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Neke nove mrežno vrednosne algebarske strukture sa komparativnom analizom različitih pristupa

Some new lattice valued algebraic structures with comparative analysis of various approaches.

-Doktorska Disertacija-

Novi Sad, 2017

Rezime

Ovaj rad bavi se komparativnom analizom različitih pristupa rasplinutim (fazi) algebarskim strukturama i odnosom tih struktura sa odgovarajućim klasičnim algebrama. Posebna pažnja posvećena je poređenju postojećih pristupa ovom problemu sa novim tehnikama i pojmovima nedavno razvijenim na Univerzitetu u Novom Sadu. U okviru ove analize, proučavana su i proširenja kao i redukti algebarskih struktura u kontekstu rasplinutih algebri. Brojne važne konkretne algebarske strukture istraživane su u ovom kontekstu, a neke nove uvedene su i ispitane. Bavili smo se detaljnim istraživanjima Ω-grupa, sa stanovišta kongruencija, normalnih podgrupa i veze sa klasičnim grupama. Nove strukture koje su u radu uvedene u posebnom delu, istražene su sa aspekta svojstava i međusobne ekvivalentnosti. To su Ω-Bulove algebre, kao i odgovarajuće mreže i Bulovi prsteni. Uspostavljena je uzajamna ekvivalentnost tih struktura analogno odnosima u klasičnoj algebri.

U osnovi naše konstrukcije su mrežno vrednosne algebarske strukture definisane na klasičnim algebrama koje ne zadovoljavaju nužno identitete ispunjene na odgovarajućim klasičnim strukturama (Bulove algebre, prsteni, grupe itd.), već su to samo algebre istog tipa. Klasična jednakost zamenjena je posebnom kompatibilnom rasplinutom (mrežno-vrednosnom) relacijom ekvivalencije.

Na navedeni način i u cilju koji je u osnovi teze (poređenja sa postojećim pristupima u ovoj naučnoj oblasti) proučavane su (već definisane) Ω-grupe. U našim istraživanju uvedene su odgovarajuće normalne podgrupe. Uspostavljena je i istražena njihova veza sa Ω-kongruencijama. Normalna podgrupa Ω-grupe definisana je kao posebna klasa Ω-kongruencije. Jedan od rezultata u ovom delu je da su količničke grupe definisane pomoću nivoa Ω-jednakosti klasične normalne podgrupe odgovarajućih količničkih podgrupa polazne Ω-grupe. I u ovom slučaju osnovna struktura na kojoj je definisana Ω-grupa je grupoid, ne nužno grupa. Opisane su osobine najmanje normalne podgrupe u terminima Ω-kongruencija, a date su i neke konstrukcije Ω-kongruencija.

Rezultati koji su izloženi u nastavku povezuju različite pristupe nekim mrežno-vrednosnim strukturama. Ω-Bulova algebra je uvedena na strukturi sa dve binarne, unarnom i dve nularne operacije, ali za koju se ne zahteva ispunjenost klasičnih aksioma. Identiteti za Bulove algebre važe kao mrežno-teoretske formule u odnosu na mrežno-vrednosnu jednakost. Klasične Bulove algebre ih zadovoljavaju, ali obratno ne važi: iz tih formula ne slede standardne aksiome za Bulove algebre. Na analogan način uveden je i Ω-Bulov prsten. Glavna svojstva ovih struktura su opisana. Osnovna osobina je da se klasične Bulove algebre odnosno Bulovi prsteni javljaju kao količničke strukture na nivoima Ω-jednakosti. Veza ove strukture sa Ω-Bulovom mrežom je pokazana.

Kao ilustracija ovih istraživanja, u radu je navedeno više primera.

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Abstract

In this work a comparative analysis of several approaches to fuzzy algebraic structures and comparison of previous approaches to the recent one developed at University of Novi Sad has been done. Special attention is paid to reducts and expansions of algebraic structures in fuzzy settings. Besides mentioning all the relevant algebras and properties developed in this setting, particular new algebras and properties are developed and investigated.

Some new structures, in particular Omega Boolean algebras, Omega Boolean lattices and Omega Boolean rings are developed in the framework of omega structures. Equivalences among these structures are elaborated in details. Transfers from Omega groupoids to Omega groups and back are demonstrated. Moreover, normal subgroups are introduced in a particular way. Their connections to congruences are elaborated in this settings. Subgroups, congruences and normal subgroups are investigated for Ω-groups. These are lattice-valued algebraic structures, defined on crisp algebras which are not necessarily groups, and in which the classical equality is replaced by a lattice-valued one. A normal Ω-subgroup is defined as a particular class in an Ω-congruence. Our main result is that the quotient groups over cuts of a normal Ω-subgroup of an Ω-group G, are classical normal subgroups of the corresponding quotient groups over G. We also describe the minimal normal Ω-subgroup of an Ω-group, and some other constructions related to Ω-valued congruences.

Further results that are obtained are theorems that connect various approaches of fuzzy algebraic structures. A special notion of a generalized lattice valued Boolean algebra is introduced. The universe of this structure is an algebra with two binary, an unary and two nullary operations (as usual), but which is not a crisp Boolean algebra in general. A main element in our approach is a fuzzy equivalence relation such that the Boolean algebras identities are approximately satisfied related to the considered fuzzy equivalence. Main properties of the new introduced notions are proved, and a connection with the notion of a structure of a generalized fuzzy lattice is provided.

Dedication

I thank God Almighty for coming to this place, I would like to thank my country Libya for its financial support and thanks to the university of the College of Sciences of Ajilat.

After that, I would like to extend my sincere thanks to professor Andreja Tepavčević and professor Branimir Šešelja who have supported and influenced me throughout my studies and to work with their very insightful instructions and knowledge. Right side by side without your dedication to my success I have not reached this point. I would also like to thank the faculty members of the University and professor Marko Nedeljkov, and the members of the Commission Board Ivana Štajner-Papuga and Miroslav Ćirić who gave very useful comments on my work.

I would like to express my sincere thanks to my husband (kasm), who stood beside me throughout my Masters and PhD studies. I say I love you. I would like to thank my father Salem and my mother and my sisters and brothers (Mom and Dad I love you).

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Fuzzy set theory is first introduced by Zadeh in [145] where the intention was to generalize the usual notion of a set by its characteristic function and in this way also to generalize the classical logic to a suitable type of multi-valued logic. Instead of a dual principle membership- non-membership of an element to a set, a notion of the **membership function** is introduced. Values of the membership function are reals from the unit interval, $[0, 1]$ and this function was sometimes denoted by μ_A. Therefore, if A is a set, then a fuzzy set is A together with a function $\mu_A : A \to [0, 1]$, showing the grade of the membership of any element from A to the fuzzy set.

Professor Lotfi Zadeh, the pioneer of fuzzy mathematics died during preparation of the final version of this thesis on 6. September 2017 in the age of 96.

Goguen, 1967 was first to introduce lattice valued fuzzy sets replacing the unit interval with a lattice and Brown in [23] introduced Boolean valued fuzzy sets replacing the unit interval with a Boolean lattice. Later also were introduced more general variants of the notion of fuzzy set, where membership functions were taking values in partially ordered set or most general in a relational system. Sanchez was first to study fuzzy relations and their composition in [115], as a special type of fuzzy sets. A fuzzy relation from a set A to a set B is defined as a fuzzy set from Cartesian product $A \times B$ into [0,1] interval or a complete lattice L. After this first definition, the study of basic types of fuzzy relations was firstly done also by Zadeh, [144], where he defined fuzzy equivalence and fuzzy ordering relations. After that a lot of pa-

pers were published on fuzzy order, e.g., [8, 51, 67, 81]. In [51], Fan defined a fuzzy poset (X, R), where R is a reflexive, antisymmetric and transitive fuzzy relation over set X. Another type of fuzzy poset (called L-ordered set) was originally introduced by Bělohlávek, [6], whose intention was to to fuzzify the fundamental theorem of concept lattices.

Following these concepts, first investigations of fuzzy algebraic structures started at seventhes years of last century by A. Rosenfeld, [109]. In these first investigations, fuzzy algebraic structures were connected to mappings from standard algebraic structures to [0,1] real interval. Later, lattice codomains were taken, but still these generalized algebras were considered to be mappings from classical algebraic structures (papers [2, 4, 11, 12, 23, 39, 58, 33, 34, 45, 60, 67, 117, 123, 124, 121, 126]). Many important aspects of fuzzy algebraic structures, and among them in particular of fuzzy semigroups and fuzzy groups have been developed (this is presented in details in two monographs, first about the fuzzy group theory by J.N. Mordeson, K.R. Bhutani, and A. Rosenfeld [93], second about the fuzzy semigroup theory by J.N. Mordeson, D.S. Malik and N. Kuroki, [94], and the third about the fuzzy commutative algebras by J.N. Mordeson and D.S. Malik, [95]). After that, still classical structures were taken as elements of fuzzy structures, but the classical equality has been replaced by a particular fuzzy equality relation (this approach is presented in details in a paper by R. Belohlavek and V. Vychodil, [10], and also in the monograph by R. Belohlavek [8].

Another approach is the one of so called vague structures introduced by Demirci. The vague groups are introduced and developed in [40]. In this approach the group operation is introduced as a kind of fuzzy ternary relation and the validity of the classical results have been demonstrated in this setting. This approach is further developed in series of papers [40, 41, 42, 44]. Finally, the approach we would accept and develop in this thesis in the one introduced most recently in papers [28, 25, 122]. These structures are called L-E-fuzzy structures or in some papers Ω-structures. Here, the starting point in generalizing some structures is an algebraic structure of the same signature, which need not satisfy given identities. A main feature of this approach is a special lattice valued equality relation (satisfying weak reflexivity). Identities are satisfied up to the given fuzzy equality relation. This approach is developed partially for groups, semigroups and lattices in papers [28],[25],[122]. Our task here is to pay special attention to reducts and expansions of algebraic structures and connections of algebras and its reducts, algebras and their expansions.

The notion of lattice valued fuzzy lattice is introduced for the first time by

Tepavčević and Trajkovski, [136], where the equivalence of two approaches: fuzzification of the membership of the carrier and fuzzification of the ordering relation in a (classical) lattice is proved. In the Ph.D. thesis [49] and in a series of papers, [47, 122, 48] Ω-lattices are introduced using the approach of an Ω-algebra, both as algebraic and as order structures. In this approach, Ω-poset is an Ω-set equipped with an Ω-valued order, which is antisymmetric with respect to the corresponding Ω-valued equality. Moreover, notions of pseudo-infimum and pseudo-supremum are introduced, and using these notions a definition of an Ω-lattice as an ordering structure is developed.

Our notion of an Ω-valued Boolean algebra is based on this approach, i.e., we proved that as in the classical case, an Ω-Boolean algebra is just an extension of the Ω-lattice as an algebraic structure. Namely, in this part first we introduce the notion of an Ω-valued Boolean algebra, where Ω is a complete lattice of membership values. Here also a lattice-valued fuzzy equality has an important role to replace the classical equality. The main reason why we use a complete lattice lattice (and not e.g., a Heyting algebra) as a co-domain is that it enables the framework of cut sets,[78], which allows main algebraic and set-theoretic properties to be generalized from classical structures to lattice-valued ones. This approach is used for dealing with algebraic topics in many papers (see e.g., [45], then also [125, 131]). A complete lattice is sometimes replaced by a complete residuated lattice, [8].

For defining an Ω-Boolean algebra, we use a classical algebra with two binary, one unary operations and two constants, but we do not assume that classical identities for Boolean algebras are satisfied in the usual way. Instead, these identities hold as lattice theoretic formulas including a fuzzy equality. From these we deduce properties of the new structure (idempotency of binary operations, absorption laws, properties of constants) which enable us to prove that an Ω-valued Boolean algebra is also an Ω-lattice (the notion of Ω-lattice is introduced in [122] and further investigated in [47]).

The approach of Ω-Boolean algebra differs from all the previous ones, because the underlying algebra here is not a crisp Boolean algebra in general, it is only an algebra of the same type. E.g., the universe of finite Ω-Boolean algebras might have a number of elements different than 2^n (as in classical Boolean algebras).

Moving on, we introduce a concept of a normal subgroup in the framework of Ω-groups, introduced in [9]. Ω is a complete lattice, hence we deal with lattice-valued structures. In this case, the underlying algebra is not necessarily a group, and the classical equality is replaced by a lattice-valued one. Therefore algebraic (group) identities hold as particular lattice-valued formu-

las. In this part, first we recall particular basic references for fuzzy groups and related structures, not pretending to present an extensive list of such references. Chronologically, fuzzy groups and related notions (semigroups, rings etc.), were introduced early within the fuzzy era (e.g., Rosenfeld [109] and Das [39], then also Mordeson and Malik [95]). Since then, fuzzy groups remain among the most studied fuzzy structures (e.g., Malik, Mordeson and Kuroki [94], Mordeson, Bhutani, and Rosenfeld [93] and [127]). Investigations of notions from general algebra followed these first studies (see e.g., Di Nola, Gerla [45] and [125, 131]). The universe of an algebra was fuzzified, while the operations remained crisp. The set of truth values was either the unit interval, or a complete, sometimes residuated lattice; generalized co-domains were also used (lattice ordered monoids, Li and Pedrycz, [82], posets or relational systems, [131]). An analysis of different co-domain lattices in the framework of fuzzy topology is presented by Höhle and Šostak in [68]. The notion of a fuzzy equality was introduced by Höhle, [64], and then used by many others. Using sheaf theory,[54], in [65], Höhle was dealing with Ω-valued sets and equalities (Ω being a complete Heyting algebra), representing many fields of fuzzy set theory in this framework. Demirci (see e.g., [40, 44]) introduced the new approach to fuzzy structures. He considered particular algebraic structures equipped with fuzzy equality relations and fuzzy operations. In this framework he developed detailed studies of fuzzy groups (vague groups, smooth subgroups) and related topics. Bělohlávek (papers [10, 9], [5] with Vychodil, the books [8, 10], the second with Vychodil) introduced and investigated algebras with fuzzy equalities. These are defined as classical algebras in which the crisp equality is replaced by a fuzzy one being compatible with the fundamental operations of the algebra. Bělohlávek develops and investigates main fuzzified universal algebraic topics. Some aspects of universal algebra in a fuzzy framework were also investigated by Kuraoki and Suzuki, [79].

In our work, we develop a concept of normal subgroups in Ω-valued settings and we have proven a natural connection with Ω- valued congruences.

1.2 Preliminaries

1.2.1 Order, lattices

The lattice will be the basic structure in this work (see e.g., [14]) and it will be considered in two ways, as an algebraic structure $(L, \wedge, \vee)$ where both operations are binary and satisfy commutativity, associativity and absorptivity

and as a special poset $(L, \leq)$ in which every two elements have a supremum (denoted by $\vee$) and an infimum (denoted by $\wedge$). The lattice is complete if every subset M of L has the infimum and the supremum, denoted by $\bigwedge M$ and $\bigvee M$, respectively. A complete lattice $(L, \wedge, \vee)$ will be a starting notion in this work and it will be denoted by L, the top element of the lattice L denoted by 1 and the bottom element by 0. Every finite lattice is a complete lattice, since supremum and infimum are defined for every subset (finite in this case). This follows from the existence of the supremum and infimum of each two element set. In general we will not request this lattice to satisfy any particular lattice properties like distributivity or modularity.

If L is a lattice, and L_1 a nonempty subset of L, then L_1 is a sublattice of L if it is closed under both operations: if $x, y \in L_1$, then $x \wedge y \in L_1$ and $x \vee y \in L_1$.

Some special sublattices are ideals and filters, [14]:

Ideal I is a subset of L closed under $\wedge$, satisfying: for all $x \in I$, if $y \leq x$, then $y \in I$.

Filter F is the dual notion: it is a subset of L closed under $\vee$, satisfying: for all $x \in I$, if $x \leq y$, then $y \in I$.

$\uparrow p = \{x \in L \mid p \leq x\}$ is a special type of filter, called the principal filter.

The dual notion is denoted by $\downarrow p$ and called the principal ideal.

1.2.2 Algebras, identities, congruences

This part contains some basic facts from Universal algebra, see e.g., [30].

In Universal algebra a language $\mathcal{L}$, also called a type, is a set $\mathcal{F}$ of functional symbols with a set of non-negative integers associated to these symbols, called arities.

An algebra in the language $\mathcal{L}$ is a pair $\mathcal{A} = (A, F^A)$, A being a nonempty set and F^A a set of operations on A.

Every operation in F^A which is n-ary, corresponds to an n-ary symbol in the language.

A subalgebra of $\mathcal{A}$ is an algebra in the same language, defined on a subset of A and which is closed under the operations in F.

Terms in a language are regular expressions constructed by the variables and operational symbols. A term $t(x_1, \ldots, x_n)$ in the language of an alge-

bra $\mathcal{A}$ and the corresponding term-operation $A^n \to A$ are often (also here) denoted in the same way.

If t_1 and t_2 are terms in a given language, then an identity in the same language is a formula $t_1 \approx t_2$.

An identity $t_1(x_1, \ldots, x_n) \approx t_2(x_1, \ldots, x_n)$ hold on an algebra $\mathcal{A} = (A, F^A)$, if for all $a_1, \ldots, a_n \in A$, the equality $t_1(a_1, \ldots, a_n) = t_2(a_1, \ldots, a_n)$ is satisfied. A congruence relation on an algebra $\mathcal{A}$ is an equivalence relation ρ on A which is compatible with all fundamental operations.

Compatibility means that

$x_i \rho y_i, i = 1, \ldots, n$ implies $f(x_1, \ldots, x_n) \rho f(y_1, \ldots, y_n)$.

For a congruence relation ρ on $\mathcal{A}$ and for $a \in A$, the congruence class of a, $[a]_\rho$, is defined by $[a]_\rho := \{x \in A \mid (a, x) \in \rho\}$; the quotient algebra $\mathcal{A}/\rho$ is $\mathcal{A}/\rho := (A/\rho, F^{A/\rho})$, where $A/\rho = \{[a]_\rho \mid a \in A\}$, and the operations in $F^{A/\rho}$ are defined in a natural way by representatives.

Next, let ϕ and θ be congruences on an algebra $\mathcal{A}$, and $\theta \subseteq \phi$. Then, *the relation*

$$\phi/\theta := \{([a]_\theta, [b]_\theta) \mid (a, b) \in \phi\}$$

is a congruence on $\mathcal{A}/\theta$.

Theorem 1.2.1. [30] [Second Isomorphism Theorem] *If ϕ and θ are congruences on an algebra $\mathcal{A}$ and $\theta \subseteq \phi$, then ϕ/θ is a congruence on $\mathcal{A}/\theta$.*

Let $\mathcal{A}$ be an algebra, θ a congruence on $\mathcal{A}$ and $B \subseteq A$. Let

$$B^\theta := \{x \in A \mid B \cap [a]_\theta \neq \emptyset\},$$

and $\mathcal{B}^\theta$ the subalgebra of $\mathcal{A}$ generated by B^θ. We denote

$$\theta{\restriction}_{B^\theta} := \theta \cap B^2$$

(the restriction of θ to B). Now, *the universe of $\mathcal{B}^\theta$ is B^θ and $\theta{\restriction}_{B^\theta}$ is a congruence on $\mathcal{B}$.*

Theorem 1.2.2. [30] [Third Isomorphism Theorem] *If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and θ a congruence on $\mathcal{A}$, then*

$$\mathcal{B}/\theta{\restriction}_B \cong \mathcal{B}^\theta/\theta{\restriction}_{B^\theta}.$$

The following version of the Axiom of Choice is used in some parts of this

text:

(AC) *For a collection $\mathcal{X}$ of nonempty subsets of a set M, there exists a function $f : \mathcal{X} \to M$, such that for every $A \in \mathcal{X}$, $f(A) \in A$.*

1.2.3   Fuzzy sets

All fuzzy sets in this work will be lattice valued (fuzzy) sets $\mu : A \to L$, where L is a complete lattice. Here all lattice valued sets will be identified with the function μ in a similar way as subsets are sometimes identified with their characteristic function.

They will be sometimes called lattice valued sets, L-valued fuzzy sets, or L-valued sets. Later A can be equipped with operational structure or with a lattice valued equivalence relation. In case when we consider a set or an algebra together with a lattice valued equality (weak equivalence), then we call such a structure Ω-valued set.

Since [0,1] interval is also a complete lattice, the lattice valued approach also contains the original Zadeh's approach.

Namely, we can consider a mapping $\mu : A \to [0, 1]$ as an L-valued set and in the same time it is a fuzzy set in the Zadeh sense, where the value of the function μ for some element $a \in A$, $\mu(a)$ is a grade of membership of element a to the (fuzzy) set.

In example, let $A = \{a, b, c\}$, then a fuzzy set μ can be defined by $\mu(a) = 1$, $\mu(b) = 0$ and $\mu(c) = 0.7$. Grade of membership of element a to the fuzzy set is 1, which means that a "totally" belong to the fuzzy set. Grade of membership of b to the fuzzy set is 0, which means that b "totally" does not belong to the fuzzy set. $\mu(c) = 0.7$ means that c belongs to the fuzzy set to an extent of 0.7.

In order to illustrate a lattice valued case which is not a fuzzy set in the classical sense, let $L = (\{p, q, r, s\}, \leq)$ be a lattice on Figure 1, where $p \leq q \leq s$ and $p \leq r \leq s$ and q and r are incomparable. Then, an L-valued set can be defined by $\mu(a) = p$, $\mu(b) = q$ and $\mu(c) = r$. Here p, q and r are also called grades of membership, although they are not real numbers.
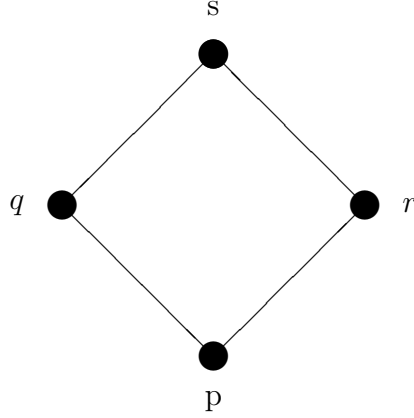
Figure 1: Lattice L

For a lattice valued set $\mu : A \to L$, the set of images is denoted as usual by $\mu(A)$

$$\mu(A) := \{p \in L : \mu(x) = p, \text{ for some } x \in X\}.$$

In the previous example, $\mu(A) = \{p, q, r\}$.

Let L be a complete lattice and $\mu : A \to L$ and $\nu : A \to L$ two lattice valued fuzzy sets (denoted by μ and ν).

Then, the notion of inclusion between fuzzy sets is defined componentwise

$$\mu \subseteq \nu \text{ if and only if } \mu(x) \leq \nu(x), \forall x \in A. \quad (1.1)$$

Similarly, the notion of equality of two fuzzy sets is defined by

$$\mu = \nu \text{ if and only if } \mu(x) = \nu(x), \forall x \in A. \quad (1.2)$$

The union (join) denoted by $\vee$ and the intersection (meet) denoted by $\wedge$ of two fuzzy sets, are also introduced componentwise:

$$(\mu \vee \nu)(x) = \mu(x) \vee \nu(x) \text{ for all } x \in A$$
$$(\mu \wedge \nu)(x) = \mu(x) \wedge \nu(x) \text{ for all } x \in A.$$

In this way two lattice valued fuzzy sets $\mu \vee \nu : A \to L$ and $\mu \wedge \nu : A \to L$ are obtained.

The support of a fuzzy set $\mu : A \to L$ is the subset of A defined by $supp(A) = \{x \in A \mid \mu(x) > 0\}$.

1.2.4   Cut sets and properties

Now a notion of cut sets (cuts) will be introduced.

Let $\mu : A \to L$ be a lattice valued fuzzy set and let $p \in L$. A cut set (p-cut) is the following subset of A:

$$\mu_p = \{x \in A \mid \mu(x) \geq p\}.$$

If $p = 0$, then the 0-cut is obtained and for each fuzzy set μ, $\mu_0 = A$.

The family of all cuts of a lattice valued set is denoted by μ_L:

$$\mu_L = \{\mu_p \mid p \in L\}.$$

A cut is a very important concept in our approach, since many properties of lattice valued structures are satisfied if and only if they are appropriately (in the crisp settings) satisfied on cuts.

Here are some propositions that characterize cuts and the family μ_L.

Firstly, every p-cut of μ is the inverse image of the principal filter of the lattice L induced by p,

$$\mu_p = \mu^{-1}(\uparrow p),$$

where

$$\uparrow p = \{q \in L \mid p \leq q\}.$$

The next proposition shows how one can calculate the values of lattice valued set from the cuts:

Proposition 1.2.3. [134] *Let $\mu : X \to L$ be a lattice-valued set on X. Then for all $x \in X$ it holds that,*

$$\mu(x) = \bigvee \{p \in L : x \in \mu_p\}$$

The following proposition directly follows from the definition of cuts:

Proposition 1.2.4. [134] *Let $\mu : X \to L$ be a lattice-valued set on X. For $p, q \in L$, if $p \leq q$ then $\mu_p \supseteq \mu_q$.*

The converse of proposition (1.2.4) does not hold in general and the next proposition gives conditions under which the converse of proposition (1.2.4) is satisfied.

Proposition 1.2.5. [123] *Let* $\mu : X \to L$ *be a lattice-valued set on* X. *Then*

1. *for* $p \in L$ *and* $a \in X$, $p \le \mu(a)$ *if and only if* $\mu_p \supseteq \mu_{\mu(a)}$.

2. *for* $a, b \in X$, $\mu(a) \ne \mu(b)$ *if and only if* $\mu_{\mu(a)} \ne \mu_{\mu(b)}$.

The following proposition gives some further properties of cuts. Namely, the family of cuts is closed under intersections and has the top element, hence it is a complete lattice under inclusion.

Proposition 1.2.6. [134] *Let* $\mu : X \to L$ *be an* L-*valued set on* X. *Then,*

1. *if* $L_1 \subseteq L$, *then* $\bigcap \{ \mu_p \mid p \in L_1 \} = \mu_{\bigvee \{ \mu_p \mid p \in L_1 \}}$.

2. $\bigcup \{ \mu_p \mid p \in L \} = X$.

3. $\forall\, x \in X,\ \bigcap \{ \mu_p \mid x \in \mu_p \} \in \mu_L$.

As a consequence it is obtained that the collection of cuts forms a (complete) lattice under set inclusion, where μ_0 is the top and μ_1 is the bottom element of this lattice:

Theorem 1.2.7. *Let* $\mu : X \to L$ *be an* L-*valued set on* X. *Then the collection* $\mu_L = \{ \mu_P : p \in L \}$ *of all cuts of* μ *forms a complete lattice under inclusion.*

Moreover, in a following representation theorem it is stated that for every complete lattice there exists a lattice-valued set such that the collection of cut sets of this lattice-valued set under set inclusion is a complete lattice anti-isomorphic with the given lattice.

Theorem 1.2.8. *(Representation Theorem)*[134] *Let* $X \ne \emptyset$ *and* $\mathcal{F}$ *a collection of subsets of* X *closed under arbitrary intersections. Let* $(L, \leqslant)$ *be a lattice dual to* $(\mathcal{F}, \subseteq)$ *and let* $\mu : X \to L$ *be defined by,*

$$\mu(x) := \bigcap \{ p \in \mathcal{F} : x \in p \}.$$

Then μ *is a lattice valued set on* X, *and each* $p \in \mathcal{F}$ *is equal to the corresponding cut* μ_p.

In the next part it is shown that starting from a lattice-valued set, a partition on lattice L is induced in a natural way.

Let $\mu : X \to L$ be an L-valued set on X. Now a relation $\approx$ on L is introduced as follows, [123]:

for $p, q \in L$,
$$p \approx q \text{ if and only if } \mu_p = \mu_q.$$

$\approx$ is an equivalence relation on L and for any $p \in L$, the corresponding equivalence class is:

$[p]_\approx := \{q \in L : p \approx q\}.$

The set of all $\approx$-classes is denoted by $L/\approx$

In the following proposition some characterizations of the above equivalence relation and equivalence classes are given.

Proposition 1.2.9. ([134]) *Let $\mu : X \to L$ be an L-valued set on X and $p, q \in L$, then*

$$p \approx q \text{ if and only if } \uparrow p \cap \mu(X) = \uparrow q \cap \mu(X).$$

Proposition 1.2.10. [123] *Let $\mu : X \to L$ be an L-valued set on X, and let $X_1 \subseteq X$. If for all $x \in X_1$, $p = \mu(x)$, then p is the top element of the $\approx$-class to which it belongs.*

The ordering relation $\leq$ on L can be extended in a natural way to an ordering on the set of $\approx$-classes $L/\approx$ by:

$$[p]_\approx \leq [q]_\approx \text{ if and only if } \uparrow q \cap \mu(X) \subseteq \uparrow p \cap \mu(X).$$

The ordering relation $\leq$ on classes is well defined and there is an anti-isomorphism among the poset of classes and the collection of cuts of μ ordered by the set inclusion, as follows:

Proposition 1.2.11. [123] *Let $\mu : X \to L$ be an L-valued set on X, then;*

$$[p]_\approx \leq [q]_\approx \text{ if and only if } \mu_p \supseteq \mu_q.$$

For the lattice $(L/\approx, \leq)$ the supremum of each $\approx$-class $[p]_\approx$ denoted by $\bigvee [p]_\approx$ exists and defined by

$$\bigvee [p]_\approx := \bigvee \{q \in L : q \in [p]_\approx\}.$$

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Now, a convenient collection of subsets of the co-domain lattice L is introduced:

Let L be a lattice, $X \neq \emptyset$ and $\mu : X \to L$ a lattice valued set, then an ordered set is defined as follows:

$$L_\mu := (\{\uparrow p \cap \ \mu(X) : p \in L\}, \subseteq).$$

It can be seen from the definition that L_μ consists of particular collections of images of μ in L and it is ordered by set inclusion.

Proposition 1.2.12. [123] *For an L-valued set μ the lattice L_μ is isomorphic to the lattice μ_L of cuts under the mapping $f(\mu_p) = \uparrow p \cap \ \mu(X)$.*

1.2.5 Lattice valued relations

A lattice valued (binary) relation R on a set A is a lattice valued set on the direct product $A \times A$:

$$R : A \times A \to L.$$

Some special properties of lattice-valued relations are introduced as follows:

- R is reflexive if for all $x \in A$, $R(x, x) = 1$,

- R is strict (weakly reflexive): If for all $x, y \in A$, $R(x, x) \geq R(x, y)$ and $R(x, x) \geq R(y, x)$,

- R is symmetric if for all $x, y \in A$, $R(x, y) = R(y, x)$,

- R is transitive if for all $x, y, z \in A, R(x, y) \wedge R(y, z)) \leq R(x, z)$.

- R is antisymmetric if for all $x, y \in A$, $R(x, y) \wedge R(y, x) = 0$, for $x \neq y$.

As usual, a lattice valued relation which is reflexive, symmetric and transitive is called a lattice valued equivalence or a lattice valued similarity relation.

A reflexive and transitive lattice valued relation R is called a lattice valued preorder, and the pair (A, R) is called a lattice valued-preordered set. A lattice valued preorder which is also antisymmetric is called a lattice valued order and the pair (A, R) is a lattice valued ordered set.

Since a lattice valued relation is a lattice valued set on a square of sets, also cuts of L-valued relations can be considered and all properties of cuts defined above are also valid.

In the next proposition it is stated that all properties of lattice valued relations are cutworthy (i.e., properties of the lattice valued structure are valid if and only if the related properties of cuts are also valid):

Proposition 1.2.13. [124] *Let $R : A \times A \to L$ be a lattice valued relation.*

R is a reflexive lattice valued relation if and only if all the cut relations R_p for all $p \in L$ are reflexive relations on A.

R is a symmetric lattice valued relation if and only if all the cut relations R_p for all $p \in L$ are symmetric relations on A.

R is an anti-symmetric lattice valued relation if and only if all the cut relations R_p for all $p \in L$ are anti-symmetric relations on A.

R is a transitive lattice valued relation if and only if all the cut relations R_p for all $p \in L$ are transitive relations on A.

As a consequence the following corollary is obtained:

Proposition 1.2.14. [124] *Let $R : A \times A \to L$ be a lattice valued relation.*

R is a lattice valued equivalence relation of and only if all the cut relations R_p for $p \in L$ are equivalence relations on A.

R is a lattice valued preorder of and only if all the cut relations R_p for $p \in L$ are preorders on A.

R is a lattice valued order of and only if all the cut relations R_p for $p \in L$ are orders on A.

Now an important connection of an L-valued set with an L-valued relation will be introduced in the sequel.

Let μ be an L-valued set on A and R an L-valued relation on A. Then, R is an L-valued relation on μ if for every $x, y \in A$

$$R(x, y) \leq \mu(x) \wedge \mu(y). \tag{1.3}$$

This condition is a generalization of the following ordinary relational property: If ρ is a binary relation on a subset Y of A, then from $x\rho y$ it follows that $x, y \in Y$.

If such a kind of connection of a lattice valued set and a lattice valued relation exists, and the lattice-valued relation is reflexive, then $\mu(x) = 1$ for every $x \in A$. Therefore, the reflexivity is not satisfied in a non-trivial case, when μ is not a constant function. Hence, in such case, when μ is an arbitrary function, the maximum value that $R(x, x)$ should take is $\mu(x)$.

Therefore, in this framework and throughout this work, we say that an L-valued relation R on an L-valued set μ is reflexive if for all $x, y \in A$

$$R(x, x) = \mu(x). \tag{1.4}$$

This notion of reflexivity as stated above is known from ([53]) and other papers.

The following lemma is true.

Lemma 1.2.15. *If $R : A^2 \to L$ is reflexive on $\mu : A \to L$, then for all $x, y \in A$*
$$R(x, x) \geq R(x, y) \text{ and } R(x, x) \geq R(y, x). \quad \square$$

1.2.6 Lattice valued algebras

The notion that will be introduced next, is the notion of the lattice valued algebra (subalgebra). This notion have been elaborated in many papers (see e.g. book [72]). Starting notion is an algebra $\mathcal{A} = (A, \mathbb{F})$, where A is a nonempty set and $\mathbb{F}$ is a set of operations on A. A lattice valued subalgebra is a special lattice valued set on A, introduced in the sequel.

Definition 1.2.16. Let $\mathcal{A} = (A, \mathbb{F})$ be an algebra. Then a lattice valued subset $\mu : A \to L$ is a lattice valued subalgebra of $\mathcal{A}$ if

1. $\mu(e) = 1$ for every nullary operation in $\mathbb{F}$

2. for every n-ary operation $f^{\mathcal{A}} \in \mathbb{F}$ for $n \neq 0$ and for every $a_1, ..., a_n \in A$,

$$\mu(a_1) \wedge ... \wedge \mu(a_n) \leq \mu(f^{\mathcal{A}}(a_1, ..., a_n))$$

Next it is stated that the notion of lattice valued algebra is cutworthy, see [125].

Theorem 1.2.17. *Let $\mathcal{A}$ be an algebra and $\mu : A \to L$ be a lattice valued set. Then μ is a lattice valued subalgebra of $\mathcal{A}$ if and only if for every $p \in L$ p-cut μ_p is a subalgebra of $\mathcal{A}$.*

Having the notions of lattice valued equivalence relations and lattice valued algebras, a notion of a lattice valued congruence is introduced, [125].

Let $\mathcal{A} = (A, F)$ be an algebra and $R : A^2 \to L$ be a lattice valued equivalence relation on A. Then R is said to be a lattice valued congruence on $\mathcal{A}$ if for $a_1, ..., a_n, b_1, ..., b_n \in A$ and for each operation symbol $f \in F$

$$R(a_1, b_1) \wedge ... \wedge R(a_n, b_n) \leq R(f^{\mathcal{A}}(a_1, ..., a_n), f^{\mathcal{A}}(b_1, ..., b_n)).$$

Obviously, lattice valued congruence relations are lattice valued equivalence relations that preserve algebraic structures.

Again the notion of lattice valued congruences are cutworthy, [125]:

Theorem 1.2.18. *Let $\mathcal{A} = (A, F)$ be an algebra and $R : A^2 \to L$ a lattice valued relation. Then R is a lattice valued congruence on $\mathcal{A}$ if and only if for each $p \in L$ the cut relation R_p is a classical congruence relation on $\mathcal{A}$.*

1.2.7 Residuated lattices

Here the definition and some basic properties of residuated lattices is introduced. They will not be used as co-domains of the structures we investigate here, but we introduce them here in order to compare different approaches to fuzzy algebraic structures (e.g., with the approaches developed in [5, 8, 10]).

Here it is the definition of a residuated lattice, which is a special extension of a lattice.

Residuated lattice is an algebra with four binary and two nullary operations

$$\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$$

such that

  (i) $(L, \wedge, \vee)$ is a lattice,

  (ii) 0 is the bottom element of the lattice and 1 is the top element,

 (iii) $(L, \otimes, 1)$ is a commutative monoid, i.e., $\otimes$ is associative and commutative operation with neutral element 1, i.e., $x \otimes 1 = x$ for all $x \in L$.

 (iv) $x \leqslant y \to z$ if and only if $x \otimes y \leqslant z$ for all $x, y, z \in L$ ($\leqslant$ is a lattice order).

A residuated lattice is complete if $(L, \wedge, \vee, 0, 1)$ is a complete lattice.

Operations $\otimes$ and $\to$ are associated since each of them can be determined by the other.

Residuate lattice satisfying

$$x \otimes y = x \wedge y$$

is called *Heyting* algebra.

Here are basic properties of a residuated lattice:

1. $x \otimes (x \to y) \leqslant y, \ y \leqslant x \to (x \otimes y), \ x \leqslant (x \to y) \to y$

2. $x \leqslant y$ if and only if $x \to y = 1$

3. $x \to x = 1, \ x \to 1 = 1, \ 0 \to x = 1$

4. $1 \to x = x$

5. $x \otimes 0 = 0$

6. $x \otimes y \leqslant x, \ x \leqslant y \to x$

7. $x \otimes y \leqslant x \wedge y$

8. $(x \otimes y) \to z = x \to (y \to z)$

9. $(x \to y) \otimes (y \to z) \leqslant (x \to z)$

Chapter 2

Comparative analysis of different approaches of lattice valued algebraic structures

Throughout the history of fuzzy sets there were several approaches to fuzzy algebraic structures. First definitions of fuzzy algebras were introduced at the beginning of fuzzy era. The underlying structures of these algebras were ordinary algebras of the same type which satisfy same crisp identities. The first ones were mappings from the universe of the algebra to [0,1] interval and then, when lattice valued structures were introduced, there were mappings to a complete lattice.

In these first definitions the approach was cutworthy, i.e., all the cuts satisfied the corresponding crisp identities.

This means that for a fuzzy group all the cuts were classical groups, for a fuzzy ring all the cuts were classical rings, for a fuzzy lattice all the cuts were classical lattices.

This first approach to fuzzy algebras (lattice valued algebras) is already presented in part **Lattice valued algebras** in chapter **Introduction and preliminaries**. All types of algebraic and relational structures are developed following this approach, and several hundreds of papers and books are published developing all aspects of this approach. In books [93, 94, 95] the results on fuzzy groups, fuzzy semigroups and fuzzy commutative algebras from many papers in the topics are systematically presented.

A version of this definition that is not cutworthy is introduced in context of residuated lattices (i.e., the codomain of the function is a residuated

lattice).

Let $\mathcal{A} = (A, F)$, be an algebra and $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ a complete residuated lattice. In the following definition, using the approach from [5, 8, 10], instead of the lattice infimum, the operation $\otimes$ in the residuated lattice is used, since it replaces the logical conjunction in this context. In the following definition $\bigotimes_{i=1}^{n} \mu(x_i)$ is the notation for a successive application of operation $\otimes$, since $\otimes$ is an associative operation.

Then, a lattice valued set $\mu : A \rightarrow L$. is a lattice valued subalgebra of algebra $\mathcal{A}$ if for all operations $f \in F$ and all $x_1, ..., x_n \in A$,

$$\mu(f(x_1, ..., x_n)) \geqslant \bigotimes_{i=1}^{n} \mu(x_i).$$

and every nullary operation $c \in F$, $\mu(c) = 1$, where 1 is the top element in L.

2.1 Algebras with fuzzy equality

An alternative approach that will also be mentioned here is developed by Belohlavek and his coworkers and presented in series of his papers and books, [6, 5, 7, 8, 9, 10].

In this approach the fuzzy equality is a fuzzy equivalence relation $R : A^2 \rightarrow L$ satisfying:

from $R(x, y) = 1$ it follows that $x = y$.

An algebra with fuzzy a equality is introduced in the sequel.

First a definition of the type in this context would be introduced.

A type is an ordered triple (E, F, σ), where $E \notin F$ and σ is a mapping

$$\sigma : F \cup \{E\} \rightarrow N_0$$

where $\sigma(E) = 2$. Every $f \in F$ is a functional symbol, E is a relational symbol that corresponds to the fuzzy equality. Mapping σ denotes arity $\sigma(f)$ of each functional symbol $f \in F$. Sometimes instead of (E, F, σ) only F is used if no ambiguity could arise.

An algebra with a fuzzy equality of the type (E, F, σ) is an ordered triple

$\mathbf{M} = (M, E_M, F_M)$ such that (M, F_M) is an algebra of the type (F, σ) and E_M is a fuzzy equality on M such that each $f_M \in F_M$ is compatible with E_M [6, 5, 7, 8, 9, 10].

An algebra with a fuzzy equality is often called shortly an L-algebra [8].

In the context above, for $a, b \in M$, $E_M(a, b)$ is called the degree of similarity between a and b.

If $L = \{0, 1\}$ is a two element residuated lattice, then the ordinary algebras are obtained, so the notion of L-algebras is a generalization of the notion of classical algebras.

In comparison to the approach accepted in this thesis, that will be developed in detail in the next parts, the difference is that in Belohlavek's approach algebras satisfy usual identities and only the equality is fuzzy. So, the underlying algebras satisfy ordinary identities. In related fuzzy algebras the identities are satisfied to some extent (they are more or less satisfied with respect to the order in L).

In our approach also a lattice valued (fuzzy) equality is used, however, the underlying crisp algebras do not satisfy the corresponding classical identities in the usual way. According to the definition, these identities hold by fulfilling particular lattice theoretic formulas. Therefore, there is no grade of satisfiability of identities as in Belohlavek's approach. They are simply satisfied or not, depending whether the corresponding lattice-theoretic formulas hold.

In the next part we present basics of our approach. This approach is introduced in several papers and PhD thesis, mostly at University of Novi Sad (with co-workers) [47, 119, 25, 26, 27, 28, 122, 49].

2.2 Ω-valued functions, Ω-valued relations and Ω-sets

In the sequel the membership values lattice will be denoted by $(\Omega, \leqslant)$, instead of L, as before. The reason is partially historical, since our approach is partially based on the theory of Ω-sets introduced by Fourmann and Scott in 1979 in category theory. Therefore, sometimes the term Ω-valued function will be used instead of a lattice valued set, and also a short notation Ω-set will be used.

So, this is a lattice valued set, and the definition will be repeated here.

An Ω-valued function μ on a nonempty set A, is a function $\mu : A \to \Omega$, where $(\Omega, \leqslant)$ is a complete lattice. This notion can be identified with the one of a fuzzy set with the lattice-codomain, or a lattice valued set on A.

An Ω-valued function μ on A is said to be nonempty, if $\mu(x) > 0$ for some $x \in A$.

A notion of the cut set is already introduced and will be used in the sequel.

Here it is again a definition of lattice valued relations and a bit different terminology in the Ω-valued setting.

An Ω-valued (binary) relation ρ on A is an Ω-valued function on A^2, i.e., it is a mapping $\rho : A^2 \to \Omega$.

$$\rho \text{ is symmetric if } \rho(x,y) = \rho(y,x) \text{ for all } x, y \in A; \tag{2.1}$$

$$\rho \text{ is transitive if } \rho(x,y) \geqslant \rho(x,z) \wedge \rho(z,y) \text{ for all } x, y, z \in A. \tag{2.2}$$

We say that a symmetric and transitive relation ρ on A is an Ω-valued equality on A.

An Ω-valued equality ρ on a set A fulfills the strictness property, see [65]:

$$\rho(x,y) \leqslant \rho(x,x) \wedge \rho(y,y). \tag{2.3}$$

Similarly as in [65], an Ω-valued equality ρ on A is separated, if it satisfies the property

$$\rho(x,y) = \rho(x,x) \text{ implies } x = y. \tag{2.4}$$

Next, the above notions are briefly connected with Ω-*valued relations on* Ω-*valued sets.*

Let $\mu : A \to \Omega$ be an Ω-valued function on A and let $\rho : A^2 \to \Omega$ be an Ω-valued relation on A. If for all $x, y \in A$, ρ satisfies

$$\rho(x,y) \leqslant \mu(x) \wedge \mu(y), \tag{2.5}$$

then ρ is an Ω-valued relation on μ (see e.g., [65]).

An Ω-valued relation ρ on $\mu : A \to \Omega$ is said to be reflexive on μ if

$$\rho(x,x) = \mu(x) \text{ for every } x \in A. \tag{2.6}$$

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A reflexive Ω-valued relation on μ is strict on A, in the sense of (2.3), since $\rho(x, y) \leqslant \mu(x) \wedge \mu(y)$.

A symmetric and transitive Ω-valued relation ρ on A, which is reflexive on $\mu : A \to \Omega$ is an Ω-valued equality on μ. In addition, if ρ is separated on A, then it is called a separated Ω-valued equality on μ.

A lattice-valued subalgebra of an algebra $\mathcal{A} = (A, F)$, here is called an Ω-valued subalgebra of $\mathcal{A}$.

Ω-valued subalgebra of $\mathcal{A}$ is a function $\mu : A \to \Omega$ which is not constantly equal to 0, and fulfilling: For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \ldots, a_n \in A$,

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \ldots, a_n)), \tag{2.7}$$

and for a nullary operation $c \in F$, $\mu(c) = 1$. $\tag{2.8}$

The next proposition states that an analogous property is valid not only for all the operations from F, but also for all the term operations.

Proposition 2.2.1. [28, 25]*Let $\mu : A \to \Omega$ be an Ω-valued subalgebra of an algebra $\mathcal{A}$ and let $t(x_1, \ldots, x_n)$ be a term in the language of $\mathcal{A}$. If $a_1, \ldots, a_n \in A$, then the following holds:*

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(t^A(a_1, \ldots, a_n)), \tag{2.9}$$

where t^A is the corresponding term operation. $\qquad\square$

An Ω-valued relation $R : A^2 \to \Omega$ on an algebra $\mathcal{A} = (A, F)$ is compatible with the operations in F if the following two conditions holds: for every n-ary operation $f \in F$,
for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leqslant R(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)); \tag{2.10}$$

$$R(c, c) = 1. \tag{2.11}$$

An Ω-set, as defined in [54], is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A.

In many cases a crucial requirement would be that an Ω-set fulfills the separation property, 2.4, but this would always be pointed out in the text.

For an Ω-set (A, E), by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x, x), \tag{2.12}$$

will be denoted.

In this context, μ is determined by E. Clearly, by the strictness property, E is an Ω-valued relation on μ, namely, it is an Ω-valued equality on μ. Hence, in an Ω-set (A, E), E is an Ω-valued equality.

The following lemma states that the notion of an Ω-valued equality is cutworthy.

Lemma 2.2.2. [28] *If (A, E) is an Ω-set and $p \in \Omega$, then the cut E_p is an equivalence relation on the corresponding cut μ_p of μ.*

Here the cut E_p is an equivalence relation on μ_p, but if E_p is considered as a relation on A, it is a weak equivalence relation (i.e., symmetric, transitive and weakly reflexive).

2.3 Ω-algebra

Next a notion of a lattice-valued algebra with a lattice valued equality is introduced.

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F. Then we say that $(\mathcal{A}, E)$ is an Ω-algebra. Algebra $\mathcal{A}$ is the underlying algebra of $(\mathcal{A}, E)$.

Now some cut properties of Ω-algebras are presented. These have been proved in [28], in the framework of groups.

Proposition 2.3.1. *Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold:*

(i) The function $\mu : A \to \Omega$ determined by Ω ($\mu(x) = E(x, x)$ for all $x \in A$), is an Ω-valued subalgebra of A.

(ii) For every $p \in \Omega$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$, and

(iii) For every $p \in \Omega$, the cut E_p of E is a congruence relation on μ_p.

There is the clear difference between two similar notions: an Ω-valued subalgebra μ of an algebra $\mathcal{A}$, and an Ω-algebra $(\mathcal{A}, E)$. An Ω-valued subalgebra μ of an algebra $\mathcal{A}$ is a function compatible with the operations on $\mathcal{A}$

in the sense of (2.9), and an Ω-algebra $(\mathcal{A}, E)$ is a pair $(\mathcal{A}, E)$, consisting of an algebra $\mathcal{A}$ and an Ω-equality E. Relationship among these two is given in the above Proposition 2.3.1.

This proposition states that the notions of a subalgebra and a congruence are cutworthy also it this context.

2.4 Identities

In this part, a notion of satisfaction of identities on Ω-algebras will be defined according to the approach in [119].

Let and $u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n)$ (briefly $u \approx v$) be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. It is assumed, as usual, that variables appearing in terms u and v are from $x_1, \ldots, x_n$ Then, $(\mathcal{A}, E)$ satisfies identity $u \approx v$ (i.e., this identity holds on $(\mathcal{A}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)), \tag{2.13}$$

for all $a_1, \ldots, a_n \in A$.

If Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, then this identity need not hold on $\mathcal{A}$. On the other hand, if the supporting algebra fulfills an identity then also the corresponding Ω-algebra does.

Proposition 2.4.1. [28] *If an identity $u \approx v$ holds on an algebra $\mathcal{A}$, then it also holds on an Ω-algebra $(\mathcal{A}, E)$.*

The fact that an Ω- algebra $(\mathcal{A}, E)$ fulfils an identity $u \approx v$, does not imply that this identity holds on $\mathcal{A}$. However, if E is a separated Ω-valued equality, then the following converse property is valid.

An idempotent identity in the language of an algebra $\mathcal{A}$ is a formula $t(x) \approx x$, where $t(x)$ is a term in the language of $\mathcal{A}$, depending on a single variable x.

Proposition 2.4.2. *Let $(\mathcal{A}, E)$ be an Ω-algebra, where E is separated. Then $(\mathcal{A}, E)$ satisfies an idempotent identity $t(x) \approx x$ if and only if the same identity holds on $\mathcal{A}$.*

Lemma 2.4.3. [25]*Let $(\mathcal{A}, E)$ be an Ω-algebra and u, v terms in the language of $\mathcal{A}$.*

(*i*) If $(\mathcal{A}, E)$ satisfies the identity $u \approx v$, then every Ω-algebra $(\mathcal{A}, F)$ with $E \leqslant F$ satisfies the same identity.

(*ii*) If $(\mathcal{A}, E_i)$, $i \in I$ is a family of Ω-algebras all of which satisfy satisfies the identity $u \approx v$, then also the Ω-algebra $(\mathcal{A}, \bigwedge_{i \in I} E_i)$ satisfies the same identity.

In the papers [27, 26] various aspects of Ω-algebras in general have been investigated, the results about homomorphisms, subalgebras and direct products of Ω-algebras and also the results on Ω-varieties. This is not a topic of this work, but in the next Chapter known results about particular types of Ω-algebras, like semigroups, quasigroups, groups and lattices will be presented.

Chapter 3

Results on various Ω-algebras - reducts and extensions

In this chapter several Ω-algebraic structures are elaborated. Some of them are reducts of other algebras and some are extensions. Also, some of the Ω-algebras introduced in the Chapter 4 are extensions of Ω-algebras in this chapter (e.g., Ω-Boolean algebras are an extension of Ω-lattices presented here.)

Our procedure is as follows. For introducing a particular Ω-algebra, we start with a classical basic structure of the same type. Then we add particular properties formulated in the lattice valued context. The first case are Ω-groupoids, for which a basic structure is an algebra with a binary operation. Then we develop extensions which are Ω-quasigroups, Ω-semigroups and Ω-groups. In the analogue way we introduce and investigate Ω- lattices, starting with an algebra with two binary operations. In Chapter 4 we start with an algebra with two binary, one unary operation and two constants and then develop Ω-Boolean algebras as an extension of Ω- lattices.

3.1 Ω-groupoid

Let $\mathcal{G} = (G, \cdot)$ be a groupoid (i.e., an algebra with a single binary operation), Ω a complete lattice and $E : G^2 \to \Omega$ a lattice valued equality on $\mathcal{G}$. Let $\mu : G \to \Omega$ be defined by $\mu(x) = E(x, x)$. Here, E is a lattice valued relation on μ.

Then $\bar{\mathcal{G}} = (\mathcal{G}, E)$ is an Ω-groupoid. Usually Ω is considered to be fixed,

so it is not mentioned in the structure.

Now some identities on Ω-groupoids are introduced, as above. This part is adapted from [25].

As above, an identity $u \approx v$ holds on an Ω-groupoid $\mathcal{G} = (\mathcal{G}, E)$, if the formula (2.13) is satisfied.

In this part it is assumed that E is separated and under this condition, it is shown that some identities with one variable hold also on the corresponding crisp groupoid $\mathcal{G}$ in the classical way.

Theorem 3.1.1. [25]*Let $\mathcal{G} = (G, \cdot)$ be a groupoid, $\bar{\mathcal{G}} = (\mathcal{G}, E)$ an Ω-groupoid, and $t(x)$ a term depending on a variable x only. Then the valued identity $t(x) \approx x$ holds on $\bar{\mathcal{G}}$ if and only if holds on $\mathcal{G}$.*

The proof of this proposition follows by the separation property [25].

In the following part, some usual groupoid identities in this context will be mentioned, [25].

An element $x_0 \in G$ is idempotent in an Ω-groupoid $\bar{\mathcal{G}} = (\mathcal{G}, E)$ if $E(x_0 \cdot x_0, x_0) \geqslant \mu(x_0)$ is satisfied.

An Ω-groupoid is idempotent if every element in G is idempotent in this Ω-groupoid.

Corollary 3.1.2. *An element $x \in G$ is idempotent in an Ω-groupoid $\bar{\mathcal{G}}$ if and only if x is idempotent in G (i.e., if $x^2 = x$).*

An Ω-groupoid $\bar{\mathcal{G}} = (\mathcal{G}, E)$ is idempotent if and only if the groupoid $\mathcal{G} = (G, \cdot)$ is idempotent.

In an Ω-groupoid $\bar{\mathcal{G}} = (\mathcal{G}, E)$, an element $e \in G$ is a **neutral element**, i.e., a unit in $\bar{\mathcal{G}}$ if for all $x \in G$

$$E(x \cdot e, x) \geqslant \mu(x) \qquad \text{and} \qquad E(e \cdot x, x) \geqslant \mu(x).$$

Similarly as for an idempotent element, the following characterization of a unit in an Ω-groupoid holds.

Theorem 3.1.3. [25]*An element $e \in G$ is a unit in an Ω-groupoid $\bar{\mathcal{G}}$ if and only if e is a unit in G.*

The unit element is a constant, but there is no nullary operations in the language of the groupoid. So, adding it in the language would be an extension of this lattice valued algebra.

Corollary 3.1.4. [25] *If a neutral element exists in an Ω-groupoid, then it is unique and idempotent.*

The absorptive element is given by the following definition from [25].

Let $\mathcal{G} = (G, \cdot)$ be a groupoid, and $\bar{\mathcal{G}} = (\mathcal{G}, E)$ an Ω-groupoid. An element $a \in G$ is said to be absorptive in $\bar{\mathcal{G}}$ if for all $x \in G$ the following hold:

$$E(x \cdot a, a) \geqslant \mu(x) \qquad \text{and} \qquad E(a \cdot x, a) \geqslant \mu(x).$$

An absorptive element in an Ω-groupoid is idempotent, and the following property is valid.

Theorem 3.1.5. [25]*If a_1 and a_2 are two absorptive elements in an Ω-groupoid $\bar{\mathcal{G}} = (\mathcal{G}, E)$, then the following holds:*

$$E(a_1, a_2) = \mu(a_1) \wedge \mu(a_2).$$

Then some further properties are introduced.

An Ω-groupoid $\bar{\mathcal{G}} = (\mathcal{G}, E)$ is commutative if the corresponding identity $x \cdot y \approx y \cdot x$ hold, i.e., if for all $x, y \in G$,

$$\mu(x) \wedge \mu(y) \leqslant E(x \cdot y, y \cdot x). \tag{3.1}$$

3.2 Ω-semigroup

Now the Ω-semigroup can be introduced in the following manner, see [25].

Let $S = (S, \cdot)$ be a groupoid and $\bar{\mathcal{S}} = (\mathcal{S}, E)$ an Ω-groupoid.

Then $\bar{\mathcal{S}}$ is an Ω-semigroup if the corresponding identity $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$ holds, i.e., if for all $x, y, z \in S$

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E^\mu(x \cdot (y \cdot z), (x \cdot y) \cdot z)). \tag{3.2}$$

As for other structures, an Ω-semigroup is defined on a crisp groupoid which need not be a semigroup.

An idempotent, commutative Ω-semigroup is called an Ω-semilattice.

An Ω-semilattice can be naturally ordered under some conditions, and this aspect will be elaborated in the sequel.

An Ω-lattice will be defined as an extension of an Ω-semilattice.

Let $\bar{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-groupoid.

A strongly compatible relation will be introduced in order to cover some further aspects of these structures.

A lattice valued relation $E : A^2 \to L$ on $\bar{\mathcal{G}}$ is said to be strongly compatible with the operation $\cdot$ on G [25], if the following formulas hold: for all $x, y, z \in G$

$$E(x, y) \leqslant E(x \cdot z, y \cdot z) \quad \text{and} \quad E(x, y) \leqslant E(z \cdot x, z \cdot y). \tag{3.3}$$

A strong compatibility together with some other properties, imply compatibility in the sense of (2.10), as it is stated in the following proposition:

Proposition 3.2.1. [25] *A strongly compatible, symmetric and transitive lattice valued relation ρ on $\bar{\mathcal{G}}$ fulfills the following: For $a_1, a_2, b_1, b_2 \in G$*

$$\rho(a_1, b_1) \wedge \rho(a_2, b_2) \leqslant \rho(a_1 \cdot a_2, b_1 \cdot b_2).$$

Further on, the strongly compatible lattice-valued equalities will be used.

An identity $u \approx v$ is said to be strongly satisfied on an Ω-groupoid $\bar{\mathcal{G}}$ if

(a) it holds on $\bar{\mathcal{G}}$ (i.e., if formula (2.13) is satisfied), and

(b) for $a, b, a_1, \ldots, a_n \in G$ and for the term-operations u^A and v^A on $\bar{\mathcal{G}}$ corresponding to terms u and v respectively,

$$E(a, u^A(a_1, \ldots, a_n)) \wedge E(v^A(a_1, \ldots, a_n), b) \leqslant E(a, b). \tag{3.4}$$

In example of associativity, it strongly holds on $\bar{\mathcal{G}}$ if (3.2) is satisfied and for all $a, b, x, y, z \in G$

$$E(a, x \cdot (y \cdot z)) \wedge E((x \cdot y) \cdot z, b) \leqslant E(a, b). \tag{3.5}$$

Clearly, if an identity $t_1 \approx t_2$ holds on $\mathcal{G}$, then it strongly holds on $\bar{\mathcal{G}}$.

Here we just recall the notion of the order on an Ω-groupoid $\bar{\mathcal{G}}$.

For an Ω-relation $R : G^2 \to \Omega$ on $\mu : G \to \Omega$ the antisymmetry with respect to an Ω-valued equality E is defined as follows ([25]):

R is antisymmetric with respect to E if for all $x, y \in G$,

$$R(x, y) \wedge R(y, x) \leqslant E(x, y). \tag{3.6}$$

Reflexivity and transitivity of an Ω-valued relation R on an Ω-valued set μ are defined before.

A reflexive, antisymmetric with respect to E and transitive Ω-valued relation R on μ is an Ω-valued order on μ, with respect to E.

In the following theorem, the natural Ω-valued ordering relation is defined on an Ω-semilattice.

Theorem 3.2.2. [25] *Let $\bar{\mathcal{S}} = (\mathcal{S}, E)$ be an Ω-semilattice on which associativity is strongly satisfied, and let E be strongly compatible. Let also $R : S^2 \to L$ be an Ω-valued relation on S, defined by*

$$R(x, y) := E(x, x \cdot y). \tag{3.7}$$

Then R is an Ω-valued order on $\bar{\mathcal{S}}$ with respect to the Ω-valued equality E.

Further we mention cancellable and regular Ω-semigroups, that are generalizations of Ω-groups (which will be defined in the sequel).

Let $\bar{\mathcal{S}} = (\mathcal{S}, E)$ be an Ω-semigroup. $\bar{\mathcal{S}}$ is cancellable [25] if

$$E(y, z) \geqslant E(x \cdot y, x \cdot z) \quad \text{and} \quad E(y, z) \geqslant E(y \cdot x, z \cdot x),$$

for all $x, y, z \in S$.

The following proposition is proved in [25].

Proposition 3.2.3. *An Ω-semigroup $\bar{\mathcal{S}}$ in which E is strongly compatible is cancellable if and only if for all $x, y, z \in S$*

$$E(y, z) = E(x \cdot y, x \cdot z) \quad \text{and} \quad E(y, z) = E(y \cdot x, z \cdot x). \tag{3.8}$$

$\square$

Next, an Ω-semigroup $\bar{\mathcal{S}} = (\mathcal{S}, \mu, E^{\mu})$ is regular [25] if for every $a \in S$ there is $x \in S$ such that

$$\mu(a) \leqslant \mu(x) \quad \text{and} \quad \mu(a) \leqslant E((a \cdot x) \cdot a, a). \tag{3.9}$$

By associativity it can be checked that a regular Ω-semigroup fulfills also condition

$$\mu(a) \leqslant E(a \cdot (x \cdot a), a). \tag{3.10}$$

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3.3 Ω-quasigroup

In the following part a notion of Ω-quasigroups developed in paper [73] will be presented. $(\Omega, \leqslant)$ is a complete lattice.

The starting notion is again an Ω- groupoid, so the Ω-quasigroups are a special type of an Ω- groupoid.

Let $(\mathcal{Q}, E)$ be an Ω groupoid, where $\mathcal{Q} = (Q, \cdot)$.

First, solutions of some equations will be defined.

If $(\mathcal{Q}, E)$ be an Ω-groupoid, then the formulas $a \cdot x = b$ and $y \cdot a = b$, where $a, b \in Q$, and x, y are variables, are called linear equations over $(\mathcal{Q}, E)$.

An equation $a \cdot x = b$ is said to be solvable over $(\mathcal{Q}, E)$ [73] if there is $c \in Q$ such that

$$\mu(a) \wedge \mu(b) \leqslant \mu(c) \wedge E(a \cdot c, b). \tag{3.11}$$

Similarly, an equation $y \cdot a = b$ is solvable over $(\mathcal{Q}, E)$ if there is $d \in Q$ such that

$$\mu(a) \wedge \mu(b) \leqslant \mu(d) \wedge E(d \cdot a, b). \tag{3.12}$$

Elements c and d in 3.11 and 3.12 are called solutions of equations $a \cdot x = b$ and $y \cdot a = b$, respectively in $(\mathcal{Q}, E)$ [73].

If c and d are solutions of $a \cdot x = b$ and $y \cdot a = b$, respectively in $(\mathcal{Q}, E)$, then for every $p \in \Omega$ satisfying $p \leqslant \mu(a) \wedge \mu(b)$,

$$p \leqslant \mu(c) \wedge E(a \cdot c, b), \tag{3.13}$$

and

$$p \leqslant \mu(d) \wedge E(d \cdot a, b). \tag{3.14}$$

Now the notion of unique solvability will be elaborated.

The above equations is E-uniquely solvable over $(\mathcal{Q}, E)$ [73] if the following hold:

If c is a solution of the equation $a \cdot x = b$ over $(\mathcal{Q}, E)$ and $c_1 \in Q$ fulfills $E(a \cdot c_1, b) \geqslant p$ for some $p \leqslant \mu(a) \wedge \mu(b)$, then

$$E(c, c_1) \geqslant p. \tag{3.15}$$

Also, if d is a solution of the equation $y \cdot a = b$ over $(\mathcal{Q}, E)$ and $d_1 \in Q$ fulfills

$E(d_1 \cdot a, b) \geqslant p$ for some $p \leqslant \mu(a) \wedge \mu(b)$, then

$$E(d, d_1) \geqslant p. \tag{3.16}$$

If c_1 and d_1 are other solutions of equations $a \cdot x = b$ and $y \cdot a = b$, respectively, then conditions (3.15) and (3.16) hold.

This means that an E-uniquely solvable equation may have several solutions. This sounds as a contradiction, but in the context of Ω-structures it is natural, since these solutions are equal up to the Ω-equality E.

More precisely, the following proposition is valid.

Theorem 3.3.1. [73]*Let $(\mathcal{Q}, E)$ be an L-groupoid. If equations $a \cdot x = b$ and $y \cdot a = b$, are E-uniquely solvable over $(\mathcal{Q}, E)$ for all $a, b \in Q$, then for every $p \in L$ the quotient groupoid μ_p/E_p is a quasigroup.*

The definition of Ω-quasigroup will be stated in the sequel, [73].

An Ω-groupoid $(\mathcal{Q}, E)$ is an Ω-quasigroup, if all equations of the form $a \cdot x = b$ and $y \cdot a = b$ are E-uniquely solvable over $(\mathcal{Q}, E)$.

The converse of Theorem 3.3.1 also holds, as follows.

Theorem 3.3.2. [73]*Let $(\mathcal{Q}, E)$ be an Ω-groupoid. If for all $a, b \in Q$ and for every $p \leqslant \mu(a) \wedge \mu(b)$ the quotient groupoid μ_p/E_p is a quasigroup, then $(\mathcal{Q}, E)$ is an Ω-quasigroup.*

As in the classical case, the Ω-quasigroup is defined as a particular type of Ω-groupoids (i.e., the underlying algebra has one binary operation).

In the following, a notion of a lattice valued quasigroup is introduced in another way, starting with the underlying algebra with three binary operations.

Here the definition of Ω-equasigroup is given, and later it will be stated that it is equivalent with the notion of Ω-quasigroup. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be an algebra in the language with three binary operations, Ω a complete lattice and $E : Q^2 \to \Omega$ an Ω-valued compatible equality over $\mathcal{Q}$. Then, $(\mathcal{Q}, E)$ is an Ω-equasigroup, if the following identities hold.

$Q1 : y = x \cdot (x \backslash y);$

$Q2 : y = x \backslash (x \cdot y);$

$Q3 : y = (y/x) \cdot x;$

$Q4 : y = (y \cdot x)/x.$

This means that the following formulas should be satisfied:

$QE1: \ \mu(x) \wedge \mu(y) \leqslant E(y, x \cdot (x \backslash y));$

$QE2: \ \mu(x) \wedge \mu(y) \leqslant E(y, x \backslash (x \cdot y));$

$QE3: \ \mu(x) \wedge \mu(y) \leqslant E(y, (y/x) \cdot x);$

$QE4: \ \mu(x) \wedge \mu(y) \leqslant E(y, (y \cdot x)/x).$

The following theorem gives connection with the cut sets.

Theorem 3.3.3. [73] *An Ω-algebra $((Q, \cdot, \backslash, /), E)$ is an Ω-equasigroup, if and only if for every $p \in L$, the quotient structure μ_p/E_p is a classical equasigroup.*

The following corollary follows directly from the fact that an Ω-quasigroup is a reduct of an Ω-equasigroup.

Corollary 3.3.4. *If $((Q, \cdot, \backslash, /), E)$ is an Ω-equasigroup, then $((Q, \cdot), E)$ is an Ω-quasigroup.*

The opposite claim concerning extension can be defined by the Axiom of Choice (AC).

Usually in case of extensions of Ω-algebras, the new operations are defined using the Axiom of Choice.

In this case also by AC new operations will be defined.

Let $((Q, \cdot), E)$ be an Ω-groupoid which is an Ω-quasigroup. For every $p \in L$, the quotient groupoid $(\mu_p/E_p, \cdot)$ is considered, which is a quasigroup. Now the operation $\cdot$ is defined by $[a]_{E_p} \cdot [b]_{E_p} = [a \cdot b]_{E_p}, \ a, b \in \mu_p.$

By Theorem 3.3.3, the structure $(\mu_p/E_p, \cdot, \backslash, /)$ is an equasigroup, where the operations $\backslash$ and $/$ are the usual ones:

$$[a]_{E_p} \backslash [b]_{E_p} = [c]_{E_p} \ \text{ if and only if } \ [a]_{E_p} \cdot [c]_{E_p} = [b]_{E_p}, \ \text{ and}$$
$$[b]_{E_p} / [a]_{E_p} = [d]_{E_p} \ \text{ if and only if } \ [d]_{E_p} \cdot [a]_{E_p} = [b]_{E_p}.$$

Now, binary operations $\backslash$ and $/$ over Q can be defined as follows by AC [73]:

For every pair $a, b \in Q$, $\ a \backslash b = c$, where c is an element chosen by AC from $[a]_{E_p} \backslash [b]_{E_p}$ in the quasigroup μ_p/E_p, where $p = \mu(a) \wedge \mu(b)$.

Similarly, $\ b/a = d$, where d is chosen by the AC from $[b]_{E_p} / [a]_{E_p}$ in μ_p/E_p, for $p = \mu(a) \wedge \mu(b)$.

If $((Q, \cdot), E)$ is an Ω-groupoid which is an Ω-quasigroup then the operations $\backslash$ and $/$ over Q are well defined.

Lemma 3.3.5. [73] *Let* $((Q, \cdot), E)$ *be an* Ω-*groupoid which is an* Ω-*quasigroup. Then for every* $q \in L$ *and for all* $a, b \in \mu_q$, *in the quasigroup* $(\mu_q/E_q, \cdot, \backslash, /)$, $[a \backslash b]_{E_q} = [a]_{E_q} \backslash [b]_{E_q}$, *and* $[a/b]_{E_q} = [a]_{E_q}/[b]_{E_q}$, *is satisfied, where the operations* $\backslash$ *and* $/$ *on the left hand sides are the ones defined on* Q *by the Axiom of Choice.*

Now, the opposite theorem is stated.

Theorem 3.3.6. *Let* $((Q, \cdot), E)$ *be an* Ω-*groupoid which is an* Ω-*quasigroup. Then the structure* $((Q, \cdot, \backslash, /), E)$ *is an* Ω-*equasigroup, where the binary operations* $\backslash$ *and* $/$ *over* Q *are defined by the Axiom of Choice as above.*

3.4 Ω- groups

Next task is to introduce Ω-groups as extensions of Ω-groupoids. The results in this section are from [28], while the original results concerning Ω- groups are presented in Chapter 4.

We deal here with Ω-algebras $\overline{\mathcal{G}} = (\mathcal{G},\ E)$, where $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $\cdot$, a unary operation $^{-1}$ and a constant e, and $E : G^2 \to \Omega$ is an Ω-valued equality on $\mathcal{G}$.

The fact that μ is a fuzzy (Ω-valued) subalgebra of $\mathcal{G}$, for all $x, y \in G$, insures that formulas (3.26) are satisfied.

Same as for other structures, $\mathcal{G}$ is not a (classical) group in general. Hence, formulas (3.26) here do not mean that μ is a fuzzy subgroup of $\mathcal{G}$, since $\mathcal{G}$ is not a group.

Let
$$\overline{\mathcal{G}} = (\mathcal{G},\ E)$$
be an Ω- algebra in which $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\cdot)$, unary operation $(^{-1})$ and a constant (e). Then $\overline{\mathcal{G}}$ is an Ω-group [28] if it satisfies the classical identities for groups

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z;$$

$$x \cdot e \approx x, \ \ e \cdot x \approx x;$$

$$x \cdot x^{-1} \approx e, \ \ x^{-1} \cdot x \approx e.$$

This means that for all x, y, z from G,

(i) $E(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z),$

(ii) $E(x \cdot e, x) \geqslant \mu(x)$ and $E(e \cdot x, x) \geqslant \mu(x),$

(iii) $E(x \cdot x^{-1}, e) \geqslant \mu(x)$ and $E(x^{-1} \cdot x, e) \geqslant \mu(x).$

As introduced above, an element e is the unit in $\overline{\mathcal{G}}$, and x^{-1} is the inverse of element x in $\overline{\mathcal{G}}$.

Algebra $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is called the underlying algebra of Ω-group $\overline{\mathcal{G}}$.

According to the definitions presented in Preliminaries, the fact that the Ω-valued set μ determined by E is a fuzzy subalgebra of $\mathcal{G}$ means that for all $x, y \in G$

$$\mu(x \cdot y) \geqslant \mu(x) \wedge \mu(y),$$

$$\mu(x^{-1}) \geqslant \mu(x),$$

$$\mu(e) = 1.$$

Analogously to the classical algebra, an Ω- group can be defined by a simplified system of axioms.

Namely, as above let $\overline{\mathcal{G}}' = (\mathcal{G}', E)$ be an Ω- algebra, in which $\mathcal{G}' = (G, \cdot, ', e')$ is an algebra with a binary operation $(\cdot)$, unary operation $(')$ and a constant (e'), and let $E : G^2 \to \Omega$ be an Ω-valued equality on $\mathcal{G}$.

The following two theorems are proved in [28].

Theorem 3.4.1. *Let* $\overline{\mathcal{G}}' = (\mathcal{G}', E)$ *be an* Ω- *algebra described above, fulfilling the following:*

(i') $E(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z),$

(ii') $E(x \cdot e', x) \geqslant \mu(x),$

(iii') $E(x \cdot x', e') \geqslant \mu(x),$

for all x, y, z *from* G. *Then,* $\overline{\mathcal{G}}'$ *is an* Ω-group.

As expected, if the basic groupoid is a classical group, then also the corresponding Ω-groupoid is an Ω-group.

Theorem 3.4.2. *Let* $\mathcal{G} = (G, \cdot, ^{-1}, e)$ *be a group, and* E *an* Ω-valued equality on $\mathcal{G}$. *Then,* $\overline{\mathcal{G}} = (\mathcal{G}, E)$ *is an* Ω-group.

3.4.1 Separated Ω-groups

If the separation property for E is also valid, the additional properties are obtained. They are presented in the sequel and proved in [28].

Proposition 3.4.3. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a separated Ω- group. Then $x \cdot e = e \cdot x = x$.*

As introduced above for groupoids, $x \in G$ is an idempotent element of an Ω- group $\overline{\mathcal{G}} = (\mathcal{G}, E)$ if

$$E(x \cdot x, x) \geqslant \mu(x). \tag{3.17}$$

When the separation property is assumed, additional statements are valid.

Proposition 3.4.4. *An element $x \in G$ of a separated Ω- group $\overline{\mathcal{G}}$ is idempotent if and only if x is idempotent in $\mathcal{G}$ (i.e., if $x^2 = x$).*

Moreover, using the separation property, the property that idempotent element is unique is obtained.

Proposition 3.4.5. *The unit e of a separated Ω-group $\overline{\mathcal{G}}$ is a unique idempotent element in $\overline{\mathcal{G}}$.*

In separated Ω-groups the operation $^{-1}$ is a classical involution, as formulated in the following proposition.

Proposition 3.4.6. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a separated Ω group. Then $(x^{-1})^{-1} = x$ for every $x \in G$.*

Also the following corollary can be obtained.

Corollary 3.4.7. *If $\overline{\mathcal{G}} = (\mathcal{G}, E)$ is a separated Ω-group, then $\mu(x) = \mu(x^{-1})$, for all $x \in G$.*

Next a particular way some identities are satisfied in case of separated Ω-groups are given.

Proposition 3.4.8. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a separated Ω-group. Let $t(x)$ be a term containing a variable x only. Then the identity*

$$t(x) \approx x$$

holds on $\overline{\mathcal{G}}$ if and only this identity holds on $\mathcal{G}$.

In a special case, the following consequence is obtained.

Corollary 3.4.9. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a separated Ω group. Then the algebra $\mathcal{G}$ satisfies identity $(x \cdot x^{-1}) \cdot x \approx x$.*

For the separated Ω-groups, cancellability holds in the same manner as in the classical case.

Theorem 3.4.10. [28] *A separated Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$ is cancellative.*

The cancellativity of a separated Ω group does not imply that the underlying algebra is cancellative, analogously as for other properties.

In the next theorem we discuss a known identity that is valid for classical groups.

Theorem 3.4.11. [28] *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be a separated Ω-group. Then*

a) $E((xy)^{-1}, y^{-1}x^{-1}) \geqslant \mu(x) \wedge \mu(y)$

b) $E((x_1 \cdots x_n)^{-1}, x_n^{-1} \cdots x_1^{-1}) \geqslant \bigwedge_{i=1}^{n} \mu(x_i)$

for all $x, y, x_1, ..., x_n \in G$.

3.4.2 Ω-subgroups

Here a notion of Ω-subgroups is introduced as in [28] and this notion is used to construct and investigate a new concept of Ω- normal subgroups in Chapter 4.

If $\nu : G \to \Omega$ is a nonempty Ω-valued subset of an Ω-valued set $\mu : G \to \Omega$, R an Ω-valued relation on μ, and $S : G^2 \to \Omega$ an Ω-valued relation on G, then, S is a restriction of R to ν if

$$S(x, y) = R(x, y) \wedge \nu(x) \wedge \nu(y). \tag{3.18}$$

If A is a nonempty set, (A, E) an Ω- set on A and E_1 the restriction of E to a nonempty lattice valued subset μ_1 of μ (where μ is determined by E), then (A, E_1) is also an Ω-set on A.

This is true also for Ω-algebras:

Proposition 3.4.12. *If $(\mathcal{A}, E)$ is an Ω-valued algebra on an algebra $\mathcal{A} = (A, F)$ and μ_1 is an Ω-valued subset of μ and a subalgebra of $\mathcal{A}$, then also $(\mathcal{A}, E_1)$ is an Ω-valued algebra on $\mathcal{A}$, where E_1 is the restriction of E to μ_1 .*

Now Ω-subgroups are defined.

If $\overline{\mathcal{G}} = (\mathcal{G}, E)$ and $\overline{\mathcal{G}}_1 = (\mathcal{G}, E_1)$ are Ω-groups over the same algebra $\mathcal{G} = (G, \cdot, ^{-1}, e)$, then $\overline{\mathcal{G}}_1$ is an Ω- subgroup of Ω-group $\overline{\mathcal{G}}$, if E_1 is a restriction of E to the Ω-valued subalgebra μ_1 of $\mathcal{G}$, determined by E_1.

Theorem 3.4.13. [28]*Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-group and $E_1 : G^2 \to L$ an Ω-valued relation on G, satisfying the formula:*

$$E_1(x, y) = E(x, y) \wedge E_1(x, x) \wedge E_1(y, y). \qquad (3.19)$$

Then the structure $\overline{\mathcal{G}}_1 = (\mathcal{G}, E_1)$ is an Ω-subgroup of the Ω-group $\overline{\mathcal{G}}$ if and only if it satisfies:

$$E_1(x, x) \wedge E_1(y, y) \leqslant E_1(x \cdot y, x \cdot y), \qquad (3.20)$$
$$E_1(x, x) \leqslant E_1(x^{-1}, x^{-1}), \qquad (3.21)$$
$$E_1(e, e) = 1. \qquad (3.22)$$

Analogously as in the classical group theory the following corollary is true.

Corollary 3.4.14. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω- group, $\mu_1 : G \to L$ a nonempty Ω-valued subset of μ, and E_1 the restriction of E to μ_1. Then the structure $\overline{\mathcal{G}}_1 = (\mathcal{G}, E_1)$ is an Ω-subgroup of $\overline{\mathcal{G}}$ if and only if it is an Ω-algebra.*

The intersection of a family of Ω-subgroups of an Ω-group is, under particular conditions, also an Ω-subgroup, which is formulated in the following theorem.

Theorem 3.4.15. *Let $\{\overline{\mathcal{G}}_i = (\mathcal{G}, E_i) \mid i \in I\}$ be a nonempty family of Ω-subgroups of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E)$, where $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is a given algebra. Further, let $\delta = \bigcap_{i \in I} \mu_i$ and let E^δ be the restriction of E to δ. Then the structure $\overline{\mathcal{G}^\delta} = (\mathcal{G}, E^\delta)$, is an Ω-subgroup of the Ω-group $\overline{\mathcal{G}}$.*

3.4.3 Cut properties of Ω-groups

As for other structures, investigating cut properties of Ω-groups it can be noted that the cuts of the Ω-valued subalgebra μ (determined by E) are not

subgroups of $\mathcal{G}$ in general (which is essentially a consequence of the fact that $\mathcal{G}$ is not always a group). However, their quotient structures with respect to the corresponding cuts of the Ω-valued equality E are groups, as already stated for algebras in general.

Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-algebra. By known properties of lattice valued structures, for every $p \in L$, the cut μ_p of the Ω-valued subalgebra μ ($\mu(x) = E(x, x)$) of $\mathcal{G}$ is a subalgebra of $\mathcal{G}$. Further, the cut relation E_p of E is a congruence relation on μ_p.

For the sake of completeness of the material, this is the formulation of the related theorem for groups:

Theorem 3.4.16. [28]*Let $\overline{\mathcal{G}} = (\mathcal{G}, E)$ be an Ω-algebra. Then, $\overline{\mathcal{G}}$ is an Ω-group if and only if for every $p \in L$ the quotient structure μ_p/E_p is a group.*

3.5 Ω-lattice

In this section results on Ω-lattices are presented. As in classical case, Ω-lattices are introduced as ordered structures and also as Ω-algebras and equivalences of two approaches are defined. Moreover, complete Ω-lattices are also introduced. Majority of notions and results are from [47] and [49].

3.5.1   An Ω-lattice as an ordered structure

Let A be a nonempty set and let E be an Ω-valued equality A.

As already mentioned, an Ω-valued relation $R : A^2 \to \Omega$ on A is E-antisymmetric, if

$$R(x, y) \wedge R(y, x) = E(x, y), \quad \text{for all} \quad x, y \in A. \tag{3.23}$$

Let (A, E) be an Ω-set. An Ω-valued relation $R : A^2 \to \Omega$ on A is an Ω-valued order on (A, E), if it fulfills the strictness property (2.3), it is E-antisymmetric, and it is transitive

An ordered triple (A, E, R) is an Ω-poset, if (A, E) is an Ω-set, and $R : A^2 \to \Omega$ is an Ω-valued order on (A, E).

By (3.23), $R(x, x) = E(x, x)$, for every $x \in M$.

As before (2.12), by μ the Ω-valued function on M, defined by $\mu(x) = E(x, x)$ is denoted.

Both E and R are reflexive relations on μ, in the sense of (1.4), i.e., $\mu(x) = E(x, x) = E(x, x)$.

As already stated in Lemma 2.2.2, every cut E_p of E is an equivalence relation on the cut μ_p of μ. As usual, by $[x]_{E_p}$ the equivalence class of $x \in \mu_p$ is denoted. μ_p/E_p is the corresponding quotient set: for $p \in \Omega$

$$[x]_{E_p} := \{y \in \mu_p \mid xE_py\}, x \in \mu_p; \quad \mu_p/E_p := \{[x]_{E_p} \mid x \in \mu_p\}.$$

As in classical case, the ordering relation on the set of equivalence classes is obtained, which is stated in the next proposition.

Proposition 3.5.1. [47] *Let (M, E, R) be an Ω-poset. Then for every $p \in \Omega$, the binary relation $\leq_p$ on μ_p/E_p, defined by*

$$[x]_{E_p} \leq_p [y]_{E_p} \quad \text{if and only if} \quad (x, y) \in R_p \tag{3.24}$$

is an ordering relation.

In the following part the notions of a pseudo-infimum and a pseudo-supremum are introduced, see [47]:

Let (M, E, R) be an Ω-poset and $a, b \in M$.

An element $c \in M$ is a pseudo-infimum of a and b, if the following holds: (i) $\mu(a) \wedge \mu(b) \leq R(c, a) \wedge R(c, b)$, and for every $p \leq \mu(a) \wedge \mu(b)$, for every $x \in \mu_p$, $\quad p \leq R(x, a) \wedge R(x, b) \implies p \leq R(x, c)$.

An element $d \in M$ is a pseudo-supremum of $a, b \in M$, if the following holds: (ii) $\mu(a) \wedge \mu(b) \leq R(a, d) \wedge R(b, d)$, and for every $p \leq \mu(a) \wedge \mu(b)$, for every $x \in \mu_p$, $\quad p \leq R(a, x) \wedge R(b, x) \implies p \leq R(d, x)$.

A pseudo-infimum (supremum) of a and b belongs to μ_p for every $p \leq \mu(a) \wedge \mu(b)$.

A pseudo-infimum and a pseudo-supremum for given $a, b \in M$, if they exist, are not unique in general.

If more pseudo-infima (suprema) of two elements a, b exist, they belong to the same equivalence class in μ_p/E_p, for $p \leq \mu(a) \wedge \mu(b)$, which is proved in the following proposition.

Proposition 3.5.2. [47] *Let (M, E, R) be an Ω-poset and $a, b, c, c_1, d, d_1 \in M$.*

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If c is a pseudo-infimum of a and b, then

$\mu(a) \wedge \mu(b) \leqslant E(c, c_1)$ if and only if c_1 is also a pseudo-infimum of a and b.

Analogously, if d is a pseudo-supremum of a and b, then

$\mu(a) \wedge \mu(b) \leqslant E(d, d_1)$ if and only if d_1 is also a pseudo-supremum of a and b.

Since for $p \leqslant q$, every equivalence class of μ_q/E_q is contained in a class of μ_p/E_p, pseudo-infima (suprema) of two elements a, b, if they exist, belong to the same equivalence class in μ_p/E_p, for $p \leqslant \mu(a) \wedge \mu(b)$.

By the above definition, in case E is a separated equality on M, for $p = \mu(a)$, a is the unique pseudo-infimum (supremum) of one element $a \in M$ (i.e., for a and b with $a = b$).

This follows from the fact that for every $a \in M$, the class $[a]_{E_{\mu(a)}}$ consists of the single element a.

An Ω-poset (M, E, R) is an Ω-lattice as an ordered structure [47], if for every $a, b \in M$ there exist a pseudo-infimum and a pseudo-supremum.

In the following, infimum and supremum of elements a and b in an ordered set if they exist, will be denoted by $\inf(a, b)$ and $\sup(a, b)$, respectively.

The following theorem gives necessary and sufficient condition that an Ω-poset is an Ω-lattice as an ordered structure.

Theorem 3.5.3. [47] *Let (M, E, R) be an Ω-poset. Then (M, E, R) is an Ω-lattice as an ordered structure if and only if for every $q \in \Omega$, the poset $(\mu_q/E_q, \leq_q)$ is a lattice, where the relation $\leq_q$ on the quotient set μ_q/E_q is defined by (3.24) and the following holds: for all $a, b \in M$, and $p = \mu(a) \wedge \mu(b)$,*

$$\inf([a]_{E_p}, [b]_{E_p}) \subseteq \inf([a]_{E_q}, [b]_{E_q}), \ \sup([a]_{E_p}, [b]_{E_p}) \subseteq \sup([a]_{E_q}, [b]_{E_q}) \quad (3.25)$$

for every $q, q \leqslant p$

According to Theorem 3.5.3, if E is a separated equality and if $\mu_1 \neq \emptyset$, then (μ_1, R_1) is a lattice. In this case E_1 is a diagonal relation and the congruence classes are one-element sets. Then also R_1 is an order on μ_1 and the posets (μ_1, R_1) and $(\mu_1/E_1, \leq_1)$ are order isomorphic. Since $(\mu_1/E_1, \leq_1)$ is a lattice by Theorem 3.5.3, (μ_1, R_1) is also a lattice.

In case for $p \in \Omega$ equivalence classes under E_p are all one-element sets, then (μ_p, R_p) is a lattice.

3.5.2 An Ω-lattice as an Ω-algebra

In this section the notion of an Ω-lattice as an Ω-algebra is developed, according to the definition in section 2.3. This algebraic approach was first introduced in [122], and further developed in [47]. In the same paper results are adapted to the wider framework of both algebraic and relational approach.

In the manner of Ω-algebraic structures that is exploited here, in order to define a lattice as an Ω-algebra, the starting notion is a bi-groupoid, a classical algebra with two binary operations. Some particular examples of bi-groupoids are lattices, bi-semilattices and rings.

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, as an algebra with two binary operations and let $E : M^2 \to \Omega$ be an Ω-valued equality on M such that (M, E) is an Ω-set.

Then $(\mathcal{M}, E)$ is an Ω-bi-groupoid, if E satisfies the following: for all $x, y, z, t \in M$,

$$E(x, y) \wedge E(z, t) \leqslant E(x \sqcap z, y \sqcap t) \quad \text{and} \quad E(x, y) \wedge E(z, t) \leqslant E(x \sqcup z, y \sqcup t).$$

As above, this property means that E is compatible with operations $\sqcap$ and $\sqcup$.

In the following proposition, some basic properties of the above notions are given.

Proposition 3.5.4. [47] *If E is a compatible Ω-valued equality on a bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$, and $\mu : M \to \Omega$ is defined by $\mu(x) = E(x, x)$, then the following hold:*

(i) For all $x, y \in M$,

$$\mu(x) \wedge \mu(y) \leqslant \mu(x \sqcap y) \quad \text{and} \quad \mu(x) \wedge \mu(y) \leqslant \mu(x \sqcup y). \qquad (3.26)$$

(ii) For every $p \in \Omega$, the cut μ_p of μ is a sub-bi-groupoid of $\mathcal{M}$.

In the sequel the notion of an Ω-lattice as an Ω-algebra will be introduced as in [122, 47].

An Ω-algebra $(\mathcal{M}, E)$ is an Ω-lattice as an Ω-algebra (Ω-lattice as an algebra), if it satisfies lattice identities:

$\ell 1 : \ x \sqcap y \approx y \sqcap x$
$\ell 2 : \ x \sqcup y \approx y \sqcup x$ $\quad (commutativity)$

$\ell 3: \ x \sqcap (y \sqcap z) \approx (x \sqcap y) \sqcap z$ (associativity)
$\ell 4: \ x \sqcup (y \sqcup z) \approx (x \sqcup y) \sqcup z$

$\ell 5: \ (x \sqcap y) \sqcup x \approx x$ (absorption)
$\ell 6: \ (x \sqcup y) \sqcap x \approx x.$

As above, let the mapping $\mu : M \to \Omega$ be defined by $\mu(x) = E(x, x)$.

As defined in the subsection 2.4, the satisfiability of the identities above is equivalent to the following conditions:

For all $x, y, z \in M$, the following formulas are satisfied, where

$$L1: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcap y, y \sqcap x) \tag{3.27}$$

$$L2: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcup y, y \sqcup x) \tag{3.28}$$

$$L3: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcap y) \sqcap z, x \sqcap (y \sqcap z)) \tag{3.29}$$

$$L4: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcup y) \sqcup z, x \sqcup (y \sqcup z)) \tag{3.30}$$

$$L5: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcap y) \sqcup x, x) \tag{3.31}$$

$$L6: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcup y) \sqcap x, x). \tag{3.32}$$

As consequences of the definition of Ω-lattices as algebras some additional properties are valid, which is formulated in the following propositions.

Lemma 3.5.5. [122] *An Ω-lattice $(\mathcal{M}, E)$ fulfills the following special absorption identities:*

$$(y \sqcap x) \sqcup x \approx x \quad and \quad (y \sqcup x) \sqcap x \approx x.$$

Proposition 3.5.6. [122] *In an Ω-lattice $(\mathcal{M}, E)$ as an algebra, the idempotent identities*

$$x \sqcup x \approx x \quad and \quad x \sqcap x \approx x$$

are valid.

Under the condition that the separation property is valid, the following

property is valid.

Proposition 3.5.7. [122] *Let $(\mathcal{M}, E)$ be an Ω-lattice, in which E is a separated Ω-valued equality. Then the idempotent law $x \sqcap x \approx x$ is valid in $(\mathcal{M}, E)$ if and only if the operation $\sqcap$ is idempotent in the bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$, and analogously $x \sqcup x \approx x$ holds in $(\mathcal{M}, E)$ if and only if $\sqcup$ is idempotent in $\mathcal{M}$.*

Similarly as for other structures, if a bi-groupoid $\mathcal{M}$ is a classical lattice, and E is an Ω-valued compatible equality on $\mathcal{M}$, then $(\mathcal{M}, E)$ is an Ω-lattice, as formulated in the following proposition.

Proposition 3.5.8. [122] *If $\mathcal{M} = (M, \sqcap, \sqcup)$ is a lattice and E is a compatible Ω-valued equality on $\mathcal{M}$, then $(\mathcal{M}, E)$ is an Ω-lattice.*

The following theorems gives the properties of cuts of Ω-lattices.

Theorem 3.5.9. [122] *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, and let E be an Ω-valued compatible equality on $\mathcal{M}$. Then, $(\mathcal{M}, E)$ is an Ω-lattice if and only if for every $p \in \Omega$, the quotient structure μ_p/E_p is a lattice.*

Proposition 3.5.10. [47] *Let $(\mathcal{M}, E)$ be an Ω-lattice and $p, q \in \Omega$, with $p \leqslant q$. Then, the mapping $f : \mu_q/E_q \to \mu_p/E_p$, defined by $f([x]_{E_q}) = [x]_{E_p}$ is a lattice homomorphism.*

3.5.3   Equivalence of two approaches

As in the classical case, two introduced approaches the Ω-lattice as an ordered structure and the Ω-lattice as an algebra are equivalent which will be demonstrated in this section.

In the first part of this section it is stated that an Ω-lattice as an ordered structure is an Ω-lattice as an algebra, using the Axiom of Choice to construct the binary operations on the related algebraic structure.

Therefore the Axiom of Choice (AC) will be assumed throughout this section.

Let (M, E, R) be an Ω-lattice as an ordered structure. Two binary operations, $\sqcap$ and $\sqcup$ on M are defined as follows:

for every pair a, b of elements from M, $a \sqcap b$ is an arbitrary, fixed pseudo-infimum of a and b, and $a \sqcup b$ is an arbitrary, fixed pseudo-supremum of a and b.

The operations $\sqcap$ and $\sqcup$ on M are well defined by Axiom of Choice. An element is chosen among all pseudo-infima (suprema) of a and b and then this element is fixed. By the definition of an Ω-lattice, for any $a, b \in M$, a pseudo-infimum and a pseudo-supremum exist.

If E is a separated Ω-valued equality, for every $a \in M$,

$a \sqcap a = a$ and $a \sqcup a = a$, so the operation is defined uniquely in this case.

Since two binary operations are defined in this way, the structure $\mathcal{M} = (M, \sqcap, \sqcup)$ is a bi-groupoid.

By the following proposition μ is an Ω-sub-bigroupoid of $\mathcal{M}$.

Proposition 3.5.11. [47] *Let (M, E, R) be an Ω-lattice, $\mu : M \to \Omega$ defined by (2.12) ($\mu(x) = E(x, x)$) and $\mathcal{M} = (M, \sqcap, \sqcup)$ a bi-groupoid, as defined above. Then, for all $x, y \in M$*

$$\mu(x) \wedge \mu(y) \leqslant \mu(x \sqcap y) \quad \text{and} \quad \mu(x) \wedge \mu(y) \leqslant \mu(x \sqcup y). \qquad (3.33)$$

Starting point here is an Ω-lattice (M, E, R) as an ordered structure, in which, by Theorem 3.5.3, for all $p \in \Omega$, the quotient structure $(\mu_p/E_p, \leq_p)$ is a lattice, where $\leq_p$ is defined by (3.24) and (3.25).

For $x, y \in \mu_p$, the infimum and supremum of $[x]_{E_p}$ and $[y]_{E_p}$ are denoted by $[x]_{E_p} \sqcap_p [y]_{E_p}$ and $[x]_{E_p} \sqcup_p [y]_{E_p}$, respectively.

Lemma 3.5.12. [47] *Let (M, E, R) be an Ω-lattice as an ordered structure, and $p \in \Omega$. Then for all $x, y \in \mu_p$, in the lattice $(\mu_p/E_p, \leq_p)$ it holds:*

$$[x]_{E_p} \sqcap_p [y]_{E_p} = [x \sqcap y]_{E_p} \quad \text{and} \quad [x]_{E_p} \sqcup_p [y]_{E_p} = [x \sqcup y]_{E_p},$$

where $\sqcap$ and $\sqcup$ are the operations on M introduced by the Axiom of Choice.

Since $\mu : M \to \Omega$ defined by $\mu(x) = E(x, x)$) is an Ω-sub-bigroupoid of $(M, \sqcap, \sqcup)$, for every $p \in \Omega$, μ_p is a (classical) sub-bigroupoid of $(M, \sqcap, \sqcup)$. The following proposition follows by this fact.

Proposition 3.5.13. [47] *Let (M, E, R) be an Ω-lattice as an ordered structure, and $\sqcap$, $\sqcup$ the corresponding binary operations on M introduced as above. Then, E is compatible with $\sqcap$ and $\sqcup$.*

Finally, the operations $\sqcap$ and $\sqcup$ satisfy lattice-theoretic identities $\ell 1, \ldots, \ell 6$, hence the formulas $L1, \ldots, L6$ hold as formulated in the following proposition.

Proposition 3.5.14. [47] *Let (M, E, R) be an Ω-lattice as an ordered structure, and $\sqcap$, $\sqcup$ the corresponding binary operations on M, introduced as above. Then, the formulas $L1 - L6$ are satisfied.*

Theorem 3.5.15. [47]*If (M, E, R) is an Ω-lattice as an ordered structure, and $\mathcal{M} = (M, \sqcap, \sqcup)$ the bi-groupoid in which operations $\sqcap$, $\sqcup$ are introduced as above, then $(\mathcal{M}, E)$ is an Ω-lattice as an algebra.*

A property of Ω-lattices as ordered structures, analogous to the well known fact about lattices: $x \leq y$ if and only if $x \wedge y = x$ is also valid:

Proposition 3.5.16. [47] *If (M, E, R) is an Ω-lattice as an ordered structure, then for all $x, y \in M$,*

$$\mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x) = R(x, y).$$

In the following part the opposite inclusion is demonstrated: the Ω-lattice as an algebra is the Ω-lattice as an ordered structure.

Now the starting point is an Ω-lattice as an algebra. First step is to introduce an Ω-valued order on an Ω-lattice (as an algebra).

Theorem 3.5.17. [47] *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $(\mathcal{M}, E)$ an Ω-lattice as an algebra, where E is a separated Ω-valued equality on $\mathcal{M}$. Then the Ω-valued relation $R : M^2 \to \Omega$, defined by*

$$R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x) \tag{3.34}$$

is an Ω-valued order on M.

Moreover the relation R on an Ω-lattice determines the order on the cut-lattices:

Proposition 3.5.18. [47] *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, $(\mathcal{M}, E)$ an Ω-lattice as an algebra, and $R : M^2 \to \Omega$ an Ω-valued relation on M defined by 3.34. For every $p \in \Omega$, for all $x, y \in \mu_p$ and $[x]_{E_p}, [y]_{E_p} \in \mu_p/E_p$,*

$$[x]_{E_p} \leq_p [y]_{E_p} \text{ if and only if } xR_py.$$

Finally, in the next theorem it is stated that an Ω-lattice as an algebra is an Ω-lattice as an ordered structure.

Theorem 3.5.19. *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, $(\mathcal{M}, E)$ an Ω-lattice as an algebra in which E is separated. Let $R : M^2 \to \Omega$ be an Ω-valued relation on M defined by $R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x)$. Then, (M, E, R) is an Ω-lattice as an ordered structure.*

3.6 Complete Ω-Lattices

In this section, the notion of complete Ω- lattices is introduced and some special notions as in the ordinary theory of lattices and ordered sets are investigated.

As defined in Preliminaries, complete lattices are partially ordered sets in which all subsets have a supremum and an infimum. Complete Ω-Lattices are developed as a special type of Ω-Lattices, [48].

In order to develop notions of the supremum and the infimum in this context, notions of upper bounds and lower bounds are introduced and their basic properties are given.

Let (M, E, R) be an Ω-poset and $A \subseteq M$. An element $u \in M$ is an upper bound of A (under R), if for every $a \in A$

$$\bigwedge(\mu(x) \mid x \in A) \le R(a, u).$$

An element $v \in M$ is a lower bound of A, if for every $a \in A$

$$\bigwedge(\mu(x) \mid x \in A) \le R(v, a).$$

An element $u \in M$ is an upper bound of $A \subseteq M$, if

$$(\forall a)(a \in A \ \Rightarrow \ (\bigwedge(\mu(x) \mid x \in A) \le R(a, u))). \qquad (3.35)$$

v is a lower bound of A, if

$$(\forall a)(a \in A \ \Rightarrow \ (\bigwedge(\mu(x) \mid x \in A) \le R(v, a))). \qquad (3.36)$$

Proposition 3.6.1. [48] *If u is an upper bound of $A \subseteq M$ in an Ω-poset* (M, E, R), *then*

$$\bigwedge(R(x, u) \mid x \in A) = \bigwedge(\mu(x) \mid x \in A).$$

Similarly, if v is a lower bound of $A \subseteq M$ in an Ω-poset (M, E, R), then

$$\bigwedge(R(v, x) \mid x \in A) = \bigwedge(\mu(x) \mid x \in A).$$

Next the definitions of a pseudo-supremum and a pseudo-infimum and the properties of these notions, [48] are introduced.

Let (M, E, R) be an Ω-poset and $A \subseteq M$. Then an element $u \in M$ is a pseudo-supremum of A, if for every $p \in \Omega$, such that $p \leq \bigwedge(\mu(x) \mid x \in A)$, the following hold:

(i) u is an upper bound of A and

(ii) if there is $u_1 \in M$ such that $p \leq R(a, u_1)$ for every $a \in A$, then $p \leq R(u, u_1)$.

Dually, an element $v \in M$ is a pseudo-infimum of A, if for every $p \in \Omega$, such that $p \leq \bigwedge(\mu(x) \mid x \in A)$, the following hold:

(j) v is a lower bound of A and

(jj) if there is $v_1 \in M$ such that $p \leq R(v_1, a)$ for every $a \in A$, then $p \leq R(v_1, v)$.

Proposition 3.6.2. [48] *Let (M, E, R) be an Ω-poset, let $A \subseteq M$ and let $u \in M$ be a pseudo-supremum (pseudo-infimum) of $A \subseteq M$. Then $v \in M$ is also a pseudo-supremum (pseudo-infimum) of $A \subseteq M$, if and only if $\bigwedge(\mu(x) \mid x \in A) \leq E(u, v)$.*

The pseudo-supremum and pseudo-infimum are not unique in general. However, they are unique up to the equivalence class: two pseudo-suprema u, v of A belong to the same equivalence class μ_p/E_p for every $p \leq \bigwedge(\mu(x) \mid x \in A)$.

In the following, the pseudo-top and bottom elements for subsets of M in an Ω-poset (M, E, R) are introduced and some of their properties are presented, [48].

A pseudo-top of A, $A \subseteq M$, is an element $t \in A$ such that for every $y \in A$

$$\bigwedge(\mu(x) \mid x \in A) \leq R(y, t).$$

Dually, a pseudo-bottom of A, $A \subseteq M$, is an element $b \in A$, such that for every $y \in A$

$$\bigwedge(\mu(x) \mid x \in A) \leq R(b, y).$$

In particular, if $A = M$, then the above elements t and b are said to be a pseudo-top and a pseudo-bottom, respectively, of the whole Ω-poset (M, E, R).

Proposition 3.6.3. [48] *Let (M, E, R) be an Ω-poset.*

An element $t \in M$ is a pseudo-top of $A \subseteq M$ if and only if

$$\bigwedge(\mu(x) \mid x \in A) = \bigwedge(R(x, t) \mid x \in A).$$

An element $b \in M$ is a pseudo-bottom of $A \subseteq M$ if and only if

$$\bigwedge(\mu(x) \mid x \in A) = \bigwedge(R(b, x) \mid x \in A).$$

Proposition 3.6.4. [48]*If t is a pseudo-top element of a subset A in an Ω-poset (M, E, R), then $t_1 \in A$ is a pseudo-top element of A if and only if*

$$E(t, t_1) \geq \bigwedge(\mu(x) \mid x \in A).$$

Analogously, if b is a pseudo-bottom element of $A \subseteq M$, then an element $b_1 \in A$ is a pseudo-bottom element of A if and only if

$$E(b, b_1) \geq \bigwedge(\mu(x) \mid x \in A).$$

Corollary 3.6.5. [48] *If t is a pseudo-top element of an Ω-poset (M, E, R), then for every $p \leq \bigwedge(\mu(x) \mid x \in M)$, the class $[t]_{E_p}$ is the top element of the poset $(\mu_p/E_p, \leq_p)$.*

Dually, if b is a pseudo-bottom element of (M, E, R), then for every $p \leq \bigwedge_{x \in M} \mu(x)$, the class $[b]_{E_p}$ is the bottom element of the poset $(\mu_p/E_p, \leq_p)$.

Similarly as for the pseudo-supremum and pseudo-infimum, pseudo-top and pseudo-bottom elements in an $\Omega - poset$ are not unique in general, but they belong to the same equivalence class of a cut of E.

Proposition 3.6.6. [48] *A pseudo top of a subset A of an Ω-poset (M, E, R), if it exists, is a pseudo-supremum of A.*

Dually, a pseudo bottom of A is a pseudo-infimum of A.

Regarding to the pseudo-suprema (infima) of the empty subset of M, for an omega poset (M, E, R), the situation is analogous as for suprema and infima of the empty set in the classical lattice. According to formulas (3.35) and (3.36), every element $u \in M$ is an upper (lower) bound of the empty set, as a subset of M.

A consequence is that in an Ω-poset (M, E, R) a pseudo-infimum of the empty subset exists if and only if this Ω-poset possesses a pseudo-top element. In this case every pseudo-top element is a pseudo-infimum of the empty set.

Dually, a pseudo-supremum of the empty subset exists in an Ω-poset if and only if it possesses a pseudo-bottom element, and every pseudo-bottom element is a pseudo-supremum of the empty set.

Let (M, E, R) be a finite Ω-poset, and $A \subseteq M$ then $a_1 \in A$ is a maximal element of A if

$$\bigwedge(\mu(a) \mid a \in A) \nleq R(a_1, b) \tag{3.37}$$

for every $b \in A$, such that $b \neq a_1$. Dually, $a_0 \in A$ is a minimal element of A if

$$\bigwedge(\mu(a) \mid a \in A) \nleq R(b, a_0)$$

for every $b \in A$, such that $b \neq a_0$.

A definition of complete Ω-lattice is given in the sequel.

An Ω-poset (M, E, R) is called a complete Ω-lattice if for every $A \subseteq M$ a pseudo-supremum and a pseudo-infimum of A exist.

Since a pseudo-supremum (infimum) is required also for the empty subset of M, thus:

Proposition 3.6.7. *A complete Ω-lattice possesses a pseudo-top and a pseudo bottom element.*

As naturally expected, quotient cut-posets of a complete Ω-lattice are complete lattices, and the classes represented by pseudo-suprema (infima) are classical suprema (infima) in these lattices. This is formulated in the next theorems.

Theorem 3.6.8. [48] *Let (M, E, R) be a complete Ω-lattice. Then, for every $p \in \Omega$, the poset $(\mu_p/E_p, \leq_p)$ is a complete lattice. In addition, for $A \subseteq M$, if c is a pseudo-infimum of A in (M, E, R), then $[c]_p$ is the infimum of $\{[a]_p \mid a \in A\}$ in the lattice $(\mu_p/E_p, \leq_p)$, for every $p \in \Omega$, such that $A \subseteq \mu_p$. Analogously, if d is a pseudo-supremum of A, then $[d]_p$ is the supremum of $\{[a]_p \mid a \in A\}$ in $(\mu_p/E_p, \leq_p)$.*

Theorem 3.6.9. [48] *Let (M, E, R) be an Ω-poset. Then it is a complete Ω-lattice if and only if for every $q \in \Omega$, the poset $(\mu_q/E_q, \leq_q)$ is a complete lattice, and the following holds: for all $A \subseteq M$, for $p = \bigwedge(\mu(a) \mid a \in A)$,*

and $q \leq p$, we have

$$\inf\{[a]_{E_p} \mid a \in A\} \subseteq \inf\{[a]_{E_q} \mid a \in A\}, \qquad (3.38)$$
$$\text{and} \quad \sup\{[a]_{E_p} \mid a \in A\} \subseteq \sup\{[a]_{E_q} \mid a \in A\}, \qquad (3.39)$$

where the infima (suprema) are considered in the corresponding posets $(\mu_q/E_q, \leq_q)$ and $(\mu_p/E_p, \leq_p)$.

The next theorem gives necessary and sufficient conditions under which an Ω-poset is a complete Ω-lattice.

Theorem 3.6.10. [48] *An Ω-poset (M, E, R) is a complete Ω-lattice, if and only if the following conditions are fulfilled:*

(*i*) *a pseudo-infimum exists for every $A \subseteq M$;*

(*ii*) *every cut μ_p, $p \in \Omega$, possesses a pseudo-top element;*

(*iii*) *for all $A \subseteq M$,*

$$p = \bigwedge(\mu(a) \mid a \in A),$$

and $q \leq p$, if

$$\sup\{[a]_{E_p} \mid a \in A\}$$

and

$$\sup\{[a]_{E_q} \mid a \in A\}$$

exist in the posets $(\mu_q/E_q, \leq_q)$ and $(\mu_p/E_p, \leq_p)$ respectively, then

$$\sup\{[a]_{E_p} \mid a \in A\} \subseteq \sup\{[a]_{E_q} \mid a \in A\}.$$

The next theorem is a kind of a representation theorem for complete Ω-lattices.

In the following the diagonal sub-relation of a binary relation f is denoted by $\Delta(f)$:

$$\Delta(f) := \{x \in M \mid (x, x) \in f\}. \qquad (3.40)$$

Theorem 3.6.11. [48] *Let $M \neq \emptyset$, and let $\mathcal{F} \subseteq \mathcal{P}(M^2)$ be a closure system over M^2 such that each $f \in \mathcal{F}$ is transitive and strict. Then the following hold.*

(*a*) *There is a complete lattice Ω and a mapping $R : M^2 \longrightarrow \Omega$ such that $\mathcal{F}$ is a collection of cuts of R and (M, E, R) is an Ω-poset, where $E : M^2 \longrightarrow \Omega$ is defined by $E(x, y) = R(x, y) \wedge R(y, x)$.*

(b) (M, E, R) is a complete Ω-lattice, if in addition, for every $f \in \mathcal{F}$ and for every $A \subseteq \Delta(f)$ there is an infimum and a supremum in the relational structure $(\Delta(f), f)$, and for $g \in \mathcal{F}$, such that $f \subseteq g$, the following hold:

if c is an infimum of A in $\Delta(f)$, then c is an infimum of A in $\Delta(g)$

if c is a supremum of A in $\Delta(f)$, then c is a supremum of A in $\Delta(g)$.

Chapter 4

New results about particular Omega algebraic structures

In this chapter new results about particular algebraic structures will be presented. This chapter consists of original results.

In section 4.1 a concept of normal Ω-subgroups will be developed and a connection with lattice valued congruences will be presented.

In section 4.2 notions of an Ω-Boolean algebra, an Ω-Boolean lattice and an Ω-rings and their connectivities will be presented as well as a possible application of Ω-Boolean algebras.

4.1 Normal Ω-Subgroups

In this part the notion of normal Ω-subgroups are introduced as a special instance of Ω-subgroups. The results in this section are published in paper [16].

The starting point in this section is the notion of an Ω-group. Since there will be more fuzzy equalities in this part, one for an Ω-group and others that determine Ω-subgroups, they will be denoted by notations E^μ or E^ν pointed at the related functions μ and ν, respectively. This notation will be used although μ is determined by the E^μ, as its diagonal: $\mu : G \to \Omega$, with $\mu(x) = E^\mu(x, x)$. So, in the beginning, μ and E^μ are fixed. An Ω-valued equality E^ν in the case of Ω-subgroups is uniquely determined by ν,

as presented in Chapter 3:

$$E^\nu(x, y) = E^\mu(x, y) \wedge \nu(x) \wedge \nu(y).$$

So E^ν and ν are determined by each other: $\nu(x) = E^\nu(x, x)$.

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group.

In order to introduce normal Ω-subgroups, first we deal with particular cut properties of Ω-subgroups. Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group. Observe that by Theorem 3.4.16, for every $p \in \Omega$, the quotient structure μ_p/E_p^μ is a classical group, where μ_p is a p-cut of $\mu : G \to \Omega$, with $\mu(x) = E^\mu(x, x)$, and E_p^μ is the corresponding cut of E^μ. Here E^μ and μ are fixed.

Theorem 4.1.1. Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ an Ω-subgroup of $\overline{\mathcal{G}}$. Then, for every $p \in \Omega$, the group ν_p/E_p^ν is, up to an isomorphism, a subgroup of the group μ_p/E_p^μ.

Proof. Consider the quotient groups ν_p/E_p^ν and μ_p/E_p^μ , for $p \in \Omega$. Observe that ν_p is a subalgebra of the algebra μ_p, and that E_p^μ is a restriction of E_p^μ to ν_p, in the sense of the starting algebras with a binary, a unary and a nullary operation.

Now, E_p^ν is a congruence on ν_p, and E_p^μ is a congruence on μ_p. Besides, E_p^ν is a restriction of E_p^μ to ν_p. Let

$$\nu_p^{E_p^\mu} = \{a \in \mu_p \mid \nu_p \cap [a]_{E_p^\mu} \neq \emptyset\}.$$

In other words, $\nu_p^{E_p^\mu}$ is a union of classes of the congruence E_p^μ having nonempty intersection with ν_p.

It is clear that $\nu_p^{E_p^\mu}$ is a subalgebra of μ_p, and that the restriction of $\nu_p^{E_p^\mu}$ to $\nu_p^{E_p^\mu}$, $E_p^\mu \!\restriction\! \nu_p^{E_p^\mu}$, is a congruence on $\nu_p^{E_p^\mu}$.

By the Third isomorphism theorem, we have that

$$\nu_p/E_p^\nu \cong \nu_p^{E_p^\mu}/(E_p^\mu \!\restriction\! \nu_p^{E_p^\mu}).$$

Since ν_p/E_p^ν is a group, we also have that the quotient structure on the righthand side, $\nu_p^{E_p^\mu}/(E_p^\mu \!\restriction\! \nu_p^{E_p^\mu})$ is a group. In addition, $\nu_p^{E_p^\mu}/(E_p^\mu \!\restriction\! \nu_p^{E_p^\mu})$ is a subset of μ_p/E_p^μ, since the former consists of some equivalence classes of μ_p/E_p^μ. Finally, $\nu_p^{E_p^\mu}/(E_p^\mu \!\restriction\! \nu_p^{E_p^\mu})$ is a group, hence it is a subgroup of μ_p/E_p^μ. $\square$

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group and μ is the mapping from G to Ω, defined by $\mu(x) = E^\mu(x, x)$, as before.

Now the definition of a congruence of the Ω-group is introduced as a special case of the congruence of an Ω-algebra.

An Ω- valued congruence on $\overline{\mathcal{G}}$ is an Ω-valued relation $\Theta : G^2 \to \Omega$ on G, which is μ-reflexive, symmetric, transitive and compatible with the operations in $\mathcal{G}$, and which also for all $x, y \in G$ fulfills $\Theta(x, y) \geq E^\mu(x, y)$.

Observe that μ-reflexivity of Θ means that for every $x \in G$, $\Theta(x, x) = E^\mu(x, x)$.

Let Θ be a congruence on a given Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$. Define $\nu : G \to \Omega$ by

$$\nu(x) := \Theta(e, x), \tag{4.1}$$

where e is a constant, neutral element in $\mathcal{G}$. Next, let $E^\nu : G^2 \to \Omega$ be defined by

$$E^\nu(x, y) := E^\mu(x, y) \wedge \nu(x) \wedge \nu(y). \tag{4.2}$$

Proposition 4.1.2. *If $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ is an Ω-group, then $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ is an Ω-subgroup of $\overline{\mathcal{G}}$.*

Proof. We prove that necessary and sufficient conditions given in Theorem 3.4.13 are fulfilled.

First, condition (3.19) is fulfilled:

$$E^\nu(x, y) = E^\nu(y, x) := E^\mu(x, y) \wedge E^\nu(x, x) \wedge E^\nu(y, y),$$

by the definition of E^ν, since $E^\nu(x, x) = E^\mu(x, x) \wedge \Theta(e, x) = \Theta(e, x)$, and similarly for $E^\nu(y, y)$.

Further, by compatibility of Θ,

$$E^\nu(x, x) \wedge E^\nu(y, y) = \Theta(e, x) \wedge \Theta(e, y) \leq \Theta(e, x \cdot y) = E^\nu(x \cdot y, x \cdot y),$$

and (3.20) holds. Analogously, conditions (3.21) and (3.22) are satisfied.

Therefore, by Theorem 3.4.13, $\overline{\mathcal{N}}$ is an Ω-subgroup of $\overline{\mathcal{G}}$. $\qquad\square$

Remark 4.1.3. Observe that in the case of crisp, classical groups, 4.1 gives a characteristic function of a normal subgroup.

The above considerations motivates the following definition.

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ an Ω-subgroup of $\overline{\mathcal{G}}$. Then, $\overline{\mathcal{N}}$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$, if there is an Ω-valued congruence Θ

on $\overline{\mathcal{G}}$, such that for all $x, y \in G$,

$$E^\nu(x, y) = E^\mu(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y). \qquad (4.3)$$

The following result is the main argument for the definition of a normal Ω-subgroup.

Theorem 4.1.4. *An Ω-subgroup $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$, if and only if for every $p \in \Omega$, ν_p/E_p^ν is a normal subgroup of the group μ_p/E_p^μ.*

Proof. Let $\overline{\mathcal{N}}$ be a normal Ω-subgroup of the Ω-group $\overline{\mathcal{G}}$. Then, by the definition, there is an Ω-valued congruence Θ on $\overline{\mathcal{G}}$, such that for all $x, y \in G$, $\theta(x, y) \geq E^\mu(x, y)$ and

$$E^\nu(x, y) = E^\mu(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y).$$

Now, for $p \in \Omega$, the cut Θ_p is considered, which is, clearly, a congruence on the subalgebra μ_p of the underlying algebra $\mathcal{G}$, since for every $x \in G$, $\Theta(x, x) = E^\mu(x, x)$, and $E_p^\mu \subseteq \Theta_p$.

From the above, it follows that all the conditions of the Second isomorphism theorem are fulfilled. Therefore, the relation Θ_p/E_p^μ, defined by

$$([x]_{E_p^\mu}, [y]_{E_p^\mu}) \in \Theta_p/E_p^\mu \quad \text{if and only if} \quad (x, y) \in \Theta_p, \qquad (4.4)$$

is a congruence on μ_p/E_p^μ (it is well defined since Θ_p is a congruence by the assumption).

In the above formula,

$$(x, y) \in \Theta_p \quad \text{if and only if} \quad \Theta(x, y) \geq p.$$

Further, by the Second isomorphism theorem,

$$\mu_p/E_p^\mu/\Theta_p/E_p^\mu \cong \mu_p/\Theta_p.$$

Now, μ_p/E_p^μ is a group, Θ_p/E_p^μ is a congruence on this group, hence μ_p/Θ_p is a group.

Next, by the definition, for every $x \in G$, $\nu(x) = \Theta(e, x)$, hence for $p \in \Omega$, $x \in \nu_p$ if and only if $\Theta(e, x) \geq p$.

By Theorem 4.1.1, ν_p/E_p^ν is, up to an isomorphism, a subgroup of μ_p/E_p^μ. By the definition, ν_p/E_p^ν consists exactly of some equivalence classes of μ_p/E_p^μ,

so it is indeed a subgroup of μ_p/E_p^μ.

Now we show that ν_p/E_p^ν is a normal subgroup of μ_p/E_p^μ. In other words, we prove that ν_p/E_p^ν is a class of a congruence on μ_p/E_p^μ, containing the neutral element.

Indeed, we have already noted that Θ_p/E_p^μ is a congruence on μ_p/E_p^μ and now we see that the class of this congruence containing the neutral element is exactly ν_p/E_p^ν.

Conversely, suppose that

$$\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$$

is an Ω-subgroup of an Ω-group

$$\overline{\mathcal{G}} = (\mathcal{G}, E^\mu).$$

By assumption, for every $p \in \Omega$, ν_p/E_p^ν is a normal subgroup of the group μ_p/E_p^μ which means that elements in ν_p/E_p^ν are exactly some classes of μ_p/E_p^μ. Now, for every $p \in \Omega$, we define a relation θ_p on μ_p/E_p^μ by

$$[x]_{E_p^\mu} \theta_p [y]_{E_p^\mu} \quad \text{if and only if} \quad [x]_{E_p^\mu} \cdot [y]_{E_p^\mu}^{-1} \in \nu_p/E_p^\nu.$$

Since ν_p/E_p^ν is a normal subgroup, θ_p is a congruence on μ_p/E_p^μ.

$$[x]_{E_p^\mu} \cdot [y]_{E_p^\mu}^{-1} \in \nu_p/E_p^\nu$$

is equivalent with

$$[x \cdot y]_{E_p^\mu} \in \nu_p/E_p^\nu,$$

which is further equivalent with

$$x \cdot y^{-1} \in \nu_p,$$

which is equivalent with

$$\nu(x \cdot y^{-1}) \geq p.$$

Now we consider a family of congruences $\{\theta_i \mid i \in I \subset \Omega\}$. Since

$$[x]_{E_i^\mu} \theta_i [y]_{E_i^\mu}$$

is equivalent with

$$\nu(x \cdot y^{-1}) \geq i,$$

we have that
$$[x]_{E_i^\mu}\theta_i[y]_{E_i^\mu}$$
for every $i \in I$ is equivalent with
$$\nu(x \cdot y^{-1}) \geq \bigvee_{i \in I} i,$$
this is further equivalent with
$$[x]_{E_i^\mu}\theta_{\bigvee_{i \in I} i}[y]_{E_i^\mu}.$$
Hence, we have that the family of congruences
$$\{\theta_i \mid i \in \Omega\}$$
is a closure system, since
$$\bigcap_{i \in I}\theta_i = \theta_{\bigvee_{i \in I} i}.$$
Now, a relation θ is defined:
$$\Theta : G^2 \to \Omega \quad \text{by} \quad \Theta(x, y) = \bigvee\{p \mid ([x]_{E_p^\mu}, [y]_{E_p^\mu}) \in \theta_p\}.$$
Note that if (x, y) does not belong to any θ_p for $p \in \Omega$, then $\Theta(x, y) = 0$ by the definition of the supremum of $\emptyset$ in the complete lattice Ω.

Now, it is straightforward to prove that Θ is a symmetric, transitive and compatible Ω-valued relation on $\overline{\mathcal{G}}$. It is also μ-reflexive: for $x \in G$
$$\Theta(x, x) = \bigvee\{p \mid ([x]_{E_p^\mu}, [y]_{E_p^\mu}) \in \theta_p\} = \bigvee\{p \mid x \in \mu_p\} = \mu(x) = E^\mu(x, x),$$
since $\mu(x)$ is one of the values over which the supremum is taken.

Finally, we prove that for all $x, y \in G$,
$$E^\mu(x, y) \leq \Theta(x, y)$$
. Let $E^\mu(x, y) = p$. Then $(x, y) \in E_p$ and hence
$$[x]_{e_p^\mu} = [y]_{e_p^\mu}.$$
Since θ_p is a congruence on μ_p/E_p^μ, it is obvious that we have
$$([x]_{e_p^\mu}, [y]_{e_p^\mu}) \in \theta_p.$$

By the definition of Θ, we get $\Theta(x, y) \geq p$.

Hence Θ is an Ω-valued congruence on $\overline{\mathcal{G}}$, and by the construction

$$\Theta(x, e) = \nu(x) = E^{\nu}(x, x).$$

By the definition (4.3), $\overline{\mathcal{N}}$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$. $\qquad\square$

Corollary 4.1.5. *If $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ is a commutative Ω-group, then all Ω-subgroups of $\overline{\mathcal{G}}$ are normal.*

Proof. Indeed, commutativity of an Ω-group is hereditary for quotient subgroups on cuts. Therefore, if $\overline{\mathcal{G}}$ is commutative, then every quotient structure μ_p / E_p, $p \in \Omega$ is an Abelian group. All subgroups of these are normal, hence by Theorem 4.1.4, all Ω-subgroups of $\overline{\mathcal{G}}$ are normal. $\qquad\square$

Example 4.1.6. The structure $(\mathcal{G}, E^{\mu})$, where $\mathcal{G} = (G, \cdot, {}^{-1}, e)$ with a binary operation $\cdot$ on $G = \{e, a, b, c, d, f, g, h, i, j\}$ is given in Table 4.1; unary operation ${}^{-1}$ is the identity function, and neutral element is e. The lattice Ω is given by the diagram in Figure 2. The Ω-valued equality E^{μ} is presented in Table 4.2.

$\cdot$	e	a	b	c	d	f	g	h	i	j
e	e	a	b	c	d	f	g	h	i	j
a	a	e	b	c	d	f	g	h	i	j
b	b	b	e	e	g	f	h	d	i	j
c	c	c	e	e	h	g	d	f	i	j
d	d	f	g	h	e	e	c	b	i	j
f	f	d	g	h	e	e	b	c	i	j
g	g	g	d	f	b	c	e	e	i	j
h	h	h	f	d	c	b	e	e	i	j
i	i	a	b	c	d	f	g	h	e	e
j	j	a	b	c	d	f	g	h	e	e

Table 4.1: Binary operation on G
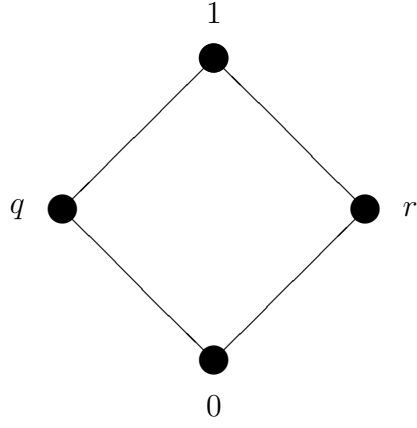
Figure 2: Lattice Ω

E^μ	e	a	b	c	d	f	g	h	i	j
e	1	r	0	0	0	0	0	0	0	0
a	r	r	0	0	0	0	0	0	0	0
b	0	0	r	r	0	0	0	0	0	0
c	0	0	r	r	0	0	0	0	0	0
d	0	0	0	0	r	r	0	0	0	0
f	0	0	0	0	r	r	0	0	0	0
g	0	0	0	0	0	0	r	r	0	0
h	0	0	0	0	0	0	r	r	0	0
i	0	0	0	0	0	0	0	0	q	q
j	0	0	0	0	0	0	0	0	q	q

Table 4.2: Ω-valued equality on G

The function $\mu : G \to \Omega$ is determined by E^μ: $\mu(x) = E^\mu(x, x)$.

x	e	a	b	c	d	f	g	h	i	j
$\mu(x)$	1	r	r	r	r	r	r	r	q	q

Table 4.3: compatible Ω-function μ

$(\mathcal{G}, E^\mu)$ is an Ω-group. Quotient cut-subgroups are:

$$\mu_r/E_r^\mu = \{\{e, a\}, \{b, c\}, \{d, f\}, \{g, h\}\} \quad \text{and} \quad \mu_q/E_q^\mu = \{\{i, j\}\}.$$

An Ω-valued congruence Θ on $(\mathcal{G}, E^\mu)$ is given in Table 4.5.

By the definition we have $\nu(x) = \Theta(e, x)$:

x	e	a	b	c	d	f	g	h	i	j
$\nu(x)$	1	r	r	r	0	0	0	0	0	0

Table 4.4: compatible Ω-function ν

Therefore, $\nu_1 = \{e\}$, and $\nu_r = \{e, a, b, c\}$ and the remaining cut ν_q is the empty set.

Consequently, $\nu_r / E_r^\nu = \{\{e, a\}, \{b, c\}\}$ is a normal subgroup of μ_r / E_r^μ, and this is the only nonempty and non-trivial cut structure.

Θ	e	a	b	c	d	f	g	h	i	j
e	1	r	r	r	0	0	0	0	0	0
a	r	r	r	r	0	0	0	0	0	0
b	r	r	r	r	0	0	0	0	0	0
c	r	r	r	r	0	0	0	0	0	0
d	0	0	0	0	r	r	r	r	0	0
f	0	0	0	0	r	r	r	r	0	0
g	0	0	0	0	r	r	r	r	0	0
h	0	0	0	0	r	r	r	r	0	0
i	0	0	0	0	0	0	0	0	q	q
j	0	0	0	0	0	0	0	0	q	q

Table 4.5: Ω-valued congruence on G

By
$$E^\nu(x, y) = E^\mu(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y),$$

E^ν is constructed and presented in Table 4.6.

E^ν	e	a	b	c	d	f	g	h	i	j
e	1	r	0	0	0	0	0	0	0	0
a	r	r	0	0	0	0	0	0	0	0
b	0	0	r	r	0	0	0	0	0	0
c	0	0	r	r	0	0	0	0	0	0
d	0	0	0	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	0	0	0
g	0	0	0	0	0	0	0	0	0	0
h	0	0	0	0	0	0	0	0	0	0
i	0	0	0	0	0	0	0	0	0	0
j	0	0	0	0	0	0	0	0	0	0

Table 4.6: Ω-valued equality determining Ω-subgroup

By Theorem 4.1.4, the structure $(\mathcal{G}, E^\nu)$ is a normal Ω-subgroup of the Ω-group $(\mathcal{G}, E^\mu)$. $\qquad\square$

Continuing with the general properties of normal Ω-subgroups, we use the fact that E^μ is also an Ω-valued congruence on $\overline{\mathcal{G}}$. Therefore, we examine a particular case when $\Theta = E^\mu$.

Theorem 4.1.7. *Let $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$ be an Ω-group, and $E^\epsilon : G^2 \to \Omega$ defined by*

$$E^\epsilon(x, y) = E^\mu(e, x) \wedge E^\mu(e, y), \tag{4.5}$$

with $\epsilon : G \to \Omega, \quad \epsilon(x) := E^\epsilon(x, x)$. Then, $\overline{\mathcal{E}} = (\mathcal{G}, E^\epsilon)$ is the smallest normal Ω-subgroup of $\overline{\mathcal{G}}$.

Proof. By (4.3), E^ϵ is an Ω-congruence on $\overline{\mathcal{G}}$:

$$E^\epsilon(x, y) = E^\mu(e, x) \wedge E^\mu(e, y) = E^\mu(x, y) \wedge E^\mu(e, x) \wedge E^\mu(e, y),$$

since by symmetry and transitivity of E^μ

$$E^\mu(e, x) \wedge E^\mu(e, y) \leq E^\mu(x, y).$$

Therefore, $\overline{\mathcal{E}}$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$. We prove that it is the smallest one. Namely, let $\overline{\mathcal{N}} = (\mathcal{G}, E^\nu)$ be an arbitrary normal Ω-subgroup of $\overline{\mathcal{G}}$; we show that $(\mathcal{G}, E^\epsilon)$ is an Ω-subgroup of $\overline{\mathcal{N}}$. Indeed, E^ϵ is a restriction of E^ν to ϵ, where $\epsilon(x) = E^\mu(e, x)$, and $E^\nu(x, y) = E^\mu(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y)$, for an Ω-congruence Θ on $\overline{\mathcal{G}}$, $E^\mu(x, y) \leq \Theta(x, y)$. So, we have

$$E^\epsilon(x, y) = E^\mu(x, y) \wedge E^\mu(e, x) \wedge E^\mu(e, y) =$$

$$E^\mu(x, y) \wedge E^\mu(e, x) \wedge E^\mu(e, y) \wedge \Theta(e, x) \wedge \Theta(e, y) =$$

$$E^\nu(x, y) \wedge E^\mu(e, x) \wedge E^\mu(e, y) = E^\nu(x, y) \wedge \epsilon(x) \wedge \epsilon(y),$$

and E^ϵ is a restriction of E^ν to ϵ. By Proposition 3.4.12, $\overline{\mathcal{E}}$ is an Ω-subgroup of an arbitrary normal Ω-subgroup $\overline{\mathcal{N}}$ of $\overline{\mathcal{G}}$, hence it is the smallest one. $\square$

The following is an explicit description of $\overline{\mathcal{E}}$ in terms of cut relations.

Corollary 4.1.8. *Let $\overline{\mathcal{E}} = (\mathcal{G}, E^\epsilon)$ be the subgroup of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^\mu)$, with E^ϵ being defined by (4.5). Then, for every $p \in \Omega$, the cut E^ϵ_p is the diagonal relation (equality) on the quotient group μ_p / E^μ_p.*

Proof. By (4.4), the relation E^ϵ_p / E^μ_p, defined by

$$([x]_{E^\mu_p}, [y]_{E^\mu_p}) \in E^\epsilon_p / E^\mu_p \quad \text{if and only if} \quad (x, y) \in E^\epsilon_p,$$

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is a congruence on μ_p/E_p^μ. By the definition of E^ϵ and by transitivity of E^μ we have

$$(x, y) \in E_p^\epsilon \quad \text{if and only if} \quad E^\epsilon(x, y) \geq p$$

which implies $E^\mu(x, y) \geq E^\mu(e, x) \wedge E^\mu(e, y) \geq p$.

Obviously, this is equivalent with $[x]_{E_p^\mu} = [y]_{E_p^\mu}$, hence E_p^ϵ is a classical equality on μ_p/E_p^μ. $\square$

Next we prove that a separated Ω-valued congruence on an Ω-group, acting as an Ω-valued equality, generates an Ω-group itself. Recall that an Ω-valued congruence Θ on an Ω-group $(\mathcal{G}, E^\mu)$ is an Ω-valued equivalence on G, compatible with the group operations and satisfying $\Theta(x, y) \geq E^\mu(x, y)$. It is separated if it fulfills

$$\Theta(x, y) = \Theta(x, x) \quad \text{implies} \quad x = y. \tag{4.6}$$

Theorem 4.1.9. *Let* $\Theta : G^2 \to \Omega$ *be an* Ω-*valued separated congruence on an* Ω- *group* $(\mathcal{G}, E^\mu)$. *Then* $(\mathcal{G}, \Theta)$ *is an* Ω-*group as well. In addition, for every* $p \in \Omega$, *the mapping* $f : \mu_p/E_p^\mu \to \mu_p/\Theta_p$, *defined by* $f([x]_{E_p^\mu}) = [x]_{\Theta_p}$ *is a classical surjective group homomorphism.*

Proof. It is obvious that $(\mathcal{G}, \Theta)$ is an Ω-algebra. We prove that the group identities are fulfilled. This follows by the fact that for every $x \in G$, $\mu(x) = \Theta(x, x)$. Hence, e.g., for Ω-associativity of the binary operation on G, we have

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E^\mu(x \cdot (y \cdot z), (x \cdot y) \cdot z) \leq \Theta(x \cdot (y \cdot z), (x \cdot y) \cdot z),$$

similarly with other group identities.

Next, let $f : \mu_p/E_p^\mu \to \mu_p/\Theta_p$, be such that $f([x]_{E_p^\mu}) = [x]_{\Theta_p}$. Then, for $x, y \in \mu_p$,

$$f([x \cdot y]_{E_p^\mu}) = [x \cdot y]_{\Theta_p} = [x]_{\Theta_p} \cdot [y]_{\Theta_p} = f([x]_{E_p^\mu}) \cdot f([y]_{E_p^\mu}),$$

hence f is a homomorphism. Analogously, one can check that f is compatible with the unary operation $^{-1}$, and that $f([e]_{E_p^\mu}) = [e]_{\Theta_p}$. It is surjective, since every class $[x]_{\Theta_p}$ is the image of $[x]_{E_p^\mu}$ under f. $\square$

4.2 Omega Boolean algebras, Omega Boolean lattices and Omega Boolean rings

This section contains part of the original results of this thesis. Notions of Ω-Boolean algebras, Ω-Boolean lattices and Ω-Boolean rings are introduced and investigated. These notions illustrate the main aspects of our work showing that it is possible to make transfers in an equivalent way from one Ω-algebraic structure to another, similarly as in classical algebras. First Ω-Boolean algebras are introduced, proving main properties and finally a consequence that an Ω-valued Boolean algebra is also an Ω-valued lattice is obtained, as expected (i.e., Ω-valued Boolean lattice is investigated).

An example of an Ω-Boolean algebra, defined on a structure which is not a crisp Boolean algebra is also given. At the end the corresponding Ω-valued order is introduced. As for other algebras, a property of Ω-algebras which explains the relationship of the new structure with classical Boolean algebras, i.e., the factor algebras on cuts, over the corresponding cuts of the Ω-valued equivalence, are classical Boolean algebras is proved.

4.2.1 Omega Boolean algebras, Omega Boolean lattices

In this part we start with $\mathcal{B} = (B, \sqcap, \sqcup, ', O, I)$, an algebraic structure with two binary, one unary and two nullary operations (constants). This is the algebra of the same type as the classical Boolean algebra. Here Ω is again a complete lattice with the top and the bottom element 1 and 0 respectively.

Let $\mu : B \to \Omega$ be a lattice valued substructure of this structure, i.e., a mapping satisfying for every $x, y \in B$:

$$\mu(x) \wedge \mu(y) \leq \mu(x \sqcap y), \ \ \mu(x) \wedge \mu(y) \leq \mu(x \sqcup y),$$

$$\mu(x) \leq \mu(x'), \ \ \mu(O) = 1, \ \ \mu(I) = 1.$$

The following auxiliary statement (an easy consequence of the definition) is used in the sequel.

Lemma 4.2.1. $\mu(x) \leq \mu(x \sqcap O)$.

Proof. $\mu(x) = \mu(x) \wedge \mu(0) \leq \mu(x \sqcap 0)$. $\qquad\qquad\qquad\qquad \square$

Further, let E be an Ω-valued equality on μ (as above, this means that $E(x, y) \leq \mu(x) \wedge \mu(y)$), compatible with all the operations of this structure.

In other words, E is reflexive on μ ($E(x,x) = \mu(x)$), symmetric and transitive Ω-valued relation, compatible with all the operations on B:

$$E(x,y) \wedge E(z,t) \leq E(x \sqcap z, y \sqcap t),$$

$$E(x,y) \wedge E(z,t) \leq E(x \sqcup z, y \sqcup t)$$

and

$$E(x,y) \leq E(x', y').$$

An ordered pair $(\mathcal{B}, E)$ is an Ω-valued Boolean algebra if the following axioms are satisfied for all $x, y, z \in B$:

$b1: \ x \sqcap y \approx y \sqcap x$
$b2: \ x \sqcup y \approx y \sqcup x$
    (*commutativity*)

$b3: \ x \sqcap (y \sqcup z) \approx (x \sqcap y) \sqcup (x \sqcap z)$
$b4: \ x \sqcup (y \sqcap z) \approx (x \sqcup y) \sqcap (x \sqcup z)$
    (*distributivity*)

$b5: \ x \sqcup O \approx x$
$b6: \ x \sqcap I \approx x$
    (*properties of constants*)

$b7: \ x \sqcap I \approx x$
$b8: \ x \sqcap x' \approx O$
    (*properties of unary operation*)

$b9: \ O \neq I.$

In the framework of Ω-structures, this means that the following lattice theoretic formulas hold:

$$\mu(x) \wedge \mu(y) \leq E(x \sqcap y, y \sqcap x) \tag{4.7}$$

$$\mu(x) \wedge \mu(y) \leq E(x \sqcup y, y \sqcup x) \tag{4.8}$$

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(x \sqcap (y \sqcup z), (x \sqcap y) \sqcup (x \sqcap z)) \tag{4.9}$$

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(x \sqcup (y \sqcap z), (x \sqcup y) \sqcap (x \sqcup z)) \tag{4.10}$$

$$\mu(x) \leq E(x \sqcup O, x) \tag{4.11}$$

$$\mu(x) \leq E(x \sqcap I, x) \tag{4.12}$$

$$\mu(x) \leq E(x \sqcap x', O) \tag{4.13}$$

$$\mu(x) \leq E(x \sqcup x', I) \qquad (4.14)$$

$$E(O, I) < 1. \qquad (4.15)$$

In this notation, we consider an Ω-Boolean algebra as an ordered pair $(\mathcal{B}, E)$, where $\mathcal{B}$ is an algebraic structure with two binary, one unary and two nullary operations, as defined above, and E is an Ω-valued equivalence compatible with these operations. The Ω-valued algebraic structure $\mu : B \to \Omega$ is implicitly contained in this notation, since it is uniquely defined by E: $\mu(x) = E(x, x)$. By the compatibility condition, it is easy to check that μ is indeed a lattice valued substructure of algebra $\mathcal{B}$.

Remark 4.2.2. From formulas 4.11 and 4.12, and by $E(x \sqcup O, x) \leq E(x, x) = \mu(x)$, it follows that

$$\mu(x) = E(x \sqcup O, x).$$

Similarly,

$$\mu(x) = E(x \sqcap I, x).$$

In case E is a separated Ω-valued equality, then $x \sqcup O = x$ and $x \sqcap I = x$ is obtained. This means that the underlying algebra in this case should have neutral elements for $\sqcup$ and $\sqcap$.

In the following, an example of an Ω-valued Boolean algebra is given, the co-domain of which is a complete lattice Ω, presented in Figure 3.

Example 4.2.3. Let $B = \{a, b, c, I, O\}$ be the universe of an algebra with two binary, one unary and two nullary operations. Let O and I be nullary operations. Binary operations $\sqcap$ and $\sqcup$ are given in Table 4.7 and Table 4.8 and an unary operation $'$ is given in Table 4.9. It is straightforward to check that the structure $(B, \sqcup, \sqcap, ', O, I)$ is not a Boolean algebra (e.g., distributivity laws are not satisfied).

$\sqcap$	a	b	c	I	O
a	a	O	O	a	O
b	O	b	O	b	O
c	O	O	c	c	O
I	a	b	c	I	O
O	O	O	O	O	O

Table 4.7: Binary operation $\sqcap$

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⊔	a	b	c	I	O
a	a	I	I	I	a
b	I	b	I	I	b
c	I	I	c	I	c
I	I	I	I	I	I
O	a	b	c	I	O

Table 4.8: Binary operation ⊔

′	a	b	c	I	O
	b	a	a	O	I

Table 4.9: Unary operation
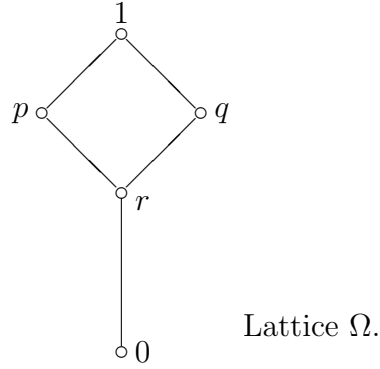
Lattice Ω.

Figure 3.

Let $\mu : B \to \Omega$ be an Ω-valued subalgebra of B, given by:

$$\mu(x) = \begin{pmatrix} a & b & c & I & O \\ p & p & 0 & 1 & 1 \end{pmatrix}.$$

Finally, in Table 4.10, an Ω-valued equality E on μ is given:

E	a	b	c	I	O
a	p	r	0	r	r
b	r	p	0	r	r
c	0	0	0	0	0
1	r	r	0	1	r
0	r	r	0	r	1

Table 4.10: Lattice valued equality

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Formulas 4.7-4.15 for Ω-Boolean algebras are satisfied, which means that the pair (B, E) is an Ω-Boolean algebra. $\qquad\square$

Remark 4.2.4. Since the formulas corresponding to axioms are dual in the sense that they appear in the dual pairs w.r.t. $\sqcap$ and $\sqcup$, also O and I, **the principle of duality** is satisfied. This means that for every statement which is true in the language of algebra $(B, \sqcap, \sqcup, ', O, I)$, the dual statement is also true. The dual statement is obtained exchanging each occurrence of $\sqcap$ with $\sqcup$ and vice versa and exchanging each occurrence of O with I and vice versa.

In the following some properties of Ω-Boolean algebras are proved.

In the next proposition it is proved that Ω-Boolean algebras can naturally be obtained from classical Boolean algebras.

Proposition 4.2.5. *Let* $\mathcal{B} = (B, \sqcap, \sqcup, ', O, I)$ *be a Boolean algebra,* Ω *a complete lattice and let* $\mu : B \to \Omega$ *be a lattice-valued algebra. If* E *is an arbitrary* Ω-valued equality on μ, then, $(\mathcal{B}, E)$ *is an* Ω-Boolean algebra.

Proof. Since the identities of Boolean algebras are valid in $(B, \sqcap, \sqcup, ', O, I)$, then also all Axioms b1.-b9. are valid. Indeed, in order to check (1), from $x \sqcap y = y \sqcap x$, we have that $E(x \sqcap y, y \sqcap x) = \mu(x \sqcap y)$. Hence, $\mu(x) \wedge \mu(y) \leq \mu(x \sqcap y) = E(x \sqcap y, y \sqcap x)$. In the same way all the axioms are checked. $\qquad\square$

Hence, Ω-Boolean algebras can be obtained if the basic structure is a classical Boolean algebra. This proves that the new notion is logically deduced from the analogue crisp structure. Still, nontrivial Ω-Boolean algebras are those in which the underlying structure is not a classical Boolean algebra.

Next some properties of the constants O and I in an Ω-Boolean algebra, in connections to the binary operations are proved.

Lemma 4.2.6. *Let* $\mathcal{B} = (B, \sqcap, \sqcup, ', O, I)$ *be an algebraic structure,* Ω *a complete lattice,* $\mu : B \to \Omega$ *a lattice valued algebra on* B, E *an* Ω-valued equality on μ and $(\mathcal{B}, E)$ *an* Ω-Boolean algebra. Then, the identity

$$x \sqcap O \approx O$$

holds on $(\mathcal{B}, E)$.

Proof. By formula (2.13), we have to prove that

$$\mu(x) \leq E(x \sqcap O, O).$$

$\mu(x) = \mu(x) \wedge \mu(O) \leq \mu(x \sqcap O) \leq E(x \sqcap O, (x \sqcap O) \sqcup O)$, by Lemma 4.2.1 and by formula 4.11 for Ω-valued Boolean algebras.

$\mu(x) \leq \mu(x) \wedge \mu(x \sqcap O) \leq E(x \sqcap O, x \sqcap O) \wedge E(O, x \sqcap x') \leq E((x \sqcap O) \sqcup O, (x \sqcap O) \sqcup (x \sqcap x'))$, by formula 4.13 for Ω-Boolean algebras.

$\mu(x) \leq \mu(x) \wedge \mu(x') \wedge \mu(O) \leq E((x \sqcap O) \sqcup (x \sqcap x'), x \sqcap (O \sqcup x'))$, by formula 4.9 for Ω-Boolean algebras.

$\mu(x) \leq \mu(x) \wedge \mu(x') \wedge \mu(O) \leq E(x, x) \wedge E(O \sqcup x', x' \sqcup O) \leq E(x \sqcap (O \sqcup x'), x \sqcap (x' \sqcup O))$, by formula 4.8 for Ω-Boolean algebras.

$\mu(x) \leq \mu(x) \wedge \mu(x') \leq E(x, x) \wedge E(x' \sqcup O, x') \leq E(x \sqcap (x' \sqcup O), x \sqcap x')$, by formula 4.11 for Ω-Boolean algebras.

$\mu(x) \leq E(x \sqcap x', O)$ by formula 4.13 for Ω-Boolean algebras.

Now, using the transitivity of relation E,

$\mu(x) \leq E(x \sqcap O, (x \sqcap O) \sqcup O) \wedge E((x \sqcap O) \sqcup O, (x \sqcap O) \sqcup (x \sqcap x')) \wedge E((x \sqcap O) \sqcup (x \sqcap x'), x \sqcap (O \sqcup x')) \wedge E(O \sqcup x', x' \sqcup O) \leq E(x \sqcap (O \sqcup x'), x \sqcap (x' \sqcup O)) \wedge E(x \sqcap (x' \sqcup O), x \sqcap x') \wedge E(x \sqcap x', O) \leq E(x \sqcap O, O)$. $\qquad \square$

The next corollary follows by the duality principle.

Corollary 4.2.7. *Let $\mathcal{B} = (B, \sqcap, \sqcup, ', O, I)$ be an algebraic structure, Ω a complete lattice, $\mu : B \to \Omega$ a lattice valued algebra on B, E an arbitrary Ω-valued equality on μ and $(\mathcal{B}, E)$ an Ω-Boolean algebra. Then, the identity*

$$x \sqcup I \approx I$$

holds on $(\mathcal{B}, E)$.

Proposition 4.2.8. *In an Ω-valued Boolean algebra $(\mathcal{B}, E)$, the absorptive law is valid:*
$$x \sqcap (x \sqcup y) \approx x.$$

Proof. $\mu(x) \wedge \mu(y) \leq E(x, x \sqcup O) \wedge E(x \sqcup y, x \sqcup y) \leq E(x \sqcap (x \sqcup y), (x \sqcup O) \sqcap (x \sqcup y))$, by the compatibility and formula 4.11.

$\mu(x) \wedge \mu(y) \leq \mu(x) \wedge \mu(y) \wedge \mu(O) \leq E((x \sqcup O) \sqcap (x \sqcup y), x \sqcup (O \sqcap y))$, by formula 4.10.

$\mu(x) \wedge \mu(y) \leq E(x, x) \wedge E(O \sqcap y, O) \leq E(x \sqcup (O \sqcap y), x \sqcup O)$,

$\mu(x) \wedge \mu(y) \leq \mu(x) \leq E(x \sqcup O, x)$, by formula 4.11.

Using the transitivity of relation E,

$\mu(x) \wedge \mu(y) \leq E(x \sqcap (x \sqcup y), (x \sqcup O) \sqcap (x \sqcup y)) \wedge E((x \sqcup O) \sqcap (x \sqcup y), x \sqcup (O \sqcap y)) \wedge E(x \sqcup (O \sqcap y), x \sqcup O) \wedge E(x \sqcup O, x) \leq E(x \sqcap (x \sqcup y), x)$. $\qquad \square$

By the duality principle, we have the following corollary:

Corollary 4.2.9. *In an Ω-valued Boolean algebra (B, E), the absorptive law is valid:*
$$x \sqcup (x \sqcap y) \approx x$$

$\square$

Using the similar technique, and the absorptive laws, the following properties are obtained.

Proposition 4.2.10. *In an Ω-valued Boolean algebra (B, E), the idempotent laws are valid:*
$$x \sqcap x \approx x.$$
$$x \sqcup x \approx x.$$

Remark 4.2.11. From the previous proposition, $\mu(x) = E(x \sqcap x, x)$ and $\mu(x) = E(x \sqcup x, x)$. In case when E is a separated Ω-valued equality, by Proposition 2.4.2 it follows that

$x \sqcap x = x$ and $x \sqcup x = x$,

i.e., both binary operations are idempotent in the underlying algebra.

Proposition 4.2.12. *In an Ω-valued Boolean algebra (B, E), the associative law is valid:*
$$x \sqcap (y \sqcap z) \approx (x \sqcap y) \sqcap z.$$

Proof. By the absorptive law:
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E((x \sqcap (y \sqcap z)) \sqcup x, x)$.
Further, by
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E(((x \sqcap y) \sqcap z) \sqcup x, ((x \sqcap y) \sqcup x) \sqcap (z \sqcup x))$,
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E(((x \sqcap y) \sqcup x) \sqcap (z \sqcup x), x \sqcap (z \sqcup x))$ and
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E(x \sqcap (z \sqcup x), x)$,
and the transitivity:
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E(((x \sqcap y) \sqcap z) \sqcup x, x)$.
Hence:
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E(((x \sqcap (y \sqcap z)) \sqcup x, ((x \sqcap y) \sqcap z) \sqcup x))$.
By similar techniques:
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E((x \sqcap (y \sqcap z)) \sqcup x', (y \sqcap z) \sqcup x')$ and
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E(((x \sqcap y) \sqcap z) \sqcup x', (y \sqcap z) \sqcup x')$, and finally,
$\mu(x) \wedge \mu(y) \wedge \mu(z) \le E(((x \sqcap (y \sqcap z)) \sqcup x', ((x \sqcap y) \sqcap z) \sqcup x'))$.

Hence, by the compatibility:

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq$

$E(((x \sqcap (y \sqcap z)) \sqcup x, ((x \sqcap y) \sqcap z) \sqcup x)) \wedge E(((x \sqcap (y \sqcap z)) \sqcup x', ((x \sqcap y) \sqcap z) \sqcup x')) \leq$

$E(((x \sqcap (y \sqcap z)) \sqcup x) \wedge ((x \sqcap (y \sqcap z)) \sqcup x'), (((x \sqcap y) \sqcap z) \sqcup x) \wedge (((x \sqcap y) \sqcap z) \sqcup x')).$

Thus,

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq$

$E(((x \sqcap (y \sqcap z)) \sqcup x) \sqcap ((x \sqcap (y \sqcap z)) \sqcup x'), (((x \sqcap y) \sqcap z) \sqcup x) \sqcap (((x \sqcap y) \sqcap z) \sqcup x'))$

(AAA)

Further:

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(((x \sqcap y) \sqcap z), ((x \sqcap y) \sqcap z) \sqcup O),$

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(((x \sqcap y) \sqcap z) \sqcup O, ((x \sqcap y) \sqcap z) \sqcup (x \sqcap x')),$

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(((x \sqcap y) \sqcap z) \sqcup (x \sqcap x'), (((x \sqcap y) \sqcap z) \sqcup x) \sqcap (((x \sqcap y) \sqcap z) \sqcup x')),$

hence

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(((x \sqcap y) \sqcap z), (((x \sqcap y) \sqcap z) \sqcup x) \sqcap (((x \sqcap y) \sqcap z) \sqcup x')).$

Similarly:

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(x \sqcap (y \sqcap z), ((x \sqcap (y \sqcap z)) \sqcup x) \sqcap ((x \sqcap (y \sqcap z)) \sqcup x')).$

Finally, by the last two formulas, and by (AAA):

$\mu(x) \wedge \mu(y) \wedge \mu(z) \leq E(((x \sqcap y) \sqcap z), (x \sqcap (y \sqcap z))).$ $\qquad \square$

By the duality principle, the dual proposition is also valid.

Proposition 4.2.13. *In an Ω-valued Boolean algebra (B, E), the associative law is valid:*

$$x \sqcup (y \sqcup z) \approx (x \sqcup y) \sqcup z.$$

Now the result which shows that an Ω-valued Boolean algebra is also an Ω-valued lattice is formulated.

Theorem 4.2.14. *Let $(B, \sqcap, \sqcup, ', O, I)$ be an algebraic structure as above, Ω a complete lattice, $\mu : B \to \Omega$ a lattice valued algebra on B, E an Ω-valued equivalence compatible with the operations on $(B, \sqcap, \sqcup, ', O, I)$ and (B, E) an Ω-valued Boolean algebra. Then (M, E) is an Ω-valued lattice, where $M = (B, \sqcap, \sqcup)$ is a bi-groupoid which is a reduct of the starting structure.*

Proof. By the definition of an Ω-valued Boolean algebra, $M = (B, \sqcap, \sqcup)$ is a bi-groupoid, such that $\mu : B \to \Omega$ is a lattice valued algebra on M which is a reduct of B, and E is an Ω-valued equality relation on μ, compatible with the two binary operations $\sqcap$ and $\sqcup$. (M, E) is an Ω-valued lattice, since

the lattice axioms are satisfied, as follows: (3.27) and (3.28) (commutative laws) are satisfied by the definition of an Ω-valued Boolean algebra, (3.29) and (3.30) (associative laws) are satisfied by Propositions 4.2.12 and 4.2.13 and finally, (3.31) and (3.32) (absorptive laws) are satisfied by Proposition 4.2.8 and Corollary 4.2.9. □

In this part cutworthy properties of introduced structures are mentioned.

These are analogous to the ones already defined for lattices, so they are just formulated here without proofs.

Firstly, if $(B, \sqcap, \sqcup, ', O, I)$ is an algebraic structure and $\mu : B \to \Omega$ a lattice valued algebra on μ, then for every $p \in \Omega$, μ_p are subalgebras of $(B, \sqcap, \sqcup, ', O, I)$. Moreover, if $E : B^2 \to \Omega$ is an Ω-valued equality on μ, then all the cut relations E_p, are congruences on μ_p for $p \in \Omega$.

Therefore, we can consider factor algebras μ_p/E_p for every $p \in \Omega$. Obviously, those algebras are of the same type as the algebraic structure $(B, \sqcap, \sqcup, ', O, I)$.

Hence, the following theorem is valid.

Theorem 4.2.15. *Let $\mathcal{B} = (B, \sqcap, \sqcup, ', O, I)$ be an algebraic structure with two binary operations, one unary and two constants, and Ω a complete lattice. Let also $\mu : B \to \Omega$ be a lattice valued algebra on B, E an Ω-valued equality on μ. Then, $(\mathcal{B}, E)$ is an Ω-Boolean algebra if and only if for every $p \in \Omega$, the quotient structure μ_p/E_p is a (classical) Boolean algebra.*

Since every Ω-Boolean algebra is an Ω-lattice, as a consequence of the results from paper [47], an Ω-valued ordering relation, in case when E is a separated Ω-valued equality is introduced as follows.

Let $(\mathcal{B}, E)$ be an Ω-Boolean algebra with $(B, \sqcap, \sqcup, ', O, I)$ being an algebraic structure as above, and $\mu(x) = E(x, x)$. Then an Ω-valued relation is defined:

$R : B^2 \to \Omega$ by $R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x)$.

Relation R is an Ω-valued order on $(\mathcal{B}, E)$, as proved in the following proposition.

Proposition 4.2.16. *Let $(\mathcal{B}, E)$ be an Ω-Boolean algebra, with $(B, \sqcap, \sqcup, ', O, I)$ being an algebraic structure with two binary, one unary and two nullary operations and E a separated Ω-valued equality on $\mathcal{B}$. Then an Ω-valued relation $R : B^2 \to \Omega$, defined by $R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x)$ is an Ω-valued order on $(\mathcal{B}, E)$.*

Proof. The proof follows from the analogous results for Ω-lattices, by

Theorem 4.2.14 and the fact that the Ω-valued relation R is defined using only the binary operation $\sqcap$. $\square$

4.2.2 Omega Boolean rings

In this part the definitions of Ω-rings and Ω-Boolean rings are introduced and its connection with Ω-Boolean algebras are proved.

Let

$$R = (\mathcal{R},\ E)$$

be an Ω-algebra in which $\mathcal{R} = (R, +, \cdot,\ ^{-1}, e)$ is an algebra with two binary operations ($+$ and $\cdot$), one unary operation ($^{-1}$) and a constant (e), and

$\mu : R \to \Omega$, such that $\mu(x) = E(x,x)$.

Then R is an Ω-ring if it satisfies the known ring identities:

$x + (y + z) \approx (x + y) + z$

$x + e \approx x, \quad e + x \approx x$

$x + x^{-1} \approx e, \quad x^{-1} + x \approx e.$

$x + y \approx y + x$

$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$

$x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)$ and

$(y + z) \cdot x \approx (y \cdot x) + (z \cdot x).$

By formula (2.13), the above identities hold if the following lattice-theoretic formulas are satisfied.

By (2.8), $\mu(e) = 1$.

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E(x + (y + z), (x + y) + z); \qquad (4.16)$$

$$\mu(x) \leqslant E(x + e, x), \quad \mu(x) \leqslant E(e + x, x); \qquad (4.17)$$

$$\mu(x) \wedge \mu(x^{-1}) \leqslant E(x \cdot x^{-1}, e), \quad \mu(x) \wedge \mu(x^{-1}) \leqslant E(x^{-1} \cdot x, e). \qquad (4.18)$$

$$\mu(x) \wedge \mu(y) \leqslant E(x \cdot y, y \cdot x); \qquad (4.19)$$

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E(x \cdot (y \cdot z), (x \cdot y) \cdot z); \qquad (4.20)$$

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$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E(x \cdot (y + z), (x \cdot y) + (x \cdot z)); \qquad (4.21)$$

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((y + z) \cdot x, (y \cdot x) + (z \cdot x)). \qquad (4.22)$$

Ω-ring $R = (\mathcal{R}, \; E)$ is an Ω-Boolean ring if the following is satisfied:

$x \cdot x \approx x$,

or as a lattice theoretic formula:

$$\mu(x) \leqslant E(x \cdot x, x). \qquad (4.23)$$

By the properties of weak reflexivity, from the definition directly follows:

$$\mu(x) = E(x, x) = E(x \cdot x, x).$$

If E is separated, it will follow that the idempotent identity $x \cdot x = x$ in algebra R is always true.

Let
$$R = (\mathcal{R}, \; E)$$

be an Ω-ring in which $\mathcal{R} = (R, +, \cdot, \; ^{-1}, e)$. Then, this ring is commutative if the following identity is true:

$x \cdot y \approx y \cdot x$, or if a lattice theoretic formula:

$$\mu(x) \wedge \mu(y) \leqslant E(x \cdot y, y \cdot x)$$

holds.

An Ω- ring with the identity in another language (with two nullary operations) is introduced as follows:

$\mathcal{R} = (R, +, \cdot, \; ^{-1}, e, 1)$ is an Ω-ring with identity if $\mathcal{R} = (R, +, \cdot, \; ^{-1}, e)$ is Ω-ring and the following is satisfied:

$x \cdot 1 \approx x, \quad 1 \cdot x \approx x$

or equivalently,

$$\mu(x) \leqslant E(x \cdot e, x), \quad \mu(x) \leqslant E(e \cdot x, x).$$

The proposition about the cuts is formulated in the sequel.

Proposition 4.2.17. *If $\mathcal{R} = (R, +, \cdot, \; ^{-1}, e, 1)$ is a commutative Ω-ring with identity, then for all $p \in \Omega$, the factor algebras $(R_p/E_p, +, \cdot, \; ^{-1}, e, 1)$ are commutative rings with identity.* $\square$

In order to define the corresponding Ω-Boolean algebra, the following operations on R are introduced.

$x \sqcup y := x + (y + x \cdot y)$ and $x' := x + 1$.

The first operation is binary and the second one is unary.

Now, the algebraic structure: $(R, \vee, \cdot, ', e, 1)$ is considered.

Lemma 4.2.18. *Let* $R = ((R, +, \cdot, \ ^{-1}, e, 1), \ E)$ *be a* Ω-*Boolean ring with the identity* 1. *Then the lattice valued equality relation* E *is compatible with the operations*

$x \sqcup y := x + (y + x \cdot y)$ and $x' := x + 1$

defined on R.

Proof. By the compatibility of E with the operations $+$ and $\cdot$, it follows that:

$E(x, y) \wedge E(z, t) \leq E(x + z, y + t)$ and $E(x, y) \wedge E(z, t) \leq E(x \cdot z, y \cdot t)$.

Hence,

$$E(x, y) \wedge E(z, t) \leq E(x, y) \wedge E(z, t) \wedge E(x \cdot z, y \cdot t) \leq$$

$$E(x, y) \wedge E(y + x \cdot z, t + y \cdot t) \leq$$

$$E(x + (y + x \cdot z), y + (t + y \cdot t)),$$

i.e., the compatibility with $\sqcup$ is proved.

Further, $E(x, y) \leq E(x, y) \wedge E(1, 1) \leq E(x + 1, y + 1) = E(x', y')$. $\qquad \square$

The next theorem gives a connection between an Ω-Boolean ring with identity and an Ω-Boolean algebra.

Theorem 4.2.19. *If* $((R, +, \cdot, \ ^{-1}, e, 1), E)$ *is a commutative* Ω-*Boolean ring with identity and the operations are defined by* $x \sqcup y := x + (y + x \cdot y)$ *and* $x' := x + 1$ *then* $((R, \sqcup, \cdot, ', e, 1), E)$ *is an* Ω-*Boolean algebra.*

Proof. $((R, +, \cdot, \ ^{-1}, e, 1), E)$ is an Ω-algebra, since we proved that E is compatible with the operations. By Proposition 4.2.17 for every p, μ_p/E_p are Boolean rings in the language of rings, and by the definition of operations $\vee$ and $'$, it is also a Boolean algebra in the language of the Boolean algebra (by the well known classical result). Hence, $((R, +, \cdot, \ ^{-1}, e, 1), E)$ is an Ω-Boolean algebra. $\qquad \square$

Now to see that from an Ω-Boolean algebra we can construct an Ω-Boolean ring, we start from an Ω-Boolean algebra $(\mathcal{B}, E)$, with $(B, \sqcap, \sqcup, ', O, I)$ being an algebraic structure with two binary, one unary and two nullary operations.

Another binary operation on B is defined by $x + y := (x \sqcap y') \sqcup (x' \sqcap y)$.

Also an unary operation is defined by: $-x := x$.

E is compatible with those two operations and the following theorem, which follows directly from the known classical results using factor algebras B_p/E_p for all $p \in \Omega$ is formulated.

Theorem 4.2.20. *Let* $(\mathcal{B}, E)$ *be an* Ω-*Boolean algebra, with* $\mathcal{B} = (B, \sqcap, \sqcup, ', O, I)$. *Then,* $((R, +, \sqcup, -, O, I), E)$ *is an* Ω-*Boolean ring where the operations are defined by* $x + y := (x \sqcap y') \sqcup (x' \sqcap y)$ *and* $-x := x$.

4.3 Application to Boolean n-tuples

An important application of this research is connected to Ω-Boolean algebras in which the basic structure is a collection of n-tuples over the two-element set $\{0, 1\}$. In other words, the concentration will be mostly on Ω-Boolean algebras $(\mathcal{B}, E)$, where

$$\mathcal{B} = (B, \sqcap, \sqcup, \bar{\ }, O, I), \quad B \subseteq \{0, 1\}^n, \tag{4.24}$$

while the operations $\sqcap$, $\sqcup$, and $\bar{\ }$ are arbitrary (two binary and a unary one, respectively) and $O = (0, 0, \dots, 0), I = (1, 1, \dots, 1)$.

As usual, by μ the function $\mu : B \to \Omega$ is denoted, such that for every n-tuple $x \in B$,

$$\mu(x) = E(x, x).$$

These finite sequences of zeros and ones are codewords in the digital technology and the above structure is usually complete (consisting of the whole set $\{0, 1\}^n$), moreover it is a classical Boolean algebra. However, in reality noise and errors have an impact to the operations, and the Boolean structure may be corrupted to some extent; in addition, some tuples might be missing. The above Ω-Boolean algebra with suitable operations and with an Ω-valued equality could be a model of such a modified structure.

Let us denote the classical Boolean algebra of all n-tuples of 0 and 1 as follows:

$$\mathcal{B}_2^n = (\{0, 1\}^n, \min, \max, ', 0, 1),$$

where, as usual, operations are defined componentwise:

for $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \{0, 1\}^n$,

$$\min((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = (\min(a_1, b_1), \ldots, \min(a_n, b_n));$$
$$\max((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = (\max(a_1, b_1), \ldots, \max(a_n, b_n));$$
$$(a_1, \ldots, a_n)' = (a_1', \ldots, a_n').$$

Clearly,

min	1	0
1	1	0
0	0	0

,

max	1	0
1	1	1
0	1	0

and

	$'$
1	0
0	1

.

An Ω-Boolean algebra is standard if it is of the form $(\mathcal{B}_2^n, H)$, $H :$ $(\{0, 1\}^n)^2 \to \Omega$ being an Ω-valued equality.

In other words, a standard Ω-Boolean algebra is the classical Boolean algebra of all n-tuples from $\{0, 1\}^n$, equipped with an Ω-valued equality.

In the sequel, we deal with Ω-Boolean algebras of the type (4.24), namely those in which $B \subseteq \{0, 1\}^n$, for some natural number n.

Such an Ω-Boolean algebra $((B, \sqcap, \sqcup, \bar{\ }, O, I), E)$, $B \subseteq \{0, 1\}^n$ is called regular, if there is a standard Ω-Boolean algebra $(\mathcal{B}_2^n, H)$, such that the following hold:

(i) The Ω-valued equality $E : B^2 \to \Omega$ is a restriction of the Ω-valued equality H, i.e., $E = H|_B$.

(ii) For all n-tuples $x, y \in B$,

(a) $E(x, x) \leqslant E(x', \bar{x})$;

(b) $E(x, x) \wedge E(y, y) \leqslant E(x \sqcap y, \min(x, y))$;

(c) $E(x, x) \wedge E(y, y) \leqslant E(x \sqcup y, \max(x, y))$.

In a regular Ω-Boolean algebra, for the underlying structure $(B, \sqcap, \sqcup, \bar{\ }, O, I)$ it is not required to be a Boolean algebra, while in a standard one this structure is the Boolean algebra $\mathcal{B}_2^n$ of all binary n-tuples.

Theorem 4.3.1. *For a regular Ω-Boolean algebra the following holds:*

(i) For every $p \in \Omega$, $E_p \subseteq \Theta$, where $E_p = E^{-1}(\uparrow p)$ is the p-cut of the Ω-valued equality E, and Θ is a congruence on a Boolean subalgebra $\mathcal{M}$ of $\{0, 1\}^n$.

(ii) For every $p \in \Omega$, the map $[a]_{E_p} \mapsto [a]_\Theta$ is an isomorphism of the quotient Boolean algebra μ_p / E_p onto the Boolean algebra $\mathcal{M}/\Theta$, with notation as in (i).

Sketch of the proof. (i) By the definition of a standard Ω-Boolean algebra, every cut of such a structure is a classical Boolean algebra. Further, by (a), (b) and (c), application of operations in a regular Ω-Boolean algebra gives elements such that on every cut E_p to which they belong, they are in the same class with the corresponding elements of the standard Boolean algebra. Since the Ω- equality E is a restriction of the Ω-equality H on the standard algebra, for every E_p, there is a congruence Θ on the subalgebra of the standard algebra, such that $E_p \subseteq \Theta$.

(ii) By the definition of a regular algebra, there is a bijective homomorphism among the corresponding classes of the regular and the standard algebra. $\qquad\square$
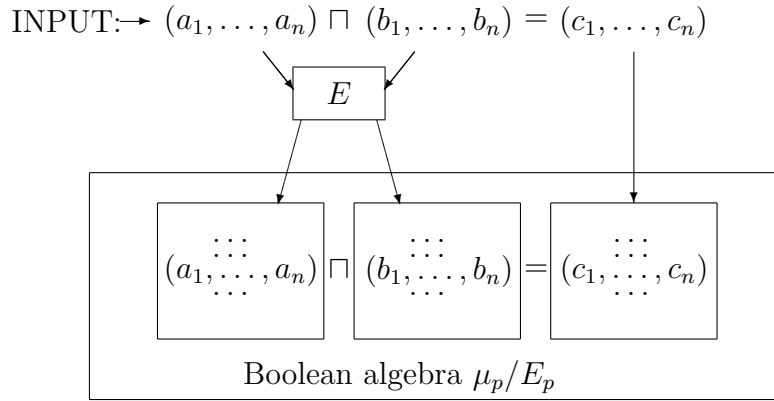
INPUT: $\rightarrowtail$ $(a_1, \ldots, a_n) \sqcap (b_1, \ldots, b_n) = (c_1, \ldots, c_n)$

Figure 4.

Example 4.3.2. Ω is a four-element chain:

$\Omega = \{0, q, p, 1\}$, $0 < q < p < 1$.

$((B, \sqcap, \sqcup, ', 0, 1), E)$ is a regular Ω-Boolean algebra, where

$B = \{000, 100, 010, 001, 110, 011, 111\}$

and operations $\sqcap$ and $\sqcup$ are those from $\mathcal{B}_2^3$ except:

$100 \vee 001 = 111$ and $010' = 111$,

E	000	100	010	001	110	011	111
000	1	q	p	q	q	q	q
100	q	q	q	q	q	q	q
010	p	q	p	q	q	q	q
001	q	q	q	q	q	q	q
110	q	q	q	q	q	q	q
011	q	q	q	q	q	q	q
111	q	q	q	q	q	q	1

.

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Table 4.11

Now the cuts are:

$\mu_1 = \{000, 111\}$.

$\mu_p = \{000, 010, 111\}$.

$\mu_q = \mu_0 = B$.

E_1	000	111
000	1	0
111	0	1

Table 4.12

E_p	000	010	111
000	1	1	0
010	1	1	0
111	0	0	1

Table 4.13

$E_0 = E_q = B^2$.

The corresponding standard Ω-Boolean algebra is $(\mathcal{B}_2^3, \Theta)$, where for $a, b \in \{0, 1\}^3$

$$\Theta = \begin{cases} E(a, b) & \text{if } a, b \in B \\ p & \text{if } (a, b) \in \{(101, 111), (101, 101), (111, 101)\} \\ q & \text{else} \end{cases}.$$

Then, $E = \Theta|_B$.

Chapter 5

Conclusion

This work, as presented above, is devoted to particular generalizations of classical algebraic and order-theoretic structures, to several classes of Ω-algebras and Ω-lattices. Concerning classical structures, we deal with groupoids, quasigroups, semigroups and groups, and in connection with order, we analyze complete lattices, Boolean algebras and Boolean rings.

It is well known that the mentioned structures can be approached in several equivalent ways, depending on the language in which these structures are defined. Quasigroups and groups are particular groupoids, and each of these structures can be defined as algebras in special languages fulfilling appropriate identities. Ordered structures like different classes of lattices are equivalently defined by operations and identities.

Generalizations that we use here originate in the fuzzy set theory, with the co-domain structure being a complete lattice without additional operations. Further generalizations replace the classical equality by the lattice-valued one and the order by the suitable lattice-valued relation.

Our research was concentrated to different approaches dealing with these generalizations.

As one of our main contributions, we have introduced, described and analyzed Ω-Boolean algebras. Then we have shown their equivalence with Ω-Boolean lattices, analogously to the classical case. To complete this relationship, we have introduced Ω-Boolean rings, comparing it with Ω-Boolean algebras, as it is the case in the classical ordering theory. In this way we have shown that these Ω-structures can be analyzed in both, order-theoretic and algebraic setting.

In the framework of Ω-groupoids, we have discussed different approaches to Ω-quasigroups and groups. Our main results in this part is introduction

and detailed description of normal Ω-subgroups. In turned out that in spite of different approaches to groups and Ω-groups, normal Ω-subgroups as defined here keep all (analogue) properties of normal subgroups as in the crisp case.

Let us mention our plans for the future work in this field. Our comparative analysis of different approaches to essentially same structures in Ω-valued setting is mostly leaded by the advantages obtained by the use of the complete lattice as the membership values structure. As indicated throughout the text, this co-domain enables classical structures and their properties to appear as quotient structures over cut relations of Ω-equality. On the other hand, logic behind fuzzy objects and topics in fuzzy and other graded frameworks should be algebraically presented by the membership valued structure. Complete lattices are not suitable for such logics, unless extended by additional operations, in which case we deal with different kinds of residuated lattices and algebras. But then we lose these classical crisp properties over cut structures and investigations should be performed by different techniques than those applied here.

From the mentioned reasons connected to logic, our future task would be investigations and comparative analysis of Ω-structures, with Ω being a particular residuated lattice.

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PRIRODNO-MATEMATIČKI FAKULTET
KLJUČNA DOKUMENTACIJSKA INFORMACIJA

Redni broj RBR	
Identifikacioni broj IBR	
Tip dokumentacije TD	Monografska dokumentacija
Tip zapisa TZ	Tekstualni štampani materijal
Vrsta rada (dipl. mag. dokt) VR	Doktorska disertacija
Ime i prezime autora AU	Omalkhear Salem Almabruk Bleblou
Mentor (titula, ime, prezime,zvanje) MN	dr Andreja Tepavčević, redovni profesor Prirodno-matematičkog fakulteta u Novom Sadu
Naslov rada NR	Neke nove mrežno vrednosne algebarske strukture sa komparativnom analizom različitih pristupa
Jezik publikacije JP	engleski
Jezik izvoda JI	srp. / eng.
Zemlja publikovanja ZP	Republika Srbija
Uže geografsko područje UGP	Vojvodina
Godina GO	2017
Izdavač IZ	Autorski reprint
Mesto i adresa MA	Novi Sad, Trg Dositeja Obradovića 3
Fizički opis rada FO	(/glava, /stranica, /slika, /grafikona, /referenci, /priloga)
Naučna oblast NO	Matematika
Naučna disciplina ND	Algebra i matematicka logika
Predmetna odrednica, ključne reči PO	rasplinuti skup, Omega-algebre, Ω-vrednosne kongruencije, Slabe kongruecije, Ω-skup; Ω-poset; Ω-mreza; Kompletna Ω mreza; Ω vrednosna jednakost;Ω-Bulova algebra
UDK	
Čuva se ČU	U biblioteci Departmana za matematiku i informatiku Prirodno-matematičkog fakulteta u Novom Sadu, Trg Dositeja Obradović a 4, Novi Sad

V.n. VN	
Izvod: IZ	Ovaj rad bavi se komparativnom analizom različitih pristupa rasplinutim (fazi) algebarskim strukturama i odnosom tih struktura sa odgovarajućim klasičnim algebrama. Posebna pažnja posvećena je poređenju postojećih pristupa ovom problemu sa novim tehnikama i pojmovima nedavno razvijenim na Univerzitetu u Novom Sadu. U okviru ove analize, proučavana su i proširenja kao i redukti algebarskih struktura u kontekstu rasplinutih algebri. Brojne važne konkretne algebarske strukture istraživane su u ovom kontekstu, a neke nove uvedene su i ispitane. Balvili smo se detaljnim istraživanjima Ω-grupa, sa stanovišta kongruencija, normalnih podgrupa i veze sa klasičnim grupama. Nove strukture koje su u radu uvedene u posebnom delu, istražene su sa aspekta svojstava i međusobne ekvivalentnosti. To su Ω-Bulove algebre, kao i odgovarajuće mreže i Bulovi prsteni. Uspostavljena je uzajamna ekvivalentnost tih struktura analogno odnosima u klasičnoj algebri.

U osnovi naše konstrukcije su mrežno vrednosne algebarske strukture definisane na klasičnim algebrama koje ne zadovoljavaju nužno identitete ispunjene na odgovarajućim klasičnim strukturama (Bulove algebre, prsteni, grupe itd.), već su to samo algebre istog tipa. Klasična jednakost zamenjena je posebnom kompatibilnom rasplinutom (mrežno-vrednosnom) relacijom ekvivalencije.

Na navedeni način i u cilju koji je u osnovi teze (poređenja sa postojećim pristupima u ovoj naučnoj oblasti) proučavane su (već definisane) Ω-grupe. U našim istraživanju uvedene su odgovarajuće normalne podgrupe. Uspostavljena je i istražena njihova veza sa Ω-kongruencijama. Normalna podgrupa Ω-grupe definisana je kao posebna klasa Ω-kongruencije. Jedan od rezultata u ovom delu je da su količničke grupe definisane pomoću nivoa Ω-jednakosti klasične normalne podgrupe odgovarajućih količničkih podgrupa polazne Ω-grupe. I u ovom slučaju osnovna struktura na kojoj je definisana Ω-grupa je grupoid, ne nužno grupa. Opisane su osobine najmanje normalne podgrupe u terminima Ω-kongruencija, a date su ineke konstrukcije Ω-kongruencija.

Rezultati koji su izloženi u nastavku povezuju različite pristupe nekim mrežno-vrednosnim strukturama. Ω-Bulova algebra je uvedena na strukturi sa dve binarne, unarnom i dve nularne operacije, ali za koju se ne zahteva ispunjenost klasičnih aksioma. Identiteti za Bulove algebre važe kao mrežno-teoretske formule u odnosu na mrežno-vrednosnu jednakost. Klasične Bulove algebre ih zadovoljavaju, ali obratno ne važi: iz tih formula ne slede standardne aksiome za Bulove algebre. Na analogan način uveden je i Ω-Bulov prsten. Glavna svojstva ovih struktura su opisana. Osnovna osobina je da se klasiňe Bulove algebre odnosno Bulovi prsteni javljaju kao količničke strukture na nivoima Ω-jeenakosti. Veza ove strukture sa Ω-Bulovom mrežom je pokazana.

Kao ilustracija ovih istraživanja, u radu je navedeno više primera. |

Datum prihvatanja teme od strane NN veća DP	maj 2016
Datum odbrane DO	
Članovi komisije (ime i prezime / titula / zvanje / naziv organizacije / status) KO	1. dr Ivana Štajner-Papuga, redovni profesor, PMF u Novom Sadu - predsednik 3. dr Andreja Tepavčević, redovni profesor PMF u Novom Sadu - mentor 2. dr Branimir Šešelja, redovni profesor PMF u Novom Sadu - član 4. dr Miroslav Ćirić, redovni profesor, PMF u Nišu - član

UNIVERSITY OF NOVI SAD
FACULTY OF SCIENCES
KEY WORD DOCUMENTATION

Accession number ANO	
Identification number INO	
Document type DT	Monograph documentation
Type of record TR	Textual printed material
Contents code CC	Doctoral Thesis
Author AU	Omalkhear Salem Almabruk Bleblou
Mentor MN	dr. Andreja Tepavcevic, professor, Fac. of Sci. Novi Sad
Title TI	Some new lattice valued algebraic structures with comparative analysis of various approaches
Language of text LT	English
Language of abstract LA	English/ Serbian
Country of publication CP	Rebublic of Srbija
Locality of publication LP	Vojvodina
Publication year PY	2017
Publisher PU	Author's reprint
Publication place PP	University of Novi Sad, Faculty of Sciences
Physical description PD	(/no of chpts, /pages, /figures, /charts, /references, /Contributions)
Scientific field SF	Mathematics
Scientific discipline SD	Algebra and mathematical logic
Subject, Key words SKW	Fuzzy set, Ω-algebras, Ω-valued homomorphisms, Ω-valued congruences, Weak congruences, Ω-set; Ω-poset; Ω-lattice; Complete Ω-lattice; Ω-valued Equality; Closure system
UC	
Holding data HD	

Note: N	
Abstract: AB	In this work a comparative analysis of several approaches to fuzzy algebraic structures and comparison of previous approaches to the recent one developed at University of Novi Sad has been done. Special attention is paid to reducts and expansions of algebraic structures in fuzzy settings. Besides mentioning all the relevant algebras and properties developed in this setting, particular new algebras and properties are developed and investigated. Some new structures, in particular Omega Boolean algebras, Omega Boolean lattices and Omega Boolean rings are developed in the framework of omega structures. Equivalences among these structures are elaborated in details. Transfers from Omega groupoids to Omega groups and back are demonstrated. Moreover, normal subgroups are introduced in a particular way. Their connections to congruences are elaborated in this settings. Subgroups, congruences and normal subgroups are investigated for Ω-groups. These are lattice-valued algebraic structures, defined on crisp algebras which are not necessarily groups, and in which the classical equality is replaced by a lattice-valued one. A normal Ω-subgroup is defined as a particular class in an Ω-congruence. Our main result is that the quotient groups over cuts of a normal Ω-subgroup of an Ω-group G, are classical normal subgroups of the corresponding quotient groups over G. We also describe the minimal normal Ω-subgroup of an Ω-group, and some other constructions related to Ω-valued congruences. Further results that are obtained are theorems that connect various approaches of fuzzy algebraic structures. A special notion of a generalized lattice valued Boolean algebra is introduced. The universe of this structure is an algebra with two binary, an unary and two nullary operations (as usual), but which is not a crisp Boolean algebra in general. A main element in our approach is a fuzzy equivalence relation such that the Boolean algebras identities are approximately satisfied related to the considered fuzzy equivalence. Main properties of the new introduced notions are proved, and a connection with the notion of a structure of a generalized fuzzy lattice is provided.

Accepted on Scientific Board on AS	
Defended DE	
Thesis Defend Board DB	1. dr Ivana Štajner-Papuga, full professor, PMF in Novom Sadu - president 3. dr Andreja Tepavčević, full professor PMF u Novom Sadu - mentor 2. dr Branimir Šešelja, full professor PMF u Novom Sadu - member 4. dr Miroslav Ćirić, full professor, PMF u Nišu - member