Since $\{X_i \mid i \in I\}$ is a chain, there exists $k \in I$ such that $Y \subseteq X_k$, so we have $x \in C(Y) \subseteq C(X_k) = X_k$, and hence $x \in \bigcup_{i\in I} X_i = Z$. Therefore, $C(Z) = Z$. ■

Theorem 1.24. *The range of an algebraic closure operator ordered by the set inclusion is an algebraic poset.*

Proof. Let C be an algebraic partial closure operator on a set S and let $\mathcal{F}$ be the range of C. Then, by Corollary 1.22, $\mathcal{F}$ is a CPO, and by Theorem 1.23 and Theorem 1.6, this poset is closed for unions of directed families. We need to prove that compact elements in $\mathcal{F}$ are exactly those sets Z for which there exists finite Y such that $Z = C(Y)$.

First assume that $Z \in \mathcal{F}$ and $Z = C(Y)$ for a finite set Y. Let $\mathcal{G}$ be a directed collection from $\mathcal{F}$ such that $Z \subseteq \bigcup \mathcal{G}$. Now we have $Y \subseteq C(Y) \subseteq \bigcup \mathcal{G}$. Since Y is finite, there exists a finite subcollection $\mathcal{G}_1 \subseteq \mathcal{G}$ such that $Y \subseteq \bigcup \mathcal{G}_1$. Since $\mathcal{G}$ is directed, we have that there exists a set $T \in \mathcal{G}$ such that $\bigcup \mathcal{G}_1 \subseteq T$, and hence $Y \subseteq T$. This implies that $C(Y) \subseteq C(T) = T$ and Z is a compact element in $\mathcal{F}$.

Now assume that $Z \in \mathcal{F}$ is compact. By Pc_6, the collection

$$\{C(Y) \mid Y \subseteq Z,\, C(Y) \text{ is defined and } Y \text{ is finite}\}$$

is directed and

$$Z = C(Z) = \bigcup\{C(Y) \mid Y \subseteq Z,\, C(Y) \text{ is defined and } Y \text{ is finite}\}.$$

Because of the fact that Z is compact and below (or equal to) a union of a directed set, it follows that Z is below some member of that union, that is, there exists a finite set Y such that $C(Y) = Z$.

Therefore, by Pc_6, $\mathcal{F}$ is an algebraic poset. ■

Theorem 1.25. *For every algebraic poset S there exists an algebraic partial closure operator whose range is isomorphic with the poset S.*

Proof. Let S be an algebraic poset and K the set of all compact elements in P. We define a partial closure operator by: if X is a directed subset of K, then

$$C(X) := \downarrow(\bigvee X) \cap K,$$

otherwise $C(X)$ is not defined.

Let $x \in X \subseteq K$. It is obvious that $x \leqslant \bigvee X$, so we have $x \in \downarrow(\bigvee X) \cap K$, hence Pc_1 holds.

Now, let X and Y be directed subsets of K such that $X \subseteq Y$, and let $x \in C(X)$. It follows that $x \in K$ and $x \leqslant \bigvee X \leqslant \bigvee Y$, hence $C(X) \subseteq C(Y)$, therefore Pc_2 holds.

To prove Pc_3, let X be a directed subset of K. By Pc_1 we have $C(X) \subseteq C(C(X))$. If $x \in C(C(X))$, then $x \in K$ i $x \leqslant \bigvee C(X) = \bigvee(\downarrow(\bigvee X) \cap K) \leqslant \bigvee(\downarrow(\bigvee X)) = \bigvee X$, hence $C(C(X)) \subseteq C(X)$, and therefore $C(C(X)) = C(X)$.

The condition Pc_4 holds, because for every $x \in K$ the singleton $\{x\}$ is directed, therefore $C(\{x\})$ is defined.

Let $\{X_i \mid i \in I\}$ be a chain of directed sets. Then $\bigcup X_i$ is directed, too, and since C is defined for every element of the chain (they are directed), we have that $C(\bigcup X_i)$ is also defined, hence Pc_5 holds.

Now let X be a directed subset of K. To show Pc_6, we first show that

$$\mathcal{Y} = \{C(Y) \mid Y \subseteq X,\, C(Y) \text{ is defined and } Y \text{ is finite}\}$$

is a directed family of sets. Let $C(Y_1)$ and $C(Y_2)$ be in this family. Then Y_1 and Y_2 are finite directed sets of compact elements. Denote by y_1 and y_2 suprema of these two sets, respectively. Since Y_i is finite, y_i is compact for $i = 1, 2$. Therefore, since X is directed and $y_i \leqslant \bigvee X$, there exists $x_i \in X$ such that $y_i \leqslant x_i$, for $i = 1, 2$. Further, there exists $x \in X$ such that $x_1, x_2 \leqslant x$. Now, for $i = 1, 2$ we have $\downarrow y_i \leqslant \downarrow x$, so $C(Y_i) \subseteq \downarrow x \cap K = C(\{x\})$. That means that $\mathcal{Y}$ is directed family.

Second, obviously we have

$$C(X) \supseteq \bigcup\{C(Y) \mid Y \subseteq X,\, C(Y) \text{ is defined and } Y \text{ is finite}\},$$

so let $x \in C(X)$ and x is a compact element in S. Then $x \in K$ and $x \leqslant \bigvee X$, and since X is directed, there exists $d \in X$ such that $x \leqslant d$. Now it follows that $x \in C(\{d\})$. Hence we have the reverse inclusion too; therefore, Pc_6 holds.

At last, define a map $f : S \to \mathcal{F}_C$ by

$$f(a) := \downarrow a \cap K.$$

The map f is surjective, since for every set $X \in \mathcal{F}_C$ we have $X = \downarrow(\bigvee X) \cap K$, and therefore $f(\bigvee X) = X$. Lastly, let $a_1 \leqslant a_2$, then $\downarrow a_1 \cap K \subseteq \downarrow a_2 \cap K$. Conversely, if $\downarrow a_1 \cap K \subseteq \downarrow a_2 \cap K$, then we have $\bigvee(\downarrow a_1 \cap K) \leqslant \bigvee(\downarrow a_2 \cap K)$, and since S is compactly generated, we have $a_1 \leqslant a_2$. Therefore, f is an isomorphism between S and the range of C. ■

1.5 Generating closure operators from partial closure operators

In this section we show how to extend a partial closure operator to a closure operator (that is, how to define closure of sets with undefined closure in such a way that the resulting operator is a closure operator). This is always possible, and, in general, such a closure operator is not uniquely determined. However, we shall show that, among all such closure operators that correspond to a given partial closure operator C on a set S, there are two, say $C^{\ominus}$ and $C^{\oplus}$, that can be considered the "smallest" and the "largest" one, in the following sense: if K is any closure operator on S such that $K(X) = C(X)$ whenever $C(X)$ is defined, then for each X we have $C^{\ominus}(X) \subseteq K(X) \subseteq C^{\oplus}(X)$.

Let C be a partial closure operator on a set S. We first define $C^{\oplus} : \mathcal{P}(S) \to \mathcal{P}(S)$ in the following way:

$$C^{\oplus}(X) = \bigcap\{C(X') \mid X' \supseteq X \text{ and } C(X') \text{ is defined}\}.$$

(Note: if the family at the right-hand side is empty, then, since everything is considered on the set S, we then have $C^{\oplus}(X) = \bigcap \emptyset = S$.)

Theorem 1.26. *For a partial closure operator C on a set S, the operator $C^{\oplus}$ is a closure operator on S, and $C^{\oplus}(X) = C(X)$ for each X such that $C(X)$ is defined.*

Proof. We first show that $C^{\oplus}$ is a closure operator.

C_1: Follows directly by definition.

C_2: Let $X \subseteq Y$. The family at the right-hand side for $C^{\oplus}(Y)$ is clearly a subfamily of the family at the left-hand side for $C^{\oplus}(X)$; therefore, $C^{\oplus}(X) \subseteq C^{\oplus}(Y)$.

C_3: Let X be given. In order to prove $C^{\oplus}(C^{\oplus}(X)) = C^{\oplus}(X)$, it is enough to show that, if $X' \supseteq X$ and $C(X')$ is defined, then $C(X') \supseteq C^{\oplus}(X)$ (the equality $C^{\oplus}(C^{\oplus}(X)) = C^{\oplus}(X)$ then follows by the definition of $C^{\oplus}$, since the corresponding right-hand sides families are equal). However, if X' is such a set, then $C^{\oplus}(X) \subseteq C(X')$ immediately follows by the definition of $C^{\oplus}(X)$. This completes the argument.

Finally, we note that, if $C(X)$ is defined, then for each X', $X' \supseteq X$, such that $C(X')$ is defined, we have $C(X') \supseteq C(X)$. Therefore, $C^{\oplus}(X) = C(X)$, which completes the proof. ■

We now show the "maximality" of $C^{\oplus}$.

Theorem 1.27. *Let C be a partial closure operator on a set S, and let K be any closure operator on S such that $K(X) = C(X)$ whenever $C(X)$ is defined. Then for each X we have $K(X) \subseteq C^{\oplus}(X)$.*

Proof. For each X' such that $X' \supseteq X$ and $C(X')$ is defined, we have $K(X) \subseteq K(X') = C(X')$; therefore, since $K(X)$ is a subset of each such $C(X')$, we have $K(X) \subseteq C^{\oplus}(X)$. ■

We now turn to the announced "smallest" closure operator. If C is a partial closure operator on a set S, we first define $D'_C : \mathcal{P}(S) \to \mathcal{P}(S)$ in the following way:

$$D'_C(X) := \bigcup \{C(X') \mid X' \subseteq X \text{ and } C(X') \text{ is defined}\}.$$

Now, let

$$D_C^{(0)}(X) := X,$$

and, for an ordinal α,

$$D_C^{(\alpha+1)}(X) := D'_C(D_C^{(\alpha)}(X));$$

$$D_C^{(\alpha)}(X) := \bigcup_{\xi<\alpha} D_C^{(\xi)}(X), \text{ if } \alpha \text{ is a limit ordinal.}$$

Finally, we define

$$C^{\ominus}(X) := \bigcup_{\alpha \in \mathbf{ON}} D_C^{(\alpha)}(X).$$

Note, since, clearly, $D_C^{(\alpha)}(X)$ is always a subset of S, we have that $C^{\ominus}(X)$ is indeed a set (not a proper class), in fact, a subset of S.

Theorem 1.28. *For a partial closure operator C on a set S, the operator $C^{\ominus}$ is a closure operator on S, and $C^{\ominus}(X) = C(X)$ for each X such that $C(X)$ is defined.*

Proof. We first show that $C^{\ominus}$ is a closure operator.

C_1: Let a set X, $X \subseteq S$, be given. For any $x \in X$, by Pc_4 we have that $C(\{x\})$ is defined, and by Pc_1 we have $x \in C(\{x\})$. Therefore, $x \in C(\{x\}) \subseteq D'_C(X) \subseteq C^{\ominus}(X)$. This proves $X \subseteq C^{\ominus}(X)$.

C_2: Let $X \subseteq Y$. Then clearly $D'_C(X) \subseteq D'_C(Y)$, which implies that for all α we have $D_C^{(\alpha)}(X) \subseteq D_C^{(\alpha)}(Y)$, and therefore $C^{\ominus}(X) \subseteq C^{\ominus}(Y)$.

C_3: Let X be given. Note that, since $X \subseteq D'_C(X)$, we have $D_C^{(\alpha)}(X) \subseteq D_C^{(\beta)}(X)$ whenever $\alpha < \beta$. Also, if $D_C^{(\alpha)}(X) = D_C^{(\alpha+1)}(X)$ for some α, then $D_C^{(\alpha)}(X) = D_C^{(\beta)}(X)$ whenever $\alpha < \beta$.

Since each $D_C^{(\alpha)}(X)$ is a subset of S, we conclude that there can be at most $|S|$ different sets among $\{D_C^{(\alpha)}(X) : \alpha \in \mathbf{ON}\}$. Altogether, we conclude that there is an ordinal α such that $C^{\ominus}(X) = D_C^{(\alpha)}(X)$; in fact, we may say $C^{\ominus}(X) = D_C^{(|S|^+)}(X)$ (where $\cdot^+$ denotes the successor cardinal). Now we have

$$D'(C^{\ominus}(X)) = D'(D_C^{(|S|^+)}(X)) = D_C^{(|S|^+ + 1)}(X) = D_C^{(|S|^+)}(X) = C^{\ominus}(X).$$

From this we obtain $C^{\ominus}(C^{\ominus}(X)) = C^{\ominus}(X)$, which was to be proved.

Finally, we note that, if $C(X)$ is defined, then for each X', $X' \subseteq X$, such that $C(X')$ is defined, we have $C(X') \subseteq C(X)$. Therefore, $D'_C(X) = C(X)$, which implies $C^{\ominus}(X) = C(X)$. This completes the proof. ∎

We now show the "minimality" of $C^{\ominus}(X)$.

Theorem 1.29. *Let C be a partial closure operator on a set S, and let K be any closure operator on S such that $K(X) = C(X)$ whenever $C(X)$ is defined. Then for each X we have $C^{\ominus}(X) \subseteq K(X)$.*

Proof. We shall prove the following: if $X \subseteq K(Y)$, then $D'_C(X) \subseteq K(Y)$. This is enough to finish the proof: indeed, in that case we clearly have $D_C^{(\alpha)}(X) \subseteq K(Y)$ for each α (by repeatedly applying the same claim), and thus $C^{\ominus}(X) \subseteq K(Y)$; the proof then follows by taking $X = Y$.

Therefore, let us prove the claim. Let $X \subseteq K(Y)$. For any $X' \subseteq X$ such that $C(X')$ is defined we have $C(X') = K(X')$. Therefore, since $X' \subseteq X \subseteq K(Y)$, we have $K(X') \subseteq K(K(Y)) = K(Y)$, that is, $C(X') \subseteq K(Y)$. From this we obtain $D'_C(X) \subseteq K(Y)$, which was to be proved. ■

1.6 Summary

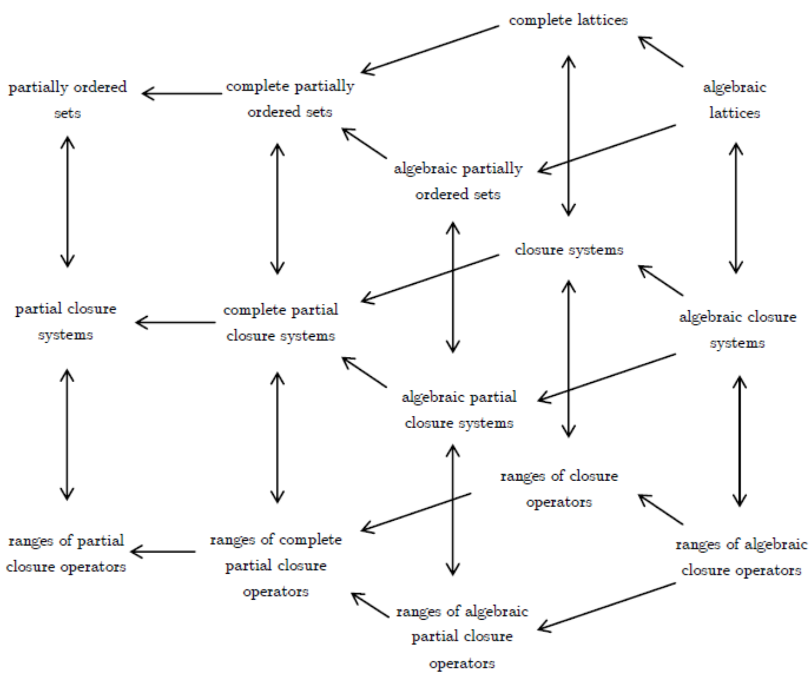

Figure 2.

Connections between posets, closure systems and closure operators established in this section are shown in diagram in Figure 2.

CHAPTER

2

POSETS AND PARTIAL CLOSURE OPERATORS

A closure system is a complete lattice under inclusion, and as a converse, the collection of principal ideals of a lattice is a closure system, which is, when equipped by inclusion, order isomorphic with the lattice itself. Still, the closure system of principal ideals is not the only closure system isomorphic to a given lattice.

Our aim in this section is to establish a particular relationship among collections of sets, operators and posets. This relationship should be analogous (as much as possible) to the one among closure operators, closure systems and complete lattices. Still, our present approach brings some new requirements, which enable essential improvements of the mentioned relationship.

This chapter presents fully original work, mostly from the paper [50].

2.1 Sharpness

We say that a partial closure operator C on S is **sharp**, if it satisfies the condition:

Pc_7: Let $B \subseteq S$. If $\bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C$, then $C(B)$ is defined and

$$C(B) = \bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\}. \quad \text{(sharpness)} \tag{2.1}$$

We also say that a partial operator on S, fulfilling properties Pc_1–Pc_4, Pc_7 is an **SPCO** on S.

Remark 2.1. In Pc_7 if C is monotone then (2.1) folows given that $C(B)$ is defined.

Notice that if in Pc_7 there does not exist a set $X \in \mathcal{F}_C$ such that $B \subseteq X$, then straightforwardly $C(B)$ is not defined (because of $B \subseteq C(B)$).

Observe also that a closure operator C on S (i.e., an operator which is a function) trivially fulfils condition Pc_7, which reduces to the condition (1.1).

Remark 2.2. By (1.1) the converse implication in the condition Pc_7 is always valid.

We note that the condition Pc_7 can not be derived from the conditions Pc_1–Pc_4, as shown by the following example.

Example 2. Let C be a partial mapping defined on $\{a, b, c\}$ with

$$C : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{a,b,c\} \\ \{a\} & \{b\} & \{a,b,c\} & \{a,b,c\} \end{pmatrix}.$$

It is straightforward to check that C satisfies conditions Pc_1–Pc_4, but the property Pc_7 does not hold because $C(\{a, b\})$ is not defined. □

From the same example, it follows that Pc_7 can neither be derived from the above conditions, to which Pc_5 and Pc_6 are added.

Further, neither of the conditions Pc_5 and Pc_6 can be derived from Pc_1–Pc_4 and Pc_7, as shown by the following example.

Example 3. Let C be a partial mapping defined on $\mathbb{N}$ by

$$C(X) = \begin{cases} X, & \text{if } X \text{ is a finite subset of } \mathbb{N}; \\ E_1, & \text{if } X \text{ is an infinite subset of } E_1; \end{cases}$$

where E is the set of all even natural numbers and $E_1 = E \cup \{1\}$. This is a sharp partial closure operator, but it is not complete. Indeed, consider the family $\{X_i \mid i \in \mathbb{N}\}$, where $X_i = \{1, 2, \ldots, i\}$. This family is a chain and $C(X_i)$ is defined for every $i \in \mathbb{N}$, but $C(\bigcup_{i \in \mathbb{N}} X_i) = C(\mathbb{N})$ is not defined.

The constructed example does not satisfy Pc_6 either. Indeed, $C(E) = E_1$, but there does not exist a finite subset of even numbers that contains 1, hence we cannot represent $C(E)$ as the union of closures of all finite subsets of E. □

The following is a refinement of a theorem from [53].

Theorem 2.3. *The range of a partial closure operator on a set S is a partial closure system.*

Conversely, for every partial closure system $\mathcal{F}$ on S, there is a unique sharp partial closure operator on S such that its range is $\mathcal{F}$.

Proof. The first part of this theorem is in Theorem 1.19, so we prove only the other direction. Let $\mathcal{F}$ be a partial closure system on a set S. We define the partial mapping $C : \mathcal{P}(S) \to \mathcal{P}(S)$ as follows:

$$C(X) := \bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\}$$

if the intersection on the right-hand side is in $\mathcal{F}$, otherwise $C(X)$ is not defined.

This partial mapping C is defined in the same way as in proof of Theorem 1.19, so we know that it is partial closure operator. Now we show that C is sharp, i.e., that also Pc_7 holds. Let $B \subseteq S$ and assume that

$$\bigcap \{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C.$$

Then, by the definition of C, this partial operator fulfills Pc_7 and the range of C is $\mathcal{F}$. It remains to show that the SPCO defined in this way is the unique partial mapping with the range $\mathcal{F}$ satisfying properties Pc_1–Pc_4 and Pc_7. Assume that there exists another partial mapping $K : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfying mentioned conditions and that the range of K is also $\mathcal{F}$. Since $\mathcal{F}_C = \mathcal{F}_K = \mathcal{F}$, by Pc_7 and Remarks 2.1 and 2.2 we get that $C(X)$ is defined, and then equals $\bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}$, if and only if this set is in $\mathcal{F}$, and the same holds for $K(X)$. Therefore $C(X)$ is defined if and only if $K(X)$ is defined, and in that case $C(X) = K(X)$, which was to be proved. ■

Example 4. Let C_s be a partial mapping defined on $\{a,b,c\}$ with

$$C_s : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\ \{a\} & \{b\} & \{a,b,c\} & \{a,b,c\} & \{a,b,c\} & \{a,b,c\} & \{a,b,c\} \end{pmatrix}.$$

This partial mapping is an SPCO on the set $\{a,b,c\}$. Note that the range $\mathcal{F}_{C_s}$ here is equal to the range $\mathcal{F}_C$ of the partial closure operator from Example 2. This implies that there is no 1-1 correspondence between partial closure operators and partial closure systems. However, as proven in Theorem 2.3, there is a bijective correspondence between SPCO's and partial closure systems. $\square$

By the above, it is clear that for a given partial closure system $\mathcal{F}$ on S, there is a collection of partial closure operators on S whose range is $\mathcal{F}$, among which, by Theorem 2.3, precisely one is sharp. In addition, the latter is maximal in the following sense.

Proposition 2.4. *Let $\mathcal{F}$ be a partial closure system on S. The sharp partial closure operator has the greatest exact domain among all partial closure operators whose range is $\mathcal{F}$. In addition, if D is a partial closure operator and C the sharp closure operator with the same domain, then $C(A) = D(A)$, for all $A \subseteq S$ for which D is defined.*

Proof. Let D be an arbitrary partial closure operator whose range is $\mathcal{F}$, and let C be the sharp one with the same range $\mathcal{F}$. Now, if $A \subseteq S$ and $D(A)$ is defined, i.e., $A \in Dom(D)$, then $C(D(A)) = D(A)$, since the ranges of C and D coincide by assumption.

We have that

$$D(A) = \bigcap\{X \in \mathcal{F} \mid A \subseteq X\} \in \mathcal{F}.$$

By the Pc_7, it directly follows that $C(A)$ is defined and $C(A) = \bigcap\{X \in \mathcal{F} \mid A \subseteq X\}$. Hence, $C(A) = D(A)$. $\blacksquare$

The sharp partial closure operator is a natural generalization of the closure operator, as follows.

Theorem 2.5. *If the range $\mathcal{F}$ of a sharp partial closure operator C on a set S forms a complete lattice with respect to set inclusion, then C is a function. Conversely, if C is a closure operator on S, then it is sharp.*

Proof. Let $X \subseteq S$. We have $X \subseteq \bigcup\{C(\{x\}) \mid x \in X\}$, and since the range $\mathcal{F}$ is a complete lattice, the supremum of the collection $\{C(\{x\}) \mid$

$x \in X\}$ exists and contains its union, which implies that $\bigvee\{C(\{x\}) \mid x \in X\} \in \mathcal{F}$. If $X \subseteq Y$ for a set Y such that $Y \subseteq \mathcal{F}$, then $\bigvee\{C(\{x\}) \mid x \in X\} \subseteq Y$. Indeed, for every $x \in X$, $C(\{x\}) \subseteq Y$. Hence, $\bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\} = \bigvee\{C(\{x\}) \mid x \in X\}$. By Pc_7 we have that $C(X)$ is defined, so C is a function and $C(X) = \bigvee\{C(\{x\}) \mid x \in X\}$.

Suppose now that C is a closure operator. Then its range forms a complete lattice with respect to a set inclusion (this is consequence of Corollary 1.7). Let $B \subseteq S$. The closure $C(B)$ is defined because C is a function, and it satisfies Pc_7 by Theorem 1.11. ∎

As shown in paper [53], a completion of a partial closure system to a closure system is equivalent to Dedekind MacNeille completion. Here we present a completion of any nonempty collection of subsets of S to a partial closure system. Clearly, by adding all singletons of S, we get a partial closure system, but then the existing centralized intersections may not be preserved. Therefore, we introduce another completion, as follows.

For an arbitrary nonempty collection $\mathcal{F}$ of subsets of a set S, we define an extension $\widehat{\mathcal{F}} \subseteq \mathcal{P}(S)$ as follows:

$$\widehat{\mathcal{F}} := \mathcal{F} \cup \{\bigcap_{x \in Y} Y \in \mathcal{F} \mid x \in S\}.$$

Example 5. Let
$S = \{a, b, c, d, e, f, g\}$ and
$\mathcal{F} = \{\{b\}, \{c\}, \{e\}, \{a, b, c\}, \{b, c, d, e, f\}, \{e, f, g\}\}$.
Then $\widehat{\mathcal{F}} = \mathcal{F} \cup \{\{e, f\}\}$. □

The following is a straightforward consequence of the definition of $\widehat{\mathcal{F}}$.

Proposition 2.6. *For an arbitrary nonempty collection $\mathcal{F}$ of a set S, the extension $\widehat{\mathcal{F}}$ is a partial closure system on S which preserves all intersections and centralized intersections existing in $\mathcal{F}$.*

Recall that the collection of all principal ideals of a complete lattice L is a closure system which is, when ordered by inclusion, order isomorphic with L under the mapping $i(x) = \downarrow x$, $x \in L$. In addition, this closure system consists of closed sets under the corresponding closure operator.

However, it is clear that not every closure system is a collection of all principal ideals of a complete lattice L.

The analogous statement is true for posets and related partial closure operators and partial closure systems.

In the following we introduce a special type of partial closure systems which are isomorphic to collections of all principal ideals in posets.

We say that a partial closure system $\mathcal{F}$ on a nonempty set S is **principal** if

Ps_5: $\emptyset \notin \mathcal{F}$ and for every $X \in \mathcal{F}$ we have

$$\left| X \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X\} \right| = 1. \tag{2.2}$$

Our main motivation for the above definition, as already mentioned, are principal ideals in a poset.

Proposition 2.7. *Let $(S, \leqslant)$ be a poset. Then the family $\{{\downarrow}x \mid x \in S\}$ of principal ideals is a principal partial closure system.*

Proof. It is easy to see that $\mathcal{F} = \{{\downarrow}x \mid x \in S\}$ is a partial closure system and that $\emptyset \notin \mathcal{F}$. Let us show that for every ${\downarrow}x \in \mathcal{F}$ we have $|{\downarrow}x \setminus \bigcup\{{\downarrow}y \in \mathcal{F} \mid {\downarrow}y \subsetneq {\downarrow}x\}| = 1$.

Obviously, $x \in {\downarrow}x \setminus \bigcup\{{\downarrow}y \in \mathcal{F} \mid {\downarrow}y \subsetneq {\downarrow}x\}$. Suppose that there is element an $z \neq x$ such that $z \in {\downarrow}x \setminus \bigcup\{{\downarrow}y \in \mathcal{F} \mid {\downarrow}y \subsetneq {\downarrow}x\}$. It follows that $z < x$, therefore ${\downarrow}z \in \{{\downarrow}y \in \mathcal{F} \mid {\downarrow}y \subsetneq {\downarrow}x\}$, which is a contradiction with $z \notin \bigcup\{{\downarrow}y \in \mathcal{F} \mid {\downarrow}y \subsetneq {\downarrow}x\}$. ■

Let $\mathcal{F}$ be a principal partial closure system on a set S. In order to prove the opposite connection of principal partial closure systems and principal ideals in a poset, we introduce a mapping:

$G : \mathcal{F} \to S$ defined by

$$G(X) = x, \text{ where } x \in X \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X\}. \tag{2.3}$$

The mapping is well defined by the definition of the principal partial closure system.

Proposition 2.8. *If $\mathcal{F}$ is a principal partial closure system on a set S then the mapping $G : \mathcal{F} \to S$ defined by (2.3) is a bijection.*

Proof. First, let $X_1, X_2 \in \mathcal{F}$ such that $G(X_1) = G(X_2)$. Therefore, there exists $x \in S$ such that $\{x\} = X_1 \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_1\} = X_2 \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_2\}$. Since $\mathcal{F}$ is a partial closure operator, a set $T = \bigcap\{Z \in \mathcal{F} \mid x \in Z\}$ is in $\mathcal{F}$. Hence, $T \subseteq X_1 \cap X_2$. Since $x \in T$, we have that $T \notin \{Y \in \mathcal{F} \mid Y \subsetneq X_1\}$. By $T \subseteq X_1 \cap X_2 \subseteq X_1$, it follows that $T = X_1$. Similarly, we have $T = X_2$ and then $X_1 = X_2$, which implies that the mapping G is injective.

Now, let $x \in S$ and denote $X_x = \bigcap\{X \in \mathcal{F} \mid x \in X\}$. Since $\mathcal{F}$ is a partial closure system, we have $X_x \in \mathcal{F}$, and we shall show that

$G(X_x) = x$. We have $x \in X_x$ and $x \notin \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}$ because X_x is the smallest set (with respect to set inclusion) in $\mathcal{F}$ that contains x. Since $|X_x \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}| = 1$, it follows that $\{x\} = X_x \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}$. Hence G is also a surjective mapping. ■

Using the introduced bijection G, an order on S can be naturally induced by the set inclusion in a principal partial closure system $\mathcal{F}$ on S, as follows: for all $x, y \in S$,

$$x \leqslant y \text{ if and only if } G^{-1}(x) \subseteq G^{-1}(y). \tag{2.4}$$

It is straightforward to check that $\leqslant$ is an order on S. Therefore, as a consequence of Proposition 2.8, we get the following.

Corollary 2.9. *Let $\mathcal{F}$ be a principal partial closure system on a set S, and $\leqslant$ the order on S, defined by* (2.4)*. Then, the function G defined by* (2.3) *is an order isomorphism from $(\mathcal{F}, \subseteq)$ to $(S, \leqslant)$. In addition, the collection of principal ideals in $(S, \leqslant)$ is $\mathcal{F}$.*

Proof. The function G is a bijection by Proposition 2.8, which is, by the definition of $\leqslant$ on S, compatible with the corresponding orders. In other words, if $X, Y \in \mathcal{F}$, we have that $X \subseteq Y$ if and only if $G(X) \leqslant G(Y)$. To prove that subsets in $\mathcal{F}$ are principal ideals, for $x \in S$, we use the denotation from Proposition 2.8, $G^{-1}(x) = X_x$. We will prove that $\downarrow x = X_x$. If $y \leqslant x$, then $G^{-1}(y) \subseteq G^{-1}(x)$ and since $y \in G^{-1}(y)$, we have that $y \in G^{-1}(x)$. On the other hand, suppose that $y \in X_x$. Then, $X_y \subseteq X_x$ and hence, $y \leqslant x$. Since G is a bijection, all the elements from $\mathcal{F}$ are in the form X_x for $x \in X$, so all of them coincides with the principal ideals of $(S, \leqslant)$. ■

We can also start from a poset, and via principal ideals we get a partial closure system, which induces the starting order, as follows.

Corollary 2.10. *Let $(S, \leqslant)$ be a poset and $\mathcal{F}$ a partial closure system consisting of its principal ideals. Then, the order on S defined by* (2.4) *coincides with $\leqslant$.*

Proof. By Proposition 2.7, principal ideals make a principal partial closure system. The function G defined by (2.3) associates to every principal ideal its generator, and by (2.4), inclusion among principal ideals induces the existing order $\leqslant$ from the poset. ■

Finally, we introduce a partial closure operator which corresponds to a principal partial closure system.

A partial closure operator C on S is **principal** if it satisfies

Pc_8: If $X = C(X)$, then there exists unique $x \in X$ such that
$x \notin \bigcup\{Y \in \mathcal{F}_C \mid Y \subsetneq X\}$.

It is easy to see that the axioms Pc_7 and Pc_8 are independent.

A connection among these notions can be explained as follows.

The range of a principal partial closure operator is a principal partial closure system and the sharp partial closure operator obtained from a principal partial closure system, as defined in Theorem 2.3, is principal.

Obviously, the empty set can not be closed under a principal partial closure operator. As an additional property, we prove that the range of a principal partial closure operator consists of closures of singletons.

Proposition 2.11. *Let C be a principal partial closure operator on S. If $X \in \mathcal{F}_C$, then there exists $x \in X$ such that $C(\{x\}) = X$.*

Proof. If X is a closed set, then by Pc_8 there exists a unique $x \in X$ such that $x \notin \bigcup\{Y \in \mathcal{F}_C \mid Y \subsetneq X\}$. From $x \in C(\{x\}) \subseteq C(X) = X$ it follows that $C(\{x\}) = X$. ■

The following is a *Representation theorem* of posets by SPCO's and by the corresponding partial closure systems.

Theorem 2.12. *Let $(S, \leqslant)$ be a poset. The partial mapping C on $\mathcal{P}(S)$ defined by*

$$C(X) = {\downarrow}(\bigvee X), \text{ if there exists } \bigvee X,$$

otherwise not defined, is a principal SPCO. The corresponding partial closure system is principal and it is isomorphic with S.

Proof. It is straightforward to check that C is a partial closure operator. In order to prove that it is sharp, suppose that $B \subseteq S$ and that

$$\bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C.$$

Then, there is a set $Z \subseteq S$, such that $\bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} = {\downarrow}(\bigvee Z)$. Consequently, for every $b \in B$, $b \leq \bigvee Z$. Suppose there is another upper bound of B, say x. Then $B \subseteq {\downarrow}x$ and $C({\downarrow}x) = {\downarrow}x$. Hence, ${\downarrow}(\bigvee Z) \subseteq {\downarrow}x$ and $\bigvee Z \leqslant x$. Therefore, $C(B) = {\downarrow}(\bigvee Z) = C(Z)$.

It is easy to see that C is principal by the definition.

Closed elements are principal ideals of S, hence the corresponding partial closure system is isomorphic with S. ■

2.2 Partial closure domains

So far we have analyzed a lot ranges of (partial) closure operators, but not much has been said about exact domains of partial closure operators (that is, collections of sets whose closure is defined). In this section, given a set S, we aim to characterize subsets of $\mathcal{P}(S)$ that can be exact domains of partial closure operators on S. It turns out that this question is fairly trivial when no limit is imposed on a type of considered partial closure operators, but if we consider only sharp partial closure operators, there is a nontrivial and (arguably) quite elegant characterization.

For a collection $\mathcal{B}$ of subsets of a nonempty set S, consider the following condition:

B_1: for every $x \in S$ we have $\{x\} \in \mathcal{B}$.

Theorem 2.13. *For a partial closure operator C on a set S, the collection $Dom(C)$ fulfills the condition B_1.*

Conversely, if $\mathcal{B}$ is any collection of subsets of S that fulfills the condition B_1, then there exists a partial closure operator C on S such that $Dom(C) = \mathcal{B}$.

Proof. If C is a partial closure operator, then B_1 holds for $Dom(C)$ directly by Pc_4.

Conversely, if $\mathcal{B}$ is a collection of subsets of S that fulfills the condition B_1, we may define $C : \mathcal{P}(S) \to \mathcal{P}(S)$ by $C(X) = X$ whenever $X \in \mathcal{B}$, and $C(X)$ is undefined otherwise. It is clear that C is indeed a partial closure operator on S. ■

Let us now consider the following condition:

B_2: for every $X \subseteq S$, we have:

$$\text{if } \bigcap\{B \in \mathcal{B} \mid X \subseteq B\} \in \mathcal{B}, \text{ then } X \in \mathcal{B}.$$

It turns out that condition B_1 and B_2 together precisely characterize exact domains of sharp partial closure operators. In other words, we have the following theorem.

Theorem 2.14. *For a sharp partial closure operator C on a set S, the collection $Dom(C)$ fulfills the conditions B_1 and B_2.*

Conversely, if $\mathcal{B}$ is any collection of subsets of S that fulfills the conditions B_1 and B_2, then there exists a sharp partial closure operator C on S such that $Dom(C) = \mathcal{B}$.

Proof. Let C be a sharp partial closure operator on S and let $\mathcal{B} = Dom(C)$. We need to show that $\mathcal{B}$ satisfies B_1 and B_2.

B_1: Since $C(\{x\})$ is defined for every $x \in S$, the condition B_1 holds.

B_2: Let $X \subseteq S$ and $\bigcap\{B \in \mathcal{B} \mid X \subseteq B\} \in \mathcal{B}$. To show that $X \in \mathcal{B}$, we need to show that $C(X)$ is defined.

Denote
$$M = \bigcap\{B \in \mathcal{B} \mid X \subseteq B\}.$$

By the assumption, we have $M \in \mathcal{B}$, and thus $C(M)$ is defined. We shall show that

$$C(M) = \bigcap\{F \in \mathcal{F}_C \mid X \subseteq F\}.$$

$(\subseteq)$: If $X \subseteq F \in \mathcal{F}_C$, then $F \in \mathcal{B}$, so $M \subseteq F$ and hence $C(M) \subseteq C(F) = F$. Therefore, $C(M) \subseteq \bigcap\{F \in \mathcal{F}_C \mid X \subseteq F\}$.

$(\supseteq)$: Since $X \subseteq M \subseteq C(M)$ and $C(M) \in \mathcal{F}_C$, we have $C(M) \in \{F \in \mathcal{F}_C \mid X \subseteq F\}$. This immediately gives $C(M) \supseteq \bigcap\{F \in \mathcal{F}_C \mid X \subseteq F\}$.

The shown equality gives $\bigcap\{F \in \mathcal{F}_C \mid X \subseteq F\} \in \mathcal{F}_C$. Now, since C is sharp, by Pc_7 we get that $C(X)$ is defined and $C(X) = \bigcap\{F \in \mathcal{F}_C \mid X \subseteq F\}$; therefore, $X \in \mathcal{B}$.

We have thus shown that $Dom(C)$ indeed satisfies both B_1 and B_2.

Conversely, if $\mathcal{B}$ is a collection of subsets of S that fulfills the conditions B_1 and B_2, we may define $C : \mathcal{P}(S) \to \mathcal{P}(S)$ by $C(X) = X$ whenever $X \in \mathcal{B}$, and $C(X)$ is undefined otherwise. It is clear that C is indeed a partial closure operator on S. We need to show that it is sharp. Assume that $X \subseteq S$ and $\bigcap\{F \in \mathcal{F}_C \mid X \subseteq F\} \in \mathcal{F}_C$. Since, by the definition of C, we have $\mathcal{F}_C = \mathcal{B}$, the previous relation is equivalent to $\bigcap\{F \in \mathcal{B} \mid X \subseteq F\} \in \mathcal{B}$. By B_2 we get $X \in \mathcal{B}$, which implies that $C(X)$ is defined and in fact $C(X) = \bigcap\{F \in \mathcal{F}_C \mid X \subseteq F\}$. This completes the proof. ■

2.3 Summary

To sum up, we have bijective correspondences among:

- posets

- principal sharp partial closure operators
- principal partial closure systems.

Indeed, correspondences are witnessed by Theorem 2.12; they are bijective by Theorem 2.3, Propositions 2.7, 2.8 and Corollaries 2.9, 2.10.

In particular, if we deal with posets which are complete lattices, then the bijective correspondence already exists among closure systems and closure operators. As mentioned, every closure operator fulfils the sharpness property. Still, to every lattice there correspond more closure operators and systems. If the closure operators and systems are principal, then we get bijective correspondences as for posets.

CHAPTER

3

P-MATROIDS, GEOMETRIC AND SEMIMODULAR POSETS

We generalize the notion of matroids to the notion of partial matroids, by replacing the closure operator from the definition of matroid by partial closure operators in the natural way. We also generalize the notion of geometric lattices to the notion of geometric posets. Then we show that, as with matroids and geometric lattices, there is a correspondence of the same kind between partial matroids and geometric posets. At the end, we generalize the notion of semimodular lattices to the notion of semimodular posets in such a way that, as in the case of lattices, a poset is geometric if and only if it is atomistic and semimodular. Sections 3.1 and 3.2 have an introductory character, and the next three sections present original work, mostly included in the paper [51].

3.1 Geometric lattices

A lattice L is **(upper) semimodular** if, for all $x, y \in L$,

$$x \wedge y \prec x \text{ implies } y \prec x \vee y. \tag{3.1}$$

A lattice L is **lower semimodular** if, for all $x, y \in L$,

$$y \prec x \vee y \text{ implies } x \wedge y \prec x. \tag{3.2}$$

Theorem 3.1. *A lattice L is semimodular if and only if for all $x, y, z \in L$ we have:*

$$x \prec y \text{ implies } x \vee z \preccurlyeq y \vee z. \tag{3.3}$$

Proof. ($\Rightarrow$): Let (3.1) be true and let $x \prec y$, where $x, y \in L$. If for arbitrary $z \in L$ we have $z \leqslant x$ or $y \leqslant x \vee z$, then it is obvious that $x \vee z \preccurlyeq y \vee z$. Now assume that $z \nleqslant x$ and $y \nleqslant x \vee z$. We have $x \leqslant (x \vee z) \wedge y < y$, and since y covers x, we have $(x \vee z) \wedge y = x$. Hence $(x \vee z) \wedge y \prec y$. Now it follows from (3.1) that $x \vee z \prec x \vee z \vee y = y \vee z$.

($\Leftarrow$): Now let (3.3) be true and $x \wedge y \prec x$. Then $(x \wedge y) \vee y \preccurlyeq x \vee y$, that is, $y \preccurlyeq x \vee y$. If we have $y = x \vee y$, then $x \leqslant y$, but that is in contradiction with $x \wedge y \prec x$. Therefore, we have $y \prec x \vee y$, so condition (3.1) holds. ■

Theorem 3.2. *Let a lattice L be such that all its chains between two arbitrary elements are finite. Then L is semimodular if and only if:*

$$x \wedge y \prec x \text{ and } x \wedge y \prec y \text{ imply } x \prec x \vee y \text{ and } y \prec x \vee y. \tag{3.4}$$

Proof. It is obvious that (3.1) implies (3.4). Therefore, we assume that (3.4) holds and let $x \wedge y \prec x$. We observe a maximal chain between $x \wedge y$ and y: $x \wedge y = z_0 \prec z_1 \cdots \prec z_n = y$. Since $x \wedge y = x \wedge z_1$, we have $x \wedge z_1 \prec x$ and $x \wedge z_1 \prec z_1$, hence by (3.4) we have $x \prec x \vee z_1$ and $z_1 \prec x \vee z_1$. In the same way we have that for all $i \in \{1, 2, \ldots, n\}$ holds $z_i \prec x \vee z_i$. Note here that $z_{i+1} \neq x \vee z_i$, as otherwise we would have $x \leqslant x \vee z_i = z_{i+1} \leqslant z_n = y$, a contradiction with $x \wedge y < x$. So from $z_i \prec z_{i+1}$ we have $(x \vee z_i) \wedge z_{i+1} \prec x \vee z_i$ and $(x \vee z_i) \wedge z_{i+1} \prec z_{i+1}$. By (3.4) and $(x \vee z_i) \vee z_{i+1} = x \vee z_{i+1}$, we get that $x \vee z_i \prec x \vee z_{i+1}$ and $z_{i+1} \prec x \vee z_{i+1}$. In particular, for $z_n = y$ we have $y \prec x \vee y$, hence semimodularity holds. ■

The **length** of a finite chain is the number of elements in chain minus 1. The length of a poset (and thus also a lattice) is the length of its largest chain, if such a number exists, and then we say that the considered poset is of finite length.

The **height** of an element x of a poset is the length of the longest descending chain (if it exists) which starts at x; we denote it by $h(x)$.

Theorem 3.3. *Given a semimodular lattice L, if all maximal chains between two elements are finite, then all of them have the same length.*

Proof. By induction on the greatest length n of maximal chains between two arbitrary elements x and y of a lattice L we prove that all the maximal chains between these two elements are of the same length.

Let $n = 1$. Then $x \prec y$ and there exists only one chain between x i y, so the statement holds.

Assume that for all $m < n$ we have: if m is the greatest length of a maximal chain between arbitrary x and y, then all maximal chains between x and y are of length m. Now, let the length of one maximal chain be equal to n. Denote that chain by: $x = x_1 \prec x_2 \prec \cdots \prec x_{n+1} = y$.

Now, let $x = y_1 \prec y_2 \prec \cdots \prec y_{m+1} = y$ be another chain between x and y, of length m. If $x_2 = y_2$, then one maximal chain between x_2 and y has length $n-1$, so by the inductive hypothesis all maximal chains between x_2 and y have the same length, so $n = m$. Let now $x_2 \neq y_2$. In this case, we observe the following maximal chain between $x_2 \vee y_2$ and y: $x_2 \vee y_2 = z_1 \prec z_2 \prec \cdots \prec z_{k+1} = y$. Since $x = x_2 \wedge y_2$, $x \prec x_2$ and $x \prec y_2$, by (3.4) we have $x_2 \prec x_2 \vee y_2$ and $y_2 \prec x_2 \vee y_2$. Therefore, $x_2 \prec z_1 \prec z_2 \prec \cdots \prec z_{k+1} = y$ is a maximal chain between x_2 and y of length $k+1$, while $x_2 \prec x_3 \prec \cdots \prec x_{n+1} = y$ is a maximal chain of length $n-1$ between the same elements, hence $k+1 = n-1$. In the same way we get $k+1 = m-1$, so $n = m$. ■

From the theorem above we directly have the following corollary.

Corollary 3.4. *If in a semimodular lattice L there exists a finite maximal chain, then every maximal chain in the lattice L has the same length and any such lattice contains the least element and the greatest element.*

We now give one more characterization of semimodular lattices of finite length by heights of elements.

Theorem 3.5. *A lattice L of finite length is semimodular if and only if for all $x, y \in L$ we have*

$$h(x \wedge y) + h(x \vee y) \leqslant h(x) + h(y). \tag{3.5}$$

Proof. ($\Rightarrow$): If L is a semimodular lattice of finite length, then by Theorem 3.2 the length of a maximal chain between $x \wedge y$ and x: $x \wedge y = x_1 \prec x_2 \prec \cdots \prec x_n = x$, equals $h(x) - h(x \wedge y)$ (since there exists the least element, from which each of these maximal chains begins). Now by (3.3) we have $y = (x \wedge y) \vee y = x_1 \vee y \preccurlyeq x_2 \vee y \preccurlyeq \cdots \preccurlyeq x_n \vee y = x \vee y$, so different elements in this chain constitute a chain of maximal length between y and $x \vee y$, hence $h(x \vee y) - h(y) \leqslant h(x) - h(x \wedge y)$, which is (3.5).

($\Leftarrow$): Let (3.5) be true and let $x \wedge y \prec x$. We have $h(x) = h(x \wedge y) + 1$, which implies $h(x \wedge y) + h(x \vee y) \leqslant h(x \wedge y) + 1 + h(y)$, so $h(x \vee y) \leqslant h(y) + 1$. From $h(x \vee y) \geqslant h(y)$ it follows that $h(x \vee y)$ is equal to either $h(y)$ or $h(y) + 1$. The former is not possible, because it implies $x \wedge y = x$. Therefore, the latter is true, which implies $y \prec x \vee y$. ■

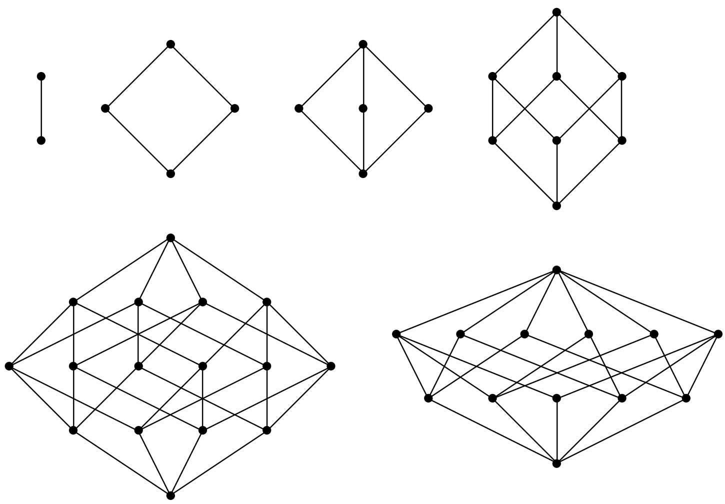

Figure 3

The elements of lattice that cover the least element are called **atoms**. A lattice is **atomistic** if every element different from the least one is supremum of a set of atoms.

A lattice that is semimodular, atomistic and which has only finite chains is called **geometric**.

It is clear that every geometric lattice is of finite length (by Corollary 3.4).

In Figure 3, the first row shows diagrams of all geometric lattices with at most 3 atoms. In the second row we show an example of a geometric lattice with 4 atoms and an example with 5 atoms.

Theorem 3.6. *Let L be a semimodular lattice of finite length. Then, if $a \in L$ is an atom and $x \in L$ is an arbitrary element, then either $a \leqslant x$ or $x \prec x \vee a$.*

Proof. Let a be an atom of a lattice L and $x \in L$. Assume that $a \nleqslant x$. Then we have $x \wedge a = 0$, so by (3.5) it follows $h(x \vee a) \leqslant h(x) + h(a) = h(x) + 1$. From $a \nleqslant x$ also follows $h(x) \neq h(x \vee a)$, so we have $h(x \vee a) = h(x) + 1$, that is, $x \prec x \vee a$. ■

This theorem helps us to deduce two more characterizations of geometric lattices of finite length, which we give hereby.

Theorem 3.7. *A lattice L of finite length is geometric if and only if for all $x, y \in L$ we have*

$$x \prec y \text{ if and only if there exists an atom } a \text{ such that } a \nleqslant x \text{ and } y = x \vee a. \tag{3.6}$$

Proof. ($\Rightarrow$): Let L be a geometric lattice and let $x, y \in L$ be such that $x \prec y$. Since L is atomistic and $x < y$, there exists an atom a which is below y but not below x. Then we have $x < x \vee a \leqslant y$, and since y covers x, we conclude $x \vee a = y$. On the other hand, if a is an atom which is not below x and such that $y = x \vee a$, then $x \wedge a = 0$, that is, $x \wedge a \prec a$, therefore by semimodularity we have $x \prec x \vee a = y$.

($\Leftarrow$): Assume that L is a lattice of finite length such that (3.6). We prove that the condition (3.3) holds, which is, by Theorem 3.1, equivalent with semimodularity. Let $x \prec y$. Then by (3.6) there exists an atom a for which we have $x \wedge a = 0$ and $x \vee a = y$. Now, if for any element $z \in L$ we have $a \leqslant x \vee z$, then $y \vee z = a \vee x \vee z = x \vee z$. If $a \nleqslant x \vee z$, then, since $x \vee z \vee a = y \vee z$, by (3.6) we have $x \vee z \prec y \vee z$.

It remains to show that L is atomistic. Let x be any element of the lattice L different from the least one. We prove that x is a supremum of atoms, using induction on height $h(x) = n$. If $h(x) = 1$, then x is an atom. Assume that every element whose height is smaller than n is a supremum of atoms. Let $h(x) = n$. Then there exists y that is covered by x; therefore, y is a supremum of atoms, and hence by (3.6) it follows that x is a supremum of atoms, too. Therefore, L is a geometric lattice. ■

Theorem 3.8. *A lattice L of finite length is geometric if and only if it is atomistic and for every two atoms a and b and $x \in L$ we have:*

$$\text{from } a < x \vee b \text{ and } a \nleqslant x \text{ it follows } b < x \vee a. \tag{3.7}$$

*(The condition (3.7) is called the **law of exchange for geometric lattices**.)*

Proof. Let L be a geometric lattice. Then, by definition, L is atomistic, so we need to show only that (3.7) holds. Let a and b be two different atoms (the case $a = b$ is trivial) and x an arbitrary element from L such that $a < x \vee b$ and $a \nleqslant x$. Since $a \nleqslant x$, by Theorem 3.7 we have $x \prec x \vee a$. The element x can not be the least element, because then we would have $a < b$, which is in contradiction with the fact that a and b are atoms. Therefore, we have two cases: $b \leqslant x$ and $x \prec x \vee b$, again by Theorem 3.7. In the first case we have $b < x \vee a$ immediately. In the second case, since x is covered by $x \vee a$ and $x \vee b$, and since $a < x \vee b$, we have $x \vee a \leqslant x \vee b$. Then it follows $x \vee a = x \vee b$, so $b \leqslant x \vee a$. Finally, the inequality is strict since $b = x \vee a$ would imply that an atom is strictly greater than another atom, which is impossible.

Conversely, let a lattice L be atomistic, of finite length and L fulfills the condition (3.7). We show that it fulfills the condition (3.6), and then by Theorem 3.7 we have that L is geometric. Let $x \prec y$. Since L is atomistic, there exists an atom a below y which is not below x (otherwise x should be equal to y). Then we have $x < x \vee a \leqslant y$, so since y covers x, we have $x \vee a = y$. On the other hand, let a be an atom such that $a \nleqslant x$. Of course, we have $x \leqslant x \vee a$. If $x = x \vee a$, it follows that a is below x; hence $x < x \vee a$. Assume that there exists y such that $x < y < x \vee a$. Since L is atomistic, there exists an atom b such that $b \leqslant y$ and $b \nleqslant x$, so we have $b < x \vee a$, and by (3.7) it follows $a < x \vee b$, hence $x \vee a \leqslant x \vee b$. Now from $x \vee b \leqslant y < x \vee a$ we get a contradiction; therefore, $x \prec x \vee a$. The proof is complete. ■

3.2 Matroids

A set A with a closure operator $\bar{\ } : \mathcal{P}(A) \to \mathcal{P}(A)$, denoted by $M(A)$, is called **matroid** on A if for all $X \subseteq A$ and for all $x, y \in A$ we have

$M_1 :\ x \notin \overline{X}$ and $x \in \overline{X \cup \{y\}}$ imply $y \in \overline{X \cup \{x\}}$; (*exchange axiom*)

M_2 : there exists a finite Y such that $Y \subseteq X$ and $\overline{Y} = \overline{X}$. (*finite basis*)

A matroid $M(A)$ is **simple** (or **combinatorial geometry** or just **geometry**) if:

M_3 : $\overline{\emptyset} = \emptyset$, and for all $x \in A$ we have $\overline{\{x\}} = \{x\}$.

Closed subsets of a matroid are often called **flats** or **subspaces** of $M(A)$. We denote the range of $M(A)$ by $L_M(A)$, and call it the **lattice of flats** of a matroid $M(A)$.

Theorem 3.9. *For every two elements U and V of the lattice of flats $L_M(A)$ of a matroid $M(A)$ we have:*

$$U \prec V \text{ if and only if } V = \overline{U \cup \{v\}} \text{ for some } v \notin U.$$

Proof. If $U \prec V$ and $v \in V \setminus U$, then, since $U < \overline{U \cup \{v\}} \leqslant V$, we have $\overline{U \cup \{v\}} = V$. On the other hand, let $v \notin U$ be such that $V = \overline{U \cup \{v\}}$. Further, let Z be such that $U < Z \leqslant V$. Then there exists an element $z \in Z \setminus U$; hence, $z \in Z \subseteq V = \overline{U \cup \{v\}}$, and therefore by M_1 we have $v \in \overline{U \cup \{z\}} \subseteq Z$. It follows that $V \subseteq Z$, so $V = Z$; hence, $U \prec V$. ■

Theorem 3.10. *The lattice of flats of a simple matriod is geometric.*

Conversely, if L is a geometric lattice and A the set of all atoms of the lattice L, then the map $\bar{\cdot}$, defined on $\mathcal{P}(A)$ by $\overline{X} = \{a \in A \mid a \leqslant \bigvee X\}$, is a closure operator which induces a simple matroid $M(A)$ on A whose lattice of flats $L_M(A)$ is isomorphic to L.

Proof. Let $M(A)$ be a simple matroid and $L_M(A)$ its lattice of flats. Let $X, Y \in L_M(A)$ be such that $X \cap Y \prec X$. Then by Theorem 3.9 there exists $x \in X \setminus Y$ such that $\overline{(X \cap Y) \cup \{x\}} = X$. Also, we have $X \vee Y = \overline{X \cup Y} = \overline{\overline{(X \cap Y) \cup \{x\}} \cup Y} = \overline{\{x\} \cup Y}$, since $x \notin Y$ and $X \cap Y \subseteq Y$. Because $\overline{\{x\} \cup Y}$ covers Y, it follows that $Y \prec X \vee Y$; therefore $L_M(A)$ is semimodular.

The lattice $L_M(A)$ is atomistic. Indeed, since $M(A)$ is a simple matroid, for all $x \in A$ we have $\overline{\{x\}} = \{x\}$, so singletons are atoms, and since for a nonempty $X \in L_M(A)$ we have

$$X = \overline{X} = \bigcup \{\{x\} \mid x \in X\} = \bigcup \left\{\overline{\{x\}} \mid x \in X\right\} = \bigvee \{\{x\} \mid x \in X\},$$

it follows that X is a supremum of atoms.

In this part of the proof, it remains to show that $L_M(A)$ does not have infinite chains. As every infinite chain (clearly) contains a countable increasing or decreasing subchain, we shall distinguish these two cases. We first consider an increasing chain $X_1 \subsetneq X_2 \subsetneq \dots$ in the lattice $L_M(A)$. By M_2 there exists a finite subset Y of the set $\bigcup_{n\in\mathbb{N}} X_n$ such that $\overline{Y} = \overline{\bigcup_{n\in\mathbb{N}} X_n}$. Since $Y \subseteq \bigcup_{n\in\mathbb{N}} X_n$ and Y is finite, there exists an element of the chain, call it X_k, such that $Y \subseteq X_k$. Now we have that $\overline{Y} \subseteq \overline{X_k} \subseteq \overline{\bigcup_{n\in\mathbb{N}} X_n}$, hence $\overline{X_k} = \overline{\bigcup_{n\in\mathbb{N}} X_n}$. Of course, since $\{X_n \mid n \in \mathbb{N}\}$ is a chain of closed sets, every set in this chain that contains X_k has to be equal with X_k, so the chain can not be infinite. Now, let $X_1 \supsetneq X_2 \supsetneq \dots$ be an infinite descending chain. Denote $X = \{x_n \mid n \in \mathbb{N}\}$, where $x_n \in X_n \setminus X_{n+1}$, and $Y_n = \{x_n, x_{n+1}, \dots\}$, for $n \in \mathbb{N}$. Notice that, since $\overline{Y_{n+1}} \subseteq \overline{X_{n+1}} = X_{n+1}$, we have $x_n \notin \overline{Y_{n+1}}$. Let $x_k \in X$. Assume that there exists some i such that $x_k \in \overline{Y_i \setminus \{x_k\}}$, and let i be the largest natural number that fulfills this. (The largest among them exists since the inequality $i < k$ holds, because for all $j \geqslant 1$ we have $x_k \notin X_{k+j} = \overline{X_{k+j}} \supseteq \overline{Y_{k+j}} = \overline{Y_{k+j} \setminus \{x_k\}}$.) By maximality of i we have $x_k \notin \overline{Y_{i+1} \setminus \{x_k\}} = \overline{(Y_i \setminus \{x_k\}) \setminus \{x_i\}}$. Now by the exchange axiom we have $x_i \in \overline{Y_i \setminus \{x_i\}} = \overline{Y_{i+1}} \subseteq X_{i+1}$, which gives a contradiction. Hence, for all x_k we have $x_k \notin \overline{Y_i \setminus \{x_k\}}$ when $i \in \mathbb{N}$. Since $Y_1 = X$, it follows that for $k \in \mathbb{N}$ we have $x_k \notin \overline{X \setminus \{x_k\}}$. On the other hand, by M_2 there exists a finite $Z \subseteq X$ such that $\overline{Z} = \overline{X}$, and by choosing any $x_k \in X \setminus Z$ we get $\overline{X \setminus \{x_k\}} = \overline{X} \ni x_k$, which is a contradiction.

Conversely, let L be a geometric lattice and A the set of all atoms of L. It easy to see that the map defined by $\overline{X} = \{a \in A \mid a \leqslant \bigvee X\}$ is a closure operator. We also have that a set is closed if and only if the considered set is the set of all atoms below some element of the lattice L. First we show that M_1 holds. Let $x \notin \overline{X}$ and $x \in \overline{X \cup \{y\}}$. It follows that $x \nleqslant \bigvee X$ and $x < y \vee \bigvee X$. By Theorem 3.8 we have $y < x \vee \bigvee X$, hence $y \in \overline{X \cup \{x\}}$.

Let K be a closed set, that is, $K = \{a \in A \mid a \leqslant x\}$ for some $x \in L$. Assume that there does not exist a finite subset H of the set K such that $\bigvee H = \bigvee K$. Then we can construct an infinite chain of atoms that belong to K in the following way: $a_1 < a_1 \vee a_2 < a_1 \vee a_2 \vee a_3 < \dots$. This gives a contradiction with the fact that a geometric lattice is of finite length; therefore, we have M_2 also. By the definition of a closure operator C it is easy to see that $\emptyset$ and $\{a\}$ are closed, hence $M(A)$ is a simple matroid.

At last, a map $f : L \to L_M(A)$ defined by $f(x) = \{a \in A \mid a \leqslant x\}$ is bijective; this follows from the fact that L is atomistic (injectivity) and the fact that the set of atoms is closed if and only if the considered set is the set of all atoms below some element of the lattice (surjectivity). Also, we have that $x \leqslant y$ is equivalent to $\downarrow\! x \cap A \subseteq \downarrow\! y \cap A$ and this is

$f(x) \subseteq f(y)$, and therefore L and $L_M(A)$ are isomorphic. ■

The previous theorem does not hold only for simple matroids; it holds for all finite matroids, since for every finite matroid there exists a corresponding simple matroid.

Theorem 3.11. *For every finite matroid there exists a simple matroid such that their lattices of flats are isomorphic.*

Proof. Let $M(A)$ be a finite matroid on a set A with the closure operator $X \mapsto \overline{X}$. Let A_1 be the set of all atoms in the lattice of flats $L_M(A)$. For clarity, we write $\overline{x}$ instead $\overline{\{x\}}$.We define a new closure operator $X \mapsto \widetilde{X}$ on the set A_1 in the following way:

$$\widetilde{X} := \left\{ \overline{a} \in A_1 \mid a \in \overline{\bigcup X} \right\}. \tag{3.8}$$

We claim that A_1 with the closure operator $\widetilde{\cdot}$ is a simple matroid and that the lattices of flats $L_M(A)$ and $L_M(A_1)$ are isomorphic.

During the proof we shall need the following claim: for an arbitrary set $X \subseteq A$, if we denote $Y = \{\overline{a} \in A_1 \mid a \in X\}$, then we have $\overline{\bigcup Y} = \overline{X}$. The inclusion ($\subseteq$) is clear ($a \in X$ implies $\overline{a} \subseteq \overline{X}$, so $\bigcup Y \subseteq \overline{X}$, and we conclude $\overline{\bigcup Y} \subseteq \overline{X}$), so we are left to show the inclusion ($\supseteq$). It is sufficient to show $X \subseteq \overline{\bigcup Y}$, since it implies $\overline{X} \subseteq \overline{\bigcup Y}$. Let $x \in X$. If we have $\overline{x} = \overline{\emptyset}$, the claim is clear. Otherwise, let $\overline{a}$ be any atom such that $\overline{a} \subseteq \overline{x}$. Since $a \notin \overline{\emptyset}$ (otherwise we have $\overline{a} = \overline{\emptyset}$) and $a \in \overline{\{x\}} = \overline{\emptyset \cup \{x\}}$, by M_1 we have $x \in \overline{\emptyset \cup \{a\}} = \overline{\{a\}} = \overline{a}$; hence, $x \in \bigcup Y \subseteq \overline{\bigcup Y}$, which proves the claim.

We prove that $\widetilde{\cdot}$ is a closure operator on A_1. Let $\overline{a} \in X$. Then $a \in \bigcup X \subseteq \overline{\bigcup X}$, so we have $X \subseteq \widetilde{X}$. If $X \subseteq Y$, then $\overline{\bigcup X} \subseteq \overline{\bigcup Y}$ and hence $\widetilde{X} \subseteq \widetilde{Y}$. Since $\widetilde{X} = \left\{ \overline{a} \in A_1 \mid a \in \overline{\bigcup X} \right\}$, by the claim from the previous paragraph we have $\overline{\bigcup \widetilde{X}} = \overline{\overline{\bigcup X}} = \overline{\bigcup X}$ and therefore $\widetilde{\widetilde{X}} = \widetilde{X}$. So the conditions C_1, C_2 and C_3 are fulfilled. Now we prove that M_1, M_2 and M_3 hold.

M_1: Let $\overline{a} \notin \widetilde{X}$ and $\overline{a} \in \widetilde{X \cup \{\overline{b}\}}$. Then $a \notin \overline{\bigcup X}$ and

$$a \in \overline{\bigcup (X \cup \{\overline{b}\})} = \overline{\bigcup X \cup \overline{b}} = \overline{\bigcup X \cup \{b\}},$$

so, since the operator $\overline{\cdot}$, satisfies the condition M_1, we have

$$b \in \overline{\bigcup X \cup \{a\}} = \overline{\bigcup X \cup \overline{a}} = \overline{\bigcup (X \cup \{\overline{a}\})},$$

that is, $\overline{b} \in \widetilde{X \cup \{\overline{a}\}}$. Therefore, the operator $\widetilde{\cdot}$ satisfies M_1.

M_2: The set A_1 is finite, since A is finite. Therefore the property M_2 holds.

M_3: First we prove $\widetilde{\emptyset} = \emptyset$. Notice that $a \in \overline{\bigcup \emptyset} = \overline{\emptyset}$ implies $\overline{a} = \overline{\emptyset}$, but then $\overline{a} \notin A_1$; therefore, $\widetilde{\emptyset} = \emptyset$. Now, since C_1 holds, in order to prove $\{\overline{a}\} = \widetilde{\{\overline{a}\}}$ we need to show that $\widetilde{\{\overline{a}\}} \subseteq \{\overline{a}\}$. If $\overline{b} \in \widetilde{\{\overline{a}\}}$, then $b \in \overline{\bigcup\{\overline{a}\}} = \overline{a}$, and it follows $\overline{b} \subseteq \overline{a}$, hence $\overline{b} = \overline{a}$ (since $\overline{a}$ and $\overline{b}$ are atoms), that is, $\overline{b} \in \{\overline{a}\}$. Therefore, $M(A_1)$ is simple matroid.

It remains to prove that the lattices $L_M(A)$ and $L_M(A_1)$ are isomorphic. We define a map $\varphi : L_M(A) \to L_M(A_1)$ as follows:

$$\varphi(B) := \{\overline{a} \in A_1 \mid a \in B\}.$$

First we show that all sets on the right-hand side are closed with respect to the operator $\widetilde{\cdot}$. By (3.8), it is enough to show that every element $\overline{a} \in A_1$ such that $a \in \overline{\bigcup \varphi(B)}$ has to be in $\varphi(B)$; but, because of the definition of the set $\varphi(B)$, we obviously have $\bigcup \varphi(B) \subseteq \overline{B}$, and then $\overline{\bigcup \varphi(B)} \subseteq \overline{B} = B$, that is, $a \in B$, and $\overline{a} \in \varphi(B)$ follows.

The map φ is injective. Indeed, by the definition of φ and the claim proved above, we have $\overline{\bigcup \varphi(B)} = \overline{B}$. Hence, from the assumption $\varphi(B_1) = \varphi(B_2)$ it follows $B_1 = \overline{B_1} = \overline{B_2} = B_2$, so φ is injective.

Now, we show that φ is surjective. Let $B_1 \in L_M(A_1)$ be arbitrary. We have

$$\varphi\left(\overline{\bigcup B_1}\right) = \left\{\overline{a} \in A_1 \mid a \in \overline{\bigcup B_1}\right\} = \widetilde{B_1} = B_1,$$

and it is obvious that $\overline{\bigcup B_1} \in L_M(A)$, so φ is surjective and therefore bijective.

At last, in order to show that φ is an isomorphism, it is enough to show the equalities $\varphi(B \cap C) = \varphi(B) \cap \varphi(C)$ and $\varphi(\overline{B \cup C}) = \widetilde{\varphi(B) \cup \varphi(C)}$. The first one is easy:

$$\begin{aligned}\varphi(B \cap C) &= \{\overline{a} \in A_1 \mid a \in B \cap C\} \\ &= \{\overline{a} \in A_1 \mid a \in B\} \cap \{\overline{a} \in A_1 \mid a \in C\} = \varphi(B) \cap \varphi(C).\end{aligned}$$

The second one also holds: indeed, notice that

$$\begin{aligned}\overline{\bigcup(\varphi(B) \cup \varphi(C))} &= \overline{\bigcup \varphi(B) \cup \bigcup \varphi(C)} \\ &= \overline{\overline{\bigcup \varphi(B)} \cup \overline{\bigcup \varphi(C)}} = \overline{\overline{B} \cup \overline{C}} = \overline{B \cup C},\end{aligned}$$

and directly by definitions of the operator $\widetilde{\cdot}$ and the map φ it follows

$$\widetilde{\varphi(B) \cup \varphi(C)} = \varphi(\overline{B \cup C}).$$

This completes the proof. ■

3.2.1 Alternative definitions of matroids

Matroids occur in different branches of mathematics and they have a few different but equivalent definitions. They are usually assumed to be finite, so in this section we work with only finite sets.

The first definition is inspired by linearly independent vectors and their properties.

- (I-definition) A matroid $(E, \mathcal{I})$ is a finite set E with a nonempty family $\mathcal{I}$ of subsets of E, called **independent** sets, with the following properties:

  I_1: Every subset of an independent set is independent.

  I_2: If X_1 and X_2 are independent sets, and $|X_1| < |X_2|$ then for some $x \in X_2 \setminus X_1$, the set $X_1 \cup \{x\}$ is independent.

Theorem 3.12. *In the previous definition condition I_2 can be replaced by*

I_2': *If $S \subseteq E$, then the maximal independent subsets of S are all equal in size.*

Proof. Let E be a finite set and $\mathcal{I}$ a nonempty family of subsets of E such that I_1 and I_2 hold. If X_1 and X_2 are maximal independent subsets of a set S, $S \subseteq E$, and if they do not have the same cardinality, say $|X_1| < |X_2|$, then by I_2 there exists $x \in X_2 \setminus X_1$ such that the set $X_1 \cup \{x\}$ is independent. Also we have $|X_1| < |X_1 \cup \{x\}|$, and thus X_1 is not a maximal independent set, a contradiction.

Now assume that $(E, \mathcal{I})$ fulfills I_1 and I_2'. If $X_1, X_2 \in \mathcal{I}$ and $|X_1| < |X_2|$, then $X_1 \cup X_2$ contains a maximal independent subset Y such that $X_1 \subseteq Y$. There exists an element $x \in Y \cap X_2$, since $Y \cap X_2 = \emptyset$ would imply that $|Y| < |X_2|$, which contradicts I_2'. Now we have $X_1 \cup \{x\} \subseteq Y$, and by I_1 the set $X_1 \cup \{x\}$ is independent. ■

In the graph theory, we arrive to the same definition: E is an edge set of a finite (undirected) graph G, and the independent subsets of E

are acyclic sets of edges (we say that a set of edges of a graph is acyclic if the induced subgraph contains no cycles).

We get the next definition of matroid from the previous one, but using the notion of rank. For a set $S \subseteq E$, a **rank** of the set S, denoted by $r(S)$, is a cardinality of a maximal independent subset of the set S; we can define rank this way because of I_2'. (Notice that this definition of rank is the same as the definition of rank in the graph theory: the number of vertices minus the number of connected components. Indeed, this last value is precisely the number of edges of any maximal subforest of a given graph: if the graph is connected, then the claim follows since in each tree the number of edges is precisely the number of vertices minus 1, while if G is not connected, then the claim follows because the same holds in each of its connected components.)

Theorem 3.13. *For a matroid $(E, \mathcal{I})$, the mapping $r : \mathcal{P}(E) \to \mathbb{Z}$ defined as above satisfies:*

R_1*: For every $S \subseteq E$ we have $0 \leqslant r(S) \leqslant |S|$.*

R_2*: If $S \subseteq T$, then $r(S) \leqslant r(T)$.*

R_3*: For all $S, T \subseteq E$ we have $r(S) + r(T) \geqslant r(S \cup T) + r(S \cap T)$.*

Proof. It is clear that the cardinality of a subset of the set S is between 0 and $|S|$, therefore R_1 holds. It is also clear that the property R_2 holds. To show R_3, let $S, T \subseteq E$ and let K be a maximal independent subset of $S \cap T$. Since K is an independent subset of $S \cup T$, there exists a maximal independent set M such that $K \subseteq M \subseteq S \cup T$. It follows that $|M| = r(S \cup T)$, $|K| = r(S \cap T)$, $r(S) \geqslant |M \cap S|$ and $r(T) \geqslant |M \cap T|$. Also $M \cap S \cap T = K$. Indeed, $K \subseteq M \cap S \cap T$, by I_1 and $M \in \mathcal{I}$ the set $M \cap S \cap T$ is independent subset of $S \cap T$, therefore, since K is a maximal independent subset of $S \cap T$, these two sets coincide. Hence we have

$$r(S)+r(T) \geqslant |M \cap S|+|M \cap T| = |M|+|M \cap S \cap T| = r(S \cup T)+r(S \cap T).$$

■

So we have the next definition of a matroid.

- (R-definition) A matroid (E, r) is a finite set E with the function $r : \mathcal{P}(E) \to \mathbb{Z}$ fulfilling the properties R_1, R_2 and R_3.

Another definition is also inspired by the graph theory. Notice that circuits in a graph are minimal sets of edges which are independent (in the sense of axioms I_1 and I_2). A generalization of this leads us to the next definition.

- (S-definition) A matroid $(E, \mathcal{C})$ is a finite set E together with a set $\mathcal{C}$ of nonempty subsets of the set E, which we call **circuits**, and which have the following properties:

  S_1: No circuit is contained in any other circuit.

  S_2: If X_1 and X_2 are two distinct circuits and $x \in X_1 \cap X_2$, then $(X_1 \cup X_2) \setminus \{x\}$ contains some other circuit.

The next definition is inspired by definition of a basis of a vector space in linear algebra.

- (B-definition) A matroid $(E, \mathcal{B})$ is a finite set E with nonempty family of its subsets $\mathcal{B}$, named **bases**, with the following properties:

  B_1: No basis is contained in any other basis.

  B_2: If X_1 i X_2 are different bases, then for every $x \in X_1 \setminus X_2$ there exists some $y \in X_2 \setminus X_1$ such that $(X_1 \setminus \{x\}) \cup \{y\}$ is a basis.

It can be noticed that, since bases are defined on a finite set, all bases have the same cardinality.

Theorem 3.14. *For a matroid (E, r) by R-definition there exists a family $\mathcal{B} \subseteq \mathcal{P}(E)$ that satisfies B_1 and B_2.*

Proof. Let E be a finite set, the function $r : \mathcal{P}(E) \to \mathbb{Z}$ satisfies R_1, R_2 and R_3 and let

$$\mathcal{B} = \{S \subseteq E \mid |S| = r(S) = r(E)\}.$$

If some set X_1 is contained in another set X_2 and $|X_1| = r(E) = |X_2|$, they must be equal. Hence B_1 holds.

Now, let $X_1, X_2 \in \mathcal{B}$ and $x \in X_1 \setminus X_2$. We have $|X_1| = r(X_1) = r(E) = r(X_2) = |X_2|$ and $r(X_1 \setminus \{x\}) \leqslant r(E) - 1$. Moreover, $r(X_1 \setminus \{x\}) = r(E) - 1$, because if $r(X_1 \setminus \{x\}) < r(E) - 1$, then

$$\begin{aligned} r(E) &= r(E) - 1 + 1 > r(X_1 \setminus \{x\}) + r(\{x\}) \\ &\geqslant r((X_1 \setminus \{x\}) \cup \{x\}) + r((X_1 \setminus \{x\}) \cap \{x\}) \\ &= r(X_1) + r(\emptyset) = r(E), \end{aligned}$$

and that is a contradiction.

Denote $X_2 \setminus X_1 = \{y_1, y_2, \dots, y_k\}$ and consider $r(X_1 \setminus \{x\}), r((X_1 \setminus \{x\}) \cup \{y_1\}), r((X_1 \setminus \{x\}) \cup \{y_1, y_2\}), \dots, r((X_1 \setminus \{x\}) \cup \{y_1, y_2, \dots, y_k\})$. This is a nondecreasing sequence of integers whose first element is $r(E)-1$ and the last element is $r(E)$ (since $r(E) \geqslant r((X_1 \setminus \{x\}) \cup \{y_1, y_2, \dots, y_k\}) \geqslant r(X_2) = r(E)$). Therefore, there exists $j \in \{1, 2, \dots, k\}$ such that $r((X_1 \setminus \{x\}) \cup \{y_1, y_2, \dots, y_{j-1}\}) = r(E) - 1$ and $r((X_1 \setminus \{x\}) \cup \{y_1, y_2, \dots, y_j\}) = r(E)$, and thus we have

$$r((X_1 \setminus \{x\}) \cup \{y_1, y_2, \dots, y_{j-1}\}) + r((X_1 \setminus \{x\}) \cup \{y_j\})$$

$$\geqslant r((X_1 \setminus \{x\}) \cup \{y_1, y_2, \dots, y_j\}) + r(X_1 \setminus \{x\}),$$

which is equivalent with

$$r(E) - 1 + r((X_1 \setminus \{x\}) \cup \{y_j\}) \geqslant r(E) + r(E) - 1.$$

This implies $r((X_1 \setminus \{x\}) \cup \{y_j\}) = r(E)$. Of course, $|(X_1 \setminus \{x\}) \cup \{y_j\}| = |X_1| = r(E)$, hence $(X_1 \setminus \{x\}) \cup \{y_j\} \in \mathcal{B}$ and B_2 holds. ■

Theorem 3.15. *For a matroid $(E, \mathcal{B})$ by B-definition there exists a family $\mathcal{I} \subseteq \mathcal{P}(E)$ which satisfies I_1 and I_2'.*

Proof. Let E be a finite set and $\mathcal{B}$ a nonempty family such that B_1 and B_2 hold. We show that $\mathcal{I} = \{I \subseteq B \mid B \in \mathcal{B}\}$ satisfies I_1 and I_2. First, if $X \subseteq Y$ and $Y \in \mathcal{I}$, then clearly $X \in \mathcal{I}$.

Second, let $S \subseteq E$ and let X_1 and X_2 be two maximal independent subsets of S. We need to show that $|X_1| = |X_2|$. Since $X_1, X_2 \in \mathcal{I}$ and they are maximal, it follows that $X_1, X_2 \in \mathcal{B}$. Since all bases have the same cardinality, the proof is completed. ■

For the next theorem, we need to define nullity of a set. For a set $S \subseteq E$, the **nullity** of S, denoted by $n(S)$, equals the difference between the cardinality and the rank of the set S, that is, $n(S) = |S| - r(S)$.

Theorem 3.16. *For a matroid $(E, \mathcal{I})$ (by $\mathcal{I}$-definiton), the nullity function satisfies the following properties:*

N_1: *For every $S \subseteq E$ we have $0 \leqslant n(S) \leqslant |S|$.*

N_2: *If $S \subseteq T$, then $n(S) \leqslant n(T)$.*

N_3: *For all $S, T \subseteq E$ we have $n(S) + n(T) \leqslant n(S \cup T) + n(S \cap T)$.*

Proof. Since Theorem 3.13 holds, the first property is trivial. To prove the second, note that for any $S \subseteq T$ the following holds:

$$\begin{aligned} r(T) &= r(T) + r(\emptyset) = r((T \setminus S) \cup S) + r((T \setminus S) \cap S) \\ &\leqslant r(T \setminus S) + r(S) \leqslant |T \setminus S| + r(S) \\ &= |T| - |S| + r(S). \end{aligned}$$

This implies

$$|S| - r(S) \leqslant |T| - r(T).$$

The last property follows from R_3 and the fact that $|S| + |T| = |S \cup T| + |S \cap T|$. ■

Theorem 3.17. *Let $(E, \mathcal{I})$ be a matroid by I-definition. Then there exists the family $\mathcal{C} \subseteq \mathcal{P}(E)$ that satisfies the conditions S_1 and S_2.*

Proof. For $(E, \mathcal{I})$ we define the set of circuits $\mathcal{C}$ as the set of all minimal subsets of E which are not independent. It is clear that S_1 holds.

Before we prove that S_2 also holds, we will show that $n(X) = 1$ for every $X \in \mathcal{C}$. Every circuit has at least one element since empty set is independent in every matroid. Hence, there exists $x \in X$. The set X is not independent and the set $X \setminus \{x\}$ is independent, therefore $r(X) < |X|$ and $r(X \setminus \{x\}) = |X| - 1$. Now by R_2 we have $|X| - 1 = r(X \setminus \{x\}) \leqslant r(X) < |X|$. This implies that $r(X) = |X| - 1$, that is, $n(X) = 1$.

Now let $X_1, X_2 \in \mathcal{C}$, $X_1 \neq X_2$ and $x \in X_1 \cap X_2$. By S_1, $X_1 \cap X_2$ is a proper subset of both circuits and hence it is independent. Therefore, by N_3 we have

$$n(X_1) + n(X_2) \leqslant n(X_1 \cup X_2) + n(X_1 \cap X_2),$$

and this is equivalent to $1 + 1 \leqslant n(X_1 \cup X_2) + 0$, that is, $n(X_1 \cup X_2) \geqslant 2$. Further, we have

$$\begin{aligned} n(X_1 \cup X_2) &= |X_1 \cup X_2| - r(X_1 \cup X_2) \\ &\leqslant |(X_1 \cup X_2) \setminus \{x\}| + 1 - r((X_1 \cup X_2) \setminus \{x\}) \\ &= n((X_1 \cup X_2) \setminus \{x\}) + 1. \end{aligned}$$

Hence, $n((X_1 \cup X_2) \setminus \{x\}) \geqslant 1$, therefore $(X_1 \cup X_2) \setminus \{x\}$ is not independent and thus it contains a circuit. ■

Theorem 3.18. *Let $(E, \mathcal{C})$ be a matroid by S-definition. Then there exists a family $\mathcal{I} \subseteq \mathcal{P}(E)$ that satisfies the conditions I_1 and I_2.*

Proof. Let $(E, \mathcal{C})$ be a matroid by S-definition. We define

$$\mathcal{I} := \{X \subseteq E \mid (\forall Y \subseteq X)(Y \notin \mathcal{C})\}.$$

By the definition of $\mathcal{I}$, it is easy to see that it satisfies the condition I_1. To prove I_2, let $X_1, X_2 \in \mathcal{I}$ and $|X_1| < |X_2|$. Assume that for all $x \in X_2 \setminus X_1$ we have $X_1 \cup \{x\} \notin \mathcal{I}$. Therefore, there exists a subset of $X_1 \cup X_2$ of greater cardinality than X_1, that is, that belongs to $\mathcal{I}$. We choose such a subset X_3 for which $|X_1 \setminus X_3|$ is minimal. The set $X_1 \setminus X_3$ is nonempty; indeed, the opposite would imply $X_1 \subseteq X_3$, and now because of $|X_1| < |X_3|$ (the cardinality of X_3 is greater than or equal to $|X_2|$) and the assumption we would have $X_3 \notin \mathcal{I}$, which contradicts $X_3 \in \mathcal{I}$. Therefore, we can choose an element $a \in X_1 \setminus X_3$. Now we define $X_b := (X_3 \cup \{a\}) \setminus \{b\}$, for every $b \in X_3 \setminus X_1$. Then we have $X_b \subseteq X_1 \cup X_2$ and $|X_1 \setminus X_b| = |X_1 \setminus ((X_3 \cup \{a\}) \setminus \{b\})| = |X_1 \setminus (X_3 \cup \{a\})| < |X_1 \setminus X_3|$. Hence, $X_b \notin \mathcal{I}$ (because of the minimality of $|X_1 \setminus X_3|$), and therefore there exists $C_b \in \mathcal{C}$ such that $C_b \subseteq X_b$. It is clear that $b \notin C_b$. Moreover, $a \in C_b$, since otherwise $C_b \subseteq X_3$, what contradicts $X_3 \in \mathcal{I}$.

Let now c be an element of $X_3 \setminus X_1$. If $C_c \cap (X_3 \setminus X_1) = \emptyset$, then we have $C_c \subseteq ((X_1 \cap X_3) \cup \{a\}) \setminus \{c\} \subseteq X_1$, contradicting the fact that $X_1 \in \mathcal{I}$. Hence, there is an element $d \in C_c \cap (X_3 \setminus X_1)$. Therefore, $a \in C_c \cap C_d$, and by S_2 we have that $(C_c \cup C_d) \setminus \{a\}$ has a subset $C \in \mathcal{C}$. Since $C_c, C_d \subseteq X_3 \cup \{a\}$, it follows $C \subseteq X_3$; this contradicts $X_3 \in \mathcal{I}$. Therefore, I_3 holds. ■

Let us now see how these definitions are connected with the definition of matroid which we use in other sections. We find it convenient to name it C-definition.

Theorem 3.19. *Let $M(E)$ be a finite matroid by C-definition. Then there exists a family $\mathcal{B} \subseteq \mathcal{P}(E)$ which satisfies B_1 and B_2.*

Proof. Let E be a nonempty finite set with a closure operator $\bar{\ }$ satisfying M_1 and M_2. We define

$$\mathcal{B} := \{B \subseteq E \mid B \text{ is minimal such that } \overline{B} = E\}.$$

The property B_1 holds because of minimality of sets in $\mathcal{B}$. Let $X, Y \in \mathcal{B}$ and $x \in X \setminus Y$. Then we have that X and Y are minimal with the property $\overline{X} = \overline{Y} = E$. We have $Y \setminus \overline{X \setminus \{x\}} \neq \emptyset$; otherwise it follows $Y \subseteq \overline{X \setminus \{x\}}$, therefore $E = \overline{Y} \subseteq \overline{X \setminus \{x\}}$, and this contradicts the minimality of X. Therefore, there exists $y \in Y$ such that $y \notin \overline{X \setminus \{x\}}$. If $y \in X$, then $y \in X \setminus \{x\}$ (since $y = x$ implies $x \in Y$) and hence $y \in \overline{X \setminus \{x\}}$. Since this gives a contradiction, we have $y \in Y \setminus X$. Now, in order to show B_2, we need to prove $(X \setminus \{x\}) \cup \{y\} \in \mathcal{B}$, that is: $(X \setminus \{x\}) \cup \{y\}$ is minimal fulfilling $\overline{(X \setminus \{x\}) \cup \{y\}} = E$.

First, since $y \notin \overline{X \setminus \{x\}}$ and $y \in E = \overline{X} = \overline{(X \setminus \{x\}) \cup \{x\}}$, by M_1 it follows that $x \in \overline{(X \setminus \{x\}) \cup \{y\}}$. Therefore, $X \subseteq \overline{(X \setminus \{x\}) \cup \{y\}}$, so $E = \overline{X} \subseteq \overline{\overline{(X \setminus \{x\}) \cup \{y\}}} = \overline{(X \setminus \{x\}) \cup \{y\}}$, that is, $\overline{(X \setminus \{x\}) \cup \{y\}} = E$.

Second, $X \setminus \{x\} \subsetneq E$ holds because of minimality of X, so let $x' \in X \setminus \{x\}$ and assume that $\overline{(X \setminus \{x, x'\}) \cup \{y\}} = E$. Then we have $x' \in \overline{(X \setminus \{x, x'\}) \cup \{y\}}$. Also, $x' \notin \overline{X \setminus \{x, x'\}}$ (otherwise we have $E = \overline{X} = \overline{(X \setminus \{x, x'\}) \cup \{x, x'\}} \subseteq \overline{\overline{X \setminus \{x, x'\}} \cup \{x, x'\}} = \overline{\overline{X \setminus \{x, x'\}} \cup \{x\}} \subseteq \overline{\overline{X \setminus \{x, x'\} \cup \{x\}}} = \overline{\overline{X \setminus \{x'\}}} = \overline{X \setminus \{x'\}}$; a contradiction with minimality of X). Therefore, by M_1 it follows that $y \in \overline{X \setminus \{x\}}$. Finally,

$$E = \overline{(X \setminus \{x\}) \cup \{y\}} \subseteq \overline{\overline{X \setminus \{x\}} \cup \{y\}} = \overline{\overline{X \setminus \{x\}}} = \overline{X \setminus \{x\}},$$

which again contradicts the minimality of X. This makes the proof complete. ■

Theorem 3.20. *Let $(E, \mathcal{I})$ be a matroid by I-definition. Then for all $X \subseteq E$ and $x \notin X$ we have:*

there exists a maximal independent $X' \subseteq X$ such that $X' \cup \{x\} \notin \mathcal{I}$

if and only if

for all maximal independent $W \subseteq X$ we have $W \cup \{x\} \notin \mathcal{I}$.

Proof. The second direction is trivial, so we prove only the first. Let $X \subseteq E$, $x \notin X$, and let there be a maximal independent $X' \subseteq X$ such that $X' \cup \{x\} \notin \mathcal{I}$. We claim that $W \cup \{x\} \notin \mathcal{I}$ for all maximal independent $W \subseteq X$. By I_2' we have $|W| = |X'|$. If $W \cup \{x\} \in \mathcal{I}$, then we have that $W \cup \{x\}$ and X' are maximal independent subsets of $X' \cup \{x\}$ (X' is maximal independent since $X' \cup \{x\} \notin \mathcal{I}$), but these two sets do not have the same cardinality, hence we have a contradiction with I_2'. ■

The previous Theorem is very useful for proving the following Theorem.

Theorem 3.21. *Let $(E, \mathcal{I})$ be a matroid by I-definition. Then there exists a closure operator $\bar{\cdot}$ on E which satisfies the conditions M_1 and M_2.*

Proof. Let $(E, \mathcal{I})$ be a matroid by I-definition. We define a mapping $\bar{\cdot} : \mathcal{P}(E) \to \mathcal{P}(E)$ by

$$\overline{X} := X \cup \{x \in E \mid \text{there exists a maximal independent } X' \subseteq X \text{ such that } X' \cup \{x\} \notin \mathcal{I}\}.$$

By the definition of the mapping $\bar{\cdot}$ we have that C_1 holds. To prove C_2, let $X \subseteq Y$ and $x \in \overline{X}$, that is $x \in X$ or there exists a maximal independent $X' \subseteq X$ such that $X' \cup \{x\} \notin \mathcal{I}$. We need to show that $x \in \overline{Y}$. If $x \in X$, then $x \in Y \subseteq \overline{Y}$. So assume that $X' \cup \{x\} \notin \mathcal{I}$. Denote by Y' a maximal independent subset of Y such that $X' \subseteq Y'$. If $Y' \cup \{x\} \in \mathcal{I}$, then by I_1 we would have $X' \cup \{x\} \in \mathcal{I}$, therefore $Y' \cup \{x\} \notin \mathcal{I}$, hence $x \in \overline{Y}$.

Now, we prove that $\overline{\overline{X}} \subseteq \overline{X}$. Together with the already proved C_1, this implies C_3. Let $x \in \overline{\overline{X}}$, that is, $x \in \overline{X}$ or $W \cup \{x\} \notin \mathcal{I}$, for all maximal independent subsets $W \subseteq \overline{X}$ (by Lemma 3.20). The first case is trivial, so assume that the second case holds. Let X' be a maximal independent subset of X and let W be a maximal independent subset of $\overline{X}$ such that $X' \subseteq W$. If there exists $w \in W \setminus X'$, then by the maximality of X' we deduce $w \notin X$. Since X' is a maximal independent subset of X and $w \in \overline{X}$, we have $X' \cup \{w\} \notin \mathcal{I}$, but this contradicts I_1 and $X' \cup \{w\} \subseteq W$. Therefore, $W = X'$. Hence, $X' \cup \{x\} = W \cup \{x\} \notin \mathcal{I}$, so $x \in \overline{X}$.

The property M_2 trivially holds since we work with finite sets, so it remains to show that M_1 holds. Let $x \notin \overline{X}$ (that is, $x \notin X$ and for all maximal independent $X' \subseteq X$ we have $X' \cup \{x\} \in \mathcal{I}$) and $x \in \overline{X \cup \{y\}}$ (that is, $x \in X \cup \{y\}$ or $Z \cup \{x\} \notin \mathcal{I}$ for all maximal independent $Z \subseteq X \cup \{y\}$). If $x \in X \cup \{y\}$, then we have $x = y$, so M_1 trivially holds, hence it remains to see if the claim is true for the latter assumption. We need to show that $y \in \overline{X \cup \{x\}}$. If $y \in X \cup \{x\}$, then the necessary property follows by the definition of $\bar{\cdot}$. So we assume that $y \notin X \cup \{x\}$ and show that there exists a maximal independent $W \subseteq X \cup \{x\}$ such that $W \cup \{y\} \notin \mathcal{I}$. Let Y be a maximal independent subset of $X \cup \{y\}$ such that $Y \cup \{x\} \notin \mathcal{I}$ (such Y exists since $x \in \overline{X \cup \{y\}}$, $x \notin X$ and $y \notin X \cup \{x\}$). Because of $x \notin \overline{X}$, we have $y \in Y$. Denote $W = (Y \setminus \{y\}) \cup \{x\}$. First, we have $W \cup \{y\} \notin \mathcal{I}$. Second, $W \in \mathcal{I}$, since $(Y \setminus \{y\}) \cup \{x\} \notin \mathcal{I}$ implies

$x \in \overline{X}$ (because $Y \setminus \{y\}$ is a maximal independent subset of X). And third, since $Y \setminus \{y\}$ is a maximal independent subset of X, we have that $(Y \setminus \{y\}) \cup \{x\}$ is maximal independent subset of $X \cup \{x\}$. This makes the proof complete. ■

All connections between mentioned different definitions of matroids are shown in Figure 4. We can conclude that for finite sets they all result in the same structure.

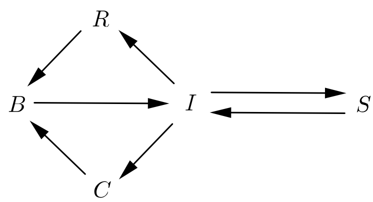

Figure 4.

3.3 Geometric posets

Our definition of a geometric poset is based on the condition from the Theorem 3.8, in fact, it generalizes that condition in a natural way.

We first prove some properties of partial matroids and geometric posets, and then show that, as with matroids and geometric lattices, there is a correspondence of the same kind between partial matroids and geometric posets.

All the sets considered until the end of the chapter are finite.

Let a poset $(P, \leqslant)$ be given. If P has the least element, we say that **atoms** of P are elements that cover the least element (this is the same definition as for atoms in lattices); if P does not have the least element, we define **atoms** of P to be all minimal elements of P. The set of all atoms is denoted by A_P or simply A (if P is clear from the context). We say that a poset is **atomistic** if every element different from the least is the supremum of a set of atoms.

We now define a geometric poset.

We say that a poset $(P, \leqslant)$ is **geometric** if and only if P is atomistic and for every $x \in P$ and atoms a and b we have:

$$\text{if } x \vee b \text{ exists, } a < x \vee b, \text{ and } a \nleqslant x, \text{ then } x \vee a \text{ exists and } b < x \vee a. \tag{3.9}$$

The following theorem gives another, equivalent definition of a geometric poset.

Theorem 3.22. *A poset $(P, \leqslant)$ is geometric if and only if P is atomistic and*

$$\begin{array}{c}\text{for } x, y \in P, \text{ if } x \nleqslant y \text{ and there is } a \in A_P \text{ such that} \\ y \vee a \text{ exists and } x \leqslant y \vee a, \text{ then } x \vee y \text{ exists and } y \prec x \vee y.\end{array} \tag{3.10}$$

In order to prove Theorem 3.22, we need the following lemma, which can be checked straightforwardly.

Lemma 3.23. *Let P be a poset satisfying the property* (3.10)*. Then the following holds:*

if $x \vee a$ exists for some element x and for an atom a such that $a \nleqslant x$, then $x \prec x \vee a$.

Proof of Theorem 3.22. Let $(P, \leqslant)$ be a geometric poset, $x, y \in P$, $a \in A_P$, such that $x \nleqslant y$, $y \vee a$ exists and $x \leqslant y \vee a$. Since P is atomistic, because of $x \nleqslant y$ we get that there exists $b \in A_P$ such that $b \leqslant x$ and $b \nleqslant y$. Further, we have $b < y \vee a$; therefore, by (3.9) it follows that $y \vee b$ exists and $a < y \vee b$, and hence $y \vee a \leqslant y \vee b$. We also have $y \vee b \leqslant y \vee a$ (since $b < y \vee a$), and therefore $y \vee a = y \vee b$. Note that for any upper bound z of x and y we have $y \vee b \leqslant z$ (because of $b \leqslant x \leqslant z$ and $y \leqslant z$); therefore, $y \vee b$ is a supremum of x and y. It remains to show $y \prec y \vee b$. Suppose otherwise: there exists t such that $y < t < y \vee b$. We can choose $c \in A_P$ such that $c \leqslant t$ and $c \nleqslant y$, and now again by (3.9) we get $b < y \vee c$ and hence $y \vee b \leqslant y \vee c \leqslant t$, which is impossible.

Now, let $(P, \leqslant)$ be an atomistic poset which satisfies (3.10). Also, let $x \in P$, $a, b \in A_P$, $a \nleqslant x$, $x \vee b$ exists and $a < x \vee b$. Since (3.10) holds, it follows that $x \vee a$ exists and $x \prec x \vee a$. We need to show that $b < x \vee a$. We know $x \vee a \leqslant x \vee b$. If $b \leqslant x$, then $a < x \vee b = x$, which contradicts $a \nleqslant x$; therefore, we conclude $b \nleqslant x$. By Lemma 3.23 we get $x \prec x \vee b$. Consequently, since $x \vee a$ and $x \vee b$ are comparable and both cover x, we get $x \vee a = x \vee b$. Hence, $b < x \vee a$, which completes the proof. ■

Now, we prove some useful lemmas.

Lemma 3.24. *Let $(P, \leqslant)$ be a poset and $a, b, x \in P$ such that all the suprema $b \vee x$, $x \vee a$ and $(b \vee x) \vee a$ are defined. Then $b \vee (x \vee a)$ is also defined and $(b \vee x) \vee a = b \vee (x \vee a) = \sup\{b, x, a\}$.*

Proof. Let $b \vee x$, $x \vee a$ and $(b \vee x) \vee a$ be defined and let y be an arbitrary upper bound for b and $x \vee a$. Therefore, $b, x, a \leqslant y$, and hence $b \vee x \leqslant y$

and also $(b \vee x) \vee a \leqslant y$. Since $b, x \vee a \leqslant (b \vee x) \vee a$, it follows that $(b \vee x) \vee a$ is the least upper bound for b and $x \vee a$, that is, $(b \vee x) \vee a = b \vee (x \vee a)$. ■

Lemma 3.25. *In every geometric poset, if $x \vee a$ is defined for some element x and atom a such that $a \nleqslant x$, then $x \prec x \vee a$.*

Proof. Let $(P, \leqslant)$ be a poset and A its set of atoms, $x \in P$, $a \in A$, $a \nleqslant x$ and $x \vee a$ is defined. Obviously, $x < x \vee a$. Suppose that there exists y that $x < y < x \vee a$. Since P is atomistic, there exists $b \in A$ such that $b \leqslant y$ and $b \nleqslant x$. Now by (3.9) we have that $x \vee b$ is defined and $a < x \vee b$. Further, $a \vee x \vee b = x \vee b$, and hence $a \vee x \leqslant x \vee b$. On the other hand, $x, b \leqslant y$, and since $x \vee b$ is defined, it follows $x \vee b \leqslant y < a \vee x \leqslant x \vee b$, a contradiction. The proof is complete. ■

3.4 Partial matroids

We define a **partial matroid** (or shorter **p-matroid**) as the pair (E, C), where E is a nonempty set and C a sharp partial closure operator on E, satisfying the following conditions:

(M) if $C(X)$ and $C(X \cup \{x\})$ are defined, then the relations $y \notin C(X)$ and $y \in C(X \cup \{x\})$ imply that $C(X \cup \{y\})$ is defined and $x \in C(X \cup \{y\})$;

(P) $C(\{x\}) = \{x\}$ for every $x \in E$.

The following two theorems show that there is a direct generalization of the correspondence between geometric lattices and matroids to a correspondence between geometric posets and partial matroids.

Theorem 3.26. *The range of a p-matroid with respect to set inclusion is a geometric poset.*

Proof. Let (E, C) be a p-matroid and $\mathcal{F}$ its range. By (P), atoms of the poset $(\mathcal{F}, \subseteq)$ are exactly all singletons $\{x\}$, where $x \in E$. Therefore, $\mathcal{F}$ is atomistic because every closed set X is equal to union of all one-element subsets of X.

Now we show that (3.9) also holds. Let $x, y \in E$ and $X \in \mathcal{F}$ be such that $X \vee \{x\}$ is defined (and therefore $C(X \cup \{x\}) = X \vee \{x\}$), $y \in X \vee \{x\}$ and $y \notin X$. Then by (M) we have $C(X \cup \{y\})$ defined, which implies that $X \vee \{y\}$ is defined and $x \in C(X \cup \{y\})$. ■

For the converse, we need the following lemma.

Lemma 3.27. *Let P be an atomistic poset with a set of atoms A. If $\{p_i, \mid i \in I\} \subseteq P$ and $\bigcap_{i\in I}(\downarrow p_i \cap A) = \downarrow q \cap A$ for some $q \in P$, then $q = \bigwedge\{p_i \mid i \in I\}$.*

Proof. First, q is a lower bound of the family $\{p_i, \mid i \in I\}$, since for every i, $\downarrow q \cap A \subseteq \downarrow p_i \cap A$, hence $q \leqslant p_i$. If $r \leqslant p_i$, then $\downarrow r \cap A \subseteq \bigcap_{i\in I}(\downarrow p_i \cap A)$, and therefore $\downarrow r \cap A \subseteq \downarrow q \cap A$, implying $r \leqslant q$. So, q is the greatest lower bound of the family. ■

Theorem 3.28. *For every geometric poset $(P, \leqslant)$ there exists a p-matroid whose range is isomorphic with $(P, \leqslant)$.*

Proof. Let $(P, \leqslant)$ be a geometric poset and A its set of atoms. We define partial mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ as follows:

$$C(X) := \{a \in A \mid a \leqslant \bigvee X\}, \tag{3.11}$$

if $\bigvee X$ exists, and otherwise $C(X)$ is not defined. It is easy to check that C is a partial closure operator. In addition, it is sharp, we prove that Pc_7 holds. A subset X of P is closed if and only if X is equal to the set of all atoms below some element of P: $X = C(X)$ if and only if $X = \downarrow p \cap A$, for some $p \in P$ (one direction follows from the definition of C and another from the fact that P is atomistic). Therefore, for $B \subseteq A$, suppose $\bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C$, in other words, $\bigcap_{i\in I}\{\downarrow p_i \cap A \mid B \subseteq \downarrow p_i \cap A\} \in \mathcal{F}_C$. By Lemma 3.27,

$$\bigcap_{i\in I}\{\downarrow p_i \cap A \mid B \subseteq \downarrow p_i \cap A\} = \downarrow \bigwedge_{i\in I} p_i \cap A.$$

Obviously, $\bigwedge_{i\in I} p_i$ is the smallest upper bound for B, hence $\bigwedge_{i\in I} p_i = \bigvee B$ and by (3.11) $C(B) \in \mathcal{F}_C$, proving that Pc_7 holds.

Now we show that (M) holds. Let $C(X)$ and $C(X \cup \{x\})$ be defined, $y \notin C(X)$ and $y \in C(X \cup \{x\})$. Hence, $\bigvee X$ and $x \vee \bigvee X$ exist, $y \not\leqslant \bigvee X$ and $y < x \vee \bigvee X$. Then by (3.9) we get that $y \vee \bigvee X$ exists and $x < y \vee \bigvee X$, which gives $x \in C(X \cup \{y\})$. By definition of the partial operator C, it is easy to see that all the sets $\{x\}$ where $x \in A$ are closed, and therefore (A, C) is a p-matroid.

At last, the function $f : P \to \mathcal{F}_C$ defined by $f(x) = \{a \in A \mid a \leqslant x\}$ is a bijection. Indeed, the injectivity follows since P is atomistic, while the surjectivity is implied by the fact that a set of atoms is closed if and only if it is the set of all atoms below an element of the poset P. By the definition, f is compatible with the order in both directions, therefore P and $\mathcal{F}_C$ are isomorphic. ■

Here we present a few examples of geometric posets and partial matroids. Easy examples of geometric posets are, of course, geometric lattices, as well as geometric lattices with its smallest and/or largest element

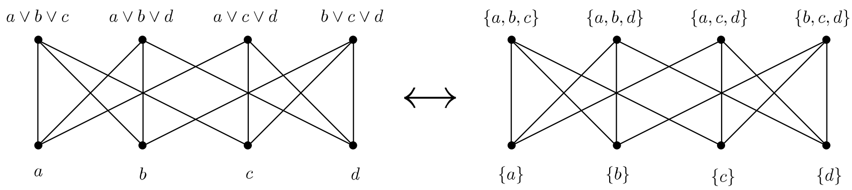

Figure 5.

removed. A less trivial example, which we denote by $(G_{4,3}, \leqslant)$, is given in Figure 5 left. Figure 5 right shows the range of the corresponding p-matroid (E, C), where $E = \{a, b, c, d\}$ and

$$C : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{d\} & \{a,b,c\} & \{a,b,d\} & \{a,c,d\} & \{b,c,d\} \\ \{a\} & \{b\} & \{c\} & \{d\} & \{a,b,c\} & \{a,b,d\} & \{a,c,d\} & \{b,c,d\} \end{pmatrix}.$$

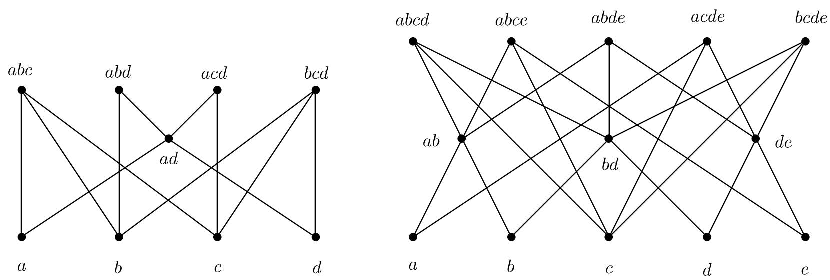

Figure 6.

This poset can be generalized to $(G_{n,k}, \leqslant)$ in the following way: we start with n minimal elements (atoms) and add suprema of all subsets of atoms of cardinality k. For $k = 1, 2$ we get a lattice with its smallest and largest elements removed, for $k = n$ we get the lattice M_n without the smallest element, while for $k \in \{3, 4, \ldots, n-1\}$ we get some less trivial examples.

There are also some examples of geometric posets not of the form $G_{n,k}$. In Figure 6 we present two of them (for brevity, we here write xy instead of $x \vee y$).

3.5 Semimodular posets

Recall that a finite lattice is geometric if and only if it is atomistic and semimodular. In this section we aim to generalize the notion of semimodularity to posets in such a way that the same equivalence holds also for posets.

The following definition, though not very natural-looking, achieves that goal.

- A poset $(P, \leqslant)$ which has the least element is **semimodular** if for every $x, y \in P$ the following holds:

$$\begin{gathered} \text{if } x \wedge y \prec x, \text{ then} \\ y \prec x \vee y \text{ or } (P \text{ is not a join-semilattice and} \\ \text{there is no atom } a \text{ such that } x \leqslant y \vee a). \end{gathered} \tag{3.12}$$

  In the above definition, $x \wedge y \prec x$ means "$x \wedge y$ exists and $x \wedge y \prec x$", and similar for the other occurrences.

- A poset $(P, \leqslant)$ which does not have the least element is **semimodular** if the poset $(P_0, \leqslant)$ is semimodular, where $(P_0, \leqslant)$ is the poset obtained by adding the least element to the poset $(P, \leqslant)$.

A generalization of semimodularity to posets appeared in Birkhoff's book [7] (the definition is by Ore [41]). A poset P is (upper) semimodular if it satisfies: if $a \neq b$ and both a and b cover c, then there exists $d \in P$ which covers both a and b. This notion is mostly accepted in the literature; see also the book [10].

Example 6. In Figure 7 left we show a poset which is semimodular by our definition but not semimodular by Ore's definition. Indeed, abc and ad both cover b, but there does not exist an element which covers both abc and ad. On the other hand, note that this poset is not a join-semilattice, since the set $\{a, b\}$ does not have supremum. In order to show that this poset is semimodular by our definition, we consider a few cases in which the implication $x \wedge y \prec x \Rightarrow y \prec x \vee y$ is not true. Therefore, for these cases we need to show that

$$x \wedge y \prec x \Rightarrow (\text{there does not exist an atom } s \text{ such that } x \leqslant y \vee s). \tag{3.13}$$

- $a \wedge b \prec a$: There does not exist an atom s such that $b \vee s$ is defined.
- $b \wedge a \prec b$: We have that $a \vee s$ is defined only for $s = d$, but $b \nleqslant ad$.
- $abc \wedge ad \prec abc$: If s is an atom, we have $ad \vee s \in \{ad, abd, acd\}$, and neither of these elements is comparable with abc.

Therefore, in these cases (3.13) is fulfilled. The remaining cases are analogous to some of the observed cases or they fulfill $x \wedge y \prec x \Rightarrow y \prec x \vee y$. $\square$

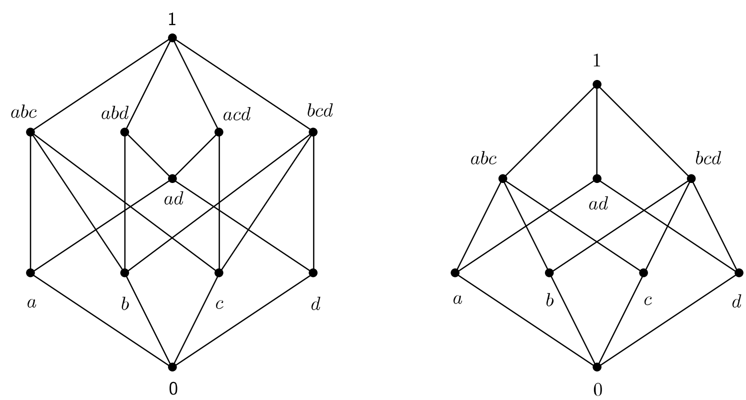

Figure 7.

Example 7. In Figure 7 right we show a poset which is not semimodular by our definition of semimodularity: we have that $b \wedge c \prec b$, but $b \vee c$ is not defined and for the atom a we have $b \leqslant abc = c \vee a$.

On the other hand, this poset is semimodular by Ore's definition. If a and b from the definition are two atoms of the given poset, then there always exists an element that covers both of these atoms (that would be one of the coatoms). If a and b from the definition are coatoms, then 1 covers them both. These are all the relevant cases. $\square$

So neither Ore's definition of semimodularity implies our definition, neither our definiton implies Ore's definition, as shown by Examples 6 and 7. We can notice that both these examples are atomistic and therefore neither implications hold for atomistic posets either.

Still, if a poset is a lattice, both definitions are equivalent with the classical notion of lattice semimodularity. The following proposition (whose proof is easy) shows that semimodularity for posets is indeed a generalization of semimodularity for lattices.

Proposition 3.29. *Let $(L, \leqslant)$ be a lattice. Then L is semimodular as a lattice if and only if it is semimodular as a poset.*

Finally, the following two theorems are the main point of this section and they support our definition of semimodularity of posets.

Theorem 3.30. *Every atomistic and semimodular poset is geometric.*

Proof. Let $(P, \leqslant)$ be a semimodular and atomistic poset and A its set of atoms. Without loss of generality, suppose that P has the least element 0. Let $a, b \in A$, $x \in P$, $a \nleqslant x$, and let $x \vee b$ exist and $a < x \vee b$.

First we show that $x \vee a$ exists. If not, then by semimodularity, since we have $a \wedge x = 0 \prec a$, it follows that there does not exist an atom c such that $x \vee c$ exists and $a \leqslant x \vee c$, which is impossible since $a < x \vee b$ by assumption.

Hence, $x \vee a$ exist. Suppose now that $b < x \vee a$ does not hold. Obviously, it is impossible that $b = x \vee a$. Therefore, $b \nleqslant x \vee a$. Now, since $(b \vee x) \vee a$ exists (because $a < x \vee b$) by Lemma 3.24 we have that $b \vee (x \vee a)$ exists and $(b \vee x) \vee a = b \vee (x \vee a)$. Hence by semimodularity and the fact that $b \vee (x \vee a)$ exists and $b \wedge (x \vee a) = 0 \prec b$, we get

$$x \vee a \prec b \vee (x \vee a). \tag{3.14}$$

Let us now show $b \nleqslant x$. Otherwise, from $b \leqslant x$ and the assumption $a < x \vee b$ it would follow $a < x$, which contradicts the assumption $a \nleqslant x$. Therefore, $b \nleqslant x$. Now by semimodularity and the fact that $x \vee b$ exists and $b \wedge x = 0 \prec b$, we get $x \prec x \vee b$. From $a < x \vee b$ and Lemma 3.24 it follows that $x \vee b = a \vee (x \vee b) = (a \vee x) \vee b$, which leads to

$$x \prec b \vee (x \vee a). \tag{3.15}$$

Taken together, (3.14) and (3.15) give $x = x \vee a$, that is, $a \leqslant x$, but this contradicts the assumption $a \nleqslant x$. This completes the proof. ∎

Theorem 3.31. *Every geometric poset is semimodular.*

Proof. Let $(P, \leqslant)$ be a geometric poset and let A be its set of atoms. Clearly, if P is a join-semilattice, then it is semimodular also as a poset.

Suppose that P is not a join-semilattice.

Let $x, y \in P$ be such that $x \wedge y$ exists and $x \wedge y \prec x$. Clearly, $x \nleqslant y$.

Suppose that $x \vee y$ does not exist and that there is an atom a such that $y \vee a$ exists and $x \leqslant y \vee a$. Since $x \nleqslant y$ and P is atomistic, there exists an atom b such that $b \leqslant x$ and $b \nleqslant y$. Now by $b \leqslant x \leqslant y \vee a$ and (3.9), $y \vee b$ exists and $a < y \vee b$, implying $y \vee a \leqslant y \vee b$. On the other hand, we get $y \vee b \leqslant y \vee a$ and together with the previous conclusion we obtain $y \vee a = y \vee b$. Let now u be any upper bound of elements x and y. From $b \leqslant x \leqslant u$ we get $b \vee y \leqslant u$, that is, $a \vee y \leqslant u$. It follows that $y \vee a$ is the least upper bound for x and y, which contradicts the assumption that $x \vee y$ does not exist.

Further, let $x \vee y$ exists and let $y \prec x \vee y$ be false, i.e., there exist u such that $y < u < x \vee y$. If x has an upper bound of the form $y \vee a$ for some $a \in A$, then we also have $x \vee y \leqslant y \vee a$. Since $y < u$ and P is atomistic, there exists an atom c such that $c \leqslant u$ and $c \nleqslant y$. Further, we have $c \leqslant u < x \vee y \leqslant y \vee a$ and by (3.9) we get $a < y \vee c$, therefore $y \vee a \leqslant y \vee c$; on the other hand we get $y \vee c \leqslant u \leqslant y \vee a$, which, together with the previous inequality, gives $y \vee c = y \vee a$, but then also $u = y \vee a$, which contradicts $u < x \vee y \leqslant y \vee a$. This contradiction implies that there does not exist $a \in A$ such that $y \vee a$ is defined and $x \leqslant y \vee a$. ■

The previous two theorems directly imply the following corollary.

Corollary 3.32. *A poset is geometric if and only if it is atomistic and semimodular.*

CHAPTER

4

APPLICATIONS

Closure operators have a wide scope of applications in different branches of science. Among them we concentrate on analyzing large amount of data. Namely, in science such as medicine, ecology and economy there are large data sets of objects and their characteristics, attributes. For example, objects can be patients, rivers, consumers and their attributes: symptoms, pollutions, products. Every patient has a set of detected symptoms, every river is polluted by certain chemical contaminants, each consumer has a list of products he buys. The problem is to draw conclusions about connections between characteristics, for example, to find which symptoms are manifesting together or which group of products imply buying a new one. One of approaches is via implications, which are subject of research of formal concept analysis. They also arise in many different areas such as data analysis, data-mining, knowledge structures, relational databases. These implications represent rules that say "an object with attributes from the set X have attribute x". It is hard to write down all implications that hold in one set of data, but we may choose some of them (called *basis*) that can be used to generate all the others. Closure operators are used in this part of problem.

In this chapter we assume that all the considered sets are finite.

Section 4.2 here is an original work.

4.1 Concepts, implications and bases

Data are often recorded in tables with rows and columns, where one of them contains objects and the other one attributes. These tables can be represented as formal contexts.

A **formal context** $\mathbb{K} = (G, M, I)$ consists of two sets, G and M, and a relation I between G and M. The elements of G are called **objects** and the elements of M are called **attributes** of the context. If an object $g \in G$ has an attribute $m \in M$, we write $(g, m) \in I$. We call this relation the **incidence relation** of the context.

The set of attributes common to the objects of a set $A \subseteq G$ is denoted by A', that is,

$$A' = \{m \in M \mid (g, m) \in I \text{ for all } g \in A\}.$$

Similarly, the set of objects which have all attributes in a set $B \subseteq M$ is

$$B' = \{g \in G \mid (g, m) \in I \text{ for all } m \in B\}.$$

A **formal concept** of the context (G, M, I) is a pair (A, B) such that $A \subseteq G$, $B \subseteq M$, $A' = B$ and $A = B'$. The set A is called the **extent** and B the **intent** of concept (A, B). The set of all concepts of a context (G, M, I) is denoted by $\mathcal{B}(G, M, I)$.

Theorem 4.1. *For a context (G, M, I) and $A, A_1, A_2 \subseteq G$, $B, B_1, B_2 \subseteq M$, we have:*

1. *$A_1 \subseteq A_2$ implies $A_2' \subseteq A_1'$, $B_1 \subseteq B_2$ implies $B_2' \subseteq B_1'$;*
2. *$A \subseteq A''$, $B \subseteq B''$;*
3. *$A' = A'''$, $B' = B'''$;*
4. *$A \subseteq B' \Leftrightarrow B \subseteq A' \Leftrightarrow A \times B \subseteq I$.*

Proof.

1. If $m \in A_2'$, then $(g, m) \in I$ for all $g \in A_2$. Therefore, $(g, m) \in I$ for all $g \in A_1$, since $A_1 \subseteq A_2$. Hence, $m \in A_1'$.
2. If $g \in A$, then $(g, m) \in I$ for all $m \in A'$, therefore $g \in A''$.
3. By 2. we have $A' \subseteq A'''$, and because of $A \subseteq A''$ and 1. it follows $A''' \subseteq A'$.

4. This holds by definition.

■

The previous theorem gives us the following corollary.

Corollary 4.2. *For a context (G, M, I), the operator $''$ is a closure operator on both sets G and M.*

For every set $A \subseteq G$, (A'', A') is a concept and A'' is the smallest extent containing A. This means that $A \subseteq G$ is an extent if and only if $A = A''$. The same holds for intents.

On the set $\mathcal{B}(G, M, I)$ we define an order $\leqslant$: $(A_1, B_1) \leqslant (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ (if and only if $B_2 \subseteq B_1$). This way we get a lattice and, because of that, we call $\mathcal{B}(G, M, I)$ a **concept lattice** of the context (G, M, I).

Theorem 4.3. *The concept lattice $\mathcal{B}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{k\in K} (A_k, B_k) = \left(\bigcap_{k\in K} A_k, \left(\bigcup_{k\in K} B_k \right)'' \right),$$

$$\bigvee_{k\in K} (A_k, B_k) = \left(\left(\bigcup_{k\in K} A_k \right)'', \bigcap_{k\in K} B_k \right).$$

Proof. First we prove that intersection of intents is again an intent, that is, for $B_k \subseteq M$, $k \in K$, we have

$$\left(\bigcup_{k\in K} B_k \right)' = \bigcap_{k\in K} B'_k.$$

Indeed,

$$\begin{aligned} g \in \left(\textstyle\bigcup_{k\in K} B_k \right)' &\Leftrightarrow (g, m) \in I \text{ for all } m \in \textstyle\bigcup_{k\in K} B_k \\ &\Leftrightarrow (g, m) \in I \text{ for all } m \in B_k \text{ for all } k \in K \\ &\Leftrightarrow g \in B'_k \text{ for all } k \in K \\ &\Leftrightarrow g \in \textstyle\bigcap_{k\in K} B'_k. \end{aligned}$$

The same holds for extents. Now, since for all $k \in K$ we have $A_k = B'_k$, by the equality above we have

$$\begin{aligned}\left(\bigcap_{k\in K} A_k, \left(\bigcup_{k\in K} B_k\right)''\right) &= \left(\bigcap_{k\in K} B'_k, \left(\bigcup_{k\in K} B_k\right)''\right)\\ &= \left(\left(\bigcup_{k\in K} B_k\right)', \left(\bigcup_{k\in K} B_k\right)''\right).\end{aligned}$$

Therefore, this is a concept. It is indeed the infimum, because the extent of this concept is the intersection of extents (A_k, B_k), $k \in K$. The second part of the theorem is analogous. ∎

An **implication** $X \to Y$ is an ordered pair (X, Y) of subsets of M. Hence the set of implications is a binary relation on $\mathcal{P}(M)$; we call it an **implicational system** (or set of functional dependencies). When the set Y in implication $X \to Y$ is a singleton, then we write $X \to y$; this type of implications are called **unit implications**. So the set of unit implications, **unit implicational system** (UIS), is a relation between $\mathcal{P}(M)$ and M.

A subset $T \subseteq M$ **respects** an implication $X \to Y$ if $X \nsubseteq T$ or $Y \subseteq T$. T **respects a set** Σ of implications if T respects every implication in Σ. $X \to Y$ **holds in a context** (G, M, I) if every object intent of the context respects $X \to Y$. We also say that X is a **premise of** Y, or that $X \to Y$ is an **implication of the context** (G, M, I).

It is easy to see that the following theorem is true.

Theorem 4.4. *An implication $X \to Y$ holds in (G, M, I) if and only if $Y \subseteq X''$. Then it holds in the set of all intents as well.*

An implication $X \to Y$ **follows** from a set of implications Σ if every subset of attributes respecting Σ also respects $X \to Y$. A set of implications Σ is **closed** (full implicational system, entail relation, full family of functional dependencies, relational databases scheme) if each implication following from Σ is already in Σ.

For an implicational system $\Sigma = \{X_1 \to Y_1, X_2 \to Y_2, \dots, X_m \to Y_m\}$, we define the **size** $s(\Sigma)$ of Σ by

$$s(\Sigma) = \sum_{i=1}^{m} (|X_i| + |Y_i|).$$

We can always replace an implicational system by a unit implicational system: an implication $X \to Y$ can be replaced by the set of unit implications $\{X \to y \mid y \in Y\}$. That is why we shall work with unit implications only.

The set X in implication $X \to y$ is called the **premise**, and y is called the **conclusion**. A subset $A \subseteq M$ **respects** the implication $X \to y$ if $X \subseteq A$ implies $y \in A$. If Σ is a set of implications, $A \subseteq M$ is **Σ-closed** if A respects all implications in Σ (in other words: all Σ-implications). By $\mathcal{F}_\Sigma$ we denote the set of all Σ-closed sets. It is easy to see that $\mathcal{F}_\Sigma$ forms a closure system on M. The corresponding closure operator is denoted by $C_{\mathcal{F}_\Sigma}$.

To a system Σ we can also associate a closure operator C_Σ: for $X \subseteq M$ let

$$\pi_\Sigma(X) = X \cup \bigcup \{b \in M \mid A \subseteq X \text{ and } A \to b \in \Sigma\}$$

and

$$\pi^n_\Sigma(X) = \pi^{n-1}_\Sigma(X) \cup \bigcup \{b \in M \mid A \subseteq \pi^{n-1}_\Sigma(X) \text{ and } A \to b \in \Sigma\};$$

then

$$C_\Sigma(X) = \pi_\Sigma(X) \cup \pi^2_\Sigma(X) \cup \pi^3_\Sigma(X) \cup \dots$$

Since M is finite, there is $n \in \mathbb{N}$ such that $\pi^n_\Sigma(X) = \pi^{n+1}_\Sigma(X)$, so $C_\Sigma(X) = \pi^n_\Sigma(X)$. In fact, we have more: $C_\Sigma = C_{\mathcal{F}_\Sigma}$.

If we start with a closure operator C on a set M, then closed sets coincide with the Σ-closed set of the following UIS:

$$\Sigma_C = \{X \to y \mid y \in M,\ X \subseteq M \text{ and } y \in C(X)\}.$$

This unit implicational system satisfies the following properties:

F_1: $X \subseteq M$ and $x \in X$ imply $X \to x$;

F_2: for each $y \in M$ and all $X, Y \subseteq M$,

$$X \to y \text{ and } (\forall x \in X)(Y \to x) \text{ imply } Y \to y.$$

Every UIS that satisfies properties F_1 and F_2 is called **full**. The full UISs are in one-to-one correspondence with closure operators.

Every UIS Σ is contained in full UIS, namely Σ_C where $C = C_\Sigma$, and this is the smallest full UIS that contains Σ. We can also generate the smallest full UIS that contains Σ by recursively applying the rules F_1 and F_2. Then we call Σ the **generating system** of Σ_C. For UISs Σ_1 and Σ_2 we say that they are **equivalent** if they generate the same full UIS.

An UIS Σ is **minimal** or **non-redundant** if $\Sigma \setminus \{X \to y\}$ is not equivalent to Σ for all $X \to y \in \Sigma$. It is also called **basis**. If for all UISs Σ' equivalent to Σ we have $|\Sigma| \leqslant |\Sigma'|$, then Σ is a **minimum basis**. If for all UISs Σ' equivalent to Σ we have $s(\Sigma) \leqslant s(\Sigma')$, then Σ is **optimal**. If for every $X \subseteq M$ holds $C_\Sigma(X) = \pi_\Sigma(X)$, then Σ is **direct** or **iteration-free**. An UIS is called **proper** if it does not contain trivial implications, that is, implications of the form $X \to x$ where $x \in X$. From every UIS we can get an equivalent proper UIS by deleting all proper implications and that is why in the following sections we assume that every UIS is replaced by equivalent proper UIS.

There are several types of generating systems for UISs. We mention a few of them.

An UIS Σ_{do} is **direct-optimal** if it is direct and if $s(\Sigma_{do}) \leqslant s(\Sigma)$ for every direct UIS Σ equivalent to Σ_{do}.

In [6] it is proved that this kind of UIS is unique and that it can be obtained from every equivalent UIS.

The **left-minimal** basis Σ_{lm} is:

$$\Sigma_{lm} = \{X \to y \mid y \in C(X) \setminus X \text{ and for every } Y \subsetneq X,\, y \notin C(Y)\}.$$

This kind of basis is the restriction of the full UIS to implications where the premise is of minimal cardinality.

Theorem 4.5. *Let C be a closure operator on a set M. Then the direct-optimal and left-minimal basis coincide.*

Proof. We prove $\Sigma_{do} = \Sigma_{lm}$. Since there is unique direct-optimal basis, we show that Σ_{lm} is direct-optimal. For Σ_{lm} to be direct, for any $X \subseteq M$ it has to be $C(X) = X \cup \bigcup\{b \in M \mid (\exists A)(A \subseteq X \text{ and } A \to b \in \Sigma_{lm})\}$. This is true, because for every $b \in C(X)\setminus X$ we can always choose minimal (with respect to the set inclusion) subset A of X such that $b \in C(A)$, so we have $A \to b \in \Sigma_{lm}$.

It remains to show that Σ_{lm} is direct-optimal. It is sufficient to prove that for arbitrary UIS Σ equivalent to Σ_{lm}, if $A \to x \in \Sigma_{lm}$, then $A \to x \in \Sigma$. Suppose the opposite. Let $A \to x \in \Sigma_{lm}$. If there exists $B \subsetneq A$ such that $B \to x \in \Sigma$, then $B \to x \in \Sigma_C$, but this is in contradiction with the fact that $A \to x \in \Sigma_{lm}$. Hence, for all $B \subsetneq A$ we have $B \to x \notin \Sigma$. Now it follows

$$x \notin A \cup \{b \in M \mid \text{there exists } B \subseteq A \text{ such that } B \to b \in \Sigma\} = \pi_\Sigma(A).$$

This gives a contradiction with the fact that Σ is direct. ■

The previous theorem is shown in [5] and more, this type of basis coincide with another three definitions of basis. Thus, these bases are called **canonical direct basis** and denoted by Σ_{cd}.

In order to define one more type of basis of implicational system, we need some additional definitions and properties.

A closure operator C on a set A is **reduced** if it fulfills:

C_5: $C(\{i\}) = C(\{j\})$ implies $i = j$, for all $i, j \in A$.

Given a closure operator C on a set A, there always exists a reduced closure operator C_r such that their ranges are isomorphic. C_r can be constructed in the following way. We define an equivalence relation $\approx$ on the set A by $x \approx y$ if and only if $C(\{x\}) = C(\{y\})$. Then we define a closure operator C_r as the restriction of the closure operator C on the set of representatives of all $\approx$-classes. Thus, from here onward we shall consider reduced closure operators.

If C is a closure operator on a set A, we define a relation $\ll$ between subsets of A as follows: $X \ll Y$ if and only if for every $x \in X$ there exists $y \in Y$ such that $x \in C(\{y\})$. We write $X \sim_{\ll} Y$ if $X \ll Y$ and $Y \ll X$.

It is easy to see that $\sim_{\ll}$ is an equivalence relation. The equivalence classes of this relation which contain X will be denoted by $[X]$. We make the following observations: $X \subseteq Y$ implies $X \ll Y$; if $Y \in [X]$, then $C(X) = C(Y)$. Natural order on $\sim_{\ll}$-classes is $\leqslant_C$ defined by: $[X] \leqslant_C [Y]$ if and only if $X \ll Y$.

Theorem 4.6. *If C is a reduced closure operator on A, then each equivalence class $[X]$ has unique minimal element with respect to the set inclusion.*

Proof. Let there be two minimal elements X_1 and X_2 in the class $[X]$ and let $x \in X_1 \setminus X_2$. Since $X_1 \sim_{\ll} X_2$, we have $X_1 \ll X_2 \ll X_1$, so there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $x \in C(\{x_2\})$ and $x_2 \in C(\{x_1\})$. We have $x \in C(\{x\}) \subseteq C(\{x_2\}) \subseteq C(\{x_1\})$. If $x = x_1$, then $C(\{x\}) \subseteq C(\{x_2\}) \subseteq C(\{x\})$; since C is reduced, we have $x = x_2$. This is impossible, since $x \notin X_2$. Therefore, $x \neq x_1$. Since $x \in C(\{x_1\})$, then $X_1 \ll X_1 \setminus \{x\}$. On the other hand, $X_1 \setminus \{x\} \subsetneq X_1$, so $X_1 \setminus \{x\} \ll X_1$, hence $X_1 \setminus \{x\} \in [X]$. This implies that X_1 can not be minimal in $[X]$. ∎

Now we define a relation between elements of A and subsets of A. We say that $X \subseteq A$ is a **cover** of $x \in A$ if $x \in C(X) \setminus \bigcup_{x' \in X} C(\{x'\})$, and we write $x \lhd X$. We call $Y \subseteq A$ a **minimal cover** of an element $x \in A$ if $x \lhd Y$ and for every other cover Z of x, $Z \ll Y$ implies $Y \subseteq Z$. Therefore, by Theorem 4.6, Y is a minimal cover of x if it is minimal in the class $[Y]$ with respect to the set inclusion.

Theorem 4.7. *Let C be a reduced closure operator on a set A, $X \subseteq A$ and $x \in A$. If $x \lhd X$, then there exists $Y \subseteq A$ such that $x \lhd Y$, $Y \ll X$ and Y is minimal cover of x.*

Proof. Denote $P_x = \{[X] \mid x \lhd X\}$. Since P_x is a subset of the set of all $\sim_{\ll}$-equivalence classes, it is ordered by $\leqslant_C$. If P_x is not empty, we choose a minimal element $[Y]$ below $[X]$ with respect to $\leqslant_C$. By Theorem 4.6, the class $[Y]$ has unique minimal element with respect to the set inclusion; denote it by Y. Then $Y \ll X$ and $x \lhd Y$. Let now Z be an arbitrary cover of x such that $Z \ll Y$. We prove $Y \subseteq Z$. From $Z \ll Y$ it follows that $[Z] \leqslant_C [Y]$. Since $[Y]$ is minimal in P_x, we have $[Y] = [Z]$. Therefore, $Y \subseteq Z$, because Y is minimal in $[Y]$. ■

Finally, we define one more type of basis.

Let C be a reduced closure operator on a set A. D**-basis** Σ_D is union of the following two sets of implications:

1. $\{y \to x \mid x \in C(\{y\}) \setminus \{y\} \text{ and } y \in A\}$;
2. $\{X \to x \mid X \text{ is minimal cover of } x\}$.

The first part is usually called the **binary part**.

Theorem 4.8. *Σ_D generates Σ_C.*

Proof. We prove that for every $x \in A$ and $X \subseteq A$ such that $x \in C(X) \setminus X$, the implication $X \to x$ follows from implications in Σ_D. (We do not consider the case $x \in X$, since we work only with proper UISs.)

If there exists $x_1 \in X$ such that $x_1 \neq x$ and $x \in C(\{x_1\})$, then $X \to x$ follows from $x_1 \to x$, which is in Σ_D. Hence, assume that $x \notin C(\{x_1\})$ for all $x_1 \in X$. Then $x \lhd X$. By Theorem 4.7, there exists $Y \ll X$ such that $x \lhd Y$ and Y is a minimal cover of x. Then $Y \to x \in \Sigma_D$ and also $x \in C(Y)$. Since $Y \ll X$, we have that for every $y \in Y \setminus X$ there exists $z \in X$ such that $y \in C(\{z\})$. Thus, $z \to y \in \Sigma_D$ and $y \in C(\{z\})$. Finally, we get the implication $X \to x$ as a consequence of $\{z \to y \mid y \in Y\}$ and $Y \to x$. This completes the proof. ■

D-basis is comparable with canonical direct basis. It is easy to check that the following theorem holds.

Theorem 4.9. *If C is a reduced closure operator on A, then D-basis is a subset of the canonical direct basis, that is $\Sigma_D \subseteq \Sigma_{cd}$.*

4.2 Generalization to partial unit implicational systems

We give one generalization of unit implicational system. A **full partial unit implicational system** Σ on a set M fulfills the following properties:

PF_1: if there exists y such that $X \to y$, then for all $x \in X$ we have $X \to x$;

PF_2: for each $y \in M$ and all $X, Y \subseteq M$,

$$X \to y \text{ and } (\forall x \in X)(Y \to x) \text{ imply } Y \to y;$$

PF_3: if for $X \subseteq M$ there exists $z \in M$ such that $X \to z$, then there exists $x \in M$ such that $\{y \in M \mid X \to y\} \to x$;

PF_4: $x \to x$, for all $x \in M$.

There is a correspondence between full partial UIS and partial closure operators.

Theorem 4.10. *Every full partial UIS on a set M defines a partial closure operator on the same set.*

Proof. Let Σ be a full partial UIS on a set M. For $X \subseteq M$ we define

$$C(X) := \{y \in M \mid X \to y\},$$

if there exists $y \in M$ such that $X \to y$, otherwise $C(X)$ is not defined. Then C is a partial closure operator on M.

PC_1 : Assume that $C(X)$ is defined. Then there exists some $y \in M$ such that $X \to y$, so by PF_1 we have that $X \to x$ for all $x \in X$, that is, $X \subseteq C(X)$.

PC_2 : Let $X \subseteq Y$ and $C(X)$, $C(Y)$ be defined. If $z \in C(X)$, then $X \to z$. Since $X \subseteq Y$, by PF_1 we have $Y \to x$ for all $x \in X$. Now by PF_2 we have that $Y \to z$, therefore $z \in C(Y)$.

PC_3 : Let $C(X)$ be defined. Hence for some $z \in M$ we have $X \to z$. By PF_3, there exists $x \in M$ such that $C(X) \to x$. Therefore $C(C(X))$ is defined; by PC_1 we have $X \subseteq C(X)$ and by PC_2 we have $C(X) \subseteq C(C(X))$. On the other hand, let $w \in C(C(X))$. Then $C(X) \to w$ and since for all $y \in C(X)$ we have $X \to y$, by PF_2 it follows that $X \to w$. Therefore, $w \in C(X)$, and thus $C(C(X)) \subseteq C(X)$.

PC_4 : This property follows directly from PF_4. ∎

Theorem 4.11. *If C is a partial closure operator on a set M, then*

$$\Sigma_C := \{X \to y \mid y \in M,\ X \subseteq M,\ C(X) \text{ is defined and } y \in C(X)\}$$

is the full partial UIS on M.

Proof. We prove that Σ_C fulfills properties PF_1–PF_4.

PF_1 : Let $X \to y$. Then we have that $C(X)$ is defined and $y \in C(X)$, therefore for all $x \in C(X)$ we have $X \to x$.

PF_2 : Let $y \in M$ and $X, Y \subseteq M$ be such that $X \to y$ and $(\forall x \in X)(Y \to x)$. Then we have that $C(X)$ and $C(Y)$ are defined, $y \in C(X)$ and $x \in C(Y)$, for all $x \in X$. Hence, $X \subseteq C(Y)$, so $C(X) \subseteq C(C(Y)) = C(Y)$, and then $y \in C(Y)$, what implies $Y \to y$.

PF_3 : Let $X \subseteq M$ be such that there exists $z \in M$ such that $X \to z$. Therefore, $C(X)$ is defined and $z \in C(X)$. We also have that $C(X) = \{y \in M \mid X \to y\}$. Since $z \in C(X) = C(C(X))$, it follows that $C(X) \to z$, that is, $\{y \in M \mid X \to y\} \to z$.

PF_4 : Since $C(\{x\})$ is defined for all $x \in M$ and $x \in C(\{x\})$, then $x \to x \in \Sigma_C$. ∎

CHAPTER

5

CONCLUSION

This thesis provides an insight into connections between closure operators, closure systems and complete lattices. These connections have been studied a lot, while their generalizations are very rarely considered. That is why this thesis extends the current state of knowledge about connections between partial closure operators, partial closure systems and ordered sets. The connections are made stronger than they have been until now. By adding and analyzing new axioms, we obtain better results than known ones on relationships between collections of sets, closure operators and ordered sets. In order to achieve unique correspondence between partial closure systems and partial closure operators, we introduce a special kind of closure operator: sharp partial closure operator. This is a partial operator on a power set that fulfills axioms analogous to the closure axioms, plus a few additional ones. We have shown the uniqueness of such a partial operator that corresponds to a given partial closure system. We further introduce partial closure systems which correspond to principal ideals in ordered set. Also, we state and prove the representation theorem for ordered sets with respect to the introduced partial closure operators and partial closure systems.

We pay a special attention to collections of sets related to finite geometries, such as matroids. They are objects that have a large number of mutually equivalent definitions, in different branches of mathematics. The main result of this thesis is generalization of axioms and establishing connection between those objects and ordered sets. After the transition from closure operators to partial closure operators, we introduce a generalization of matroids: p-matroids, as well as a generalization of geometric lattices: geometric ordered sets. These generalizations emerge from the relations between geometric lattices and matroids. We then research and define an analogue to the notion of semimodularity for ordered sets that are not lattices.

Relations between closure systems, closure operators and complete lattices are in the foundations of the theory of ordered sets and the lattice theory. Closure systems and closure operators are among the basic tools used in research of ordered sets, topology, universal algebra, logic etc. All this leads to the conclusion that generalizations of these notions, considered in this thesis, have a very high potential of being applied in different branches of mathematics. In this thesis we have shown an example of how they can be applied on implicational systems, which is used a lot in big data analysis. We hope that this is one of the applications of partial operators than can be greatly developed, and this is one of directions we plan to work on in the future.

PROŠIRENI IZVOD

1 Uvod

1.1 Parcijalno uređeni skupovi

Uređeni skup je uređen par $(P, \leqslant)$, gde je $\leqslant$ binarna relacija na nepraznom skupu P koja je refleksivna, antisimetrična i tranzitivna. Strukturu $(P, \leqslant)$ zovemo i **parcijalno uređen skup** ili **poset** (od enlgeskog naziva *partially ordered set*). Kada je jasno sa kojim posetom radimo, često pišemo samo P umesto $(P, \leqslant)$.

Kažemo da je x **pokriveno sa** y (ili y **pokriva** x) ako i samo ako $x < y$ i $\neg(\exists z)(x < z < y)$, i pišemo $x \prec y$. Ako važi $x \prec y$ ili $x = y$, onda pišemo $x \preceq y$.

Podskup T skupa P nazivamo **lanac** (resp. **antilanac**), ako su svaka dva različita elementa skupa T uporediva (resp. neuporediva).

Element $a \in P$ je:

– **najmanji** (resp. **najveći**), ako za sve $x \in P$ važi $a \leqslant x$ (resp. $x \leqslant a$); ovi elementi su jedinstveni u uređenom skupu (ukoliko postoje) i obično ih označavamo sa 0 i 1, respektivno;

– **minimalni** (resp. **maksimalni**), ako za sve $x \in P$, $x \leqslant a$ (resp. $a \leqslant x$) implicira $x = a$.

Ako $p \in P$, onda skup

$$\downarrow p = \{x \in P \mid x \leqslant p\}$$

nazivamo **glavni ideal** generisan elementom p.

Ako je P uređeni skup i $Q \subseteq P$, onda skup svih donjih ograničenja skupa Q, u oznaci Q^d, definišemo sa

$$Q^d = \{a \in P \mid a \leqslant b, \text{ za sve } b \in Q\},$$

a skup svih gornjih ograničenja skupa Q, u oznaci Q^g, sa

$$Q^g = \{a \in P \mid b \leqslant a, \text{ za sve } b \in Q\}.$$

Podskup D skupa P je **usmeren** ako svaki konačan podskup skupa D ima gornje ograničenje u skupu D. Uređeni skup $(P, \leqslant)$ nazivamo **potpun** ako ima najmanji element 0 i ako svaki njegov usmeren podskup D ima supremum. Umesto celog naziva obično koristimo skrećenicu **CPO** (od engl. complete partial ordered set).

Element a potpuno uređenog skupa P je **kompaktan** ako za svaki njegov usmeren podskup D takav da $a \leqslant \bigvee D$ postoji $d \in D$ da $a \leqslant d$.

Neka je $(P, \leqslant)$ poset i C skup njegovih kompaktnih elemenata. Tada za P kažemo da je **algebarski** ako je CPO i ako je za svako $x \in P$ skup $\downarrow x \cap C$ usmeren, i važi $x = \bigvee(\downarrow x \cap C)$.

Uređeni skup $(L, \leqslant)$ u kojem svaki par elemenata ima supremum (resp. infimum) se naziva **$\vee$-polumreža** (resp. **$\wedge$-polumreža**). Uređeni skup $(L, \leqslant)$ u kojem svaki par elemenata ima supremum i infimum nazivamo **mreža**. Uređeni skup u kojem svaki podskup ima supremum i infimum nazivamo **potpuna mreža**.

Dobro je poznato da za uređene skupova i mreže važi *princip dualnosti*, tj. ako je $(P, \leqslant)$ uređen skup/mreža/potpuna mreža, onda je njegov dual $(P, \geqslant)$ takođe uređen skup/mreža/potpuna mreža.

Još jedna poznata osobina mrežaje sledeća teorema.

Teorema 1.2. *Uređeni skup $(P, \leqslant)$ u kojem svaki podskup ima infimum (supremum) je potpuna mreža.*

Element a mreže L je **kompaktan** ako važi sledeće: iz $a \leqslant \bigvee A$ za svaki podskup A skupa L, sledi da postoji konačan podskup B skupa A takav da $a \leqslant \bigvee B$. Kako je mreža uređen skup, prirodno je zapitati se da li se kompaktni elementi mreže i kompaktni elementi mreže kao poseta poklapaju. Odgovor na ovo pitanje je potvrdan.

Mreža je **kompaktno generisana** ako je svaki njen element supremum kompaktnih elemenata. **Algebarska** mreža je potpuna i kompaktno generisana.

Teorema 1.3. *Neka je L uređen skup. Tada svaki od sledećih uslova povlači naredni:*

1. *L je algebarska mreža,*
2. *L je potpuna mreža,*
3. *L je potpun uređeni skup.*

Teorema 1.4. *Neka je L uređen skup. Tada svaki od sledećih uslova povlači naredni:*

1. *L je algebarska mreža,*
2. *L je algebarski uređen skup,*
3. *L je potpun uređeni skup.*

Veze koje važe između ovih uređenih skupova prikazane su na slici 1 (strana 16).

Sada navodimo nekoliko tvrđenja koja su korisna kasnije.

Teorema 1.5 (Iwamurina lema). *Uređeni skup je CPO ako i samo ako svaki njegov lanac ima supremum.*

Teorema 1.6. *Ako je $(\mathcal{F}, \subseteq)$ familija podskupova nekog skupa uređena inkluzijom, onda je $\mathcal{F}$ zatvorena za unije lanaca ako i samo ako je zatvorena za unije usmerenih familija.*

1.2 Sistemi zatvaranja

Familija $\mathcal{F}$ podskupova nepraznog skupa A koja je zatvorena za presek i sadrži ceo skup A se naziva **sistem zatvaranja** (ili **Murova familija**). Za sistem zatvaranja kažemo da je **algebarski** (ili **algebarska Murova familija**) ako sadrži uniju svake svoje usmerene potfamilije.

Teorema 1.7. *Sistem zatvaranja uređen inkluzijom je potpuna mreža.*

Familija $\mathcal{F}$ podkupova nepraznog skupa S se naziva **parcijalni sistem zatvaranja** na S (u literaturi se još koristi naziv **centralizovan sistem** ili **tačkasti sistem zatvaranja**, npr. [23, 21]) ako zadovoljava sledeće uslove:

Ps_1: $\bigcup \mathcal{F} = S$,

Ps_2: za svako $x \in S$ važi $\bigcap\{X \in \mathcal{F} \mid x \in X\} \in \mathcal{F}$.

Kažemo da je skup $\bigcap\{X \in \mathcal{F} \mid x \in X\}$ **centralizovan presek** za $x \in S$.

Teorema 1.8. *Svaki uređen skup $(P, \leqslant)$ je izomorfan nekom parcijalnom sistemu zatvaranja na P uređenom inkluzijom.*

Parcijalni sistem zatvaranja $\mathcal{F}$ na nepraznom skupu S je **potpun** ako ispunjava:

Ps_3: svaki lanac u $\mathcal{F}$ ima supremum.

Teorema 1.9. *Svaki potpun parcijalni sistem zatvaranja je CPO. Obratno, za svaki CPO postoji njemu izomorfan potpun parcijalni sistem zatvaranja.*

Neka je $\mathcal{F}$ parcijalni sistem zatvaranja na skupu S koji ispunjava sledeća dva uslova:

Ps_3': $\mathcal{F}$ je zatvoren za unije lanaca;

Ps_4': za svako $X \in \mathcal{F}$, kolekcija

$$\{Y \subseteq X \mid Y \text{ je kompaktan element u } (\mathcal{F}, \subseteq)\}$$

je usmerena u $(\mathcal{F}, \subseteq)$.

Tada $\mathcal{F}$ nazivamo **algebarski** parcijalni sistem zatvaranja na S.

Teorema 1.10. *Algebarski parcijalni sistem zatvaranja je algebarski uređeni skup u odnosu na inkluziju i obratno, za svaki algebarski uređeni skup postoji njemu izomorfan algebarski parcijalni sistem zatvaranja uređen inkluzijom.*

1.3 Operatori zatvaranja

Operator zatvaranja na nepraznom skupu A je unarna operacija $X \mapsto \overline{X}$ na partitivnom skupu $P(\mathcal{A})$, koja za sve $X, Y \subseteq A$ ispunjava sledeće uslove:

C_1: $X \subseteq \overline{X}$;

C_2: $X \subseteq Y$ implicira $\overline{X} \subseteq \overline{Y}$;

C_3: $\overline{\overline{X}} = \overline{X}$.

Ako $X \subseteq A$ i $\overline{X} = X$, onda je X **zatvoren** skup i $\overline{X}$ je **zatvorenje** skupa X. Familija zatvorenih skupova $\mathcal{F}$ je **opseg** operatora zatvaranja.

Teorema 1.11. *Opseg operatora zatvaranja na skupu A jeste sistem zatvaranja na istom skupu.*

Teorema 1.12. *Ako je $\mathcal{F}$ sistem zatvaranja na skupu A, onda je preslikavanje $X \mapsto \overline{X}$ iz $\mathcal{P}(A)$ u $\mathcal{P}(A)$, takvo da se X preslikava u presek svih elemenata $\mathcal{F}$ koji sadrže X, operator zatvaranja na skupu A.*

Korespondencija između sistema zatvaranja i odgovarajućih operatora zatvaranja je jedinstvena.

Teorema 1.13. *U mreži $\mathcal{F}$ zatvorenih skupova nekog operatora zatvaranja važi: $\{X_i \mid i \in I\} \subseteq \mathcal{F}$ povlači*

$$\bigvee\{X_i \mid i \in I\} = \overline{\bigcup\{X_i \mid i \in I\}}.$$

Teorema 1.14. *Za svaku potpunu mrežu L postoji skup i operator zatvaranja na njemu takav da je L izomorfna opsegu ovog operatora zatvaranja.*

Operator zatvaranja $X \mapsto \overline{X}$ na skupu A je **algebarski** ako ispunjava sledeće:

C_4: za sve $X \subseteq A$ važi $\overline{X} = \bigcup\{\overline{Y} \mid Y \subseteq X$ i Y je konačan$\}$.

Teorema 1.15. *Neka je $\mathcal{F}$ sistem zatvaranja na A. Tada je kolekcija $\mathcal{F}$ opseg algebarskog operatora zatvaranja ako i samo ako je $\mathcal{F}$ algebarski sistem zatvaranja.*

Teorema 1.16. *Neka je $\mathcal{F}$ algebarski sistem zatvaranja na skupu A, $X \mapsto \overline{X}$ operator zatvaranja na A takav da je $\mathcal{F}$ njegov opseg i $B \in \mathcal{F}$. Tada je B kompaktan u mreži $(\mathcal{F}, \subseteq)$ ako i samo ako $B = \overline{Y}$ za neki konačan podskup Y skupa A.*

Teorema 1.17. *Mreža $(\mathcal{F}, \subseteq)$, gde je $\mathcal{F}$ opseg algebarskog operatora zatvaranja na A, jeste algebarska.*

Teorema 1.18. *Svaka algebarska mreža $(L, \leqslant)$ je izomorfna mreži $(\mathcal{F}, \subseteq)$, gde je $\mathcal{F}$ opseg nekog algebarskog operatora zatvaranja.*

1.4 Parcijalni operatori zatvaranja

Neka je za neprazan skup S parcijalno preslikavanje $C : \mathcal{P}(S) \to \mathcal{P}(S)$ takvo da važe sledeći uslovi:

Pc_1: Ako je $C(X)$ definisano, onda $X \subseteq C(X)$.

Pc_2: Ako su $C(X)$ i $C(Y)$ definisani, onda $X \subseteq Y$ implicira $C(X) \subseteq C(Y)$.

Pc_3: Ako je $C(X)$ definisano, onda je $C(C(X))$ takođe definisano i važi $C(C(X)) = C(X)$.

Pc_4: $C(\{x\})$ je definisano za sve $x \in S$.

Kako je definisano u [53], parcijalno preslikavanje C koje ispunjava uslove Pc_1–Pc_4 nazivamo **parcijalni operator zatvaranja** na S.

Kao i kod operatora zatvaranja, ako je $X \subseteq S$ i $C(X) = X$, onda X nazivamo **zatvoren** skup. Familiju zatvorenih skupova $\mathcal{F}_C$ u odnosu na parcijalni operator C nazivamo **opseg** parcijalnog operatora zatvaranja C. **Strogi domen** parcijalnog operatora C na skupu S označavamo sa $Dom(C)$:

$$Dom(C) := \{X \mid X \subseteq S \text{ i } C(X) \text{ je definisano}\}.$$

Neka je C parcijalni operator zatvaranja na S. Ako je $C(X)$ definisano, onda se lako proverava da, ekvivalentno istoj osobini operatora zatvaranja (koji nije parcijalan),

$$C(X) = \bigcap\{Y \in \mathcal{F}_C \mid X \subseteq Y\}. \tag{5.1}$$

Takođe, lako se vidi da parcijalni operator jeste uopštenje operatora zatvaranja.

Teorema 1.19. *Opseg parcijalnog operatora zatvaranja na skupu S jeste parcijalni sistem zatvaranja.*

Obratno, za svaki parcijalni sistem zatvaranja $\mathcal{F}$ na skupu S postoji parcijalni operator na skupu S takav da je njegov opseg upravo $\mathcal{F}$.

Teorema 1.20. *Neka je $\mathcal{F}$ parcijalni sistem zatvaranja i C parcijalni operator zatvaranja na S takav da je $\mathcal{F}$ njegov opseg. Neka je dalje*

$$\widehat{\mathcal{F}} := \{S\} \cup \{X \subseteq S \mid X = \bigcap \mathcal{G}, \text{ za } \mathcal{G} \subseteq \mathcal{F}\}.$$

Tada je $\widehat{\mathcal{F}}$ sistem zatvaranja i za odgovarajući operator zatvaranja $\widehat{C}$ važi $\widehat{C}(X) = C(X)$, kada je $C(X)$ definisano.

Parcijalni operator zatvaranja C na skupu S je *potpun*, ako zadovoljava sledeći uslov:

Pc_5: ako je $\{X_i \mid i \in I\}$ lanac i $C(X_i)$ je definisano za sve $i \in I$, onda je $C(\bigcup_{i \in I} X_i)$ takođe definisano.

Teorema 1.21. *Opseg potpunog parcijalnog operatora zatvaranja jeste potpun parcijalni sistem zatvaranja.*

Obratno, parcijalni operator zatvaranja, čiji je opseg potpun parcijalni sistem zatvaranja, jeste potpun.

Sledeća posledica direktno sledi iz teorema 1.9 i 1.21.

Posledica 1.22. *Opseg potpunog parcijalnog operatora zatvaranja uređen inkluzijom je CPO, i obratno, svaki CPO je izomorfan opsegu nekog potpunog parcijalnog operatora zatvaranja.*

Potpun parcijalni operator zatvaranja C na skupu S je **algebarski** ako ispunjava sledeće:

Pc_6: ako je $C(X)$ definisano, onda je skup

$$\{C(Y) \mid Y \subseteq X,\ C(Y) \text{ je definisano i } Y \text{ je konačan}\} \tag{5.2}$$

usmeren i

$$C(X) = \bigcup \{C(Y) \mid Y \subseteq X,\ C(Y) \text{ je definisano i } Y \text{ je konačan}\}. \tag{5.3}$$

Teorema 1.23. *Ako je C algebarski parcijalni operator zatvaranja, onda je njegov opseg zatvoren za unije lanaca.*

Teorema 1.24. *Opseg algebarskog operatora zatvaranja uređen inkluzijom jeste algebarski uređen skup.*

Teorema 1.25. *Za svaki algebarski poset S postoji algebarski parcijalni operator zatvaranja takav da je njegov opseg izomorfan sa posetom S.*

1.5 Generisanje operatora zatvaranja parcijalnim operatorima zatvaranja

U ovom delu pokazujemo kako proširiti parcijalni operator zatvaranja do operatora zatvaranja (to jest, kako definisati zatvorenje skupova sa nedefinisanim zatvorenjem na takav način da dobijeni operator bude operator zatvaranja). Ovo je uvek moguće i u opštem slučaju takav operator nije jedinstveno određen. Međutim, mi ćemo pokazati da, od svih operatora zatvaranja koji proširuju dati parcijalni operator zatvaranja C na skupu S, postoje dva, označimo ih sa $C^{\ominus}$ i $C^{\oplus}$, koje možemo smatrati „najmanjim“ i „najvećim“, u sledećem smislu: ako je K proizvoljan operator zatvaranja na S takav da $K(X) = C(X)$ kad god je $C(X)$ definisano, onda za sve X važi $C^{\ominus}(X) \subseteq K(X) \subseteq C^{\oplus}(X)$.

Neka je C parcijalni operator zatvaranja na skupu S. Najpre definišemo $C^{\oplus} : \mathcal{P}(S) \to \mathcal{P}(S)$ na sledeći način:

$$C^{\oplus} = \bigcap\{C(X') \mid X' \supseteq X \text{ i } C(X') \text{ je definisano}\}.$$

(Primetimo, ako je familija sa desne strane jednakosti prazna, onda dobijamo $C^{\oplus}(X) = \bigcap \emptyset = S$.)

Teorema 1.26. *Za parcijalni operator zatvaranja C na skupu S, operator $C^{\oplus}$ je operator zatvaranja na S i važi $C^{\oplus}(X) = C(X)$ za sve X za koje je $C(X)$ definisano.*

Sledeća teorema nam daje „maksimalnost" operatora $C^{\oplus}$.

Teorema 1.27. *Neka je C parcijalni operator zatvaranja na skupu S i K proizvoljan operator zatvaranja na S takav da $K(X) = C(X)$, kad god je $C(X)$ definisano. Tada za sve $X \subseteq S$ važi $K(X) \subseteq C^{\oplus}(X)$.*

Vratimo se najavljenom „najmanjem" operatoru. Ako je C parcijalni operator zatvaranja na skupu S, najpre definišemo $D'_C : \mathcal{P}(S) \to \mathcal{P}(S)$ na sledeći način:

$$D'_C(X) := \bigcup\{C(X') \mid X' \subseteq X \text{ i } C(X') \text{ je definisano}\}.$$

Sada, neka

$$D_C^{(0)}(X) := X,$$

i, za svaki ordinal α,

$$D_C^{(\alpha+1)}(X) := D'_C(D_C^{(\alpha)}(X));$$

$$D_C^{(\alpha)}(X) := \bigcup_{\xi<\alpha} D_C^{(\xi)}(X), \text{ ako je } \alpha \text{ granični ordinal.}$$

Konačno, definišimo

$$C^{\ominus}(X) := \bigcup_{\alpha\in\mathbf{ON}} D_C^{(\alpha)}(X).$$

Primetimo, pošto, naravno, $D_C^{(\alpha)}(X)$ je uvek podskup skupa S, imamo da $C^{\ominus}(X)$ jeste skup (a ne prava klasa), koji je pritom podskup skupa S.

Teorema 1.28. *Za parcijalni operator zatvaranja C na skupu S, operator $C^{\ominus}$ je operator zatvaranja na S i važi $C^{\ominus}(X) = C(X)$ za sve X za koje je $C(X)$ definisano.*

Sada sledi „minimalnost“ operatora $C^{\ominus}$.

Teorema 1.29. *Neka je C parcijalni operator zatvaranja na skupu S i K proizvoljan operator zatvaranja na S takav da $K(X) = C(X)$ kad god je $C(X)$ definisano. Tada za sve $X \subseteq S$ važi $C^{\ominus}(X) \subseteq K(X)$.*

2 Poseti i parcijalni operatori zatvaranja

Sistem zatvaranja uređen inkluzijom je potpuna mreža i obrnuto, kolekcija glavnih ideala mreže je sistem zatvaranja, koji je, uređen inkluzijom, izomorfan sa datom mrežom. Ipak, sistem zatvaranja glavnih ideala nije jedini sistem zatvaranja koji je izomorfan datoj mreži.

Naš cilj u ovom delu rada jeste uspostaviti posebnu vezu među familijama skupova, parcijalnih operatora i uređenih skupova. Ova veza je analogna (koliko je to moguće) onoj između operatora zatvaranja, sistema zatvaranja i potpunih mreža.

2.1 Oštri parcijalni operatori zatvaranja

Za parcijalni operator zatvaranja C na skupu S kažemo da je **oštar** ako ispunjava sledeći uslov:

Pc_7: Neka $B \subseteq S$. Ako je $\bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C$, onda je $C(B)$ definisano i

$$C(B) = \bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\}. \tag{5.4}$$

Za parcijalni operator na skupu S koji ispunjava uslove Pc_1–Pc_4 i Pc_7 korisitmo i skraćenicu **SPCO** na S koja potiče od engleskog naziva *sharp partial closure operator.*

Sledi profinjenje teoreme iz [53].

Teorema 2.3. *Opseg parcijalnog operatora zatvaranja na skupu S je parcijalni sistem zatvaranja.*

Obratno, za svaki parcijalni sistem zatvaranja $\mathcal{F}$ na S postoji jedinstveni oštar parcijalni operator zatvaranja na S takav da je $\mathcal{F}$ njegov opseg.

Zbog prethodno pokazanog, jasno je da za dati parcijalni sistem zatvaranja $\mathcal{F}$ na S, postoji kolekcija parcijalnih operatora zatvaranja na S čiji je opseg $\mathcal{F}$, među kojima je, prema teoremi 2.3, samo jedan oštar. Dodatno, pomenuti operator je i maksimalan u sledećem smislu.

Propozicija 2.4. *Neka je $\mathcal{F}$ parcijalni sistem zatvaranja na skupu S. Oštar parcijalni operator zatvaranja ima najveći strogi domen među svim parcijalnim operatorima zatvaranja čiji je opseg $\mathcal{F}$. Dodatno, ako je D parcijalni operator zatvaranja i C oštar parcijalni operator zatvaranja sa istim strogim domenom, onda $C(A) = D(A)$, za sve $A \subseteq S$ za koje je D definisano.*

Oštar parcijalni operator zatvaranja je prirodno uopštenje operatora zatvaranja, kao što sledi.

Teorema 2.5. *Ako opseg $\mathcal{F}$ oštrog parcijalnog operatora zatvaranja C na skupu S čini potpunu mrežu u odnosu na inkluziju, onda je C funkcija.*

Obratno, ako je C operator zatvaranja na S, onda je oštar.

Svaku nepraznu familiju podskupova skupa S možemo dopuniti tako da dobijemo parcijalni sistem zatvaranja. Jasno, dodavanjem svih singltona elemenata iz S dobijamo parcijalni sistem zatvaranja, no tim dodavanjem se može desiti da nećemo očuvati centralizovane preseke. Stoga, uvodimo sledeće proširenje.

Za proizvoljnu familiju $\mathcal{F}$ podskupova skupa S proširenje $\widehat{\mathcal{F}} \subseteq \mathcal{P}(S)$ definišemo na sledeći način:

$$\widehat{\mathcal{F}} := \mathcal{F} \cup \{ \bigcap_{x \in Y} Y \in \mathcal{F} \mid x \in S \}.$$

Sledeća propozicija je direktna posledica definicije $\widehat{\mathcal{F}}$.

Propozicija 2.6. *Za proizvoljnu nepraznu kolekciju $\mathcal{F}$ podskupova skupa S, proširenje $\widehat{\mathcal{F}}$ jeste parcijalni sistem zatvaranja na S koji očuvava sve preseke i centralizovane preseke koje postoje u $\mathcal{F}$.*

U nastavku navodimo specijalan tip parcijalnog sistema zatvaranja koji je izomorfan kolekciji svih glavnih ideala u posetu.

Kažemo da je parcijalni sitem zatvaranja $\mathcal{F}$ na nepraznom skupu S **glavni** ako

Ps_5: $\emptyset \notin \mathcal{F}$ i za sve $X \in \mathcal{F}$ važi

$$\left| X \setminus \bigcup \{ Y \in \mathcal{F} \mid Y \subsetneq X \} \right| = 1. \tag{5.5}$$

Motivacija za gornju definiciju jesu glavni ideali u uređenom skupu.

Propozicija 2.7. *Neka je $(S, \leqslant)$ uređen skup. Tada je familija $\{\downarrow x \mid x \in S\}$ glavni parcijalni sistem zatvaranja.*

Neka je $\mathcal{F}$ glavni parcijalni sistem zatvaranja na skupu S. Da bismo pokazali obrnutu vezu između glavnih parcijalnih sistema zatvaranja i glavnih ideala u uređenom skupu, uvodimo sledeće preslikavanje:

$G : \mathcal{F} \to S$ definisano sa

$$G(X) = x, \text{ gde } x \in X \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X\}. \tag{5.6}$$

Ovo preslikavanje je dobro definisano zbog definicije glavnog parcijalnog sistema zatvaranja.

Propozicija 2.8. *Ako je $\mathcal{F}$ glavni parcijalni sistem zatvaranja na skupu S, onda je preslikavanje $G : \mathcal{F} \to S$ definisano sa* (5.6) *bijekcija.*

Na S se može prirodno indukovati poredak, koristeći uvedenu bijekciju G i glavni parcijalni sitem zatvaranja $\mathcal{F}$ na S, na sledeći način: za sve $x, y \in S$,

$$x \leqslant y \text{ ako i samo ako } G^{-1}(x) \subseteq G^{-1}(y). \tag{5.7}$$

Direktno se proverava da je $\leqslant$ poredak na S. Dakle, kao posedicu propozicije 2.8, dobijamo sledeće tvrđenje.

Posledica 2.9. *Neka je $\mathcal{F}$ glavni parcijalni sistem zatvaranja na skupu S i $\leqslant$ poredak na S, definisan sa* (5.7)*. Tada funkcija G definisana sa* (2.3) *jeste izomorfizam iz $(\mathcal{F}, \subseteq)$ u $(S, \leqslant)$. Još više, kolekcija glavnih ideala u $(S, \leqslant)$ jeste $\mathcal{F}$.*

Takođe, možemo krenuti od uređenog skupa i preko glavnih ideala doći do parcijalnog sitema zatvaranja, koji indukuje polazni poredak.

Posledica 2.10. *Neka je $(S, \leqslant)$ uređen skup i $\mathcal{F}$ parcijalni sistem zatvaranja kojeg čine glavni ideali datog uređenog skupa. Tada, poredak na S definsan sa* (5.7) *se poklapa sa $\leqslant$.*

Konačno, uvodimo parcijalni operator zatvaranja koji odgovara glavnom parcijalnom sistemu zatvaranja.

Parcijalni operator zatvaranja C na S je **glavni** ako ispunjava sledeći uslov

Pc_8: Ako $X = C(X)$, tada postoji jedinstveno $x \in X$ takvo da $x \notin \bigcup\{Y \in \mathcal{F}_C \mid Y \subsetneq X\}$.

Jednostavno se pokazuje da su uslovi Pc_7 i Pc_8 nezavisni.

Veza između ovih pojmova može biti objašnjena na sledeći način.

Opseg glavnog operatora zatvaranja je glavni parcijalni sistem zatvaranja i oštar parcijalni operator zatvaranja koji odgovara glavnom parcijalnom sistemu zatvaranja kako je definisan u teoremi 2.3 jeste glavni.

Očigledno, prazan skup ne može biti zatvoren u odnosu na glavni parcijalni operator zatvaranja. Dodatno, imamo da se opseg glavnog parcijalnog operatora zatvaranja sastoji od zatvorenja singltona.

Propozicija 2.11. *Neka je C glavni parcijalni operator zatvaranja na skupu S. Ako $X \in \mathcal{F}_C$, onda postoji $x \in X$ takvo da $C(\{x\}) = X$.*

Naredna teorema je *teorema reprezentacije* uređenih skupova preko oštrih parcijalnih operatora zatvaranja i odgovarajućih parcijalnih sistema zatvaranja.

Teorema 2.12. *Neka je $(S, \leqslant)$ uređen skup. Parcijalno preslikavanje $C : \mathcal{P}(S) \to \mathcal{P}(S)$ definisano sa*

$$C(X) = \downarrow(\bigvee X), \text{ ako postoji } \bigvee X,$$

inače nije definisano, jeste glavni SPCO. Odgovarajući parcijalni sistem zatvaranja je glavni i izomorfan je sa S.

2.2 Parcijalni domeni zatvaranja

Dosad smo analizirali opsege različitih (parcijalnih) operatora zatvaranja, ali o strogim domenim parcijalnih operatora nije mnogo rečeno. U ovom delu cilj je okarakterisati podfamilije familije $\mathcal{P}(S)$, za dati skup S, koje mogu biti strogi domen nekog parcijalnog operatora zatvaranja na S.

Za famliju $\mathcal{B}$ podskupova nepraznog skupa S razmotrimo sledeći uslov:

B_1: za svako $x \in S$ važi $\{x\} \in \mathcal{B}$.

Teorema 2.13. *Za parcijalni operator zatvaranja C na skupu S kolekcija $Dom(C)$ ispunjava uslov B_1.*

Obratno, ako je B proizvoljna kolekcija podskupova skupa S koja ispunjava uslov B_1, onda postoji parcijalni operator zatvaranja na skupu S, takav da $Dom(C) = \mathcal{B}$.

Dalje, uvedimo sledeći uslov:

B_2: za svako $X \subseteq S$ važi:

$$\text{ako } \bigcap\{B \in \mathcal{B} \mid X \subseteq B\} \in \mathcal{B}, \text{ onda } X \in \mathcal{B}.$$

Teorema 2.14. *Za oštar parcijalni operator zatvaranja C na skupu S, kolekcija $Dom(C)$ ispunjava uslova B_1 i B_2.*

Obratno, ako je B proizvoljna kolekcija podskupova skupa S koja ispunjava uslove B_1 i B_2, onda postoji oštar parcijalni operator zatvaranja na skupu S takav da $Dom(C) = \mathcal{B}$.

3 p-matroidi, geometrijski i polumodularni poseti

U ovom delu rada uopštavamo pojam matroida na pojam parcijalnih matroida tako što operatore zatvaranja iz definicije matroida menjamo parcijalnim operatorima zatvaranja. Takođe uopštavamo pojam geometrijskih mreža na pojam geometrijskih uređenih skupova. Zatim pokazujemo da, kao i kod matroida i geometrijskih mreža, postoji korespondencija istog tipa među parcijalnim matroidima i geometrijskim uređenim skupovima. Na kraju još uopštavamo pojam polumodularnih mreža na pojam polumodularnih uređenih skupova na takav način da, kao i u slučaju mreža, uređen skup jeste geometrijski tada i samo tada kad je atomarno generisan i polumodularan.

3.1 Geometrijske mreže

Mreža L je **polumodularna** (nagore) ako za sve $x, y \in L$

$$\text{iz } x \wedge y \prec x \text{ sledi } y \prec x \vee y, \tag{5.8}$$

dok je **polumodularna nadole** ako za sve $x, y \in L$ važi

$$\text{iz } y \prec x \vee y \text{ sledi } x \wedge y \prec x. \tag{5.9}$$

Teorema 3.1. *Mreža L je polumodularna ako i samo ako za sve $x, y, z \in L$ važi:*

$$\textit{iz } x \prec y \textit{ sledi } x \vee z \preceq y \vee z. \tag{5.10}$$

Teorema 3.2. *Neka je mreža L takva da su svi njeni lanci između dva proizvoljna elementa konačni. Tada je L polumodularna ako i samo ako važi:*

$$\textit{iz } x \wedge y \prec x \textit{ i } x \wedge y \prec y \textit{ sledi } x \prec x \vee y \textit{ i } y \prec x \vee y. \tag{5.11}$$

Dužina konačnog lanca se definiše kao broj elemenata u lancu umanjen za jedan a dužina uređenog skupa (pa i mreže) kao najveći broj među dužinama njegovih lanaca (ukoliko takav broj postoji) i tada kažemo da je **konačne dužine**.

Teorema 3.3. *Ako su u polumodularnoj mreži L svi maksimalni lanci između dva elementa konačni, onda su oni iste dužine.*

Kao posledicu prethodnog tvrđenja dobijamo sledeće.

Posledica 3.4. *Ako u polumodularnoj mreži L postoji konačan maksimalan lanac, onda su svi maksimalni lanci u mreži L iste dužine.*

Elemente mreže koji pokrivaju najmanji elemenat nazivamo **atomi**. Mreža je **atomarno generisana** ako se u njoj svaki elemenat različit od najmanjeg može predstaviti kao supremum atoma.

Mreža koja je polumodularna, atomarno generisana i čiji su svi lanci konačni se naziva **geometrijska**.

Jasno je da je svaka geometrijska mreža konačne dužine (zbog tvrđenja 3.4). Na slici 3 (strana 52) u prvom redu su prikazani dijagrami svih geometrijskih mreža sa najviše 3 atoma a u drugom redu je jedan primer geometrijske mreže sa četiri i jedan primer sa pet atoma.

Teorema 3.6. *Neka je L polumodularna mreža konačne dužine. Tada, ako je $a \in L$ atom i $x \in L$ proizvoljan element, onda važi ili $a \leqslant x$ ili $x \prec x \vee a$.*

Ovo tvrđenje nam pomaže da izvedemo još neke karakterizacije geometrijskih mreža, koje navodimo u nastavku.

Teorema 3.7. *Mreža L konačne dužine je geometrijska ako i samo ako za sve $x, y \in L$ važi*

$$x \prec y \text{ ako i samo ako postoji atom } a \text{ takav da } a \nleqslant x \text{ i } y = x \vee a. \tag{5.12}$$

Teorema 3.8. *Mreža konačne dužine L je geometrijska ako i samo ako je atomarno generisana i za proizvoljne atome a i b i element x važi:*

$$\text{iz } a < x \vee b \text{ i } a \nleqslant x \text{ sledi } b < x \vee a. \tag{5.13}$$

(Uslov (5.13) se naziva i **zakon zamene za geometrijske mreže***.)*

3.2 Matroidi

Skup A sa operatorom zatvaranja $\bar{\cdot} : \mathcal{P}(A) \to \mathcal{P}(A)$, u oznaci $M(A)$, nazivamo **matroid** na A ako za sve $X \subseteq A$ i za sve $x, y \in A$ važi

M_1 : iz $x \notin \overline{X}$ i $x \in \overline{X \cup \{y\}}$ sledi $y \in \overline{X \cup \{x\}}$; (*aksiom zamene*)

M_2 : postoji konačan Y tako da je $Y \subseteq X$ i $\overline{Y} = \overline{X}$. (*aksiom konačne baze*)

Matroid je **prost** (ili **kombinatorna geometrija** ili samo **geometrija**) ako važi još i:

M_3 : $\overline{\emptyset} = \emptyset$ i za sve $x \in A$ je $\overline{\{x\}} = \{x\}$.

Zatvorene skupove matroida nazivamo **potprostori** a opseg označavamo sa $L_M(A)$ i nazivamo ga **mreža potprostora** matroida $M(A)$.

Teorema 3.9. *Za proizvoljne elemente U i V mreže potprostora $L_M(A)$ matroida $M(A)$ na skupu A važi:*

$$U \prec V \text{ ako i samo ako } V = \overline{U \cup \{v\}}, \text{ za neko } v \notin U.$$

Teorema 3.10. *Mreža potprostora prostog matroida je geometrijska.*

Obratno, ako je L geometrijska mreža i A skup svih njenih atoma, onda je preslikavanje $\bar{\cdot}$, definisano na $\mathcal{P}(A)$ sa $\overline{X} = \{a \in A \mid a \leqslant \bigvee X\}$, operator zatvaranja u odnosu na koji je A prost matroid, a njegova mreža potprostora $L_M(A)$ je izomorfna sa L.

Prethodna teorema ne važi samo za proste matroide već i za sve konačne matroide, s obzirom na to da svakom konačnom matroidu odgovara prost matroid.

Teorema 3.11. *Za svaki konačan matroid postoji prost matroid takav da su odgovarajuće mreže potprostora izomorfne.*

Matroidi se javljaju u različitim oblastima matematike i imaju nekoliko, ekvivalentnih, definicija. Obično se definišu kao konačni, te ćemo u ovom delu raditi samo sa konačnim skupovima.

Prva definicija je inspirisana linearno nezavisnim vektorima i njihovim osobinama.

- (I-definicija) Matroid $(E, \mathcal{I})$ je konačan skup E sa nepraznom familijom $\mathcal{I}$ podskupova skupa E, koje nazivamo **nezavisni skupovi**, sa sledećim osobinama:

  I_1: Svaki podskup nezavisnog skupa je nezavisan.

  I_2: Ako su X_1 i X_2 nezavisni skupovi i $|X_1| < |X_2|$, onda je za neko $x \in X_2 \setminus X_1$ skup $X_1 \cup \{x\}$ nezavisan.

U prethodnoj definiciji uslov I_2 može biti zamenjen uslovom

I_2': Ako je $S \subseteq E$, onda svi maksimalni nezavisni podskupovi skupa S imaju isti broj elemenata.

U teoriji grafova se dolazi do iste definicije: E je skup grana konačnog (neusmerenog) grafa i nezavisni podkupovi skupa E su aciklični skupovi grana (za skup grana kažemo da je acikličan ako indukovan graf ne sadrži konture).

Sledeću definiciju matroida dobijamo od prethodne, ali koristeći definiciju ranga. Za skup $S \subseteq E$ **rang** skupa S, u oznaci $r(S)$, jeste kardinalnost maksimalnog nezavisnog podskupa skupa S; rang možemo definisati na ovakav način zbog I_2'.

Dakle, imamo sledeću definiciju matroida.

- (R-definicija) Matroid (E, r) je konačan skup E sa funkcijom $r : \mathcal{P}(E) \to \mathbb{Z}$ koja zadovoljava uslove:

  R_1: Za svaki $S \subseteq E$ važi $0 \leqslant r(S) \leqslant |S|$.

  R_2: Ako je $S \subseteq T$, onda $r(S) \leqslant r(T)$.

  R_3: Za sve $S, T \subseteq E$ važi $r(S) + r(T) \geqslant r(S \cup T) + r(S \cap T)$.

Još jedna definicija je inspirisana teorijom grafova. Primetimo da su konture u grafu minimalni skupovi grana koji su nezavisni (u smislu aksioma I_1 i I_2). Uopštenjem ovoga dobijamo sledeću definiciju.

- (S-definicija) Matroid $(E, \mathcal{C})$ je konačan skup E sa familijom $\mathcal{C}$ nepraznih podskupova skupa E, koje zovemo **ciklusi**, i koji imaju sledeće osobine:

  S_1: Nijedan podskup ciklusa nije ciklus.

  S_2: Ako dva različita ciklusa X_1 i X_2 sadrže x, onda $(X_1 \cup X_2) \setminus \{x\}$ sadrži ciklus.

Dalje, navodimo definiciju matroida inspirisanu pojmom baze vektorskog prostora u linearnoj algebri.

- (B-definicija) Matroid je konačan skup E sa nepraznom familijom njegovih podskupova $\mathcal{B}$, koje nazivamo **baze**, sa narednim osobinama:

  B_1: Nijedna baza nije sadržana u drugoj bazi.

  B_2: Ako su X_1 i X_2 baze, onda za svako $x \in X_1 \setminus X_2$ postoji neko $y \in X_2 \setminus X_1$ takvo da je $(X_1 \setminus \{x\}) \cup \{y\}$ baza.

Sve veze između pomenutih definicija matroida i definicije matroida koju koristimo u ostalim odeljcima rada, koju ćemo sada radi preglednosti nazvati C-definicija, prikazane su na slici 4 (strana 67). Možemo zaključiti da sve one daju istu strukturu.

3.3 Geometrijski uređeni skupovi

Naša definicija geometrijskog uređenog skupa se zasniva na teoremi 3.8. Zapravo, ona je prirodno uopštenje uslova te teoreme.

Najpre pokazujemo neke osobine parcijalnih matroida i geometrijskih poseta, a zatim pokazujemo da, kao i kod matroida i geometrijskih mreža, postoji korespondencija istog tipa među parcijalnim matroidima i geometrijskim uređenim skupovima.

Svi skupovi koje razmatramo odavde do kraja glave su konačni.

Neka je dat uređeni skup $(P, \leqslant)$. Ako P ima najmanji element, elemente koji pokrivaju najmanji nazivamo **atomi** (ovo je isto kao definicija atoma u mreži); ako P nema najmanji element, onda su **atomi** svi minimalni elementi u P. Skup svih atoma označavamo sa A_P ili samo A (ukoliko je jasno iz konteksta iz kojeg uređenog skupa su atomi). Kažemo da je uređeni skup **atomarno generisan** ako je svaki njegov element različit od najmanjeg supremum nekog skupa atoma.

Sada definišemo geometrijski uređeni skup.

Uređeni skup $(P, \leqslant)$ nazivamo **geometrijski** ako i samo ako je P atomarno generisan i ako za sve atome a i b i svako $x \in P$ važi:

$$\text{ako je } x \vee b \text{ definisano, } a < x \vee b \text{ i } a \nleqslant x, \text{ onda } x \vee a \text{ postoji i } b < x \vee a. \tag{5.14}$$

Sledeća teorema daje još jednu, ekvivalentnu definiciju geometrijskog poseta.

Teorema 3.22. *Uređeni skup $(P, \leqslant)$ je geometrijski ako i samo ako je atomarno generisan i za sve $x, y \in P$ takve da $x \nleqslant y$, važi:*

$$\textit{ako postoji } a \in A_P \textit{ takvo da je } y \vee a \textit{ definisano i } x \leqslant y \vee a, \tag{5.15}$$

$$\textit{onda } x \vee y \textit{ postoji i } y \prec x \vee y.$$

3.4 Parcijalni matroidi

Parcijalni matroid (ili skraćeno p**-matroid**) definišemo kao uređeni par (E, C), gde je E neprazan skup a C oštar parcijalni operator zatvaranja na E koji ispunjava sledeće uslove:

(M) ako su $C(X)$ i $C(X \cup \{x\})$ definisani, onda $y \notin C(X)$ i $y \in C(X \cup \{x\})$ povlači da je $C(X \cup \{y\})$ definisano i $x \in C(X \cup \{y\})$;

(P) $C(\{x\}) = \{x\}$, za sve $x \in E$.

Naredne dve teoreme pokazuju da postoji direktno uopštenje korespondencije između geometrijskih mreža i matroida na korespondenciju između geometrijskih poseta i parcijalnih matroida.

Teorema 3.26. *Opseg p-matroida u odnosu na inkluziju je geometrijski uređen skup.*

Teorema 3.28. *Za svaki geometrijski uređen skup $(P, \leqslant)$ postoji p-matroid čiji je opseg izomorfan sa $(P, \leqslant)$.*

3.5 Polumodularni uređeni skupovi

Podsetimo se da je konačna mreža geometrijska ako i samo ako je atomarno generisana i polumodularna. U ovom odeljku cilj nam je da uopštimo pojam polumodularnosti na uređene skupove na takav način da ista ekvivalencija važi i za posete.

Naredna definicija, iako izgleda pomalo neprirodno, postiže taj cilj.

- Uređeni skup $(P, \leqslant)$ koji ima najmanji element je **polumodularan** ako za sve $x, y \in P$ važi sledeće:

$$\begin{gathered}\text{ako } x \wedge y \prec x, \text{ onda} \\ y \prec x \vee y \text{ ili } (P \text{ nije } \vee\text{-polumreža i} \\ \text{ne postoji atom } a \text{ takav da } x \leqslant y \vee a).\end{gathered} \tag{5.16}$$

- Uređeni skup $(P, \leqslant)$ koji nema najmanji element je **polumodularan** ako je uređeni skup $(P_0, \leqslant)$ polumodularan, gde je $(P_0, \leqslant)$ uređeni skup dobijen dodavanjem najmanjeg elementa uređenom skupu $(P, \leqslant)$.

Polumodularnost za uređene skupove je uopštenje polumodularnosti za mreže, to jest, mreža je polumodularna kao mreža ako i samo ako je polumodularna kao uređeni skup.

Naredne dve teoreme su suština ovog odeljka.

Teorema 3.30. *Svaki atomarno generisan i polumodularan uređeni skup je geometrijski.*

Teorema 3.31. *Svaki geometrijski uređeni skup je polumodularan.*

Direktna posledica prethodne dve teoreme je sledeće tvrđenje.

Posledica 3.32. *Uređeni skup je geometrijski ako i samo ako je atomarno generisan i polumodularan.*

4 Primene

Operatori zatvaranja imaju široku primenu u nauci. Mi ćemo se koncentrisati na analiziranje velikih količina podataka. Ovde se iz podataka o objektima sa puno osobina izvlače povezanosti njihovih karakteristika. Jedan pristup ovom problemu jeste preko implikacija, koje su predmet izučavanja formalne koncept analize. One se takođe pojavljuju i mnogim drugim oblastima kao što su analiza podatka, data-mining, relacione baze podataka. Ove implikacije predstavljaju pravila koja govore „objekat sa atributima iz skupa X ima atribut x“. Teško je zapisati sve implikacije koje važe na jednom skupu podataka, ali ako možemo naći neku vrstu baze ovih implikacija, onda ih možemo sve generisati pomoću te baze. Operatori zatvaranja se koriste u ovom delu problema.

S obzirom da je većina rezultata u ovoj glavi dokazana za konačne skupove, smatraćemo da radimo samo sa konačnim skupovima.

4.1 Koncepti, implikacije i baze

Formalni kontekst $\mathbb{K} = (G, M, I)$ se sastoji od dva skupa G i M i relacije I između G i M. Elemente skupa G nazivamo **objekti** a elemente skupa M **atributi** konteksta. Ako element konteksta $g \in G$ ima osobinu $m \in M$, onda to zapisujemo kao $(g, m) \in I$. Ovu relaciju nazivamo **relacija incidencije** konteksta.

Skup atributa zajedničkih za sve objekte skupa $A \subseteq G$ zapisujemo kao A', to jest

$$A' = \{m \in M \mid (g, m) \in I \text{ za sve } g \in A\}.$$

Slično, skup objekata koji imaju sve atribute u skupu $B \subseteq M$ jeste

$$B' = \{g \in G \mid (g, m) \in I \text{ za sve } m \in B\}.$$

Formalni koncept konteksta (G, M, I) je uređeni par (A, B) takav da $A \subseteq G$, $B \subseteq M$, $A' = B$ i $A = B'$. Skup S nazivamo **ekstent** a B **intent** koncepta (A, B). Skup svih koncepata konteksta (G, M, I) označavamo $\mathcal{B}(G, M, I)$.

Teorema 4.1. *Za koncept (G, M, I) i $A, A_1, A_2 \subseteq G$, $B, B_1, B_2 \subseteq M$ važi:*

1. *iz $A_1 \subseteq A_2$ sledi $A_2' \subseteq A_1'$, iz $B_1 \subseteq B_2$ sledi $B_2' \subseteq B_1'$;*

2. *$A \subseteq A''$, $B \subseteq B''$;*

3. $A' = A'''$, $B' = B'''$;

4. $A \subseteq B' \Leftrightarrow B \subseteq A' \Leftrightarrow A \times B \subseteq I$.

Prethodna teorema daje nam sledeću posledicu.

Posledica 4.2. *Ako je (G, M, I) kontekst, onda je operator $''$ operator zatvaranja na oba skupa G i M.*

Za svaki skup $A \subseteq G$, (A'', A') jeste koncept a A'' je najmanji ekstent koji sadrži A. Ovo znači da je $A \subseteq G$ ekstent ako i samo ako $A = A''$. Isto važi i za intente.

Na skupu $\mathcal{B}(G, M, I)$ se definiše poredak $\leqslant$: $(A_1, B_1) \leqslant (A_2, B_2)$ ako i samo ako $A_1 \subseteq A_2$ (ako i samo ako $B_2 \subseteq B_1$). Na ovakav način se dobija mreža i zato $\mathcal{B}(G, M, I)$ nazivamo **mreža koncepata** konteksta (G, M, I).

Teorema 4.3. *Mreža koncepata $\mathcal{B}(G, M, I)$ jeste kompletna mreža u kojoj su infimum i supremum definisani sa:*

$$\bigwedge_{k \in K} (A_k, B_k) = \left(\bigcap_{k \in K} A_k, \left(\bigcup_{k \in K} B_k \right)'' \right),$$

$$\bigvee_{k \in K} (A_k, B_k) = \left(\left(\bigcup_{k \in K} A_k \right)'', \bigcap_{k \in K} B_k \right).$$

Implikacija $X \to Y$ je uređeni par (X, Y) podskupova skupa M. Stoga, skup implikacija jeste binarna relacija na $\mathcal{P}(M)$; zovemo ga **implikacioni sistem**. Kada je skup Y u implikaciji $X \to Y$ singlton, onda pišemo $X \to y$, ova vrsta implikacije se naziva **unitarna implikacija**. Stoga je skup unitarnih implikacija, **unitarni implikacioni sistem** (UIS), relacija između $\mathcal{P}(M)$ i M.

Implikacija $X \to Y$ **važi na skupu** $T \subseteq M$ ako je $X \nsubseteq T$ ili $Y \subseteq T$. Skup implikacija Σ **važi na skupu** T ako svaka implikacija iz Σ važi na T. $X \to Y$ **važi u kontekstu** (G, M, I) ako važi u svakom intentu datog konteksta. Takođe kažemo da je X **premisa** od Y ili da je $X \to Y$ **implikacija konteksta** (G, M, I).

Lako se vidi da važi sledeća teorema.

Teorema 4.4. *Implikacija $X \to Y$ važi u (G, M, I) ako i samo ako $Y \subseteq X''$. Tada takođe važi u svim intentima.*

Implikacija $X \to Y$ **sledi** iz skupa implikacija Σ ako važi u svakom podskupu atributa na kojem važi i Σ. Skup implikacija Σ je **zatvoren** (potpun implikacioni sistem) ako je svaka implikacija koja sledi iz Σ takođe u Σ.

Za implikacioni sistem $\Sigma = \{X_1 \to Y_1, X_2 \to Y_2, \ldots, X_m \to Y_m\}$ definišemo **veličinu** $s(\Sigma)$ kao

$$s(\Sigma) = \sum_{i=1}^{m} (|X_i| + |Y_i|).$$

Implikacioni sistem možemo uvek zameniti unitarnim implikacionim sistemom. Implikacija $X \to Y$ može biti zamenjena skupom unitarnih implikacija $\{X \to y \mid y \in Y\}$. Zbog toga u nastavku radimo sa unitarnim implikacijama.

Implikacija $X \to y$ **važi** na podskupu $A \subseteq M$ ako $X \subseteq A$ implicira $y \in A$. Ako je Σ skup implikacija, $A \subseteq M$ je **Σ-zatvoren** kada sve implikacije iz Σ (to jest Σ-implikacije) važe na A. Skup svih Σ-zatvorenih skupova označavamo sa $\mathcal{F}_\Sigma$. Lako se vidi da $\mathcal{F}_\Sigma$ čini sistem zatvaranja na skupu M. Odgovarajući operator zatvaranja označavamo sa $C_{\mathcal{F}_\Sigma}$.

Sistemu Σ možemo pridružiti operator C_Σ: za $X \subseteq M$ neka

$$\pi_\Sigma(X) = X \cup \bigcup \{b \in M \mid A \subseteq X \text{ i } A \to b \in \Sigma\}$$

i

$$\pi_\Sigma^n(X) = \pi_\Sigma^{n-1}(X) \cup \bigcup \{b \in M \mid A \subseteq \pi_\Sigma^{n-1}(X) \text{ i } A \to b \in \Sigma\};$$

tada

$$C_\Sigma(X) = \pi_\Sigma(X) \cup \pi_\Sigma^2(X) \cup \pi_\Sigma^3(X) \cup \ldots$$

Kako je M konačan, postoji $n \in \mathbb{N}$ takvo da $\pi_\Sigma^n(X) = \pi_\Sigma^{n+1}(X)$, stoga imamo $C_\Sigma(X) = \pi_\Sigma^n(X)$. Važi još više: $C_\Sigma = C_{\mathcal{F}_\Sigma}$.

Ako krenemo od operatora zatvaranja C na skupu M, onda se zatvoreni skupovi poklapaju sa Σ-zatvorenim skupovima sledećeg UIS:

$$\Sigma_C = \{X \to y \mid y \in M,\ X \subseteq M \text{ i } y \in C(X)\}.$$

Ovaj unitarni implikacioni sistem ispunjava sledeća svojstva:

F_1: iz $X \subseteq M$ i $x \in X$ sledi $X \to x$;

F_2: za svako $y \in M$ i sve $X, Y \subseteq M$

$$\text{iz } X \to y \text{ i } (\forall x \in X)(Y \to x) \text{ sledi } Y \to y.$$

Svaki UIS koji zadovoljava osobine F_1 i F_2 nazivamo **potpun**. Potpuni unitarni implikacioni sistemi su u jedan-na-jedan korespondenciji sa operatorima zatvaranja.

Svaki UIS Σ je sadržan u nekom potpunom UIS. To je UIS Σ_C, gde $C = C_\Sigma$ i ovo je najmanji potpun UIS koji sadrži Σ. Drugi način za generisanje najmanjeg potpunog UIS koji sadrži Σ jeste rekurzivna primena pravila F_1 i F_2. Tada Σ nazivamo **generišući sistem** za Σ_C. Za unitarne implikacione sisteme Σ_1 i Σ_2 kažemo da su **ekvivalentni**, ako generišu isti potpun UIS.

UIS Σ je **minimalan** ili **neredundantan** ako $\Sigma \setminus \{X \to y\}$ nije ekvivalentno sa Σ, za sve $X \to y \in \Sigma$. Nazivamo ga **baza**. Ako za sve Σ' ekvivalentne sa Σ važi $|\Sigma| \leqslant |\Sigma'|$, onda je Σ **minimalna baza**. Ako za svako UIS Σ' koje je ekvivalentno sa Σ važi $s(\Sigma) \leqslant s(\Sigma')$, onda je Σ **optimalno**. Ako za sve $X \subseteq M$ važi $C_\Sigma(X) = \pi_\Sigma(X)$, onda Σ je **direktna**. UIS nazivamo **pravi** ako ne sadrži trivijalne implikacije; to su implikacije $X \to x$ gde $x \in X$. Od svakog UIS možemo dobiti ekvivalentan UIS tako što izostavimo trivijalne implikacije. Zbog toga u nastavku ćemo podrazumevati da radimo sa pravim unitarnim implikacionim sistemima.

Postoje više vrsta generišućih sitema za unitarne implikacione sisteme. Mi ćemo pomenuti samo nekoliko vrsta.

UIS Σ_{do} je **direktno-optimalan** ako je direktan i ako $s(\Sigma_{do}) \leqslant s(\Sigma)$, za svaki direktan UIS Σ koji je ekvivalentan sa Σ_{do}.

Može se pokazati da je ovakav sistem jedinstven i da se za svaki UIS može odrediti ekvivalentan direktno-optimalan UIS.

Levo-minimalna baza Σ_{lm} je:

$$\Sigma_{lm} = \{X \to y \mid y \in C(X) \setminus X \text{ i za sve } Y \subsetneq X,\, y \notin C(Y)\}.$$

Ova vrsta baze jeste restrikcija potpunog implikacionog sistema na implikacije čije su premise minimalne kardinalnosti.

Teorema 4.5. *Neka je C operator zatvaranja na skupu M. Tada se direktno-optimalna i levo-minimalna baza poklapaju.*

Stoga, ovakvu vrstu baze nazivamo **kanonička direktna baza** i označavamo je sa Σ_{cd}.

4.2 Uopštenje na parcijalne unitarne implikacione sisteme

Ovde dajemo uopštenje unitarnih implikacionih sistema. **Potpun parcijalni unitarni implikacioni sistem** Σ na skupu M ispunjava sledeće uslove:

PF_1: ako postoji y takvo da $X \to y$, onda za sve $x \in X$ važi $X \to x$;

PF_2: za svako $y \in M$ i sve $X, Y \subseteq M$

$$X \to y \text{ i } (\forall x \in X)(Y \to x) \text{ povlači } Y \to y;$$

PF_3: ako za $X \subseteq M$ postoji $z \in M$ takvo da $X \to z$, onda postoji $x \in M$ takvo da $\{y \in M \mid X \to y\} \to x$;

PF_4: $x \to x$, za sve $x \in M$.

Postoji korespondencija između potpunih parcijalnih UIS i parcijalnih operatora zatvaranja.

Teorema 4.6. *Svaki potpun parcijalni UIS na skupu M definiše parcijalni operator zatvaranja na istom skupu.*

Teorema 4.7. *Ako je C parcijalni operator zatvaranja na M, onda*

$$\Sigma_C := \{X \to y \mid y \in M,\ X \subseteq M,\ C(X) \textit{ je deinisano i } y \in C(X)\}$$

jeste potpun parcijalni UIS na M.

5 Zaključak

Ova teza daje uvid u veze između operatora zatvaranja, sistema zatvaranja i potpunih mreža. Ove veze su izučavane puno, dok su njihova uopštenja retko razmatrana. Zato ova teza proširuje znanje o vezama između parcijalnih operatora zatvaranja, parcijalnih sistema zatvaranja i uređenih skupova. Povezanost je pojačana u odnosu na to kako je bilo dosad. Dodavanjem i analiziranjem novih aksioma stečeni su bolji rezultati od onih poznatih koji povezuju kolekcije skupova, operatore zatvaranja i uređene skupove. U cilju postizanja jedinstvene korespondencije između parcijalnih sistema zatvaranja i parcijalnih operatora zatvaranja uveli smo novi tip parcijalnog operatora zatvaranja: oštar parcijalni operator zatvaranja. Ovo je parcijalni operator zatvaranja na partitivnom skupu koji ispunjava aksiome analogne aksiomama zatvaranja zajedno sa nekoliko dodatnih aksioma. Pokazali smo jedinstvenost takvog parcijalnog operatora koji odgovara datom parcijalnom sistemu zatvaranja. Dalje uvodimo parcijalni sistem zatvaranja, koji odgovara glavnim idealima u uređenom skupu. Takođe smo formulisali i dokazali teoremu reprezentacije za uređene skupove u odnosu na uvedene parcijalne operatore zatvaranja i parcijalne sisteme zatvaranja.

Posebna pažnja je posvećena kolekcijama skupova povezanih sa konačnim geometrijama, kao što su matroidi. To su objekti sa velikim brojem međusobno ekvivalentnih definicija, iz različitih oblasti matematike. Glavni rezultat ove teze jeste uopštenje njihovih aksioma i uspostavljanje veza između ovih objekata i uređenih skupova. Posle prelaska sa operatora zatvaranja na parcijalne operatore zatvaranja, uvedeli smo uopštenje matroida: p-matroide, kao i uopštenje geometrijskih mreža: geometrijski uređene skupove. Ova uopštenja potiču od odnosa između matroida i geometrijskih mreža. Dalje smo istraživali i definisali analogon pojma polumodularnosti za uređene skupove koji nisu mreže.

Odnosi između sistema zatvaranja, operatora zatvaranja i potpunih mreža su u osnovama teorije uređenih skupova i teorije mreža. Sistemi zatvaranja i operatori zatvaranja su među osnovnim alatima koji se koriste u istraživanjima u uređenim skupovima, topologiji, univerzalnoj algebri, logici, i tako dalje. Sve ovo vodi do zaključka da uopštenja ovih pojmova, koja su razmatrana u ovoj tezi, imaju veliki potencijal primene u različitim oblastima matematike. U ovoj tezi smo prikazali primer kako mogu biti primenjena na implikacionim sistemima, koji se u velikom meri koriste u analizi velikih podataka. Nadamo se da ova primena parcijalnih operatora može biti znatno razvijena, i ovo je jedan od pravaca u kojem planiramo da nastavimo da radimo.

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Anna Slivková je rođena 21. 11. 1986. u Pančevu. Odrasla je u Padini, gde je i završila Osnovnu školu „Maršal Tito“ sa Vukovom diplomom kao najbolja u svojoj generaciji. Te iste godine, 2001, upisala se u Gimnaziju „Mihajlo Pupin“ u Kovačici. Gimnaziju je završila 2005. godine i ponovo je bila đak generacije, kao i nosilac Vukove diplome. Na Prirodno-matematički fakultet u Novom Sadu, Departman za matematiku i informatiku, upisala se na smer diplomirani matematičar. Diplomirala je 2010. godine odbranivši diplomski rad pod nazivom *Problem crva.* Iste godine upisala se na master studije na istom fakultetu. Master diplomu je stekla odbranom master rada *O nekim klasama $\mathcal{I}$-ultrafiltera na ω*. Tokom studija je stekla nekoliko nagrada za postignut uspeh a takođe je bila stipendista Fonda za mlade talente i Ministarstva prosvete. Na doktorskim studijama je položila sve ispite i time stekla uslov za odbranu doktorske disertacije.

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Novi Sad, mart 2018 Anna Slivková

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Izvod: U ovoj tezi uopštavamo dobro poznate veze između operatora zatvaranja, sistema zatvaranja i potpunih mreža. Uvodimo posebnu vrstu parcijalnog operatora zatvaranja, koji nazivamo oštar parcijalni operator zatvaranja, i pokazujemo da svaki oštar parcijalni operator zatvaranja jedinstveno korespondira parcijalnom sistemu zatvaranja. Dalje uvodimo posebnu vrstu parcijalnog sistema zatvaranja, nazvan glavni parcijalni sistem zatvaranja, a zatim dokazujemo teoremu reprezentacije za posete u odnosu na uvedene parcijalne operatore zatvaranja i parcijalne sisteme zatvaranja.
Dalje, s obzirom na dobro poznatu vezu između matroida i geometrijskih mreža, a budući da se pojam matroida može na prirodan način uopštiti na parcijalne matroide (definišući ih preko parcijalnih operatora zatvaranja umesto preko operatora zatvaranja), definišemo geometrijske uređene skupove i pokazujemo da su povezani sa parcijalnim matroidima na isti način kao što su povezani i matroidi i geometrijske mreže. Osim toga, definišemo polumodularne uređene skupove i pokazujemo da su oni zaista uopštenje polumodularnih mreža i da ista veza postoji između polumodularnih i geometrijskih poseta kao što imamo između polumodularnih i geometrijskih mreža.
Konačno, konstatujemo da definisani pojmovi mogu biti primenjeni na implikacione sisteme, koji imaju veliku primenu u realnom svetu, posebno u analizi velikih podataka.
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Abstract: In this thesis we generalize the well-known connections between closure operators, closure systems and complete lattices. We introduce a special kind of a partial closure operator, named sharp partial closure operator, and show that each sharp partial closure operator uniquely corresponds to a partial closure system. We further introduce a special kind of a partial closure system, called principal partial closure system, and then prove the representation theorem for ordered sets with respect to the introduced partial closure operators and partial closure systems.

Further, motivated by a well-known connection between matroids and geometric lattices, given that the notion of matroids can be naturally generalized to partial matroids (by defining them with respect to a partial closure operator instead of with respect to a closure operator), we define geometric poset, and show that there is a same kind of connection between partial matroids and geometric posets as there is between matroids and geometric lattices. Furthermore, we then define semimodular poset, and show that it is indeed a generalization of semimodular lattices, and that there is a same kind of connection between semimodular and geometric posets as there is between semimodular and geometric lattices.

Finally, we note that the defined notions can be applied to implicational systems, that have many applications in real world, particularly in big data analysis.

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