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Marko S. Đikić

COHERENT AND PRECOHERENT OPERATORS

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Abstract:	In this thesis we introduce and investigate new classes of operators which we call coherent and precoherent operators. These operators appear as solutions of some problems in the literature, but they also represent a generalization of some frequently studied classes of operators. After we study different properties of these new classes, we continue by considering a few interesting problems in operator theory. We consider problems about the Moore-Penrose inverse and arbitrary reflexive inverse of the sum of operators, range additivity of operators, lattice properties of the star and core partial orders on Hilbert space operators, the connection about the parallel sum of operators and their infimum in different partial orders, and one special type of operators, inspired by recently introduced disjoint range operators. Accordingly, we generalize and improve a number of results from the existing literature. One part of the thesis is dedicated to Rickart *-rings and generalizations of some presented results in the algebraic setting. We included a number of examples in order to demonstrate our statements and their possible extent: reduction of conditions, proving opposite directions, etc. In the end, we propose few problems for further research on these topics.
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Contents

Preface

This thesis represents the result of a research regarding certain problems in operator theory concerned with, generally speaking, mutual relationship of two Hilbert space operators. Particularly, we present solutions of some problems in the theory of partial orders between Hilbert space operators, and extensions of some results about generalized inverses of the sum of two operators. It turned out that the operators which appeared as the solutions of the problems under discussion have many properties in common, and over time we have become convinced that they deserve a study independent of the context of concrete problems. The thesis was written indulging this idea: the central notions in it are operators that we call coherent and precoherent, and after we give a study of such operators, we present our results in the areas mentioned above, as an application of the given study.

The thesis is organized in five chapters, which are further divided into sections. Let us describe the content of every chapter in a few lines.

Chapter 1 is an introductory chapter with a purpose of making the presentation more self–contained and elegant. We use the first section to establish our notation and terminology, and in subsequent sections we give short expositions of certain topics, some of which are more general than others. More general topics are described in Sections 1.2 and 1.3, former giving a compilation of results about closed subspaces and ranges of Hilbert space operators, and latter giving some basic information about theory of generalized inverses. Sections 1.4 and 1.5 cover two specific notions in operator theory, namely, the problem of range additivity of two operators, and the parallel summation of operators, respectively. This chapter contains no new results, but some statements are included with proofs, if those proofs also carry an important information: whether it is an idea, or the proof is in a spirit of our own research, or it is a proof of a well-known fact with a less-known source, or it is just a beautiful proof.

In Chapter 2 we describe our central notions, the relation of coherence and precoherence for operators between Hilbert spaces. We gather general properties of coherent and precoherent pairs of operators inside this chapter, most of them being interesting independently of the following chapters, but some of them are clearly motivated by the study yet to be presented. The definitions, some introductory discussion and few (counter)examples are placed in Section 2.1. In Section 2.2 we develop further properties of such operators, while Section 2.3 is reserved for the study of range additivity properties. The last section of this chapter, Section 2.4, describes one special case of precoherent operators, that we named operators with compatible ranges (CoR operators for short). This section was made as an answer to a recent study of the so called disjoint range operators.

In Chapter 3 we extend some known results about the generalized inverses of the sum of operators. The extensions we give are twofold: our results are derived for a wider class of operators than the original results, while the underlying spaces are of arbitrary, possibly infinite, dimension. The motivation and a quick overview of the results that we are going to extend are given in Section 3.1. In Section 3.2 we give a more generalized version of a formula by Fill and Fishkind expressing the Moore-Penrose inverse of the sum of two operators under certain conditions. Furthermore, in Section 3.3 we give a formula for arbitrary reflexive inverse of the sum extending some old results, and also consider arbitrary linear combinations of operators. The results that we are extending are originally given for pairs of rectangular matrices which column spaces are virtually disjoint, as well as the column spaces of their adjoint matrices. Our results are given for pairs of operators which are precoherent, as well as their adjoints, which is a condition more general than the one previously mentioned.

Chapter 4 contains our results regarding partial orders on Hilbert space operators. This is an interesting field of research that can be approached from different angles. For us the most interesting problems were those regarding the lattice properties of partial orders, so they make the biggest part of this chapter. Thus in Section 4.3 we give our treatment of the star partial order, in Section 4.4 we study the core partial order, while in Section 4.5 we present an interesting relation between infimums in these orders and the parallel summation of operators. However, in our introductory section of this chapter, Section 4.1, we present a couple of new results regarding the definition of the minus partial order and the range additivity, where the main result was told to us by Alejandra Maestripieri. For the sake of completeness, we also included Section 4.2 presenting known results about the lattice properties of the star and minus partial orders. Problems regarding lattice properties are directly connected with the notions of coherence and precoherence: this connection can be obvious, giving no more than a reformulation of a problem, but it can also be hidden and surprising. Nevertheless, a solution to every problem begins with a convenient reformulation.

Finally in Chapter 5 we show that an interesting theory of coherence can be developed in an algebraic setting of Rickart *-rings, since the structure of such rings is very similar to the algebra of bounded operators on a Hilbert space. The first section of this chapter is an introductory section, laying out basics of the theory of Rickart *-rings. Then, in Section 5.2 we introduce coherent and precoherent elements in Rickart *-rings, following our definitions from Chapter 2. Some nice properties of coherent and precoherent operators stay true in this algebraic setting as well. In the end, in Section 5.3 we again study the lattice properties of the star partial order, this time on Rickart *-rings, improving some recent results on this subject.

We finish the thesis with concluding remarks, summarizing our results and giving some final comments.

The results of this thesis are published in international mathematical journals included in the Thomson Reuters citation index SCIe (see [25–29]), and they were presented to mathematical community in two international conferences. Some results are given here in a slightly improved form, but we also presented results which are not included in the existing publications. For example, entire Section 3.3 appears for the first time in this thesis.

We should make the following remark on our choice of the term *coherent*. Coincidentally, the same name was used in at least two other occasions. One of them is in certain considerations in quantum mechanics, connected with so called coherent states. In this situation, coherence is not a relation, like in our case, but a property of a single operator, and there are no (obvious) connections with our work. On the other hand, the term *coherent elements* was also used by Cirulis in [20] which we noticed some time after this term became customary in our study. As it turns out, coherent elements as defined in this thesis are more general than ones from [20], but the motivation came from elsewhere.

It gives me a great pleasure to express my sincere gratitude to professors Gustavo Corach and Alejandra Maestripieri, with whom I had very interesting discussions, offering me a different perspective on some matters which previously seemed final. I am grateful to prof. Maestripieri also for providing some unpublished results which contributed to the thesis in more than one way. I will remain grateful to my thesis supervisor, professor Dragan Đorđević, for all the advices, discussions, ideas, suggestions, remarks, corrections, encouragement, patience, kindness, etc. not only during this research, but for as long as I knew him.

Chapter 1

Introduction

In this chapter we will describe the framework for the study presented in this thesis and introduce the notation which is used throughout. The amount of details presented is chosen with a hope that it properly complements the content of the thesis. Many of the results of this chapter originate from papers published since 2000 even though they have a 'classical flavour'. Such results offer the best invitation for further research in those areas of operator theory.

1.1 Operators on Hilbert spaces

By an *operator* T between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ we mean a bounded linear map $T : \mathcal{H} \to \mathcal{K}$, and all Hilbert spaces in this thesis are over complex field. By a subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ we always mean a linear subspace, which is not necessarily closed in the topology induced by the scalar product on $\mathcal{H}$ (thus, the word subspace refers only to the linear structure of $\mathcal{H}$, and not to the topological structure, as some authors prefer). We will always emphasize if $\mathcal{M}$ is a closed subspace, in which case it inherits the Hilbert space structure as well. Unlike for ordinary sets, when we say that subspaces $\mathcal{M}$ and $\mathcal{N}$ are *disjoint* we mean in fact $\mathcal{M} \cap \mathcal{N} = \{0\}$. The sum of disjoint subspaces $\mathcal{M}$ and $\mathcal{N}$, which are not necessarily orthogonal, is denoted by $\mathcal{M} \oplus \mathcal{N}$, while for arbitrary subspaces $\mathcal{M}$ and $\mathcal{N}$, with $\mathcal{M} \ominus \mathcal{N}$ we denote $\mathcal{M} \cap \mathcal{N}^{\perp}$.

The set of all operators between $\mathcal{H}$ and $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, or by $\mathcal{B}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$. We will clearly state when we work with an unbounded operator with a domain which is a proper subset of $\mathcal{H}$, but we will never work with nonlinear operators. For (bounded or unbounded) operator T we will denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range of T and the null-space of T, respectively. The scalar product and norm on any Hilbert space will be denoted respectively: $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$, and if there is a need for clarification on which Hilbert space they refer to, we add a symbol in the subscript (e.g. $\langle a, b \rangle_{\mathcal{H}}$). If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{M} \subseteq \mathcal{H}$, $\mathcal{N} \subseteq \mathcal{K}$ are two subspaces such that $T(\mathcal{M}) \subseteq \mathcal{N}$, the reduction of T between $\mathcal{M}$ and $\mathcal{N}$ will be denoted by $T|_{\mathcal{M},\mathcal{N}}$, or just by $T|_{\mathcal{M}}$ if $\mathcal{N} = \mathcal{K}$, or $\mathcal{N} = \mathcal{R}(T)$. The adjoint of an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ will be denoted by $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. The terms describing an operator: normal, self-adjoint, unitary, partial isometry will have the usual meaning. The term positive operator will be used for such $T \in \mathcal{B}(\mathcal{H})$ that $\langle Tx, x \rangle \geq 0$ for every $x \in \mathcal{H}$ (it is also common to call such operators non-negative).

All positive operators make a convex cone in $\mathcal{B}(\mathcal{H})$, and the partial order induced by this cone is called Löwner order, which we will denote by $\leq$. Operator $T \in \mathcal{B}(\mathcal{H})$ which is an idempotent element of $\mathcal{B}(\mathcal{H})$, i.e. for which $T^2 = T$ holds, is called a projection. We say that T is an orthogonal projection if $T^2 = T = T^*$. The projection with the range and null-space, respectively, $\mathcal{M}$ and $\mathcal{N}$ will be denoted by $P_{\mathcal{M},\mathcal{N}}$, while $P_{\mathcal{M}}$ denotes the orthogonal projection with the range $\mathcal{M}$.

For every positive operator T, there is a unique positive operator S such that $T = S^2$. We denote S by $T^{1/2}$. Since operator A^*A is positive for every $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the operator $(A^*A)^{1/2}$ is well-defined, and it is called the modulus of A, denoted by $|A|$. More generally, since the spectrum of a positive operator is contained in $[0, +\infty)$, where the function $x \mapsto x^\alpha$ is continuous for any $\alpha > 0$, by means of the continuous functional calculus we can define arbitrary positive power of a positive operator: T^α.

For $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, we will say that $T = VP$ is the polar decomposition of an operator T if P is positive, V is a partial isometry, and $\mathcal{N}(T) = \mathcal{N}(V) = \mathcal{N}(P)$, in which case $P = |T|$. In that case $T = V|T| = |T^*|V$ (the proof can be found in [58]).

For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, with $\mathcal{H} \times \mathcal{K}$ we denote a Hilbert space of ordered pairs (x, k), $x \in \mathcal{H}$, $k \in \mathcal{K}$ with the scalar product defined as: $\langle (x_1, k_1), (x_2, k_2)\rangle = \langle x_1, x_2\rangle + \langle k_1, k_2\rangle$. In that way, $\mathcal{H}$ and $\mathcal{H} \times \{0\}$ are isometrically isomorphic, as well as $\mathcal{K}$ and $\{0\} \times \mathcal{K}$, and $\mathcal{H} \times \{0\} \perp \{0\} \times \mathcal{K}$.

Let us now say something about adjoints of densely defined, possibly unbounded, operators between Hilbert spaces. For every densely defined linear transformation $T : D(T) \to \mathcal{K}$, $\overline{D(T)} = \mathcal{H}$ we can define an operator T^* in the following manner: we define $D(T^*)$ as the collection of all those $y \in \mathcal{K}$ for which there exists $v(y) \in \mathcal{H}$, such that:

$$\langle Tx, y\rangle = \langle x, v(y)\rangle, \quad \text{for all } x \in D(T);$$

since $D(T)$ is dense, for any y there is at most one $v(y)$, and $0 \in D(T^*)$ so it is nonempty; in this way we obtain a set $D(T^*)$ and a map $y \mapsto v(y)$ defined on it; it is not difficult to see that $D(T^*)$ is a subspace and that mapping $y \mapsto v(y)$ is linear, so this mapping is the adjoint of T with the domain $D(T^*)$. Obviously, for any other map $S : D(S) \to \mathcal{H}$, $D(S) \subseteq \mathcal{K}$ which satisfies: $\langle Tx, y\rangle = \langle x, Sy\rangle$ for every $x \in D(T)$ and $y \in D(S)$, we have that $D(S) \subseteq D(T^*)$ and that S is the restriction of T^* on $D(S)$ (in other words, $S \subseteq T^*$). Using Riesz representation theorem, it is not difficult to see that $D(T^*)$ contains exactly those $y \in \mathcal{K}$ for which the mapping $x \mapsto \langle Tx, y\rangle$ is a bounded functional on $D(T)$, which in turns shows that T^* is always a closed operator. In order to present one proof in Chapter 2 more elegantly, we also give the following theorem. The proof of statement 1. can be found in [80], while 2. is proved in [61].

Theorem 1.1.1 (See [61, 80])**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces, and $T : D(T) \to \mathcal{K}$ a densely defined linear transformation, $\overline{D(T)} = \mathcal{H}$. Then:*

1. *If $D(T^*)$ is dense in $\mathcal{K}$, then T is closable and $\overline{T}^* = T^*$;*

2. *If $D(T^*) = \mathcal{K}$ then T is bounded on $D(T)$.*

Let us note in the end that we interpret matrices from $\mathbb{C}^{m\times n}$ as operators between finite-dimensional Hilbert spaces $\mathbb{C}^n$ and $\mathbb{C}^m$, in a standard fashion: we identify a matrix

A with an operator for which A is the matrix representation in the standard basis. Thus for $A \in \mathbb{C}^{m\times n}$ all the notation introduced before makes sense: $\mathcal{R}(A)$ is the column space of A, A^* is exactly the conjugate-transpose of A, etc.

1.2 Geometry of subspaces and operator ranges

The first lemma we present in this section is a simple application of the open mapping theorem, but nevertheless a useful fact which is sometimes overlooked.

Lemma 1.2.1. *Let $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$ be closed subspaces of a Hilbert space $\mathcal{H}$, such that $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_k$ is closed. Then for every nonempty $J \subseteq \{1, 2, ..., k\}$, the sum $\bigoplus_{i\in J} \mathcal{H}_i$ is closed.*

Proof. Suppose first that $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_k = \mathcal{H}$. Define on $\mathcal{H} \times \mathcal{H}$ a mapping f such that

$$f(x_1 + x_2 + ... + x_k, y_1 + y_2 + ... + y_k) = \sum_{i=1}^{k} \langle x_i, y_i\rangle, \quad x_i, y_i \in \mathcal{H}_i, \quad i = 1, 2, ..., k$$

where $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathcal{H}$. Such a mapping is well-defined and it is a scalar product on $\mathcal{H}$, so we denote by $\mathcal{K}$ a unitary space with respect to the new scalar product f, and with vectors from $\mathcal{H}$, to avoid confusion. The scalar product induced on $\mathcal{H}_i$ in $\mathcal{K}$ is the same as the scalar product induced on $\mathcal{H}_i$ in $\mathcal{H}$. This is why $\mathcal{H}_i$ are also closed in $\mathcal{K}$. It is not difficult to see now that every Cauchy sequence in $\mathcal{K}$ is convergent, thus $\mathcal{K}$ is a Hilbert space. Finally, identity $I : \mathcal{K} \to \mathcal{H}$ is a bounded operator, since $||I(x_1 + x_2 + ... + x_k)||^2_{\mathcal{H}} = ||x_1 + x_2 + ... + x_k||^2_{\mathcal{H}} \le (||x_1||_{\mathcal{H}} + ||x_2||_{\mathcal{H}} + ... + ||x_k||_{\mathcal{H}})^2 = ||x_1 + x_2 + ... + x_k||^2_{\mathcal{K}}$. By the open mapping theorem, I is a closed mapping. It is a straightforward fact that for every nonempty $J \subseteq \{1, 2, ..., k\}$, the sum $\bigoplus_{i\in J} \mathcal{H}_i$ is closed in $\mathcal{K}$, and since I is closed, it is also closed in $\mathcal{H}$.
Now if $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_k = \mathcal{H}' \subsetneq \mathcal{H}$, then $\mathcal{H}'$ is a Hilbert space also, and $\mathcal{M} \subseteq \mathcal{H}'$ is closed in $\mathcal{H}'$ if and only if it is closed in $\mathcal{H}$, so the assertion follows from the already proved part. □

The previous lemma is true in Banach spaces also. For a thorough discussion on this subject, the reader is referred to [78]. We presented a proof in the setting of Hilbert spaces in order to get the following conclusion: even if the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_k = \mathcal{H}$ is not orthogonal, we can always define a new scalar product on $\mathcal{H}$ with respect to which this sum becomes orthogonal (note that in the proof of the previous lemma, $\mathcal{H}_i$ are orthogonal in $\mathcal{K}$), and the norm induced by the new scalar product is equivalent to the old norm.

If $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_k, \quad \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus ... \oplus \mathcal{K}_l \tag{1.1}$$

and $\mathcal{H}_j, \mathcal{K}_i$ are closed for every $j \in \{1, 2, ..., k\}$ and $i \in \{1, 2, ..., l\}$, then from Lemma 1.2.1 it follows that $P_j = P_{\mathcal{H}_j, \bigoplus_{n\neq j} \mathcal{H}_n}$ and $Q_i = P_{\mathcal{K}_i, \bigoplus_{n\neq i} \mathcal{K}_n}$ are bounded idempotents. If

$A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, for every $j \in \{1, 2, ..., k\}$ and $i \in \{1, 2, ..., l\}$ we have a bounded operator $A_{ij} = Q_i A P_j$ which can be seen as an operator from $\mathcal{B}(\mathcal{H}_j, \mathcal{K}_i)$. In this way we obtain *an operator matrix* $A = [A_{ij}]_{i,j}$ which corresponds to the decompositions in (1.1). This matrix represents the mapping A by means of formal multiplication, in the following sense: if $x = x_1 + x_2 + ... + x_k$, where $x_i \in \mathcal{H}_i$ and $Ax = y = y_1 + y_2 + ... + y_l$, where $y_j \in \mathcal{K}_j$, then $y_i = \sum\limits_j A_{ij} x_j$, i.e.

$$y = Ax = \begin{bmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{ik} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{l1} & \cdots & A_{lj} & \cdots & A_{lk} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_l \end{bmatrix}. \tag{1.2}$$

It is important however that every matrix filled with arbitrary bounded operators gives a bounded operator overall. Indeed, if $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{K}_i)$ are arbitrary, then $Q_i A_{ij} P_j$ is well defined bounded operator from $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and so the mapping defined with formal multiplication by the matrix $[A_{ij}]_{i,j}$ like in (1.2), is just the sum of bounded operators $\sum\limits_{i,j} Q_i A_{ij} P_j$, which is again bounded.

If the decompositions in (1.1) are orthogonal, the operator matrix of A^* with respect to these decompositions is just the conjugate transpose of the operator matrix of A, i.e. $A^* = [A_{ji}^*]_{j,i}$.

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{M} \subseteq \mathcal{H}$ is a closed subspace, such that $\mathcal{M} \oplus \mathcal{N}(A) = \mathcal{H}$, while $\mathcal{N} \subseteq \mathcal{K}$ is a closed subspace, such that $\mathcal{K} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}$, then the operator matrix of A with respect to these decompositions has the form:

$$A = \begin{bmatrix} A|_{\mathcal{M}} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{N} \end{bmatrix}.$$

Here $A|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M}, \overline{\mathcal{R}(A)})$ is an injection with a dense range. For $\mathcal{M}$ and $\mathcal{N}$ we can always choose: $\mathcal{M} = \overline{\mathcal{R}(A^*)}$, $\mathcal{N} = \mathcal{N}(A^*)$ in which case we obtain orthogonal decompositions.

It is an important fact that in an infinite-dimensional Hilbert space, the sum of closed subspaces is not necessarily closed. The counterexample can be found in a separable Hilbert space and thus in every infinite-dimensional Hilbert space. The example we present is due to Halmos [47].

Example 1 (See [47]). Let $\mathcal{H}$ be $l^2(\mathbb{N})$ and denote by e_n a sequence which has 1 on nth coordinate, and 0 elsewhere. Let $\mathcal{M}$ be a subspace containing sequences of the form: $(a_1, 0, a_3, 0, a_5, 0, ...)$, i.e. sequences in which all entries on even coordinates are equal to 0. Define $\mathcal{N}$ as

$$\mathcal{N} = \overline{\operatorname{span}\{\cos \frac{1}{k} e_{2k-1} + \sin \frac{1}{k} e_{2k} \ : \ k \in \mathbb{N}\}}.$$

Thus, $\mathcal{N}$ is the closure of the span of sequences:

$$(\cos 1, \sin 1, 0, 0, 0, 0, ...), \quad (0, 0, \cos(1/2), \sin(1/2), 0, 0, ...), \quad ...$$

By definition, both $\mathcal{M}$ and $\mathcal{N}$ are closed, but $\mathcal{M}+\mathcal{N}$ is not closed. To see this, let $y = (0, \sin 1, 0, \sin(1/2), 0, \sin(1/3), 0, ...) = \sum_k \sin(1/k)e_{2k}$. In that case, $y \in l^2(\mathbb{N})$ and also $y \in \overline{\mathcal{M}+\mathcal{N}}$. On the other hand, if $y = m + n$ with $m \in \mathcal{M}$ and $n \in \mathcal{N}$, then $m = -(\cos 1, 0, \cos(1/2), 0, \cos(1/3), 0, ...)$, which is not possible since $\cos(1/n) \to 1,\ n \to \infty$, so such m does not belong to $l^2(\mathbb{N})$.

When we introduce the notion of angle between subspaces, we will make a further remark on this construction, clarifying the motivation behind it. $\diamond$

The collection of all closed subspaces of $\mathcal{H}$ ordered with the inclusion $\subseteq$ becomes a lattice, and if $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces, then $\inf\{\mathcal{M},\mathcal{N}\} = \mathcal{M}\cap\mathcal{N}$ and $\sup\{\mathcal{M},\mathcal{N}\} = \overline{\mathcal{M}+\mathcal{N}}$. If we denote by $\mathcal{C}_{\mathcal{H}}$ the collection of all closed subspaces in $\mathcal{H}$, and by $\mathcal{P}(\mathcal{H})$ the collection of all orthogonal projections on $\mathcal{H}$, then the mapping $\mathcal{M} \mapsto P_{\mathcal{M}}$ from $(\mathcal{C}_{\mathcal{H}}, \subseteq)$ to $(\mathcal{P}(\mathcal{H}), \leq)$ is an order isomorphism, due to the following well-known fact.

Theorem 1.2.2 (See [79])**.** *If P and Q are two orthogonal projections on a Hilbert space $\mathcal{H}$, then the following statements are equivalent:*

(i) $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$*;*

(ii) $PQ = P$*;*

(iii) $QP = P$*;*

(iv) $P \leq Q$*.*

If we denote by $\wedge$ and $\vee$ the infimum and supremum, respectively, in the order $\leq$ on $\mathcal{P}(\mathcal{H})$ then we have:

$$P_{\mathcal{M}} \wedge P_{\mathcal{N}} = P_{\mathcal{M}\cap\mathcal{N}}, \quad P_{\mathcal{M}} \vee P_{\mathcal{N}} = P_{\overline{\mathcal{M}+\mathcal{N}}}.$$

An important question is, when is $\mathcal{M}+\mathcal{N}$ closed. Any finite-dimensional subspace of a Hilbert space is closed, and the sum of a closed subspace with a finite-dimensional one is again closed. This is stated in the following lemma, which proof can be found, for example, in [49, Problem 11].

Lemma 1.2.3 (See [49])**.** *If $\mathcal{M}$ is a closed subspace of a Hilbert space $\mathcal{H}$ and $\mathcal{N}$ is a finite-dimensional subspace of $\mathcal{H}$, then $\mathcal{M}+\mathcal{N}$ is closed.*

Another occasion when we are sure that the sum of two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ is closed is when $\mathcal{M}$ and $\mathcal{N}$ are orthogonal. In that case we can be more precise.

Lemma 1.2.4. *Let $\mathcal{M},\mathcal{N} \subseteq \mathcal{H}$ be two subspace such that $\mathcal{M}\perp\mathcal{N}$. Then $\mathcal{M}+\mathcal{N}$ is closed if and only if $\mathcal{M}$ and $\mathcal{N}$ are closed.*

Proof. Subspaces $\mathcal{M}$ and $\mathcal{N}$ are disjoint, since they are orthogonal.

If $\mathcal{M}$ and $\mathcal{N}$ are both closed and orthogonal, then $\mathcal{M}\oplus\mathcal{N}$ as a normed space is isometrically isomorphic to a Hilbert space $\mathcal{M}\times\mathcal{N}$, so it is complete and hence closed in $\mathcal{H}$.

Conversely, if $\mathcal{M}\oplus\mathcal{N}$ is closed, then it is a Hilbert space. In this Hilbert space, it is not difficult to see that $\mathcal{M}^{\perp}$ is exactly $\mathcal{N}$, so $\mathcal{N}$ is closed in it, and so it is closed in $\mathcal{H}$. The same goes for $\mathcal{M}$. $\square$

The case of two orthogonal closed subspaces is only a special case in a more general study regarding the notion of an angle between subspaces, which we are going to present now, following the classical survey paper by Deutsch [24]. We only include some proofs to illustrate required techniques which we find very interesting.

Definition 1.2.5. If $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$ are two closed subspaces of a Hilbert space $\mathcal{H}$, then the Friedrichs angle between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ which cosine is equal to:

$$c(\mathcal{M}, \mathcal{N}) := \sup\{|\langle x, y\rangle| \ \ x \in \mathcal{M} \ominus (\mathcal{M} \cap \mathcal{N}),\ y \in \mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N}),\ ||x|| = ||y|| = 1\}.$$

The Dixmier angle between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ which cosine is equal to:

$$c_0(\mathcal{M}, \mathcal{N}) := \sup\{|\langle x, y\rangle| \ \ x \in \mathcal{M},\ y \in \mathcal{N},\ ||x|| = ||y|| = 1\}.$$

The following lemma is trivial.

Lemma 1.2.6 (See [24])**.** *If $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$ are closed subspaces, then:*

a) $0 \le c(\mathcal{M}, \mathcal{N}) \le c_0(\mathcal{M}, \mathcal{N}) \le 1$;

b) $c(\mathcal{M}, \mathcal{N}) = c(\mathcal{N}, \mathcal{M})$ *and* $c_0(\mathcal{M}, \mathcal{N}) = c_0(\mathcal{N}, \mathcal{M})$;

c) $c(\mathcal{M}, \mathcal{N}) = c_0(\mathcal{M} \ominus (\mathcal{M} \cap \mathcal{N}), \mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N}))$;

d) If $\mathcal{M} \cap \mathcal{N} = \{0\}$, *then* $c(\mathcal{M}, \mathcal{N}) = c_0(\mathcal{M}, \mathcal{N})$;

e) If $\mathcal{M} \cap \mathcal{N} \neq \{0\}$, *then* $c_0(\mathcal{M}, \mathcal{N}) = 1$.

Theorem 1.2.7 (See [24])**.** *If $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$ are closed subspaces, the following statements are equivalent:*

(i) $c_0(\mathcal{M}, \mathcal{N}) < 1$;

(ii) $\mathcal{M} \cap \mathcal{N} = \{0\}$ *and* $\mathcal{M} + \mathcal{N}$ *is closed;*

(iii) There exists $\rho > 0$ *such that* $||x + y|| \ge \rho ||y||$ *for all* $x \in \mathcal{M}, y \in \mathcal{N}$.

Proof. (i) $\Rightarrow$ (ii) Denote by $c_0 = c_0(\mathcal{M}, \mathcal{N})$. Since $c_0 < 1$, it is clear that $\mathcal{M} \cap \mathcal{N} = \{0\}$. From the definition of c_0, for every $x \in \mathcal{M}$ and $y \in \mathcal{N}$ we have $|\langle x, y\rangle| \le c_0 ||x||\ ||y||$, and since

$$||x + y||^2 = ||x||^2 + ||y||^2 + 2\mathrm{Re}(\langle \mathrm{x}, \mathrm{y}\rangle) \ge ||\mathrm{x}||^2 + ||\mathrm{y}^2|| - 2|\langle \mathrm{x}, \mathrm{y}\rangle|,$$

we find that

$$||x + y||^2 \ge (||x|| - ||y||)^2 + 2(1 - c_0)||x||\ ||y||. \tag{1.3}$$

In order to prove that $\mathcal{M} + \mathcal{N}$ is closed, let $(z_n) \subseteq \mathcal{M} + \mathcal{N}$ be an arbitrary convergent sequence and $z_n \to z$. For every $n \in \mathbb{N}$ there are $x_n \in \mathcal{M}$ and $y_n \in \mathcal{N}$ such that $x_n + y_n = z_n$ and by (1.3) it holds:

$$||z_n|| = ||x_n + y_n||^2 \ge (||x_n|| - ||y_n||)^2 + 2(1 - c_0)||x_n||\ ||y_n||.$$

Given that $c_0 < 1$, and that (z_n) is a bounded sequence, we get that $(||x_n|| - ||y_n||)$ and $(||x_n||\ ||y_n||)$ are bounded, which in turns gives that (x_n) and (y_n) are bounded. Using famous Banach-Alaoglu theorem, we conclude that every bounded sequence has a weakly convergent subsequence, so without loss of generality, we may assume that sequences (x_n) and (y_n) are weakly convergent to x and y, respectively. But every (strongly) closed subspace is weakly closed, so $x \in \mathcal{M}$ and $y \in \mathcal{N}$. On the other hand, being a strong limit of $(x_n + y_n)$, vector z is also a weak limit of $(x_n + y_n)$, and so $z = x + y \in \mathcal{M} + \mathcal{N}$, showing that $\mathcal{M} + \mathcal{N}$ is closed.

(ii) $\Rightarrow$ (iii) Since $\mathcal{M} \oplus \mathcal{N}$ is closed, it is a Hilbert space (i.e. a Banach space), and so the projection P with the range $\mathcal{N}$ and the null-space $\mathcal{M}$ is bounded. We can take ρ to be $||P||^{-1}$.

(iii) $\Rightarrow$ (i) If (i) is not satisfied then there exist sequences $(x_n) \subseteq \mathcal{M}$ and $(y_n) \subseteq \mathcal{N}$ of unit vectors such that $|\langle x_n, y_n\rangle| \to 1$, i.e. $\mathrm{Re}(\langle \mathrm{x_n}, \mathrm{y_n}\rangle) \to 1$. From (iii) we have that $||x_n - y_n||^2 \geq \rho^2 > 0$, but

$$||x_n - y_n||^2 = 2 - 2\mathrm{Re}(\langle \mathrm{x_n}, \mathrm{y_n}\rangle) \to 0,$$

which is a contradiction. Thus (i) is satisfied. □

From Theorem 1.2.7 follows directly that the sum of two closed orthogonal subspaces is closed. The idea behind Example 1 is also more clear now: using the notation from this example, we have $e_{2k-1} \in \mathcal{M}, \cos(1/k)e_{2k-1} + \sin(1/k)e_{2k} \in \mathcal{N}$, they are both unit vectors, and $|\langle e_{2k-1}, \cos(1/k)e_{2k-1} + \sin(1/k)e_{2k}\rangle| = \cos(1/k) \to 1,\ k \to \infty$; it is clear that $\mathcal{M} \cap \mathcal{N} = \{0\}$, thus $c_0(\mathcal{M}, \mathcal{N}) = c(\mathcal{M}, \mathcal{N}) = 1$ and so $\mathcal{M} + \mathcal{N}$ is not closed.

Theorem 1.2.8 (See [24]). *If $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$ are closed subspaces, the following statements are equivalent:*

(i) $\mathcal{M} + \mathcal{N}$ *is closed;*

(ii) $\mathcal{M}^\perp + \mathcal{N}^\perp$ *is closed;*

(iii) $[\mathcal{M} \ominus (\mathcal{M} \cap \mathcal{N})] \oplus [\mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N})]$ *is closed;*

(iv) $c(\mathcal{M}, \mathcal{N}) < 1$.

Theorem 1.2.9 (See [24]). *Let $\mathcal{H}$ be a Hilbert space and $A, B \in \mathcal{B}(\mathcal{H})$ be operators with closed ranges. Then the following statements are equivalent:*

(i) $\mathcal{R}(AB)$ *is closed;*

(ii) $c(\mathcal{R}(B), \mathcal{N}(A)) < 1$*;*

(iii) $\mathcal{N}(A) + \mathcal{R}(B)$ *is closed;*

(iv) $\mathcal{N}(B^*) + \mathcal{R}(A^*)$ *is closed.*

In the rest of this section we will give some results regarding special kind of subspaces of a Hilbert space: operator ranges. It is somewhat surprising that not every subspace of a Hilbert space $\mathcal{H}$ can be the range of a bounded operator. Of course every closed subspace of $\mathcal{H}$ is the range of a bounded operator (e.g. an orthogonal projection), but there are also non-closed subspaces which are ranges of bounded operators, i.e. some bounded operators have non-closed ranges. Such operator ranges also provide a classical example of two closed subspaces with a non-closed sum.

Example 2. Let $\mathcal{H} = l^2(\mathbb{N})$. Define $K : \mathcal{H} \to \mathcal{H}$ as:

$$K : (a_1, a_2, a_3, ...) \mapsto (\frac{1}{1}a_1, \frac{1}{2}a_2, \frac{1}{3}a_3, ...).$$

Obviously $K \in \mathcal{B}(\mathcal{H})$. Since every sequence of the form $(0, 0, ..., 0, 1, 0, 0, ...)$ is in $\mathcal{R}(K)$ we have $\overline{\mathcal{R}(K)} = \mathcal{H}$. On the other hand, $(1, 1/2, 1/3, ...) \in \mathcal{H}$ but $(1, 1/2, 1/3, ...) \notin \mathcal{R}(K)$, since $(1, 1, 1, ...) \notin \mathcal{H}$. Thus $\mathcal{R}(K) \neq \overline{\mathcal{R}(K)} = \mathcal{H}$.

We can note that the operator K is a positive, injective, compact operator with a dense range.

In a Hilbert space $\mathcal{H}\times\mathcal{H}$, the subspace $\mathcal{R}(K)\times\mathcal{H}$ is not closed, given that $\mathcal{R}(K)$ is not closed. On the other hand, the subspace $\mathcal{M} = \{(Kx, x) \ : \ x \in \mathcal{H}\}$ is closed as the graph of a bounded operator, and $\mathcal{N} = \{0\} \times \mathcal{H}$ is obviously closed, but $\mathcal{M} + \mathcal{N} = \mathcal{R}(K) \times \mathcal{H}$ which is not closed. This gives another example of a non-closed sum of two closed subspaces in a Hilbert space. $\diamond$

The following theorem gathers some basic but important relations about the range of an operator. We include the proof for completeness.

Theorem 1.2.10. *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then:*

1. *If $\mathcal{H} = \mathcal{K}$ and A is positive then $\mathcal{R}(A) \subseteq \mathcal{R}(A^\alpha)$, for every $\alpha \in (0, 1)$. Moreover $\mathcal{R}(A) = \mathcal{R}(A^\alpha)$ for some $\alpha \in (0, 1)$, if and only if $\mathcal{R}(A)$ is closed, in which case $\mathcal{R}(A) = \mathcal{R}(A^\alpha)$ for every $\alpha \in (0, 1)$;*

2. *$\mathcal{R}(A) = \mathcal{R}(|A^*|)$;*

3. *$\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(AA^*)}$ and $\mathcal{R}(A) = \mathcal{R}(AA^*)$ if and only if $\mathcal{R}(A)$ is closed, if and only if $\mathcal{R}(AA^*)$ is closed.*

4. *$\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A^*)$ is closed.*

Proof. 1. Let us prove first that for every $\alpha > \beta > 0$ we have $\mathcal{N}(A^\alpha) = \mathcal{N}(A^\beta)$. From $A^\alpha = A^{\alpha-\beta}A^\beta$ we have that $\mathcal{N}(A^\beta) \subseteq \mathcal{N}(A^\alpha)$. The proof will be completed if we find γ such that $\gamma > \alpha$ and $\mathcal{N}(A^\gamma) = \mathcal{N}(A^\beta)$. Since for every positive operator T we have $\mathcal{N}(T) = \mathcal{N}(T^2)$, then $\mathcal{N}(A^\beta) = \mathcal{N}(A^{2^k\cdot\beta})$ for every $k \in \mathbb{N}$, and for suitable k we can take $\gamma = 2^k\beta$.

We now go back to the proof. If $\alpha \in (0, 1)$ then $A = A^\alpha A^{1-\alpha}$ showing that $\mathcal{R}(A) \subseteq \mathcal{R}(A^\alpha)$. From $\mathcal{N}(A) = \mathcal{N}(A^\alpha)$ it follows $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^\alpha)}$ which leads to:

$$\mathcal{R}(A) \subseteq \mathcal{R}(A^\alpha) \subseteq \overline{\mathcal{R}(A)}.$$

Hence, if $\mathcal{R}(A)$ is closed, then all ranges $\mathcal{R}(A^\alpha)$ are equal to $\mathcal{R}(A)$.

Suppose now that $\mathcal{R}(A) = \mathcal{R}(A^\alpha)$ for some $\alpha \in (0,1)$ and let us prove that $\mathcal{R}(A)$ is closed. From $A = A^\alpha A^{1-\alpha}$ we have $\mathcal{R}(A) = A^\alpha(\mathcal{R}(A^{1-\alpha}))$. On the other hand,

$$\mathcal{R}(A) = \mathcal{R}(A^\alpha) = A^\alpha(\overline{\mathcal{R}(A^\alpha)}) = A^\alpha(\overline{\mathcal{R}(A^{1-\alpha})}),$$

since the closure of the range is the same regardless of the power. Thus we obtain:

$$A^\alpha(\mathcal{R}(A^{1-\alpha})) = A^\alpha(\overline{\mathcal{R}(A^{1-\alpha})}),$$

but A^α is an injection on $\overline{\mathcal{R}(A^{1-\alpha})} = \overline{\mathcal{R}(A^\alpha)}$ so the spaces $\overline{\mathcal{R}(A^{1-\alpha})}$ and $\mathcal{R}(A^{1-\alpha})$ must not be different. Hence, $\mathcal{R}(A^{1-\alpha})$ is closed, but then so is $\mathcal{R}(A^{(1-\alpha)\cdot 2^k})$ for every $k \in \mathbb{N}$ (if for positive operator T, $\mathcal{R}(T)$ is closed, then $T|_{\mathcal{R}(T),\mathcal{R}(T)}$ is an isomorphism). For some k we have $\beta = (1-\alpha)\cdot 2^k > 1$ and denote $B = A^\beta$. Since $\mathcal{R}(B)$ is closed, by the already proved part, we have that $\mathcal{R}(B) = \mathcal{R}(B^{1/\beta}) = \mathcal{R}(A)$, so $\mathcal{R}(A)$ is closed.

2. Since $AA^* = |A^*||A^*|^*$ the equality of ranges follows from the famous Douglas' theorem, which will be given in Section 1.3 as Theorem 1.3.2.

3. From 2. we have that $\mathcal{R}(A) = \mathcal{R}(|A^*|) = \mathcal{R}((AA^*)^{1/2})$, which together with statement 1. gives: $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}((AA^*)^{1/2})} = \overline{\mathcal{R}(AA^*)}$. We have that $\mathcal{R}(A)$ is closed iff $\mathcal{R}(|A^*|)$ is closed, and by statement 1. this is iff $\mathcal{R}(AA^*)$ is closed, which is iff $\mathcal{R}(AA^*) = \mathcal{R}((AA^*)^{1/2})$, i.e. $\mathcal{R}(AA^*) = \mathcal{R}(A)$.

4. If the range of A is closed, then the reduction A_1 of A onto $\overline{\mathcal{R}(A^*)}$ is an isomorphism between Hilbert spaces $\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(A)$. If $B : \mathcal{R}(A) \to \overline{\mathcal{R}(A^*)}$ is defined as $Bx = A^*x$, for every $x \in \mathcal{R}(A)$, then B is a well-defined operator and $B = A_1^*$. Since A_1 is an isomorphism, so is B (this is due to the bounded inverse theorem), thus $\mathcal{R}(B) = \overline{\mathcal{R}(A^*)}$, but $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$, showing that $\mathcal{R}(A^*) = \overline{\mathcal{R}(A^*)}$. □

The following result is contributed to Crimmins, while a beautiful proof that we present is due to Fillmore and Williams, and can be found in their classical paper about operator ranges [38].

Theorem 1.2.11 (Crimmins, see [38]). *If $\mathcal{H}$ is a Hilbert space, and $A, B \in \mathcal{B}(\mathcal{H})$ then:*

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{1/2}).$$

Proof. Consider an operator $T = \begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix}$ defined on the space $\mathcal{H} \times \mathcal{H}$ according to orthogonal decomposition $(\mathcal{H} \times \{0\}) \oplus (\{0\} \times \mathcal{H})$. We have that $\mathcal{R}(T) = (\mathcal{R}(A) + \mathcal{R}(B)) \times \{0\}$, but from Theorem 1.2.10 we know that $\mathcal{R}(T) = \mathcal{R}(|T^*|)$, while $|T^*| = \begin{bmatrix} (AA^* + BB^*)^{1/2} & 0 \\ 0 & 0 \end{bmatrix}$, so $\mathcal{R}(|T^*|) = \mathcal{R}((AA^* + BB^*)^{1/2}) \times \{0\}$. Hence $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{1/2})$. □

The following corollary for positive operators is particularly useful.

Corollary 1.2.12 (See [38]). *If $\mathcal{H}$ is a Hilbert space and $A, B \in \mathcal{B}(\mathcal{H})$ are positive, then $\mathcal{R}(A^{1/2}) + \mathcal{R}(B^{1/2}) = \mathcal{R}((A+B)^{1/2})$. Consequently $\mathcal{R}(A) \subseteq \mathcal{R}((A+B)^{1/2})$.*

The following theorem, originating from [38] as well, shows that two disjoint operator ranges can sum up to a closed subspace only if they are both closed. Of course, the converse is not true, since any two closed subspaces are operator ranges, and their sum is not necessarily closed, even if they are disjoint.

Theorem 1.2.13 (See [38]). *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, then $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed.*

In the end, we note that for operator ranges, the inclusion $\overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \subseteq \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$, always holds, and it is proper in general. Moreover, the following example shows that $\overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ can be equal to $\{0\}$ while $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is the whole space.

Example 3 (See [38]). In [38, Corollary 1] it is proved that for a non-closed operator range $\mathcal{R}$ in a separable Hilbert space $\mathcal{H}$, there is a family of unitary operators $\{U_t\}_{t\in\mathbb{R}}$ such that $U_t(\mathcal{R})$ and $U_s(\mathcal{R})$ are disjoint whenever $t \neq s$.

Now take for example $\mathcal{R} = \mathcal{R}(K)$, where K is defined as in Example 2. Since $\mathcal{R}$ is dense, so is $U_t(\mathcal{R})$, for every t, and of course, $U_t(\mathcal{R}) = \mathcal{R}(U_tK)$, or we can take $\mathcal{R}(U_tKU_t^*)$ if we need positive operators (as we will). In this way we obtain a family of mutually disjoint dense operator ranges, i.e. $\overline{\mathcal{R}(U_tK) \cap \mathcal{R}(U_sK)} = \{0\}$, while $\overline{\mathcal{R}(U_tK)} \cap \overline{\mathcal{R}(U_sK)} = \mathcal{H}$ for every $s \neq t$.

If A_1 and A_2 are two positive operators with dense disjoint ranges, consider the operators $A = A_1 + A_2$ and $B = 2A_1 + A_2$. It is straightforward to show that A and B also have disjoint ranges. Then from $A \leq B \leq 2A$ and Theorem 1.3.2 which we give later, we see that $\mathcal{R}(A^{1/2}) = \mathcal{R}(B^{1/2})$, and finally, since $A_1 \leq A, B$, A and B both have dense ranges. In this way we obtain two positive operators A and B with disjoint dense ranges, such that $\mathcal{R}(A^{1/2}) = \mathcal{R}(B^{1/2})$. ◇

1.3 Generalized inverses

The invertibility of an operator is a very important and useful property, but the condition of invertibility is too strong, and in many cases can be replaced by a weaker condition. The theory of generalized inverses studies different ways in which we can define an 'inverse' of a non-bijective operator, as well as the applications of such inverses and their properties. It is an important part of operator theory, and it has been developed over the last sixty years. For historical background, thorough study and many results from this area, the reader is referred to [15, 31, 72]. Most of the results and notions presented in this section are well-known, except the notion of the core generalized inverse, which was introduced recently in [13] and [75].

Almost all expositions of generalized inverses begin with the following equations given by Penrose [74]:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

As we can see, the first two equations make sense in any semigroup, and the other two as soon as some involution is defined. Although a rich theory of generalized inverses can be developed even on such sets with only algebraic structure, we are going to restrict our exposition only on operators between Hilbert spaces, but the terminology is the same everywhere.

Thus, throughout this chapter, $\mathcal{H}$ and $\mathcal{K}$ will denote arbitrary Hilbert spaces, $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and we are looking for the solution of above equations in $\mathcal{B}(\mathcal{K}, \mathcal{H})$. The set of common solutions of equations $i, j, ..., k$ is denoted by $A\{i, j, ..., k\}$, and some of them have special names. For example, the set $A\{1\}$ is the set of inner inverses, $A\{2\}$ is the set of outer inverses, and $A\{1, 2\}$ is the set of reflexive inverses of A.

It is well-known that A has some inner inverse if and only if its range $\mathcal{R}(A)$ is closed. If A^- is an arbitrary inner inverse of A, then AA^- and A^-A are projections, and $\mathcal{R}(AA^-) = \mathcal{R}(A)$, while $\mathcal{N}(A^-A) = \mathcal{N}(A)$. However, it is not difficult to see that any operator $A \neq 0$ has some outer inverse $X \neq 0$. The following property of outer inverses is well known, and the proof can be found in [31].

Theorem 1.3.1. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \setminus \{0\}$, and $\mathcal{M} \subseteq \mathcal{H}$ and $\mathcal{N} \subseteq \mathcal{K}$ be two subspaces of $\mathcal{H}$ and $\mathcal{K}$. The following statements are equivalent:*

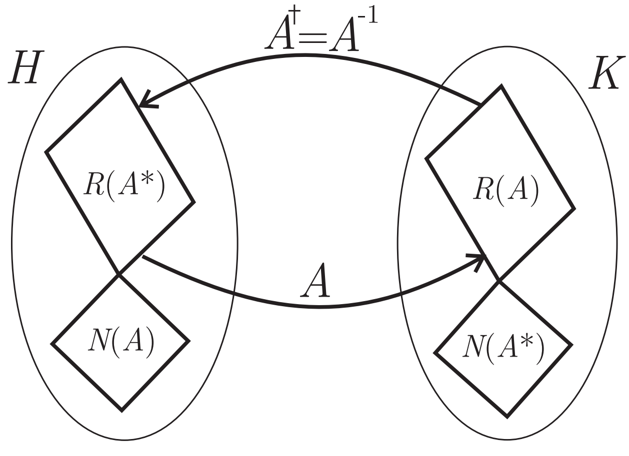
(i) There exists $X \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \setminus \{0\}$ such that $XAX = X$ and $\mathcal{R}(X) = \mathcal{M}$, $\mathcal{N}(X) = \mathcal{N}$;

(ii) Subspaces $\mathcal{M}, \mathcal{N}$ and $A(\mathcal{M})$ are closed, $A(\mathcal{M}) \oplus \mathcal{N} = \mathcal{K}$ and $\mathcal{N}(A) \cap \mathcal{M} = \{0\}$.

In that case, such X is unique.

The unique outer inverse with the predefined range $\mathcal{M}$ and $\mathcal{N}$ described in the previous theorem will be denoted by $A^{(2)}_{\mathcal{M},\mathcal{N}}$.

When the range of an operator A is closed, the set $A\{1, 2, 3, 4\}$ contains a unique element which is denoted by $A^\dagger$ and called the Moore-Penrose inverse of A. We can define the Moore-Penrose inverse of an operator in a fashion that better suits the Hilbert space setting. Namely, if A has a closed range, then $\mathcal{R}(A^*)$ is also closed, and an operator $A|_{\mathcal{R}(A^*),\mathcal{R}(A)} : \mathcal{R}(A^*) \to \mathcal{R}(A)$ is an isomorphism. The (bounded) operator which is defined as $(A|_{\mathcal{R}(A^*),\mathcal{R}(A)})^{-1}$ on $\mathcal{R}(A)$, and as the null-operator on $\mathcal{N}(A^*)$ is exactly the Moore-Penrose inverse of A. Note also that, when $\mathcal{R}(A)$ is closed, then $A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A)}$.



There are other important generalized inverses which are not defined only by Penrose equations. We are going to describe here the so called group inverse, and the core inverse of an operator. They can also be described as unique solution to a certain system of equations, but we do it in a manner more appropriate for us. The two mentioned generalized inverses are defined only for operators from $\mathcal{B}(\mathcal{H})$, which are of index at most 1, i.e. belong to a set $\mathcal{B}^1(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \ : \ \mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{H}\}$. This definition implicitly contains the fact that all operators from $\mathcal{B}^1(\mathcal{H})$ have closed ranges (see, e.g. Theorem 1.2.13). Hence, if $A \in \mathcal{B}^1(\mathcal{H})$, then the subspace $\mathcal{R}(A)$ reduces A to an isomorphism $A|_{\mathcal{R}(A)} : \mathcal{R}(A) \to \mathcal{R}(A)$.

The operator defined as $(A|_{\mathcal{R}(A)})^{-1}$ on $\mathcal{R}(A)$ and as the null-operator on $\mathcal{N}(A)$ is called the group inverse of A and is denoted by $A^\sharp$. Group inverse is obviously a reflexive inverse of A, and also $AA^\sharp = A^\sharp A = P_{\mathcal{R}(A),\mathcal{N}(A)}$.

The operator defined as $(A|_{\mathcal{R}(A)})^{-1}$ on $\mathcal{R}(A)$, but as the null-operator on $\mathcal{N}(A^*)$ is called the core inverse of A and is denoted by $A^{\textcircled{\#}}$. For the core inverse, we have $AA^{\textcircled{\#}} = P_{\mathcal{R}(A)}$ and $A^{\textcircled{\#}}A = P_{\mathcal{R}(A),\mathcal{N}(A)}$, since $\mathcal{R}(A^{\textcircled{\#}}) = \mathcal{R}(A)$ and $\mathcal{N}(A^{\textcircled{\#}}) = \mathcal{N}(A^*)$. Given the 'asymmetric' definition of the core inverse, we see that it does not obey some classic duality rules like the group or Moore-Penrose inverse. E.g. in general $(A^{\textcircled{\#}})^* \neq (A^*)^{\textcircled{\#}}$, $(A^{\textcircled{\#}})^{\textcircled{\#}} \neq A$, etc.

We should note at the end that for every operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, there is always a linear transformation $X : \mathcal{K} \to \mathcal{H}$ which satisfies $AXA = A$. Some authors also call such transformations inner inverses of A, emphasizing that there is a *bounded* inner inverse if and only if the range of A is closed. The Moore-Penrose inverse $A^\dagger$ can also be constructed for every $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ (in the sense that it satisfies all Penrose equations), but with the domain $D(A^\dagger) = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$. Namely, every A reduced between $\overline{\mathcal{R}(A^*)}$ and $\mathcal{R}(A)$ is a bijection, so $A^\dagger$ is defined as the inverse of this bijection on $\mathcal{R}(A)$, and as the null-operator on $\mathcal{N}(A^*)$. Thus $\mathcal{R}(A^\dagger) = \overline{\mathcal{R}(A^*)}$, $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$ and $A^\dagger$ is densely defined closed operator. It is true that $A^\dagger$ is bounded if and only if $\mathcal{R}(A)$ is closed (see [15]).

Since $A^\dagger$ is a closed operator, it is not difficult to show that the composition $A^\dagger B$, for any $B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, is also a closed operator. From a closed graph theorem it follows that $A^\dagger B$ is in fact bounded. This is a very important observation, and one of its applications is in the proof of the famous Douglas' theorem:

Theorem 1.3.2. *(See [33]) Let $\mathcal{H},\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces, $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{K})$. The following statements are equivalent:*

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$;

(ii) There exists $X \in \mathcal{B}(\mathcal{L},\mathcal{H})$ such that $B = AX$;

(iii) There exists $\lambda > 0$ such that $BB^* \leq \lambda^2 AA^*$.

In that case, the equation $AX = B$ has a unique solution X such that $\mathcal{R}(X) \subseteq \overline{\mathcal{R}(A^*)}$.

Proof. (i) $\Rightarrow$ (ii) From the discussion before the theorem, we have that $A^\dagger B \in \mathcal{B}(\mathcal{L},\mathcal{H})$, and clearly $X = A^\dagger B$ is a solution of $AX = B$.

(ii) $\Rightarrow$ (i) This is clear.

(ii) $\Rightarrow$ (iii) If $B = AX$, then for every $x \in \mathcal{K}$ we have:

$$\langle BB^*x, x\rangle = ||B^*x||^2 = ||X^*A^*x||^2 \leq ||X^*||^2||A^*x||^2 = ||X^*||^2\langle AA^*x, x\rangle.$$

This proves (iii).

(iii) $\Rightarrow$ (ii) From (iii) we have that for every $x \in \mathcal{K}$: $||B^*x|| \leq \lambda||A^*x||$, so $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$. Hence a map $D : \mathcal{R}(A^*) \to \mathcal{R}(B^*)$ defined as $D(A^*x) = B^*x$ is a well-defined, linear and bounded. This map can be uniquely extended by continuity on $\overline{\mathcal{R}(A^*)}$, and defined as null-operator on $\mathcal{N}(A)$. In this way, $D \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ and $B^* = DA^*$. In other words $B = AD^*$, for $D^* \in \mathcal{B}(\mathcal{L}, \mathcal{H})$.

To prove the other assertion of the statement, first note that the both solutions of the equation $AX = B$ constructed above satisfy $\mathcal{R}(X) \subseteq \overline{\mathcal{R}(A^*)}$, i.e. $\mathcal{N}(A) \subseteq \mathcal{N}(X^*)$. Assume that Y is also a solution satisfying this condition. Then also $\mathcal{N}(A) \subseteq \mathcal{N}(Y^*)$, so X^* and Y^* coincide on $\mathcal{N}(A)$. Since $B^* = X^*A^* = Y^*A^*$, we have that X^* and Y^* coincide on $\overline{\mathcal{R}(A^*)}$ also. Hence X^* and Y^* coincide on whole $\mathcal{H}$, i.e. $X = Y$. $\square$

The unique solution described in the theorem of Douglas is usually called the reduced solution of the equation $B = AX$.

1.4 Range additivity

If $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces, and $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that:

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B), \tag{1.4}$$

we say that A and B are *range additive*. The condition of range additivity appears naturally in some problems in linear algebra and operator theory, and we will see some of them in this thesis. In infinite-dimensional Hilbert spaces, the question of range additivity is not only a question about algebraic properties of operator ranges, but also a question about their topological properties, which will be apparent after we give a few results along these lines. That being said, besides (1.4), one could also consider conditions $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A + B)}$, or $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} = \overline{\mathcal{R}(A + B)}$, which have more a topological flavour. Observe that $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} = \overline{\mathcal{R}(A) + \mathcal{R}(B)}$, but we prefer to keep the closures of ranges within, whenever we do not know if ranges are closed.

In case of matrices, together with range additivity, the relation of rank additivity is also interesting: we say that two matrices $A, B \in \mathbb{C}^{n\times m}$ are rank additive if

$$\mathrm{r}(A) + \mathrm{r}(B) = \mathrm{r}(A + B). \tag{1.5}$$

It is obvious however that the rank additivity is symmetric with respect to taking adjoints, while the range additivity is not (Example 4). The rank additivity is also much stronger condition, since it implies direct range additivity $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{R}(A + B)$ and $\mathcal{R}(A^*) \oplus \mathcal{R}(B^*) = \mathcal{R}(A + B)$. For this result, any many other interesting and recent results, the reader is referred to [7, 9–11, 38, 62], and furthered references therein.

In this section, we give a compilation of results for later reference, but also to illustrate some technique of this interesting topic. We start by noticing an obvious fact, we always have:

$$\mathcal{R}(A+B) \subseteq \mathcal{R}(A) + \mathcal{R}(B).$$

The difference between $\mathcal{R}(A+B)$ and $\mathcal{R}(A)+\mathcal{R}(B)$ in general can be drastic, e.g. take $B = -A$. The following lemma is based on simple algebraic manipulations, and we will use it without referencing it.

Lemma 1.4.1. *(See [10]) If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ then $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A+B)$ if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$.*

Example 4. (See [10]) Even on $\mathbb{C}^2$ we can find an example of two operators A and B such that A and B are range additive, but A^* and B^* are not. For example, take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. ◇

The following proposition appeared in [9] where it was attributed to A. Maestripieri. In Theorem 2.3.5 we study such relations for precoherent operators.

Proposition 1.4.2 (A. Maestripieri)**.** *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, then $\mathcal{R}(A+B) = \mathcal{R}(A) + \mathcal{R}(B)$ if and only if $\mathcal{N}(A) + \mathcal{N}(B) = \mathcal{H}$.*

Having in mind Proposition 1.4.2 and Theorem 1.2.8, the proof of the following proposition is derived easily. In this proposition we also see a connection between a topological and an algebraical condition.

Proposition 1.4.3 (See [10])**.** *For $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ consider the following statements:*

1. *$\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)} = \{0\}$ and $\overline{\mathcal{R}(A^*)} \oplus \overline{\mathcal{R}(B^*)}$ is closed;*
2. *$\mathcal{N}(A) + \mathcal{N}(B) = \mathcal{H}$;*
3. *$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A+B)$.*

The following implications hold 1. ⇔ 2. ⇒ 3. *If $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ then* 3. ⇒ 2. *also holds.*

The sum of two positive real numbers can not be smaller than those two numbers, and the same happens with ranges of positive operators: the range $\mathcal{R}(A+B)$ can not be 'significantly' smaller than the ranges $\mathcal{R}(A)$ and $\mathcal{R}(B)$. A hint for this was given in Corollary 1.2.12, but the refinement we present here is from a recent paper [7] (for example, statement 2. of the following theorem was already known under the assumption that A and B have closed ranges, but in [7] this was proved without such an assumption). We also include the proof.

Theorem 1.4.4 (See [7])**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators. Then:*

1. *$\overline{\mathcal{R}(A+B)} = \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$;*

2. *$\mathcal{R}(A)+\mathcal{R}(B)$ is closed if and only if $\mathcal{R}(A+B)$ is closed. In that case: $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$;*

3. *If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, then $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$ if and only if $\mathcal{R}(A+B)$ is closed (if and only if $\mathcal{R}(A)+\mathcal{R}(B)$ is closed).*

4. *If $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}$ then $\mathcal{R}(A)+\mathcal{R}(B)$ is closed if and only if $\mathcal{R}(A)$, $\mathcal{R}(B)$ are closed, and $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$.*

Proof. 1. From Theorem 1.2.7 and Corollary 1.2.12 we have $\overline{\mathcal{R}(A+B)}=\overline{\mathcal{R}((A+B)^{1/2})}=\overline{\mathcal{R}(A^{1/2})+\mathcal{R}(B^{1/2})}=\overline{\overline{\mathcal{R}(A^{1/2})}+\overline{\mathcal{R}(B^{1/2})}}=\overline{\overline{\mathcal{R}(A)}+\overline{\mathcal{R}(B)}}$.

2. We have the following sequence of inclusions and equalities:

$$\mathcal{R}(A+B)\subseteq\mathcal{R}(A)+\mathcal{R}(B)\subseteq\mathcal{R}(A^{1/2})+\mathcal{R}(B^{1/2})=\mathcal{R}((A+B)^{1/2})\subseteq$$

$$\subseteq\overline{\mathcal{R}(A+B)}=\overline{\mathcal{R}(A)+\mathcal{R}(B)},$$

in which we used Theorem 1.2.7, Corollary 1.2.12 and previously proved fact. If $\mathcal{R}(A)+\mathcal{R}(B)$ is closed, then $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}((A+B)^{1/2})$ showing that $\mathcal{R}((A+B)^{1/2})$ is closed, thus $\mathcal{R}(A+B)$ is closed, and $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$. If $\mathcal{R}(A+B)$ is closed, then $\mathcal{R}((A+B)^{1/2})=\mathcal{R}(A+B)$, and so $\mathcal{R}(A+B)=\mathcal{R}(A)+\mathcal{R}(B)$ showing that $\mathcal{R}(A)+\mathcal{R}(B)$ is also closed.

3. If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, and $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$ then we have

$$\mathcal{R}((A+B)^{1/2})=\mathcal{R}(A^{1/2})+\mathcal{R}(B^{1/2})=\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B),$$

showing that $\mathcal{R}(A+B)$ is also closed. The converse is contained in 2.

4. Follows from previous statements and Theorem 1.2.13. □

Example 5. Observe that, if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are not closed, then $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$ can hold without $\mathcal{R}(A+B)$ being closed. Just take $A=B$, a positive operator with a non-closed range. ◇

Interplay between topological and algebraic aspect of range additivity is nicely demonstrated by the following example as well.

Example 6. If $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces, and P and Q are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$ respectively, then P and Q are positive operators with closed ranges. Statement 3. of Theorem 2.4.10 tells that $\mathcal{R}(P)+\mathcal{R}(Q)=\mathcal{R}(P+Q)$ exactly when $\mathcal{M}+\mathcal{N}$ is a closed subspace, i.e. exactly when $\mathcal{R}(P+Q)$ is closed. In that case: $\mathcal{R}(P_{\mathcal{M}+\mathcal{N}})=\mathcal{M}+\mathcal{N}=\mathcal{R}(P+Q)$. ◇

The following lemma for orthogonal projections appeared in [60].

Lemma 1.4.5 (See [60]). *If P and Q are orthogonal projections on $\mathcal{H}$, the following statements are equivalent:*

(i) *$\mathcal{R}(P-Q)$ is closed;*

(ii) *$\mathcal{R}(P+Q)$ is closed;*

(iii) $\mathcal{R}(P) + \mathcal{R}(Q)$ *is closed;*

(iv) $\mathcal{N}(P) + \mathcal{N}(Q)$ *is closed;*

(v) $\mathcal{R}(P(I - Q))$ *is closed;*

(vi) $\mathcal{R}((I - P)Q)$ *is closed.*

If any of these statements hold, then $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{R}(P + Q)$.

Equivalence of statements (ii)-(vi) of the previous lemma follows from the presented study of positive operators, and Theorems 1.2.8 and 1.2.9, but the equivalence of (i) and (ii) is more intriguing. In [60], this was proved by referring to spectrum of operators derived from P and Q, and in [7] one can find a proof without using spectrum of an operator.

In the end, we give one short lemma which seems interesting, but we didn't notice it in the existing literature.

Lemma 1.4.6. *If* $A, B \in \mathcal{B}(\mathcal{H})$ *are positive, commuting operators, then* $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B)$.

Proof. By Douglas' theorem we have that $\mathcal{R}(A) \subseteq \mathcal{R}(A + B)$ if and only if $A^2 \leq \lambda(A + B)^2 = \lambda(A^2 + B^2 + AB + BA)$, for some $\lambda \geq 0$. If A and B commute, then $AB = BA$ is a positive operator, and so $A^2 \leq (A + B)^2$ showing $\mathcal{R}(A) \subseteq \mathcal{R}(A + B)$, i.e. $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B)$. $\square$

1.5 Shorted operators and parallel sums

The notions of parallel summation of operators and the shorted operator originated from the theory of electrical networks[1], although over time they have proved useful in other areas as well. These notions are closely related and each of them can be defined in terms of the other. For a short but very informative historical survey, the reader is referred to [6] and the references therein. Here we are going to present only a few moments from the development of this theory, and state a few properties in the end of the section.

Parallel addition of positive-semidefinite matrices was introduced by Anderson in [2]. The definition is as follows: if A and B are two p.s.d. matrices of the same order, their parallel sum is

$$A : B = A(A + B)^{\dagger}B.$$

One of the important feature of positive-semidefinite matrices for the study of the parallel sum is the range additivity. Namely, any two positive-semidefinite matrices are range additive, but the same doesn't hold for arbitrary positive operators (see Section 1.2).

[1]The term parallel summation refers to the parallel connection of two electrical networks. Under some conditions, the impedance matrix of the new network is obtained exactly as the parallel sum of old impedance matrices. On the other hand, the term 'shorted' refers to a short circuit in a network: if some nodes get short circuited, then under certain conditions an impedance matrix will become exactly the matrix we here define as the shorted matrix, i.e. operator, with suitably chosen subspace. A detailed presentation on this subject can be found in [69].

The first extension of the parallel summation for arbitrary positive operators A and B, given by Anderson and Schreiber in [3], covered only a case when $\mathcal{R}(A+B)$ is closed, assuring the range additivity. Using a more sophisticated device, Fillmore and Williams in [38] gave a satisfactory definition of the parallel sum of arbitrary positive operators. Namely, from Corollary 1.2.12 and Theorem 1.2.10, for every positive operators A and B we have $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2}) \subseteq \mathcal{R}((A+B)^{1/2})$, $\mathcal{R}(B) \subseteq \mathcal{R}(B^{1/2}) \subseteq \mathcal{R}((A+B)^{1/2})$ and so by Douglas' theorem, equations:

$$A^{1/2} = (A+B)^{1/2}X, \quad B^{1/2} = (A+B)^{1/2}Y,$$

are solvable. If $X = C$ and $Y = D$ are their reduced solutions, then in fact: $C = ((A+B)^{1/2})^{\dagger}A^{1/2}$ and $D = ((A+B)^{1/2})^{\dagger}B^{1/2}$, where the appearing Moore-Penrose inverse is not necessarily bounded, but operators C and D are well-defined and are bounded (see Section 1.3). The parallel sum is then defined as:

$$A : B = A^{1/2}C^{*}DB^{1/2}.$$

They also show that if the operators A and B are range additive, this definition coincides with $A : B = A(A+B)^{\dagger}B$ (again, the Moore-Penrose inverse is not necessarily bounded, but $\mathcal{R}(B) \subseteq \mathcal{R}(A+B)$). A different approach was offered by Anderson and Trapp in [4], and in order to present it, we should pause here and say something about the shorted operator.

For a positive operator A on a Hilbert space $\mathcal{H}$, and a closed subspace $\mathcal{S}$, the set $\{B \;:\; 0 \le B \le A \;\text{ and }\; \mathcal{R}(B) \subseteq \mathcal{S}\}$ contains the maximum – this is the result which was known but rediscovered by Anderson [1] for matrices and by Anderson and Trapp [4] in general case. This maximum is called a shorted operator of A by the subspace $\mathcal{S}$:

$$A_{/\mathcal{S}} = \max\{B \;:\; 0 \le B \le A \;\text{ and }\; \mathcal{R}(B) \subseteq \mathcal{S}\}.$$

It is shown in [4] that if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is the matrix form of A with respect to the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$, then $\mathcal{R}(A_{21}) \subseteq \mathcal{R}(A_{22}^{1/2})$ and if C is the reduced solution of the equation $A_{21} = A_{22}^{1/2}X$, then the shorted operator $A_{/\mathcal{S}}$ is equal to:

$$A_{/\mathcal{S}} = \begin{bmatrix} A_{11} - C^{*}C & 0 \\ 0 & 0 \end{bmatrix}.$$

Again, we have here in fact $C = (A_{22}^{1/2})^{\dagger}A_{12}$, so in case that $\mathcal{R}(A_{21}) \subseteq \mathcal{R}(A_{22})$, then $A_{11} - C^{*}C = A_{11} - A_{12}A_{22}^{\dagger}A_{21}$, where the Moore-Penrose inverse is not necessarily bounded (the relation with the generalized Schur complement is obvious and the reader is referred to cited references for further information).

The parallel sum of two positive operators A and B is now defined as follows: consider the Hilbert space $\mathcal{H} \times \mathcal{H}$, the operator $T = \begin{bmatrix} A & A \\ A & A+B \end{bmatrix}$, and let $\mathcal{S} = \mathcal{H} \times \{0\}$; then $A : B$ is the operator appearing in the $(1,1)$-entry of $T_{/\mathcal{S}}$:

$$\begin{bmatrix} A : B & 0 \\ 0 & 0 \end{bmatrix} = T_{/\mathcal{S}}.$$

Of course, it is shown that this definition coincides with the previous one. One feature of parallel summation is particularly interesting and we put it in a theorem:

Theorem 1.5.1 (See [38]). *If P and Q are two orthogonal projections on a Hilbert space $\mathcal{H}$, then $2(P:Q) = P \wedge Q$, i.e. $2(P:Q)$ is the orthogonal projection onto $\mathcal{R}(P) \cap \mathcal{R}(Q)$.*

Let us present now how the notions of parallel summation and shorting of an operator were generalized for arbitrary operators between different Hilbert spaces. For considerations in the finite-dimensional case see e.g. [70], and we will here present a more general approach given by Antezana, Corach and Stojanoff in [6]. With this we conclude our exposition.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces, $\mathcal{S} \subseteq \mathcal{H}$ and $\mathcal{T} \subseteq \mathcal{K}$ two closed subspaces, and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with the decomposition:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{S} \\ \mathcal{S}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{T} \\ \mathcal{T}^\perp \end{bmatrix}.$$

We say that operator A is $(\mathcal{S}, \mathcal{T})$-weakly complementable if:

$$\mathcal{R}(A_{21}) \subseteq \mathcal{R}(|A_{22}^*|^{1/2}) \quad \text{and} \quad \mathcal{R}(A_{12}^*) \subseteq \mathcal{R}(|A_{22}|^{1/2}).$$

In that case, if E and F are the reduced solutions of the equations $A_{21} = |A_{22}^*|^{1/2}UX$ and $A_{12}^* = |A_{22}|^{1/2}Y$ respectively, where $A_{22} = U|A_{22}| = |A_{22}^*|U$ is the polar decomposition of A_{22}, then the bilateral shorted operator of A with the subspaces $\mathcal{S}$ and $\mathcal{T}$ is defined as:

$$A_{/(\mathcal{S},\mathcal{T})} = \begin{bmatrix} A_{11} - F^*E & 0 \\ 0 & 0 \end{bmatrix}.$$

If the stronger condition holds: $\mathcal{R}(A_{21}) \subseteq \mathcal{R}(A_{22})$ and $\mathcal{R}(A_{12}^*) \subseteq \mathcal{R}(A_{22}^*)$, then A is said to be $(\mathcal{S}, \mathcal{T})$-complementable.

The parallel sum is now defined accordingly: if $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we say that A and B are weakly parallel summable if the operators $T_A = \begin{bmatrix} A & A \\ A & A+B \end{bmatrix}$ and $T_B = \begin{bmatrix} B & B \\ B & A+B \end{bmatrix}$ from $\mathcal{H} \times \mathcal{H}$ to $\mathcal{K} \times \mathcal{K}$ are $(\mathcal{H} \times \{0\}, \mathcal{K} \times \{0\})$-weakly complementable. In other words, if the following inclusions hold:

$$\mathcal{R}(A) \subseteq \mathcal{R}(|A^* + B^*|^{1/2}), \quad \mathcal{R}(B) \subseteq \mathcal{R}(|A^* + B^*|^{1/2})$$

$$\mathcal{R}(A^*) \subseteq \mathcal{R}(|A + B|^{1/2}), \quad \mathcal{R}(B^*) \subseteq \mathcal{R}(|A + B|^{1/2}).$$

The parallel sum can then be defined in any of the following four ways:

$$\begin{bmatrix} A:B & 0 \\ 0 & 0 \end{bmatrix} = T_{A_{/(\mathcal{H}\times\{0\}, \mathcal{K}\times\{0\})}}, \quad \begin{bmatrix} A:B & 0 \\ 0 & 0 \end{bmatrix} = T_{B_{/(\mathcal{H}\times\{0\}, \mathcal{K}\times\{0\})}},$$

$$A:B = F_A^* E_B, \quad A:B = F_B^* E_A,$$

where E_A, E_B, F_A, F_B are the reduced solutions of the equations $A = |A^* + B^*|^{1/2}UX$, $B = |A^* + B^*|^{1/2}X$, $A^* = |A+B|^{1/2}X$, $B^* = |A+B|^{1/2}X$, and $A + B = U|A+B|$ is the polar decomposition. Operators for which range additivity holds, i.e. $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$

and $\mathcal{R}(A^*) \subseteq \mathcal{R}(A^* + B^*)$, are called parallel summable. This condition is obviously stronger, and if $\mathcal{R}(A+B)$ is closed, then the two notions coincide.

Note that if T_A is $(\mathcal{H} \times \{0\}, \mathcal{K} \times \{0\})$-weakly complementable, so is T_B, and there is no need to check all four range inclusions. Indeed, since $\mathcal{R}(A+B) = \mathcal{R}(|A^* + B^*|) \subseteq \mathcal{R}(|A^* + B^*|^{1/2})$, if $\mathcal{R}(A) \subseteq \mathcal{R}(|A^* + B^*|^{1/2})$, then for every $x \in \mathcal{H}$ we have $Bx = (A+B)x - Ax \in \mathcal{R}(|A^* + B^*|^{1/2})$, so $\mathcal{R}(B) \subseteq \mathcal{R}(|A^* + B^*|^{1/2})$.

As we can see this definition generalizes the previous one for positive operators, and two positive operators are always weakly parallel summable. In the end, we gather some basic facts about the parallel sum.

Proposition 1.5.2 (See [6])**.** *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are weakly parallel summable operators, then:*

1. *$A : B = B : A$, A^* and B^* are weakly parallel summable and $(A : B)^* = A^* : B^*$;*
2. *If $x, y \in \mathcal{H}$ are such that $Ax = By \in \mathcal{R}(A) \cap \mathcal{R}(B)$, then $Ax = By = (A : B)(x+y)$;*
3. *$\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \mathcal{R}(A : B) \subseteq \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$;*
4. *If A and B are positive then $\mathcal{R}((A : B)^{1/2}) = \mathcal{R}(A^{1/2}) \cap \mathcal{R}(B^{1/2})$.*

Theorem 1.5.3 (See [6])**.** *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are parallel summable and $\mathcal{R}(A+B)$ is closed, then $A : B = A(A+B)^\dagger B = A - A(A+B)^\dagger A$ and $\mathcal{R}(A : B) = \mathcal{R}(A) \cap \mathcal{R}(B)$.*

Chapter 2

Coherent and precoherent operators

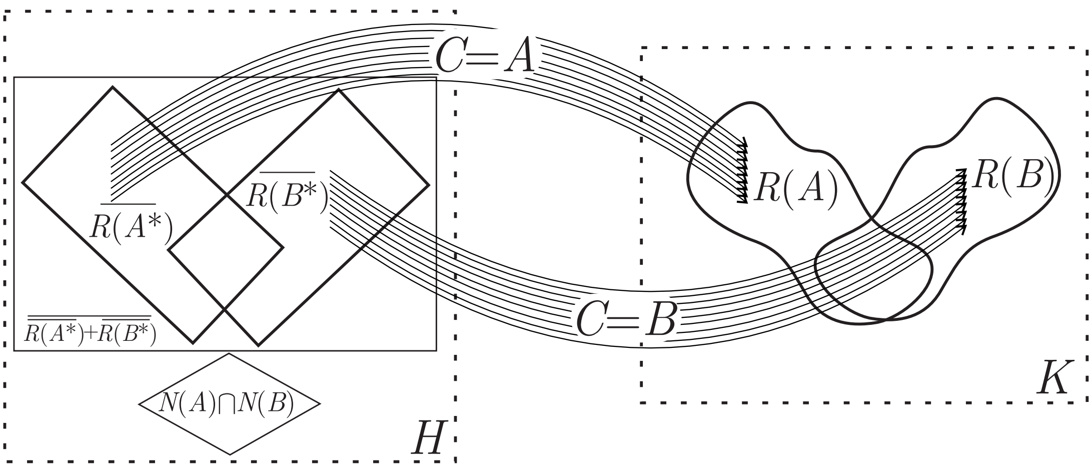
In this chapter we introduce the main notion of the thesis: coherent and precoherent operators. We will prove interesting geometric results of such operators, many of which will be used in the subsequent chapters. The definition of coherent operators first appeared in [28], while the term precoherent was used for the first time in [29], although such operators were used in all the papers [25–29].

2.1 Definition, motivation and examples

We will first give the definition of coherent and precoherent operators, and then we will explain our motivation to study such operators.

Definition 2.1.1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces of $\mathcal{H}$. Pairs $(A, \mathcal{M})$ and $(B, \mathcal{N})$ are precoherent if A and B coincide on $\mathcal{M} \cap \mathcal{N}$. Operators A and B are precoherent if pairs $(A, \overline{\mathcal{R}(A^*)})$ and $(B, \overline{\mathcal{R}(B^*)})$ are precoherent.

Definition 2.1.2. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces of $\mathcal{H}$. Pairs $(A, \mathcal{M})$ and $(B, \mathcal{N})$ are coherent if there exists $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that A and C coincide on $\mathcal{M}$ and B and C coincide on $\mathcal{N}$. Operators A and B are coherent if pairs $(A, \overline{\mathcal{R}(A^*)})$ and $(B, \overline{\mathcal{R}(B^*)})$ are coherent.



These definitions can be reformulated in a few different ways. For example, Definition 2.1.1 can be stated as: pairs $(A, \mathcal{M})$ and $(B, \mathcal{N})$ are precoherent if $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{N}(A - B)$. Algebraically, conditions of precoherence and coherence can be stated like this: $(A, \mathcal{M})$ and $(B, \mathcal{N})$ are precoherent if $(A - B)(P_{\mathcal{M}} \wedge P_{\mathcal{N}}) = 0$; $(A, \mathcal{M})$ and $(B, \mathcal{N})$ are coherent if system of equations $AP_{\mathcal{M}} = CP_{\mathcal{M}}$, $BP_{\mathcal{N}} = CP_{\mathcal{N}}$ has a solution. Particularly, if $\mathcal{M} = \overline{\mathcal{R}(A^*)}$ and $\mathcal{N} = \overline{\mathcal{R}(B^*)}$ this system is equivalent to $AA^* = CA^*$ and $BB^* = CB^*$ (we can do a similar reformulation whenever $\mathcal{M}$ and $\mathcal{N}$ are closures of operator ranges). In Chapter 5 we will use such definitions, since we are going to work in an algebraic setting. It is clear that, in order for pairs $(A, \mathcal{M})$ and $(B, \mathcal{N})$ to be coherent, they have to be precoherent. However, in general it is not sufficient for them to be precoherent. Before a discussion along these lines, let us first explain where such operators appear.

Coherent operators appear naturally in the study of common upper bounds of two operators in some partial orders. We will discuss this subject in detail in Chapter 4, but we present only a small excerpt here. For example, the so called star partial order on the algebra $\mathcal{B}(\mathcal{H})$ is defined in a following manner: we say that $A \overset{\star}{\leq} C$ if C coincides with A on $\overline{\mathcal{R}(A^*)}$, while $C(\mathcal{N}(A)) \subseteq \mathcal{N}(A^*)$. Obviously, if A and B have a common upper bound C in this partial order, then A and B are coherent. It is the same situation with any other partial order: minus, left star, right star, sharp, core, etc. only the underlying spaces are not always $\overline{\mathcal{R}(A^*)}$ and $\overline{\mathcal{R}(B^*)}$, so we need also a more general notion of coherence, for different pairs $(A, \mathcal{M})$ and $(B, \mathcal{N})$.

Precoherent operators, although a superset of coherent operators, appear in other situations as well. For example, precoherence of A and B is a generalization of range disjointness condition: $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)} = \{0\}$, and simultaneous precoherence of A and B, and A^* and B^* is a generalization of rank additivity condition (1.5). One can find quite a few interesting results for rank additive matrices, and range disjoint operators (see Section 2.4 and Chapter 3) and it is of interest to see if such results can be extended to precoherent operators as well. We will say more about this in subsequent sections, and now we just give one short example of an important class of precoherent operators.

Example 7. If $P, Q \in \mathcal{B}(\mathcal{H})$ are orthogonal projections, then P and Q are obviously precoherent, since on $\overline{\mathcal{R}(P^*)} \cap \overline{\mathcal{R}(Q^*)} = \mathcal{R}(P) \cap \mathcal{R}(Q)$ they are both equal to identity. Of course, P and Q do not have to be range disjoint, nor range additive. It is also obvious that P and Q are coherent: identity operator I is a bounded operator which coincides with P and with Q on appropriate subspaces.

It is interesting that operators PQ and QP are also precoherent. This is also true for any two operators obtained by alternating two orthogonal projections P and Q, even with a different number of factors, as soon as the last projections in the products are different: $PQ \cdots Q$ and $PQ \cdots P$, or $PQ \cdots Q$ and $QP \cdots P$. We will prove this in Theorem 2.4.11. $\diamond$

Let us now return to the question whether precoherence of two pairs $(A, \mathcal{M})$ and $(B, \mathcal{N})$ imply their coherence. For two pairs $(A, \mathcal{M})$ and $(B, \mathcal{N})$, the existence of a linear, not necessarily bounded, transformation $C : \mathcal{H} \to \mathcal{K}$ which coincides with A on $\mathcal{M}$ and with B on $\mathcal{N}$ is equivalent to coincidence of A and B on $\mathcal{M} \cap \mathcal{N}$, i.e. to precoherence of these pairs. On the other hand, the existence of such bounded operator can not be guaranteed only by coincidence of A and B on the intersection of $\mathcal{M}$ and

$\mathcal{N}$. However, if $\mathcal{M}+\mathcal{N}$ is closed, then the pairs $(A,\mathcal{M})$ and $(B,\mathcal{N})$ are coherent if and only if they are precoherent. In fact, in this way we can characterize the situation when $\mathcal{M}+\mathcal{N}$ is closed, as presented in Proposition 2.1.3. Recall the fact from Theorem 1.2.7, condition (iii): if $\mathcal{M}\cap\mathcal{N}=\{0\}$, then $\mathcal{M}\oplus\mathcal{N}$ is closed if and only if the linear idempotent P on the normed space $\mathcal{M}\oplus\mathcal{N}$ defined as $P(m+n)=m$ is bounded.

Proposition 2.1.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of a Hilbert space $\mathcal{H}$. The following statements are equivalent:*

(i) Subspace $\mathcal{M}+\mathcal{N}$ is closed;

(ii) For every Hilbert space $\mathcal{K}$, and $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and $B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ such that $(A,\mathcal{M})$ and $(B,\mathcal{N})$ are precoherent, the pairs $(A,\mathcal{M})$ and $(B,\mathcal{N})$ are coherent.

Proof. Suppose that $\mathcal{M}+\mathcal{N}$ is closed, and A and B are arbitrary operators coinciding on $\mathcal{M}\cap\mathcal{N}$. Then $\mathcal{H}$ can be expressed as $\mathcal{H}=(\mathcal{M}+\mathcal{N})^\perp\oplus(\mathcal{M}+\mathcal{N})$, while the subspaces $\mathcal{M}$ and $\mathcal{N}$ can be further decomposed as $\mathcal{M}=(\mathcal{M}\cap\mathcal{N})\oplus(\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp)$ and $\mathcal{N}=(\mathcal{M}\cap\mathcal{N})\oplus(\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp)$. Thus, $\mathcal{H}$ can be expressed as the sum of four closed subspaces $\mathcal{H}=(\mathcal{M}+\mathcal{N})^\perp\oplus(\mathcal{M}\cap\mathcal{N})\oplus(\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp)\oplus(\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp)$. If we define C to be the null-operator on $(\mathcal{M}+\mathcal{N})^\perp$, to coincide with A on $(\mathcal{M}\cap\mathcal{N})$ and $(\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp)$, and to coincide with B on $(\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp)$, then C is bounded as explained in Section 1.2, and C demonstrates that the pairs $(A,\mathcal{M})$ and $(B,\mathcal{N})$ are coherent.

Now suppose that $\mathcal{M}+\mathcal{N}$ is not closed and take $\mathcal{K}=\mathcal{H}$. From Theorem 1.2.8 we have that $(\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp)\oplus(\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp))$ is not closed. Let A be the identity on $\mathcal{M}$ and null-operator on $\mathcal{M}^\perp$, while B is the identity on $\mathcal{M}\cap\mathcal{N}$ and the null-operator on $\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp$ and $\mathcal{N}^\perp$. Then $A,B\in\mathcal{B}(\mathcal{H})$. If $C\in\mathcal{B}(\mathcal{H})$ is such an operator that coincides with A on $\mathcal{M}$ and with B on $\mathcal{N}$, then seen as an operator on $(\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp)\oplus(\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp))$, C is bounded projection onto $\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp$ along $\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp)$, which is not possible. Thus, the pairs $(A,\mathcal{M})$ and $(B,\mathcal{N})$ are not coherent. $\square$

As we can see, if operators $A,B\in\mathcal{B}(\mathcal{H})$ are such that $\overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)}$ is closed, then they are coherent if and only if they are precoherent. If we wish to find an example of precoherent operators A and B which are not coherent, we have to do so when $\overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)}$ is not closed, which is only possible if $\mathcal{H}$ is infinite-dimensional. Observe that operators A and B constructed in the proof of Proposition 2.1.3 can not be used now, since in the case $\mathcal{M}=\overline{\mathcal{R}(A^*)}$ and $\mathcal{N}=\overline{\mathcal{R}(B^*)}$ we have the additional condition that A and B must be injections on $\mathcal{M}$ and $\mathcal{N}$, respectively.

Example 8. Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of an infinite-dimensional Hilbert space $\mathcal{H}$ such that $\mathcal{M}+\mathcal{N}$ is not closed. We can assume that $\mathcal{M}\cap\mathcal{N}=\{0\}$, or else we can replace $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp$ and $\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp$, respectively. Let $A=2P_{\mathcal{M}}$ and $B=P_{\mathcal{N}}$. Suppose that A and B are coherent, and $C\in\mathcal{B}(\mathcal{H})$ coincides with A on $\overline{\mathcal{R}(A^*)}=\mathcal{R}(A)=\mathcal{M}$ and with B on $\overline{\mathcal{R}(B^*)}=\mathcal{R}(B)=\mathcal{N}$. On the normed space $\mathcal{M}\oplus\mathcal{N}$, C is equal to $I+P$, where P is the projection onto $\mathcal{M}$ along $\mathcal{N}$. If C is bounded, then so is $P=I-C$ which is not possible, and so A and B are not coherent. $\diamond$

If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are coherent, then every $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ coinciding with A and B on appropriate subspaces, due to continuity, is uniquely determined on $\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$. In other words, $C_0(A, B) = CP_{\overline{\overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)}}}$, is the same for every C, and has the matrix form:

$$C_0(A, B) = \begin{bmatrix} S(A, B) & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}} \\ \mathcal{N}(A) \cap \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} \\ \mathcal{N}(A^*) \cap \mathcal{N}(B^*) \end{bmatrix}. \tag{2.1}$$

Although A and B are injective on $\overline{\mathcal{R}(A^*)}$ and $\overline{\mathcal{R}(B^*)}$, respectively, we can not expect for operator $S(A, B)$ from (2.1) to be injective. Also, the coherence of A and B does not imply even the precoherence of A^* and B^*. This is all demonstrated by the following example.

Example 9. Let $\mathcal{H}$ and $\mathcal{K}$ be arbitrary (possibly finite-dimensional) Hilbert spaces. We define operators A and B on $\mathcal{H} \times \mathcal{H} \times \mathcal{K}$. Corresponding to this decomposition, let

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have $\overline{\mathcal{R}(A^*)} = \mathcal{R}(A^*) = \mathcal{N}(A)^\perp = \mathcal{H} \times \{0\} \times \{0\}$ and $\overline{\mathcal{R}(B^*)} = \mathcal{R}(B^*) = \mathcal{N}(B)^\perp = \{0\} \times \mathcal{H} \times \{0\}$. Operators A and B are coherent, and

$$C_0(A, B) = \begin{bmatrix} I & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can see that operator C_0 is not injective on $\mathcal{R}(A^*) \oplus \mathcal{R}(B^*)$.

Also, we have that $\mathcal{R}(A) = \mathcal{R}(B)$, while A^* and B^* are not the same operators, so they do not coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$. This means that A^* and B^* are not precoherent. $\diamond$

The following example shows that A and B can be coherent and operator $S(A, B)$ is injective on $\overline{R(A^*)} + \overline{\mathcal{R}(B^*)}$, while it is still not injective on the whole of its domain $\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$.

Example 10. Choose a Hilbert space $\mathcal{H}$ and its closed subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} + \mathcal{N} \neq \overline{\mathcal{M} + \mathcal{N}} = \mathcal{H}$. Let x be a vector not contained in $\mathcal{M} + \mathcal{N}$, and S be the orthogonal projection with the range $\operatorname{span}\{x\}^\perp$. In that case, S is injective on $\mathcal{M} + \mathcal{N}$, but not on its closure. Define operators A and B in such a way that they coincide with S on $\mathcal{M}$ and $\mathcal{N}$ respectively while they are the null-operators on $\mathcal{M}^\perp$ and $\mathcal{N}^\perp$, respectively. Then $\mathcal{M} = \mathcal{N}(A)^\perp = \overline{\mathcal{R}(A^*)}$, $\mathcal{N} = \mathcal{N}(B)^\perp = \overline{\mathcal{R}(B^*)}$ and so $S(A, B)$ is exactly the operator S, which is injective on $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$, but not on $\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$. $\diamond$

It is an interesting fact that A and B can be coherent, and in the same time, A^* and B^* can be coherent, while $S(A, B)$ and $S(A^*, B^*)$ are not adjoints of each other. The reason why such an example is important will be clear in Chapter 4.

Example 11. Let $\mathcal{M}$ and $\mathcal{N}$ be nontrivial closed subspaces of a Hilbert space $\mathcal{H}$ such that $\mathcal{M}\cap\mathcal{N}=\{0\}$, $\mathcal{M}+\mathcal{N}$ is closed, and $\mathcal{M}$ is not contained in $\mathcal{N}^{\perp}$ (e.g. such subspaces are easily constructed in $\mathbb{C}^2$). Let $A=2P_{\mathcal{M}}$ and $B=P_{\mathcal{N}}$. From $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}=\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$ and the fact that $\mathcal{M}+\mathcal{N}$ is closed, we have that $A=A^*$ and $B=B^*$ are coherent. Also $S(A,B)=S(A^*,B^*)$, but $S(A,B)$ is not self-adjoint, and so $S(A,B)^*\neq S(A^*,B^*)$. To see this, take $m\in\mathcal{M}\setminus\mathcal{N}^{\perp}$ and $n\in\mathcal{N}$ such that $\langle m,n\rangle\neq 0$. Then $2\langle m,n\rangle\neq\langle m,n\rangle$, but $2\langle m,n\rangle=\langle S(A,B)m,n\rangle$ and $\langle m,n\rangle=\langle m,S(A,B)n\rangle$. So $S(A,B)$ is not self-adjoint. $\diamond$

If operators A and B are from $\mathcal{B}^1(\mathcal{H})$, then it is also natural to ask for (pre)coherence of pairs $(A,\mathcal{R}(A))$ and $(B,\mathcal{R}(B))$. Such condition will be crucial in Sections 4.4 and 4.5. One could compare precoherence of $(A,\mathcal{R}(A))$ and $(B,\mathcal{R}(B))$ with $(A^\sharp,\mathcal{R}(A))$ and $(B^\sharp,\mathcal{R}(B))$, or $(A^{\textcircled{\#}},\mathcal{R}(A))$ and $(B^{\textcircled{\#}},\mathcal{R}(B))$. It is fairly obvious that if $\mathcal{R}(A)\cap\mathcal{R}(B)$ is finite-dimensional, then $(A,\mathcal{R}(A))$ and $(B,\mathcal{R}(B))$ are precoherent if and only if $(A^\sharp,\mathcal{R}(A))$ and $(B^\sharp,\mathcal{R}(B))$ are precoherent if and only if $(A^{\textcircled{\#}},\mathcal{R}(A))$ and $(B^{\textcircled{\#}},\mathcal{R}(B))$ are precoherent (this can be seen from Lemma 2.2.10, statement 3.). In general, this equivalence is false.

Example 12. Let us show that if $\mathcal{R}(A)\cap\mathcal{R}(B)$ is not finite-dimensional, pairs $(A,\mathcal{R}(A))$ and $(B,\mathcal{R}(B))$ can be precoherent, while pairs $(A^\sharp,\mathcal{R}(A))$ and $(B^\sharp,\mathcal{R}(B))$ are not. Let $\mathcal{H}_1$ be an infinite-dimensional separable Hilbert space with an orthonormal basis $\{e_1,e_2,e_3,...\}$ and let $\mathcal{H}=\mathcal{H}_1\times\mathcal{H}_1\times\mathcal{H}_1$. If $A,B:\mathcal{H}\to\mathcal{H}$ are maps defined in the following manner:

$$A\ :\ (\sum_{i=1}^{\infty}x_ie_i,\sum_{i=1}^{\infty}y_ie_i,\sum_{i=1}^{\infty}z_ie_i)\ \mapsto\ (\sum_{i=1}^{\infty}x_{i+1}e_i,x_1e_1+\sum_{i=2}^{\infty}y_{i-1}e_i,0),$$

$$B\ :\ (\sum_{i=1}^{\infty}x_ie_i,\sum_{i=1}^{\infty}y_ie_i,\sum_{i=1}^{\infty}z_ie_i)\ \mapsto\ (0,z_1e_1+\sum_{i=2}^{\infty}y_{i-1}e_i,\sum_{i=1}^{\infty}z_{i+1}e_i),$$

it is not difficult to see that $A,B\in\mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(A)=\mathcal{H}_1\times\mathcal{H}_1\times\{0\}$ and $\mathcal{R}(B)=\{0\}\times\mathcal{H}_1\times\mathcal{H}_1$. Moreover, A and B coincide on $\mathcal{R}(A)\cap\mathcal{R}(B)$. Furthermore, we have:

$$A^{\textcircled{\#}}=A^\sharp\ :\ (\sum_{i=1}^{\infty}x_ie_i,\sum_{i=1}^{\infty}y_ie_i,\sum_{i=1}^{\infty}z_ie_i)\ \mapsto\ (y_1e_1+\sum_{i=2}^{\infty}x_{i-1}e_i,\sum_{i=1}^{\infty}y_{i+1}e_i,0),$$

$$B^{\textcircled{\#}}=B^\sharp\ :\ (\sum_{i=1}^{\infty}x_ie_i,\sum_{i=1}^{\infty}y_ie_i,\sum_{i=1}^{\infty}z_ie_i)\ \mapsto\ (0,\sum_{i=1}^{\infty}y_{i+1}e_i,y_1e_1+\sum_{i=2}^{\infty}z_{i-1}e_i).$$

Thus, pairs $(A^\sharp,\mathcal{R}(A))$ and $(B^\sharp,\mathcal{R}(B))$ are not precoherent, given that $A^{\textcircled{\#}}(0,e_1,0)\neq B^{\textcircled{\#}}(0,e_1,0)$. $\diamond$

The definition of coherence of two pairs can be naturally extended to coherence of arbitrary family of pairs which will be used in one point.

Definition 2.1.4. Let I be an arbitrary nonempty set, and for every $i \in I$ let $\mathcal{M}_i \subseteq \mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{H}$, and $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The pairs $\{(A_i, \mathcal{M}_i) \ : \ i \in I\}$ are coherent if there exists $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that A_i and C coincide on $\mathcal{M}_i$ for every $i \in I$. The family of operators $\{A_i \ : \ i \in I\}$ is coherent if pairs $\{(A_i, \overline{\mathcal{R}(A_i^*)}) \ : \ i \in I\}$ are coherent.

The coherence of a family of pairs can also be stated in terms of the solvability of a certain system of equations. The following example shows that the coherence of two by two pairs can not replace a kind of coherence described in Definition 2.1.4.

Example 13. Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of an infinite-dimensional Hilbert space $\mathcal{H}$ such that $\mathcal{M} + \mathcal{N}$ is not closed. Then $A = 2P_{\mathcal{M}}$ and $B = P_{\mathcal{N}}$ are not coherent (Example 8). Let $\{e_i \mid i \in I\}$ be an algebraic base of $\mathcal{N}$, and P_i the orthogonal projection onto the one-dimensional space spanned by e_i. Since $\mathcal{R}(P_i) \cap \mathcal{R}(A) = \{0\}$ and $\mathcal{R}(P_i) + \mathcal{R}(A)$ is closed (Lemma 1.2.3), P_i and A are coherent (Proposition 2.1.3). Of course, any P_i and P_j are coherent, so the set $\{A\} \cup \{P_i \mid i \in I\}$ has the property of two-by-two coherence. Now assume that there is some X such that $AA^* = XA^*$ and $P_iP_i^* = XP_i^*$ for every $i \in I$. Let us show that such X must also fulfill $BB^* = XB^*$, which yields a contradiction. Take any $x \in \mathcal{R}(B^*) = \mathcal{R}(B) = \mathcal{N}$. Then there are $\alpha_{i_1}, ..., \alpha_{i_k}$ such that $x = \alpha_{i_1}e_{i_1} + ... + \alpha_{i_k}e_{i_k} = \alpha_{i_1}P_{i_1}e_{i_1} + ... + \alpha_{i_k}P_{i_k}e_{i_k}$. Then $Xx = X(\alpha_{i_1}P_{i_1}e_{i_1} + ... + \alpha_{i_k}P_{i_k}e_{i_k}) = \alpha_{i_1}P_{i_1}e_{i_1} + ... + \alpha_{i_k}P_{i_k}e_{i_k} = x = Bx$. Thus $BB^* = XB^*$, which is a contradiction. $\diamond$

In the end of this section, we note that the relation of (pre)coherence of two pairs is obviously reflexive, symmetric, but it is not transitive.

Example 14. Let $\mathcal{H} = \mathbb{C}^3$, and choose

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we have $A = P_{\mathcal{R}(A^*)}$, $B = P_{\mathcal{R}(B^*)}$ and $P_{\mathcal{R}(C^*)}$ equals to a matrix that has 1 at $(2,2)$-entry and $(3,3)$-entry, and 0 elsewhere. So

$$P_{\mathcal{R}(A^*) \cap \mathcal{R}(B^*)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_{\mathcal{R}(B^*) \cap \mathcal{R}(C^*)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{\mathcal{R}(A^*) \cap \mathcal{R}(C^*)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that A and B are (pre)coherent, as well as B and C, but A and C are not. $\diamond$

2.2 Properties of coherent and precoherent operators

It is particularly convenient to work with operators A, B such that A and B are precoherent and in the same time A^* and B^* are precoherent, as shown by the following statements. We always denote by $\mathcal{H}$ and $\mathcal{K}$ arbitrary Hilbert spaces.

Lemma 2.2.1. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that A^* and B^* coincide on a set $S \subseteq \mathcal{K}$, then for every $x \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, it holds $Ax \in \mathcal{R}(A) \cap S^{\perp}$ and $Bx \in \mathcal{R}(B) \cap S^{\perp}$.*

Proof. If $x \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ and $s \in S$ is an arbitrary element, then: $\langle Ax, s\rangle = \langle x, A^*s\rangle = 0$, since $A^*s = B^*s \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$. The same is true for B. □

The following proposition in the case of operators with closed ranges appeared in [27], with a comment for non-closed range operators.

Proposition 2.2.2. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are operators such that A and B are precoherent, and A^* and B^* are precoherent, then:*

1. *If $x \in \mathcal{H}$ is such that $x \perp \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$, then $Ax \in \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$;*
2. *$A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $A(\overline{\mathcal{R}(A^*)} \ominus (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})) = \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$;*
3. *If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, then $A^\dagger(\mathcal{R}(A) \cap \mathcal{R}(B)) = \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ and $A^\dagger(\mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))) = \mathcal{R}(A^*) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$.*

Proof. 1. Follows from Lemma 2.2.1.
2. Since A and B are precoherent, we have that $A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$ and from statement 1. also $A(\overline{\mathcal{R}(A^*)} \ominus (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})) \subseteq \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. Now take arbitrary $y \in \mathcal{R}(A) \cap \mathcal{R}(B)$. There is $x_0 \in \overline{\mathcal{R}(A^*)}$ such that $Ax_0 = y$. We can write $x_0 = x_1 + x_2$, where $x_1 \in \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$ and $x_2 \in \overline{\mathcal{R}(A^*)} \ominus (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})$. Then $y = Ax_0 = Ax_1 + Ax_2$, where $Ax_1 \in \mathcal{R}(A) \cap \mathcal{R}(B)$, while $Ax_2 \in \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. Thus $Ax_2 = 0$, and so $x_2 = 0$, which proves that $x_0 \in \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$. Thus $\mathcal{R}(A) \cap \mathcal{R}(B) = A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})$. The other equality is proved similarly.
3. Follows directly from statement 2. and the fact that $A^\dagger$ is a usual inverse of A reduced on $\mathcal{R}(A^*)$ and $\mathcal{R}(A)$. □

Recall that for bounded operators A and B the sets $\overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ and $\overline{R(A)} \cap \overline{\mathcal{R}(B)}$ are not the same (Example 3). However, if A and B are precoherent and A^* and B^* are precoherent, then they are the same, and this will be very important in a few occasions.

Lemma 2.2.3. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that A and B are precoherent, and A^* and B^* are precoherent, then $\overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$.*

Proof. Let T be the reduction of A on $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$, and S be the reduction of A^* on $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. Then, $T \in \mathcal{B}(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}, \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})$, $S \in \mathcal{B}(\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}, \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})$ and $T^* = S$. Operator S is injective and so T has a dense range. Thus: $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(T)} \subseteq \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$, which proves the wanted equality. □

Our main concern in Section 4.3 will be, stated in the most concise possible terms, the solvability of the following system of equations:

$$AA^* = XA^*, \;\; A^*A = A^*X, \;\; BB^* = XB^*, \;\; B^*B = B^*X. \tag{2.2}$$

If such a system has a solution, then for A and B we would have $AA^*B = AX^*B = AB^*B$ and $BA^*A = BX^*A = BB^*A$. Let us consider these equalities more carefully.

Lemma 2.2.4. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $AA^*B = AB^*B$ and $BA^*A = BB^*A$. Then A and B are precoherent, and A^* and B^* are precoherent.*

Proof. From $BA^*A = BB^*A$ we get $A^*(A-B)B^* = 0$. This means that $(A-B)(\mathcal{R}(B^*)) \subseteq \mathcal{N}(A^*)$, and so $(A-B)(\overline{\mathcal{R}(B^*)}) \subseteq \mathcal{N}(A^*)$. Analogously, $(A-B)(\overline{\mathcal{R}(A^*)}) \subseteq \mathcal{N}(B^*)$. Thus $(A-B)(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) \subseteq \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$. On the other hand, $\mathcal{R}(A-B) \subseteq \overline{\mathcal{R}(A) + \mathcal{R}(B)} \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*))^{\perp}$. So, $(A-B)(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) = \{0\}$, and A and B coincide on $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$. The same is true for A^* and B^* since the equalities $AA^*B = AB^*B$ and $BA^*A = BB^*A$ after taking adjoints, give exactly the same equalities for B^* and A^*. $\square$

From the previous lemma and Proposition 2.1.3 it is clear that, when $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ is closed, then $AA^*B = AB^*B$ and $BA^*A = BB^*A$ imply the coherence of A and B. In the following theorem, we show that, in general, under these equalities there is certainly a closed densely defined operator which coincides with A and B on desired subspaces. We also characterize the equalities $AA^*B = AB^*B$ and $BA^*A = BB^*A$.

Theorem 2.2.5. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The following statements are equivalent:*

(i) $AA^*B = AB^*B$ *and* $BA^*A = BB^*A$*;*

(ii) There exist linear transformations $S : \overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)} \to \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ *and* $T : \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)} \to \overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ *such that:*

a) S *coincides with* A *and* B *on* $\overline{\mathcal{R}(A^*)}$ *and* $\overline{\mathcal{R}(B^*)}$ *respectively;*

b) T *coincides with* A^* *and* B^* *on* $\overline{\mathcal{R}(A)}$ *and* $\overline{\mathcal{R}(B)}$ *respectively;*

c) For every $a \in \overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ *and* $b \in \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ *we have* $\langle Sa, b\rangle = \langle a, Tb\rangle$.

In this case, transformations S and T are injective.

Proof. (i) $\Rightarrow$ (ii): From Lemma 2.2.4 we get that A and B are precoherent, and A^* and B^* are precoherent. Thus, the linear transformations S and T with properties a) and b) exist. If $y \in \mathcal{R}(B^*)$ and $\alpha \in \mathcal{R}(A)$, then $y = B^*y'$ and $\alpha = A\alpha'$ and so

$$\langle y, A^*\alpha\rangle = \langle y', BA^*A\alpha'\rangle = \langle y', BB^*A\alpha'\rangle = \langle BB^*y', A\alpha'\rangle = \langle By, \alpha\rangle.$$

From continuity of B, A^* and the scalar product we obtain that $\langle y, A^*\alpha\rangle = \langle By, \alpha\rangle$ for every $y \in \overline{\mathcal{R}(B^*)}$ and every $\alpha \in \overline{\mathcal{R}(A)}$. In the same way we obtain that for every $x \in \overline{\mathcal{R}(A^*)}$ and every $\beta \in \overline{\mathcal{R}(B)}$ it holds $\langle Ax, \beta\rangle = \langle x, B^*\beta\rangle$. Finally, if we take arbitrary $x \in \overline{\mathcal{R}(A^*)}, y \in \overline{\mathcal{R}(B^*)}, \alpha \in \overline{\mathcal{R}(A)}, \beta \in \overline{\mathcal{R}(B)}$, then

$$\langle S(x+y), \alpha+\beta\rangle = \langle Ax + By, \alpha+\beta\rangle = \langle Ax, \alpha\rangle + \langle By, \beta\rangle + \langle Ax, \beta\rangle + \langle By, \alpha\rangle =$$

$$= \langle x, A^*\alpha\rangle + \langle y, B^*\beta\rangle + \langle x, B^*\beta\rangle + \langle y, A^*\beta\rangle = \langle x+y, A^*\alpha + B^*\beta\rangle = \langle x+y, T(\alpha+\beta)\rangle,$$

proving that property c) is satisfied as well.

(ii) $\Rightarrow$ (i): From (ii) it follows that for every $y \in \mathcal{R}(B^*)$ and every $\alpha \in \mathcal{R}(A)$ we have $\langle By, \alpha\rangle = \langle y, A^*\alpha\rangle$, i.e. $\langle y, (B^* - A^*)\alpha\rangle = 0$. And so for every $u, v \in \mathcal{H}$, denoting

$y = B^*u$ and $\alpha = Av$ we have that $\langle u, B(B^* - A^*)Av\rangle = 0$. Hence $B(B^* - A^*)A = 0$. The other equality is proved in the same fashion.

We will now prove that S is injective, and the injectivity of T follows by symmetry. Suppose that $Ax = By$, for $x \in \overline{\mathcal{R}(A^*)}$ and $y \in \overline{\mathcal{R}(B^*)}$. Then there exists a sequence of vectors $(\alpha_n) \subseteq \mathcal{H}$ such that $A^*\alpha_n \to x$. We have that $B^*Bx = \lim B^*BA^*\alpha_n = \lim B^*AA^*\alpha_n = B^*Ax = B^*By$. So $x - y \in \mathcal{N}(B)$. In the same way we get that $x-y \in \mathcal{N}(A)$, and so $x-y \in \mathcal{N}(A)\cap\mathcal{N}(B)$, but at the same time $x-y \in \overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)} \subseteq (\mathcal{N}(A) \cap \mathcal{N}(B))^{\perp}$. Hence $x = y \in \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$. Finally, if $S(a + b) = 0$ for some $a \in \overline{\mathcal{R}(A^*)}$ and $b \in \overline{\mathcal{R}(B^*)}$, then $Aa = B(-b)$ which shows that $a = -b$, i.e. $a + b = 0$. Hence S is injective. □

Corollary 2.2.6. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $AA^*B = AB^*B$ and $BA^*A = BB^*A$ if and only if there exists a closed densely defined operator $S : D(S) \to \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$, $\overline{D(S)} = \overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$ such that: its domain contains $\overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)}$, S coincide with A and B on $\overline{\mathcal{R}(A^*)}$ and $\overline{\mathcal{R}(B^*)}$ respectively, while the domain of S^* contains $\overline{\mathcal{R}(A)}+\overline{\mathcal{R}(B)}$ and S^* coincides with A^* and B^* on $\overline{\mathcal{R}(A)}$ and $\overline{\mathcal{R}(B)}$ respectively.*

Proof. Suppose that $AA^*B = AB^*B$ and $BA^*A = BB^*A$, and let S_1 and T_1 denote the linear transformations defined in Theorem 2.2.5 (within Theorem 2.2.5 , they were denoted by S and T, respectively). The transformation S_1 is densely defined in $\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$ and from Theorem 2.2.5 we see that $T_1 \subseteq S_1^*$. Thus S_1^* is densely defined in $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$, which means, by Theorem 1.1.1, that S_1 is closable. If $S = \overline{S_1}$ then $T_1 \subseteq S_1^* = \overline{S_1}^* = S^*$, so S fulfils all the required conditions.

The other implication of the statement follows directly from Theorem 2.2.5. □

Corollary 2.2.7. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $AA^*B = AB^*B$ and $BA^*A = BB^*A$. Operators A and B are coherent if and only if operators A^* and B^* are coherent. In this case, $S(A, B)^* = S(A^*, B^*)$, where $S(A, B)$ is defined as in* (2.1).

Proof. Let: A and B be coherent, operator $S(A, B)$ be defined as in (2.1), and linear transformations S and T be defined as in Theorem 2.2.5. In that case, $S(A, B)$ is an extension of S on $\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$ and due to continuity of $S(A, B)$, we can easily deduce that the adjoint of densely defined linear transforation T is exactly $S(A, B)$. Since its domain is whole space $\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$, from Theorem 1.1.1 we deduce that T is bounded on $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ so we can extend it to continuity on $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$ showing that A^* and B^* are also coherent. It is also clear that $S(A, B)^* = S(A^*, B^*)$. □

Corollary 2.2.8. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $AA^*B = AB^*B$ and $BA^*A = BB^*A$. If $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ or $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ is closed, then A and B, as well as A^* and B^* are coherent.*

Proof. Since $AA^*B = AB^*B$ and $BA^*A = BB^*A$, from Lemma 2.2.4 we have that A and B are precoherent, and A^* and B^* are precoherent. If $\overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)}$ is closed, from Proposition 2.1.3 it follows that A and B are coherent, so by Corollary 2.2.7, A^* and B^* are also coherent. If $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ is closed, then we first conclude that A^* and B^* are coherent, and then again by Corollary 2.2.7, that A and B are coherent as well. □

Note that the equality $BA^*A = BB^*A$ can be written in a few equivalent forms. For example:

$$BA^*A = BB^*A \quad \Leftrightarrow \quad P_{\overline{\mathcal{R}(B^*)}}(A^* - B^*)P_{\overline{\mathcal{R}(A)}} = 0 \quad \Leftrightarrow \quad AP_{\overline{\mathcal{R}(B^*)}} = P_{\overline{\mathcal{R}(A)}}B. \quad (2.3)$$

If A and B are closed range operators, then it is equivalent to $AA^\dagger B = AB^\dagger B$, etc. The same holds for $AA^*B = AB^*B$. We will use the form convenient for the problem in question.

In one part of Section 4.4 we will be concerned with the following system of equations:

$$AA^{\textcircled{\ddagger}} = XA^{\textcircled{\ddagger}},\ A^{\textcircled{\ddagger}}A = A^{\textcircled{\ddagger}}X,\ BB^{\textcircled{\ddagger}} = XB^{\textcircled{\ddagger}},\ B^{\textcircled{\ddagger}}B = B^{\textcircled{\ddagger}}X,$$

for $A, B \in \mathcal{B}^1(\mathcal{H})$. Similarly like before, if such a system has a solution, then for A and B we would have $A^{\textcircled{\ddagger}}BB^{\textcircled{\ddagger}} = A^{\textcircled{\ddagger}}AB^{\textcircled{\ddagger}}$ and $B^{\textcircled{\ddagger}}AA^{\textcircled{\ddagger}} = B^{\textcircled{\ddagger}}BA^{\textcircled{\ddagger}}$. We give some results regarding these equalities and (pre)coherence which are going to be useful later. T statement shows that we can write these equalities without generalized inverses.

Lemma 2.2.9. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$. The following statements are equivalent:*

(i) $A^{\textcircled{\ddagger}}BB^{\textcircled{\ddagger}} = A^{\textcircled{\ddagger}}AB^{\textcircled{\ddagger}}$;

(ii) $AA^{\textcircled{\ddagger}}B = AB^{\textcircled{\ddagger}}B$;

(iii) $A^*AB = A^*BB$.

Proof. The assertion follows directly from the convenient multiplications from the left and right, and the fact that (i) is equivalent to $(A - B)\mathcal{R}(B) \subseteq \mathcal{N}(A^*)$. ☐

Lemma 2.2.10. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$ be such that $A^{\textcircled{\ddagger}}BB^{\textcircled{\ddagger}} = A^{\textcircled{\ddagger}}AB^{\textcircled{\ddagger}}$ and $B^{\textcircled{\ddagger}}AA^{\textcircled{\ddagger}} = B^{\textcircled{\ddagger}}BA^{\textcircled{\ddagger}}$. Then:*

1. *$(A, \mathcal{R}(A))$ and $(B, \mathcal{R}(B))$ are precoherent;*

2. *if $(A^{\textcircled{\ddagger}}, \mathcal{R}(A))$ and $(B^{\textcircled{\ddagger}}, \mathcal{R}(B))$ are precoherent, then $(\mathcal{R}(A) + \mathcal{R}(B)) \cap [\mathcal{N}(A) \cap \mathcal{N}(B)] = \{0\}$;*

3. *if $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional, then $(A^{\textcircled{\ddagger}}, \mathcal{R}(A))$ and $(B^{\textcircled{\ddagger}}, \mathcal{R}(B))$ are precoherent.*

Proof. 1. From $A^{\textcircled{\ddagger}}BB^{\textcircled{\ddagger}} = A^{\textcircled{\ddagger}}AB^{\textcircled{\ddagger}}$ we get that $(A - B)(\mathcal{R}(B)) \subseteq \mathcal{N}(A^*)$ (see Lemma 2.2.9). Similarly, from $B^{\textcircled{\ddagger}}AA^{\textcircled{\ddagger}} = B^{\textcircled{\ddagger}}BA^{\textcircled{\ddagger}}$ we get $(A-B)(\mathcal{R}(A)) \subseteq \mathcal{N}(B^*)$. Hence $(A-B)(\mathcal{R}(A) \cap \mathcal{R}(B)) \subseteq \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$, but on the other hand, $(A-B)(\mathcal{R}(A) \cap \mathcal{R}(B)) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$. So $(A - B)(\mathcal{R}(A) \cap \mathcal{R}(B)) = \{0\}$, i.e. A and B coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$.

2. Let $n \in (\mathcal{R}(A) + \mathcal{R}(B)) \cap [\mathcal{N}(A) \cap \mathcal{N}(B)]$ be arbitrary. There are $r_A \in \mathcal{R}(A)$ and $r_B \in \mathcal{R}(B)$ such that $n = r_A - r_B$. Since $n \in \mathcal{N}(A) \cap \mathcal{N}(B)$ we get that $Ar_A = Ar_B$ and $Br_A = Br_B$. Let us prove that $Ar_A = Br_B$. From $A^{\textcircled{\ddagger}}BB^{\textcircled{\ddagger}} = A^{\textcircled{\ddagger}}AB^{\textcircled{\ddagger}}$, in the same way as before, we get that $(A - B)r_B \in \mathcal{N}(A^*)$. Since $Ar_B = Ar_A$ this

means that $Ar_A - Br_B \in \mathcal{N}(A^*)$. Similarly we have that $Ar_A - Br_B \in \mathcal{N}(B^*)$, and so $Ar_A - Br_B \in \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$. On the other hand $Ar_A - Br_B \in \mathcal{R}(A) + \mathcal{R}(B)$, and so $Ar_A - Br_B = 0$, i.e. $Ar_A = Br_B \in \mathcal{R}(A) \cap \mathcal{R}(B)$. Since $A^{\textcircled{\dagger}}$ and $B^{\textcircled{\dagger}}$ coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$: $r_A = A^{\textcircled{\dagger}} Ar_A = B^{\textcircled{\dagger}} Br_B = r_B$, and $n = 0$. This completes the proof.

3. From 1. we have that A and B map $\mathcal{R}(A)\cap\mathcal{R}(B)$ into itself and they are injective on this space, so if $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional, they are also bijective. Thus, for every $y \in \mathcal{R}(A) \cap \mathcal{R}(B)$ there is $x \in \mathcal{R}(A) \cap \mathcal{R}(B)$ such that $Ax = Bx = y$. This means that $A^{\textcircled{\dagger}} y$, as well as, $B^{\textcircled{\dagger}} y$ are exactly equal to x. Thus $A^{\textcircled{\dagger}}$ and $B^{\textcircled{\dagger}}$ coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$. □

Lemma 2.2.11. *If $A, B \in \mathcal{B}^1(\mathcal{H})$ are such that $A^{\textcircled{\dagger}} BB^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} AB^{\textcircled{\dagger}}$ and $B^{\textcircled{\dagger}} AA^{\textcircled{\dagger}} = B^{\textcircled{\dagger}} BA^{\textcircled{\dagger}}$, and if $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, then $(A, \mathcal{R}(A))$ and $(B, \mathcal{R}(B))$ are coherent pairs. Furthermore, if $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional, then $(A^{\textcircled{\dagger}}, \mathcal{R}(A))$ and $(B^{\textcircled{\dagger}}, \mathcal{R}(B))$ are also coherent pairs.*

Proof. From Lemma 2.2.10 we have that $(A, \mathcal{R}(A))$ and $(B, \mathcal{R}(B))$ are precoherent pairs. Since $\mathcal{R}(A)+\mathcal{R}(B)$ is closed, from Proposition 2.1.3 we get that $(A, \mathcal{R}(A))$ and $(B, \mathcal{R}(B))$ are coherent. If $\mathcal{R}(A)\cap\mathcal{R}(B)$ is finite-dimensional, again from Lemma 2.2.10 and Proposition 2.1.3 we get that $(A^{\textcircled{\dagger}}, \mathcal{R}(A))$ and $(B^{\textcircled{\dagger}}, \mathcal{R}(B))$ are coherent. □

2.3 Range additivity of precoherent operators

In this section we study, as the title suggests, range additivity properties of precoherent operators. We will see that any two precoherent operators satisfy range additivity condition: $\overline{\mathcal{R}(A + B)} = \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$, so in a sense, the range $\mathcal{R}(A + B)$ can not be significantly smaller than the ranges $\mathcal{R}(A)$ and $\mathcal{R}(B)$. However equalities $\mathcal{R}(A+B) = \mathcal{R}(A)+\mathcal{R}(B)$, or $\overline{\mathcal{R}(A + B)} = \overline{\mathcal{R}(A)}+\overline{\mathcal{R}(B)}$ do not hold in general (see Example 6). One more time we recall that $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$ is the same as $\overline{\mathcal{R}(A) + \mathcal{R}(B)}$, but we prefer to write $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$, to highlight the fact that the ranges are not necessarily closed.

We begin with few lemmas concerning arbitrary operators.

Lemma 2.3.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a Hilbert space $\mathcal{H}$, such that $\mathcal{M}$ is closed, and $\mathcal{M}^\perp \subseteq \mathcal{N}$. Then $\mathcal{M} \cap \overline{\mathcal{N}} = \overline{\mathcal{M} \cap \mathcal{N}}$.*

Proof. It is clear that $\overline{\mathcal{M} \cap \mathcal{N}} \subseteq \mathcal{M} \cap \overline{\mathcal{N}}$. Now take arbitrary $x \in \mathcal{M} \cap \overline{\mathcal{N}}$. We have a sequence $(x_n) \subseteq \mathcal{N}$ such that $x_n \to x$. Then $P_\mathcal{M} x_n \to P_\mathcal{M} x = x$. Given that: $P_\mathcal{M} x_n = -(I - P_\mathcal{M})x_n + x_n$, and $\mathcal{R}(I - P_\mathcal{M}) = \mathcal{M}^\perp \subseteq \mathcal{N}$, we have that $(P_\mathcal{M} x_n) \subseteq \mathcal{N}$, but in the same time $(P_\mathcal{M} x_n) \subseteq \mathcal{M}$. Thus, we found a sequence from $\mathcal{M} \cap \mathcal{N}$ that converges to x, i.e. $x \in \overline{\mathcal{M} \cap \mathcal{N}}$. So we have $\mathcal{M} \cap \overline{\mathcal{N}} \subseteq \overline{\mathcal{M} \cap \mathcal{N}}$, which concludes the proof. □

Lemma 2.3.2. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $A(\mathcal{N}(A) + \mathcal{N}(B)) \subseteq \mathcal{R}(A + B)$.*

Proof. If we take $x \in \mathcal{N}(A)+\mathcal{N}(B)$, then $x = n_A + n_B$, where $n_A \in \mathcal{N}(A)$ and $n_B \in \mathcal{N}(B)$. We have that $Ax = An_B$, and so $(A+B)n_B = An_B = Ax$, thus $Ax \in \mathcal{R}(A+B)$, for every $x \in \mathcal{N}(A)+\mathcal{N}(B)$. □

Lemma 2.3.3. *If $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$, then*

$$A(\overline{\mathcal{R}(A^*)} \cap (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})^{\perp}) \subseteq \overline{\mathcal{R}(A+B)}.$$

Proof. First note that

$$\overline{\mathcal{R}(A^*)} \cap (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})^{\perp} = \overline{\mathcal{R}(A^*)} \ \cap \ \overline{\mathcal{N}(A)+\mathcal{N}(B)} = \overline{\overline{\mathcal{R}(A^*)} \cap (\mathcal{N}(A)+\mathcal{N}(B))},$$

according to Lemma 2.3.1. From Lemma 2.3.2 we have that $A(\overline{\mathcal{R}(A^*)} \cap (\mathcal{N}(A)+\mathcal{N}(B))) \subseteq \mathcal{R}(A+B)$, and so $A(\overline{\overline{\mathcal{R}(A^*)} \cap (\mathcal{N}(A)+\mathcal{N}(B))}) \subseteq \overline{\mathcal{R}(A+B)}$. □

Let us shed a bit more light on Lemma 2.3.3. The range of an operator A is equal to $A(\overline{\mathcal{R}(A^*)})$, while $\overline{\mathcal{R}(A^*)}$ is decomposed as an orthogonal sum of subspaces $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$ and $\overline{\mathcal{R}(A^*)} \ominus (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})$. Thus $\mathcal{R}(A) = A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) + A(\overline{\mathcal{R}(A^*)} \ominus (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}))$. Lemma 2.3.3 shows that the part of range of A: $A(\overline{\mathcal{R}(A^*)} \ominus (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}))$ is always contained in $\overline{\mathcal{R}(A+B)}$. The same goes for B. So, as far as we are concerned with inclusions $\mathcal{R}(A), \mathcal{R}(B) \subseteq \overline{\mathcal{R}(A+B)}$, it is only important how A and B act on $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$.

Lemma 2.3.4. *Let $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then:*

1. $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} = \overline{\mathcal{R}(A+B)}$ *if and only if* $A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) \subseteq \overline{\mathcal{R}(A+B)}$;
2. *If A and B are precoherent, then* $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} = \overline{\mathcal{R}(A+B)}$.
3. *If A and B are precoherent and $\mathcal{R}(A+B)$ is closed, then $\mathcal{R}(A)+\mathcal{R}(B) = \mathcal{R}(A+B)$. Consequently, $\mathcal{R}(A)+\mathcal{R}(B)$ is also closed.*

Proof. 1. If $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} = \overline{\mathcal{R}(A+B)}$ then obviously $\mathcal{R}(A) \subseteq \overline{\mathcal{R}(A+B)}$, so $A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) \subseteq \overline{\mathcal{R}(A+B)}$. If $A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) \subseteq \overline{\mathcal{R}(A+B)}$, from Lemma 2.3.3, we have $\mathcal{R}(A) \subseteq \overline{\mathcal{R}(A+B)}$. But then $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(A+B)}$, since for every $x \in \mathcal{H}$ we have $Bx = (A+B)x - Ax \in \overline{\mathcal{R}(A+B)}$. Thus $\mathcal{R}(A), \mathcal{R}(B) \subseteq \overline{\mathcal{R}(A+B)}$ giving $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} \subseteq \overline{\mathcal{R}(A+B)}$. The opposite inclusion is obvious, so $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} = \overline{\mathcal{R}(A+B)}$.

2. If A and B are precoherent, then for every $x \in \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$, $Ax = \frac{1}{2}(A+B)x \in \mathcal{R}(A+B)$, so $A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) \subseteq \mathcal{R}(A+B)$. Together with 1. this gives $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} = \overline{\mathcal{R}(A+B)}$.

3. If $\mathcal{R}(A+B)$ is closed, then from Lemma 2.3.3 and the fact that A and B are precoherent, it follows $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$, i.e. $\mathcal{R}(A)+\mathcal{R}(B) = \mathcal{R}(A+B)$. □

Statement 1. of the following theorem appeared in [27], but statements 2. and 3. originated from [39].

Theorem 2.3.5. *Let $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then:*

1. *If A and B coincide on $\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$, and A^* and B^* coincide on $\mathcal{R}(A)\cap\mathcal{R}(B)$, and $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$, then $\mathcal{N}(A)+\mathcal{N}(B)$ is a closed subspace.*

2. *If A and B are precoherent and $\mathcal{N}(A)+\mathcal{N}(B)$ is closed, then $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$.*

3. *If A and B are precoherent, and A^* and B^* are precoherent, then $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$ if and only if $\mathcal{N}(A)+\mathcal{N}(B)$ is closed.*

Proof. 1. Let $x\in(\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)})^\perp=\overline{\mathcal{N}(A)+\mathcal{N}(B)}$ be arbitrary. We will prove that $x\in\mathcal{N}(A)+\mathcal{N}(B)$, and so $\mathcal{N}(A)+\mathcal{N}(B)$ is closed. Write $x=a_1+a_2$, with $a_1\in\mathcal{N}(A)$ and $a_2\in\overline{\mathcal{R}(A^*)}$. We also have that $a_1\perp\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$, and so is $x-a_1=a_2\perp\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$. Now we have $(A+B)x=Aa_2+Ba_1+Ba_2$. From Lemma 2.2.1, we see that $Aa_2,Ba_1,Ba_2\perp\mathcal{R}(A)\cap\mathcal{R}(B)$. Given the equality $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$, we can write $Aa_2+Ba_1=(A+B)c$, for some $c\in\mathcal{H}$. Hence $(A+B)c\perp\mathcal{R}(A)\cap\mathcal{R}(B)$. If $c=t_1+t_2$, with $t_1\in\overline{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}$ and $t_2\in\overline{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}^\perp$, then $(A+B)t_1\in\mathcal{R}(A)\cap\mathcal{R}(B)$ (A and B also coincide on $\overline{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}$) and $(A+B)t_2\perp\mathcal{R}(A)\cap\mathcal{R}(B)$. So it must be $(A+B)t_1=0$, i.e. $2At_1=0$, and $t_1=0$, while $c=t_2\perp\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$. Then, it is also true that $Ac,Bc\perp\mathcal{R}(A)\cap\mathcal{R}(B)$.

Finally, going back to $(A+B)x=(A+B)c+Ba_2$, and rewriting it as $A(x-c)=B(c+a_2-x)$, we conclude that $A(x-c)=B(c+a_2-x)\in\mathcal{R}(A)\cap\mathcal{R}(B)$, but in the same time $A(x-c)=B(c+a_2-x)\perp\mathcal{R}(A)\cap\mathcal{R}(B)$. So $A(x-c)=0=B(c+a_2-x)$, from where it follows $x-c\in\mathcal{N}(A)$ and $c+a_2-x\in\mathcal{N}(B)$. Given that $a_2-x=-a_1\in\mathcal{N}(A)$, we see that $c\in\mathcal{N}(A)+\mathcal{N}(B)$, and so $x\in\mathcal{N}(A)+\mathcal{N}(B)$.

2. If $\mathcal{N}(A)+\mathcal{N}(B)$ is closed, then $\mathcal{H}=(\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)})\oplus(\mathcal{N}(A)+\mathcal{N}(B))$. From Lemma 2.3.2 and the fact that A and B are precoherent, we see that $\mathcal{R}(A)\subseteq\mathcal{R}(A+B)$. In other words, $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$.

3. Follows from 1. and 2. $\square$

If A and B are precoherent, and A^* and B^* are precoherent it is interesting that $\mathcal{R}(A+B)$ being closed forces $\mathcal{R}(A)$ and $\mathcal{R}(B)$ to be closed.

Theorem 2.3.6. *Let $A,B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ such that A and B are precoherent and A^* and B^* are precoherent. The following statements are equivalent:*

(i) $\mathcal{R}(A+B)$ *is closed;*

(ii) $\mathcal{R}(A)+\mathcal{R}(B)$ *is closed and* $\mathcal{R}(A^*)+\mathcal{R}(B^*)$ *is closed;*

(iii) $\mathcal{R}(A)+\mathcal{R}(B)$ *is closed and* $\overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)}$ *is closed.*

If any of conditions (i)-(iii) is satisfied, then $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, and $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{R}(A+B)$.

Proof. Since A and B are precoherent, and having in mind Lemma 2.3.3, the following inclusions hold:

$$\mathcal{R}(A+B)\subseteq\mathcal{R}(A)+\mathcal{R}(B)\subseteq\overline{\mathcal{R}(A)}+\overline{\mathcal{R}(B)}\subseteq\overline{\mathcal{R}(A+B)}. \tag{2.4}$$

(i) $\Rightarrow$ (ii) This follows from Lemma 2.3.4, since $\mathcal{R}(A^* + B^*)$ is also closed, and A^* and B^* are also precoherent.

(ii) $\Rightarrow$ (iii) If $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ is closed, from (2.4) written for A^* and B^* we obtain that $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is also closed.

(iii) $\Rightarrow$ (i) If $\mathcal{R}(A)+\mathcal{R}(B)$ is closed, from (2.4) we obtain $\overline{\mathcal{R}(A+B)} = \mathcal{R}(A)+\mathcal{R}(B) = \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$. In (iii) we also have the assumption that $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is closed, i.e. that $\mathcal{N}(A) + \mathcal{N}(B)$ is closed (Theorem 1.2.8). Together with Theorem 2.3.5 this gives gives $\mathcal{R}(A + B) = \mathcal{R}(A) + \mathcal{R}(B)$ so $\mathcal{R}(A + B)$ is closed.

Now suppose that conditions (i)-(iii) are satisfied. From (2.4) it follows $\mathcal{R}(A) + \mathcal{R}(B) = \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)} = \mathcal{R}(A + B)$. Now from Proposition 2.2.2 and Lemma 2.2.3 we have $A(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $A(\overline{\mathcal{R}(A^*)} \ominus (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})) = \mathcal{R}(A) \ominus (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})$ (we use Lemma 2.2.3 to replace $\mathcal{R}(A) \ominus (\overline{\mathcal{R}(A) \cap \mathcal{R}(B)})$ with $\mathcal{R}(A) \ominus (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})$). Hence $\mathcal{R}(A) = (\overline{\mathcal{R}(A) \cap \mathcal{R}(B)}) \oplus [\mathcal{R}(A) \ominus (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})]$. Similarly for $\mathcal{R}(B)$. But from $\mathcal{R}(A) + \mathcal{R}(B) = \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ we see that:

$$(\mathcal{R}(A) \cap \mathcal{R}(B)) \oplus \ [\mathcal{R}(A) \ominus (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})] \ \oplus \ [\mathcal{R}(B) \ominus (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})] =$$

$$(\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}) \ \oplus \ [\overline{\mathcal{R}(A)} \ominus (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})] \ \oplus \ [\overline{\mathcal{R}(B)} \ominus (\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})]. \qquad (2.5)$$

From this we conclude the equalities between appropriate subspaces, i.e. $\mathcal{R}(A) = \overline{\mathcal{R}(A)}$ and $\mathcal{R}(B) = \overline{\mathcal{R}(B)}$. □

Implicit in the proof of the previous theorem was the statements of the following lemma.

Lemma 2.3.7. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that A and B are precoherent and A^* and B^* are precoherent.*

1. *$\mathcal{R}(A) + \mathcal{R}(B) = \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ if and only if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed;*
2. *If $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, then $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed.*

Proof. 1. If $\mathcal{R}(A) + \mathcal{R}(B) = \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$, the same argument as in (2.5) shows that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed.

2. If $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, then (2.4) shows that $\mathcal{R}(A) + \mathcal{R}(B) = \overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$, so $\mathcal{R}(A)$ and $\mathcal{R}(B)$ have to be closed. □

To demonstrate the extent of Theorem 2.3.6 and Lemma 2.3.7 we give following examples.

Example 15. Let us show that condition:

$\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ is closed and $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ is closed;

is not equivalent to conditions (i)-(iii) of Theorem 2.3.6. It is enough to take $A = B$ be an operator with non-closed range (obviously, A and B are precoherent). On the other hand, this condition for precoherent operators implies range additivity as shown in Theorem 2.3.5. ◇

Example 16. Condition (ii) in Theorem 2.3.6 can not be changed to:

$\mathcal{R}(A) + \mathcal{R}(B)$ is closed.

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces of $\mathcal{H}$ such that $\mathcal{M} \oplus \mathcal{N}$ is not closed, while $\mathcal{K}$ and $\mathcal{L}$ are two closed subspaces such that $\mathcal{K} \oplus \mathcal{L}$ is closed. Let A be defined as an isomorphism between $\mathcal{M}$ and $\mathcal{K}$ on $\mathcal{M}$, and as the null-operator on $\mathcal{M}^\perp$, while B is an isomorphism between $\mathcal{N}$ and $\mathcal{L}$ on $\mathcal{N}$, and the null-operator on $\mathcal{N}^\perp$. In that case $\mathcal{R}(A) = \mathcal{L}$, $\mathcal{R}(B) = \mathcal{K}$, $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp = \mathcal{M}$, $\mathcal{R}(B^*) = \mathcal{N}(B)^\perp = \mathcal{N}$. Operators A and B, as well as A^* and B^* are precoherent, $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, while $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ is not. Range $\mathcal{R}(A+B)$ is also not closed, and $\mathcal{R}(A) + \mathcal{R}(B) \neq \mathcal{R}(A+B)$.

However, as Lemma 2.3.7 shows, if $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, we are certain that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are also closed. $\diamond$

Example 17. Statement 1 in Lemma 2.3.7 is not true in general. Let $A \in \mathcal{B}(\mathcal{H})$ be an operator with a non-closed range $\mathcal{R}(A)$, $x \in \mathcal{R}(A)$, and $B = P_{\{x\}^\perp}$. Then $\mathcal{H} = \mathcal{R}(A) + \mathcal{R}(B)$ and so $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)} = \mathcal{H}$, but $\mathcal{R}(A)$ is non-closed. $\diamond$

2.4 A special case: CoR operators

It is well-known that interesting properties of a real or complex square matrix A can be described through certain geometric relations between its column space and the column space of its adjoint matrix A^*. For example, the column spaces $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ coincide if and only if the matrix A commutes with its Moore-Penrose generalized inverse $A^\dagger$. Such matrices are known as EP matrices, and they were the subject of many research papers (see also [15, Chapter 4]). Quite opposite, if $\mathcal{R}(A) \oplus \mathcal{R}(A^*)$ is equal to whole space, then and only then $AA^\dagger - A^\dagger A$ is nonsingular. Such matrices are called co-EP matrices, and they were introduced and studied by Benítez and Rakočević [16]. Werner [82], and later Fill and Fishkind [37] and Groß [44], studied the pairs of matrices A and B with conveniently positioned column spaces: $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and $\mathcal{R}(A^*) \cap \mathcal{R}(B^*) = \{0\}$, which is known to be equivalent to rank additivity condition (1.5). It turns out that such matrices are particulary useful with joint systems of equations $Ax = a$, $Bx = b$, and we discuss such results in Chapter 3.

As a generalization of a class of co-EP matrices, Baksalary and Trenkler [14] introduced a new class of matrices which merits its own name: *disjoint range* matrices. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a disjoint range (or DR) matrix if $\mathcal{R}(A) \cap \mathcal{R}(A^*) = \{0\}$. They proved many properties of such matrices, of their functions and appropriate Moore-Penrose inverses. However, their study was based on linear algebra techniques, which are not appropriate for infinite-dimensional Hilbert spaces. The study of DR matrices, i.e. operators on arbitrary Hilbert spaces was conducted by Deng et al. [23]. Among others, the authors in [23] studied the classes of operators described in the following definition.

Definition 2.4.1. Let $\mathcal{H}$ be a Hilbert space, and T a closed range operator on $\mathcal{H}$. Then T is:

1) DR if $\mathcal{R}(T) \cap \mathcal{R}(T^*) = \{0\}$;

2) EP if $\mathcal{R}(T) = \mathcal{R}(T^*)$;

3) SR if $\mathcal{R}(T) + \mathcal{R}(T^*) = \mathcal{H}$;

4) co-EP if $\mathcal{R}(T) \oplus \mathcal{R}(T^*) = \mathcal{H}$;

5) weak-EP if $P_{\overline{\mathcal{R}(T)}} P_{\overline{\mathcal{R}(T^*)}} = P_{\overline{\mathcal{R}(T^*)}} P_{\overline{\mathcal{R}(T)}}$.

However, one very important class of operators is not fully contained in the union of the classes from Definition 2.4.1. Namely, if P and Q are two orthogonal projections on a Hilbert space $\mathcal{H}$, the operator PQ need not to belong to any of the mentioned classes, and not only because its range need not to be closed.

Example 18. Let $\mathcal{H} = \mathbb{C}^4$ and:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} \frac{3}{4} & 0 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{3}}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then P and Q are orthogonal projections, while for $T = PQ$ we have:

$$T = \begin{bmatrix} \frac{3}{4} & 0 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T^* = \begin{bmatrix} \frac{3}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{3}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We readily check that T does not belong to any of the classes EP, DR, SR, co-EP, weak-EP. ◇

This is our main motivation to extend the DR class in the following way. Note that we do not ask for T to have a closed range, although most of the presented results will deal with closed range operators.

Definition 2.4.2. Let $\mathcal{H}$ be a Hilbert space, and T a bounded linear operator on $\mathcal{H}$. We say that T is a *compatible range operator* (CoR)[1] if T and T^* coincide on the set $\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$, i.e. if T and T^* are precoherent.

The main framework for studying DR matrices and DR operators was established through certain space and operator decompositions. In [14] the Hartwig-Spindelböck decomposition of matrix is used (see [53]), and in case of operators on arbitrary Hilbert space, the appropriate operator decomposition is used: if $T \in \mathcal{B}(\mathcal{H})$ then

$$T = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T^*) \end{bmatrix}. \tag{2.6}$$

The reader is referred to [30, Lemma 1.2] and the discussion therein for further properties of such decompositions.

[1]We abbreviated *compatible range* as CoR, and not as CR, since CR is commonly used for the class of closed range operators.

If T is a closed range operator, [23, Theorem 3.5] gives necessary and sufficient conditions for T to be DR, SR and co-EP operator, under the additional assumption that $\mathcal{R}(TT^\dagger - T^\dagger T)$ is closed (which will be the subject of Lemma 2.4.4). The main tool in that proof is the famous Halmos' two projections theorem (see [19, 48]). However, this assumption is dispensable if we apply a more direct approach.

Theorem 2.4.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a closed range operator, with operators A and B defined as in* (2.6)*. Then:*

(1) T is DR if and only if $\overline{\mathcal{R}(B)} = \mathcal{R}(T)$;

(2) T is SR if and only if $\mathcal{R}(B^*) = \mathcal{N}(T^*)$;

(3) T is co-EP if and only if B is invertible.

Proof. (1) Since

$$T^* = \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}$$

then $\mathcal{R}(T) \cap \mathcal{R}(T^*) = \{0\}$ if and only if for every $x \in \mathcal{R}(T)$ the implication $B^*x = 0 \implies A^*x = 0$ holds. This is equivalent to $\overline{\mathcal{R}(A)} \subseteq \overline{\mathcal{R}(B)}$, which is equivalent to $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(B)}$.

The subspace $\mathcal{R}(T)$ is closed and $\mathcal{R}(T) = \mathcal{R}(A) + \mathcal{R}(B)$, so we have $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)} \subseteq \overline{\mathcal{R}(A) + \mathcal{R}(B)} = \overline{\mathcal{R}(T)} = \mathcal{R}(T) = \mathcal{R}(A) + \mathcal{R}(B)$. Hence $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(B)}$ if and only if $\overline{\mathcal{R}(B)} = \mathcal{R}(T)$ and the statement (1) is proved.

(2) First let us prove that $\mathcal{R}(T) + \mathcal{R}(T^*) = \mathcal{R}(T) \oplus \mathcal{R}(B^*)$. For every $x \in \mathcal{R}(T)$ we have that $B^*x = -A^*x + (A^*x + B^*x)$, where $A^*x \in \mathcal{R}(T)$ and $A^*x + B^*x \in \mathcal{R}(T^*)$. Thus $\mathcal{R}(B^*) \subseteq \mathcal{R}(T) + \mathcal{R}(T^*)$ and so $\mathcal{R}(T) \oplus \mathcal{R}(B^*) \subseteq \mathcal{R}(T) + \mathcal{R}(T^*)$. The other implication is clear, since $\mathcal{R}(T^*) \subseteq \mathcal{R}(T) \oplus \mathcal{R}(B^*)$. Thus $\mathcal{H} = \mathcal{R}(T) + \mathcal{R}(T^*)$ if and only if $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{R}(B^*)$, and $\mathcal{R}(B^*) \subseteq \mathcal{N}(T^*)$, so this is equivalent to $\mathcal{R}(B^*) = \mathcal{N}(T^*)$.

(3) If T is co-EP then T is DR and SR, so $\overline{\mathcal{R}(B)} = \mathcal{R}(T)$ and $\mathcal{R}(B^*) = \mathcal{N}(T^*)$. Thus $\mathcal{R}(B^*)$, i.e. $\mathcal{R}(B)$ is closed, $\mathcal{R}(B) = \mathcal{R}(T)$ and $\mathcal{N}(B) = \mathcal{R}(B^*)^\perp = \{0\}$, showing that B is invertible.

If B is invertible, from (1) and (2) we conclude that T is in the same time DR and SR, so it is co-EP. □

Lemma 2.4.4. *If $T \in \mathcal{B}(\mathcal{H})$ is a closed range operator, then $\mathcal{R}(TT^\dagger - T^\dagger T)$ is closed if and only if $\mathcal{R}(T) + \mathcal{R}(T^*)$ is closed, if and only if $\mathcal{R}(B)$ is closed, where B is as in* (2.6)*.*

Proof. Operators $TT^\dagger$ and $T^\dagger T$ are orthogonal projections, so from Lemma 1.4.5 we have that $\mathcal{R}(TT^\dagger - T^\dagger T)$ is closed iff $\mathcal{R}(TT^\dagger) + \mathcal{R}(T^\dagger T)$ is closed, iff $\mathcal{R}(T) + \mathcal{R}(T^*)$ is closed.

As in the proof of statement (2) in Theorem 2.4.3 we have that $\mathcal{R}(T) + \mathcal{R}(T^*) = \mathcal{R}(T) \oplus \mathcal{R}(B^*)$. Since $\mathcal{R}(T)$ is closed and $\mathcal{R}(B^*) \subseteq (\mathcal{R}(T))^\perp$ we have that $\mathcal{R}(T) \oplus \mathcal{R}(B^*)$ is closed iff $\mathcal{R}(B^*)$ is closed, i.e. iff $\mathcal{R}(B)$ is closed (Lemma 1.2.4). □

It is clear from Lemma 2.4.4 that [23, Theorem 3.5, (i)] follows from Theorem 2.4.3, while the other statements of [23, Theorem 3.5] hold verbatim without additional assumptions.

A natural connection between CoR and DR operators is described by the following statements.

Lemma 2.4.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a closed range CoR operator. Then $T(\mathcal{R}(T)\cap\mathcal{R}(T^*)) = \mathcal{R}(T)\cap\mathcal{R}(T^*)$, $T(\mathcal{R}(T^*)\ominus(\mathcal{R}(T)\cap\mathcal{R}(T^*))) = \mathcal{R}(T)\ominus(\mathcal{R}(T)\cap\mathcal{R}(T^*))$, and consequently, $T((\mathcal{R}(T)\cap\mathcal{R}(T^*))^\perp) \subseteq (\mathcal{R}(T)\cap\mathcal{R}(T^*))^\perp$.*

Proof. Follows directly from Proposition 2.2.2, applied to T and T^*. □

Theorem 2.4.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a closed range CoR operator. There exists a Hilbert space $\mathcal{H}_1$, a bounded linear surjection $\pi : \mathcal{H} \to \mathcal{H}_1$ and an operator $T_1 \in \mathcal{B}(\mathcal{H}_1)$ such that:*

(1) T_1 has a closed range and it is DR;

(2) For every $x \in \mathcal{H}$, $\pi(Tx) = T_1\pi(x)$, and $\pi(T^*x) = T_1^*\pi(x)$;

(3) $\mathcal{N}(\pi) = \mathcal{R}(T)\cap\mathcal{R}(T^*)$;

(4) For every $x \in \mathcal{H}$, $||\pi(x)|| = ||(I - P_{\mathcal{R}(T)\cap\mathcal{R}(T^*)})x||$.

If π satisfies these conditions, and $\mathcal{M}$ is a subspace of $\mathcal{H}$ such that $\mathcal{R}(T)\cap\mathcal{R}(T^*) \subseteq \mathcal{M}$ then $\mathcal{M}$ is closed in $\mathcal{H}$ if and only if $\pi(\mathcal{M})$ is closed in $\mathcal{H}_1$.

Proof. Let $\mathcal{H}_1$ be the orthogonal complement of $\mathcal{R}(T)\cap\mathcal{R}(T^*)$ in $\mathcal{H}$ and $\pi : \mathcal{H} \to \mathcal{H}_1$ defined as $\pi(x) = (I - P_{\mathcal{R}(T)\cap\mathcal{R}(T^*)})x$. In that case π is a bounded linear surjection which satisfies (3) and (4).

Usgin Lemma 2.4.5 it is not difficult to see that the operator $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$ defined as $T_1x = Tx$, for every $x \in \mathcal{H}_1$, is a well–defined operator, with a closed range, satisfying all the given conditions. This is easily seen from $T((\mathcal{R}(T)\cap\mathcal{R}(T^*))^\perp) \subseteq (\mathcal{R}(T)\cap\mathcal{R}(T^*))^\perp$, $\mathcal{R}(T_1) = \pi(\mathcal{R}(T)) = \mathcal{R}(T)\ominus(\mathcal{R}(T)\cap\mathcal{R}(T^*))$, etc.

To prove the last statement, note that if $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{R}(T)\cap\mathcal{R}(T^*) \subseteq \mathcal{M}$ and $\mathcal{N} = (\mathcal{R}(T)\cap\mathcal{R}(T^*))^\perp \cap \mathcal{M}$, then $\mathcal{M} = (\mathcal{R}(T)\cap\mathcal{R}(T^*)) \oplus \mathcal{N}$, $\mathcal{N}\perp\mathcal{R}(T)\cap\mathcal{R}(T^*)$ and according to (4), π is an isometry on $\mathcal{N}$. So $\mathcal{M}$ is closed iff $\mathcal{N}$ is closed, iff $\pi(\mathcal{N})$ is closed, iff $\pi(\mathcal{M})$ is closed, since according to (3) we have $\pi(\mathcal{N}) = \pi(\mathcal{M})$. □

Remark 2.4.7. The converse of Theorem 2.4.6 is not true: if there exist such $\mathcal{H}_1, \pi$ and T_1, the operator T need not to be CoR. However, in that case we can conclude that $T(\mathcal{R}(T)\cap\mathcal{R}(T^*)) \subseteq \mathcal{R}(T)\cap\mathcal{R}(T^*)$, and similarly for T^*, and so the decomposition $\mathcal{H} = (\mathcal{R}(T)\cap\mathcal{R}(T^*)) \oplus (\mathcal{R}(T)\cap\mathcal{R}(T^*))^\perp$ completely reduces both T and T^*. This further yields that T is an isomorphism on $\mathcal{R}(T)\cap\mathcal{R}(T^*)$ although it is not necessarily self–adjoint.

In order to state the characterization of CoR operators similar to that in Theorem 2.4.3, let:

$$P_{\overline{\mathcal{R}(T)}\cap\overline{\mathcal{R}(T^*)}} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(T)} \\ \mathcal{N}(T^*) \end{bmatrix}, \tag{2.7}$$

where $P \in \mathcal{B}(\overline{\mathcal{R}(T)})$ is the orthogonal projection with the range $\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$ and the null-space $\overline{\mathcal{R}(T)} \ominus (\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)})$. Also, if $\mathcal{R}(T)$ is closed and A and B defined as in (2.6), then $AA^* + BB^* \in \mathcal{B}(\mathcal{R}(T))$ is invertible, and as in [23] we denote $\Delta = (AA^* + BB^*)^{-1}$.

Theorem 2.4.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator with A and B defined as in (2.6). The operator T is CoR if and only if A and A^* coincide on $\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$ and $\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)} \subseteq \mathcal{N}(B^*)$. In that case, we have:*

(1) $PAP = AP = A^*P = PA^*P$. If $\mathcal{R}(T)$ is closed, then also $P\Delta P = \Delta P$;

(2) $\mathcal{N}(B^*) = \overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$, i.e. $\overline{\mathcal{R}(B)} = \overline{\mathcal{R}(T)} \ominus (\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)})$.

Proof. The operator T is CoR if and only if $(T - T^*)P_{\overline{\mathcal{R}(T)}\cap\overline{\mathcal{R}(T^*)}} = 0$, i.e.

$$\begin{bmatrix} (A - A^*)P & 0 \\ -B^*P & 0 \end{bmatrix} = 0.$$

From here the first statement of the theorem follows directly.

Suppose now that T is a CoR operator.

(1) We already have $AP = A^*P$, and $\mathcal{R}(P) \subseteq \mathcal{N}(B^*)$. If $x \in \mathcal{R}(P) = \overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$ is arbitrary, then:

$$T^*x = \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A^*x \\ 0 \end{bmatrix} \in \overline{\mathcal{R}(T)}.$$

Since $T^*x \in \overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$, we have that $A^*x \in \overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$. This proves $PA^*P = A^*P$, but $A^*P = AP$, so $PAP = AP$ also. From here we also obtain $(I-P)A(I-P) = A(I-P)$, and $(I-P)A^*(I-P) = A^*(I-P)$, so $\overline{\mathcal{R}(T)} \ominus (\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)})$ is also invariant for A and A^*. The equality $B^*P = 0$ implies $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(T)} \ominus (\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)})$. Finally, if $\mathcal{R}(T)$ is closed, we see that the subspaces $\mathcal{R}(T) \cap \mathcal{R}(T^*)$ and $\mathcal{R}(T) \ominus (\mathcal{R}(T) \cap \mathcal{R}(T^*))$ are invariant also for $AA^* + BB^*$ which is an isomorphism. Therefore $P\Delta P = \Delta P$.

(2) If $x \in \overline{\mathcal{R}(T)} \ominus (\overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)})$ is such that $B^*x = 0$, then $T^*x \in \overline{\mathcal{R}(T)}$, i.e. $T^*x \in \overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$. Therefore, $TT^*x \in \overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$, and so $0 = \langle x, TT^*x \rangle = ||T^*x||^2$, giving $x = 0$. Thus $\mathcal{N}(B^*) = \overline{\mathcal{R}(T)} \cap \overline{\mathcal{R}(T^*)}$. $\square$

In order to give a formula for $(T + T^*)^\dagger$ when T is CoR, we first prove the following result regarding range additivity, explaining when does $(T + T^*)^\dagger$ exist. Note that the following theorem is a special case of a more general Theorem 2.3.6. Nevertheless, we present a different proof which uses Lemma 2.3.4, Proposition 1.4.3 and Theorem 2.4.6 which enables us to pass on to disjoint range operators.

Theorem 2.4.9. *Let $T \in \mathcal{B}(\mathcal{H})$ be a closed range CoR operator. Then $\mathcal{R}(TT^\dagger - TT^\dagger)$ is closed if and only if $\mathcal{R}(T) + \mathcal{R}(T^*)$ is closed if and only if $\mathcal{R}(T + T^*)$ is closed. In that case $\mathcal{R}(T) + \mathcal{R}(T^*) = \mathcal{R}(T + T^*)$.*

Proof. The first equivalence follows from Lemma 2.4.4, so we prove the second equivalence.

Suppose first that $\mathcal{R}(T)+\mathcal{R}(T^*)$ is closed. Let $\mathcal{H}_1, \pi$ and T_1 be defined as in the proof of Theorem 2.4.6. Then T_1 is a closed range DR operator. Note that $\mathcal{R}(T_1) \oplus \mathcal{R}(T_1^*) = \pi(\mathcal{R}(T) + \mathcal{R}(T^*))$, and $\mathcal{R}(T) \cap \mathcal{R}(T^*) \subseteq \mathcal{R}(T) + \mathcal{R}(T^*)$, so using Theorem 2.4.6 we have that $\mathcal{R}(T_1) \oplus \mathcal{R}(T_1^*)$ is also closed. According to Proposition 1.4.3, we have that $\mathcal{R}(T_1 + T_1^*) = \mathcal{R}(T_1) \oplus \mathcal{R}(T_1^*)$, so $\mathcal{R}(T_1 + T_1^*)$ is also closed. We can easily prove that $\mathcal{R}(T_1 + T_1^*) = \pi(\mathcal{R}(T + T^*))$, and since T is CoR, $\mathcal{R}(T) \cap \mathcal{R}(T^*) = T(\mathcal{R}(T) \cap \mathcal{R}(T^*)) \subseteq \mathcal{R}(T + T^*)$ (Lemma 2.4.5). Thus, again from Theorem 2.4.6 we get that $\mathcal{R}(T + T^*)$ is also closed.

Suppose now that $\mathcal{R}(T+T^*)$ is closed. From Lemma 2.3.4 we have that $\mathcal{R}(T+T^*) = \mathcal{R}(T) + \mathcal{R}(T^*)$, so $\mathcal{R}(T) + \mathcal{R}(T^*)$ is also closed. This also proves the second statement of the theorem. $\square$

Thus the range additivity $\mathcal{R}(T+T^*) = \mathcal{R}(T)+\mathcal{R}(T^*)$ which appears in [23, Theorem 3.9 (ii)] is also present in the case when operators are DR and not necessarily SR. For matrices, this was noted in [14, p. 1229], but the technique used therein relies on notions which are not accessible in infinite-dimensional Hilbert spaces.

Theorem 2.4.10. *If T is a closed range CoR operator and if any of the (equivalent) conditions is satisfied: $\mathcal{R}(B)$ is closed, $\mathcal{R}(T + T^*)$ is closed, $\mathcal{R}(T) + \mathcal{R}(T^*)$ is closed, or $\mathcal{R}(TT^\dagger - T^\dagger T)$ is closed, then:*

$$(T + T^*)^\dagger = \begin{bmatrix} \frac{1}{2}A^*\Delta P & (B^*)^\dagger \\ B^\dagger & -B^\dagger(A + A^*)(B^*)^\dagger \end{bmatrix}, \tag{2.8}$$

where operators A, B, Δ and P are defined as in the previous discussion.

Proof. Denote by X the operator on the right in (2.8). By direct multiplication, we obtain:

$$(T + T^*)X = \begin{bmatrix} \frac{1}{2}(A + A^*)A^*\Delta P + BB^\dagger & (A + A^*)(B^*)^\dagger - BB^\dagger(A + A^*)(B^*)^\dagger \\ \frac{1}{2}B^*A^*\Delta P & (B^*)(B^*)^\dagger \end{bmatrix}.$$

From Theorem 2.4.8 we have $B^*P = 0$, $BB^\dagger = I - P$, $P(B^*)^\dagger = 0$, $P(A + A^*) = (A + A^*)P = 2AP$, $AA^*P = (AA^* + BB^*)P$, $\Delta P = P\Delta P$. Hence:

$$\begin{aligned} \frac{1}{2}(A + A^*)A^*\Delta P + BB^\dagger &= \frac{1}{2}(A + A^*)PA^*P\Delta P + I - P \\ &= AA^*P\Delta P + I - P \\ &= (AA^* + BB^*)P\Delta P + I - P \\ &= I, \end{aligned}$$

also $(A + A^*)(B^*)^\dagger - BB^\dagger(A + A^*)(B^*)^\dagger = (A + A^*)(B^*)^\dagger - (I - P)(A + A^*)(B^*)^\dagger = 0$, and $\frac{1}{2}B^*A^*\Delta P = \frac{1}{2}B^*PA^*P\Delta P = 0$. So we conclude:

$$(T + T^*)X = \begin{bmatrix} I & 0 \\ 0 & P_{\mathcal{R}(B^*)} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}.$$

From Theorem 2.4.9 and the proof of statement (2) in Theorem 2.4.3 we have that $\mathcal{R}(T+T^*) = \mathcal{R}(T)+\mathcal{R}(T^*) = \mathcal{R}(T)\oplus\mathcal{R}(B^*)$. So $(T+T^*)X$ is the orthogonal projection onto $\mathcal{R}(T+T^*)$. It is also true that X is self–adjoint. To see this, note that Δ is self–adjoint and that $A^*P = PA^* = PA^*P$ commutes with $(AA^*+BB^*)P = P(AA^*+BB^*) = P(AA^*+BB^*)P$, and so it commutes with ΔP. Thus $A^*\Delta P = \Delta A^*P = \Delta AP = P\Delta A$. Hence, $X(T+T^*)$ is also the orthogonal projection onto $\mathcal{R}(T+T^*)$. This proves $X = (T+T^*)^\dagger$. □

Formula (2.8) generalizes the result from [23, Theorem 3.9] regarding the formula for $(T+T^*)^\dagger$, and we have $T(T+T^*)^\dagger T = T - \frac{1}{2}P_{\mathcal{R}(T)\cap\mathcal{R}(T^*)}TP_{\mathcal{R}(T)\cap\mathcal{R}(T^*)}$, while $2T(T+T^*)^\dagger T^* = 2T^*(T+T^*)^\dagger T = P_{\mathcal{R}(T)\cap\mathcal{R}(T^*)}TP_{\mathcal{R}(T)\cap\mathcal{R}(T^*)}$. In fact, the last expression gives the parallel sum of T and T^*, and in the same time the infimum of T and T^* with respect to the star partial order on $\mathcal{B}(\mathcal{H})$ (see Section 4.5).

There are few results from [14] for DR matrices that can be easily proved for CoR operators in the Hilbert space setting. For example, [14, Theorem 4, Theorem 5] are also true for CoR operators, and [14, Theorem 8] can be extended using Theorem 3.2.4. However, we can not have elegant characterizations as the one in [14, Theorem 1], since the CoR class is not defined only by mutual positioning of the ranges of appropriate operators. When we make a transition from operators T and T^* to the orthogonal projections $P = P_{\overline{\mathcal{R}(T)}}$ and $Q = P_{\overline{\mathcal{R}(T^*)}}$, we lose the information of the way T and T^* act on these subspaces which determines whether T is CoR.

In the end of this section, we give some results regarding products of orthogonal projections, and draw reader's attention to the so called factorization problems. If $\mathfrak{A}$ and $\mathfrak{B}$ are two classes of operators from $\mathcal{B}(\mathcal{H})$ there is a natural problem of characterizing operators which belong to the class $\mathfrak{A}\cdot\mathfrak{B} = \{A\cdot B \mid A\in\mathfrak{A},\ B\in\mathfrak{B}\}$, or to the class $\mathfrak{A}^\infty = \bigcup_k \mathfrak{A}^k$, where $\mathfrak{A}^k$ stands for $\mathfrak{A}\cdot\mathfrak{A}\cdot\ldots\cdot\mathfrak{A}$. Such problems are commonly known as factorization problems, and the reader is referred to [8, 22, 73, 83] for some prominent results and further reference on this subject.

Let $\mathcal{P}$ denote the class of all orthogonal projections from $\mathcal{B}(\mathcal{H})$. We have the following results regarding the factors from $\mathcal{P}$.

Theorem 2.4.11. *Let $P, Q \in \mathcal{P}$ and $T = PQP...P$ or $T = PQP...Q$. Then T is a CoR operator. More generally, $A = PQP\cdots P$ and $B = PQP\cdots Q$, as well as $A = PQP\cdots P$ and $B = QPQ\cdots Q$ are precoherent, where A and B do not necessarily have the same number of factors P and Q.*

Proof. Assume that $A = PQP\cdots P$ and $B = PQP\cdots Q$ and the other case is proved similarly. Then $\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)} \subseteq \overline{\mathcal{R}(P)}\cap\overline{\mathcal{R}(Q)} = \mathcal{R}(P)\cap\mathcal{R}(Q)$. On the other hand, $\mathcal{R}(P)\cap\mathcal{R}(Q)$ is obviously a subspace of $\mathcal{R}(A^*)$ and $\mathcal{R}(B^*)$. Thus $\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)} = \mathcal{R}(P)\cap\mathcal{R}(Q)$, and A and B coincide on this subspace, both being equal to identity on it. Hence A and B are precoherent. □

Corollary 2.4.12. *The class $\mathcal{P}^2$ belongs to the class of CoR operators.*

Proof. Follows directly from Theorem 2.4.11. □

Corollary 2.4.13. *If P and Q are orthogonal projections, then $\mathcal{R}(PQP\cdots Q+QPQ\cdots P)$ is closed if and only if $\mathcal{R}(PQP\cdots Q)+\mathcal{R}(QPQ\cdots P)$ is closed. In that case $\mathcal{R}(PQP\cdots Q)$ is also closed and $\mathcal{R}(PQ\cdots PQ)+\mathcal{R}(QP\cdots QP)=\mathcal{R}(PQ\cdots PQ+QP\cdots QP)$. (Here $PQ...PQ$ and $QP...QP$ have the same number of factors.)*

Proof. Directly from Theorem 2.4.11 and Theorem 2.3.6. □

Note that Corollary 2.4.13 generalizes [12, Corollary 4] in infinite-dimensional setting and for products of arbitrary length.

Example 19. We will show now that the class $\mathcal{P}^3$ is not contained in the CoR class. Let $\mathcal{H}=\mathbb{C}^4$ and:

$$P=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad Q=\begin{bmatrix}\frac{1}{2}&0&0&\frac{1}{2}\\0&1&0&0\\0&0&0&0\\\frac{1}{2}&0&0&\frac{1}{2}\end{bmatrix},\quad R=\begin{bmatrix}1&0&0&0\\0&\frac{3}{4}&0&\frac{\sqrt{3}}{4}\\0&0&1&0\\0&\frac{\sqrt{3}}{4}&0&\frac{1}{4}\end{bmatrix}.$$

Then $P,Q,R\in\mathcal{P}$, while for $T=PQR\in\mathcal{P}^3$ we have:

$$T=\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{8}&0&\frac{1}{8}\\0&\frac{3}{4}&0&\frac{\sqrt{3}}{4}\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad T^*=\begin{bmatrix}\frac{1}{2}&0&0&0\\\frac{\sqrt{3}}{8}&\frac{3}{4}&0&0\\0&0&0&0\\\frac{1}{8}&\frac{\sqrt{3}}{4}&0&0\end{bmatrix}.$$

We can now check that $x=(1,0,0,0)\in\mathcal{R}(T)\cap\mathcal{R}(T^*)$, however $Tx\neq T^*x$, and so T is not CoR. ◇

Chapter 3

Inverting the sum of precoherent operators

This chapter is devoted to the study of generalized inverses of the sum of two precoherent operators. We begin with some notes on the famous Sherman-Morrison-Woodburry formula, and then present a few results of Mitra, Werner, Fill and Fishkind, Groß, on inverting the sum of a special kind of matrices. Our goal is to study these results in a more general framework: for Hilbert space operators which are precoherent. In this way we extend the existing results not only in an infinite-dimensional setting, but also for a wider class of operators (matrices), e.g. orthogonal projections and their products.

3.1 Motivation and some results

Inverting a nonsingular matrix and computing generalized inverses of a singular matrix are expensive procedures from a computational point of view. Suppose that we have a matrix A and its inverse A^{-1} already computed. It would be wasteful to repeat the whole process of computing the inverse after every 'small change' of the matrix A to $A + A'$. It is therefore convenient to develop a procedure that would compute the inverse of $A + A'$ using the inverse of A. Of course, a 'small change', often referred to as a *matrix update*, can have at least two meanings: only a couple of entries of A change, or the whole change A' is small, in the sense of a norm. A classical result regarding such a problem is given by the Sherman-Morrison-Woodburry formula (SMW). For historical survey of this result, and many examples of applications, the reader is referred to [46, 55]. One form of this formula is given in the following theorem.

Theorem 3.1.1 (SMW)**.** *Let $A \in \mathbb{C}^{n\times n}$, and $G \in \mathbb{C}^{m\times m}$ be invertible matrices, and let $Y \in \mathbb{C}^{n\times m}$ and $Z \in \mathbb{C}^{m\times n}$. Then $A + YGZ$ is invertible if and only if $G^{-1} + ZA^{-1}Y$ is invertible, in which case:*

$$(A + YGZ)^{-1} = A^{-1} - A^{-1}Y(G^{-1} + ZA^{-1}Y)^{-1}ZA^{-1}.$$

Observe that the inverse on the right hand side: $(G^{-1} + ZA^{-1}Y)^{-1}$ is the inverse of an $m \times m$ matrix, so this formula is particularly useful when m is much smaller than n. For example, if the matrix A changes by some matrix A' of rank 1, then Y, Z and G

can be chosen so that $Y \in \mathbb{C}^{n\times 1}$, $Z \in \mathbb{C}^{1\times n}$ and $G = [1]$, and $A' = YZ$. The only new inverse that has to be computed in order to compute $(A+A')^{-1}$ is the inverse of a scalar $1 + ZA^{-1}Y$.

A natural next step in developing SMW formula is to discard the condition of invertibility. Generalized inverses of matrices are known to have many applications, so efficient computing of generalized inverses of an updated matrix is as equally important task as for the usual inverse. In general, expressing the generalized inverses of $A + B$ via generalized inverses of A and B is a difficult problem. However, for a special kind of matrices, and generalized inverses defined by Penrose equations, this problem has an elegant solution. Such matrices were already mentioned in Section 2.4, and they are defined by any of the following equivalent conditions:

1. $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} = \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$;
2. $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{R}(A + B)$ and $\mathcal{R}(A^*) \oplus \mathcal{R}(B^*) = \mathcal{R}(A^* + B^*)$;
3. $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{R}(A + B)$;
4. $\mathrm{r}(A) + \mathrm{r}(B) = \mathrm{r}(A + B)$;

Let us first say a few words about these matrices. Mitra in [65] refers to them as disjoint matrices, while Werner in [82] calls such matrices weakly bicomplementary, as we will here. They were also studied by other authors, and we are also going to mention the results of Fill and Fishkind [37] and Groß [44]. In [65] and [82] some characterizations and applications of such matrices are given, mainly concerned with systems of linear equations. Werner describes the following problem: when is it possible to change two systems of equations $Ax = \alpha$ and $Bx = \beta$ with a single system $(A+B)x = \alpha+\beta$. More precisely he proves the following theorem.

Theorem 3.1.2 (See [82]). *If $A, B \in \mathbb{C}^{m\times n}$ the following statements are equivalent:*

(i) A and B are weakly bicomplementary;

(ii) For every $a, b \in \mathbb{C}^m$ such that the equations $A\alpha = a$ and $B\beta = b$ have a solution, the equation $(A + B)x = a + b$ also has a solution, and every its solution is a common solution to the equations $A\alpha = a$, $B\beta = b$ (i.e. it holds $Ax = a$ and $Bx = b$).

One more characterization of such matrices is given in terms of inner generalized inverses.

Theorem 3.1.3 (See [65, 82]). *If $A, B \in \mathbb{C}^{m\times n}$ the following statements are equivalent:*

(i) A and B are weakly bicomplementary;

(ii) Every inner inverse $(A + B)^-$ of $A + B$ is also an inner inverse of A and B.

Now back to the problem of expressing generalized inverses of $(A+B)$ as the sum of inverses of A and B. Werner gives the following formula.

Theorem 3.1.4 (See [82]). *If A and B are pair of rectangular weakly bicomplementary matrices, and $\mathcal{M}$ and $\mathcal{N}$ are arbitrary direct complements of $\mathcal{N}(A+B)$ and $\mathcal{R}(A+B)$ respectively, then:*

$$(A+B)^{(1,2)}_{\mathcal{M},\mathcal{N}} = A^{(1,2)}_{\mathcal{M}\cap\mathcal{N}(B),\mathcal{N}\oplus\mathcal{R}(B)} + B^{(1,2)}_{\mathcal{M}\cap\mathcal{N}(A),\mathcal{N}\oplus\mathcal{R}(A)}. \tag{3.1}$$

Choosing different subspaces $\mathcal{M}$ and $\mathcal{N}$ provides formulas for Moore-Penrose inverse of the sum, or the usual inverse when it exists. Thus, in [82], one can also find formulas as:

$$(A+B)^\dagger = \bar{A} + \bar{B},$$

where

$$\bar{A} = A^{(1,2)}_{(\mathcal{R}(A^*)+\mathcal{R}(B^*))\cap\mathcal{N}(B),((\mathcal{N}(A^*)\cap\mathcal{N}(B^*))\oplus\mathcal{R}(B))}$$
$$\bar{B} = B^{(1,2)}_{(\mathcal{R}(A^*)+\mathcal{R}(B^*))\cap\mathcal{N}(A),((\mathcal{N}(A^*)\cap\mathcal{N}(B^*))\oplus\mathcal{R}(A))},$$

and:

$$(A+B)^{-1} = A^{(1,2)}_{\mathcal{N}(B),\mathcal{R}(B)} + B^{(1,2)}_{\mathcal{R}(A),\mathcal{N}(A)}.$$

The following formula was first derived by Fill and Fishkind [37] for square matrices. Groß [44] proved the same formula for arbitrary rectangular matrices using a different technique. He also related this new formula with the one given by Werner.

Theorem 3.1.5 (See [37, 44]). *If $A, B \in \mathbb{C}^{m\times n}$ are disjoint matrices, then:*

$$(A+B)^\dagger = (I-S)A^\dagger(I-T) + SB^\dagger T,$$

with

$$S = (P_{\mathcal{N}(B)^\perp}P_{\mathcal{N}(A)})^\dagger \quad \textit{and} \quad T = (P_{\mathcal{N}(A^*)}P_{\mathcal{N}(B^*)^\perp})^\dagger.$$

Our goal in this chapter is to explore if the similar formulas can be derived, and under what (minimal) conditions, for operators between arbitrary Hilbert spaces. First we should ask what weakly bicomplementary should mean in the operator case. Conditions 1.-4. listed above are not equivalent for arbitrary Hilbert space operators (let alone that condition 4. now has no meaning). Characterizations expressed in Theorems 3.1.2 and 3.1.3 should then serve as guidelines for the right generalization on Hilbert space operators.

It is not difficult to see that condition (ii) in Theorem 3.1.2 is equivalent to $\mathcal{R}(A)\oplus\mathcal{R}(B) = \mathcal{R}(A+B)$, for arbitrary Hilbert space operators A and B. Furthermore, if by inner inverse of $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ we mean only a linear transformation $S : \mathcal{K} \to \mathcal{H}$ such that $TST = T$, and so do not impose any closed range conditions, then (ii) in Theorem 3.1.3 is also equivalent to $\mathcal{R}(A)\oplus\mathcal{R}(B) = \mathcal{R}(A+B)$. We give a proof for the sake of completeness.

Lemma 3.1.6. *If $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ then $\mathcal{R}(A)\oplus\mathcal{R}(B) = \mathcal{R}(A+B)$ if and only if every inner inverse $(A+B)^-$ of $A+B$ is also an inner inverse of A and B.*

Proof. Suppose that $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{R}(A+B)$, and let $(A+B)^-$ and $x \in \mathcal{H}$ be arbitrary. Then $Ax \in \mathcal{R}(A+B)$ and so $Ax = (A+B)y$, thus $A(A+B)^-Ax = (A+B-B)(A+B)^-(A+B)y = (A+B)y - B(A+B)^-Ax = Ax - B(A+B)^-Ax$, giving $A(x-(A+B)^-Ax) = B(A+B)^-Ax \in \mathcal{R}(A) \cap \mathcal{R}(B)$. So $A(x-(A+B)^-Ax) = 0$, i.e. $Ax = A(A+B)^-Ax$. This shows that $A = A(A+B)^-A$. The same is true for B.

Suppose now that every inner inverse $(A+B)^-$ of $A+B$ is an inner inverse of A and B. If $\mathcal{R}(A) \nsubseteq \mathcal{R}(A+B)$, then there is some $Ax = y \in \mathcal{R}(A) \setminus \mathcal{R}(A+B)$ and there is an algebraic complement of $\mathcal{R}(A+B)$, say $\mathcal{M}$, containing y. We can define an inner inverse $(A+B)^-$ such that $\mathcal{N}((A+B)^-) = \mathcal{M}$, and so $A(A+B)^-Ax = 0 \neq Ax$. Hence, $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$ and similarly $\mathcal{R}(B) \subseteq \mathcal{R}(A+B)$ giving $\mathcal{R}(A)+\mathcal{R}(B) = \mathcal{R}(A+B)$. If $0 \neq y \in \mathcal{R}(A) \cap \mathcal{R}(B)$ then there are a, b, c such that $y = Aa = Bb = (A+B)c$, and so $y = (A+B)c = (A+B)(A+B)^-(A+B)c = A(A+B)^-(A+B)c + B(A+B)^-(A+B)c = A(A+B)^-Aa + B(A+B)^-Bb = 2y$, which is a contradiction. Thus $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{R}(A+B)$. □

In both cases, we come to the condition: $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{R}(A+B)$. However, this condition for A and B is not equivalent to the same condition for A^* and B^*. Preferably, A^* and B^* should be weakly bicomplementary whenever A and B are (in fact, this is what the prefix bi means in Werners paper). Thus, we will say that A and B are weakly bicomplementary whenever $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{R}(A+B)$ and $\mathcal{R}(A^*) \oplus \mathcal{R}(B^*) = \mathcal{R}(A^*+B^*)$.[1] Such operators already appeared in the paper by Arias, Corach and Maestripieri [10] which gives the generalization of the Fill-Fishkind formula from Theorem 3.1.5. This speaks in favour of our definition.

Theorem 3.1.7 (See [10])**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be arbitrary Hilbert spaces, and $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. If:*

(1) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed;

(2) $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, $\mathcal{R}(A^*) \cap \mathcal{R}(B^*) = \{0\}$;

(3) $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A+B)$;

(4) $\mathcal{R}(A^*) + \mathcal{R}(B^*) = \mathcal{R}(A^*+B^*)$.

then:

$$(A+B)^\dagger = (I-S)A^\dagger(I-T) + SB^\dagger T,$$

with

$$S = (P_{\mathcal{N}(B)^\perp} P_{\mathcal{N}(A)})^\dagger \quad \text{and} \quad T = (P_{\mathcal{N}(A^*)} P_{\mathcal{N}(B^*)^\perp})^\dagger,$$

where all of the appearing Moore-Penrose inverses are bounded.

The study of generalized inverses of the sum of two weakly bicomplementary operators can now be conducted like in [82]. However, we offer such a study for a more general class of operators. Namely, from Proposition 1.4.3 we see that for weakly bicomplementary operators the relations $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)} = \{0\} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ hold. In other words,

[1] In the following chapter we will see that this is exactly the information that A is bellow $A+B$ in the so called minus partial order.

weakly bicomplementary operators are a special case of precoherent operators. In the following sections we generalize the formulas presented here for the class of precoherent operators.

To conclude, we note that throughout the following sections, $\mathcal{H}$ and $\mathcal{K}$ will stand for arbitrary Hilbert spaces. All the results are about operators $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that A and B are precoherent, and in the same time A^* and B^* are precoherent. Following Werner's terminology, such operators will be called *bi-precoherent*.

3.2 An extension of the Fill-Fishkind formula

For the sake of clarity, within this section and the next one, an oblique projection onto $\mathcal{M}$ parallel to $\mathcal{N}$ will be denoted as $Q(\mathcal{M}, \mathcal{N})$.

One of the main ingredients of the Fill-Fishkind formula, as well as of this extended version is the result regarding the Moore-Penrose inverse of the product of two orthogonal projections.

Theorem 3.2.1 (See [21]). *If $Q \in \mathcal{B}(\mathcal{H})$ is a projection, then $Q^\dagger = P_{\mathcal{N}(Q)^\perp} P_{\mathcal{R}(Q)}$. Conversely, if $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{H}$ such that $\mathcal{R}(P_\mathcal{M} P_\mathcal{N})$ is closed, then $(P_\mathcal{M} P_\mathcal{N})^\dagger = Q(\mathcal{R}(P_\mathcal{N} P_\mathcal{M}), \mathcal{N}(P_\mathcal{N} P_\mathcal{M}))$.*

Lemma 3.2.2. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that A and B are bi-precoherent and $\mathcal{R}(A+B)$ is closed, then $\mathcal{R}(P_{\mathcal{N}(B)^\perp} P_{\mathcal{N}(A)})$ and $\mathcal{R}(P_{\mathcal{N}(A^*)} P_{\mathcal{N}(B^*)^\perp})$ are closed.*

Proof. From Theorem 2.3.6 we know that A and B are closed range operators. A simple observation assures us that $\mathcal{R}(P_\mathcal{N} P_\mathcal{M}) = \mathcal{N} \cap (\mathcal{M} + \mathcal{N}^\perp)$ (as well as $\mathcal{N}(P_\mathcal{N} P_\mathcal{M}) = \mathcal{M}^\perp \oplus (\mathcal{M} \cap \mathcal{N}^\perp)$), where $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of a Hilbert space. Now, having in mind Theorem 2.3.6 and Theorem 1.2.8, the assertion follows. $\square$

The following theorem is a main theorem of this section, from which our extended version of the Fill-Fishkind formula follows directly. We will use Proposition 2.2.2 so for the sake of convenience

Theorem 3.2.3. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that A and B are bi-precoherent and $\mathcal{R}(A+B)$ is closed, then:*

(i) A and B are closed range operators, and moreover $\mathcal{H} = ([\mathcal{R}(A^*) + \mathcal{R}(B^*)] \ominus [\mathcal{R}(A^*) \cap \mathcal{R}(B^*)]) \oplus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*)) \oplus (\mathcal{N}(A) \cap \mathcal{N}(B))$ and $\mathcal{K} = ([\mathcal{R}(A) + \mathcal{R}(B)] \ominus [\mathcal{R}(A) \cap \mathcal{R}(B)]) \oplus (\mathcal{R}(A) \cap \mathcal{R}(B)) \oplus (\mathcal{N}(A^*) \cap \mathcal{N}(B^*))$;

(ii) The operator $A+B$, with respect to the decompositions in (i), is equal to

$$A + B = \begin{bmatrix} C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{3.5}$$

the operators C and D are invertible, provided that underlying spaces are nontrivial;

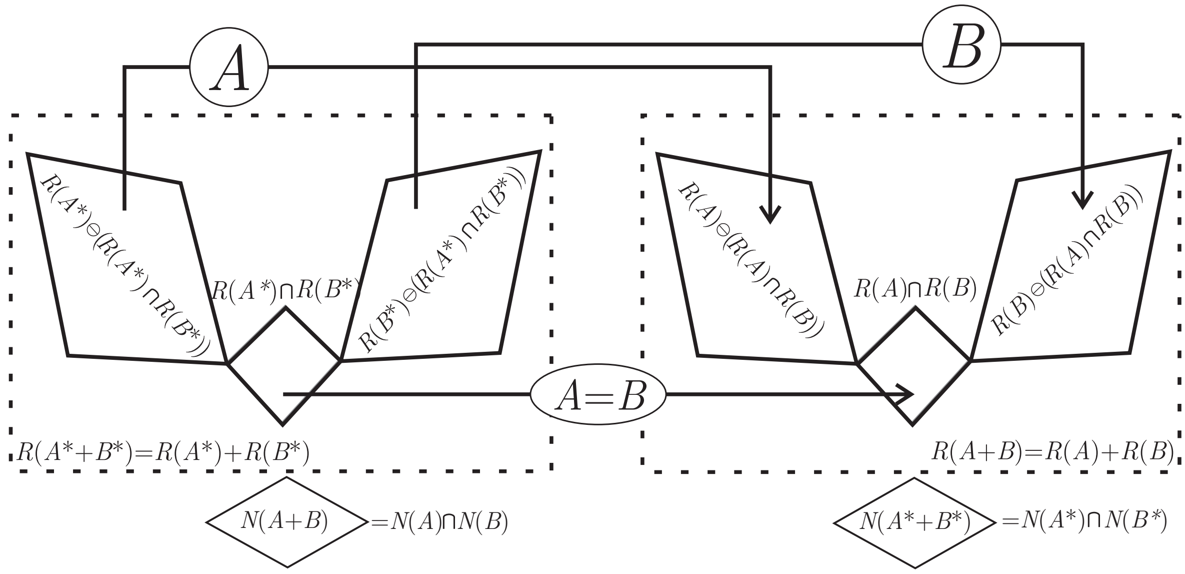
(iii) $S = (P_{\mathcal{N}(B)^\perp} P_{\mathcal{N}(A)})^\dagger$ and $T = (P_{\mathcal{N}(A^*)} P_{\mathcal{N}(B^*)^\perp})^\dagger$ are bounded;

(iv) The operator $L = (I-S)A^\dagger(I-T) + SB^\dagger T$, with respect to the decompositions in (i), is equal to

$$L = \begin{bmatrix} C^{-1} & 0 & 0 \\ 0 & 2D^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.6}$$

Proof. (i) Follows from Theorem 2.3.6, whence $\mathcal{H} = (\mathcal{R}(A^*)+\mathcal{R}(B^*)) \oplus (\mathcal{N}(A) \cap \mathcal{N}(B))$ and similarly for $\mathcal{K}$.
(ii) If we take $x \in (\mathcal{R}(A^*) + \mathcal{R}(B^*)) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$, then $x \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, so by Lemma 2.2.1, $Ax, Bx \perp \mathcal{R}(A) \cap \mathcal{R}(B)$, and $Ax + Bx \in \mathcal{R}(A) + \mathcal{R}(B)$. Thus $(A+B)([\mathcal{R}(A^*)+\mathcal{R}(B^*)] \ominus [\mathcal{R}(A^*) \cap \mathcal{R}(B^*)]) \subseteq (\mathcal{R}(A)+\mathcal{R}(B)) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. The operators A and B are precoherent, so $(A+B)(\mathcal{R}(A^*) \cap \mathcal{R}(B^*)) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$, and trivially $(A+B)(\mathcal{N}(A) \cap \mathcal{N}(B)) = \{0\}$. Hence, the operator matrix of $A+B$ is indeed diagonal and has the form as in (3.5). We will prove that C and D are invertible.

Operators A and B, as well as A^* and B^* are precoherent, so using Proposition 2.2.2 it is clear that D is invertible. Regarding the operator C, take two vectors $v_1, v_2 \in (\mathcal{R}(A^*)+\mathcal{R}(B^*)) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$ such that $Cv_1 = Cv_2$, i. e. $(A+B)v_1 = (A+B)v_2$. Then we get $A(v_1 - v_2) = B(v_2 - v_1)$. We have that $v_1, v_2 \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, so following Proposition 2.2.2, we get $A(v_1 - v_2) \in \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$ and $B(v_2 - v_1) \in \mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. The intersection of these two subspaces is equal to $\{0\}$, and so $Av_1 = Av_2$ and $Bv_1 = Bv_2$. So $v_1 - v_2 \in \mathcal{N}(A) \cap \mathcal{N}(B)$, but in the same time $v_1 - v_2 \in \mathcal{R}(A^*) + \mathcal{R}(B^*) = (\mathcal{N}(A) \cap \mathcal{N}(B))^\perp$. Thus $v_1 = v_2$, and so C is injective. From the fact that $\mathcal{R}(A+B)$ is closed, and that $\mathcal{R}(A+B) = \mathcal{R}(C) \oplus \mathcal{R}(D)$, as well as $\mathcal{R}(D) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{R}(C) \subseteq \mathcal{R}(A+B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$, we get that $\mathcal{R}(C) = \mathcal{R}(A+B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$, so C is also onto, i.e. invertible.



(iii) Follows from Lemma 3.2.2.
(iv) First of all, according to Theorem 3.2.1 and former discussion, we have that

$$S = Q(\mathcal{N}(A) \cap (\mathcal{R}(A^*) + \mathcal{R}(B^*)), \mathcal{N}(B) \oplus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))), \tag{3.7}$$

$$T = Q(\mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B)), \mathcal{R}(A) \oplus (\mathcal{N}(A^*) \cap \mathcal{N}(B^*))). \tag{3.8}$$

To begin with, we show that the operator matrix of L is indeed diagonal. We break the subspace $(\mathcal{R}(A)+\mathcal{R}(B)) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$ into two subspaces: $\mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$ and $\mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$, and consider four possibilities for $x \in \mathcal{K}$.

Case 1: $x \in \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$. In this case $Tx = 0$, and so $Lx = (I-S)A^\dagger x = 0$. Hence $Lx = 0$.

Case 2: $x \in \mathcal{R}(A) \cap \mathcal{R}(B)$. Again $Tx = 0$, and $Lx = (I-S)A^\dagger x$. According to Proposition 2.2.2, we have that $A^\dagger x \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, and so $SA^\dagger x = 0$. Hence $Lx = A^\dagger x \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$.

Case 3: $x \in \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. Still $Tx = 0$, and once more $Lx = (I-S)A^\dagger x = A^\dagger x - SA^\dagger x$, where $A^\dagger x \in \mathcal{R}(A^*) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$ (Proposition 2.2.2). Now observe that $A^\dagger x - SA^\dagger x \in \mathcal{R}(A^*) + \mathcal{R}(B^*)$, but also $A^\dagger x - SA^\dagger x \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, since $SA^\dagger x \in \mathcal{N}(A)$. Hence $Lx \in (\mathcal{R}(A^*) + \mathcal{R}(B^*)) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$.

Case 4: $x \in \mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. In this case $Tx = x$ and $Lx = SB^\dagger x$. Since $SB^\dagger x \in \mathcal{R}(A^*) + \mathcal{R}(B^*)$ and in the same time $SB^\dagger x \in \mathcal{N}(A)$, making it orthogonal to $\mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, we get $Lx \in (\mathcal{R}(A^*) + \mathcal{R}(B^*)) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$.

By now we have proved that

$$A + B = \begin{bmatrix} C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} E & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with respect to the decomposition as in (i). First consider the operators D and F. If we take $x \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ then $Dx = (A+B)x = 2Ax$, and if we take $y \in \mathcal{R}(A) \cap \mathcal{R}(B)$, then $Fy = Ly = A^\dagger y$, as in Case 2 above. Thus $F = (\frac{1}{2}D)^{-1} = 2D^{-1}$. Now we consider the operators C and E. We have already proved that C is invertible and now we only need to show that $CE = I_{\mathcal{H}_1}$ where $\mathcal{H}_1 = (\mathcal{R}(A) + \mathcal{R}(B)) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$.

Case 1.1: $x \in \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. This is the same as Case 3 above, but we now need one more detail. We know that $Ex = Lx = (I-S)A^\dagger x \in \mathcal{N}(S) = \mathcal{N}(B) \oplus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$. Since any vector from $\mathcal{N}(B)$ is orthogonal to $\mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, as well as Lx, we conclude that the part of Lx from $\mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ is equal to 0, i.e. $Lx \in \mathcal{N}(B)$. Then $CEx = (A+B)(I-S)A^\dagger x = AA^\dagger x + B(I-S)A^\dagger x = x$, where we used that $AS = 0$.

Case 1.2: $x \in \mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. Similarly as in Case 4 we have $Ex = Lx = SB^\dagger x$. Now $CEx = (A+B)SB^\dagger x = BSB^\dagger x$, since $AS = 0$. Note that $B^\dagger x \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, and also $SB^\dagger x \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, because $SB^\dagger x \in \mathcal{N}(A)$. Thus $(I-S)B^\dagger x \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, but $(I-S)B^\dagger x \in \mathcal{N}(B) \oplus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$. Since the vectors from $\mathcal{N}(B)$ are also orthogonal on $\mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, it is not difficult to note that $(I-S)B^\dagger x \in \mathcal{N}(B)$. So $BSB^\dagger x = BB^\dagger x = x$.

In this way we have proved that $CE = I_{\mathcal{H}_1}$. Finally, we see that:

$$L = \begin{bmatrix} C^{-1} & 0 & 0 \\ 0 & 2D^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which completes the proof. □

The extended version of the Fill-Fishkind formula, from which Theorem 3.1.7 follows as a direct corollary, is given in the following theorem.

Theorem 3.2.4. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfy:*

(1) $\mathcal{R}(A)$ *and* $\mathcal{R}(B)$ *are closed;*

(2') A *and* B *are bi-precoherent;*

(3) $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B)$;

(4) $\mathcal{R}(A^*) + \mathcal{R}(B^*) = \mathcal{R}(A^* + B^*)$.

then

$$(A + B)^\dagger = (I - S)A^\dagger(I - T) + SB^\dagger T - A^\dagger X \tag{3.9}$$

with

$$S = (P_{\mathcal{N}(B)^\perp} P_{\mathcal{N}(A)})^\dagger, \quad T = (P_{\mathcal{N}(A^*)} P_{\mathcal{N}(B^*)^\perp})^\dagger, \quad X = \frac{1}{2} P_{\mathcal{R}(A)\cap\mathcal{R}(B)},$$

where all of the appearing Moore-Penrose inverses are bounded.

Proof. From Theorem 2.3.5 we have that A and B satisfy (3'). Using Theorem 3.2.3 and the operator L defined as before, we see that it holds:

$$(A + B)^\dagger = L - \begin{bmatrix} 0 & 0 & 0 \\ 0 & D^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} = L - M.$$

The operator M is in fact equal to $(A+B)^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$ but this is also equal to $\frac{1}{2}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$ since D^{-1} is the same as $\frac{1}{2}A^\dagger$ on $\mathcal{R}(A) \cap \mathcal{R}(B)$. □

Note also that, according to Theorems 2.3.5 and 2.3.6, conditions:

(1) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed;

(2') A and B are bi-precoherent;

(3) $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B)$;

(4) $\mathcal{R}(A^*) + \mathcal{R}(B^*) = \mathcal{R}(A^* + B^*)$.

are equivalent to conditions:

(2') A and B are bi-precoherent;

(3') $\mathcal{R}(A + B)$ is closed.

3.3 Extensions of Werner's formulas

In this section we study arbitrary reflexive inverse $(A+B)^{(1,2)}_{\mathcal{M},\mathcal{N}}$ where $\mathcal{M} \subseteq \mathcal{H}$ and $\mathcal{N} \subseteq \mathcal{K}$ are complements of $\mathcal{N}(A+B)$ and $\mathcal{R}(A+B)$ respectively. Under the same conditions as in the previous section, we will prove that every reflexive inverse $(A+B)^{(1,2)}_{\mathcal{M},\mathcal{N}}$ can be expressed as a sum involving outer inverses of A and B and a 'correction' similar to $A^\dagger X$ in formula (3.9), but adjusted according to $\mathcal{M}$ and $\mathcal{N}$. We will also explain connections with formula (3.9) and then specialize to the usual inverse of $A+B$. Theorem 2.3.6 and Proposition 2.2.2 are used throughout without a warning. We start by a technical lemma.

Lemma 3.3.1. *Let $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be bi-precoherent operators such that $\mathcal{R}(A+B)$ is closed. Then subspace $\mathcal{N}(A)+\mathcal{N}(B)$ is closed. Let $\mathcal{M} \subseteq \mathcal{H}$ and $\mathcal{N} \subseteq \mathcal{K}$ be closed subspaces such that $\mathcal{M} \oplus \mathcal{N}(A+B) = \mathcal{H}$ and $\mathcal{N} \oplus \mathcal{R}(A+B) = \mathcal{K}$, and denote*

$$\tilde{M} = \mathcal{M} \cap ((\mathcal{R}(A^*) \cap \mathcal{R}(B^*)) \oplus (\mathcal{N}(A) \cap \mathcal{N}(B))), \text{ and}$$

$$\tilde{N} = \mathcal{N} \oplus (\mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))) \oplus (\mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))).$$

1. *The following relations hold:*

$$\mathcal{N}(A) + \mathcal{N}(B) = (\mathcal{N}(A) \cap \mathcal{N}(B)) \oplus (\mathcal{M} \cap \mathcal{N}(A)) \oplus (\mathcal{M} \cap \mathcal{N}(B)), \tag{3.10}$$

$$\mathcal{M} \cap (\mathcal{N}(A) + \mathcal{N}(B)) = (\mathcal{M} \cap \mathcal{N}(A)) \oplus (\mathcal{M} \cap \mathcal{N}(B)), \tag{3.11}$$

$$\mathcal{M} = (\mathcal{M} \cap \mathcal{N}(A)) \oplus (\mathcal{M} \cap \mathcal{N}(B)) \oplus \tilde{M}, \tag{3.12}$$

$$\mathcal{H} = (\mathcal{N}(A) + \mathcal{N}(B)) \oplus \tilde{M}, \text{ and} \tag{3.13}$$

$$\mathcal{K} = (\mathcal{R}(A) \cap \mathcal{R}(B)) \oplus \tilde{N}. \tag{3.14}$$

   The subspaces $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$ are closed.

2. *Subspaces $\mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ and $\tilde{M}$ are isomorphic, and $P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}(\mathcal{R}(A^*) \cap \mathcal{R}(B^*)) = \tilde{M}$.*

3. *For every $x \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, $Ax = AP_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}x$.*

4. *$A(\mathcal{M} \cap \mathcal{N}(B)) = \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$, $A(\tilde{M}) = \mathcal{R}(A) \cap \mathcal{R}(B)$.*

Proof. The fact that $\mathcal{N}(A)+\mathcal{N}(B)$ is closed follows from Theorem 1.2.8.

1. Since $\mathcal{M}$ is a direct complement of $\mathcal{N}(A+B) = \mathcal{N}(A) \cap \mathcal{N}(B)$ in $\mathcal{H}$, then $\mathcal{M} \cap \mathcal{N}(A)$ is its direct complement in $\mathcal{N}(A)$ and $\mathcal{M} \cap \mathcal{N}(B)$ is its direct complement in $\mathcal{N}(B)$. Now relations (3.10) and (3.11) are derived easily. Regarding (3.12), note first that every $m \in \mathcal{M}$ can be written as a sum $n + r$ for $n \in \mathcal{N}(A) + \mathcal{N}(B)$ and $r \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, while n can be further decomposed as $n = n_1 + n_2 + n_3$, where $n_1 \in \mathcal{M} \cap \mathcal{N}(A)$, $n_2 \in \mathcal{M} \cap \mathcal{N}(B)$ and $n_3 \in \mathcal{N}(A) \cap \mathcal{N}(B)$. Thus $m = n_1 + n_2 + n_3 + r$, and so $n_3 + r$ are in the same time in $\mathcal{M}$ and $(\mathcal{R}(A^*) \cap \mathcal{R}(B^*)) \oplus (\mathcal{N}(A) \cap \mathcal{N}(B))$. This shows that $\mathcal{M} \subseteq ((\mathcal{M} \cap \mathcal{N}(A)) \oplus (\mathcal{M} \cap \mathcal{N}(B))) + \tilde{M}$, while the other inclusion is direct. The subspaces $(\mathcal{M} \cap \mathcal{N}(A)) \oplus (\mathcal{M} \cap \mathcal{N}(B))$ and $\tilde{M}$ are disjoint, which can be seen from

(3.10) and $(\mathcal{N}(A)+\mathcal{N}(B))\oplus(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))=\mathcal{H}$. Relation (3.13) follows from (3.10), (3.12) and $\mathcal{H}=(\mathcal{N}(A)\cap\mathcal{N}(B))\oplus\mathcal{M}$, while relation (3.14) is straightforward.

The subspace $(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))\oplus(\mathcal{N}(A)\cap\mathcal{N}(B))$ is closed as a sum of two closed orthogonal subspaces, so also is $\tilde{M}$. Since $\mathcal{K}$ is decomposed as the sum of four closed subspaces $\mathcal{R}(A)\cap\mathcal{R}(B)$, $\mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$, $\mathcal{R}(B)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$ and $\mathcal{N}$, from Lemma 1.2.1, we have that $\tilde{N}$ is closed.

2. Subspaces $\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$ and $\tilde{M}$ are isomorphic, both being direct complements of $\mathcal{N}(A)+\mathcal{N}(B)$ (they are both isomorphic to the quotient space $\mathcal{H}/(\mathcal{N}(A)+\mathcal{N}(B))$). From here we also have $P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))=\tilde{M}$.

3. Let $x\in\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$, and $x=m+n$, where $m\in\tilde{M}$ and $n\in\mathcal{N}(A)+\mathcal{N}(B)$, so that $P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}x=m$. We can further decompose m as $m=r+n_1$ where $r\in\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$ and $n_1\in\mathcal{N}(A)\cap\mathcal{N}(B)$. Then $x=r+n_1+n$, i.e. $x-r=n_1+n$. On the other hand, $x-r\perp n_1+n$, showing that $x=r$, and $n=-n_1$. Hence $Ax=A(m+n)=A(m-n_1)=Am=AP_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}x$.

4. The first equality follows like this: $\mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))=A(\mathcal{N}(A)+\mathcal{N}(B))$ $=A(\mathcal{N}(B))=A((\mathcal{N}(A)\cap\mathcal{N}(B))\oplus(\mathcal{M}\cap\mathcal{N}(B)))=A(\mathcal{M}\cap\mathcal{N}(B))$. The other follows from 2. and 3. $\square$

In the following theorem we prove the main result of this section.

Theorem 3.3.2. *Let $A,B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ be bi-precoherent operators such that $\mathcal{R}(A+B)$ is closed. Let $\mathcal{M}\subseteq\mathcal{H}$ and $\mathcal{N}\subseteq\mathcal{K}$ be closed subspaces such that $\mathcal{M}\oplus\mathcal{N}(A+B)=\mathcal{H}$ and $\mathcal{N}\oplus\mathcal{R}(A+B)=\mathcal{K}$. Then:*

1. *Outer inverses $\bar{A}=A^{(2)}_{\mathcal{M}\cap\mathcal{N}(B),\mathcal{N}\oplus\mathcal{R}(B)}$ and $\bar{B}=B^{(2)}_{\mathcal{M}\cap\mathcal{N}(A),\mathcal{N}\oplus\mathcal{R}(A)}$ exist.*

2. *If $\tilde{M}$ and $\tilde{N}$ are defined as in Lemma 3.3.1, then:*

$$(A+B)^{(1,2)}_{\mathcal{M},\mathcal{N}}=\bar{A}+\bar{B}+\frac{1}{2}P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B),\tilde{N}}. \tag{3.15}$$

Proof. 1. We should check conditions of Theorem 1.3.1, and we do so only for A, since for B everything follows similarly. Since $(\mathcal{R}(A)+\mathcal{R}(B))\oplus\mathcal{N}=\mathcal{H}$, then $(\mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B)))\oplus\mathcal{R}(B)\oplus\mathcal{N}=\mathcal{H}$ and from Lemma 1.2.1, $\mathcal{R}(B)\oplus\mathcal{N}$ is a closed subspace. From Lemma 3.3.1 it follows that $A(\mathcal{M}\cap\mathcal{N}(B))=\mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$ which is a closed subspace, and also $A(\mathcal{M}\cap\mathcal{N}(B))\oplus(\mathcal{R}(B)\oplus\mathcal{N})=\mathcal{K}$. Finally, $\mathcal{N}(A)\cap(\mathcal{M}\cap\mathcal{N}(B))=\{0\}$, showing that $\bar{A}$ exists.

2. Denote by $C=\frac{1}{2}P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B),\tilde{N}}$. Let us prove first that $\mathcal{R}(\bar{A}+\bar{B}+C)=\mathcal{M}$ and $\mathcal{N}(\bar{A}+\bar{B}+C)=\mathcal{N}$. If we denote by $\mathcal{H}_1=\mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$, $\mathcal{H}_2=\mathcal{R}(A)\cap\mathcal{R}(B)$ and $\mathcal{H}_3=\mathcal{R}(B)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$, we see that $\bar{A}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)=\bar{A}(\mathcal{H}_1)=\mathcal{M}\cap\mathcal{N}(B)$, $\bar{B}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)=\bar{B}(\mathcal{H}_3)=\mathcal{M}\cap\mathcal{N}(A)$ and $\bar{C}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)=P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))=\tilde{M}$, according to Lemma 3.3.1. This shows that $\mathcal{R}(\bar{A}+\bar{B}+C)=(\mathcal{M}\cap\mathcal{N}(B))\oplus(\mathcal{M}\cap\mathcal{N}(A))\oplus\tilde{M}=\mathcal{M}$, but also that $\mathcal{N}(\bar{A}+\bar{B}+C)=\mathcal{N}$, given that $\bar{A}$ is an injection on $\mathcal{H}_1$, $\bar{B}$ is an injection on $\mathcal{H}_3$ and C is an injection on $\mathcal{H}_2$.

We should now prove that $\bar{A}+\bar{B}+C$ is indeed a reflexive inverse of $A+B$. Note that:

$$(A+B)(\bar{A}+\bar{B}+C)=A\bar{A}+B\bar{B}+(A+B)C.$$

We have $\mathcal{R}(A\bar{A}) = A(\mathcal{R}(\bar{A})) = \mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$, according to Lemma 3.3.1. Thus $A\bar{A} = P_{\mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B)),\mathcal{R}(B)\oplus\mathcal{N}}$. Similarly $B\bar{B} = P_{\mathcal{R}(B)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B)),\mathcal{R}(A)\oplus\mathcal{N}}$. Since A and B coincide on $\tilde{M}$ we have $(A+B)C = AP_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B),\tilde{N}}$. From Lemma 3.3.1, statement 3. we conclude that this is equal to $P_{\mathcal{R}(A)\cap\mathcal{R}(B),\tilde{N}}$. It is now obvious that $(A+B)(\bar{A}+\bar{B}+C) = P_{\mathcal{R}(A+B),\mathcal{N}}$. On the other hand:

$$(\bar{A}+\bar{B}+C)(A+B) = \bar{A}A + \bar{B}B + C(A+B),$$

and since $\mathcal{R}(A)\cap\mathcal{N}(\bar{A}) = \mathcal{R}(A)\cap\mathcal{R}(B)$, we get $\mathcal{N}(\bar{A}A) = (\mathcal{R}(A^*)\cap\mathcal{R}(B^*))\oplus\mathcal{N}(A)$, showing $\bar{A}A = P_{\mathcal{M}\cap\mathcal{N}(B),(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))\oplus\mathcal{N}(A)}$. In the same way, $\bar{B}B = P_{\mathcal{M}\cap\mathcal{N}(A),(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))\oplus\mathcal{N}(B)}$. Note that for $x \in (\mathcal{R}(A^*)\cap\mathcal{R}(B^*))^\perp$ we have $C(A+B)x = 0$, so

$$\begin{aligned} C(A+B) &= C(A+B)P_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)} = 2CAP_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)} \\ &= P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B),\tilde{N}}P_{\mathcal{R}(A)\cap\mathcal{R}(B)}AP_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)} \\ &= P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)}AP_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)} \\ &= P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}P_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)} \\ &= P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}. \end{aligned}$$

This leads us to:

$$(\bar{A}+\bar{B}+C)(A+B) =$$

$$= P_{\mathcal{M}\cap\mathcal{N}(B),(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))\oplus\mathcal{N}(A)} + P_{\mathcal{M}\cap\mathcal{N}(A),(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))\oplus\mathcal{N}(B)} + P_{\tilde{M},\mathcal{N}(A)+\mathcal{N}(B)}$$

which is exactly equal to $P_{\mathcal{M},\mathcal{N}(A)\cap\mathcal{N}(B)}$, according to (3.12). To conclude, we proved that: $(A+B)(\bar{A}+\bar{B}+C) = P_{\mathcal{R}(A+B),\mathcal{N}(\bar{A}+\bar{B}+C)}$ and $(\bar{A}+\bar{B}+C)(A+B) = P_{\mathcal{R}(\bar{A}+\bar{B}+C,\mathcal{N}(A+B)}$, showing that $\bar{A}+\bar{B}+C$ is indeed the reflexive inverse of $A+B$ with the range $\mathcal{M}$ and the null-space $\mathcal{N}$. □

Observe that Theorem 3.3.2 generalizes Werner's result from Theorem 3.1.4, but generalized inverses of A and B appearing in Theorem 3.3.2 are not reflexive. By specializing subspaces $\mathcal{M}$ and $\mathcal{N}$ we can now obtain Moore-Penrose inverse of $A+B$ or the usual inverse, provided it exists.

Corollary 3.3.3. *Let $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be bi-precoherent operators such that $\mathcal{R}(A+B)$ is closed. Then the following generalized inverses exist:*

$$\bar{A} = A^{(2)}_{(\mathcal{R}(A^*)+\mathcal{R}(B^*))\cap\mathcal{N}(B),(\mathcal{N}(A^*)\cap\mathcal{N}(B^*))\oplus\mathcal{R}(B)},$$

$$\bar{B} = B^{(2)}_{(\mathcal{R}(A^*)+\mathcal{R}(B^*))\cap\mathcal{N}(A),(\mathcal{N}(A^*)\cap\mathcal{N}(B^*))\oplus\mathcal{R}(A)}$$

and:

$$(A+B)^\dagger = \bar{A}+\bar{B}+\frac{1}{2}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)}. \tag{3.16}$$

Proof. Directly from Theorem 3.3.2, since now $\mathcal{M} = \mathcal{R}(A^*)+\mathcal{R}(B^*)$, $\mathcal{N} = \mathcal{N}(A)\cap\mathcal{N}(B)$, $\tilde{M} = \mathcal{R}(A^*)\cap\mathcal{R}(B^*)$ and $\tilde{N} = (\mathcal{R}(A)\cap\mathcal{R}(B))^\perp$. □

Relations (3.16) and (3.9) both give a formula for computing $(A+B)^\dagger$. In a sense, these formulas are the same, just (3.9) gives an explicit way to calculate the outer inverses appearing in (3.16).

Proposition 3.3.4. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be bi-precoherent operators such that $\mathcal{R}(A+B)$ is closed. Then:*

$$A^{(2)}_{(\mathcal{R}(A^*)+\mathcal{R}(B^*))\cap\mathcal{N}(B),(\mathcal{N}(A^*)\cap\mathcal{N}(B^*))\oplus\mathcal{R}(B)} = (I-S)A^\dagger(I-T) - A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)},$$

$$B^{(2)}_{(\mathcal{R}(A^*)+\mathcal{R}(B^*))\cap\mathcal{N}(A),(\mathcal{N}(A^*)\cap\mathcal{N}(B^*))\oplus\mathcal{R}(A)} = SB^\dagger T,$$

where $S = (P_{\mathcal{N}(B)^\perp} P_{\mathcal{N}(A)})^\dagger$, $\quad T = (P_{\mathcal{N}(A^*)} P_{\mathcal{N}(B^*)^\perp})^\dagger$.

Proof. For convenience, let us rewrite relations (3.7) and (3.8).

$$S = Q((\mathcal{R}(A^*) + \mathcal{R}(B^*)) \cap \mathcal{N}(A), \mathcal{N}(B) \oplus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))),$$

$$T = Q(\mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B)), \mathcal{R}(A) \oplus (\mathcal{N}(A^*) \cap \mathcal{N}(B^*))).$$

Denote by $P = P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$. The following relations are straightforward: $AS = 0$, $TA = 0$, $PT = 0$, $TB(I-S) = 0$, $BB^\dagger T = T$, $SA^\dagger P = 0$. A simple calculation now shows that $X = (I-S)A^\dagger(I-T) - A^\dagger P$ is indeed an outer inverse of A, while $Y = SB^\dagger T$ is an outer inverse of B.

If $x \in \mathcal{R}(A)\cap\mathcal{R}(B)$ then $x = Px = (I-T)x$ and we easily get that $Xx = (I-S)A^\dagger x - A^\dagger x = 0$. If $x \in \mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$, then $Px = (I-T)x = 0$, and again $Xx = 0$. If $x \in \mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$, again $Px = 0$, and now $Xx = (I-S)A^\dagger x$, so if $Xx = 0$, then $A^\dagger x \in \mathcal{N}(A)$ which is possible only if $A^\dagger x = 0$, i.e. $x = 0$. If $x \in \mathcal{R}(B)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B))$, $Px = x$, and $Xx = 0$ as well. Finally, if $x \in \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$ we obtain $Xx = 0$ easily. These considerations show that $\mathcal{N}(X) = (\mathcal{R}(A) \cap \mathcal{R}(B)) \oplus (\mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))) \oplus (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) = \mathcal{R}(B) \oplus (\mathcal{N}(A^*) \cap \mathcal{N}(B^*))$, while $\mathcal{R}(X) = (I-S)A^\dagger(\mathcal{R}(A) \ominus (\mathcal{R}(A)\cap\mathcal{R}(B))) = (I-S)((\mathcal{R}(A^*)\ominus(\mathcal{R}(A^*)\cap\mathcal{R}(B^*))))$. To conclude, let us prove that this subspace is exactly $(\mathcal{R}(A^*) + \mathcal{R}(B^*)) \cap \mathcal{N}(B) = (\mathcal{N}(A) \cap \mathcal{N}(B))^\perp \cap \mathcal{N}(B)$.

If $x \in \mathcal{R}(A^*) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$, and $x = (n_B + r) + n_A$ where $n_B \in \mathcal{N}(B)$, $r \in (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$ and $n_A \in (\mathcal{R}(A^*) + \mathcal{R}(B^*)) \cap \mathcal{N}(B) = (\mathcal{N}(A) \cap \mathcal{N}(B))^\perp \cap \mathcal{N}(B)$, then since x, n_B and n_A are all orthogonal to r, we conclude that $r = 0$, but now since x and n_A are orthogonal to $\mathcal{N}(A) \cap \mathcal{N}(B)$, so is n_B. Hence $(I-S)x = n_B \in (\mathcal{N}(A) \cap \mathcal{N}(B))^\perp \cap \mathcal{N}(B)$. This shows $(I-S)((\mathcal{R}(A^*)\ominus(\mathcal{R}(A^*)\cap\mathcal{R}(B^*)))) \subseteq (\mathcal{N}(A)\cap\mathcal{N}(B))^\perp \cap \mathcal{N}(B)$. On the other hand, every $n_B \in (\mathcal{N}(A) \cap \mathcal{N}(B))^\perp \cap \mathcal{N}(B)$ can be decomposed as $n_B = r_1 + r_2 + n_A$ where $r_1 \in \mathcal{R}(A^*) \ominus (\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$, $r_2 \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ and $n_A \in \mathcal{N}(A)$. Then n_A is also orthogonal to $\mathcal{N}(A) \cap \mathcal{N}(B)$, and so $(I-S)r_1 = (I-S)n_B + (I-S)(-r_2 - n_A) = (I-S)n_B = n_B$. This proves the other inclusion as well.

Thus $\mathcal{R}(X) = (\mathcal{R}(A^*)+\mathcal{R}(B^*))\cap\mathcal{N}(B)$, which finishes the proof for X. The relations $\mathcal{N}(Y) = (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{R}(A)$ and $\mathcal{R}(Y) = (\mathcal{R}(A^*) + \mathcal{R}(B^*)) \cap \mathcal{N}(A)$ are proved in a similar but more direct fashion. $\square$

Observe that Proposition 3.3.4 and Theorem 3.3.2 offer an alternative way of proving Theorem 3.2.3. In the following theorem we study usual invertibility for bi-precoherent operators, and exhibit a simple criteria for invertibility of $A + B$.

Theorem 3.3.5. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that A and B are bi-precoherent, the following conditions are equivalent:*

(i) $A+B$ *is invertible.*

(ii) $\mathcal{R}(A)+\mathcal{R}(B)=\mathcal{K}$ *and* $\mathcal{R}(A^*)+\mathcal{R}(B^*)=\mathcal{H}$.

In that case, $\mathcal{R}(A)$ *and* $\mathcal{R}(B)$ *are closed, generalized inverses* $A^{(2)}_{\mathcal{N}(B),\mathcal{R}(B)}$ *and* $B^{(2)}_{\mathcal{N}(A),\mathcal{R}(A)}$ *exist, and:*

$$(A+B)^{-1}=A^{(2)}_{\mathcal{N}(B),\mathcal{R}(B)}+B^{(2)}_{\mathcal{N}(A),\mathcal{R}(A)}+\frac{1}{2}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)}, \tag{3.17}$$

Moreover $A^{(2)}_{\mathcal{N}(B),\mathcal{R}(B)}=(I-S)A^\dagger(I-T)-A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$ *and* $B^{(2)}_{\mathcal{N}(A),\mathcal{R}(A)}=SB^\dagger T$*, where* $S=(P_{\mathcal{N}(B)^\perp}P_{\mathcal{N}(A)})^\dagger$, $\quad T=(P_{\mathcal{N}(A^*)}P_{\mathcal{N}(B^*)^\perp})^\dagger$.

Proof. The equivalence of (i) and (ii) follows from Theorem 2.3.6, since both of the conditions (i) and (ii) imply range additivity of A and B.

The remaining part of the assertion follows from Corollary 3.3.3 and 3.3.4. □

Remark 3.3.6. Formula (3.17) could be derived more directly, if we weren't interested in explicit expressions giving generalized inverses appearing there. There are far less technical details now than in Theorem 3.3.2, and behaviour of the operators is easily controlled. Namely:

$$(A+B)(A^{(2)}_{\mathcal{N}(B),\mathcal{R}(B)}+B^{(2)}_{\mathcal{N}(A),\mathcal{R}(A)}+\frac{1}{2}A^\dagger P_{\mathcal{R}(A)\cap\mathcal{R}(B)})=$$

$$=AA^{(2)}_{\mathcal{N}(B),\mathcal{R}(B)}+BB^{(2)}_{\mathcal{N}(A),\mathcal{R}(A)}+P_{\mathcal{R}(A)\cap\mathcal{R}(B)}=I,$$

since $AA^{(2)}_{\mathcal{N}(B),\mathcal{R}(B)}=P_{\mathcal{R}(A)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B)),\mathcal{R}(B)}$, and $BB^{(2)}_{\mathcal{N}(A),\mathcal{R}(A)}=P_{\mathcal{R}(B)\ominus(\mathcal{R}(A)\cap\mathcal{R}(B)),\mathcal{R}(A)}$.

To conclude this section, we give some results regarding linear combinations of bi-precoherent operators $\lambda_1A+\lambda_2B$, for $\lambda_1,\lambda_2\in\mathbb{C}\setminus\{0\}$ and $\lambda_1+\lambda_2\neq 0$. We will see that these results resemble the well-known properties of idempotent operators (cf. [35, 36]).

Lemma 3.3.7. *If* $A,B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ *are such that* A *and* B *are bi-precoherent, and* $\lambda\in\mathbb{C}\setminus\{-1\}$*, then* $\mathcal{N}(A+\lambda B)=\mathcal{N}(A)\cap\mathcal{N}(\lambda B)$*. Moreover,* $\mathcal{N}(A-B)=(\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)})\oplus(\mathcal{N}(A)\cap\mathcal{N}(B))$.

Proof. If for $x\in\mathcal{H}$ we have $(A+\lambda B)x=0$, then $Ax=B(-\lambda x)=\beta\in\mathcal{R}(A)\cap\mathcal{R}(B)$. So there exists $\alpha\in\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)}$ such that $A\alpha=B\alpha=\beta$, and so $A(x-\alpha)=0=B(\alpha+\lambda x)$. From here we see that $(1+\lambda)x=(x-\alpha)+(\alpha+\lambda x)\in\mathcal{N}(A)+\mathcal{N}(B)$, and so $x\in\mathcal{N}(A)+\mathcal{N}(B)$. But then $Ax\in\mathcal{R}(A)\ominus(\overline{\mathcal{R}(A)}\cap\overline{\mathcal{R}(B)})$ and $B(-\lambda x)\in\mathcal{R}(B)\ominus(\overline{\mathcal{R}(A)}\cap\overline{\mathcal{R}(B)})$, showing that $Ax=0$ and $B(-\lambda x)=0$. Hence, $\mathcal{N}(A+\lambda B)\subseteq\mathcal{N}(A)\cap\mathcal{N}(\lambda B)$, but the other inclusion is trivial, so the equality holds.

The other equality follows similarly. □

Theorem 3.3.8. *If* $A,B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ *are such that* A *and* B *are bi-precoherent, and* $\lambda_1,\lambda_2\in\mathbb{C}\setminus\{0\}$, $\lambda_1+\lambda_2\neq 0$*, then* $\overline{\mathcal{R}(A+B)}=\overline{\mathcal{R}(\lambda_1A+\lambda_2B)}$*. Moreover, the following conditions are equivalent:*

(i) $\mathcal{R}(A+B)$ *is closed;*

(ii) $\mathcal{R}(\lambda_1 A + \lambda_2 B)$ *is closed, for every* $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$*,* $\lambda_1 + \lambda_2 \neq 0$*;*

(iii) $\mathcal{R}(\lambda_1 A + \lambda_2 B)$ *is closed, for some* $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$*,* $\lambda_1 + \lambda_2 \neq 0$*.*

Proof. The first statement of the theorem follows from Lemma 3.3.7 with A^* and B^* in place of A and B.

We prove the second part.

(i) $\Rightarrow$ (ii) From Theorem 2.3.6 we have that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and so is $\mathcal{R}(A) + \mathcal{R}(B)$, and $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B)$. The same holds for A^* and B^*.

For the sake of convenience, we will prove that $\mathcal{R}(A + cB)$ is closed, for arbitrary $c \notin \{0, -1\}$, from where the general case follows. We will show in fact that $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B) \subseteq \mathcal{R}(A + cB)$, so $\mathcal{R}(A + B) = \mathcal{R}(A + cB)$, since the other inclusion is trivial.

Take arbitrary $y \in \mathcal{R}(A + B) = \mathcal{R}(A) + \mathcal{R}(B)$ and let $y = y_1 + y_2 + y_3$, where $y_1 \in \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$, $y_2 \in \mathcal{R}(A) \cap \mathcal{R}(B)$ and $y_3 \in \mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. Between Hilbert spaces $\mathcal{R}(A^* + B^*)$ and $\mathcal{R}(A + B)$, operator $A + B$ is an isomorphism which maps $\mathcal{R}(A^*) \cap \mathcal{R}(B^*)$ bijectively to $\mathcal{R}(A) \cap \mathcal{R}(B)$ and $(\mathcal{R}(A^*) \cap \mathcal{R}(B^*))^\perp$ to $(\mathcal{R}(A) \cap \mathcal{R}(B))^\perp$ (Theorem 3.2.3, Statement 1.). Hence if we denote by $x' = \frac{2}{(1+c)}(A + B)^\dagger y_2$ and $x'' = (A + B)^\dagger (y_1 + \frac{1}{c} y_3)$, then $x' \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, and $x'' \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$. We have now $(A + cB)x' = (1 + c)Ax' = \frac{1+c}{2}(A + B)x' = y_2$. Also, $Ax'' + Bx'' = y_1 + \frac{1}{c} y_3$, i.e. $Ax'' - y_1 = \frac{1}{c} y_3 - Bx''$, but $x'' \perp \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, so $Ax'' - y_1 \in \mathcal{R}(A) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$, while $\frac{1}{c} y_3 - Bx'' \in \mathcal{R}(B) \ominus (\mathcal{R}(A) \cap \mathcal{R}(B))$. These subspaces have only 0 in common, so $Ax'' = y_1$ and $Bx'' = \frac{1}{c} y_3$, showing that $(A + cB)x'' = y_1 + y_3$. Finally, this shows $(A + cB)(x' + x'') = y$. Hence $\mathcal{R}(A + cB) = \mathcal{R}(A + B)$ is closed.

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (i) From Lemma 3.3.7 it follows that $\mathcal{N}(\lambda_1 A^* + \lambda_2 B^*) = \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$, so considering their orthogonal complements, we have that $\overline{\mathcal{R}(\lambda_1 A + \lambda_2 B)} = \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$, but $\mathcal{R}(\lambda_1 A + \lambda_2 B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$. Thus $\mathcal{R}(A) + \mathcal{R}(B) = \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$ showing that $\mathcal{R}(A) + \mathcal{R}(B)$ is closed. The same holds for $\mathcal{R}(A^*) + \mathcal{R}(B^*)$, and so the implication follows from Theorem 2.3.6. $\square$

Corollary 3.3.9. *If* $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *are such that* A *and* B *are bi-precoherent, the following conditions are equivalent:*

(i) $A + B$ *is invertible;*

(ii) $\lambda_1 A + \lambda_2 B$ *is invertible for every* $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$*,* $\lambda_1 + \lambda_2 \neq 0$*;*

(iii) $\lambda_1 A + \lambda_2 B$ *is invertible for some* $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$*,* $\lambda_1 + \lambda_2 \neq 0$*.*

In that case, for every $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$*,* $\lambda_1 + \lambda_2 \neq 0$*, we have:*

$$(\lambda_1 A + \lambda_2 B)^{-1} = \frac{1}{\lambda_1} A^{(2)}_{\mathcal{N}(B), \mathcal{R}(B)} + \frac{1}{\lambda_2} B^{(2)}_{\mathcal{N}(A), \mathcal{R}(A)} + \frac{1}{\lambda_1 + \lambda_2} A^\dagger P_{\mathcal{R}(A) \cap \mathcal{R}(B)}.$$

Proof. The equivalence of conditions (i)-(iii) follows from Theorems 3.3.5 and 3.3.8. The second part of the assertion follows by direct verification, similarly as in Remark 3.3.6. $\square$

Chapter 4

Partial orders on Hilbert space operators

A specific class of partial orders that will be described in this chapter, and which have been studied in detail during the past forty years, have much in common with the notion of coherent operators. This becomes obvious in the study of the lattice properties of these partial orders. We will give new, or different but improved, solutions to some standing problems in this area, regarding the star partial order, and recently introduced, core partial order. Results and the whole spirit of Chapter 2 plays a crucial role in this chapter.

4.1 Definitions of different partial orders: old and new

The study of partial orders started with papers of Drazin [34] (introducing the star order), Hartwig [51] and Nambooripad [71] (introducing the minus order) and Mitra [67] (introducing the sharp order). Although the star and minus orders were defined on structures more general than the algebra of bounded operators, the definitions were closely related to the notion of generalized inverses, which originates from linear algebra and operator theory. It was natural then that this theory developed mainly as a part of linear algebra, but with considerable results in general rings, rings with involution, Rickart rings, etc.

We begin with the definition of the star order which was introduced the first. We state this definition verbatim as the definition Drazin gave in arbitrary semigroups with proper involution, only we do it for Hilbert space operators. In this section, $\mathcal{H}$ and $\mathcal{K}$ stand for arbitrary Hilbert spaces.

Definition 4.1.1. The $\star$-partial order on $\mathcal{B}(\mathcal{H},\mathcal{K})$, denoted by $\overset{\star}{\leq}$, is defined as:

$$A \overset{\star}{\leq} B \quad \Leftrightarrow \quad AA^* = BA^* \quad \text{and} \quad A^*A = A^*B. \tag{4.1}$$

There are many different ways to reformulate this definition. One reformulation that includes generalized inverses and which was emphasized in the original paper [34], states that:

$$A \overset{\star}{\leq} B \quad \Leftrightarrow \quad AA^\dagger = BA^\dagger \quad \text{and} \quad A^\dagger A = A^\dagger B, \tag{4.2}$$

with an assumption that A has closed range. In what it seems to be the first detailed study of $\star$-order on $\mathcal{B}(\mathcal{H})$, Antezana et. al. [5] give the following characterization:

$$A \overset{\star}{\leq} B \quad \Leftrightarrow \quad A = PB = BQ \quad \text{for some orthogonal projections } P \text{ and } Q, \tag{4.3}$$

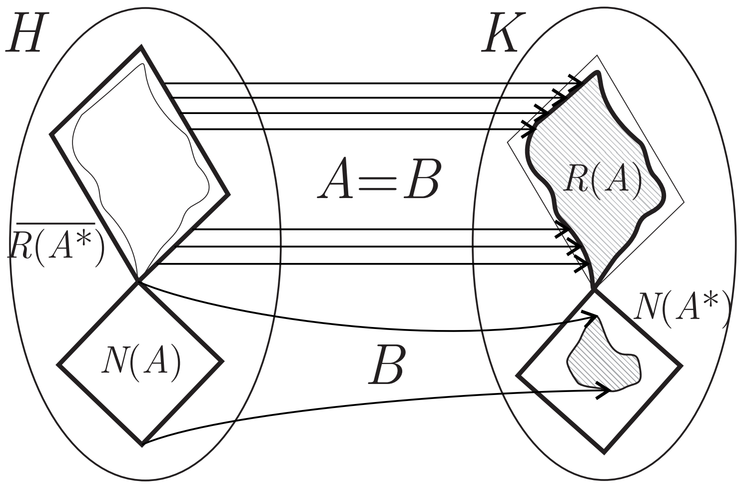
while Dolinar and Marovt [32] for example also offer:

$$A \overset{\star}{\leq} B \quad \Leftrightarrow \quad \overline{\mathcal{R}(A)} \perp \overline{\mathcal{R}(B-A)} \quad \text{and} \quad \overline{\mathcal{R}(A^*)} \perp \overline{\mathcal{R}(B^*-A^*)}. \tag{4.4}$$

However, in the spirit of the subject of the thesis, most convenient reformulation of the definition is:

$$A \overset{\star}{\leq} B \quad \Leftrightarrow \quad A \text{ and } B \text{ coincide on } \overline{\mathcal{R}(A^*)} \text{ and } B(\mathcal{N}(\mathcal{A})) \subseteq \mathcal{N}(A^*), \tag{4.5}$$

illustrated on the figure bellow.



The equivalences between these definitions are all proved readily. This partial order imposes a nice structure on $\mathcal{B}(\mathcal{H},\mathcal{K})$, which will be apparent when we say more about its lattice properties. It should be mentioned that Gudder in [45] introduced this partial order in the set of self-adjoint operators independently of the existing study originating from Drazin's paper. The main argument was that its structural properties are more convenient than the ones of the Löwner order.

Let us now describe the minus order. In his paper from 1980 Hartwig defined, what he called, the plus partial order in any semigroup, which will later be renamed to minus partial order[1]. The idea was to generalize the usual order for idempotents $e \leq f \Leftrightarrow$

[1]It is interesting that the name 'plus' was chosen since the reflexive inverse of an element a, which appears in the definition, was denoted by a^+. It was renamed to 'minus' partial order in [54] after the realization that the reflexive inverse can be changed by any inner inverse, commonly denoted with the minus in the superscript: a^-, and in order to avoid confusion since a^+ often denotes the Moore-Penrose inverse, which is used in the star order. The fact that this order is the same as the rank subtractivity: $\mathrm{r}(B-A) = \mathrm{r}(B) - \mathrm{r}(A)$ in the set of rectangular matrices further justified this renaming.

$ef = fe = e$, in the set of all von Neumann regular elements. So unlike the order Drazin defined, this was defined only for pairs (a, b) where a has an inner inverse.

Definition 4.1.2. If S is a semigroup and $a, b \in S$ such that a has an inner inverse, then we define $a \overset{-}{\leq} b$ if $aa^- = ba^-$ and $a^-a = a^-b$ for some inner inverse a^- of a.

The original definition used a reflexive inverse instead, but it is of course equivalent to the one stated here. It is obvious that we can not use this definition verbatim if we wish to define the minus order as the partial order on the set $\mathcal{B}(\mathcal{H}, \mathcal{K})$. However, in the set of rectangular matrices, this definition is used as stated, giving rise to a fruitful theory. We gather some of its properties in the following theorem. The connection with the topics presented in the previous chapters are obvious.

Theorem 4.1.3. *If $A, B \in \mathbb{C}^{m\times n}$, the following statements are equivalent:*

(i) $AA^- = BA^-$ *and* $A^-A = A^-B$ *for some* $A^- \in A\{1\}$ *(i.e.* $A \overset{-}{\leq} B$*);*

(ii) $\mathrm{r}(B - A) = \mathrm{r}(B) - \mathrm{r}(A)$*;*

(iii) $\mathcal{R}(B) = \mathcal{R}(B - A) \oplus \mathcal{R}(A)$*;*

(iv) $\mathcal{R}(B^*) = \mathcal{R}(B^* - A^*) \oplus \mathcal{R}(A^*)$*;*

(v) $B\{1\} \subseteq A\{1\}$*.*

The proofs of these equivalences can be found in a very comprehensive reference [69] by Mitra. That being said, let us note that probably Mitra is the one who contributed to the study of matrix partial orders the most. He also gave the unified theory of matrix partial orders in [68], of which we are not going to give more details here, except to note that a similar unified theory on rings was given in [76].

Returning to the minus partial order, it seems that the first generalization to the set $\mathcal{B}(\mathcal{H}, \mathcal{K})$ was given by Antezana, Corach and Stojanoff in [6]. Their definition was the following (recall the notation c_0 given in Definition 1.2.5).

Definition 4.1.4. If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we define $A \overset{-}{\leq} B$ if $c_0(\overline{\mathcal{R}(A)}, \overline{\mathcal{R}(B - A)}) < 1$ and $c_0(\overline{\mathcal{R}(A^*)}, \overline{\mathcal{R}(B^* - A^*)}) < 1$.

Having in mind Theorem 1.2.7 we see that this definition is in fact:

$$A \overset{-}{\leq} B \quad \Leftrightarrow \quad \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B - A)} \text{ and } \overline{\mathcal{R}(B^*)} = \overline{\mathcal{R}(A^*)} \oplus \overline{\mathcal{R}(B^* - A^*)}. \tag{4.6}$$

A different look at this definition is provided by the following proposition.

Proposition 4.1.5 (See [6])**.** *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $A \overset{-}{\leq} B$ if and only if there exist projections $P \in \mathcal{B}(\mathcal{K})$ and $Q \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{R}(P) = \overline{\mathcal{R}(A)}$, $\mathcal{N}(Q) = \mathcal{N}(A)$ and $A = PB = BQ$.*

It is not difficult to see that for closed range operators A:

$$A \overset{-}{\leq} B \quad \Leftrightarrow \quad AA^- = BA^- \text{ and } A^-A = A^-B \text{ for some } A^- \in A\{1\}.$$

From Proposition 4.1.5 it is obvious how we can reformulate the definition of the minus order in a fashion suitable for us:

$$\begin{aligned} A \overset{-}{\leq} B \quad \Leftrightarrow \quad & \text{there exist closed subspaces } \mathcal{M} \subseteq \mathcal{H} \text{ and } \mathcal{N} \subseteq \mathcal{K} \text{ such that:} \\ & \mathcal{H} = \mathcal{N}(A) \oplus \mathcal{M} \text{ and } \mathcal{K} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}, \\ & \text{and } A \text{ and } B \text{ coincide on } \mathcal{M}, \text{ while } B(\mathcal{N}(A)) \subseteq \mathcal{N}. \end{aligned} \tag{4.7}$$

It is interesting that in (4.6) there is no need to take closures of ranges. The minus partial order can be defined with an ordinary range additivity exactly as in the matrix case. This is the content of the following theorem, communicated to us by A. Maestripieri.

Theorem 4.1.6 (A. Maestripieri)**.** *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ then $A \overset{-}{\leq} B$ if and only if:*

$$\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B - A) \quad \text{and} \quad \mathcal{R}(B^*) = \mathcal{R}(A^*) \oplus \mathcal{R}(B^* - A^*). \tag{4.8}$$

Proof. Suppose that $A \overset{-}{\leq} B$. By (4.6) we see that $\overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B-A)}$ and $\overline{\mathcal{R}(A^*)} \oplus \overline{\mathcal{R}(B^*-A^*)}$ are closed, so according to Proposition 1.4.3 we have the range additivity, and it is also direct. This proves (4.8).

Now suppose that (4.8) holds. Then $\overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A) \oplus \mathcal{R}(B-A)} = \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B-A)}}$, and similarly for A^* and B^*. Proposition 1.4.3 and (4.8) show that $\overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B-A)}$ and $\overline{\mathcal{R}(A^*)} \oplus \overline{\mathcal{R}(B^*-A^*)}$ are closed. We now directly obtain (4.6). $\square$

Note also the following equivalence.

Lemma 4.1.7. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the following conditions are equivalent:*

(i) $\mathcal{R}(A) \oplus \mathcal{R}(B-A) = \mathcal{R}(B)$ *and* $\mathcal{R}(A^*) \oplus \mathcal{R}(B^*-A^*) = \mathcal{R}(B^*)$

(ii) $\mathcal{R}(A) \oplus \mathcal{R}(B-A) = \mathcal{R}(B)$ *and* $\overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B-A)} = \overline{\mathcal{R}(B)}$.

Proof. (i) $\Rightarrow$ (ii) From Theorem 4.1.6 we have $A \overset{-}{\leq} B$, and now (ii) follows from (4.6).

(ii) $\Rightarrow$ (i) Since $\overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B-A)}$ is closed, from Proposition 1.4.3 we have that $\mathcal{R}(A^*) + \mathcal{R}(B^*-A^*) = \mathcal{R}(B^*)$. From the same proposition and $\mathcal{R}(A) \oplus \mathcal{R}(B-A) = \mathcal{R}(B)$ we have that $\mathcal{R}(A^*) \cap \mathcal{R}(B^*-A^*) = \{0\}$, thus $\mathcal{R}(A^*) \oplus \mathcal{R}(B^*-A^*) = \mathcal{R}(B^*)$. $\square$

We should note that in an independent study, Šemrl [81] introduced the minus partial order on $\mathcal{B}(\mathcal{H})$ taking the statement of Proposition 4.1.5 as a definition, and equalities from (4.6) were derived as another characterization of this order. Building up to such a definition, he tentatively proposes three range relations to define $A \overset{-}{\leq} B$:

$$\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B-A); \tag{4.9}$$

$$\overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B-A)}; \tag{4.10}$$

$$\overline{\mathcal{R}(B)} = \overline{\overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B-A)}}. \tag{4.11}$$

From Lemma 4.1.7 we see that the proposed definition of the minus partial order is the conjunction of (4.9) and (4.10). This is exactly the definition of quasidirect additivity of A and $B-A$, as given by Lešnjak and Šemrl in [62].

We note one more interesting fact. The same partial order is defined by taking (4.9) together with such relation for adjoints, and by taking (4.10) together with such relation for adjoints. Naturally we can ask what happens if we take (4.11) with the relation for adjoints as well. Such a discussion originated from [39], with the following conclusion.

Proposition 4.1.8. *For $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ define $A \prec B$ if $\overline{\mathcal{R}(B)} = \overline{\overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(B-A)}}$ and $\overline{\mathcal{R}(B^*)} = \overline{\overline{\mathcal{R}(A^*)} \oplus \overline{\mathcal{R}(B^*-A^*)}}$. Then $A \prec B$ if and only if $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B-A)} = \{0\}$ and $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*-A^*)} = \{0\}$, relation $\prec$ is reflexive and antisymmetric, but it is not transitive in general.*

Proof. The first part of the assertion follows from Lemma 2.3.4. It is obvious that this relation is reflexive. To see that it is antisymmetric, suppose that $A \prec B$ and $B \prec A$. From $B \prec A$ we have $\overline{\mathcal{R}(B-A)} \subseteq \overline{\mathcal{R}(A)}$, and from $A \prec B$ we have $\overline{\mathcal{R}(B-A)} \cap \overline{\mathcal{R}(A)} = \{0\}$. Hence $\mathcal{R}(B-A) = \{0\}$, i.e. $A = B$. Finally, to show that it is not transitive, we provide a counterexample.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{N}$ and $\mathcal{L}$ two closed subspaces, such that $\mathcal{N} \oplus \mathcal{L} \neq \overline{\mathcal{N} \oplus \mathcal{L}}$. Pick arbitrary $x \in \overline{\mathcal{N} \oplus \mathcal{L}} \setminus (\mathcal{N} \oplus \mathcal{L})$ and let $\mathcal{M}$ be the one-dimensional subspace spanned by x. The sum of subspaces $\mathcal{M}, \mathcal{N}$ and $\mathcal{L}$ is direct, but it is not closed: $\mathcal{M} \oplus \mathcal{N} \oplus \mathcal{L} \neq \overline{\mathcal{M} \oplus \mathcal{N} \oplus \mathcal{L}} = \overline{\mathcal{N} \oplus \mathcal{L}}$. Finally, let $A = P_{\mathcal{M}}$, $B = P_{\mathcal{M}} + P_{\mathcal{N}}$ and $C = P_{\mathcal{M}} + P_{\mathcal{N}} + P_{\mathcal{L}}$. We have $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B-A)} = \mathcal{M} \cap \mathcal{N} = \{0\}$, so $A \prec B$ ($A^* = A$ and $B^* = B$). Also $\overline{\mathcal{R}(B)} \cap \overline{\mathcal{R}(C-B)} = \overline{\mathcal{M} \oplus \mathcal{N}} \cap \mathcal{L} = (\mathcal{M} \oplus \mathcal{N}) \cap \mathcal{L} = \{0\}$, since $\overline{\mathcal{M} \oplus \mathcal{N}} = \mathcal{M} \oplus \mathcal{N}$. Thus $B \prec C$. However, $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(C-A)} = \mathcal{M} \cap \overline{\mathcal{R}(P_{\mathcal{N}} + P_{\mathcal{L}})} = \mathcal{M} \cap \overline{\mathcal{N} \oplus \mathcal{L}} = \mathcal{M} \neq \{0\}$, so $A \not\prec C$. $\square$

By strengthening some conditions in the definitions of minus and star partial order, one obtain different partial orders. We are going to present here two more orders important for our study: sharp and core order, both being interesting due to the fact that they are defined for group invertible elements. An interested reader can find more information on partial orders in [69] and [78].

As we mentioned before, sharp order was defined by Mitra in [67], by changing the inner inverse appearing in the definition of the minus order by the group inverse. The same definition was used on arbitrary Hilbert spaces.

Definition 4.1.9. Let $A, B \in \mathcal{B}(\mathcal{H})$, such that $A \in \mathcal{B}^1(\mathcal{H})$. We define $A \leq^\sharp B$ if $AA^\sharp = BA^\sharp$ and $A^\sharp A = A^\sharp B$.

Obviously the pair (A, B) belongs to this relation only if $A \in \mathcal{B}^1(\mathcal{H})$, while B do not have to be from $\mathcal{B}^1(\mathcal{H})$. However, since we wish for $\leq^\sharp$ to be a partial order relation, i.e. to be reflexive, we can accomplish this by adjoining to $\leq^\sharp$ all the pairs (B, B), for every $B \in \mathcal{B}(\mathcal{H})$. More naturally, we can restrict our considerations only on $\mathcal{B}^1(\mathcal{H})$, where this relation is a partial order without any additional conditions. This is the case in the most of the existing literature on this subject.

Clearly, 'our' definition would be:

$$A \leq^{\sharp} B \quad \Leftrightarrow \quad A \text{ and } B \text{ coincide on } \mathcal{R}(A) \text{ and } B(\mathcal{N}(\mathcal{A})) \subseteq \mathcal{N}(A).$$

In the end, let us introduce a recently defined partial order called the core partial order, which is based on the core generalized inverse, described in Section 1.3. This partial order was introduced by Baksalary and Trenkler in [13] in the matrix setting, by Rakić, Djordjević and Dinčić in [75] and independently by Jose and Sivakumar [57] in Hilbert space setting, and on general rings with involution by Rakić and Djordjević in [77]. We give the definition on $\mathcal{B}(\mathcal{H})$.

Definition 4.1.10. For $A, B \in \mathcal{B}(\mathcal{H})$, such that $A \in \mathcal{B}^1(\mathcal{H})$, the relation $\leq^{\textcircled{\#}}$ is defined as:

$$A \leq^{\textcircled{\#}} B \quad \Leftrightarrow \quad A^{\textcircled{\#}}A = A^{\textcircled{\#}}B \quad \text{and} \quad AA^{\textcircled{\#}} = BA^{\textcircled{\#}}.$$

The same remark should be made here, as the one regarding the $\sharp$-order: we consider this relation only when both A and B are from $\mathcal{B}^1(\mathcal{H})$.

We do not need to compute any generalized inverses in order to check whether $A \leq^{\textcircled{\#}} B$, since [75, Eq. (26)] gives:

$$A \leq^{\textcircled{\#}} B \quad \Leftrightarrow \quad A^*A = A^*B \quad \text{and} \quad A^2 = BA. \tag{4.12}$$

It is convenient to state the following properties of the $\textcircled{\#}$-partial order in form of lemmas, for the later reference, the first one being our reformulation of the definition, as before. We only include the proof of Lemma 4.1.13, which seems to be scattered through the existing literature. The proofs of the other two lemmas are easily derived from the definition.

Lemma 4.1.11. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$. Then*

$$A \leq^{\textcircled{\#}} B \quad \Leftrightarrow \quad A \textit{ and } B \textit{ coincide on } \mathcal{R}(A) \textit{ and } B(\mathcal{N}(\mathcal{A})) \subseteq \mathcal{N}(A^*).$$

Moreover, if $A \leq^{\textcircled{\#}} B$, then A^* and B^* coincide on $\mathcal{R}(A)$.

Lemma 4.1.12. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$ be such that $A \leq^{\textcircled{\#}} B$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(A) \supseteq \mathcal{N}(B)$. Moreover, $A = B$ if and only if $\mathcal{R}(A) = \mathcal{R}(B)$ if and only if $\mathcal{N}(A) = \mathcal{N}(B)$.*

Lemma 4.1.13. *If $B \in \mathcal{B}(\mathcal{H})$ is a projection and $A \in \mathcal{B}^1(\mathcal{H})$ is such that $A \leq^{\textcircled{\#}} B$, then A is a projection. Moreover, if B is an orthogonal projection, so is A.*

Proof. Since the $\textcircled{\#}$-partial order induces the minus-partial order, the first statement is contained in [6, Corollary 4.14]. For the second statement, it is enough to show that $A \leq^{\textcircled{\#}} I$ if and only if A is an orthogonal projection. This can be directly obtained from the definition. □

We mention that, together with the $\textcircled{\#}$-inverse and $\textcircled{\#}$-partial order, one could consider the dual $\textcircled{\#}$-inverse and dual $\textcircled{\#}$-partial order (see [75]). Namely, if $A \in \mathcal{B}^1(\mathcal{H})$, the operator $A_{\textcircled{\#}}$ defined as $(A|_{\mathcal{R}(A^*),\mathcal{R}(A)})^{-1}$ on $\mathcal{R}(A)$ and as the null-operator on $\mathcal{N}(A)$ is called the dual $\textcircled{\#}$-inverse of A. The dual $\textcircled{\#}$-partial order is defined as:

$$A \le_{\textcircled{\#}} B \quad \Leftrightarrow \quad A_{\textcircled{\#}}A = A_{\textcircled{\#}}B \quad \text{and} \quad AA_{\textcircled{\#}} = BA_{\textcircled{\#}}.$$

Similarly to (4.12) we can obtain (see also [13, p. 693]):

$$A \le_{\textcircled{\#}} B \quad \Leftrightarrow \quad AA^* = BA^* \quad \text{and} \quad A^2 = AB. \tag{4.13}$$

Lemma 4.1.14. *If $A, B \in \mathcal{B}^1(\mathcal{H})$, then*

$$A \le^{\textcircled{\#}} B \quad \Leftrightarrow \quad A^* \le_{\textcircled{\#}} B^*.$$

Proof. Directly from (4.12) and (4.13). □

The previous lemma could also be derived from the fact that if $A \in \mathcal{B}^1(\mathcal{H})$ then the dual $\textcircled{\#}$-inverse of A is $A_{\textcircled{\#}} = ((A^*)^{\textcircled{\#}})^*$ (see [75, Theorem 3.4 and Theorem 6.1]). Therefore we focus our study only on a 'regular' $\textcircled{\#}$-partial order.

4.2 Lattice properties of the star and minus order

Studying lattice properties of partial orders is only one possible direction in which this theory can be developed. However, this direction was the most interesting to us. In this section we will present some results on this subject from the existing literature regarding the $\star$-order and the minus-order, and our results are divided in the following three sections. We will see from this section that the structure imposed by the $\star$-order on the set of operators is more strict than the 'rather loose structure of the minus partial order' [51].

We begin with the $\star$-order. It is rather easy to notice that $\mathcal{B}(\mathcal{H})$ with the $\star$-partial order is not an upper semi-lattice, i.e. there are $A, B \in \mathcal{B}(\mathcal{H})$ for which the $\star$-supremum: $A \overset{\star}{\vee} B$ does not exist. For example, for $A \neq B$ which are invertible, there is no common $\star$-upper bound at all. In fact, using (4.5) we see that the maximal elements in this partial order are those and exactly those A for which $\mathcal{N}(A) = \{0\}$ or $\mathcal{N}(A^*) = \{0\}$. The question is, which operators A and B have the $\star$-supremum.

On the other hand, $\mathcal{B}(\mathcal{H})$ is a lower semi-lattice: for every A and B the $\star$-infimum $A \overset{\star}{\wedge} B$ exists, and this was noticed in the first study on the subject given by Hartwig and Drazin [52], although for matrices. They proved that for any $A, B \in \mathbb{C}^{m\times n}$ the infimum $A \overset{\star}{\wedge} B$ exists, but they propose a problem of describing all matrices for which the $\star$-infimum attains, in a way, a maximal possible value. We will describe this problem precisely in the following section and see a direct relationship with the notion of precoherence. However, in [52], the $\star$-supremum was considered only for some special kind of matrices, and there was no characterization of matrices for which $\star$-supremum exists. In [50] Hartwig studies the $\star$-supremum in a very general setting and gives the first result along these lines. In order to present his result, we first recall some notions for general rings with involution. For an element a of a ring R with involution, we say that a is star regular if the system of equations: $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$ has a solution, in which case that solution is necessarily unique. Of course, this solution is called the Moore-Penrose inverse of a and it is denoted by $a^\dagger$. Hartwig proved the following theorem.

Theorem 4.2.1 (See [50]). *Let R be a ring with involution and $a, b \in R$ such that $a, b, (1 - aa^\dagger)b$ and $b(1 - a^\dagger a)$ are all star regular. Then a and b have a common $\star$-upper bound if and only if the following hold:*

1. $b(b^* - a^*)a = 0 = a(b^* - a^*)b;$
2. $b(b^* - a^*) \in b(1 - a^\dagger a)R;$
3. $(b^* - a^*)b \in R(1 - aa^\dagger)b.$

In that case, $a \overset{\star}{\vee} b$ exists and:

$$a \overset{\star}{\vee} b = a + (1 - aa^\dagger)bb^*[(1 - a^\dagger a)b]^\dagger. \tag{4.14}$$

This theorem proves that $\mathbb{C}^{m\times n}$ has the so-called upper bound property: the existence of one common $\star$-upper bound for A and B assures the existence of the supremum $A \overset{\star}{\vee} B$. The condition 1. from the theorem above is in fact a trivial necessary condition (see (2.2)). We will see in the following section that, in the case of matrices (and in some more general cases), the condition 1. is also a sufficient condition for the existence of $A \overset{\star}{\vee} B$.

Mitra also contributed to the study of lattice properties, especially for the minus partial order, as we will see later. However, he highlighted an interesting relation between the $\star$-infimum of two matrices and their parallel sum. The following theorem appeared in the paper by Mitra [66], and he attributed it to P. Holladay. In Section 4.5 we discuss this relation with more details.

Theorem 4.2.2 (Holladay, See [66]). *If A and B are complex matrices, then*

$$A \overset{\star}{\wedge} B \overset{\star}{\leq} 2A(A + B)^\dagger B, \quad \text{and} \quad A \overset{\star}{\wedge} B \overset{\star}{\leq} 2B(A + B)^\dagger A.$$

If $A \overset{\star}{\vee} B$ exists, then:

$$A \overset{\star}{\wedge} B = 2A(A + B)^\dagger B = 2B(A + B)^\dagger A.$$

The results of Hartwig and Drazin from [54] can not be used on arbitrary Hilbert spaces, since they are based on linear algebra techniques suitable for finite-dimensional spaces. The first study with the results greatly applicable on $\mathcal{B}(\mathcal{H})$ was given by Janowitz in [56]. In fact, he studies the $\star$-partial order on structures more general than $\mathcal{B}(\mathcal{H})$, the so called Rickart *-rings and Baer *-rings (see Chapter 5 for more details). Translated on $\mathcal{B}(\mathcal{H})$, he proves that $\mathcal{B}(\mathcal{H})$ is a complete lower semi-lattice, i.e. that any family of operators $\{A_i \,:\, i \in I\}$ has the $\star$-infimum, while $\mathcal{B}(\mathcal{H})$ has the already mentioned upper bound property. Janowitz's results also give an answer to a question when $A \overset{\star}{\vee} B$ exists only under the assumptions similar to the ones from Theorem 4.2.1.

Recently, some authors rediscovered the results from [56] in the set $\mathcal{B}(\mathcal{H})$. We already mentioned that a thorough study of the $\star$-partial order in $\mathcal{B}(\mathcal{H})$ was given by Antezana et. al. in [5]. We present here their result about the $\star$-infimum. Note on the notation: if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ with $\mathcal{M}'$ we denote the set $\{T \in \mathcal{B}(\mathcal{H}) \,:\, TM = MT, \text{ for all } M \in \mathcal{M}\}$, so called commutant of $\mathcal{M}$.

Theorem 4.2.3 (See [5, 56])**.** *For every $A, B \in \mathcal{B}(\mathcal{H})$ the infimum $A \overset{\star}{\wedge} B$ exists, and $A \overset{\star}{\wedge} B = PA = PB$, where P is the maximum of the set:*

$$\{P \,:\, P = P^2 = P^*,\ P \in \{AA^*, BB^*\}',\ \mathcal{R}(P) \subseteq \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \cap \mathcal{N}(B^* - A^*)\}$$

in the usual order for orthogonal projections. Moreover, $P = P_{\mathcal{R}(A \overset{\star}{\wedge} B)}$.

Some basic facts about the $\star$-infimum are gathered in the following proposition.

Proposition 4.2.4 (See [5])**.** *If $A, B, C \in \mathcal{B}(\mathcal{H})$ then:*

1. *$A \overset{\star}{\wedge} B = B \overset{\star}{\wedge} A$ and $(A \overset{\star}{\wedge} B)^* = A^* \overset{\star}{\wedge} B^*$;*

2. *$(A \overset{\star}{\wedge} B) \wedge C = A \overset{\star}{\wedge} (B \overset{\star}{\wedge} C)$;*

3. *$(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* \overset{\star}{\leq} AA^* \overset{\star}{\wedge} BB^*$ and $|A \overset{\star}{\wedge} B| \overset{\star}{\leq} |A| \overset{\star}{\wedge} |B|$. Inequalities can be strict;*

4. *If A or B is positive, than $A \overset{\star}{\wedge} B$ is also positive.*

In [5] authors also present an interesting relation between the $\star$-partial order and the functional calculus. One direct corollary of such relation is that $A \overset{\star}{\leq} B$ implies:

$$|A| \overset{\star}{\leq} |B| \quad \text{and} \quad |A^*| \overset{\star}{\leq} |B^*|. \tag{4.15}$$

We should notice that there are examples of structures which are not a lower semi-lattice in the $\star$-partial order. For example see [18].

Finally, in [84] Xu et. al. study the problem of $\star$-supremum for arbitrary Hilbert space operators. In this paper, they rediscovered that $\mathcal{B}(\mathcal{H})$ has an upper bound property, and gave necessary and sufficient conditions for two arbitrary operators to have the $\star$-supremum. Their result is the following.

Theorem 4.2.5 (See [84])**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let: $\mathcal{H}_1 = \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$, $\mathcal{H}_2 = \overline{\mathcal{R}(A^*)} \cap \mathcal{N}(B)$, $\mathcal{H}_3 = \overline{\mathcal{R}(A^*)} \ominus (\mathcal{H}_1 + \mathcal{H}_2)$, $\mathcal{H}_4 = \mathcal{N}(A) \cap \overline{\mathcal{R}(B^*)}$, $\mathcal{H}_5 = \mathcal{N}(A) \cap \mathcal{N}(B)$, $\mathcal{H}_6 = \mathcal{N}(A) \ominus (\mathcal{H}_4 \oplus \mathcal{H}_5)$; $\mathcal{H}'_1 = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$, $\mathcal{H}'_2 = \overline{\mathcal{R}(A)} \cap \mathcal{N}(B^*)$, $\mathcal{H}'_3 = \overline{\mathcal{R}(A)} \ominus (\mathcal{H}'_1 + \mathcal{H}'_2)$, $\mathcal{H}'_4 = \mathcal{N}(A^*) \cap \overline{\mathcal{R}(B)}$, $\mathcal{H}'_5 = \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$, $\mathcal{H}_6 = \mathcal{N}(A^*) \ominus (\mathcal{H}'_4 \oplus \mathcal{H}'_5)$. Then A and B have the $\star$-supremum if and only if the following conditions hold:*

1. *According to decompositions* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5 \oplus \mathcal{H}_6$ *and* $\mathcal{H} = \mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \mathcal{H}_3' \oplus \mathcal{H}_4' \oplus \mathcal{H}_5' \oplus \mathcal{H}_6'$*, respectively, operators A and B are equal to:*

$$A = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{33} & 0 & 0 & B_{36} \\ 0 & 0 & 0 & B_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{63} & 0 & 0 & B_{66} \end{bmatrix},$$

where $B_{33}^*(A_{33} - B_{33}) = B_{63}^* B_{63}$ *and* $(A_{33} - B_{33})B_{33}^* = B_{36}B_{36}^*$*;*

2. *There exists* $W \in \mathcal{B}(\mathcal{H}_6, \mathcal{H}_6')$ *such that* $B_{33}^* B_{36} = B_{63}^* W$ *and* $B_{63}B_{33}^* = W B_{36}^*$.

From (4.5) a relation between the problem of $\star$-supremum and the notion of coherent operators is obvious: if operators A and B have some common $\star$-upper bound, they have to be coherent. It is needless to say that our approach to this problem in the following section will be to employ everything that we can from Chapter 2. We also prove other new properties of both $\star$-supremum and $\star$-infimum.

Solving similar problems for the minus-order is considerably more difficult. This difficulty is caused by the arbitrariness of the complements $\mathcal{M}$ and $\mathcal{N}$ in (4.7). The lack of structure for the minus-order was noticed in the very first paper where it appeared. We give the example from this paper.

Example 20 (See [54])**.** It is clear from (4.7) that if $A \overset{-}{\leq} B$ then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. If A and B are matrices, then from $A \overset{-}{\leq} B$ and $\mathrm{r}(A) = \mathrm{r}(B)$ we would get $A = B$.

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have $\mathrm{r}(C) = 1$ for every c, and $C \overset{-}{\leq} A$, $C \overset{-}{\leq} B$. Thus the minus-infimum $A \overset{-}{\wedge} B$ does not exist. ◇

The majority of the results about the lattice properties of the minus-order are related to the parallel summation and the following construction resembling operator shorting (see Section 1.5). For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{S} \subseteq \mathcal{H}$, $\mathcal{T} \subseteq \mathcal{K}$, define the set:

$$\overline{\mathfrak{M}}(A, \mathcal{S}, \mathcal{T}) = \{B \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \ : \ B \overset{-}{\leq} A, \ \mathcal{R}(B) \subseteq \mathcal{T}, \ \mathcal{R}(B^*) \subseteq \mathcal{S}\}. \tag{4.16}$$

This set was first considered by Mitra in [66] in the matrix case, and in [6] for arbitrary Hilbert space operators. The following theorem gives a very interesting relation between the maximum of this set and bilateral shorting of an operator.

Theorem 4.2.6. *(See [6]) Let* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *be* $(\mathcal{S}, \mathcal{T})$*-complementable. Then the set* $\overline{\mathfrak{M}}(A, \mathcal{S}, \mathcal{T})$ *defined in* (4.16) *has the maximum and this maximum is exactly* $A_{/(\mathcal{S},\mathcal{T})}$.

In [66] the author derives many interesting results regarding minus-supremum and minus-infimum of matrices under the conditions related with the set $\bar{\mathfrak{M}}(A, \mathcal{S}, \mathcal{T})$ and the parallel summation. We gather some of them into one theorem.

Theorem 4.2.7 (See [66])**.** *Let $A, B \in \mathbb{C}^{m\times n}$. Then:*

1. *If both $\bar{\mathfrak{M}}(A, \mathcal{R}(B), \mathcal{R}(B^*))$ and $\bar{\mathfrak{M}}(B, \mathcal{R}(A), \mathcal{R}(A^*))$ have the maximum, then these maximums are the same if and only if A and B have a common upper bound in the minus order;*

2. *If at least one of the sets $\bar{\mathfrak{M}}(A, \mathcal{R}(B), \mathcal{R}(B^*))$ and $\bar{\mathfrak{M}}(B, \mathcal{R}(A), \mathcal{R}(A^*))$ does not have the maximum, $A \overset{-}{\vee} B$ does not exist;*

3. *If the sets $\bar{\mathfrak{M}}(A, \mathcal{R}(B), \mathcal{R}(B^*))$ and $\bar{\mathfrak{M}}(B, \mathcal{R}(A), \mathcal{R}(A^*))$ have the maximums and these maximums are the same, then $A \overset{-}{\wedge} B = 2(A : B)$;*

4. *If the sets $\bar{\mathfrak{M}}(A, \mathcal{R}(B), \mathcal{R}(B^*))$ and $\bar{\mathfrak{M}}(B, \mathcal{R}(A), \mathcal{R}(A^*))$ have unequal maximums, then $A \overset{-}{\wedge} B = 0$, or $A \overset{-}{\wedge} B$ does not exist.*

With this theorem we conclude the section and we are now ready to present our results on the subject. In Section 4.3 we study the $\star$-partial order and we already announced some results which will be presented there. In Section 4.4 we study the lattice properties of the ⊕-partial order, which hasn't been studied before, to the best of our knowledge. Finally, in Section 4.5 we exhibit interesting relations between infimums in these partial orders and the parallel summation. Note that throughout the following sections we only consider the algebra $\mathcal{B}(\mathcal{H})$, but our main results and their proofs remain exactly the same in the set $\mathcal{B}(\mathcal{H}, \mathcal{K})$. The main reason for this is that the majority of the results we invoke are proved in $\mathcal{B}(\mathcal{H})$ and we wish to avoid a tedious rereading to check whether they still hold on $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

4.3 Results on the star partial order

We start this section by giving necessary and sufficient conditions for the existence of a common $\star$-upper bound for arbitrary operators A and B, and we describe all these $\star$-upper bounds. As we have already mentioned before, these conditions also present necessary and sufficient conditions for $A \overset{\star}{\vee} B$ to exist.

Theorem 4.3.1. *Given $A, B \in \mathcal{B}(\mathcal{H})$ there exists a common $\star$-upper bound for A and B if and only if the following conditions are fulfilled:*

(i) $BA^*A = BB^*A$;

(ii) $AA^*B = AB^*B$;

(iii) There exists $S \in \mathcal{B}(\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}, \mathcal{H})$, such that

$$S|_{\overline{\mathcal{R}(A^*)}} = A|_{\overline{\mathcal{R}(A^*)}}$$

$$S|_{\overline{\mathcal{R}(B^*)}} = B|_{\overline{\mathcal{R}(B^*)}}.$$

In this case, every common $\star$-upper bound of A and B is given by:

$$C_X = \begin{bmatrix} S & 0 \\ 0 & X \end{bmatrix} : \begin{bmatrix} \overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}} \\ \mathcal{N}(A) \cap \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} \\ \mathcal{N}(A^*) \cap \mathcal{N}(B^*) \end{bmatrix}, \tag{4.17}$$

for $X \in \mathcal{B}(\mathcal{N}(A) \cap \mathcal{N}(B), \mathcal{N}(A^*) \cap \mathcal{N}(B^*))$, while $A \overset{\star}{\vee} B = C_0$ is given by:

$$C_0 = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}} \\ \mathcal{N}(A) \cap \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}} \\ \mathcal{N}(A^*) \cap \mathcal{N}(B^*) \end{bmatrix}. \tag{4.18}$$

Proof. Suppose that A and B have a common $\star$-upper bound C. Then, as we mentioned earlier, we have $AA^*B = AB^*B$ and $BA^*A = BB^*A$ (see (2.2)). Operators A and C coincide on the subspace $\overline{\mathcal{R}(A^*)}$ and operators B and C coincide on the subspace $\overline{\mathcal{R}(B^*)}$. If $S = C|_{\overline{\overline{\mathcal{R}(A^*)}+\overline{\mathcal{R}(B^*)}}}$, then S is from $\mathcal{B}(\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}, \mathcal{H})$ and satisfies required equalities. Let us prove now that C is of the form (4.17). We have that $C(\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}) \subseteq \overline{C(\overline{\mathcal{R}(A^*)}) + C(\overline{\mathcal{R}(B^*)})} \subseteq \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$. On the other hand, if $x \in \mathcal{N}(A) \cap \mathcal{N}(B)$, from $A^*A = A^*C$ we get $Cx \in \mathcal{N}(A^*)$. Likewise, $Cx \in \mathcal{N}(B^*)$, so $C(\mathcal{N}(A) \cap \mathcal{N}(B)) \subseteq \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$. Thus, C is given like in (4.17).

Now, suppose that conditions $(i), (ii)$ and (iii) are fulfilled. Let $C = C_X$ be given like in (4.17), for arbitrary X. We will show that $A \overset{\star}{\leq} C$, and by symmetry, we will also have $B \overset{\star}{\leq} C$. By definition of C, A and C coincide on $\overline{\mathcal{R}(A^*)}$. So, by (iii), we have $AA^* = CA^*$. Let us prove that $\mathcal{R}(A - C) \subseteq \mathcal{N}(A^*)$, i.e. $A^*A = A^*C$. It is enough to prove that $(A - C)(\mathcal{N}(A) \cap \mathcal{N}(B)) \subseteq \mathcal{N}(A^*)$, $(A - C)(\overline{\mathcal{R}(A^*)}) \subseteq \mathcal{N}(A^*)$ and $(A - C)(\overline{\mathcal{R}(B^*)}) \subseteq \mathcal{N}(A^*)$. The first relation follows from (4.17), the second relation follows from (iii), and the third follows from (iii) and (i): from (iii) we have that $(A-C)(\overline{\mathcal{R}(B^*)}) = (A-B)(\overline{\mathcal{R}(B^*)})$ and from (i) we have that $(A-B)(\mathcal{R}(B^*)) \subseteq \mathcal{N}(A^*)$, and so $(A - B)(\overline{\mathcal{R}(B^*)}) \subseteq \mathcal{N}(A^*)$. Thus: $A \overset{\star}{\leq} C$, so C is a $\star$-upper bound for A and B.

We have proved that conditions $(i), (ii)$ and (iii) are indeed equivalent to the existence of a common $\star$-upper bound for A and B, and that in this case, every common $\star$-upper bound is given like in (4.17), for arbitrary X. It is obvious that $C_0 \overset{\star}{\leq} C_X$, for every X, so $A \overset{\star}{\vee} B = C_0$. $\square$

We state Theorem 4.3.1 and the following two corollaries in terms of the coherent operators.

Corollary 4.3.2. *If $A, B \in \mathcal{B}(\mathcal{H})$ then $A \overset{\star}{\vee} B$ exists if and only if the following conditions are fulfilled:*

(i) $A(A^* - B^*)B = 0 = B(A^* - B^*)A$;

(ii) A *and* B *are coherent.*

Proof. Directly from Theorem 4.3.1 and the definition of coherent operators. □

Corollary 4.3.3. *Let* $A, B \in \mathcal{B}(H)$*. The following statements are equivalent:*

(1) $A \overset{\star}{\vee} B$ *exists;*

(2) Operators A *and* B*, as well as operators* A^* *and* B^* *are coherent and* $S(A,B)^* = S(A^*, B^*)$*, where* $S(A,B)$ *is defined as in Section 2.1.*

Proof. (1) ⇒ (2) It is clear that the existence of $A \overset{\star}{\vee} B$ forces A and B to be coherent, but due to symmetry with respect to taking adjoints, it also forces A^* and B^* to be coherent. Now since $A \overset{\star}{\vee} B$ exist, we have $AA^*B = AB^*B$ and $BA^*A = BB^*A$, so from Corollary 2.2.7 we have that $S(A,B)^* = S(A^*, B^*)$.
(2) ⇒ (1) From Theorem 2.2.5 it follows that $AA^*B = AB^*B$ and $BA^*A = BB^*A$, and so from Theorem 4.3.1 we have that $A \overset{\star}{\vee} B$ exists. □

For operators on infinite-dimensional space it is often easier to verify the equality $P_{\overline{\mathcal{R}(A)}}B = AP_{\overline{\mathcal{R}(B^*)}}$ than the equality $BA^*A = BB^*A$, because it does not involve handling with operator A^* and B^* but only with their ranges, which are orthogonal complements of $\mathcal{N}(A)$ and $\mathcal{N}(B)$ (up to a closure). On the other hand, for given matrices, equality $BA^*A = BB^*A$ is readily verified.

Recalling Corollary 2.2.8, we see that in the case when $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ is closed, or $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ is closed, the condition (ii) of Corollary 4.3.2 is superfluous. It is not difficult to see that this happens, for example, if any of the subspaces $\mathcal{N}(A)$, $\mathcal{N}(A^*)$, $\mathcal{N}(B)$, $\mathcal{N}(B^*)$, $\mathcal{R}(A)$, $\mathcal{R}(A^*)$, $\mathcal{R}(B)$, $\mathcal{R}(B^*)$ turns out to be finite-dimensional. Thus, we obtain the following result, which is our main result in the study of $\star$-supremum.

Theorem 4.3.4. *Let* $A, B \in \mathcal{B}(\mathcal{H})$*, such that at least one of the subspaces* $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ *and* $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ *is a closed subspace of* $\mathcal{H}$*. There exists* $A \overset{\star}{\vee} B$ *iff* $AA^*B = AB^*B$ *and* $BA^*A = BB^*A$*.*

Proof. Directly from Corollaries 4.3.2 and 2.2.8. □

Finally, we note that we can employ this approach without any changes for $\star$-supremum for arbitrary rectangular matrices. So if $A, B \in \mathbb{C}^{m \times n}$, then

$$A \overset{\star}{\vee} B \quad \text{exists} \quad \Leftrightarrow \quad A(A^* - B^*)B = 0 = B(A^* - B^*)A.$$

We now give two remarks on the results of Hartwig given here as Theorem 4.2.1 and Janowitz from [56]. For the sake of convenience, we state each result translated on $\mathcal{B}(\mathcal{H})$.

Theorem 4.3.5 (See [50])**.** *If* $A, B \in \mathcal{B}(\mathcal{H})$ *are such that* $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}((I - P_{\overline{\mathcal{R}(A)}})B)$ *and* $\mathcal{R}(B(I - P_{\overline{\mathcal{R}(A^*)}}))$ *are closed, then the set* $\{C \mid A \overset{\star}{\leq} C,\ B \overset{\star}{\leq} C\}$ *is nonempty iff:*

(i) $B(A^* - B^*)A = 0 = A(B^* - A^*)B$;

(ii) $B(B^* - A^*) = B(I - P_{\overline{\mathcal{R}(A^*)}})X$, *for some* $X \in \mathcal{B}(H)$;

(iii) $(B^* - A^*)B = Y(I - P_{\overline{\mathcal{R}(A)}})B$ *for some* $Y \in \mathcal{B}(H)$.

Remark 4.3.6. If conditions of Theorem 4.3.5 are satisfied, then using Theorem 1.2.9 we see that $\mathcal{R}(A) + \mathcal{R}(B)$ and $\mathcal{R}(A^*) + \mathcal{R}(B^*)$ are also closed. Hence, by Theorem 4.3.4, we conclude that conditions (ii) and (iii) of Theorem 4.3.5 are superfluous. From (4.14) we also get a formula for computing the $\star$-supremum:

$$A \overset{\star}{\vee} B = A + P_{\mathcal{N}(A^*)} BB^* (P_{\mathcal{N}(A)} B)^\dagger.$$

Theorem 4.3.7 (See [56]). *If* $A, B \in \mathcal{B}(\mathcal{H})$ *are such that* $\mathcal{R}((I - P_{\overline{\mathcal{R}(A^*)}})B^*)$ *is closed, then the set* $\{C \mid A \overset{\star}{\leq} C,\ B \overset{\star}{\leq} C\}$ *is nonempty iff:*

(i) $A^*B = P_{\overline{\mathcal{R}(A^*)}} B^*B$;

(ii) $AB^* = P_{\overline{\mathcal{R}(A)}} BB^*$;

(iii) $(B - A)B^* = X(I - P_{\overline{\mathcal{R}(A^*)}})B^*$, *for some* $X \in \mathcal{B}(\mathcal{H})$.

Remark 4.3.8. Condition (i) of Theorem 4.3.7 can be written as $P_{\overline{\mathcal{R}(A^*)}}(A^* - B^*)B = 0$, which is equivalent to $AA^*B = AB^*B$. Similarly, condition (ii) is equivalent to $BA^*A = BB^*A$. If $\mathcal{R}((I - P_{\overline{\mathcal{R}(A^*)}})B^*)$ is closed, since $\overline{\mathcal{R}(B^*)} = \mathcal{R}(P_{\overline{\mathcal{R}(B^*)}})$ we have $\mathcal{R}((I - P_{\overline{\mathcal{R}(A^*)}})B^*) = (I - P_{\overline{\mathcal{R}(A^*)}})(\mathcal{R}(B^*)) \subseteq (I - P_{\overline{\mathcal{R}(A^*)}})(\mathcal{R}(P_{\overline{\mathcal{R}(B^*)}})) \subseteq \overline{(I - P_{\overline{\mathcal{R}(A^*)}})(\mathcal{R}(B^*))} = \mathcal{R}((I - P_{\overline{\mathcal{R}(A^*)}})B^*)$. Thus $(I - P_{\overline{\mathcal{R}(A^*)}})(\mathcal{R}(P_{\overline{\mathcal{R}(B^*)}}))$ is closed, i.e. $\mathcal{R}((I - P_{\overline{\mathcal{R}(A^*)}})P_{\overline{\mathcal{R}(B^*)}})$ is closed. From Theorem 1.2.9 we deduce that $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ is closed, and as before, from Theorem 4.3.4 we see that condition (iii) in Theorem 4.3.7 is superfluous.

Unfortunately, in general, we still do not have necessary and sufficient conditions for the existence of $A \overset{\star}{\vee} B$ which do not involve checking solvability of some (system) of equations (see condition (iii) in Theorem 4.3.1, or condition (2.) in Theorem 4.2.5). A question remains whether condition *(iii)* of Theorem 4.3.1 is superfluous in general, when we have conditions *(i)* and *(ii)* fulfilled. In other words, do equalities $AA^*B = AB^*B$ and $BA^*A = BB^*A$ imply that operators A and B are coherent. If the condition *(iii)* can be omitted, we would have that only a trivial necessary condition should be satisfied for the existence of $\star$-supremum to be assured, just like for rectangular matrices.

We now continue by examining some properties of $\star$-supremums. If $A \overset{\star}{\leq} C$, from the definition of $\star$-order we have $\mathcal{R}(A) \subseteq \mathcal{R}(C)$ and $\mathcal{N}(C) \subseteq \mathcal{N}(A)$. So for every common $\star$-upper bound C for A and B, provided it exists, we have $\mathcal{N}(C) \subseteq \mathcal{N}(A) \cap \mathcal{N}(B)$, and so C is injective on $\overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$. Theorem 4.3.1 gives us a simple criterion for distinguishing $\star$-supremum among $\star$-upper bounds for two operators, as stated in the next lemma.

Lemma 4.3.9. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *such that* $A \overset{\star}{\vee} B$ *exists. If* C *is a common* $\star$*-upper bound for* A *and* B*, then the following statements are equivalent:*

(1) $C = A \overset{\star}{\vee} B$;

(2) $\mathcal{N}(C) = \mathcal{N}(A) \cap \mathcal{N}(B)$;

(3) $\overline{\mathcal{R}(C)} = \overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$.

Proof. Clear from Theorem 4.3.1 and the discussion preceding this lemma. □

Some basic properties of the $\star$-supremum are contained in the next theorem. Observe that the statements contained in (3) and (4) of the following lemma are seemingly stronger than the dual statements in Proposition 4.2.4 for the $\star$-infimum. In the end of this section we will prove that if $A \overset{\star}{\vee} B$ exists, we also have the equality in inequalities: $(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* \overset{\star}{\leq} AA^* \overset{\star}{\wedge} BB^*$ and $|A \overset{\star}{\wedge} B| \overset{\star}{\leq} |A| \overset{\star}{\wedge} |B|$. In fact, we will prove this under a weaker condition than the existence of $A \overset{\star}{\vee} B$.

Theorem 4.3.10. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $A \overset{\star}{\vee} B$ exists. Then:*

(1) $A^* \overset{\star}{\vee} B^*$ *exists and:* $A^* \overset{\star}{\vee} B^* = (A \overset{\star}{\vee} B)^*$;

(2) $(\lambda A) \overset{\star}{\vee} (\lambda B)$ *exists and:* $(\lambda A) \overset{\star}{\vee} (\lambda B) = \lambda(A \overset{\star}{\vee} B)$, *for any* $\lambda \in \mathbb{C}$;

(3) $A^*A \overset{\star}{\vee} B^*B$ *exists and* $A^*A \overset{\star}{\vee} B^*B = (A \overset{\star}{\vee} B)^*(A \overset{\star}{\vee} B)$;

(4) $|A| \overset{\star}{\vee} |B|$ *exists and* $|A| \overset{\star}{\vee} |B| = |A \overset{\star}{\vee} B|$.

Proof. (1) and (2) are clear from $X \overset{\star}{\leq} Y \Leftrightarrow X^* \overset{\star}{\leq} Y^*$, and $X \overset{\star}{\leq} Y \Leftrightarrow (\lambda X) \overset{\star}{\leq} (\lambda Y)$, for any two operators $X, Y \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

(3) If $X \overset{\star}{\leq} Y$ then $X^*X \overset{\star}{\leq} Y^*Y$, so if $A \overset{\star}{\vee} B$ exists, so does $A^*A \overset{\star}{\vee} B^*B$. If $C = A \overset{\star}{\vee} B$, then C^*C is a common $\star$-upper bound of A^*A and B^*B. From Lemma 4.3.9 we have that $\mathcal{N}(A^*A) \cap \mathcal{N}(B^*B) = \mathcal{N}(A) \cap \mathcal{N}(B) = \mathcal{N}(C) = \mathcal{N}(C^*C)$, so $C^*C = A^*A \overset{\star}{\vee} B^*B$.

(4) If $C = A \overset{\star}{\vee} B$, then from (4.15) we have that $|C|$ is a common $\star$-upper bound for $|A|$ and $|B|$ so $|A| \overset{\star}{\vee} |B|$ exists. As $\mathcal{N}(|A|) \cap \mathcal{N}(|B|) = \mathcal{N}(A) \cap \mathcal{N}(B) = \mathcal{N}(C) = \mathcal{N}(|C|)$, we have $|C| = |A| \overset{\star}{\vee} |B|$. □

From Theorem 4.3.10 we directly obtain the next conclusion.

Corollary 4.3.11. *If $A, B \in \mathcal{B}(\mathcal{H})$ are normal (self-adjoint, positive), and $A \overset{\star}{\vee} B$ exists, then it is normal (self-adjoint, positive).*

We now address one question regarding the $\star$-infimum. First, note that $\overline{\mathcal{R}(A \overset{\star}{\wedge} B)}$ is always contained in $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ while $\mathcal{N}(A \overset{\star}{\wedge} B)$ always contains $\overline{\mathcal{N}(A) + \mathcal{N}(B)}$. In this way we see what are the extremal values for range and null-space of the $\star$-infimum of two operators. Hartwig and Drazin in [50] noted that these extremal values are obtained for orthogonal projection matrices and proposed a problem of finding all matrices having this property. We will show that $\overline{\mathcal{R}(A \overset{\star}{\wedge} B)} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ and $\mathcal{N}(A \overset{\star}{\wedge} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$ if and only if A and B are precoherent, and A^* and B^* are precoherent.

Recall that from Theorem 4.2.3 we have $A \overset{\star}{\wedge} B = PA = PB$, where P is the maximum of the set:

$$\{P \ : \ P = P^2 = P^*, \ P \in \{AA^*, BB^*\}', \ \mathcal{R}(P) \subseteq \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \cap \mathcal{N}(B^* - A^*)\} \quad (4.19)$$

(maximum w.r.t $\star$-order, or the classic $\leq$-order, they coincide on this set). In our next theorem, we prove that, if A and B are precoherent, and A^* and B^* are precoherent, then $P_{\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}}$ is an element of the set in (4.19), moreover it is its maximal element, and so $A \overset{\star}{\wedge} B = P_{\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}} A = P_{\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}} B$.

Theorem 4.3.12. *If $A, B \in \mathcal{B}(\mathcal{H})$ then $\overline{\mathcal{R}(A \overset{\star}{\wedge} B)} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ and $\mathcal{N}(A \overset{\star}{\wedge} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$ if and only if A and B are precoherent, and A^* and B^* are precoherent.*

Proof. Suppose first that A and B are precoherent, and A^* and B^* are precoherent and let $S = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. Then we have that $S \subseteq \mathcal{N}(B^* - A^*)$.

For $x \in S$, we have $A^*x = B^*x \in \mathcal{R}(A^*) \cap \mathcal{R}(B^*)$, but then $AA^*x = BA^*x$, so $AA^*x \in S$. Thus $AA^*P_S = P_SAA^*P_S$, and $P_SAA^*P_S$ is self-adjoint, so AA^*P_S is also self-adjoint. From here, we get that AA^* commutes with P_S. In the same way, we get that BB^* commutes with P_S. Hence, P_S is the maximal element of the set in (4.19), and $A \overset{\star}{\wedge} B = P_SA = P_SB$.

Now from Lemma 2.2.3 we see that S is in fact $\overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ and so $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \subseteq \overline{\mathcal{R}(P_SA)} = \overline{\mathcal{R}(A \overset{\star}{\wedge} B)}$. Since the other inclusion is clear, we have $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A \overset{\star}{\wedge} B)}$.

It is left to prove that $\mathcal{N}(A \overset{\star}{\wedge} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$. Since $(A \overset{\star}{\wedge} B)^* = A^* \overset{\star}{\wedge} B^*$ (Proposition 4.2.4), and also from the already proved part of the theorem: $\overline{\mathcal{R}(A^* \overset{\star}{\wedge} B^*)} = \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$, we have that $\mathcal{N}(A \overset{\star}{\wedge} B) = \overline{\mathcal{R}((A \overset{\star}{\wedge} B)^*)}^{\perp} = (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})^{\perp} = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$.

Now we prove the opposite implication. Since $A^* \overset{\star}{\wedge} B^* \leq A^*, B^*$, operators A^* and B^* coincide together with $A^* \overset{\star}{\wedge} B^*$ on $\mathcal{R}((A^* \overset{\star}{\wedge} B^*)^*)$, but $\overline{\mathcal{R}((A^* \overset{\star}{\wedge} B^*)^*)} = \overline{\mathcal{R}(A \overset{\star}{\wedge} B)} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. From $\mathcal{N}(A \overset{\star}{\wedge} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$ we get that also $\overline{\mathcal{R}(A^* \overset{\star}{\wedge} B^*)} = \mathcal{N}(A \overset{\star}{\wedge} B)^{\perp} = \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$ and in the same way as for A^* and B^*, we get that A and B coincide on $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$. $\square$

Note that under the assumptions of the preceding theorem, from $A \overset{\star}{\wedge} B = P_SA = P_SB$, we also have $\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \mathcal{R}(A \overset{\star}{\wedge} B)$, and so $\mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{R}(A \overset{\star}{\wedge} B)$, since the other inclusion is trivial.

Corollary 4.3.13. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that A and B are precoherent, and A^* and B^* are precoherent. Then $A \overset{\star}{\wedge} B = (P_{\overline{\mathcal{R}(A)}} \wedge P_{\overline{\mathcal{R}(B)}})A = (P_{\overline{\mathcal{R}(A)}} \wedge P_{\overline{\mathcal{R}(B)}})B$, and $P_{A \overset{\star}{\wedge} B} = P_{\overline{\mathcal{R}(A)}} \wedge P_{\overline{\mathcal{R}(B)}}$.*

Proof. Directly from Theorem 4.3.12. $\square$

The existence of $\star$-supremum for A and B clearly implies that A and B are precoherent, and A^* and B^* are precoherent. On the other hand, the equalities $\mathcal{R}(A \overset{\star}{\wedge} B) = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ and $\mathcal{N}(A \overset{\star}{\wedge} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$ do not imply the existence of $A \overset{\star}{\vee} B$. Not even in the finite-dimensional case can these equalities force the conditions (i) and (ii) of Theorem 4.3.1.

Example 21. Recall the setting of Example 11. Operators A and B defined within this example are such that $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \{0\} = \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$ and $\mathcal{N}(A) + \mathcal{N}(B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)} = (\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})^{\perp} = \mathcal{H}$ (we used here the fact that $\mathcal{M} + \mathcal{N}$ is closed, and so is $\mathcal{M}^{\perp} + \mathcal{N}^{\perp} = \mathcal{N}(A) + \mathcal{N}(B)$, Theorem 1.2.8). From $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ we conclude that $\mathcal{R}(A \overset{\star}{\wedge} B) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, i.e. $A \overset{\star}{\wedge} B = 0$. So we have $\mathcal{R}(A \overset{\star}{\wedge} B) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(A \overset{\star}{\wedge} B) = \mathcal{N}(A) + \mathcal{N}(B)$. We already proved that $S(A,B)^* \neq S(A^*, B^*)$ in this case, so according to Corollary 4.3.3, $A \overset{\star}{\vee} B$ does not exist.

To underline one more time, the coherence of A and B, together with coherence of A^* and B^* is not enough to assure the existence of a common $\star$-upper bound for A and B, i.e. to assure that $AA^*B = AB^*B$ and $BA^*A = BB^*B$. $\diamond$

Finally, we prove that, under the assumption of Theorem 4.3.12 we also have: $(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* = AA^* \overset{\star}{\wedge} BB^*$ and $|A \overset{\star}{\wedge} B| = |A| \overset{\star}{\wedge} |B|$. For the sake of convenience, we again state that:

$$A \overset{\star}{\leq} B, \quad \text{then} \quad A = B \quad \Leftrightarrow \quad \mathcal{N}(B) = \mathcal{N}(A) \quad \Leftrightarrow \quad \mathcal{R}(B) = \mathcal{R}(A). \tag{4.20}$$

Theorem 4.3.14. *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that A and B are precoherent, and A^* and B^* are precoherent. Then $(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* = AA^* \overset{\star}{\wedge} BB^*$ and $|A \overset{\star}{\wedge} B| = |A| \overset{\star}{\wedge} |B|$.*

Proof. From Proposition 4.2.4 we have that $(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* \overset{\star}{\leq} AA^* \overset{\star}{\wedge} BB^*$ and $|A \overset{\star}{\wedge} B| \overset{\star}{\leq} |A| \overset{\star}{\wedge} |B|$, so according to the fact in (4.20), it is enough to prove the equalities $\mathcal{N}(AA^* \overset{\star}{\wedge} BB^*) = \mathcal{N}((A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^*) = \mathcal{N}(A^* \overset{\star}{\wedge} B^*)$ and $\mathcal{N}(|A| \overset{\star}{\wedge} |B|) = \mathcal{N}(|A \overset{\star}{\wedge} B|) = \mathcal{N}(A \overset{\star}{\wedge} B)$. Note first that if A and B are precoherent, and A^* and B^* are precoherent, then AA^* and BB^* coincide on $\overline{\mathcal{R}(AA^*)} \cap \overline{\mathcal{R}(BB^*)}$ while A^*A and B^*B coincide on $\overline{\mathcal{R}(A^*A)} \cap \overline{\mathcal{R}(B^*B)}$. Moreover, subspace $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$ is invariant for A^*A and B^*B, and so is $(\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)})^{\perp}$, since they are self-adjoint. Thus we conclude that $(A^*A)^{1/2}|_{\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}} = (A^*A|_{\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}})^{1/2} = (B^*B|_{\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}})^{1/2} = (B^*B)^{1/2}|_{\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}}$. Hence, $|A|$ and $|B|$ also coincide on $\overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)} = \overline{\mathcal{R}(|A|)} \cap \overline{\mathcal{R}(|B|)}$. Now from Theorem 4.3.12 we have: $\mathcal{N}(AA^* \overset{\star}{\wedge} BB^*) = \overline{\mathcal{N}(AA^*) + \mathcal{N}(BB^*)} = \overline{\mathcal{N}(A^*) + \mathcal{N}(B^*)} = \mathcal{N}(A^* \overset{\star}{\wedge} B^*)$, and $\mathcal{N}(|A| \overset{\star}{\wedge} |B|) = \overline{\mathcal{N}(|A|) + \mathcal{N}(|B|)} = \overline{\mathcal{N}(A) + \mathcal{N}(B)} = \mathcal{N}(A \overset{\star}{\wedge} B)$. $\square$

It is obvious that in the set of all orthogonal projections, usual ordering of projections and the $\star$-partial order coincide. Any two orthogonal projections have one common $\star$-upper bound, namely: the orthogonal projection on the closure of the sum of their ranges. This is in fact also their $\star$-supremum. This can be seen from the fact: if $X \overset{\star}{\leq} E$, and E is an orthogonal projection, then X must be an orthogonal projection too (see for example [20, Lemma 3.1]), or we can just employ Theorem 4.3.1. Either way, we have that for orthogonal projections E and F:

$$E \overset{\star}{\vee} F = P_{\overline{\mathcal{R}(E)+\mathcal{R}(F)}}. \tag{4.21}$$

Since $E \overset{\star}{\wedge} F = P_{\mathcal{R}(E)\cap\mathcal{R}(F)}$, in the case of arbitrary Hilbert space, as noted in [52] for matrices, we obtain the same formula:

$$E \overset{\star}{\vee} F = I - ((I-E) \overset{\star}{\wedge} (I-F)). \tag{4.22}$$

Also, the Proposition 1 from [52] is still valid in the Hilbert space setting:

Proposition 4.3.15. *Let E and F be two orthogonal projections such that $\mathcal{R}(E+F)$ is closed. Then*

$$E \overset{\star}{\vee} F = (E+F)^\dagger(E+F) = (E+F)(E+F)^\dagger.$$

Proof. Follows from (4.21), since when $\mathcal{R}(E+F)$ is closed, then $\mathcal{R}(E+F) = \mathcal{R}(E)+\mathcal{R}(F)$ (Theorem 1.4.4). $\square$

The next theorem describes the structure of all common $\star$-upper bounds of two orthogonal projections.

Theorem 4.3.16. *Let $\mathcal{U}$ be the set of all common $\star$-upper bounds for two orthogonal projections E and F. Then $\mathcal{U}$ is a closed unital subalgebra of $\mathcal{B}(\mathcal{H})$. Moreover, there is an order isomorphism between partially ordered sets $(\mathcal{U}, \overset{\star}{\leq})$ and $(\mathcal{B}(\mathcal{N}(E)\cap\mathcal{N}(F)), \overset{\star}{\leq})$ which is linear and bounded.*

Proof. Follows directly from Theorem 4.3.1, with mapping: $\varphi : \mathcal{U} \to \mathcal{N}(E)\cap\mathcal{N}(F)$ defined by $\varphi : C_X \mapsto X$ (in the notation from Theorem 4.3.1). $\square$

If U and V are partial isometries, then $U \overset{\star}{\vee} V$ need not exist, as seen if we take two different unitary operators U and V. But when the set of all $\star$-upper bounds of U and V is not empty, its minimum belongs to the class of partial isometries, resembling properties of orthogonal projections. This is contained in [56, Theorem 12], but since we need to be precise about the initial and final space of $U \overset{\star}{\vee} V$, and for the sake of completeness, we give the following proposition.

Proposition 4.3.17. *If U and V are partial isometries such that $U\overset{\star}{\vee}V$ exists, then $U\overset{\star}{\vee}V$ is a partial isometry with initial space $\overline{\mathcal{R}(U^*)+\mathcal{R}(V^*)}$ and final space $\overline{\mathcal{R}(U)+\mathcal{R}(V)}$.*

Proof. Let $W = U \overset{\star}{\vee} V$. Then $W^* = U^* \overset{\star}{\vee} V^*$. According to Theorem 4.3.1, we conclude that bounded operator W^*W from $\overline{\mathcal{R}(U^*)+\mathcal{R}(V^*)}$ to itself is equal to identity on a dense subspace $\mathcal{R}(U^*)+\mathcal{R}(V^*)$ of this space. Similar conclusion follows for WW^*. Thus, W is an isometry from $\overline{\mathcal{R}(U^*)+\mathcal{R}(V^*)}$ to $\overline{\mathcal{R}(U)+\mathcal{R}(V)}$, and the null operator on $\overline{\mathcal{R}(U^*)+\mathcal{R}(V^*)}^\perp$, so it is the partial isometry with initial and final spaces as stated. $\square$

In the following theorem we describe the natural correlation between the $\star$-supremum and the polar decomposition of operators. This theorem gives us a more precise statement than (4) of Theorem 4.3.10 and one occasion where such statements can be useful is, for example, when studying properties of the range of $A \overset{\star}{\vee} B$. In that case we can assume, without loss of generality, that A and B are positive and there is a variety of interesting properties regarding ranges of positive operators (see, for example, [38], [7]). We will invoke the following theorem from [5].

Theorem 4.3.18. *[See [5]] If $A, B \in \mathcal{B}(\mathcal{H})$ then $A \overset{\star}{\leq} B$ if and only if $|A| \overset{\star}{\leq} |B|$ and $U_A \overset{\star}{\leq} U_B$, where $A = U_A|A|$ and $B = U_B|B|$ are the polar decompositions of A and B.*

Theorem 4.3.19. *Let $A = U|A|$, $B = V|B|$ and $C = W|C|$ be polar decompositions of A, B and C such that the initial spaces of partial isometries U, V and W are $\overline{\mathcal{R}(A^*)}, \overline{\mathcal{R}(B^*)}$ and $\overline{\mathcal{R}(C^*)}$, respectively. Then $A \overset{\star}{\vee} B$ exists if and only if $|A| \overset{\star}{\vee} |B|$ exists and $U \overset{\star}{\vee} V$ exists. In this case, $A \overset{\star}{\vee} B = C$ if and only if $|A| \overset{\star}{\vee} |B| = |C|$ and $U \overset{\star}{\vee} V = W$.*

Proof. If $A \overset{\star}{\vee} B$ exists and $Z|A \overset{\star}{\vee} B|$ is its polar decomposition, with $\mathcal{N}(Z)^\perp = \mathcal{N}(A \overset{\star}{\vee} B)^\perp$, then $|A| \overset{\star}{\vee} |B|$ exists, as stated in Theorem 4.3.10 and $U \overset{\star}{\vee} V$ exists too, because Z is one common $\star$-upper bound for U and V (Theorem 4.3.18).

Suppose now that $|A| \overset{\star}{\vee} |B|$ and $U \overset{\star}{\vee} V$ exist. Having in mind that $P_{\overline{\mathcal{R}(A^*)}} = P_{\overline{\mathcal{R}(|A|)}} = P_{\overline{\mathcal{R}(|A|^*)}} = P_{\mathcal{R}(U^*)}$, $P_{\overline{\mathcal{R}(B^*)}} = P_{\overline{\mathcal{R}(|B|)}} = P_{\overline{\mathcal{R}(|B|^*)}} = P_{\mathcal{R}(V^*)}$, $P_{\overline{\mathcal{R}(A)}} = P_{\mathcal{R}(U)}$ and $P_{\overline{\mathcal{R}(B)}} = P_{\mathcal{R}(V)}$, the fact that $|A| \overset{\star}{\vee} |B|$ and $U \overset{\star}{\vee} V$ exist and using equivalent forms of equalities, as in (2.3), we have the following sequence of equalities:

$$P_{\overline{\mathcal{R}(A)}}B = P_{\overline{\mathcal{R}(A)}}V|B| = (P_{\mathcal{R}(U)}V)|B| = (UP_{\mathcal{R}(V^*)})|B| =$$

$$= U|B| = U(P_{\overline{\mathcal{R}(|A|)}}|B|) = U(|A|P_{\overline{\mathcal{R}(|B|^*)}}) = AP_{\overline{\mathcal{R}(B^*)}}.$$

In a similar fashion we prove that $P_{\overline{\mathcal{R}(B)}}A = BP_{\overline{\mathcal{R}(A^*)}}$. If $Z = U \overset{\star}{\vee} V$, from Proposition 4.3.17 we have that Z is a partial isometry with initial space $\overline{\mathcal{R}(U^*) + \mathcal{R}(V^*)} = \overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$ and final space $\overline{\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}}$. We also have that $\mathcal{N}(|A| \overset{\star}{\vee} |B|)^\perp = \overline{\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}}$. If $D = Z(|A| \overset{\star}{\vee} |B|)$, we see that D coincides with $A = U|A|$ on $\overline{\mathcal{R}(A^*)}$ and with $B = V|B|$ on $\overline{\mathcal{R}(B^*)}$ and so, by Theorem 4.3.1, $A \overset{\star}{\vee} B$ exists.

We prove the second part of the theorem. If $A \overset{\star}{\vee} B$ exists, then $|A \overset{\star}{\vee} B| = |A| \overset{\star}{\vee} |B|$ and from Proposition 4.3.17 we have that $U \overset{\star}{\vee} V = Z$ is the partial isometry with initial space $\overline{\mathcal{R}((A \overset{\star}{\vee} B)^*)}$ and final space $\overline{\mathcal{R}(A \overset{\star}{\vee} B)}$. We easily conclude now that $C = A \overset{\star}{\vee} B$ if and only if $|C| = |A| \overset{\star}{\vee} |B|$ and $W = Z$. $\square$

We now investigate on modularity and distributivity of the $\star$-order. The fact that $\mathcal{B}(\mathcal{H})$ equipped with the $\star$-order is not a lattice does not stop us from asking the questions like, does: $A \overset{\star}{\vee} (B \overset{\star}{\wedge} C) = (A \overset{\star}{\vee} B) \overset{\star}{\wedge} (A \overset{\star}{\vee} C)$, or whether $A \overset{\star}{\leq} C$ implies $C \overset{\star}{\wedge} (A \overset{\star}{\vee} B) = A \overset{\star}{\vee} (C \overset{\star}{\wedge} B)$, in the case when the expressions on the left and right sides of these equalities make sense (note that $A \overset{\star}{\vee} (C \overset{\star}{\wedge} B)$ make sense for every B, given that C is a $\star$-upper bound for A and $C \overset{\star}{\wedge} B$).

If A, B and C are orthogonal projections, from (4.21) we have that expressions mentioned above always exist. For such A, B and C, the questions asked above are equivalent to the questions whether the lattice of all orthogonal projections with a usual order $\leq$ is distributive, or modular, which is in final equivalent to the question of whether lattice of all closed subspaces of $\mathcal{H}$ with inclusion order is distributive, or modular. It is known that the lattice of closed subspaces of $\mathcal{H}$ is modular if and only if $\mathcal{H}$ is finite-dimensional, and distributive only in the trivial case: $\dim \mathcal{H} \leq 1$ ([49, Problem 14.]). So we abandon our investigation on distributivity of $\star$-order, and we have the following result regarding modularity.

Theorem 4.3.20. *Let $\mathcal{H}$ be a Hilbert space. The following statements are equivalent:*

(1) $\mathcal{H}$ is finite-dimensional;

(2) For every $A, C \in \mathcal{B}(\mathcal{H})$ such that $A \overset{\star}{\leq} C$, and every $B \in \mathcal{B}(\mathcal{H})$ such that $C \overset{\star}{\vee} B$ exists (and so $A \overset{\star}{\vee} B$ also exists) we have: $C \overset{\star}{\wedge} (A \overset{\star}{\vee} B) = A \overset{\star}{\vee} (C \overset{\star}{\wedge} B)$.

Proof. If we have that (2) holds then choosing for A, B and C orthogonal projections, we obtain that the lattice of all closed subspaces of $\mathcal{H}$ is modular, which implies (1), as we explained. We will prove that (1) implies (2).

Let $A, C \in \mathcal{B}(\mathcal{H})$ be arbitrary operators such that $A \overset{\star}{\leq} C$ and $B \in \mathcal{B}(\mathcal{H})$ be such that $C \overset{\star}{\vee} B$ exists. In every partially ordered set, as in this one, we have $A \overset{\star}{\vee} (C \overset{\star}{\wedge} B) \overset{\star}{\leq} C \overset{\star}{\wedge} (A \overset{\star}{\vee} B)$. It is enough to prove that $\mathcal{N}(C \overset{\star}{\wedge} (A \overset{\star}{\vee} B)) = \mathcal{N}(A \overset{\star}{\vee} (C \overset{\star}{\wedge} B))$, having in mind the equivalence in (4.20). From Lemma 4.3.9 and Theorem 4.3.12, given that C and B, as well as, C and $A \overset{\star}{\vee} B$ have a common $\star$-upper bound and that $\mathcal{H}$ is finite-dimensional, we have that $\mathcal{N}(C \overset{\star}{\wedge} (A \overset{\star}{\vee} B)) = \mathcal{N}(C) + (\mathcal{N}(A) \cap \mathcal{N}(B))$ and $\mathcal{N}(A \overset{\star}{\vee} (C \overset{\star}{\wedge} B)) = \mathcal{N}(A) \cap (\mathcal{N}(C) + \mathcal{N}(B))$. Now from $\mathcal{N}(C) \subseteq \mathcal{N}(A)$, equality $\mathcal{N}(C) + (\mathcal{N}(A) \cap \mathcal{N}(B)) = \mathcal{N}(A) \cap (\mathcal{N}(C) + \mathcal{N}(B))$ follows readily. □

Note that the only step where we used finite-dimensionality of $\mathcal{H}$ is to express null-spaces of $\star$-infimums without closure operator. In every vector space $\mathcal{X}$, if $\mathcal{U}, \mathcal{W}$ and $\mathcal{V}$ are subspaces, we have the modular law: $\mathcal{U} \subseteq \mathcal{W} \ \Rightarrow \ \mathcal{U} + (\mathcal{W} \cap \mathcal{V}) = \mathcal{W} \cap (\mathcal{U} + \mathcal{V})$. But if $\mathcal{H}$ is infinite-dimensional, desired equality would be $\overline{\mathcal{N}(C) + (\mathcal{N}(A) \cap \mathcal{N}(B))} = \mathcal{N}(A) \cap \overline{\mathcal{N}(C) + \mathcal{N}(B)}$ and this is not true in general. In the case when the last equality holds, we also have the intended equality of operators $C \overset{\star}{\wedge} (A \overset{\star}{\vee} B) = A \overset{\star}{\vee} (C \overset{\star}{\wedge} B)$.

In the end we describe some properties of the $\star$-supremum regarding the convergence of the sequence of operators, motivated by the results from [5]. Namely, in [5], it is proved that every $\star$-decreasing (or $\star$-increasing and $\star$-bounded) sequence has a strong limit which is one $\star$-lower bound ($\star$-upper bound) for operators in the sequence:

Lemma 4.3.21 (See [5]). *If (A_n) is a sequence of $\star$-decreasing operators from $\mathcal{B}(\mathcal{H})$, then there exists $A \in \mathcal{B}(\mathcal{H})$ such that $A_n \xrightarrow{s} A$. Moreover, A is the $\star$-infimum of the set $\{A_n \ : \ n \in \mathbb{N}\}$.*

If (A_n) is a sequence of $\star$-increasing operators from $\mathcal{B}(\mathcal{H})$, which has a $\star$-upper bound, then there exists $A \in \mathcal{B}(\mathcal{H})$ such that $A_n \xrightarrow{s} A$. Moreover, A is the $\star$-supremum of the set $\{A_n \ : \ n \in \mathbb{N}\}$.

It is also proved in [5], that $\star$-infimum agrees with $\star$-decreasing sequences, in the following sense: if (A_n) and (B_n) are two $\star$-decreasing sequences such that $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$, then $A_n \overset{\star}{\wedge} B_n \xrightarrow{s} A \overset{\star}{\wedge} B$. A counterexample is provided for the dual statement about $\star$-increasing sequences.

We here note that, as expected, $\star$-supremum agrees well with $\star$-increasing sequences, and not with $\star$-decreasing. Of course, we have to pay special attention on the existence of $\star$-supremum. The existence of one common $\star$-upper bound implies the existence of $\star$-supremum for two operators. Thus, for two $\star$-increasing sequences, it is enough to assume that the $\star$-supremum exists only for their limits. The proof then goes automatically from Lemma 4.3.21.

Theorem 4.3.22. *Let (A_n) and (B_n) be two $\star$-increasing sequences, such that $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$. If $A \overset{\star}{\vee} B$ exists, then for every $n \in \mathbb{N}$, $A_n \overset{\star}{\vee} B_n$ exists and $A_n \overset{\star}{\vee} B_n \xrightarrow{s} A \overset{\star}{\vee} B$.*

Proof. From the definition of the $\star$-order we conclude that A and B are $\star$-upper bounds for sequences (A_n) and (B_n) respectively, and from the existence of $A \overset{\star}{\vee} B$ we conclude the existence of $A_n \overset{\star}{\vee} B_n$, for all $n \in \mathbb{N}$. The sequence $(A_n \overset{\star}{\vee} B_n)$ is a $\star$-increasing sequence with $\star$-upper bound: $A \overset{\star}{\vee} B$. So, from Lemma 4.3.21, $A_n \overset{\star}{\vee} B_n \xrightarrow{s} D$ and $D \overset{\star}{\leq} A \overset{\star}{\vee} B$. On the other hand, also from Lemma 4.3.21, $A \overset{\star}{\leq} D$ and $B \overset{\star}{\leq} D$, and so $A \overset{\star}{\vee} B \overset{\star}{\leq} D$, i.e. $A \overset{\star}{\vee} B = D$. □

To see that $\star$-supremum does not agree with $\star$-decreasing sequences, we can use [5, Example 3.9], only with $I - P$ instead of P, for every projection P appearing in this example, and having in mind formula (4.22).

We finish with two theorems describing conditions when the existence of $A_n \overset{\star}{\vee} B_n$ for every n will force the existence of $A \overset{\star}{\vee} B$. One assumes $\star$-monotonicity of the sequences, and the other has no such assumptions, but assumes normality of operators.

Theorem 4.3.23. *Let (A_n) and (B_n) be two $\star$-increasing sequences, such that $A_n \overset{\star}{\vee} B_n$ exists for every n. If $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$, and $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ or $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ is closed, then $A \overset{\star}{\vee} B$ exists and $A_n \overset{\star}{\vee} B_n \xrightarrow{s} A \overset{\star}{\vee} B$.*

Proof. As noted before, A is also the $\star$-supremum for the sequence (A_n), then $P_{\overline{\mathcal{R}(A)}}$ is the strong limit of the sequence (P_{A_n}) and $P_{\overline{\mathcal{R}(A^*)}}$ is the strong limit of the sequence (Q_{A_n}) (see [5]). Analogous conclusions holds for sequence (B_n) and B. Thus $P_{\overline{\mathcal{R}(A)}}B$ is the strong limit of $(P_{A_n}B_n)$ while $AP_{\overline{\mathcal{R}(B^*)}}$ is the strong limit of $(A_nQ_{B_n})$. We have that $A_n \overset{\star}{\vee} B_n$ exists for every n, and so $P_{A_n}B_n = A_nQ_{B_n}$. In this way we get that $P_{\overline{\mathcal{R}(A)}}B = AP_{\overline{\mathcal{R}(B^*)}}$, and in a similar way we get $P_{\overline{\mathcal{R}(B)}}A = BP_{\overline{\mathcal{R}(A^*)}}$. From Theorem 4.3.4 we obtain that $A \overset{\star}{\vee} B$ exists, and from Theorem 4.3.22 that $A_n \overset{\star}{\vee} B_n \xrightarrow{s} A \overset{\star}{\vee} B$. □

Theorem 4.3.24. *Let (A_n) and (B_n) be sequences such that $A_n \overset{\star}{\vee} B_n$ exists for every n. If $A_n \overset{s}{\to} A$, $B_n \overset{s}{\to} B$, operators A_n, B_n, A and B are normal, and $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ is closed, then $A \overset{\star}{\vee} B$ exists.*

Proof. From the normality of operators A_n and A we have that $A_n^* \overset{s}{\to} A^*$, and in the same way $B_n^* \overset{s}{\to} B^*$. From the existence of $A_n \overset{\star}{\vee} B_n$ for every n, we have that $A_n(B_n^* - A_n^*)B_n = 0$ and $B_n(B_n^* - A_n^*)A_n = 0$, for every n. Thus $A(B^* - A^*)B = 0 = B(B^* - A^*)A$, and from Theorem 4.3.4, we have that $A \overset{\star}{\vee} B$ exists. $\square$

4.4 Results on the core partial order

In this section, we will prove that the set $\mathcal{B}^1(\mathcal{H})$ with respect to the $\textcircled{\#}$-partial order is in fact a complete lower semi-lattice, meaning that an arbitrary subset of $\mathcal{B}^1(\mathcal{H})$ has the $\textcircled{\#}$-infimum. This will follow from the fact proved in Theorem 4.4.3 stating that $\mathcal{B}^1(\mathcal{H})$ has the so called upper bound property: for any subset $\{A_j \mid j \in J\} \subseteq \mathcal{B}^1(\mathcal{H})$, the existence of the $\textcircled{\#}$-supremum is equivalent to the existence of one common $\textcircled{\#}$-upper bound. However, it is easy to see that not all $A, B \in \mathcal{B}^1(\mathcal{H})$ have a common $\textcircled{\#}$-upper bound (for example, take $A \neq B$ to be invertible). We will also give some necessary and sufficient conditions for the existence of the $\textcircled{\#}$-supremum of two operators. Henceforth, we denote the lattice operations in this partial order with $\wedge^{\textcircled{\#}}$ and $\vee^{\textcircled{\#}}$.

In the following statements, let $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$ denote a family of operators with a common $\textcircled{\#}$-upper bound $A \in \mathcal{B}^1(\mathcal{H})$. Denote by $\mathcal{R}_1$ the vector space spanned by the set of vectors $\bigcup_{i\in I} \mathcal{R}(A_i)$, i.e. $\mathcal{R}_1 = \{x_{i_1} + ... + x_{i_n} \mid x_{i_1} \in \mathcal{R}(A_{i_1}), ..., x_{i_n} \in \mathcal{R}(A_{i_n}),\ i_1, ..., i_n \in I,\ n \in \mathbb{N}\}$, and put $\mathcal{R} = \overline{\mathcal{R}_1}$. Let $\mathcal{N}$ denote $\bigcap_{i\in I} \mathcal{N}(A_i)$ and $\mathcal{N}^*$ denote $\mathcal{R}^\perp = \bigcap_{i\in I} \mathcal{N}(A_i^*)$.

Lemma 4.4.1. *It holds $\mathcal{R} \subseteq \mathcal{R}(A)$, the reduction $A : \mathcal{R} \to \mathcal{R}$ is well-defined and it is a bijection. Moreover, the reduction $A' : \mathcal{R} \to \mathcal{R}$ is the same for any common $\textcircled{\#}$-upper bound $A' \in \mathcal{B}^1(\mathcal{H})$ of the family $\{A_i \mid i \in I\}$.*

Proof. On every subspace $\mathcal{R}(A_i)$ the operators A and A_i coincide, and so $A(\mathcal{R}(A_i)) = \mathcal{R}(A_i)$. Thus $A(\mathcal{R}_1) = \mathcal{R}_1$ which gives $A(\mathcal{R}) \subseteq \mathcal{R}$, showing that this reduction is well-defined. Also, from $A(\mathcal{R}_1) = \mathcal{R}_1$, we conclude that $\mathcal{R} \subseteq \mathcal{R}(A)$, showing that this reduction is injective.

Let $y \in \mathcal{R}$ be arbitrary. Then there is some $x \in \mathcal{R}(A)$ such that $Ax = y$, and let us prove that $x \in \mathcal{R}$. Since $y \in \mathcal{R}$, there is a sequence $(y_n) \subseteq \mathcal{R}_1$ such that $y_n \to y$. For every $y_n \in \mathcal{R}_1$ there is a finite sequence of indices $i_{n,1}, i_{n,2}, ..., i_{n,k_n}$ and vectors $b_{i_{n,1}}, b_{i_{n,2}}, ..., b_{i_{n,k_n}}$, such that $y_n = b_{i_{n,1}} + b_{i_{n,2}} + ... + b_{i_{n,k_n}}$, where $b_{i_{n,1}} \in \mathcal{R}(A_{i_{n,1}})$, $b_{i_{n,2}} \in \mathcal{R}(A_{i_{n,2}})$, ..., $b_{i_{n,k_n}} \in \mathcal{R}(A_{i_{n,k_n}})$. Operators A_i are of index at most 1, so there are $a_{i_{n,1}} \in \mathcal{R}(A_{i_{n,1}})$, $a_{i_{n,2}} \in \mathcal{R}(A_{i_{n,2}})$, ..., $a_{i_{n,k_n}} \in \mathcal{R}(A_{i_{n,k_n}})$ such that $A_{i_{n,1}} a_{i_{n,1}} = b_{i_{n,1}}$, $A_{i_{n,2}} a_{i_{n,2}} = b_{i_{n,2}}$, ..., $A_{i_{n,k_n}} a_{i_{n,k_n}} = b_{i_{n,k_n}}$. Denote by $x_n = a_{i_{n,1}} + a_{i_{n,2}} + ... + a_{i_{n,k_n}}$. Then $x_n \in \mathcal{R}_1$, and since A coincides with A_i on $\mathcal{R}(A_i)$, we have that $Ax_n = y_n$.

We have now that $A(x_n - x) = y_n - Ax \to y - y = 0$. Since $x_n - x \in \mathcal{R}(A) = \overline{\mathcal{R}(A)}$, we conclude that $x_n - x \to 0$, i.e. $x \in \mathcal{R}$. Thus, the reduction is also surjective.

To prove the last part of the statement, note that A and A' coincide on every $\mathcal{R}(A_i)$, and so on $\mathcal{R}_1$, but due to continuity, they also coincide on $\mathcal{R}$. □

Theorem 4.4.2. *It holds $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$.*

Proof. Suppose that $x \in \mathcal{R} \cap \mathcal{N}$. From Lemma 4.4.1, we have that $Ax \in \mathcal{R}$. On the other hand, since $x \in \mathcal{N}$, from Lemma 4.1.11, we have that $Ax \in \mathcal{N}^* = \mathcal{R}^\perp$. This yields $Ax = 0$, but $x \in \mathcal{R} \subseteq \mathcal{R}(A)$. Thus $x = 0$, showing that $\mathcal{R} \cap \mathcal{N} = \{0\}$.

It remains to prove that $\mathcal{R} + \mathcal{N} = \mathcal{H}$. Let us first prove that the (well-defined) reduction $A : \mathcal{R}(A) \cap \mathcal{N} \to \mathcal{R}(A) \cap \mathcal{N}^*$ is a bijection. This reduction is injective, since A is injective on $\mathcal{R}(A)$. To show that it is surjective, pick any $y \in \mathcal{R}(A) \cap \mathcal{N}^*$. There is $x \in \mathcal{R}(A)$ such that $Ax = y$. For every $i \in I$ we have $A_i^{\textcircled{\#}} Ax = A_i^{\textcircled{\#}} y = 0$ and since $A_i \leq^{\textcircled{\#}} A$ we deduce $0 = A_i^{\textcircled{\#}} Ax = A_i^{\textcircled{\#}} A_i x$, i.e. $x \in \mathcal{N}(A_i)$. Thus $x \in \mathcal{R}(A) \cap \mathcal{N}$, and so this reduction is also surjective.

Denote by $S = \mathcal{R}(A) \cap \mathcal{N}$. Since $\mathcal{N}(A)$ is a part of $\mathcal{N}$ and $\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{H}$, we can easily conclude that $S \oplus \mathcal{N}(A) = \mathcal{N}$. We have that $\mathcal{R} \cap S = \{0\}$ so if we prove that $\mathcal{R} \oplus S = \mathcal{R}(A)$, we have that:

$$\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{R} \oplus S \oplus \mathcal{N}(A) = \mathcal{R} \oplus \mathcal{N}.$$

Denote by $S_1 = \mathcal{R}(A) \cap \mathcal{N}^*$. Since $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}^*$, we can easily conclude (in the same way as before) that $\mathcal{R}(A) = \mathcal{R} \oplus S_1$.

Now take any $x \in \mathcal{R}(A)$ and let $y = Ax$. Then $y = r + s_1$ where $r \in \mathcal{R}$ and $s_1 \in S_1$. From Lemma 4.4.1 it follows that there is $\rho \in \mathcal{R}$ such that $A\rho = r$, and since $A : S \to S_1$ is a bijection, it follows that there is $\sigma \in S$ such that $A\sigma = s_1$. So $Ax = y = A(\rho + \sigma)$, while $x, \rho + \sigma \in \mathcal{R}(A)$. So $x = \rho + \sigma \in \mathcal{R} \oplus S$. Thus $\mathcal{R}(A) = \mathcal{R} \oplus S$, and the theorem is proved. □

Now we prove the upper bound property of the structure $(\mathcal{B}^1(\mathcal{H}), \leq^{\textcircled{\#}})$.

Theorem 4.4.3. *If $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$ then the following statements are equivalent:*

(i) There exists $A \in \mathcal{B}^1(\mathcal{H})$ such that $A_i \leq^{\textcircled{\#}} A$ for every $i \in I$;

(ii) There exists $\overset{\textcircled{\#}}{\bigvee\limits_{i \in I}} A_i$.

Proof. Since (ii) ⇒ (i) is clear, we prove (i) ⇒ (ii).

Denote by $\mathcal{R}$ and $\mathcal{N}$ the subsets defined by the family $\{A_i \mid i \in I\}$ as before, and let $P = P_{\mathcal{R},\mathcal{N}}$, which exists by Theorem 4.4.2. We will prove that $B = AP$ is the $\textcircled{\#}$-supremum of this family. From Theorem 4.4.2 and Lemma 4.4.1, it follows that $B \in \mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(B) = \mathcal{R}$ and $\mathcal{N}(B) = \mathcal{N}$. Using $A_i P = A_i$ and $PA_i^{\textcircled{\#}} = A_i^{\textcircled{\#}}$ for every $i \in I$ (the first equality follows from $\mathcal{N}(P) \subseteq \mathcal{N}(A_i)$ and the second one from $\mathcal{R}(A_i^{\textcircled{\#}}) \subseteq \mathcal{R}(P)$), from $A_i A_i^{\textcircled{\#}} = AA_i^{\textcircled{\#}}$ and $A_i^{\textcircled{\#}} A_i = A_i^{\textcircled{\#}} A$, respectively we get,

$A_iA_i^{\textcircled{\#}} = BA_i^{\textcircled{\#}}$ and $A_i^{\textcircled{\#}}A_i = A_i^{\textcircled{\#}}B$. Thus, B is indeed one $\textcircled{\#}$-common upper bound for $\{A_i \mid i \in I\}$. Suppose that B_1 is another one and let us prove that $B \leq^{\textcircled{\#}} B_1$.

From Lemma 4.4.1 we know that B and B_1 are the same on $\mathcal{R} = \mathcal{R}(B)$. Hence, we have $BB^{\textcircled{\#}} = B_1B^{\textcircled{\#}}$. We already know that operators $B^{\textcircled{\#}}B$ and $B^{\textcircled{\#}}B_1$ are the same on $\mathcal{R}$, while on $\mathcal{N}$ both of them are equal to the null-operator: the first one because $\mathcal{N}(B) = \mathcal{N}$, and the second one since $B_1(\mathcal{N}) \subseteq \mathcal{N}^*$ (see Lemma 4.1.11), while $\mathcal{N}^* = \mathcal{R}^\perp = \mathcal{R}(B)^\perp = \mathcal{N}(B^{\textcircled{\#}})$. So from $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$ (Theorem 4.4.2) we get that $B^{\textcircled{\#}}B = B^{\textcircled{\#}}B_1$. This completes the proof. □

Previous considerations can be summarized in the next corollary.

Corollary 4.4.4. *If a family $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$ has some common $\textcircled{\#}$-upper bound $A \in \mathcal{B}^1(\mathcal{H})$, then $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$, operator $AP_{\mathcal{R},\mathcal{N}}$ does not depend on the choice of A and it is the $\textcircled{\#}$-supremum of the family $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$. Moreover, $\mathcal{R}(\overset{\textcircled{\#}}{\bigvee_{i\in I}} A_i) = \mathcal{R}$ and $\mathcal{N}(\overset{\textcircled{\#}}{\bigvee_{i\in I}} A_i) = \mathcal{N}$.*

Theorem 4.4.5. *If $\{A_j \mid j \in J\} \subseteq \mathcal{B}^1(\mathcal{H})$ is an arbitrary family, then $\overset{\textcircled{\#}}{\bigwedge_{j\in J}} A_j$ exists.*

Proof. Since the set of all common $\textcircled{\#}$-lower bounds of $\{A_j \mid j \in J\}$ is nonempty (it contains the null-operator), and has at least one common $\textcircled{\#}$-upper bound (any A_j will suffice), from Theorem 4.4.3 we conclude that it has the $\textcircled{\#}$-supremum. Now, by a simple order-theoretic argument, it follows that this $\textcircled{\#}$-supremum is in fact the $\textcircled{\#}$-infimum for $\{A_j \mid j \in J\}$. □

Theorem 4.4.2 gives one necessary condition for the existence of a common $\textcircled{\#}$-upper bound of a family of operators. We will derive a necessary and sufficient condition for an arbitrary family $\{A_i \mid i \in I\}$ to have at least one common $\textcircled{\#}$-upper bound, i.e. to have $\textcircled{\#}$-supremum. Special attention will be given to the families $\{A_1, A_2\}$, where under some restrictions, these conditions are simplified. For example, if A_1 and A_2 are square matrices, we only need to check these simplified conditions.

Theorem 4.4.6. *Let $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$. Then $\overset{\textcircled{\#}}{\bigvee_{i\in I}} A_i$ exists if and only if the following conditions are satisfied:*

(1) $\{(A_i, \mathcal{R}(A_i)) \mid i \in I\}$ are coherent pairs and $\{(A_i^{\textcircled{\#}}, \mathcal{R}(A_i)) \mid i \in I\}$ are coherent pairs;

(2) For every $i, j \in I$, it holds $A_i^{\textcircled{\#}}A_jA_j^{\textcircled{\#}} = A_i^{\textcircled{\#}}A_iA_j^{\textcircled{\#}}$;

(3) $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$, where $\mathcal{R}$ is the closure of the subspace spanned by the set $\bigcup_{i\in I} \mathcal{R}(A_i)$, while $\mathcal{N} = \bigcap_{i\in I} \mathcal{N}(A_i)$.

Proof. If $A = \overset{\textcircled{\#}}{\bigvee_{i\in I}} A_i$ exists, then A coincides with A_i on $\mathcal{R}(A_i)$, for every $i \in I$, and $A^{\textcircled{\#}}$ coincides with $A_i^{\textcircled{\#}}$ on $\mathcal{R}(A_i)$, for every $i \in I$, so condition (1) is satisfied. Conditions (2) and (3) follow from Lemma 2.2.9 and Theorem 4.4.2.

Now suppose that (1),(2) and (3) are fulfilled. Denote by $A_1 \in \mathcal{B}(\mathcal{H})$ respectively $B_1 \in \mathcal{B}(\mathcal{H})$, the operator that coincides with A_i on $\mathcal{R}(A_i)$ for every $i \in I$, respectively with $A_i^{\textcircled{\#}}$ on $\mathcal{R}(A_i)$ for every $i \in I$. Let $P = P_{\mathcal{R},\mathcal{N}}$, and $A = A_1P$, $B = B_1P$, and as before, let $\mathcal{R}_1$ be the subspace spanned by the set $\bigcup_{i\in I} \mathcal{R}(A_i)$. In that case we have $A(\mathcal{R}_1) = \mathcal{R}_1$ and $\mathcal{R}(A) = \mathcal{R}(AP) = A(\mathcal{R}) \subseteq \overline{A(\mathcal{R}_1)} = \mathcal{R}$. Similarly $\mathcal{R}(B) \subseteq \mathcal{R}$, since $B(\mathcal{R}(A_i^{\textcircled{\#}})) = \mathcal{R}(A_i)$ for every $i \in I$. Thus we can take reductions $\tilde{A}$ and $\tilde{B}$ of A and B on $\mathcal{R}$. Operator $\tilde{B}\tilde{A}$ is equal to identity on every $\mathcal{R}(A_i)$, thus on $\mathcal{R}_1$. Since it is bounded, it is equal to identity on whole $\mathcal{R}$. Similarly, $\tilde{A}\tilde{B} = I$. This means that $\tilde{A}$ and $\tilde{B}$ are both injective and surjective, which leads us to the conclusion that $\mathcal{N}(A) = \mathcal{N}$, then $A \in \mathcal{B}^1(\mathcal{H})$ and $B = A^{\sharp}$.

We will complete the proof by showing that A is one $\textcircled{\#}$-common upper bound for $\{A_i \mid i \in I\}$. Operators A and A_i coincide on $\mathcal{R}(A_i)$, and so $A_iA_i^{\textcircled{\#}} = AA_i^{\textcircled{\#}}$, for every $i \in I$. The equality $A_i^{\textcircled{\#}}A_i = A_i^{\textcircled{\#}}A$ obviously holds on $\mathcal{N}$, but also on every $\mathcal{R}(A_j)$, $j \in I$: if we take any $y \in \mathcal{R}(A_j)$ then $Ay = A_jy$ and there is some x such that $y = A_j^{\textcircled{\#}}x$, so: $A_i^{\textcircled{\#}}(A - A_i)y = A_i^{\textcircled{\#}}(A_j - A_i)y = A_i^{\textcircled{\#}}(A_j - A_i)A_j^{\textcircled{\#}}x = 0$, by (2). By continuity and (3), we have that $A_i^{\textcircled{\#}}A_i = A_i^{\textcircled{\#}}A$. Therefore A is indeed one $\textcircled{\#}$-common upper bound for $\{A_i \mid i \in I\}$, and by Theorem 4.4.3, $\overset{\textcircled{\#}}{\bigvee_{i\in I}} A_i$ exists. □

In what follows, we deal with the case $\{A_i \mid i \in I\} = \{A, B\}$. We are going to use Lemmas 2.2.9, 2.2.10 and 2.2.11 from Section 2.2. The following theorem simplifies the conditions of Theorem 4.4.2 in the case when $\mathcal{R}(A) + \mathcal{R}(B)$ is closed and $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional.

Theorem 4.4.7. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$ be such operators that $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, while $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional. Then $A \vee^{\textcircled{\#}} B$ exists if and only if the following conditions are satisfied:*

(1') $A^{\textcircled{\#}}BB^{\textcircled{\#}} = A^{\textcircled{\#}}AB^{\textcircled{\#}}$ and $B^{\textcircled{\#}}AA^{\textcircled{\#}} = B^{\textcircled{\#}}BA^{\textcircled{\#}}$;

(2') $\mathcal{H} = (\mathcal{R}(A) + \mathcal{R}(B)) + [\mathcal{N}(A) \cap \mathcal{N}(B)]$.

Proof. We only need to prove that (1') and (2') imply conditions (1), (2) and (3) of Theorem 4.4.6. Clearly, (2) holds. From Lemma 2.2.11 we have that (1) also holds. To see that (3) holds we use Lemma 2.2.10, the fact that $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, and (2'). □

The following example shows that condition (2') of Theorem 4.4.7 can not be omitted.

Example 22. Let $A \in \mathcal{B}(\mathcal{H})$ be some (not necessarily orthogonal) projection. In that case, $A^{\text{\textcircled{\#}}} = P_{\mathcal{R}(A)}$, $AA^{\text{\textcircled{\#}}} = A^{\text{\textcircled{\#}}} = P_{\mathcal{R}(A)}$ and $A^{\text{\textcircled{\#}}}A = A$.

So if $A, B \in \mathcal{B}(\mathcal{H})$ are projections such that $\mathcal{R}(A) = \mathcal{R}(B)$ we certainly have $A^{\text{\textcircled{\#}}}BB^{\text{\textcircled{\#}}} = A^{\text{\textcircled{\#}}}AB^{\text{\textcircled{\#}}}$ and $B^{\text{\textcircled{\#}}}AA^{\text{\textcircled{\#}}} = B^{\text{\textcircled{\#}}}BA^{\text{\textcircled{\#}}}$.

We can easily choose two projections A and B with the same range and such that $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$, as long as the dimension of $\mathcal{H}$ is greater than 1. Thus, in general, $A^{\text{\textcircled{\#}}}BB^{\text{\textcircled{\#}}} = A^{\text{\textcircled{\#}}}AB^{\text{\textcircled{\#}}}$ and $B^{\text{\textcircled{\#}}}AA^{\text{\textcircled{\#}}} = B^{\text{\textcircled{\#}}}BA^{\text{\textcircled{\#}}}$ do not imply that $\overline{\mathcal{R}(A) + \mathcal{R}(B)} + (\mathcal{N}(A) \cap \mathcal{N}(B)) = \mathcal{H}$. $\diamond$

We now refer to the case when A and B are two square matrices. If A and B are two square matrices of appropriate sizes and with the index at most 1, instead of checking condition (1') of Theorem 4.4.7, we readily check an equivalent condition as the one in (iii) in Lemma 2.2.9. In order to give a more computation-ready character to the condition (2') we present the following proposition. If X and Y are two square $n \times n$ matrices, then with $[X \ \ Y]$, we denote the matrix obtained by adjoining the columns of the matrix Y to the columns of the matrix X.

Proposition 4.4.8. *Let A and B be two complex $n \times n$ matrices such that $(\mathcal{R}(A) + \mathcal{R}(B)) \cap [\mathcal{N}(A) \cap \mathcal{N}(B)] = \{0\}$. The following statements are equivalent:*

(i) $(\mathcal{R}(A) + \mathcal{R}(B)) \oplus [\mathcal{N}(A) \cap \mathcal{N}(B)] = \mathbb{C}^n$;

(ii) $\mathrm{r}([A \ \ B]) = \mathrm{r}([A^* \ \ B^*])$;

(iii) $\mathrm{r}(AA^* + BB^*) = \mathrm{r}(A^*A + B^*B)$.

Proof. Since $(\mathcal{R}(A) + \mathcal{R}(B)) \cap [\mathcal{N}(A) \cap \mathcal{N}(B)] = \{0\}$, then $(\mathcal{R}(A) + \mathcal{R}(B)) \oplus [\mathcal{N}(A) \cap \mathcal{N}(B)] = \mathbb{C}^n$ if and only if $\dim(\mathcal{R}(A)+\mathcal{R}(B)) + \dim(\mathcal{N}(A)\cap\mathcal{N}(B)) = n$. We already know that $(\mathcal{R}(A^*) + \mathcal{R}(B^*)) \overset{\perp}{\oplus} [\mathcal{N}(A) \cap \mathcal{N}(B)] = \mathbb{C}^n$, thus $\dim(\mathcal{R}(A) + \mathcal{R}(B)) + \dim(\mathcal{N}(A) \cap \mathcal{N}(B)) = n$ if and only if $\dim(\mathcal{R}(A) + \mathcal{R}(B)) = \dim(\mathcal{R}(A^*) + \mathcal{R}(B^*))$. In this way, we obtain (i) $\Leftrightarrow$ (ii).
To show that (ii) $\Leftrightarrow$ (iii) recall the result of Theorems 1.2.11 and 1.2.10: $\mathcal{R}(A)+\mathcal{R}(B) = \mathcal{R}((AA^*+BB^*)^{1/2}) = \mathcal{R}(AA^*+BB^*)$, since $\mathbb{C}^n$ is finite-dimensional, and every subspace is closed. Now (ii) $\Leftrightarrow$ (iii) is clear. $\square$

Corollary 4.4.9. *If $A, B \in \mathbb{C}^{n\times n}$ are two matrices of indices at most 1, then $A \vee^{\text{\textcircled{\#}}} B$ exists if and only if the following conditions are satisfied:*

(1") $A^*AB = A^*B^2$ *and* $B^*BA = B^*A^2$;

(2") $\mathrm{r}(AA^* + BB^*) = \mathrm{r}(A^*A + B^*B)$.

Proof. From Lemmas 2.2.9 and 2.2.10 we have that (1") implies $(\mathcal{R}(A)+\mathcal{R}(B))\cap(\mathcal{N}(A)\cap\mathcal{N}(B)) = \{0\}$. Thus, according to Proposition 4.4.8, we have that condition (2") implies (2'). Hence if (1") and (2") are fulfilled, then so are (1') and (2'), showing that $A \vee^{\text{\textcircled{\#}}} B$ exists. The other implication is clear with Theorem 4.4.2, Lemmas 2.2.9 and 2.2.9, and Proposition 4.4.8 at our hands. $\square$

The $\textcircled{\#}$-supremum can exist for some A and B, while it does not exist for any of the pairs: (A^*, B^*), $(A^{\textcircled{\#}}, B^{\textcircled{\#}})$, $(A^\sharp, B^\sharp)$, $(A^{\textcircled{\#}}A, B^{\textcircled{\#}}B)$, (A^*A, B^*B), (AA^*, BB^*), $(|A|, |B|)$. It is also possible that $(A \wedge^{\textcircled{\#}} B)^\bullet$ differs from $A^\bullet \wedge^{\textcircled{\#}} B^\bullet$, where $\bullet$ can stand for *,$^{\textcircled{\#}}$,$^\sharp$, etc. This is due to the fact that the $\textcircled{\#}$-partial order is not transferable from $A \leq^{\textcircled{\#}} B$ to $A^\bullet \leq^{\textcircled{\#}} B^\bullet$. These observations, demonstrated in the following example, are unlike the ones for the star partial order, where we can expect this kind of duality (cf. Section 4.3).

Example 23. Let $\mathcal{H} = \mathbb{C}^3$ and A and B defined as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3/4 & 0 & \sqrt{3}/4 \\ 1 & 2 & 3 \\ \sqrt{3}/4 & 0 & 1/4 \end{bmatrix}.$$

Using Corollary 4.4.9 we readily check that $A \vee^{\textcircled{\#}} B$ exists. On the other hand, we have:

$$A^{\textcircled{\#}} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^{\textcircled{\#}} = \begin{bmatrix} 3/4 & 0 & \sqrt{3}/4 \\ (-3-3\sqrt{3})/8 & 1/2 & (-3-\sqrt{3})/8 \\ \sqrt{3}/4 & 0 & 1/4 \end{bmatrix}$$

$$A^\sharp = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 3/4 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^\sharp = \begin{bmatrix} 3/4 & 0 & \sqrt{3}/4 \\ (-5-9\sqrt{3})/16 & 1/2 & (3-3\sqrt{3})/16 \\ \sqrt{3}/4 & 0 & 1/4 \end{bmatrix}.$$

So we can see that the $\textcircled{\#}$-supremum does not exist for any of the above mentioned pairs. Moreover, if $D = A \wedge^{\textcircled{\#}} B$, then:

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \; D^{\textcircled{\#}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; D^\sharp = \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 1/2 & 3/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $D^\bullet \not\leq^{\textcircled{\#}} A^\bullet$, where $\bullet$ can be any of the following: *,$^{\textcircled{\#}}$,$^\sharp$. $\diamond$

If A and B are orthogonal projections, Lemma 4.1.13 shows that the $\textcircled{\#}$-supremum and $\textcircled{\#}$-infimum of A and B coincide with the regular supremum and infimum of A and B in the lattice of all orthogonal projections on $\mathcal{H}$. Namely, $A \wedge^{\textcircled{\#}} B = A \wedge B = P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$ and $A \vee^{\textcircled{\#}} B = A \vee B = P_{\overline{\mathcal{R}(A)+\mathcal{R}(B)}}$. However, for oblique projections the $\textcircled{\#}$-supremum need not exist, which we can see from Examples 22 and 26.

Observe that from Lemma 4.1.12 we have the following inclusions: $\mathcal{R}(A \wedge^{\textcircled{\#}} B) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\overline{\mathcal{N}(A) + \mathcal{N}(B)} \subseteq \mathcal{N}(A \wedge^{\textcircled{\#}} B)$. Equality is obtained if, for example, A and B are orthogonal projections. In the following theorem we describe the pairs of operators for which these inclusions become equalities. Again, the condition of precoherence is crucial, as with the analogous problem for the $\star$-partial order discussed in Section 4.3, but now with different underlying subspaces.

Theorem 4.4.10. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$ and $C = A \wedge^{\textcircled{\#}} B$. Then $\mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(C) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$ if and only if the following conditions are satisfied:*

(1) $(A, \mathcal{R}(A))$ *and* $(B, \mathcal{R}(B))$ *are precoherent;*

(2) $(A^*, \mathcal{R}(A))$ *and* $(B^*, \mathcal{R}(B))$ *are precoherent;*

(3) $\mathcal{H} = (\mathcal{R}(A) \cap \mathcal{R}(B)) + \overline{\mathcal{N}(A) + \mathcal{N}(B)}$.

Proof. Suppose first that $\mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(C) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$. Since $C \in \mathcal{B}^1(\mathcal{H})$ we have that condition (3) is satisfied. Condition (1) and (2) follow from Lemma 4.1.11, since both A and B are $\textcircled{\#}$-larger than C.

Now suppose that conditions (1), (2) and (3) are satisfied. If $n \in \mathcal{N}(B)$, and $y \in \mathcal{R}(A) \cap \mathcal{R}(B)$ then $\langle An, y\rangle = \langle n, A^*y\rangle = \langle n, B^*y\rangle = \langle Bn, y\rangle = 0$. This shows that $A(\overline{\mathcal{N}(A) + \mathcal{N}(B)}) \subseteq (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp$, while from (1) we have that $A(\mathcal{R}(A) \cap \mathcal{R}(B)) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$. From these conclusions we get that the sum in (3) is direct and also that $A(\mathcal{R}(A) \cap \mathcal{R}(B)) = \mathcal{R}(A) \cap \mathcal{R}(B)$. The same stands for B. Thus, we have:

$$A = \begin{bmatrix} D_{11} & 0 \\ 0 & A_1 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A) + \mathcal{N}(B)} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix},$$

$$B = \begin{bmatrix} D_{11} & 0 \\ 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A) + \mathcal{N}(B)} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix},$$

where D_{11} is an isomorphism. Let us define D in the following way:

$$D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A) + \mathcal{N}(B)} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix},$$

in which case we have $D \in \mathcal{B}^1(\mathcal{H})$ and:

$$D^{\textcircled{\#}} = \begin{bmatrix} D_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A) + \mathcal{N}(B)} \end{bmatrix}.$$

A direct calculation now shows that $D \leq^{\textcircled{\#}} A$ and $D \leq^{\textcircled{\#}} B$, and so $D \leq^{\textcircled{\#}} C$. On the other hand, from Lemma 4.1.12, since C is $\textcircled{\#}$-smaller than A and B, we have $\mathcal{R}(C) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{R}(D) \subseteq \mathcal{R}(C)$, implying that $D = C$. ☐

For the efficiency sake, the operators satisfying conditions of Theorem 4.4.10 will be called *core-parallel*, (or $\textcircled{\#}$-parallel).

Example 24. We should note that $\mathcal{R}(A \wedge^{\textcircled{\#}} B) = \mathcal{R}(A) \cap \mathcal{R}(B)$ is not equivalent to $\mathcal{N}(A \wedge^{\textcircled{\#}} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$.

We can take two non-null operators $A, B \in \mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and $\mathcal{N}(A) = \mathcal{N}(B)$, as long as $\dim \mathcal{H} \geq 2$. Then from $\mathcal{R}(A \wedge^{\textcircled{\#}} B) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$ we get $\mathcal{R}(A \wedge^{\textcircled{\#}} B) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(A \wedge^{\textcircled{\#}} B) = \mathcal{H} \neq \overline{\mathcal{N}(A) + \mathcal{N}(B)}$.

On the other hand, we can also take two rank-one operators $A, B \in \mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(A) = \mathcal{R}(B)$, $\mathcal{N}(A) \neq \mathcal{N}(B)$ and such that A and B do not coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$. Since condition (1) from Theorem 4.4.10 is not satisfied, we have $\mathcal{R}(A \wedge^{\textcircled{\#}} B) \subsetneq \mathcal{R}(A) \cap \mathcal{R}(B)$, which together with $\dim \mathcal{R}(A) \cap \mathcal{R}(B) = 1$ gives $\mathcal{R}(A \wedge^{\textcircled{\#}} B) = \{0\}$, i.e. $A \wedge^{\textcircled{\#}} B = 0$. Now we have $\mathcal{N}(A \wedge^{\textcircled{\#}} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)} = \mathcal{H}$, but $\mathcal{R}(A \wedge^{\textcircled{\#}} B) \neq \mathcal{R}(A) \cap \mathcal{R}(B)$. ◇

Example 25. Let us demonstrate that none of the conditions (1), (2) and (3) in Theorem 4.4.10 can be omitted.

The pair of operators A and B described in Example 24 with $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, shows that condition (1) and (2) can hold, while condition (3) does not hold.

If $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then the pair $(A, B) = (C, D)$ satisfies conditions (1) and (3) (in fact, the sum in (3) is direct), but it does not satisfy (2), while the pair $(A, B) = (C^*, D)$ satisfies (2) and (3) (again, the sum is direct) and does not satisfy (1). $\diamond$

One 'computational version' of Theorem 4.4.10 is contained in the following proposition.

Proposition 4.4.11. *Let $A, B \in \mathbb{C}^{n\times n}$ be two matrices of indices at most 1 and let $C = 2I - AA^\dagger - BB^\dagger$. Then A and B are $\textcircled{\#}$-parallel if and only if the following conditions are satisfied:*

(1') $\mathrm{r}([A^* - B^* \;\; C]) = \mathrm{r}(C)$*;*

(2') $\mathrm{r}([A - B \;\; C]) = \mathrm{r}(C)$*;*

(3') $\mathrm{r}(AA^* + BB^*) = \mathrm{r}(A^*A + B^*B)$*.*

Proof. Recall that if P and Q are two orthogonal projections such that $\mathcal{R}(P + Q)$ is closed, then $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{R}(P + Q)$ (Theorem 1.4.4).

Condition (1) of Theorem 4.4.10 is equivalent to $\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \mathcal{N}(A - B)$. Since $\mathcal{R}(A) \cap \mathcal{R}(B) = (\mathcal{R}(I - AA^\dagger) + \mathcal{R}(I - BB^\dagger))^\perp = \mathcal{R}(2I - AA^\dagger - BB^\dagger)^\perp$, we have that (1) is equivalent to $\mathcal{R}(C)^\perp \subseteq \mathcal{N}(A - B)$, i.e. with $\mathcal{R}(A^* - B^*) \subseteq \mathcal{R}(C)$. This is exactly (1').

Similarly, (2) is equivalent to (2').

Observe that implicit in the proof of Theorem 4.4.10 was the fact that (1) and (2) imply $(\mathcal{R}(A) \cap \mathcal{R}(B)) \cap (\mathcal{N}(A) + \mathcal{N}(B)) = \{0\}$. Under the condition $(\mathcal{R}(A) \cap \mathcal{R}(B)) \cap (\mathcal{N}(A) + \mathcal{N}(B)) = \{0\}$, the equality $\mathcal{H} = (\mathcal{R}(A) \cap \mathcal{R}(B)) \oplus (\mathcal{N}(A) + \mathcal{N}(B))$ holds if and only if $\dim(\mathcal{R}(A) \cap \mathcal{R}(B)) = \dim(\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$. From the relation $\dim(\mathcal{R}(A)+\mathcal{R}(B)) = \dim \mathcal{R}(A)+\dim \mathcal{R}(B)-\dim(\mathcal{R}(A)\cap\mathcal{R}(B))$, and likewise for A^* and B^*, and the fact $\dim \mathcal{R}(T) = \dim \mathcal{R}(T^*)$, we see that the equality $\dim(\mathcal{R}(A) \cap \mathcal{R}(B)) = \dim(\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$ is equivalent to $\dim(\mathcal{R}(A) + \mathcal{R}(B)) = \dim(\mathcal{R}(A^*) + \mathcal{R}(B^*))$, which is, like in Proposition 4.4.8, equivalent to (3'). Thus, under (1) and (2) of Theorem 4.4.10, (3) is equivalent to (3'). Since we already proved that (1) is equivalent to (1') and (2) is equivalent to (2'), we see that conditions (1), (2) and (3) of Theorem 4.4.10 are simultaneously satisfied if and only if conditions (1'), (2') and (3') are simultaneously satisfied, which proves the assertion of the theorem. $\square$

For orthogonal projections P and Q, such that $\mathcal{R}(P+Q)$ is closed, with $2P(P+Q)^\dagger Q$ we obtain an operator (in fact, the orthogonal projection) with the range $\mathcal{R}(P) \cap \mathcal{R}(Q)$ (Theorem 1.5.1). Thus, condition (1') of Proposition 4.4.11 can be replaced with $(A - B)AA^\dagger(AA^\dagger + BB^\dagger)^\dagger BB^\dagger = 0$, and similarly for condition (2').

Although any two orthogonal projections are $\textcircled{\sharp}$-parallel (the conditions (1) and (2) are obviously satisfied when A and B are orthogonal projections, and (3) would follow from $\mathcal{R}(A)\cap\mathcal{R}(B)=\mathcal{R}(A^*)\cap\mathcal{R}(B^*)=\overline{\mathcal{N}(A)+\mathcal{N}(B)}^{\perp}$), if one of the projections is not orthogonal, it is fairly obvious that, in general, they are not $\textcircled{\sharp}$-parallel. This can also be seen from Example 26 below. In the following proposition we give another example of $\textcircled{\sharp}$-parallel operators.

Proposition 4.4.12. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be orthogonal projections. Then $\mathcal{R}(PQ)$ is closed if and only if $\mathcal{R}(QP)$ is closed, in which case PQ and QP are from $\mathcal{B}^1(\mathcal{H})$ and they are $\textcircled{\sharp}$-parallel.*

Proof. Since $(PQ)^* = QP$, the first statement is clear. The second statement follows from Theorem 3.2.1. The final statement follows from $\mathcal{R}(PQ)=\mathcal{R}(P)\cap(\mathcal{R}(Q)+\mathcal{N}(P))$ and $\mathcal{N}(PQ)=\mathcal{N}(Q)\oplus(\mathcal{R}(Q)\cap\mathcal{N}(P))$, and similarly for QP (note that $\mathcal{R}(Q)+\mathcal{N}(P)$ and $\mathcal{N}(Q)+\mathcal{R}(P)$ are closed, Theorem 1.2.9). In fact, $\mathcal{R}(PQ)\cap\mathcal{R}(QP)=\mathcal{R}(P)\cap\mathcal{R}(Q)$ and $\overline{\mathcal{N}(PQ)+\mathcal{N}(QP)}=\overline{\mathcal{N}(P)+\mathcal{N}(Q)}$ (thus $PQ\wedge^{\textcircled{\sharp}} QP = P\wedge Q$). □

In the end, we prove certain commutativity properties of the $\textcircled{\sharp}$-supremum and $\textcircled{\sharp}$-infimum. Recall that if $\mathcal{M}\subseteq\mathcal{B}(\mathcal{H})$ with $\mathcal{M}'$ we denote the commutant of $\mathcal{M}'=\{T\in\mathcal{B}(\mathcal{H}) : TM=MT, \text{ for all } M\in\mathcal{M}\}$. Double commutant of $\mathcal{M}$ is $(\mathcal{M}')'=\mathcal{M}''$. We will prove that $A\vee^{\textcircled{\sharp}} B\in\{A,B\}''$, when this supremum exists, and $A\wedge^{\textcircled{\sharp}} B\in\{A,B\}''$, when A and B are $\textcircled{\sharp}$-parallel. We also show that if A and B are not $\textcircled{\sharp}$-parallel, in general $A\wedge^{\textcircled{\sharp}} B\notin\{A,B\}''$.

Theorem 4.4.13. *Let $A, B\in\mathcal{B}^1(\mathcal{H})$.*

1) If $A\vee^{\textcircled{\sharp}} B$ exists, then $A\vee^{\textcircled{\sharp}} B\in\{A,B\}''$.

2) If A and B are $\textcircled{\sharp}$-parallel, then $A\wedge^{\textcircled{\sharp}} B\in\{A,B\}''$.

Proof. Let $T\in\{A,B\}'$ be arbitrary. In that case, $T(\mathcal{N}(A))\subseteq\mathcal{N}(A)$, $T(\mathcal{R}(A))\subseteq\mathcal{R}(A)$, and similarly for B.

1) If $A\vee^{\textcircled{\sharp}} B$ exists, we easily obtain that both of the operators $(A\vee^{\textcircled{\sharp}} B)T$ and $T(A\vee^{\textcircled{\sharp}} B)$ are the null-operator on $\mathcal{N}(A)\cap\mathcal{N}(B)$. If $x\in\mathcal{R}(A)$ then $Tx\in\mathcal{R}(A)$, and so $(A\vee^{\textcircled{\sharp}} B)Tx = ATx = TAx = T(A\vee^{\textcircled{\sharp}} B)x$, since A and $A\vee^{\textcircled{\sharp}} B$ coincide on $\mathcal{R}(A)$. Similarly, $(A\vee^{\textcircled{\sharp}} B)T$ and $T(A\vee^{\textcircled{\sharp}} B)$ coincide on $\mathcal{R}(B)$, which gives $(A\vee^{\textcircled{\sharp}} B)T = T(A\vee^{\textcircled{\sharp}} B)$ (Corollary 4.4.4). Thus $A\vee^{\textcircled{\sharp}} B\in\{A,B\}''$.

2) If A and B are $\textcircled{\sharp}$-parallel, the proof is similar, with only one difference: we prove that operators $(A\wedge^{\textcircled{\sharp}} B)T$ and $T(A\wedge^{\textcircled{\sharp}} B)$ coincide on $\mathcal{N}(A), \mathcal{N}(B)$ and $\mathcal{R}(A)\cap\mathcal{R}(B)$. Of course, we have in mind Theorem 4.4.10. □

Example 26. Let $\mathcal{H}=\mathbb{C}^4$ and A and B defined as follows:

$$A=\begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&-1\\0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}.$$

In that case

$$A \wedge^{\textcircled{\#}} B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and operators A and B are not $\textcircled{\#}$-parallel. If we take:

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we can see that $T \in \{A, B\}'$, but $(A \wedge^{\textcircled{\#}} B)T \neq T(A \wedge^{\textcircled{\#}} B)$. Hence, if we just remove the condition that A and B are $\textcircled{\#}$-parallel from part b) of Theorem 4.4.13, the statement would not hold. On the other hand, we can easily find two operators A and B which are not $\textcircled{\#}$-parallel and $A \wedge^{\textcircled{\#}} B = 0$, so trivially we would have $A \wedge^{\textcircled{\#}} B \in \{A, B\}''$. Thus A and B being $\textcircled{\#}$-parallel is not a necessary condition for $A \wedge^{\textcircled{\#}} B \in \{A, B\}''$. $\diamond$

From Theorem 4.4.13 we can see that the $\textcircled{\#}$-supremum of two operators, as well as their $\textcircled{\#}$-infimum, if they are $\textcircled{\#}$-parallel, belong to the von Neumann algebra generated by these two operators.

4.5 Infimums and the parallel sum

The main results of this section show that, in the presence of precoherence condition, the infimum of two operators in $\star$ and $\textcircled{\#}$-partial order can be expressed as twice their parallel sum, provided that their parallel sum exists (for example, for positive operators A and B the parallel sum always exists). Such results extend the well-known formula $P \wedge Q = 2(P : Q)$ for orthogonal projections P and Q and offer a better understanding of all three notions: precoherence, partial orders and parallel sum.

Lemma 4.5.1. *If A and B are weakly parallel summable operators on a Hilbert space $\mathcal{H}$, then:*

$$A \overset{\star}{\wedge} B \overset{\star}{\leq} 2(A : B).$$

Proof. First, we will prove that:

$$(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* = 2(A : B)(A \overset{\star}{\wedge} B)^*. \tag{4.23}$$

These operators coincide on $\mathcal{N}((A \overset{\star}{\wedge} B)^*)$ so we should prove that they coincide on $\mathcal{R}(A \overset{\star}{\wedge} B)$. If we take $x \in \mathcal{R}(A \overset{\star}{\wedge} B)$, then $(A \overset{\star}{\wedge} B)^* x = A^* x = B^* x = y \in \mathcal{R}((A \overset{\star}{\wedge} B)^*)$, and so $(A \overset{\star}{\wedge} B)y = Ay = By$. From Proposition 1.5.2 we see that, if $Aa = Ba$ for some $a \in \mathcal{H}$, then $2(A : B)a = Aa = Ba$. So we have:

$$2(A : B)(A \overset{\star}{\wedge} B)^* x = 2(A : B)y = Ay = (A \overset{\star}{\wedge} B)y = (A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* x.$$

Thus we have coincidence on $\mathcal{R}(A\overset{\star}{\wedge}B)$ and by continuity, also on $\overline{\mathcal{R}(A\overset{\star}{\wedge}B)}$. This proves (4.23). To prove an analogous equality $(A\overset{\star}{\wedge}B)^*(A\overset{\star}{\wedge}B) = 2(A:B)^*(A\overset{\star}{\wedge}B)$ we just interchange A and B with A^* and B^* in (4.23). $\square$

Theorem 4.5.2. *Let A and B be weakly parallel summable operators on a Hilbert space $\mathcal{H}$ such that A and B are precoherent, and A^* and B^* are precoherent. Then:*

$$2(A:B) = A\overset{\star}{\wedge}B.$$

Proof. First we will show that $2(A:B)\overset{\star}{\le}A$ and $2(A:B)\overset{\star}{\le}B$. We should prove that $4(A:B)^*(A:B) = 2A^*(A:B)$ and $4(A:B)(A:B)^* = 2A(A:B)^*$ in order to get $2(A:B)\overset{\star}{\le}A$. The operators $4(A:B)^*(A:B)$ and $2A^*(A:B)$ already coincide on $\mathcal{N}(A:B)$ so it is enough to prove that they coincide on $\overline{\mathcal{R}((A:B)^*)} \subseteq \overline{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}$. Now take arbitrary $\gamma\in\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)}$. From Proposition 1.5.2 and the precoherence we see that $A\gamma = B\gamma = (A:B)(2\gamma)$. Note that $A\gamma\in\mathcal{R}(A)\cap\mathcal{R}(B)$, and so again: $A^*(A\gamma) = (A^*:B^*)(2A\gamma)$. Hence we get: $4(A:B)^*(A:B)\gamma = A^*A\gamma = 2A^*(A:B)\gamma$. This proves that $4(A:B)^*(A:B) = 2A^*(A:B)$. The other equality: $4(A:B)(A:B)^* = 2A(A:B)^*$ follows by symmetry. So $2(A:B)\overset{\star}{\le}A$, and similarly, $2(A:B)\overset{\star}{\le}B$. This shows that $2(A:B)\overset{\star}{\le}A\overset{\star}{\wedge}B$. To prove that they are equal we use Lemma 4.5.1. $\square$

We give one remark and one example considering the last result. Remark considers the statement of Theorem 4.5.2 (and Theorem 4.5.7 mutatis mutandis) with condition of precoherence replaced by a weaker condition: when A and B coincide only on $\mathcal{R}(A^*)\cap\mathcal{R}(B^*)$ instead of $\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)}$, and similarly for A^* and B^*. In the example, we comment on a possible opposite implication in Theorem 4.5.2.

Remark 4.5.3. Example 3 shows that there exist positive operators A and B such that $\{0\}\neq\mathcal{R}(A^{1/2}) = \mathcal{R}(B^{1/2})$ and $\mathcal{R}(A)\cap\mathcal{R}(B) = \{0\}$. So $2(A:B)$ is not the null–operator, since $\mathcal{R}((A:B)^{1/2}) = \mathcal{R}(A^{1/2})\cap\mathcal{R}(B^{1/2})$ (Proposition 1.5.2), while $A\overset{\star}{\wedge}B = 0$, and A and B coincide on $\overline{\mathcal{R}(A)\cap\mathcal{R}(B)}$. Thus, in Theorem 4.5.2 we can not assume that A and B coincide on $\overline{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}$ while A^* and B^* coincide on $\overline{\mathcal{R}(A)\cap\mathcal{R}(B)}$. Maybe the conditions of Theorem 4.5.7, yet to be stated, can be weakened, since we have that $\mathcal{R}(A+B)$ is closed. However, the present proof of Theorem 4.5.7 is strongly based on the fact that coincidence is happening on bigger sets $\overline{\mathcal{R}(A)}\cap\overline{\mathcal{R}(B)}$ and $\overline{\mathcal{R}(A^*)}\cap\overline{\mathcal{R}(B^*)}$.

Example 27. The opposite implication in Theorem 4.5.2 is true if $\mathcal{H}$ is finite–dimensional, but in general, it is not. If A and B are weakly parallel summable and $2(A:B) = A\overset{\star}{\wedge}B$, we get that $\mathcal{R}(A)\cap\mathcal{R}(B)\subseteq\mathcal{R}(A:B) = \mathcal{R}(A\overset{\star}{\wedge}B)\subseteq\mathcal{R}(A)\cap\mathcal{R}(B)$. So $\mathcal{R}(A\overset{\star}{\wedge}B) = \overline{\mathcal{R}(A)\cap\mathcal{R}(B)}$, and similarly for A^* and B^*. Hence, we have that A and B coincide on $\overline{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}$ while A^* and B^* coincide on $\overline{\mathcal{R}(A)\cap\mathcal{R}(B)}$. If $\mathcal{H}$ is finite–dimensional, then this it is the same as the precoherence of these pairs, but in general, we can not obtain these precoherences. The same example mentioned in Remark 4.5.3 gives us positive operators A and B with dense ranges, such that $\mathcal{R}(A)\cap\mathcal{R}(B) = \{0\}$. Consider the operators A^2 and B^2. We have that: $\mathcal{R}((A^2:B^2)^{1/2}) = \mathcal{R}(A)\cap\mathcal{R}(B) = \{0\}$, so

$A^2 : B^2 = 0$, but also $A^2 \overset{\star}{\wedge} B^2 = 0$, since $\mathcal{R}(A^2) \cap \mathcal{R}(B^2) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$. On the other hand, A^2 and B^2 coincide on $\mathcal{R}(A^2) \cap \mathcal{R}(B^2) = \{0\}$, but not on $\overline{\mathcal{R}(A^2)} \cap \overline{\mathcal{R}(B^2)} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \mathcal{H}$ (for equality $\overline{\mathcal{R}(A^2)} = \overline{\mathcal{R}(A)}$ see Theorem 1.2.10), so they are not precoherent. $\diamond$

Theorem 4.5.4. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are such that A and B are precoherent, A^* and B^* are precoherent, and $\mathcal{R}(A+B)$ is closed, then A and B are parallel summable, their parallel sum is equal to $A(A+B)^\dagger B$ which is equal to $B(A+B)^\dagger A$ and $\mathcal{R}(A(A+B)^\dagger B) = \mathcal{R}(A) \cap \mathcal{R}(B)$.*

Proof. From Theorem 2.3.6 we have that $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(A^* + B^*)$ ($A^* + B^*$ also has a closed range), and so the operators A and B are parallel summable. The rest of the statement follows from Theorem 1.5.3. $\square$

The following lemma is valid for any bounded operators A and B, and will be needed for the proof of Theorem 4.5.6, which generalizes one statement from Theorem 4.2.2.

Lemma 4.5.5. *If $A, B \in \mathcal{B}(\mathcal{H})$, then $\mathcal{R}(A \overset{\star}{\wedge} B) \subseteq \mathcal{R}(A+B)$ and $\mathcal{R}((A \overset{\star}{\wedge} B)^*) \subseteq \mathcal{R}(A^* + B^*)$.*

Proof. The operators A and B coincide, together with the operator $A \overset{\star}{\wedge} B$, on $\overline{\mathcal{R}((A \overset{\star}{\wedge} B)^*)}$. So we have $\mathcal{R}(A \overset{\star}{\wedge} B) = (A \overset{\star}{\wedge} B)(\mathcal{R}((A \overset{\star}{\wedge} B)^*)) = A(\mathcal{R}((A \overset{\star}{\wedge} B)^*)) \subseteq \mathcal{R}(A+B)$, since for every $x \in \mathcal{R}((A \overset{\star}{\wedge} B)^*)$ we have $Ax = Bx = (A+B)\frac{x}{2} \in \mathcal{R}(A+B)$. In the same way, $\mathcal{R}((A \overset{\star}{\wedge} B)^*) \subseteq \mathcal{R}(A^* + B^*)$. $\square$

Theorem 4.5.6. *If $A, B \in \mathcal{B}(\mathcal{H})$ are such that $\mathcal{R}(A+B)$ is closed, then*

$$A \overset{\star}{\wedge} B \overset{\star}{\leq} 2A(A+B)^\dagger B.$$

Proof. First we will prove that $(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* = 2A(A+B)^\dagger B (A \overset{\star}{\wedge} B)^*$. From $A \overset{\star}{\wedge} B \overset{\star}{\leq} B$ we have that $B(A \overset{\star}{\wedge} B)^* = (A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^*$, and due to the star cancellation property, the desired equality is equivalent to

$$A \overset{\star}{\wedge} B = 2A(A+B)^\dagger (A \overset{\star}{\wedge} B).$$

The operators $A \overset{\star}{\wedge} B$ and $2A(A+B)^\dagger (A \overset{\star}{\wedge} B)$ coincide on $\mathcal{N}(A \overset{\star}{\wedge} B)$, so it is sufficient to show that they coincide on $\mathcal{R}((A \overset{\star}{\wedge} B)^*)$, so by continuity, also on $\overline{\mathcal{R}((A \overset{\star}{\wedge} B)^*)}$.

If $x \in \mathcal{R}((A \overset{\star}{\wedge} B)^*)$, then from Lemma 4.5.5, we have $x = (A+B)^* y$, for some $y \in \mathcal{H}$, while $(A \overset{\star}{\wedge} B)x = Ax = Bx = (A+B)\frac{x}{2}$. And so:

$$2A(A+B)^\dagger (A \overset{\star}{\wedge} B)x = A(A+B)^\dagger (A+B)(A+B)^* y = A(A+B)^* y = Ax = (A \overset{\star}{\wedge} B)x.$$

In this way, we proved that $(A \overset{\star}{\wedge} B)(A \overset{\star}{\wedge} B)^* = 2A(A+B)^\dagger B (A \overset{\star}{\wedge} B)^*$. Applying the same argument with A replaced by B^* and B replaced by A^*, we obtain $(A \overset{\star}{\wedge} B)^*(A \overset{\star}{\wedge} B) = (2A(A+B)^\dagger B)^*(A \overset{\star}{\wedge} B)$. Thus $A \overset{\star}{\wedge} B \overset{\star}{\leq} 2A(A+B)^\dagger B$. $\square$

Finally we obtain the following result easily from the given discussion which proves the other statement from Theorem 4.2.2 under weaker conditions and in an infinite-dimensional setting.

Theorem 4.5.7. *If $A, B \in \mathcal{B}(\mathcal{H})$ are such that A and B are precoherent, A^* and B^* are precoherent, and $\mathcal{R}(A+B)$ is closed then:*

$$A \overset{\star}{\wedge} B = 2A(A+B)^\dagger B = 2B(A+B)^\dagger A.$$

Proof. Follows from Theorem 4.5.4 and Theorem 4.5.2. □

Note that the conditions of Theorem 4.5.7 (as well as Theorem 4.5.2) are indeed weaker than the existence of the $\star$-supremum for operators A and B (Example 21).

Similar results hold also for the $\text{\textcircled{$\sharp$}}$-order, but with different precoherence condition (i.e. the one we called $\text{\textcircled{$\sharp$}}$-parallel in Section 4.4).

Lemma 4.5.8. *If $A, B \in \mathcal{B}^1(\mathcal{H})$ are weakly parallel summable operators such that $A : B \in \mathcal{B}^1(\mathcal{H})$ then $A \wedge^{\text{\textcircled{$\sharp$}}} B \leq^{\text{\textcircled{$\sharp$}}} 2(A : B)$.*

Proof. In order to show that $(A \wedge^{\text{\textcircled{$\sharp$}}} B)(A \wedge^{\text{\textcircled{$\sharp$}}} B)^{\text{\textcircled{$\sharp$}}} = 2(A : B)(A \wedge^{\text{\textcircled{$\sharp$}}} B)^{\text{\textcircled{$\sharp$}}}$ we act in the same way as in the proof of Lemma 4.5.1. Since $(A \wedge^{\text{\textcircled{$\sharp$}}} B)^{\text{\textcircled{$\sharp$}}}(A \wedge^{\text{\textcircled{$\sharp$}}} B) = (A \wedge^{\text{\textcircled{$\sharp$}}} B)^{\text{\textcircled{$\sharp$}}} \cdot 2(A : B)$ is equivalent to $(A \wedge^{\text{\textcircled{$\sharp$}}} B)^*(A \wedge^{\text{\textcircled{$\sharp$}}} B) = (A \wedge^{\text{\textcircled{$\sharp$}}} B)^* \cdot 2(A : B)$ i.e. with $(A \wedge^{\text{\textcircled{$\sharp$}}} B)^*(A \wedge^{\text{\textcircled{$\sharp$}}} B) = 2(A : B)^*(A \wedge^{\text{\textcircled{$\sharp$}}} B)$, the proof of this equality follows in the same manner, given that A^* and B^* coincide with $(A \wedge^{\text{\textcircled{$\sharp$}}} B)^*$ on $\mathcal{R}(A \wedge^{\text{\textcircled{$\sharp$}}} B)$ (Lemma 4.1.11). □

Theorem 4.5.9. *If $A, B \in \mathcal{B}^1(\mathcal{H})$ are weakly parallel summable operators then the following statements are equivalent:*

(i) $A : B \in \mathcal{B}^1(\mathcal{H})$ and $2(A : B) = A \wedge^{\text{\textcircled{$\sharp$}}} B$;

(ii) A and B are $\text{\textcircled{$\sharp$}}$-parallel.

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 1.5.2 and the fact that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed.
(ii) $\Rightarrow$ (i) From Proposition 1.5.2 we first note that $A : B \in \mathcal{B}^1(\mathcal{H})$. From Lemma 4.5.8 we have that $A \wedge^{\text{\textcircled{$\sharp$}}} B \leq^{\text{\textcircled{$\sharp$}}} 2(A : B)$ which together with $\mathcal{R}(A \wedge^{\text{\textcircled{$\sharp$}}} B) = \mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{R}(2(A : B))$ and Lemma 4.1.12 gives $A \wedge^{\text{\textcircled{$\sharp$}}} B = 2(A : B)$. □

Chapter 5

Coherence on Rickart *-rings

In this chapter, we introduce the notions of coherent and precoherent elements in a Rickart *-ring, generalizing this concept from the ring of bounded operators on a Hilbert space. Some interesting properties of such elements are demonstrated, resembling those of bounded operators, e.g. the range additivity and the parallel summation. As an application, we solve some problems regarding the star partial order on Rickart *-rings.

5.1 Basic properties of Rickart *-rings

Rickart *-rings, although a completely algebraic notion, originated from the studies in the field of functional analysis, as an abstraction of the von Neumann algebras. The most common references regarding such structures are the books by S. K. Berberian [17] and I. Kaplansky [59]. The setting of a Rickart *-ring seems rich enough to allow developing a theory which contains generalizations of many properties of coherent and precoherent operators. The purpose of the subsequent sections is to prove this claim. We start by introducing some notation, defining Rickart *-rings, and recalling some basic properties of such rings.

If R is a ring with involution $x \mapsto x^*$, we say that $e \in R$ is a projection if $e = e^2 = e^*$ (we omit 'orthogonal'). The set of all projections of a ring R is denoted by $P(R)$.

Definition 5.1.1. If R is a ring with involution, then R is a Rickart *-ring if the right annihilator of any element $x \in R$ is a principal right ideal of R generated by a projection, i.e.

$$x^\circ = \{y \in R \mid xy = 0\} = eR, \quad \text{for some} \quad e \in P(R).$$

It is not difficult to note that $e \in P(R)$ from this definition has to be unique, and so we denote it by x'. Also, for every $f \in P(R)$ we have $f' = 1 - f$, and consequently $x'' := (x')' = 1 - x'$.

Every Rickart *-ring R has the unit $1 = 0'$ ($0'$ is a left unit, but it is a ring with involution, so it is the unit). Also, the involution has to be proper (in the sense that $a^*a = 0$ implies $a = 0$). In fact, these conclusions hold even in a more general setting of, so called, Baer *-semigroups, introduced and studied by Foulis (see for example [40–

42]). Even in this semigroup setting, Foulis[1] has proved many structural properties of the projections x'.

We denote by $\leq$ the partial order on $P(R)$: $e \leq f$ if and only if $ef = fe = e$. The set $P(R)$ forms a lattice (even in the case of Baer *-semigroups there is an analogy with this property), which can be seen from [17, Chapter 1, §3] (or from [41, Theorem 1]), and we denote the lattice operations with $\wedge$ - for the infimum, and $\vee$ - for the supremum. Basic properties regarding the structure of a Rickart *-ring are contained in the following lemma. We include a reference, comment, or a complete proof of the statement in question, for the sake of completeness.

Lemma 5.1.2. *If $a, x \in R$, $e, f \in P(R)$ and $\{e_i \mid i \in I\} \subseteq P(R)$ then:*

1) *$ax = 0$ if and only if $a''x = 0$; $xa = 0$ if and only if $x(a^*)'' = 0$;*

2) *$aa'' = a = (a^*)''a$;*

3) *$(aa^*)'' = (a^*)''$, $(a^*a)'' = a''$;*

4) *$ae = a$ if and only if $a'' \leq e$; if $ae = a$ and $e \leq f$, then $af = a$;*

5) *$ae = 0$ if and only if $e \leq a'$; if $ae = 0$ and $f \leq e$, then $af = 0$;*

6) *$(ax)'' = (a''x)''$;*

7) *$e \wedge f = 1 - (1 - e) \vee (1 - f)$; moreover, there exists the greatest lower bound for $\{e_i \mid i \in I\}$ if and only if there exists the least upper bound for $\{1 - e_i \mid i \in I\}$, and in that case: $\bigwedge_{i \in I} e_i = 1 - \bigvee_{i \in I} (1 - e_i)$;*

8) *if there exists $\bigvee_{i \in I} e_i = e$, then $ex = 0$ if and only if $e_i x = 0$ for every $i \in I$; consequently, $ea = ex$ if and only if $e_i a = e_i x$ for every $i \in I$.*

Proof. 1) Follows from the definition of Rickart *-rings and projection $a'' = 1 - a'$;

2) Follows from the definition of a''.

3) Since the involution is proper, this follows from the fact that the right annihilators of x^*x and x are the same.

4) The first statement follows from 1) after subtracting a on both sides. The second statement follows from: $af = (ae)f = a(ef) = ae = a$.

5) Similarly as 3).

6) From 1) we see that the right annihilators of ax and $a''x$ are the same, hence the statement follows.

7) Straightforward from the obvious fact: if $p, q \in P(R)$ then $p \leq q$ if and only if

[1]It is particularly interesting that Foulis in [41] gave necessary and sufficient conditions for the reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ not only when A and B are matrices, but in a more general setting which also covers operators between Hilbert spaces. The first result along these lines is attributed to Greville [43] by many authors, but as we can see, the paper of Foulis predates Greville's paper by a few years.

$1-p \geq 1-q$. Thus if p is one lower bound for $\{e_i \mid i \in I\}$ then $1-p$ is one upper bound for $\{1-e_i \mid i \in I\}$, etc.

8) This is proved in [17, Proposition 6, p.14]. □

Recall that an element x of a ring with involution is called *-regular if there exists an element y such that:

$$xyx = x, \quad yxy = y, \quad (xy)^* = xy, \quad (yx)^* = yx.$$

Such an element y is necessarily unique, and it is called the Moore–Penrose generalized inverse of x, denoted by $x^\dagger$. For example, any projection in a ring with involution has the Moore–Penrose inverse - itself. Our next lemma contains some basic results regarding the Moore–Penrose inverse. We prove only those parts related to Rickart *-rings, the others hold in a general ring with involution (see [31]).

Lemma 5.1.3. *Let R be a Rickart *-ring and $x \in R$ be *-regular. Then:*

1) $(x^\dagger)^\dagger = x$;

*2) x^* is also *-regular and $(x^*)^\dagger = (x^\dagger)^*$;*

3) $xx^\dagger$ and $x^\dagger x$ are projections;

4) $x'' = x^\dagger x$, $(x^*)'' = xx^\dagger$;

5) $(x^\dagger)'' = (x^*)''$;

Proof. 4) We can note that the right annihilator of x is in fact $(1-x^\dagger x)R$, thus $x'' = x^\dagger x$. In the same way $(x^*)'' = (x^*)^\dagger (x^*) = (xx^\dagger)^* = xx^\dagger$.

5) This follows from 1) and 4). □

If in a Rickart *-ring, the right annihilator of arbitrary subset, rather than only of one element, is a principal right ideal generated by a projection, then such a ring is called Baer *-ring. In fact, a Rickart *-ring is a Baer *-ring if and only if the lattice of projections $P(R)$ forms a complete lattice (see [17, Chapter 1, §4]).

5.2 Coherent and precoherent elements

We begin this section by defining our central notions.

Definition 5.2.1. Let R be a Rickart *-ring, and $a, b \in R$. We say that a and b are *precoherent* if $a(a'' \wedge b'') = b(a'' \wedge b'')$. We say that a and b are *coherent* if there exists $x \in R$ such that $aa^* = xa^*$ and $bb^* = xb^*$.

Example 28. The most prominent example of a Rickart *-ring is a ring of all bounded operators $R = \mathcal{B}(\mathcal{H})$ on a real or complex Hilbert space $\mathcal{H}$ of finite or infinite dimension. If $A \in R$, then A' is the orthogonal projection onto the null-space of A, and so A'' is the orthogonal projection onto the closure of the range of A^*: $\overline{\mathcal{R}(A^*)}$. Analogously, $(A^*)''$ is the orthogonal projection onto the $\overline{\mathcal{R}(A)}$.

In this Rickart *-ring, the infimum of $P, Q \in P(R)$ is the orthogonal projection onto $\mathcal{R}(P) \cap \mathcal{R}(Q)$, while their supremum is the orthogonal projection onto $\overline{\mathcal{R}(P) + \mathcal{R}(Q)}$.

To say that operators A and B are precoherent means that A and B coincide on $\mathcal{R}(A'') \cap \mathcal{R}(B'') = \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}$, which is the same as Definition 2.1.2. To say that they are coherent means that there is some $C \in R$ such that C coincides with A on $\overline{\mathcal{R}(A^*)}$ while in the same time C coincides with B on $\overline{\mathcal{R}(B^*)}$, of course the same as in Definition 2.1.1. $\diamond$

We already know that in $\mathcal{B}(\mathcal{H})$, in order for two operators to be coherent it is necessary that they are precoherent. This is true for elements in any Rickart *-ring. On the other hand, Example 8 shows that two elements of a Rickart *-ring can be precoherent, but not coherent. Henceforth, R denotes Rickart *-ring, unless stated otherwise.

Lemma 5.2.2. *If $a, b \in R$ are coherent, then they are also precoherent.*

Proof. Let $x \in R$ be such that $aa^* = xa^*$ and $bb^* = xb^*$. Then, from Lemma 5.1.2 we get that $(a-x)a'' = 0$, and since $a''(a'' \wedge b'') = a'' \wedge b''$, then we also have $(a-x)(a'' \wedge b'') = 0$. Similarly, $(b-x)(a'' \wedge b'') = 0$. Subtraction of these equalities yields $(a-b)(a'' \wedge b'') = 0$, i. e. $a(a'' \wedge b'') = b(a'' \wedge b'')$. $\square$

Obviously, any two elements from $P(R)$ are coherent (and also precoherent). On the other hand, idempotents that are not self–adjoint need not to be (pre)coherent.

Example 29. If we take $R = \mathbb{C}^{2\times 2}$,

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

then a and b are idempotents, $b'' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and a and b are not precoherent. $\diamond$

If $a, b \in R$ are coherent elements, then every x such that $aa^* = xa^*$ and $bb^* = xb^*$ has the same "part": $x(a'' \vee b'')$. This is shown in our next lemma.

Lemma 5.2.3. *Let $a, b, x, y \in R$ be such that $aa^* = xa^* = ya^*$ and $bb^* = xb^* = yb^*$. Then $x(a'' \vee b'') = y(a'' \vee b'')$.*

Proof. Since $(x-y)a^* = 0$ from Lemma 5.1.2, 1) we get that $(x-y)a'' = 0$. Similarly $(x-y)b'' = 0$, and so from Lemma 5.1.2, 8) we have $(x-y)(a'' \vee b'') = 0$, which concludes the proof. $\square$

Elements a and b can be precoherent, while elements a^* and b^* are not. This can be seen from Example 9. However, when a and b are precoherent and in the same time a^* and b^* are precoherent, many results from previous chapters can be extended to elements of a Rickart *-ring.

Theorem 5.2.4. *If $a, b \in R$ are such that a and b are precoherent, and a^* and b^* are precoherent, then $a^*a(a'' \wedge b'') = b^*b(a'' \wedge b'')$. Consequently, a^*a and b^*b are precoherent, aa^* and bb^* are precoherent and $((1 - a'') \vee (1 - b''))a^*a(a'' \wedge b'') = 0$.*

Proof. From $a(a''\wedge b'') = b(a''\wedge b'')$ we get that $(1-(a^*)'')a(a''\wedge b'') = 0 = (1-(b^*)'')a(a''\wedge b'')$. Together with Lemma 5.1.2, 8) this gives $((1 - (a^*)'') \vee (1 - (b^*)''))a(a'' \wedge b'') = 0$, or in other words, using Lemma 5.1.2, 7), $((a^*)'' \wedge (b^*)'')a(a'' \wedge b'') = a(a'' \wedge b'')$. Now from $a^*((a^*)'' \wedge (b^*)'') = b^*((a^*)'' \wedge (b^*)'')$ we see that $a^*a(a'' \wedge b'') = a^*((a^*)'' \wedge (b^*)'')a(a'' \wedge b'') = b^*((a^*)'' \wedge (b^*)'')a(a'' \wedge b'') = b^*a(a'' \wedge b'') = b^*b(a'' \wedge b'')$. Since $a'' = (a^*a)''$ (Lemma 5.1.2, 2)), this means that a^*a and b^*b are also precoherent. By symmetry, aa^* and bb^* are also precoherent.

Finally, since $(1-a'')a^*a(a''\wedge b'') = 0$, and $(1-b'')a^*a(a''\wedge b'') = (1-b'')b^*a(a''\wedge b'') = 0$, by Lemma 5.1.2, 8) we also get that $((1 - a'') \vee (1 - b''))a^*a(a'' \wedge b'') = 0$. □

In the following theorem we prove one additive result, resembling the range additivity property from Lemma 2.3.4. In one step in the proof we want for $s + s = t + t$ to imply $s = t$. This is why we make an additional assumption on the structure of a Rickart *-ring, which will be present in some results, while in Example 30 we show that this assumption is not redundant.

Definition 5.2.5. If R is a ring such that for every $x \in R$ it holds $2x = 0$ if and only if $x = 0$, then we call R a *standard* ring.

Theorem 5.2.6. *Let R be a standard Rickart *-ring. If $a, b \in R$ are precoherent then $(a^*)'' \vee (b^*)'' = (a^* + b^*)''$.*

Proof. Since $(a^* + b^*)((a^*)'' \vee (b^*)'') = a^* + b^*$, (Lemma 5.1.2, 4)) we have that $(a^*)'' \vee (b^*)'' \geq (a^* + b^*)''$ (also Lemma 5.1.2, 4)). Note that $2a(a'' \wedge b'') = (a + b)(a'' \wedge b'') = (a^* + b^*)''(a + b)(a'' \wedge b'') = 2(a^* + b^*)''a(a'' \wedge b'')$, and so:

$$a(a'' \wedge b'') = (a^* + b^*)''a(a'' \wedge b''), \tag{5.1}$$

since R is standard. Further we have $a(1 - a'') = 0 = (a^* + b^*)''a(1 - a'')$, and also $a(1-b'') = (a+b)(1-b'') = (a^*+b^*)''(a+b)(1-b'') = (a^*+b^*)''a(1-b'')$. So using Lemma 5.1.2, 8) (with multiplication of projections from the left) we get $a((1 - a'') \vee (1 - b'')) = (a^* + b^*)''a((1 - a'') \vee (1 - b''))$, i.e.

$$a(1 - a'' \wedge b'') = (a^* + b^*)''a(1 - a'' \wedge b''). \tag{5.2}$$

From (5.1) and (5.2) we obtain $a = (a^* + b^*)''a$ and so by Lemma 5.1.2, 4), $(a^* + b^*)'' \geq (a^*)''$. Similarly, $(a^* + b^*)'' \geq (b^*)''$, giving $(a^* + b^*)'' \geq (a^*)'' \vee (b^*)''$. Hence $(a^* + b^*)'' = (a^*)'' \vee (b^*)''$. □

Corollary 5.2.7. *If R is a standard Rickart *-ring, and $e, f \in P(R)$ are arbitrary, then $e \vee f = (e + f)''$.*

Proof. Directly from Theorem 5.2.6. □

If $a, b \in R$ are such that $a+b$ is *-regular, then the expression $a(a+b)^\dagger b$, resembling the parallel sum of operators on a Hilbert space has some noteworthy properties even in a Rickart *-ring, when it is a standard ring, which we are going to demonstrate in Theorem 5.3.13, Theorem 5.3.14, Corollary 5.3.15. We will need the following property of standard Rickart *-rings.

Lemma 5.2.8. *If R is a standard Rickart *-ring and $x \in R$, then $(2x)'' = x''$.*

Proof. We have that $(2x)x'' = 2x$, while if $e \in P(R)$ is such that $(2x)e = 2x$, then $2x(1-e) = 0$, and since R is standard, we have $x(1-e) = 0$, which shows that $e \geq x''$ (Lemma 5.1.2, 5)). So $(2x)'' = x''$. □

Theorem 5.2.9. *Let R be a standard Rickart *-ring and $a, b \in R$ such that $a+b$ is *-regular, a and b are precoherent, and a^* and b^* are precoherent. Denote by $x : y = x(x+y)^\dagger y$. Then:*

1) $a : b = b : a$;

2) $(a : b)^* = (a^* : b^*)$;

3) $(a : b)'' = a'' \wedge b''$.

Proof. 1) Using Theorem 5.2.6 and Lemma 5.1.3, 4), we see that $(a+b)^\dagger(a+b) = (a+b)'' = a'' \vee b''$ and $(a+b)(a+b)^\dagger = (a^*+b^*)'' = (a^*)'' \vee (b^*)''$. Having in mind Lemma 5.1.2, 4), this gives $b(a+b)^\dagger(a+b) = b$ and $(a+b)(a+b)^\dagger a = a$. Straightforward calculation gives $a : b = (a+b-b)(a+b)^\dagger(a+b-a) = b : a$.

2) Directly from 1) and the fact from Lemma 5.1.3, 2).

3) From 1) we have that $(a : b)b'' = a : b$, and $(a : b)a'' = (b : a)a'' = b : a = a : b$. So from Lemma 5.1.2, 4), we see that both a'' and b'' are greater than $(a : b)''$ and so $a'' \wedge b'' \geq (a : b)''$. Again, from Lemma 5.1.2, 4), we now conclude that $(a : b)'' = ((a : b)(a'' \wedge b''))''$. On the other hand, since $a(a'' \wedge b'') = b(a'' \wedge b'')$, we have that $2(a : b)(a'' \wedge b'') = a(a+b)^\dagger(a+b)(a'' \wedge b'') = a(a'' \vee b'')(a'' \wedge b'') = a(a'' \wedge b'')$. Thus $(2(a : b)(a'' \wedge b''))'' = (a(a'' \wedge b''))'' = a'' \wedge b''$ (Lemma 5.1.2, 6)). From the last equality, together with $(a : b)'' = ((a : b)(a'' \wedge b''))''$, and Lemma 5.2.8, we get that $(a : b)'' = a'' \wedge b''$. □

The following lemma generalizes Lemma 2.2.4.

Lemma 5.2.10. *If $a, b \in R$ satisfy $aa^*b = ab^*b$ and $ba^*a = bb^*a$ then a and b, as well as a^* and b^* are precoherent.*

Proof. From $aa^*b = ab^*b$ we get $b^*aa^* = b^*ba^*$, and so $b^*(a-b)a^* = 0$. The last equality is the same as $(b^*)''(a-b)a'' = 0$ (Lemma 5.1.2, 1)). Since $a''(a'' \wedge b'') = a'' \wedge b''$ we have $(b^*)''(a-b)(a'' \wedge b'') = 0$. In the same way, starting from $ba^*a = bb^*a$ we obtain $(a^*)''(a-b)(a'' \wedge b'') = 0$. So from Lemma 5.1.2, 8), it follows that $((a^*)'' \vee (b^*)'')(a-b)(a'' \wedge b'') = 0$, i. e. $(a-b)(a'' \wedge b'') = 0$, where we used Lemma 5.1.2, 4).

Conditions in the statement of the lemma are symmetric with respect to the involution, so we also have $(a^* - b^*)((a^*)'' \wedge (b^*)'') = 0$. □

Recall that a and b can be coherent, and also a^* and b^* can be coherent, while the condition of the preceding lemma is not satisfied, as in Example 21 . So it is certainly not equivalent to the simultaneous precoherence of a and b, and a^* and b^*.

In the end we present an example showing that the fact the Rickart *-ring R is standard is important in our results.

Example 30. Suppose that R is a Rickart *-ring which contains some nonzero element x such that $x + x = 0$. First of all, since $xx'' = x$, we get that x'' is also nonzero. But from $x(x'' + x'') = 0$ and Lemma 5.1.2, 1), we get that $x''(x'' + x'') = 0$, i.e. $x'' + x'' = 0$. Since $x^* \neq 0$ and $x^* + x^* = 0$ as well, the same holds for $(x^*)''$. Now if we take $a = b = x$, then a and b satisfy the conditions of Theorem 5.2.6, but $(a^* + b^*)'' = (x^* + x^*)'' = 0 \neq (x^*)'' = (a^*)'' \vee (b^*)''$. Of course, $(2x)'' = 0'' = 0 \neq x''$, so the statement of Lemma 5.2.8 is also not true, $(a : b)'' \neq a'' \wedge b''$ and $2(a : b) = 0 \neq a \overset{\star}{\wedge} b = a$, etc.

There exist Rickart *-rings of infinite cardinality which are not standard. For example, let X be an arbitrary nonempty set, $\mathcal{P}(X)$ its partitive set, and take the Boolean ring $R = (\mathcal{P}(X), \triangle, \cap, 0 = \varnothing, 1 = X)$. With trivial involution $x^* = x$, R becomes Rickart *-ring (see [17, p. 19]). If we take arbitrary $x \in R$, we have $x + x = 0$. $\diamond$

5.3 Star partial order

The $\star$-partial order in Rickart *-rings has been studied by different authors, for example [20, 56, 63, 64]. Our interest in this section lies in the lattice properties of the $\star$-partial order. On Rickart *-rings, such properties were studied in some detail in [56] and [20]. Janowitz [56] proved that every Baer *-ring represents a lower semi-lattice, while a Rickart *-ring has an upper bound property. Later, Cirulis [20] noted that this conclusion by Janowitz was based on some wrong observations, but proved that the statement is nevertheless correct. Following Hartwig's [50] results, Janowitz gave an analogous result for the existence of $\star$-supremum in Rickart *-rings, in the presence of some *-regularity (i.e. the existence of the Moore-Penrose generalized inverse). However, as highlighted by Janowitz, the problem of the existence of $\star$-supremum for arbitrary elements remained open. We present a solution to this problem within this section, and give more detailed results regarding the $\star$-infimum, thus generalizing some results from [20].

First to recall some properties of the $\star$-partial order on Rickart *-rings. It was noted by Drazin in [34] that the relation $\overset{\star}{\leq}$ defined as:

$$a \overset{\star}{\leq} b \quad \Leftrightarrow \quad aa^* = ba^* \quad \text{and} \quad a^*a = a^*b,$$

is a partial order on every semigroup with proper involution (in a semigroup this means that $aa^* = ab^* = bb^*$ imply $a = b$). Thus, this is certainly a partial order on the Rickart *-rings. For $a, b \in R$ their $\star$-supremum and $\star$-infimum, if exist, are denoted by $a \overset{\star}{\vee} b$ and $a \overset{\star}{\wedge} b$ respectively. We gather some basic properties in one lemma.

Lemma 5.3.1. *If $a, b, c \in R$ and $e \in P(R)$ then:*

1) $a \overset{\star}{\leq} b$ if and only if $a^* \overset{\star}{\leq} b^*$;

2) If $a \overset{\star}{\leq} b$ then $aa^* \overset{\star}{\leq} bb^*$ and $a^*a \overset{\star}{\leq} b^*b$;

3) If $a \overset{\star}{\leq} b$ and $a \overset{\star}{\leq} c$, then $2a \overset{\star}{\leq} b + c$;

4) $a \overset{\star}{\leq} b$ if and only if $a = ba'' = (a^*)''b$;

5) If $a \overset{\star}{\leq} b$ then $a'' \leq b''$;

6) If $a \overset{\star}{\leq} e$ then $a \in P(R)$.

Proof. 1), 2) and 3) Directly from the definition, by a simple calculation, regardless of the special structure of R.

4) See [64] (this is in fact the definition of $\star$-partial order in [64]) or [20, Theorem 3.3].

5) See [20, Corollary 3.4]

6) See [64, Theorem 8]. □

It is clear that on $P(R)$ partial orders $\overset{\star}{\leq}$ and $\leq$ coincide, and we will always write $e \leq f$ rather than $e \overset{\star}{\leq} f$, for $e, f \in P(R)$. The following statement seems to be implicit in the present literature on the subject, but not pointed out (except in some special cases). This is why we place it in a theorem.

Theorem 5.3.2. *If $e, f \in P(R)$, then $e \overset{\star}{\wedge} f$ and $e \overset{\star}{\vee} f$ exist and they are equal to $e \wedge f$ and $e \vee f$ respectively.*

Proof. From part 6) of Lemma 5.3.1 we can conclude that $e \overset{\star}{\wedge} f$ always exists, and it is equal to $e \wedge f$, since all $\star$-lower bounds of e and f are in $P(R)$. Moreover, we can also conclude that $e \overset{\star}{\vee} f$ exists even though not all $\star$-upper bounds for e and f need to be from $P(R)$: one common $\star$-upper bound for e and f is $e \vee f$ and if there was any other which is $\star$-smaller than $e \vee f$, it would again be from $P(R)$ and thus had to coincide with $e \vee f$. □

Henceforth we will always write $e \wedge f$ and $e \vee f$ in place of $e \overset{\star}{\wedge} f$ and $e \overset{\star}{\vee} f$, when working with projections e and f.

If two elements a and b have one common $\star$-upper bound, then Cirulis [20] proved the following theorem and gave the following corollary. We will generalize these results later in this section.

Theorem 5.3.3 (See [20]). *If $a, b, x \in R$ are such that $a, b \overset{\star}{\leq} x$, then:*

1) $a \overset{\star}{\wedge} b$ exists and it is equal to $x(a'' \wedge b'')$;

2) $a \overset{\star}{\vee} b$ exists and it is equal to $x(a'' \vee b'')$.

If, moreover, a, b and x are self-adjoint ($a = a^*$, etc.) then so are $a \overset{\star}{\wedge} b$ and $a \overset{\star}{\vee} b$.

Corollary 5.3.4 (See [20])**.** *If $a, b, x \in R$ are such that $a, b \overset{\star}{\leq} x$, then:*

1) $(a \overset{\star}{\vee} b)'' = a'' \vee b''$ *and* $(a \overset{\star}{\wedge} b)'' = a'' \wedge b''$;

2) $a(a'' \wedge b'') = b(a'' \wedge b'') = a \overset{\star}{\leq} b$.

We now give a necessary and sufficient condition for two elements a and b of a Rickart *-ring to have a common $\star$-upper bound. It is an extension of Theorem 4.3.1 in the Rickart *-ring setting. As we stated before, it is known that Rickart *-ring has the upper–bound property with regard to the $\star$-partial order (Theorem 5.3.3), and we will use this in one part of the proof.

Theorem 5.3.5. *If $a, b \in R$, then the following statements are equivalent:*

(i) The set $\{x \in R \mid a \overset{\star}{\leq} x, \quad b \overset{\star}{\leq} x\}$ is nonempty;

(ii) There exists $a \overset{\star}{\vee} b$;

(iii) It holds $aa^*b = ab^*b$, $ba^*a = bb^*a$, and a and b are coherent.

Proof. (i) $\Rightarrow$ (ii): From Theorem 5.3.3.

(ii) $\Rightarrow$ (iii): Let $c = a \overset{\star}{\vee} b$. From $aa^* = ca^*$ and $bb^* = cb^*$ we obtain that a and b are coherent. Also $aa^*b = ac^*b = ab^*b$, and likewise $ba^*a = bb^*a$.

(iii) $\Rightarrow$ (i): Let $c = x(a'' \vee b'')$, where x is such an element of R which satisfies $aa^* = xa^*$ and $bb^* = xb^*$. We will show that c is one $\star$ - upper bound for a and b. Since $(a'' \vee b'')a^* = a^*$ and $(a'' \vee b'')b^* = b^*$ (Lemma 5.1.2, 4)), we have that $aa^* = ca^*$ and $bb^* = cb^*$. We will prove that $a^*a = a^*c$, and the dual equality $b^*b = b^*c$ will follow by symmetry.

First we will prove that $a^*(a - c)(a'' \vee b'') = 0$. From $aa^* = xa^*$ we obtain $aa'' = xa'' = ca''$ (Lemma 5.1.2, 1)) and so we have that $a^*(a - c)a'' = 0$. Similarly, $cb'' = bb''$, and so $a^*(a - c)b'' = a^*(a - b)b'' = 0$, since $a^*(a - b)b^* = 0$ (Lemma 5.1.2, 1)). Using Lemma 5.1.2, 8) we have that $a^*(a - c)(a'' \vee b'') = 0$. From Lemma 5.1.2, 7), we know that $1 - a'' \vee b'' = (1 - a'') \wedge (1 - b'')$, whence, multiplying with $(1 - a'') \wedge (1 - b'')$, we get $(a'' \vee b'')((1 - a'') \wedge (1 - b'')) = 0$. This yields $c((1 - a'') \wedge (1 - b'')) = 0$, and of course $a((1 - a'') \wedge (1 - b'')) = 0$, having in mind that $a(1 - a'') = 0$ (Lemma 5.1.2, 2)). Thus $a^*(a - c)((1 - a'') \wedge (1 - b'')) = 0$. Adding the last equality to $a^*(a - c)(a'' \vee b'') = 0$ we obtain $a^*(a - c) = 0$, i.e. $a^*a = a^*c$. □

Remark 5.3.6. In Section 4.3 some special cases in which condition (iii) can be reduced only to equalities $aa^*b = ab^*b$ and $ba^*a = bb^*b$ are described. For example, if R is a ring of square matrices then these equalities are sufficient. More generally, if $R = \mathcal{B}(\mathcal{H})$ is the ring of bounded operators on a Hilbert space $\mathcal{H}$, and $A, B \in R$ are such that $\overline{\mathcal{R}(A) + \mathcal{R}(B)}$ is closed or $\overline{\mathcal{R}(A^*)} + \overline{\mathcal{R}(B^*)}$ is closed, again these equalities are sufficient (Theorem 4.3.4). Having in mind Remark 4.3.8, it is possible that condition (iii) in [56, Theorem 11] is dispensable, i.e. that in the case when $a'b^*$ is *-regular we have that $aa^*b = ab^*b$ and $ba^*a = bb^*b$ imply the existence of $a \overset{\star}{\vee} b$.

In Baer *-rings we get a similar result with an arbitrary set of elements.

Theorem 5.3.7. *If I is an arbitrary set, and $\{a_i \mid i \in I\} \subseteq B$, then the following statements are equivalent:*

(i) The set $\{x \in B \mid a_i \overset{\star}{\leq} x,\ i \in I\}$ is nonempty;

(ii) There exists $\overset{\star}{\bigvee\limits_{i\in I}} a_i$;

(iii) For any $i, j \in I$ we have $a_i a_i^* a_j = a_i a_j^* a_j$ and there exists $x \in B$ such that for every $i \in I$, $a_i a_i^* = x a_i^*$.

Proof. (i) $\Rightarrow$ (ii): Follows from [20, Theorem 4.4].

(ii) $\Rightarrow$ (iii): Similarly as in Theorem 5.3.5.

(iii) $\Rightarrow$ (i): Similarly as in Theorem 5.3.5, using Lemma 5.1.2, 8) and Lemma 5.1.2, 7). □

Some basic properties of the $\star$-supremum are contained in the following theorem. It gives an extension of similar results in the Hilbert space setting from Section 4.3. It was already noted in Theorem 5.3.3 that the $\star$-supremum of two self–adjoint elements, if it exists, is again a self–adjoint element. Statement 4) of the following theorem extends this property to normal elements of a ring, and we can note that the same is valid for unitary elements.

Theorem 5.3.8. *Let $a, b \in R$ such that $a \overset{\star}{\vee} b$ exists, and let $x \in R$ be an arbitrary $\star$-upper bound for a and b. Then:*

1) $x = a \overset{\star}{\vee} b$ if and only if $x'' = a'' \vee b''$ if and only if $x' = a' \wedge b'$;

2) $a^* \overset{\star}{\vee} b^*$ exists, and $a^* \overset{\star}{\vee} b^* = (a \overset{\star}{\vee} b)^*$;

3) $aa^* \overset{\star}{\vee} bb^*$ exists, and $aa^* \overset{\star}{\vee} bb^* = (a \overset{\star}{\vee} b)(a^* \overset{\star}{\vee} b^*)$;

4) If $aa^* = a^*a$ and $bb^* = b^*b$, then $(a \overset{\star}{\vee} b)(a \overset{\star}{\vee} b)^* = (a \overset{\star}{\vee} b)^*(a \overset{\star}{\vee} b)$.

Proof. 1) If $x = a \overset{\star}{\vee} b$, then from Corollary 5.3.4 we have $x'' = a'' \vee b''$. In the opposite direction, if $x'' = a'' \vee b''$, then $x = xx'' = x(a'' \vee b'')$, and so $x = a \overset{\star}{\vee} b$, as we have showed in Theorem 5.3.5 (or we can take Theorem 5.3.3). Since from Lemma 5.1.2, 7), we have that $x' = a' \wedge b'$ is equivalent to $x'' = 1 - a' \wedge b' = a'' \vee b''$, we have completed the proof of 1).

2) From Lemma 5.3.1, 1) we see that $a \overset{\star}{\vee} b$ exists if and only if $a^* \overset{\star}{\vee} b^*$ exists, and in that case $(a \overset{\star}{\vee} b)^* = a^* \overset{\star}{\vee} b^*$.

3) From Lemma 5.3.1, 2) we see that, if $c = a \overset{\star}{\vee} b$, then cc^* is one $\star$-upper bound for aa^* and bb^*, and so from Theorem 5.3.5 (or [20, Theorem 4.4]) we have that $aa^* \overset{\star}{\vee} bb^*$ exists.

From Lemma 5.1.2, 3) we have that $(cc^*)'' = (c^*)''$, and from part 1) and 2) it follows that $(c^*)'' = (a^*)'' \vee (b^*)'' = (aa^*)'' \vee (bb^*)''$, so we have that $cc^* = aa^* \overset{\star}{\vee} bb^*$, as proved in 1).

4) Directly from 3). □

There is an obvious distinction between the coherence condition in statements (iii) of Theorem 5.3.5 and Theorem 5.3.7. Recall that the coherence of two-by-two elements will not imply the simultaneous coherence of all elements, as we showed in Example 13.

The following example deals with the case of an arbitrary subset of a special Rickart *-ring $R = \mathbb{C}^{n\times n}$, but we first prove one lemma.

Lemma 5.3.9. *Let $a, b, d \in R$ such that $c = a \overset{\star}{\vee} b$ exists and $aa^*d = ad^*d$, $da^*a = dd^*a$, $bb^*d = bd^*d$ and $db^*b = dd^*b$. Then $cc^*d = cd^*d$ and $dc^*c = dd^*c$.*

Proof. Note that from Lemma 5.3.1, 4) we have $a''(c^* - d^*)d = a''(a - d^*)d = 0$ (Lemma 5.1.2, 1)), and also $b''(c^*-d^*)d = 0$. Thus by Lemma 5.1.2, 7), we have $(a''\vee b'')(c^*-d^*)d = 0$, and by Corollary 5.3.4 this is the same as $c''(c^* - d^*)d = 0$, i. e. $c(c^* - d^*)d = 0$. The other equality is proved similarly, by using the fact that $c^* = a^* \overset{\star}{\vee} b^*$ (Theorem 5.3.8). □

Example 31. Suppose that $S = \{A_i \mid i \in I\} \subseteq \mathbb{C}^{n\times n}$, where I is an arbitrary set. Let us prove that $\overset{\star}{\bigvee\limits_{i\in I}} A_i$ exists if and only if for any $i, j \in I$ we have $A_iA_i^*A_j = A_iA_j^*A_j$.

Any two matrices A_i and A_j have the $\star$-supremum. Moreover, according to Lemma 5.3.9, $A_i \overset{\star}{\vee} A_j$ and A_k have the $\star$-supremum for any $i, j, k \in I$. In that case, if we take any m matrices $A_1, A_2, ..., A_m$, then the expression $((((A_1 \overset{\star}{\vee} A_2) \overset{\star}{\vee} A_3) \overset{\star}{\vee} ...) \overset{\star}{\vee} A_m)$ exists and by a simple order–theoretic argument, it is equal to $\overset{\star}{\bigvee\limits_{i=1,2,...,m}} A_i$.

Now suppose first that I is finite. By the previous discussion, we get that $\overset{\star}{\bigvee\limits_{i\in I}} A_i$ exists, taking $S = \{A_1, A_2, ..., A_m\}$.

If I is infinite, then it is important to note that '$\star$-augmentation' of a matrix can happen only finitely many times. Namely, if $A \overset{\star}{\leq} B$ and $A \neq B$, then $\mathcal{R}(A) \subsetneq \mathcal{R}(B)$, and so $\mathrm{r}(A) < \mathrm{r}(B)$ (see for example (4.5)). Take $A_1 \in I$. If A_1 is $\star$-larger than any other matrix from S, then A_1 is the $\star$-supremum of S. If there is some B_1 in this set such that A_1 is not $\star$-larger than B_1, then take $A_2 = A_1 \overset{\star}{\vee} B_1$. As we explained, the set $S' = (S \setminus \{A_1, B_1\}) \cup \{A_2\}$ has the same property: $AA^*B = AB^*B$ for any $A, B \in S'$. Now if there is no $\star$-larger matrix than A_2 in S', then A_2 is the $\star$-supremum of S, but if there is, denote it by B_2 and let $A_3 = A_2 \overset{\star}{\vee} B_2$, and so on. After at most n steps we will get our $\star$-supremum. ◇

We now address our attention to the $\star$-infimum. It was first noted in [56] that a Baer *-ring forms a complete lower semi-lattice, while [20, Theorem 5.2] gives necessary and sufficient conditions for the existence of $a \overset{\star}{\wedge} b$, when a and b are elements of a Rickart *-ring. We state this result as a theorem.

Theorem 5.3.10 (See [20])**.** *If $a, b \in R$, then $a \overset{\star}{\wedge} b$ exists if and only if the set $\{u'' \mid u \overset{\star}{\leq} a,\ u \overset{\star}{\leq} b\}$ has the greatest element m. In that case $a \overset{\star}{\wedge} b = am = bm$. Furthermore, the set $\{u'' \mid u \overset{\star}{\leq} a,\ u \overset{\star}{\leq} b\}$ is equal to the set:*

$$L_{a,b} = \{e \in P(R) \mid e \text{ commutes with } a^*a \text{ and } b^*b,\ e \leq a'' \wedge b'' \wedge (a-b)'\}.$$

Properties of the $\star$-infimum do not always resemble those of the $\star$-supremum. For example, Theorem 5.3.8 contains some relatively natural and simple properties for the $\star$-supremum and only statement 2) holds also for the $\star$-infimum: according to Lemma 5.3.1, 1) we have that $a \overset{\star}{\wedge} b$ exists if and only if $a^* \overset{\star}{\wedge} b^*$ exists, in which case

$$(a \overset{\star}{\wedge} b)^* = (a^* \overset{\star}{\wedge} b^*). \tag{5.3}$$

Statements 1), 3) and 4) of Theorem 5.3.8 do not hold for the $\star$-infimum in general.

Example 32. Let $R = \mathbb{C}^{3\times 3}$, and:

$$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

then we can note that

$$a \overset{\star}{\wedge} b = c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This can be done by a direct, but tedious calculation, or more elegantly, by examining the set $L_{a,b}$. Either way we see that the only elements of R that are common $\star$-lower bounds for a and b are 0 and c. Hence $a'' \wedge b'' = b'' \neq c''$, $aa^* \overset{\star}{\wedge} bb^* = bb^* \neq cc^*$, and a and b are normal, while c is not. ⋄

One way to assure that $a \overset{\star}{\wedge} b$ exists is to assume that $a \overset{\star}{\vee} b$ exists (cf. Theorem 5.3.3, Corollary 5.3.4). In that case we also have $(a \overset{\star}{\wedge} b)'' = a'' \wedge b''$ as well as $a \overset{\star}{\wedge} b = a(a'' \wedge b'') = b(a'' \wedge b'')$, and the same holds for a^* and b^*. Such equalities are obviously closely related to precoherence of a and b, as well as a^* and b^*. The following theorem explains this relation.

Theorem 5.3.11. *If $a, b \in R$, the following statements are equivalent:*

(i) a and b are precoherent, and a^* and b^* are precoherent;

(ii) $a \overset{\star}{\wedge} b$ exist, and $a(a'' \wedge b'') = a \overset{\star}{\wedge} b = b(a'' \wedge b'')$, as well as $a^*((a^*)'' \wedge (b^*)'') = a^* \overset{\star}{\wedge} b^* = b^*((a^*)'' \wedge (b^*)'')$.

Proof. (i) $\Rightarrow$ (ii): Denote by $x = a(a'' \wedge b'')$. We will first prove that x is one lower bound for a and b. It is clear that $xx^* = ax^*$, so we should prove that $x^*x = x^*a$. The last equality is equivalent to $(1 - a'' \wedge b'')a^*a(a'' \wedge b'') = 0$, i. e. with $((1 - a'') \vee (1 -$

$b''))a^*a(a'' \wedge b'') = 0$ (Lemma 5.1.2, 7)). But this follows from Theorem 5.2.4. Hence we have $x \overset{\star}{\leq} a$. Of course, in the same way we prove $x \overset{\star}{\leq} b$ and x is indeed one $\star$-lower bound for a and b.

To show $x = a \overset{\star}{\wedge} b$ note that, according to Lemma 5.1.2, 6), $x'' = (a(a'' \wedge b''))'' = a'' \wedge b''$, and since from $t \overset{\star}{\leq} s \Rightarrow t'' \leq s''$ (Lemma 5.3.1, 5)), we have that $\max\{u'' \mid u \overset{\star}{\leq} a,\ u \overset{\star}{\leq} b\} = x'' = a'' \wedge b''$. So $a \overset{\star}{\wedge} b = a(a'' \overset{\star}{\wedge} b'') = x = b(a'' \overset{\star}{\wedge} b'')$. Analogously, the same holds for a^* and b^*.

(ii) $\Rightarrow$ (i): This is evident. □

Theorem 5.3.11 generalizes Theorem 4.3.12, and the results of Theorem 5.3.3, Corollary 5.3.4, regarding the $\star$-infimum. Namely, condition (i) can be satisfied while a and b do not have a common $\star$-upper bound, which can be seen from Example 21. Hence, condition (i) of Theorem 5.3.11 is a weaker condition than the existence of $a \overset{\star}{\vee} b$ assuring that $a \overset{\star}{\wedge} b$ exists.

Condition (i) in Theorem 5.3.11 can not be reduced to the precoherence of a and b, since then the infimum $a \overset{\star}{\wedge} b$ need not to be equal to $a(a'' \wedge b'')$. Also, condition (ii) can not be reduced only to the equality $a(a'' \wedge b'') = a \overset{\star}{\wedge} b = b(a'' \wedge b'')$, since then a^* and b^* need not to be precoherent. This is shown by the following example.

Example 33. If $R = \mathbb{C}^{3\times 3}$, and

$$a = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$a'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad a'' \wedge b'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is true that $a(a'' \wedge b'') = b(a'' \wedge b'')$, so a and b are precoherent, although $a(a'' \wedge b'') \overset{\star}{\nleq} a$, hence $a \overset{\star}{\wedge} b \neq a(a'' \wedge b'')$.

If $R = \mathbb{C}^{2\times 2}$, and

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then

$$a'' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b'' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (a^*)'' = (b^*)'' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have $a \overset{\star}{\wedge} b = 0$, as well as $a'' \wedge b'' = 0$, and so $a(a'' \wedge b'') = a \overset{\star}{\wedge} b = b(a'' \wedge b'')$. On the other hand, a^* and b^* are not precoherent (and of course, $a^*((a^*)'' \wedge (b^*)'') = a^* \overset{\star}{\wedge} b^* = b^*((a^*)'' \wedge (b^*)'')$ is not satisfied). ◇

We now give a theorem for the $\star$-infimum similar to Theorem 5.3.8.

Theorem 5.3.12. *Let $a, b \in R$ be such that a and b are precoherent, and a^* and b^* are precoherent. Then $a \overset{\star}{\wedge} b$ exists, and if x is an arbitrary $\star$-lower bound for a and b, we moreover have:*

1) $x = a \overset{\star}{\wedge} b$ if and only if $x'' = a'' \wedge b''$ if and only if $x' = a' \vee b'$;

2) $aa^* \overset{\star}{\wedge} bb^*$ exists, and $aa^* \overset{\star}{\wedge} bb^* = (a \overset{\star}{\wedge} b)(a \overset{\star}{\wedge} b)^*$;

3) If $aa^* = a^*a$ and $bb^* = b^*b$, then $(a \overset{\star}{\wedge} b)(a \overset{\star}{\wedge} b)^* = (a \overset{\star}{\wedge} b)^*(a \overset{\star}{\wedge} b)$.

Proof. From Theorem 5.3.11 we see that $a \overset{\star}{\wedge} b$ exists.

1) If $x = a \overset{\star}{\wedge} b$, then from Theorem 5.3.11 and Lemma 5.1.2, 6), it follows that $x'' = a'' \wedge b''$. The opposite direction is true regardless of the condition on a and b, as we have mentioned before. Namely, if for some $\star$-lower bound x of a and b, the equality $x'' = a'' \wedge b''$ holds, then x'' is certainly the maximum of the set $\{u'' : u \overset{\star}{\leq} a,\ u \overset{\star}{\leq} b\}$ (Lemma 5.3.1, 5)) and consequently, $x = ax'' = a \overset{\star}{\wedge} b$. Of course, we also have $1 - a'' \wedge b'' = a' \vee b'$, thus the part 1) is proved.

2) Denote by $c = (a \overset{\star}{\wedge} b)(a \overset{\star}{\wedge} b)^*$. Then by Lemma 5.3.1, 2), c is one $\star$-lower bound for aa^* and bb^*. On the other hand, by Lemma 5.1.2, 3), we have $c'' = ((a \overset{\star}{\wedge} b)^*)''$, which is, by equality (5.3) and Theorem 5.3.11 equal to $(a^*)'' \wedge (b^*)''$, i. e. to $(aa^*)'' \wedge (bb^*)''$. Now using part 1) we see that c is in fact $aa^* \overset{\star}{\wedge} bb^*$.

3) Directly from 2), since the assumptions of the theorem are symmetric with respect to involution. □

Statement 2) of the preceding theorem could also be proved by Theorem 5.3.11 and Theorem 5.2.4.

The following results are concerned with the expression $a(a+b)^\dagger b = a : b$, and are derived in the standard Rickart *-rings. We point out that we generalize Theorems 4.5.6 and 4.5.7.

Theorem 5.3.13. *Let R be a standard Rickart $*$-ring and $a, b \in R$ such that $a + b$ is *-regular. If $x \overset{\star}{\leq} a$ and $x \overset{\star}{\leq} b$, then $x \overset{\star}{\leq} 2(a : b)$, and also $x \overset{\star}{\leq} 2(b : a)$.*

Proof. First we will prove that $x \overset{\star}{\leq} 2(a : b)$, i.e. that $xx^* = 2a(a+b)^\dagger bx^*$ and $x^*x = x^*2a(a+b)^\dagger b$. We have that $x \overset{\star}{\leq} a$ and $x \overset{\star}{\leq} b$. Then by Lemma 5.3.1, 3), we have that $2x \overset{\star}{\leq} a + b$ and so by Lemma 5.3.1, 5) and Lemma 5.2.8 we have $x'' \leq (a+b)''$. Analogously, from Lemma 5.3.1, 1), we also get $(x^*)'' \leq (a^* + b^*)''$. Thus $2(a : b)x^* = a(a+b)^\dagger(a+b)x^* = a(a+b)''x^* = ax^* = xx^*$. Similarly, $x^*2(a : b) = x^*(a+b)(a+b)^\dagger b = x^*(a^* + b^*)''b = x^*b = x^*x$. In this way we proved that $x \overset{\star}{\leq} 2(a : b)$, and by symmetry, $x \overset{\star}{\leq} 2(b : a)$ also follows. □

Theorem 5.3.14. *Let R be a standard Rickart *-ring and $a, b \in R$ such that $a + b$ is *-regular, a and b are precoherent, and a^* and b^* are precoherent. Then $a \overset{\star}{\wedge} b$ exists, and moreover $2(a : b) = a \overset{\star}{\wedge} b$.*

Proof. From Theorem 5.3.11 we see that $a \overset{\star}{\wedge} b$ exists and moreover $(a \overset{\star}{\wedge} b)'' = a'' \wedge b''$. From Theorem 5.3.13 we also have that $a \overset{\star}{\wedge} b \overset{\star}{\leq} 2(a : b)$. Now from Lemma 5.3.1, 4), we get: $a \overset{\star}{\wedge} b = (2(a : b))(a \overset{\star}{\wedge} b)'' = (2(a : b))(a'' \wedge b'') = (2(a : b))(a : b)'' = 2(a : b)$. □

One direct consequence of Theorem 5.3.14 is the famous relation for the infimum of two orthogonal projections P and Q on a Hilbert space: $P \wedge Q = 2P(P+Q)^{\dagger}Q$, when $\mathcal{R}(P+Q)$ is closed.

Corollary 5.3.15. *Let R be a standard Rickart *-ring. If e and f are projections such that $e+f$ is *-regular, then $e \wedge f = 2e(e+f)^{\dagger}f$.*

Proof. Directly from Theorem 5.3.14. □

Conclusion

Let us go through all the chapters (except Chapter 1), emphasizing one more time our main results, and overall contribution of this thesis. For convenience, we will use the term bi-precoherent here, as in Chapter 3.

Chapter 2. In this chapter we study coherent and precoherent operators, developing an interesting theory around them. Pairs of precoherent operators give a generalization of some frequently studied pairs: they generalize pairs of orthogonal projections, and what is more, pairs of alternating projections; furthermore, pairs of bi-precoherent operators are a generalization of pairs of weakly bicomplementary operators (i.e. matrices, see Chapter 3). Statements proved in Section 2.3 show a surprising similarity of bi-precoherent operators and pairs of positive operators, even though positive operators are not bi-precoherent, nor vice versa. Theorem 2.3.6 and Lemma 2.3.7 give a very interesting property of bi-precoherent operators, which is to the best of our knowledge, new even in the case $A = PQ$ and $B = QP$, where P and Q are orthogonal projections. In the end, let us mention that Section 2.4 directly generalizes and improves the results of the papers:

[23] C. Deng, et. al. On disjoint range operators in a Hilbert space. Linear Algebra Appl. (2012)

[14] O. M. Baksalary and G. Trenkler. On disjoint range matrices. Linear Algebra Appl. (2011)

as well as some results from:

[12] O. M. Baksalary and G. Trenkler. Revisitation of the product of two orthogonal projections. Linear Algebra Appl. (2009)

Chapter 3. We start this chapter with a discussion about the right way to generalize the relation of rank additivity for matrices in the setting of arbitrary Hilbert space operators (we will see later that our proposal can be stated exactly like for matrices, in terms of minus partial order, which is proved in an unpublished result, borrowed from a private communication [39]). After that, we direct our research to generalization of the results of the following papers:

[10] M. L. Arias, G. Corach and A. Maestripieri. Range additivity, shorted operator and the Sherman–Morrison–Woodbury formula. Linear Algebra Appl. (2015)

[82] H. J. Werner. Generalized Inversion and Weak Bi-Complementarity. Linear Multilinear Algebra (1986)

from bicomplementary operators, to bi-precoherent operators and from finite-dimensional spaces, to arbitrary Hilbert spaces (this refers only to the results from [82]). Results of Chapter 2 give us precise control over pairs of bi-precoherent operators, so we are able to 'calculate' the Moore-Penrose inverse of their sum, and even arbitrary reflexive inverse of the sum. For example, we can use formulas derived in this section to express an arbitrary reflexive inverse of $P+Q$, or even $PQ+QP$ via generalized inverses of P and Q, or PQ and QP respectively. We finish this chapter proving some results about linear combination of bi-precoherent operators, resembling results for oblique projections.

Chapter 4. In the first section of this chapter we give a few results regarding the definition of the minus order on Hilbert space operators, further clarifying the discussion from:

[81] P. Šemrl. Automorphisms of $B(H)$ with respect to minus partial order. J. Math. Anal. Appl. (2010)

about the possibility of defining the minus order for Hilbert space operators through range additivity relations, as for matrices. These results are a part of work which is momentarily in progress.

Our solution to the problem of the existence of $A \overset{\star}{\vee} B$ that we give in Section 4.3 stands out from the solutions given in:

[50] R. E. Hartwig. Pseudo Lattice Properties of the Star-Orthogonal Partial Ordering for Star-Regular Rings. Proc. Amer. Math. Soc. (1979)

[56] M. F. Janowitz. On the *-order for Rickart *-rings. Algebra Univers. (1983)

[84] X. M. Xu, et. al. The supremum of linear operators for the *-order. Linear Algebra Appl. (2010)

since it shows that in some cases, as for rectangular matrices, trivial necessary condition: $A(A^*-B^*)B = 0 = B(A^*-B^*)A$ is also sufficient for the existence of $A \overset{\star}{\vee} B$. In order to draw such a conclusion from the results of [50] or [56] we in fact need the results which we derived studying coherent operators. We also believe that the condition $A(A^*-B^*)B = 0 = B(A^*-B^*)A$ is sufficient for the existence of $A \overset{\star}{\vee} B$ when A and B are arbitrary Hilbert space operators, but this problem remained unsolved.

In Chapter 4 we also give an answer to a question about 'maximal infimum' from:

[52] R. E. Hartwig and M. P. Drazin. Lattice Properties of the *-Order for Complex Matrices. J. Math. Anal. Appl. (1982)

where we encounter another manifestation of bi-precoherent operators. Namely, in [52] authors noticed that for two orthogonal projections the range and null-space of their $\star$-infimum are as large as they can possibly be. They proposed a problem of finding all such pairs of matrices, tentatively suggesting partial isometries as another matrices satisfying this condition. We prove that the right generalization of the pairs of orthogonal projections which still satisfies this condition are exactly bi-precoherent operators.

In Section 4.4 we give a study of lattice properties of the $\textcircled{\sharp}$-order. The condition of coherence which appears in studying $\textcircled{\sharp}$-order is different than in the rest of the thesis, more suitable for operators from $\mathcal{B}^1(\mathcal{H})$. We show that $\mathcal{B}^1(\mathcal{H})$ is a complete lower semi-lattice, which is an interesting outcome, since the sharp order does not have this property. We also give necessary and sufficient conditions for the existence of $\textcircled{\sharp}$-supremum in general, with an elegant sepcialization to the matrix case.

Finally, another interesting property of bi-precoherent operators is presented in Section 4.5. Namely, in:

[3] W. N. Anderson and M. Schreiber. The infimum of two projections. Acta Sci. Math. (Szeged) (1972)

authors prove what is now a classical result, that the infimum of two orthogonal projections is equal to twice their parallel sum. We extend this result for bi-precoherent operators, with the generalized notion of parallel sum described in Section 1.5. In this way, we generalize the results mentioned in:

[66] S. K. Mitra. The minus partial order and the shorted matrix. Linear Algebra Appl. (1986)

We prove these results under weaker conditions, and on arbitrary Hilbert spaces instead only for rectangular matrices.

Chapter 5. The research presented in this chapter was inspired by some recent studies of the $\star$-partial order on Rickart *-rings, and especially by papers of Janowitz [56] we already mentioned, and Cirulis:

[20] J. Cīrulis. Lattice operations on Rickart *-rings under the star order. Linear Multilinear Algebra (2015)

from which we improve certain results. Namely, we found a weaker sufficient condition for the existence of $\star$-infimum than the one given in [20]. When we started to work with coherent and precoherent elements in Rickart *-rings we did not expect that we would be able to derive that many results for such elements. However, the most interesting results are only true in special Rickart *-rings, that we called standard Rickart *-rings. We consider Theorem 5.2.6, giving 'range additivity' for bi-precoherent elements, to be a very nice result in the study of coherence on Rickart *-rings, with a surprising Corollary 5.2.7, which we did not notice in the existing literature.

We included over 30 examples in the thesis, illustrating given statements and their possible extents: reduction of conditions, opposite directions, etc.

In the end, let us say something about a possible further research. We can not say that we are completely satisfied with our results about necessary and sufficient conditions for the existence of supremums *in general*. For example, the existence of $A \overset{\star}{\vee} B$ is only a solvability of the following system:

$$AA^* = XA^*,\ A^*A = A^*X,\ BB^* = XB^*,\ B^*B = B^*X.$$

What Theorem 4.3.1 does is to reduce this solvability to: $A(B^* - A^*)B = 0 = B(B^* - A^*)A$ together with the solvability of a part of the system:

$$AA^* = XA^*,\ BB^* = XB^*.$$

Our study of coherent operators shows that, in some cases, there is no need to check the solvability of this shorter system. In general, the farthest we get in proving that $A(B^* - A^*)B = 0 = B(B^* - A^*)A$ implies the solvability of the shorter system (i.e. implies coherence of A and B) is Theorem 2.2.5, i.e. Corollary 2.2.6. So the question that we are most curious about is: if we have $A(B^* - A^*)B = 0 = B(B^* - A^*)A$, are the operators A and B coherent?

Coherence is a very intriguing condition, and we believe that it can be interesting to study it for some special classes of operators. For example, orthogonal projections are obviously coherent, but even their scalar multiples are not (not even in the case of disjoint ranges, see Example 8). What about oblique projections A and B? In that case, pairs $(A, \mathcal{R}(A))$ and $(B, \mathcal{R}(B))$ are obviously coherent, but this is not the same as the coherence of A and B. Also, if A and B are partial isometries, are they necessarily coherent?

Considering precoherent, and especially bi-precoherent operators, the most convincing concrete examples of such operators that we could think of are, as we mentioned several times, orthogonal projections and their products. We developed an efficient mechanism for studying bi-precoherent operators, but we need more concrete examples of such operators to inspire, direct and justify their further development. So one more question, which is not so specific: where else can we find precoherent operators?

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Biography

Marko Đikić was born in Leskovac, Serbia, on 6th August 1989. He completed "Josif Kostić" Elementary School in Leskovac, and "Svetozar Marković" Grammar School, the specialized mathematical class, in Niš, both with the highest marks. He enrolled the Faculty of Sciences and Mathematics, University of Niš, in 2008, where he earned a bachelor's degree in 2011 and master's degree in 2013, both in mathematics. He started the PhD studies at the same faculty in 2013. His grade point average on bachelor and master studies, as well as on PhD studies so far is 10/10.

He participated in problem-solving competitions in mathematics, physics and informatics, where he won prizes on republic and federal level. In the period 2008–2015 he was one of the organizers of the competition trainings for students from the specialized mathematical class in "Svetozar Marković" Grammar School, and in 2013–2015 he was the member of the National Committee for mathematical competitions in Serbia. From 2008 he is an assistant at mathematics programme in Petnica Science Center, and from 2015 he is the head of this programme. He is one of the coordinators of a programme for continuous professional improvement of mathematics teachers, which obtained the license from the National Bureau for the Improvement of Education of the Republic of Serbia for academic years 2016/2017 and 2017/2018.

Since 2013 he works at the Faculty of Sciences and Mathematics, first as a teaching associate, and then as a teaching assistant. From 2013 he is also employed as a researcher on a project: Functional analysis, stochastic analysis and applications, supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia. He is the author and coauthor of six research papers published in international mathematical journals included in the Thomson Reuters citation index SCIe, and coauthor of one university textbook. He participated on several conferences, congresses and workshops, and communicated his results on two occasions: at International Conference on Theory and Applications of Mathematics and Informatics, in Alba Iulia, Romania, 2015 and at The 26th International Conference in Operator Theory, in Timisoara, Romania, 2016.

Author's publications

Research papers:

- M. S. Djikić. Properties of the star supremum for arbitrary Hilbert space operators. J. Math. Anal. Appl., 441:446–461, 2016.

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ИЗЈАВА О ИСТОВЕТНОСТИ ЕЛЕКТРОНСКОГ И ШТАМПАНОГ ОБЛИКА ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

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