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Abstract

This thesis has been written under the supervision of my mentor, Prof. dr. Miloš Arsenović at the University of Belgrade academic, and my co-mentor dr. Vladimir Božin in year 2013. The thesis consists of three chapters. In the first chapter we start from definition of harmonic functions (by mean value property) and give some of their properties. This leads to a brief discussion of homogeneous harmonic polynomials, and we also introduce subharmonic functions and subharmonic behaviour, which we need later. In the second chapter we present a simple derivation of the explicit formula for the harmonic Bergman reproducing kernel on the ball in euclidean space and give a proof that the harmonic Bergman projection is L^p *bounded, for* $1 < p < \infty$*, we furthermore discuss duality results. We then extend some of our previous discussion to the weighted Bergman spaces. In the last chapter, we investigate the Bergman space for harmonic functions* b^p*,* $0 < p < \infty$ *on* $\mathbb{R}^n \backslash \mathbb{Z}^n$*. In the planar case we prove that* $b^p \neq \{0\}$ *for all* $0 < p < \infty$*. Finally we prove the main result of this thesis* $b^q \subset b^p$ *for* $n/(k+1) \leq q < p < n/k$*,* $(k = 1, 2, ...)$*. This chapter is based mainly on the published paper [44]. M. Arsenović, D. Kečkić,[5] gave analogous results for analytic functions in the planar case. In the plane the logarithmic function* $\log|x|$*, plays a central role because it makes a difference between analytic and harmonic case, but in the space the function* $|x|^{2-n}, n > 2$ *hints at the contrast between harmonic function in the plane and in higher dimensions.*

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Contents

Chapter 1

Harmonic functions and subharmonic functions in space

1.1 Harmonic functions in space

1.1.1 Introduction

Harmonic functions are important functions in complex analysis, partial differential equations, electromagnetics, fluids, etc. Over the years many methods have been discovered to prove the existence of a solution of the Dirichlet problem for Laplace equation. Quote collection of proofs is based on representations of the Green's function in terms of the Bergman kernel function or some equivalent linear operator.

Throughout this thesis $n \geq 2$ denotes a positive integer and $\mathbb{R}^n$ denote n-dimensional Euclidean space so that $\mathbb{R}^1$ is the line, $\mathbb{R}^2$ is the plane, etc.
Let Ω be an open nonempty subset of $\mathbb{R}^n$, where n is a fixed. If Ω is open and connected, then Ω is called a domain. A real valued function u on an open set $\Omega \subseteq \mathbb{R}^n$ is called harmonic on Ω if u is twice continuously differentiable function on Ω (that is, all first and second partial derivatives of u exist and are continuous on Ω), and $\Delta u :\equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$ on Ω. The operator Δ is called the Laplace operator or Laplacian, and the equation $\Delta u \equiv 0$ is called Laplace's equation. We shall derive some further important facts about harmonic functions, including the Poisson kernel for the unit ball in $\mathbb{R}^n$ where $n \geq 2$.
We denote the Euclidean open ball in $\mathbb{R}^n$ of center $a \in \mathbb{R}^n$ and radius $r > 0$ by $B(a,r) := \{x \in \mathbb{R}^n : |x - a| < r\}$ (which we will sometimes write $B^n(a,r)$ to emphasize that its dimension is n), its closure is the closed ball $\overline{B}(a,r)$, the corresponding sphere by $S(a,r) \equiv \partial B(a,r) = \{\xi \in \mathbb{R}^n : |\xi - a| = r\}$, the unit ball $B(0,1)$ by B, and its boundary (unit sphere) by $S \equiv \partial B$.

1.1.2 Mean value property

Let ω_n denotes volume of the unit ball in $\mathbb{R}^n$ which define by

$$\omega_n = \begin{cases} \frac{\pi^{n/2}}{(n/2)!} & \text{if } n \text{ is even} \\ \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{1.3.5...n} & \text{if } n \text{ is odd.} \end{cases}$$

And let ω_{n-1}^* denotes the (unnormalized) surface area of the unite sphere in $\mathbb{R}^n$ define by $\omega_{n-1}^* = n\omega_n$. Then the volume measure of the ball $B(a,r)$ in $\mathbb{R}^n$ is $V(B^n(a,r)) = r^n\omega_n$, and the surface area of the sphere $S(a,r)$ in $\mathbb{R}^n$ is $Area(S^{n-1}(a,r)) = r^{n-1}n\omega_n$.

Definition 1. (Mean values). Let u be a Borel function on $\overline{B}(a,r)$ which is bounded above or below, the mean value of u over the sphere is :

$$\frac{1}{Area(S(a,r))}\int\limits_{S(a,r)} u(\xi)ds(\xi),$$

and over the ball is

$$\frac{1}{V(B(a,r))}\int\limits_{B(a,r)} u(x)dV(x).$$

where ds denotes surface-area measure, $dV = dV_n = dx_1...dx_n$ denotes Lebesgue volume measure on $\mathbb{R}^n$.

The first expression gives u as an average over the boundary of the ball, and the second as an average over the ball.

Now we may write the mean value properties in the following equivalent ways:
A continuous real valued function u in a domain $\Omega \subset \mathbb{R}^n$ has mean value property over spheres, if

$$u(a) = \frac{1}{n\omega_n}\int\limits_{S} u(a+r\xi)ds(\xi) := \int\limits_{S} u(a+r\xi)d\sigma(\xi),$$

for every ball $B(a,r) \subset \Omega$, where σ denotes the normalized surface-area measure on S (so that $\sigma(S) = 1$).
And u has the mean value property for balls, if

$$u(a) = \frac{1}{\omega_n}\int\limits_{B} u(a+rx)dV(x), \tag{1.1}$$

for every ball $\overline{B}(a,r) \subset \Omega$. In particular (when n=2):

$$u(a) = \frac{1}{2\pi}\int\limits_{0}^{2\pi} u(a+re^{i\theta})d\theta.$$

for every disk $\mathbb{D}(a,r) \subset \Omega \subset \mathbb{R}^2$.

It will be convenient to use polar coordinates to integrate functions over balls.

Lemma 1. *([4], [41] **Polar coordinates formula for integration on** $\mathbb{R}^n$). For a Borel measurable, integrable function f on $\mathbb{R}^n$, we have*

$$\int\limits_{\mathbb{R}^n} f dV = n\omega_n \int\limits_{0}^{\infty} r^{n-1}\int\limits_{S} f(r\xi)d\sigma(\xi)dr.$$

Choosing f to be the characteristic function of B, we have

$$\frac{d}{dr}\int\limits_{B(0,r)} f(x)dV(x) = \int\limits_{rS} f ds = nr^{n-1}\omega_n \int\limits_{S} f(r\xi)d\sigma(\xi).$$

Integrating with respect to r we obtain the polar coordinates formula for integration on ball

$$\int\limits_{B(0,\rho)} f(x)dV(x) = n\omega_n \int\limits_{0}^{\rho} r^{n-1} \int\limits_{S} f(r\xi)d\sigma(\xi)dr. \tag{1.2}$$

More generally

$$\int\limits_{B(a,\rho)} f(x)dV(x) = \int\limits_{0}^{\rho}\int\limits_{S} f(a+r\xi)r^{n-1}ds(\xi)dr.$$

1.1.3 Harmonic functions

Suppose that u has continuous second order derivatives on a domain $\Omega \subset \mathbb{R}^n$. Recall that a function $u : \Omega \to \mathbb{R}$ is harmonic on Ω if and only if $\Delta u = 0$ in Ω. We can also consider complex valued harmonic functions $f = u + iv : \Omega \to \mathbb{C}$, we say that such function f is harmonic if both $u = Ref$ and $v = Imf$ are harmonic functions in Ω. Since the definition of harmonicity involves taking second order partial derivatives, we must of course impose smoothness conditions on such a function f.

It is an important fact that the Mean value properties are equivalent to harmonicity of real harmonic functions

Definition 2. (Harmonic function). Let Ω be an open subset of $\mathbb{R}^n$ and $u \in C(\Omega, \mathbb{R})$. u is harmonic on Ω if and only if u satisfies the mean value equality

$$u(a) = \frac{1}{V(B(a,r))}\int\limits_{B(a,r)} u(x)dV(x); \tag{1.3}$$

for all $\overline{B}(a,r) \subset \Omega$.

Or equivalently

$$u(a) = \frac{1}{Area(S(a,r))}\int\limits_{S(a,r)} u(x)ds(x); \tag{1.4}$$

for all $\overline{B}(a,r) \subset \Omega$. In fact, if u is harmonic on Ω and $\overline{B}(a,r) \subset \Omega$, then

$$u(a) = \frac{1}{V(B(a,r))}\int\limits_{B(a,r)} u(x)dV(x) =$$

$$= \frac{n}{r^n.Area(S(B(a,r))}\int\limits_{0}^{r}\int\limits_{S(a,\rho)} u(\xi)\rho^{n-1}ds(\xi)d\rho.$$

Which implies

$$r^n u(a) = n \int_0^r \frac{1}{Area(S(B(a,r))} \int_{S(a,\rho)} \rho^{n-1} u(\xi) ds(\xi) d\rho.$$

Taking derivatives with respect to r on both sides it follows that

$$nr^{n-1} u(a) = \frac{nr^{n-1}}{Area(S(B(a,r))} \int_{S(a,r)} u(\xi) ds(\xi).$$

Hence

$$u(a) = \frac{1}{Area(S(a,r))} \int_{S(a,r)} u(\xi) ds(\xi).$$

This means $u(a)$ equals the average of u over the sphere $S(a,r)$.

Example 1. Let $u(x) = \begin{cases} |x|^{2-n} & \text{if } n > 2 \\ \log|x| & \text{if } n = 2 \end{cases}$
Then u is harmonic in $\mathbb{R}^n \backslash \{0\}$

The function $u(x) = |x|^{2-n}, n > 2$ is called the fundamental solution of the Laplacian(or the potential Kernel, or Newtonian potential), and the function $\log|x|$ plays the same role when $n = 2$ that $|x|^{2-n}$ plays when $n > 2$. Notice that $\log|x| \to \infty$ as $x \to \infty$; but $\log|x|^{2-n} \to 0$ as $x \to \infty$, note also that $\log|x|$ is neither bounded above nor below, but $|x|^{2-n}$ is always positive. This fact hints at the contrast between harmonic function theory in the plane and in higher dimensions .

Note that sums and scalar multiples of harmonic functions are harmonic, but in general multiples of harmonic functions need not be harmonic. In fact if u and v are real-valued harmonic functions then $u.v$ is harmonic function if and only if $\nabla u.\nabla v \equiv 0$. Also note that every partial derivative of a harmonic function is harmonic. So every harmonic function has continuous partial derivatives of all orders.

1.1.4 Green's identity

Green's Identity form an important tool in the analysis of Laplace equation. It is derived from divergence theorem. we start from the divergence theorem now.

Let $w = (w_1, ..., w_n)$ be a vector field in $\mathbb{R}^n$, we define the divergence of w by $\sum_{m=1}^{n} D_m w_m$, and denotes by $divw$. Let u be function, we define gradient of u by $(D_1 u, ..., D_n u) = \nabla u$. Suppose ν denotes unit outward normal vector, then the directional derivative in the direction ν is $D_\nu := \partial/\partial\nu$ such that $D_\nu u = \nabla u.\nu$.

Theorem 1. ***([25] Divergence theorem in $\mathbb{R}^n$ -Greens Theorem).*** *Let Ω be a bounded domain with C^1 boundary $\partial\Omega$ in $\mathbb{R}^n$. Then for any vector field $w \in C^1(\overline{\Omega})$ we have*

$$\int_\Omega div(w) dV = \int_{\partial\Omega} w.\nu ds$$

where ν the unit outward normal vector to $\partial\Omega$, ds is the area element in $\partial\Omega$.

Theorem 2. ***(Greens identity [3]).*** *Let Ω be a bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let u and v are C^2-functions on a neighborhood of $\overline{\Omega}$. Then*

$$\int_{\Omega}(u\Delta v - v\Delta u)dV = \int_{\partial\Omega}(uD_\nu v - vD_\nu u)ds$$

Proof. Apply Greens theorem with $w = u\nabla v - v\nabla u$ and compute. □

Corollary 1. *Suppose $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Let $u \in C^2(\overline{\Omega})$, Then*

$$\int_{\Omega} \Delta u dV = \int_{\partial\Omega} D_\nu u ds. \tag{1.5}$$

Corollary 2. *Suppose $u \in C^2(\Omega)$. Then u is harmonic on Ω if and only if*

$$\int_{S(a,r)} D_\nu u ds = 0$$

for every closed ball $\overline{B}(a,r) \subset \Omega$.

1.1.5 Maximum/Minimum Principle

If u is harmonic in the region Ω, then it does not have a weak relative maximum or minimum in Ω. A special yet important case of the above maximum/minimum principle is obtained when considering bounded regions. The maximum principle says if u is a real-valued harmonic function on Ω and $u \leq M$ at the boundary of Ω, then $u \leq M$ on Ω.

Theorem 3. ***(Weak maximum and minimum Principles).*** *If u is harmonic function on a bounded domain $\Omega \subset \mathbb{R}^n$ and continuous on $\overline{\Omega}$. Then its extreme values must occur on the boundary*

$$\min_{\overline{\Omega}} = \min_{\partial\Omega} u, \quad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Proof. Let $a \in \Omega$ such that $u(a) \geq u(x)$ for all $x \in \Omega$. Choose $B(a,r) \subset \Omega$. By volume mean value property we get

$$u(a) = \int_{B(a,r)} u(x)dV(x);$$

here V is normalised (so $V(B(a,r) = 1)$. Which implies

$$\int_{B(a,r)} (u(a) - u(x))dV(x) = 0.$$

But $u(a)$ is maximum, then we have

$$\int_{B(a,r)} (u(a) - u(x))dV(x) \geq 0,$$

and hence must vanish, consequently $u(x) = u(a)$ on $B(a,r)$. So we have shown that

$$A = \{x \in \Omega : u(x) = \sup_{\Omega} u\},$$

is open. But it is also closed because u is continuous. Since Ω is connected, then $A = \phi$ or $A = \Omega$. Hence $\sup_{\Omega} u$ is achieved on the boundary $\partial\Omega$. $\square$

Corollary 3. *If u, v are harmonic in the bounded domain Ω, which are continuous on $\overline{\Omega}$ and agree on the boundary $\partial\Omega$, then $u(x) = v(x)$ for all $x \in \Omega$.*

1.1.6 The Poisson Kernel

The (extended) Poisson kernel will play a major role when we study Bergman space.

Lemma 2. *([20]) If $x, y \in \mathbb{R}^n, x \neq 0$, and $|y| = 1$, then*

$$|x - y| = ||x|^{-1}x - |x|y|$$

Recall that, if u be harmonic on an open set containing $\overline{B}$, then by the mean-value property (1.4) , we have

$$u(0) = \int_S u(\xi)d\sigma(\xi)$$

Now we need to show that for any $x \in B$, $u(x)$ is a weighted average of u over S. In other words, we will show there exists a function namely P on $B \times S$ such that

$$u(x) = \int_S u(\xi)P(x,\xi)d\sigma(\xi).$$

For $n > 2$, suppose that u is harmonic on $\overline{B} \subset \mathbb{R}^n$. Fix $x \in B\backslash\{0\}$, choose $0 < r < 1 - |x|$, and let $\Omega = \{y \in \mathbb{R}^n : r < |y - x| < 1\}$. Put $v(y) = |y - x|^{2-n}$, and $w(y) = |x|^{2-n}|y - x/|x|^2|^{2-n}$. Since w is harmonic on $\overline{B}$, and $w = v$ on S. Thus from Greens identity Theorem 2 (with u, w), we obtain

$$\int_S uD_\nu w ds = \int_S wD_\nu u ds = \int_S vD_\nu u ds.$$

Since v is harmonic on $\mathbb{R}^n\backslash\{x\}$, and $\nabla v(y) = (2 - n)|y - x|^{-n}(y - x)$, consequently $D_\nu v = (2 - n)r^{1-n}$ on $S(x,r)$, then by Greens identity

$$\begin{aligned}
\int_S (uD_\nu v - uD_\nu w)ds &= \int_S (uD_\nu v - vD_\nu u)ds \\
&= \int_{S(x,r)} (uD_\nu v - vD_\nu u)ds \\
&= (2-n)r^{1-n} \int_{S(x,r)} uds = (2-n)nV(B)u(x)
\end{aligned}$$

Hence

$$u(x) = \frac{1}{2-n} \int_S u.(D_\nu v - D_\nu w) d\sigma.$$

Setting

$$P(x, \xi) = \frac{D_\nu v - D_\nu w}{2-n}$$

we obtain the required formula:

$$u(x) = \int_S u(\xi) P(x, \xi) d\sigma(\xi) \tag{1.6}$$

By calculation of D_ν, we get the formula

$$P(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^n}, \tag{1.7}$$

for $(x, \xi) \in B \times S$. This function is called the Poisson kernel for the ball B.

Now for $n = 2$, suppose that u is a real-valued harmonic on the closed unit disk $\overline{\mathbb{D}} \subset \mathbb{R}^2$, then there exists a function f analytic in $\mathbb{D}$, such that $Ref = u$. therefore u can be represented in the form

$$u(r\xi) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} \xi^m,$$

with $0 \leq r \leq 1$, and $|\xi| = 1$. Taking $r = 1$, and then integrate over the unit circle, we get

$$a_k = \int_S u(\xi) \xi^{-k} d\sigma(\xi)$$

For any point $x \in \mathbb{D}$, we can write x as $x = r\eta$ with $0 \leq r < 1$ and $|\eta| = 1$. Then

$$\begin{aligned} u(r\eta) &= \sum_{m=-\infty}^{\infty} \left(\int_S u(\xi) \xi^{-m} d\sigma(\xi) \right) r^{|m|} \eta^m \\ &= \int_S u(\xi) \left(\sum_{m=-\infty}^{\infty} r^{|m|} \eta^m \xi^{-m} \right) d\sigma(\xi) \end{aligned}$$

We conclude that

$$u(x) = \int_S u(\xi) \frac{1 - r^2}{|r\eta - \xi|^2} d\sigma(\xi)$$

Setting

$$P(x,\xi)=\frac{1-|x|^2}{|x-\xi|^2}.$$

This function is called the Poisson kernel for the disk $\mathbb{D}$. We obtain the required formula:

$$u(x)=\int\limits_S u(\xi)P(x,\xi)d\sigma(\xi).$$

Remark 1. *For $\xi\in S$, we can write the Poisson kernel for the unit ball B^n in form*

$$P(x,\xi)=\frac{1-|x|^2}{(1-2x.\xi+|x|^2)^{n/2}}. \tag{1.8}$$

For $x\in B(a,r)$ and $\xi\in S(a,r)$, the Poisson kernel for the ball $B^n(a,r)$ in $\mathbb{R}^n$ is defined by

$$P(x,\xi)=\frac{r^2-|x-a|^2}{rn\omega_n|x-\xi|^n}. \tag{1.9}$$

1.1.6.1 Some properties of the Poisson kernel

Proposition 1. *The Poisson kernel P for the ball $B(a,r)$ has the following properties:*
(i)- The Poisson kernel P is a positive function in $B(a,r)$.
(ii)- $\int\limits_{S(a,r)} P(x,\xi)d\sigma(\xi)=1$ for all $x\in B(a,r)$.
(iii)- for every $\eta\in S$ and every $\varepsilon>0$

$$\int\limits_{|\xi-\eta|>\varepsilon} P(x,\xi)d\sigma(\xi)\to 0 \qquad as \quad x\to\eta,$$

from within B (here P is the Poisson kernel for the unit ball B).

Proof. Part (i) is clear from definition of P. part (ii) follows formula (1.6) with $u\equiv 1$. To prove (iii), fix $\eta\in S$ and $\varepsilon>0$. Consider x with $|x-\eta|<\varepsilon/2$. Thus for $|\xi-\eta|>\varepsilon$ and $|x-\xi|>\varepsilon/2$, we have $P(x,\xi)\leq(n\omega_n)^{-1}(2/\varepsilon)^n(1-|x|^2)$. Hence

$$\int\limits_{|\xi-\eta|>\varepsilon} P(x,\xi)d\sigma(\xi)\leq(n\omega_n)^{-1}(2/\varepsilon)^n(1-|x|^2)\sigma(S)\to 0 \qquad as \quad x\to\eta.$$ □

Lemma 3. *The Poisson kernel P is a harmonic function of x in $B(a,r)$.*

Proposition 2. *([4]Proposition 1.18) Suppose $\xi\in S$. Then the Poisson kernel $P(x,\xi)$ for the unit ball B is harmonic on $\mathbb{R}^n\backslash\{\xi\}$.*

1.1.6.2 Extended Poisson kernel

We extend the domain of Poisson kernel P to a function on $B \times B$ by setting $P(x, y) = P(x/|x|, |x|y)$, for $x, y \in B$ we get

$$P(x,y) = \frac{1 - |x|^2|y|^2}{(1 - 2x.y + |x|^2|y|^2)^{n/2}} \quad for \quad x, y \in B. \tag{1.10}$$

This function is called the extended Poisson kernel.

Remark 2. *For all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have*

$$P(x,y) = \frac{1 - |x|^2|y|^2}{(1 - 2x.y + |x|^2|y|^2)^{n/2}}$$

provided the denominator above is not zero.

1.1.6.3 Some properties of the extended Poisson kernel

1-The extended Poisson kernel $P(x, y)$ is a symmetric function on $B \times B$.
2-$P(x, y) = P(|x|y, x|x|)$
3-For x fixed, the function $P(x, y)$ is harmonic. In particular, for any fixed $x \in B$ the function $F : y \to P(x, y)$ is harmonic on $\overline{B}$.

1.1.7 Dirichlet problem

Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 2)$, let u be a continuous real-valued function on its boundary $\partial\Omega$. The classical Dirichlet problem consists in the determination of a harmonic function u on Ω which can be continuously extended into $\partial\Omega$ by u. Thus the Dirichlet problem is a boundary value problem for Laplace's equation.

Suppose that v is a given continuous function on the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$. The Dirichlet problem is to extend v (which is only defined on the boundary $\partial\Omega$ of Ω) to a function u defined inside the domain Ω such that $(i)\Delta u = 0$ in Ω (i.e. u is harmonic) and $(ii)u = v$ on $\partial\Omega$ (more precisely for all $y \in \partial\Omega$ we want $u(x) \to v(y)$ as $x \to y$ where $x \in \Omega$). The Poisson kernel plays a key role in Dirichlet problem .

The Dirichlet problem can always be solved on domains with smooth boundaries, like the ball (or disk). Now we introduce a famous problem:

1.1.7.1 Dirichlet problem for the ball

Let v be integrable in the sphere $S(a, r)$ with respect to the surface measure ds, we define the Poisson integral of v in $B(a, r)$, denoted $P[v](x)$, to be the function given by

$$P[v](x) = \int_{S(a,r)} P(x, \xi)v(\xi)dS(\xi),$$

for all $x \in B(a, r)$, where $P(x, \xi)$ is the Poisson kernel for the ball $B(a, r)$.

Now we can state the Dirichlet problem on the unit ball: suppose that v is a continuous function on S. Then the Dirichlet problem on the unit ball B is given by :

$$\begin{cases} \Delta u = 0 & \text{in} \quad B \\ u = v & \text{on} \quad S \end{cases}$$

and the solution of the above problem in B is given in the fowling theorem :

Theorem 4. ***(Dirichlet problem for the ball ([9] Theorem 1.2.6)).*** *Suppose v is continuous on S, Define u on $\overline{B}$ by*

$$u(x) = \begin{cases} P[v](x) & \text{if} \quad x \in B \\ v(x) & \text{if} \quad x \in S. \end{cases}$$

Then u is continuous on $\overline{B}$, harmonic on B, and $u\mid_S = v$ where $u\mid_S$ denote the restriction of the function u to the boundary S. The function u is said to solve the Dirichlet Problem with boundary data v.

Proof. For $x \in B$, set $u(x) = (n\omega_n)^{-1} \int\limits_S |x-\xi|^{-n}(1-|x|^2)v(\xi)d\sigma(\xi)$. For $x \in B$, all derivatives (of first or higher order) of $|x-\xi|^{-n}(1-|x|^2)$ with respect to x are continuous functions of ξ on S, and for each derivative, the family of all such functions, as x ranges over a compact subset of B, is uniformly bounded. So we may differentiate under the integral and use Lemma 3 to conclude v is harmonic on B. To prove that u is continuous on $\overline{B}$, fix $\eta \in S$ and let $\varepsilon > 0$. Choose $\delta > 0$ such that $|v(\xi) - v(\eta)| < \varepsilon$ whenever $|\xi - \eta| \leq \delta$, $\xi \in S$. Then for $x \in B$, we have

$$\begin{aligned} |u(x) - u(\eta)| &= |u(x) - v(\eta)| \\ &= \left| \int\limits_S P(x,\xi)v(\xi)d\sigma(\xi) - \int\limits_S P(x,\xi)v(\eta)d\sigma(\xi) \right| \\ &\leq \int\limits_S P(x,\xi)|v(\xi) - v(\eta)|d\sigma(\xi) \\ &= \int\limits_{S\cap\{\xi:|\xi-\eta|\leq\delta\}} P(x,\xi)|v(\xi) - v(\eta)|d\sigma(\xi) + \\ &+ \int\limits_{S\cap\{\xi:|\xi-\eta|>\delta\}} P(x,\xi)|v(\xi) - v(\eta)|d\sigma(\xi) \\ &\leq \varepsilon \int\limits_S P(x,\xi)d\sigma(\xi) + 2\|v\|_\infty \int\limits_{S\cap\{\xi:|\xi-\eta|>\delta\}} P(x,\xi)d\sigma(\xi), \end{aligned}$$

where $\|v\|_\infty$ denotes the supremum of of $|v|$ on S. Choose a neighborhood E of η thus by Proposition 1(iii) with $x \in E$, we have

$$\int\limits_{S\cap\{\xi:|\xi-\eta|>\delta\}} P(x,\xi)d\sigma(\xi) < \frac{\varepsilon}{2\|v\|_\infty}.$$

Hence $|u(x) - u(\eta)| < 2\varepsilon$ whenever $x \in E$, this shows that u is continuous at η. □

Corollary 4. *Suppose u is continuous on $\overline{B}$ and harmonic on B. Then $u = P[u\mid_S]$.*

Theorem 5. *([9] Theorem 1.2.8) Suppose u is continuous on a domain Ω. if for each $a \in \Omega$, there is a sequence of positive numbers $r_m \to 0$ (which may depend on a) such that*

$$u(a) = \frac{1}{area(S(a, r_m))} \int\limits_{S(a,r_m)} u(\xi) ds(\xi)$$

for each m, then u is harmonic on Ω.

Proof. Without loss of generality, we can assume that u is real valued. Pick $a \in \Omega$, $r > 0$ such that $\overline{B}(a, r) \subseteq \Omega$. Let $v = P[u \mid_{S(a,r)}]$. We need to show that $u = v$ on $B(a, r)$. Suppose that $v - u$ is positive at some point of $\overline{B}(a, r)$. Let $E = \{x \in \overline{B}(a, r) : (v - u)(x) = \sup\{(v - u)(y) : y \in \overline{B}(a, r)\}\}$. Since $v - u$ is continuous, $E \neq \phi$. Since E is compact, we may choose $x_0 \in E$ with $|x_0 - a|$ a maximum, that is, $|x_0 - a| = \max\{|x - a| : x \in E\}$. Then $x_0 \in B(a, r)$ since $v - u = 0$ on $S(a, r)$ and the supposition that $v - u$ is positive at some point of $\overline{B}(a, r)$. So there exists an $r_0 > 0$ ($r_0 = r_m$ for some m) such that $B(x_0, r_0) \subseteq B(a, r)$ and so that

$$u(x_0) = \frac{1}{area(S(x_0, r_0))} \int\limits_{S(x_0,r_0)} u(\xi) ds(\xi).$$

Since v is harmonic, such an inequality also holds for v, hence

$$(v - u)(x_0) = \frac{1}{area(S(x_0, r_0))} \int\limits_{S(x,r_0)} (u - v)(\xi) ds(\xi) \tag{1.11}$$

But $(v - u)(x_0) = \max\{(v - u)(\xi) : \xi \in S(a, r)\}$ and by choice of x_0, there exists points $\eta \in S(x_0, r_0)$ at which $(v - u)(\eta) < \max\{(v - u)(\xi) : \xi \in S(a, r)\}$. So the equality above (1.11) is not possible. Thus, $v - u$ cannot be positive in $\overline{B}(a, r)$, reasoning similarly one concludes it cannot be negative either. So $v = u$ on $\overline{B}(a, r)$, that is, u is harmonic on a neighborhood of a. Since a is arbitrary. Hence u is harmonic throughout Ω. ☐

1.1.8 Interior derivative estimates for harmonic functions

Recall, that harmonic functions are C^∞. By using the mean value formula, we can obtain good estimates for the derivatives of harmonic function. We start from definition for multi-index. A multi-index $\alpha \in \mathbb{N}^n$ is an n-tuple of nonnegative integers $(\alpha_1, ..., \alpha_n)$, with $|\alpha| = \alpha_1 + \alpha_2 + ...\alpha_n$, and $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$. If $\beta = (\beta_1, ..., \beta_n) \in \mathbb{R}^n$, and $x = (x_1, ..., x_n) \in \mathbb{R}^n$, then we define $\alpha+\beta = (\alpha_1+\beta_1, ..., \alpha_n+\beta_n)$, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$. Let D_j denote the partial derivative with respect to the j^{th} coordinate variable, then the partial differentiation operator D^α is defined to be $D_1^{\alpha_1}...D_n^{\alpha_n} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$, (D_j^0 denotes the identity operator).

Recall, that if u is a continuous on $\overline{B}$ and harmonic on B, then

$$u(x) = \int\limits_S u(\xi) P(x, \xi) d\sigma(\xi)$$

for every $x \in B$, where $P(x, \xi)$ is the Poisson kernel for the ball B. Moreover, for every multi-index α the formula

$$D^{\alpha}u(x) = \int\limits_{S} u(\xi) D^{\alpha}P(x, \xi) d\sigma(\xi) \tag{1.12}$$

holds, whenever $x \in B$.

1.1.8.1 Interior derivative estimates

Lemma 4. *([27]Lemma 1.10) If u is continuous function on $\overline{B}(a, r) \subset \mathbb{R}^n$, and harmonic on $B(a, r)$, then*

$$|Du(a)| \leq \frac{n}{r} \max_{\overline{B}(a,r)} |u|.$$

Proof. Since the gradient of harmonic function is also harmonic, then by the mean value of Du (1.3), and divergence theorems 1 we have

$$Du(a) = \frac{1}{r^n \omega_n} \int\limits_{B(a,r)} Du dx = \frac{1}{r^n \omega_n} \int\limits_{\partial B(a,r)} u \nu ds.$$

Hence

$$|Du(a)| \leq \frac{1}{r^n \omega_n} . \max_{\partial B(a,r)} |u|.r^{n-1} n \omega_n \leq \frac{n}{r} \max_{\overline{B}(a,r)} |u|.$$

□

Corollary 5. *If u is harmonic function on $\Omega \subset \mathbb{R}^n$, then*

$$|Du(x)| \leq \frac{n}{dist(x, \partial\Omega)} \sup_{\Omega} |u| \tag{1.13}$$

for all $x \in \Omega$.

Proof. For each $x \in \Omega$, apply the lemma above with $r = dist(x, \partial\Omega)$. □

1.1.8.2 Cauchy's estimates for harmonic function

The Cauchy's estimates for analytic function stated as follows: if f is analytic function and bounded by M on a disk $D(a, r) \subset \mathbb{C}$, then

$$|f^{(m)}(a)| \leq \frac{m!}{r^m} M.$$

The next theorem gives comparable results for harmonic functions defined on a ball in $\mathbb{R}^n$.

Theorem 6. ***(Cauchy's Estimates for harmonic function [4]).*** *For every multi-index α there exists a positive constant C_α such that*

$$|D^{\alpha}u(a)| \leq \frac{C_\alpha}{r^{|\alpha|}} M \tag{1.14}$$

for all functions u harmonic and bounded by M on $B(a, r)$.

Proof. We can assume that $a = 0$. If u is harmonic and bounded by M on $\overline{B}$, then by (1.12) we have

$$|D^\alpha u(0)| = \left| \int\limits_S u(\xi) D^\alpha P(0,\xi) d\sigma(\xi) \right| \leq$$
$$\leq M \int\limits_S |D^\alpha P(0,\xi)| d\sigma(\xi) = C_\alpha M,$$

where $C_\alpha = \int\limits_S |D^\alpha P(0,\xi)| d\sigma(\xi)$.

If u is harmonic and bounded by M on $\overline{B}(0,r)$, then applying the result in the previous paragraph to the r-dilate u_r shows that

$$|D^\alpha u(0)| \leq \frac{C_\alpha}{r^{|\alpha|}} M.$$

Replacing r by $r - \varepsilon$ and letting ε decrease to 0, we obtain the same conclusion if u is harmonic on the open ball $B(0,r)$ and bounded by M there. □

Corollary 6. *Let α be a multi-indexLet, and let u a harmonic bounded function on Ω. Then there exists a constant C such that*

$$|D^\alpha u(x)| \leq \frac{C}{dist(x, \partial\Omega)^{|\alpha|}}$$

for all $x \in \Omega$.

Theorem 7. *([25] Theorem 2.10). Suppose that Ω is a subset of $\mathbb{R}^n$, and let K be any compact subset of Ω. If u is harmonic function on Ω, Then for any multi-index α we have*

$$\sup_K |D^\alpha u| \leq \left(\frac{n|\alpha|}{dist(K, \partial\Omega)} \right)^{|\alpha|} \sup_\Omega |u|$$

Proof. By successive application of the estimate Corollary 1.13 in equally spaced nested balls we obtain an estimate for higher order derivatives: □

1.1.9 Some properties of harmonic function

We have observed, amongst other properties, that the mean value property and the maximum principle play very important roles in the theory of harmonic functions. Many of the general properties of harmonic functions on $\mathbb{R}^n$ are more easily proved by using the mean-value property of harmonic functions than by using the definition of harmonicity directly. Properties such as the mean-value theorem, the maximum modulus principle and the infinite-differentiability of two-variable harmonic functions are also true in the n dimensional case.

Many basic properties of harmonic functions follow from Green's identity (which we will need mainly in the special case when Ω is a ball). The mean-value theorem which characterises harmonic functions is in turn a consequence of an n-dimensional version of Greens theorem that plays a similar role in n dimensions to Cauchys theorem in the two dimensional case.

Theorem 8. *([46] Theorem7) If $f = u + iv$ is analytic in a domain $\Omega \subset \mathbb{C}$, then each of the functions u and v is harmonic in Ω*

In this case the imaginary part of a analytic function f is called a harmonic conjugate of the real part of f.

Theorem 9. *([4] Theorem 1.28) If u is harmonic on a domain $\Omega \subseteq \mathbb{R}^n$, then u is real analytic in Ω.*

Suppose that Ω is simple connected domain and let u be harmonic on Ω. Then there is an analytic function f on Ω with $Ref = u$. This means that for such a function u there exists a harmonic function v defined on Ω such that $f = u + iv$ is analytic on Ω. Now we can prove next theorem.

Theorem 10. *([7] Theorem 4.31). If $f = u + iv$ is harmonic in a simply-connected domain Ω, then $f = g + \overline{h}$, where g and h are analytic.*

Proof. Since u and v are real harmonic functions on a simply-connected domain, then the discussion before the statement of this theorem shows that there exists analytic functions f_1 and f_2 such that $u = Ref_1$ and $v = Imf_2$. Hence,

$$f = u + iv = Ref_1 + iImf_2 = \frac{f_1 + \overline{f_1}}{2} + i\frac{f_2 - \overline{f_2}}{2i} = \frac{f_1 + f_2}{2} + \frac{\overline{f_1 - f_2}}{2} = g + \overline{h}$$

□

Example 2. The harmonic function $f(x, y) = x + \frac{x^2-y^2}{2} + iy(1 - x)$ in the unit disck $\mathbb{D}$ can also be written in the form

$$f(x, y) = x + iy + \frac{x^2 - y^2}{2} - ixy = x + iy + \left(\overline{\frac{x^2 - y^2}{2} + ixy}\right)$$

Theorem 11. *Let $\Omega, \Omega' \subset \mathbb{C}$. Suppose $f : \Omega \to \Omega'$ is an analytic function and $u : \Omega' \to \mathbb{R}$ is a harmonic function. Then the function $h := u \circ f$ is harmonic on Ω.*

Proof. Take $z \in \Omega$ and let $z' = f(z)$. By continuity of f, there exist open disks $B \subset \Omega, B' \subset \Omega'$, around z, z' respectively such that $f(B) \subset B'$. Choose a conjugate v to u in B' so that $g = u + iv$ is analytic on B'. But then $g \circ f$ is analytic at z so $u \circ f := h$ is harmonic at z. □

Theorem 12. ***(Liouville's Theorem).*** *If u is harmonic on $\mathbb{R}^n$ and bounded from above or below then u is a constant function on $\mathbb{R}^n$.*

Theorem 13. ***(Identity principle-harmonic version).*** *Let u, v be harmonic on a domain $\Omega \subset \mathbb{R}^n$. If $u = v$ on an open, non-empty set $E \subset \Omega$, then $u = v$ throughout Ω.*

Proof. Assume that $h = u - v$. Let $x \in E$, then $D^\alpha h(x) = 0$ for all multi-indices α. This means that all derivatives vanish identically as well on E. Define $A_1 = \{x \in \Omega : D^\alpha h(x) = 0; \quad \forall\alpha\}$ and $A_2 = \Omega \backslash A_1$. If $y \in A_2$ then $D^\alpha h(y) \neq 0$ for some multi-index α and by continuity of all derivatives of h there is also an open neighborhood of y where $D^\alpha h(x) \neq 0$, hence A_2 is open. If $y \in A_1$, then by using a Taylor expansion of h around y that $h = 0$ in some neighborhood of y, thus A_1 is open. Since $A_1 \cap A_2 = \phi$, $A_1 \cup A_1 = \Omega$, and Ω is connected, thus either $A_1 = \phi$ or $A_2 = \phi$. But $h = 0$ on E, we have $A_1 \neq \phi$, and hence $h = 0$ on Ω As desired. □

Corollary 7. ***(Identity principle-harmonic version,*** $n = 2$***).*** *Let u be harmonic on a domain $\Omega \subset \mathbb{C}$. If $u = 0$ on an open, non-empty set $E \subset \Omega$, then $u = 0$ throughout Ω.*

Theorem 14. ***(Closure under uniform limits)([4]Theorem 1.23).*** *Suppose that u_m is a sequence of harmonic functions on an open subset $\Omega \subset \mathbb{R}^n$ such that u_m converges uniformly to a function u on each compact subsets of Ω . Then u is harmonic on Ω. Moreover, for every multi-index α , $D^\alpha u_m$ converges uniformly to $D^\alpha u$ on each compact subsets of Ω.*

Proof. First assume that $\overline{B} \subset \Omega$. Then for every $x \in B$ and every m, we have

$$u_m(x) = \int_S u_m(\xi) P(x,\xi) d\sigma(\xi).$$

Because u_m uniformly to a function u on $\overline{B}$, so (after take the limit of both sides), we obtain

$$u(x) = \int_S u(\xi) P(x,\xi) d\sigma(\xi)$$

for all $x \in B$, and hence u is harmonic on B. Let α be a multi-index, then fore every $x \in B$, we get

$$D^\alpha u_m(x) = \int_S u_m(\xi) D^\alpha P(x,\xi) d\sigma(\xi),$$

which converges to

$$\int_S u(\xi) D^\alpha P(x,\xi) d\sigma(\xi).$$

By the argument used before this theorem the last integral equals to $D^\alpha u(x)$. Suppose K is a compact subset of B, then $D^\alpha P$ is uniformly bounded on $K \times S$, and the convergence of $D^\alpha u_m$ to $D^\alpha u$ is uniform on K.
Now If $\overline{B}(a,r) \subset \Omega$, we can use the same argument to prove that u is harmonic on $B(a,r)$, and $D^\alpha u_m$ converges uniformly to $D^\alpha u$ on any compact subset of $B(a,r)$, which completes the proof. □

1.1.10 The Classical Harnack inequality

Nonnegative harmonic functions satisfy an important inequality which restates the maximum principle in strong terms, call Harnack's inequality. It tells us that a positive harmonic functions cannot oscillate too much on a compact subset of connected set.
We begin with Harnack's inequality for the ball:

Theorem 15. ***(Harnack's Inequality for ball) ([14] Theorem 1.1).*** *Let u be a nonnegative harmonic function on an open set $\Omega \subset \mathbb{R}^n$. Let $B(a,r) \subset B(a,R) \subset \Omega$. Then for all $x \in B(a,r)$ we have*

$$\frac{R^{n-2}(R-r)}{(R+r)^{n-1}} u(a) \le u(x) \le \frac{R^{n-2}(R+r)}{(R-r)^{n-1}} u(a).$$

Proof. . Set $\rho = |x-a|$, and choose r' with $r < r' < R$. Since u is continuous on $\overline{B}(a,r')$, the Poisson formula for harmonic function can be applied, yielding

$$u(x) = \frac{{r'}^2 - \rho^2}{n\omega_n R_0} \int\limits_{S(a,r')} u(y)|x-y|^{-n} d\sigma(y), \tag{1.15}$$

where $d\sigma$ denotes the surface measure on $S(a,r')$

$$\frac{{r'}^2 - \rho^2}{(r'+\rho)^n} \leq \frac{{r'}^2 - \rho^2}{|x-y|^n} \leq \frac{{r'}^2 - \rho^2}{(r'-\rho)^n} \tag{1.16}$$

combining (1.15)-(1.16), and using the mean value of harmonic function we have

$$\frac{{r'}^{n-2}(r'-\rho)}{(r'+\rho)^{n-1}} u(a) \leq u(x) \leq \frac{{r'}^{n-2}(r'+\rho)}{(r'-\rho)^{n-1}} u(a).$$

Letting $r' \to R$ and realizing that the bounds are monotone in ρ which complete the proof. □

Corollary 8. ***(Harnack's Inequality for ball).*** *If u is a positive harmonic function on $\overline{B}(a,r) \subset \mathbb{R}^n$. Then*

$$\frac{r^{n-2}(r-|x-a|)}{(r+|x-a|)^{n-1}} u(a) \leq u(x) \leq \frac{r^{n-2}(r+|x-a|)}{(r-|x-a|)^{n-1}} u(a),$$

for $|x-a| < r$.

1.1.10.1 Harnack's Inequality for Ω

Let Ω be any domain(connected open set) and x, y be points of Ω, then there exists a constant $C^* = C^*(x,y)$ such that $u(x) \leq C^* u(y)$ for every positive function u harmonic on Ω. In fact such an inequality holds uniformly on compact sets, although the constant will blow up if we hold x fixed and let y approach the boundary. If $x, y \in K$, where K is a compact subset of Ω, then there is a constant $C = C(\Omega, K) < \infty$ may depend upon Ω and K, but that C is independent of x, y, and u such that $C^* \leq C$.

Theorem 16. ***(Harnack's Inequality for Ω).*** *Let Ω be connected and that K is a compact subset of Ω. Then there is a constant $C \in (1, \infty)$ such that*

$$\frac{1}{C} \leq \frac{u(x)}{u(y)} \leq C$$

for all points $x, y \in K$ and all positive harmonic functions u on Ω.

1.1.10.2 Harnack's Principle

Harnack's Inequality leads to an important convergence theorem for harmonic functions known as Harnack's Principle which: An increasing sequence of harmonic functions either tends to infinity or converges to a harmonic function (in either case the convergence is uniform on compact sets). Consider a monotone sequence of continuous functions on Ω . The pointwise limit of such a sequence need not behave well it could be infinite at some points and finite at other points. Even if it is finite everywhere, there is no reason to expect that our sequence converges uniformly on every compact subset of Ω.

Harnack's Principle shows that none of this bad behavior can occur for a monotone sequence of harmonic functions.

Theorem 17. ***(Harnack's Principle [4]).*** *Suppose Ω is connected and u_m is a pointwise increasing sequence of harmonic functions on Ω. Then either u_m converges uniformly on compact subsets of Ω to a function harmonic on Ω or $u_m(x) \to \infty$ for every $x \in \Omega$.*

Proof. Suppose first that $u_m \geq 0$. Let $u(x) = \lim_{m\to\infty} u_m$ for each $x \in \Omega$. If $u(x) = \infty$ for some $x \in \Omega$. Let $y \in \Omega$, then by Harnack's Inequality with the compact set $K = \{x, y\}$ there exist a constant $C \in (1, \infty)$ such that $u_m(x) \leq Cu_m(y)$ for all m. Because $u_m(x) \to \infty$, we also have $u_m(y) \to \infty$, and hence $u(y) = \infty$, which implies that $u(x) = \infty$ for all $x \in \Omega$ and that convergence is uniform on compact sets.
If $u(x) < \infty$ for all $x \in \Omega$, we assume that $K \subset \Omega$ is compact. Fix $y \in K$. Then by Harnack's Inequality there exist a constant $C \in (1, \infty)$ such that $u_m(x) - u_k(x) \leq C(u_m(y) - u_k(y))$ for all $x \in K$, whenever $m > k$. This implies u_m is uniformly Cauchy on K and hence uniformly convergence on K. By Theorem 14 the limit function u is harmonic on Ω. That finishes the proof in the case $u_m \geq 0$. In general, let $v_m = u_m - u_1$. Then v_m is an increasing sequence of non-negative harmonic functions, so we have either $v_m \to \infty$ uniformly on compact sets, in which case $u_m \to \infty$ as well, or v_m converges to a harmonic function v uniformly on compact sets, in which case $u_m \to v + u_1$. □

1.1.11 Homogeneous harmonic polynomials

Suppose that m is nonnegative integer, and α denotes multi-index. A polynomial p of the form

$$p(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha$$

is said to be homogeneous of degree m. Equivalently, a polynomial p is homogeneous of degree m if for every $\lambda \in \mathbb{R}$ and every $x \in \mathbb{R}^n$, we have

$$p(\lambda x) = \lambda^m p(x).$$

Theorem 18. *([4]Theorem 1.31) Suppose u is a harmonic function on Ω and $a \in \Omega$. Then there exist harmonic homogeneous polynomials p_m of degree m such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x - a) \tag{1.17}$$

for all x near a, the series converging absolutely and uniformly near a.

Proof. We may assume u is harmonic near 0, then by Theorem 9 u is real analytic, and it has a homogenous expansion converging to u in a neighborhood of 0.

$$u(x) = \sum_{m=0}^{\infty} p_m(x)$$

where

$$p_m(x) = \sum_{|\alpha|=m} \frac{D^\alpha u(0)}{\alpha!} x^\alpha$$

for x near 0. Since u is harmonic near 0, thus

$$\Delta u(x) = \sum_{m=0}^{\infty} \Delta p_m(x) = 0$$

for x near 0. Since Δp_m is homogeneous of degree $m-2$ for $m \geq 2$, and $\Delta p_m = 0$ for $m < 2$, then by uniqueness of homogeneous expansions , we have $\Delta p_m = 0$ for every m, that is meaning p_m is harmonic for every m, and hence we can represent the harmonic function u near 0 as an infinite sum of homogeneous harmonic polynomials. Translating this local result from 0 to the point $a \in \Omega$, we obtain the desired expansion. □

The expression (1.17) is called a homogeneous expansions of the function u at the point $a \in \mathbb{R}^n$.

Suppose that $\mathcal{H}_m(\mathbb{R}^n)$ denoted the space of all complex-valued homogeneous harmonic polynomials of degree m in $\mathbb{R}^n$. The next corollary states any function harmonic on the ball may be expressed uniquely as a sum of homogeneous harmonic polynomials.

Corollary 9. *: Suppose u is a harmonic function on $B(a, r)$. Then there exist $p_m \in \mathcal{H}_m(\mathbb{R}^n)$ such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) \tag{1.18}$$

for all $x \in B(a, r)$, the series converging absolutely and uniformly on compact subsets of $B(a, r)$.

1.2 Subharmonic functions in space

1.2.1 Introduction.

In mathematics, subharmonic functions are important classes of functions used extensively in partial differential equations, complex analysis and potential theory.
A fundamental example of subharmonic function is given by the Newton kernel (the elementary solution of the usual Laplace operator $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^n$), which is actually harmonic on $\mathbb{R}^n \backslash \{0\}$.
Intuitively, subharmonic functions are related to harmonic function as follows. If the values of a subharmonic function are no larger than the values of a harmonic function on the boundary of a ball, then the values of the subharmonic function are no larger than the values of the harmonic function also inside the ball. Subharmonic functions are of a particular importance in complex analysis, where they are intimately connected to holomorphic functions.

Before we can define subharmonic functions we need to recall upper semi-continuous function.

Definition 3. (Upper semi-continuous function). Let $\Omega \subset \mathbb{R}^n$, a function $u : \Omega \to [-\infty, +\infty)$ is said to be upper semicontinuous at a point $a \in \Omega$ if for any number $C > u(a)$ there exists a number $\delta = \delta(a, C)$ such that $u(x) < C$ whenever $|x - a| < \delta$ and $x \in \Omega$. A function u is said to be semicontinuous on the set Ω if it is upper semicontinuous at each point of Ω

An equivalent definition for u to be upper semicontinuous on Ω is to require the sets $\{x \in \Omega : u(x) < C\}$ be open in Ω for every $C \in \mathbb{R}$. Another equivalent definition for upper semicontinuous $\lim\limits_{x \to a} \sup u(x) \leq u(a)$ for all $a \in \Omega$.

Remark 3. *Note that upper semi-continuous functions are allowed to take value $-\infty$.*

Clearly if f, g are upper semicontinuous functions, and C is a non-negative constant, then all the functions $Cf, f+g, \max\{f, g\}$, and $\min\{f, g\}$ are also upper semicontinuous. Upper semi-continuity implies local boundedness from above.

Theorem 19. *Let f be upper semi-continuous. Then f is bounded above on compact sets and attains its upper bound in every compact set.*

1.2.2 Definition for subharmonic

Definition 4. (Subharmonic). Let Ω be an open subset of $\mathbb{R}^n$, and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function. We say that u is subharmonic function on Ω if u satisfy the following mean value inequality:

$$u(a) \leq \frac{1}{Area(S(a,r))} \int\limits_{S(a,r)} u(\xi) ds(\xi), \tag{1.19}$$

for all $\overline{B}(a, r) \subset \Omega$.

An equivalent definition is obtained using property:

$$u(a) \leq \frac{1}{V(B(a,r))} \int\limits_{B(a,r)} u(x) dV(x), \tag{1.20}$$

for all $\overline{B}(a, r) \subset \Omega$.

Remark 4. *Note that from the definition the subharmonic functions are allowed to take value $-\infty$, for an important example the function $\log |z - a|$. Also note that from the definition follows that every harmonic function is subharmonic.*

Example 3. Let $u(x) = \begin{cases} -|x|^{2-n} & \text{if } n > 2 \\ \log |x| & \text{if } n = 2 \end{cases}$, $u(0) = -\infty$.
Then u is subharmonic in $\mathbb{R}^n$

Example 4. Suppose that f is analytic on a domain Ω in $\mathbb{C}$. Then $\ln |f|$ is subharmonic on Ω. Since $\ln |f|$ is upper semi-continuous, so one only needs to verify the local sub-mean property. Let $z \in \Omega$. If $f(z) \neq 0$, then $\ln |f|$ is harmonic near z and from the mean value property of harmonic functions we obtain

$$f(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} u(z + \rho e^{i\theta}) d\theta; \quad 0 \leq r < \rho;$$

for some $\rho > 0$. If $f(z) = 0$ then $\ln|f| = -\infty$ and hence the following inequality is satisfies anyway

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + \rho e^{i\theta}) d\theta \quad 0 \leq r < \rho.$$

Notice that, in general, $|f|$ will not be harmonic when f is harmonic. For example, take $f(z) = z^k, k \in \mathbb{N}$.

Theorem 20. *Let $u \in C^2(\Omega)$. If $\Delta u \geq 0$, then*

$$u(a) \leq \frac{1}{Area(S(a,r))} \int_{S(a,r)} u(\xi) ds(\xi),$$

for every $\overline{B}(a,r) \subset \Omega$.

Proof. Note that the outward normal ν at $\xi \in S(a,r)$ is $\eta = \frac{\xi - a}{|\xi - a|}$. Then

$$\begin{aligned} \int_{S(a,r)} D_\nu u(\xi) ds(\xi) &= r^{n-1} \frac{d}{dr} \int_{|\eta|=1} u(a + r\eta) d\eta \\ &= r^{n-1} \frac{d}{dr} \left(r^{1-n} \int_{S(a,r)} u(\xi) ds(\xi) \right) \\ &= r^{n-1} n\omega_n \frac{d}{dr} \int_{S(a,r)} u(\xi) d\sigma(\xi). \end{aligned}$$

By Corollary 1, we have

$$r^{n-1} n\omega_n \frac{d}{dr} \int_{S(a,r)} u(\xi) d\sigma(\xi) = \int_{B(a,r)} \Delta u(x) dV(x) \geq 0.$$

Thus, the mean value $\int_{S(a,r)} u(\xi) d\sigma(\xi)$ is increasing. Since u is continuous, we have $\lim_{r \to 0} \int_{S(a,r)} u(\xi) d\sigma(\xi) = u(a)$. This proves the theorem. □

Corollary 10. *Let $u \in C^2(\Omega)$. If $\Delta u \geq 0$, then*

$$u(a) \leq \frac{1}{V(B(a,r))} \int_{B(a,r)} u(x) dV(x),$$

for every $\overline{B}(a,r) \subset \Omega$.

1.2.3 Some properties of subharmonic function

The subharmonic functions are a much more flexible tool than holomorphic, or even harmonic functions. An immediate consequence of the sub-mean value property is the maximum principle for subharmonic functions. There is no minimum principle for subharmonic functions, in other words subharmonic functions do not satisfy the minimum principle, for example $u(x) = |x|^2$ is subharmonic function on $\mathbb{R}^n$ which attains it's minimum at $x = 0$, but it is not harmonic .

1- If u is subharmonic on Ω, then Cu is subharmonic in Ω for any constant $C \geq 0$.
2- If the functions $u_1(x), ..., u_m(x)$ are subharmonic in a domain $\Omega \subset \mathbb{R}^n$, then the functions $\sum\limits_{i=1}^{m} u_i$, and $\max\limits_{1\leq i\leq m} u_i(x)$ are also subharmonic in Ω .
3- The limit of a uniformly convergent sequence of subharmonic functions is subharmonic function.
4- The limit of a monotone decreasing sequence of subharmonic function is subharmonic function.

Theorem 21. ***(Weak maximum Principle for subharmonic functions).*** *If u is subharmonic function on a bounded domain $\Omega \subset \mathbb{R}^n$, and continuous on $\overline{\Omega}$. then*

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

Proof. If this is not so, there exists $x_0 \in \Omega$ such that $u(x_0) > \max\limits_{x\in\partial\Omega} u(x)$. Since Ω is bounded there exists $\varepsilon > 0$ such that $u + \varepsilon|x|^2$ also has its maximum in Ω. If this is not so, there exists a sequence, $\{\varepsilon_n\}$ of positive numbers converging to zero and a point $x_{\varepsilon_n} \in \partial\Omega$ such that $u(x_{\varepsilon_n}) + \varepsilon_n|x_{\varepsilon_n}|^2 \geq u(x) + \varepsilon_n|x|^2$ for all $x \in \overline{\Omega}$. Then using compactness of $\partial\Omega$, there exists a subsequence, still denoted by ε_n such that $x_{\varepsilon_n} \to x_1 \in \partial\Omega$ and so, taking the limit, we obtain $u(x_1) \geq u(x)$ for all $x \in \overline{\Omega}$, contrary to what was assumed about x_0.
Now let x_1 be the point in Ω at which $u(x) + \varepsilon|x|^2$ achieves its maximum. Therefore, we must have $2n\varepsilon \leq 2n\varepsilon + \Delta u(x_1) \leq 0$ a contradiction. This proves the theorem. □

Lemma 5. . *Let Ω, Ω' be domains in $\mathbb{C}$. If $f : \Omega \to \Omega'$ is analytic, one-one and $g : \Omega' \to \mathbb{R}$ is subharmonic. Then the function $g \circ f$ is subharmonic.*

The next proposition will allow us to identify many subharmonic functions which are only continuous, not C^2, so that $\nabla u \geq 0$ criterion is not applicable. But if u is not of class C^2, then u is subharmonic if and only if it is the limit of a decreasing sequence of subharmonic functions of class C^2.
Also there are discontinuous subharmonic functions, for example: $u(z) = \sum\limits_{k=1}^{\infty} 2^{-k} \log|z - 2^{-k}|$ is subharmonic in the entire plane and is discontinuous at zero.

Proposition 3. *Suppose that $\Omega \subset \mathbb{C}$, and $u : \Omega \to \mathbb{R}$ be continuous function. If*

$$u(x) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + re^{i\theta}) d\theta, \tag{1.21}$$

for every disk $\overline{\mathbb{D}}(x, r) \subset \Omega$, then u is subharmonic. Conversely, if a continuous function $u : \Omega \to \mathbb{R}$ is subharmonic, then the inequality above (1.21) holds for $\overline{\mathbb{D}}(x, r) \subset \Omega$.

1.2.4 Subharmonic behavior of $|u|^p, 0 < p < 1$

The importance of subharmonic functions for spaces of analytic and harmonic functions lies in the fact that if f is analytic (resp.harmonic), then $|f|^p$ is subharmonic for every $p > 0$ (resp. $p > 1$).

Definition 5. (Subharmonic behaviour). An upper semicontinuous function u on a domain $\Omega \subset \mathbb{R}^n$ is said to be have subharmonic behaviour, if there exists a constant $C = C_{n,\Omega}$, depending only on n and Ω, such that

$$u(x) \leq \frac{C}{r^n} \int\limits_{B(x,r)} u(y)dy \tag{1.22}$$

for all $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$.

Clearly every subharmonic function u has subharmonic behaviour. Furthermore if $u \geq 0$ has subharmonic behaviour, then u^p also has subharmonic behaviour for all $p \in (0,\infty)$. We can use the subharmonicity of $|u|^p$, where u is harmonic to prove the harmonic Bergman space is complete for $0 < p < \infty$.

Let u is harmonic function on a domain Ω in Euclidean space $\mathbb{R}^n$, and suppose that $p > 0$. If $p \geq 1$, then the function $|u|^p$ is subharmonic on Ω, and therefore has the sub-mean value property over balls

$$|u(a)|^p \leq \frac{1}{V(B(a,r))} \int\limits_{B(a,r)} |u(x)|^p dx,$$

where $B(a,r) \subset \Omega$. If $0 < p < 1$, then the function $|u|^p$ need not be subharmonic, but it behaves like subharmonic function. This fact was established by Hardy and Littlewood [26] for $n = 2$ and generalized by Fefferman and Stein [[22], Section 9, Lemma 2] for $n > 2$. This " subharmonic behaviour " of $|u|^p$ is given by

Theorem 22. *[26], [22]. Let $0 < p < \infty$. Then there exists a positive constant $C_{n,p}$ such that*

$$|u(a)|^p \leq C_{n,p} \frac{1}{V(B(a,r))} \int\limits_{B(a,r)} |u(x)|^p dx \tag{1.23}$$

for every real harmonic function u in $B(a,r) \subset \mathbb{R}^n$.

Proof. Of course $C_{n,p} = 1$ for $p \geq 1$, so we assume $0 < p < 1$. We can assume $a = 0$ and $r = 1$.

$$u(x) = \frac{\varepsilon^2 - |x|^2}{n\omega_n \varepsilon} \int\limits_{S_\varepsilon} \frac{u(y)}{|x-y|^n} d\sigma(y),$$

for $x \in B_\varepsilon = B(0,\varepsilon)$, $0 < \varepsilon < 1$. Hence

$$|u(x)| \leq \frac{\varepsilon^2 - \rho^2}{n\omega_n \varepsilon} \int\limits_{S_\varepsilon} \frac{|u(y)|}{(\varepsilon - \rho)^n} d\sigma(y) = \frac{\varepsilon + \rho}{n\omega_n \varepsilon (\varepsilon - \rho)^{n-1}} \int\limits_{S_\varepsilon} |u(y)| d\sigma,$$

for $0 \leq |x| = \rho < \varepsilon < 1$. Hence

$$M_\infty(\rho) \leq (1 + \frac{\rho}{\varepsilon}) \frac{M_1(\varepsilon)}{(1 - \frac{\rho}{\varepsilon})^{n-1}} \leq 2(1 - \frac{\rho}{\varepsilon})^{1-n} M_1(\varepsilon), \quad 0 < \rho < \varepsilon < 1 \tag{1.24}$$

$$M_1(t) \leq M_\infty^{1-p}(t) M_p^p(t), \quad 0 < t < 1 \tag{1.25}$$

where $M_p(\rho) = \left(\frac{1}{n\omega_n\rho^{n-1}} \int\limits_{S_\rho} |u|^p d\sigma\right)^{1/p}, \quad M_\infty(\rho) = \sup\limits_{S_\rho} |u|$
it suffices to obtain :$|u(0)| \leq C_{p,n}$ under normalization condition

$$\int_0^1 \rho^{n-1} M_p^p(\rho) d\rho \leq 1. \tag{1.26}$$

Using (1.25) and (1.24) we get, for $0 < t < \varepsilon < 1$:

$M_1(t) \leq M_p^p(t) M_\infty^{1-p}(t) \leq M_p^p(t) M_1^{1-p}(\varepsilon) 2^{1-p}(1 - \frac{t}{\varepsilon})^{(1-p)(1-n)}$,

so

$\log M_1(t) \leq p \log M_p(t) + (1-p)\log M_1(\varepsilon) + (1-p)\log 2 + (1-p)(1-n)\log(1 - \frac{t}{\varepsilon})$.

We get $\varepsilon = t^a$ for some $0 < a < 1$ (then $t < \varepsilon$) and obtain

$\log M_1(t) \leq p \log M_p(t) + (1-p)\log M_1(t^a) + (1-p)\log 2 + (1-p)(1-n)\log(1 - t^{1-a})$,

hence

$$\begin{aligned}
\int_{1/2}^1 \log M_1(t)\frac{dt}{t} &\leq (1-p)\log^2 2 + (1-p)(1-n)\int_{1/2}^1 \log(1-t^{1-a})\frac{dt}{t} \\
&\quad + p\int_{1/2}^1 \log M_p(t)\frac{dt}{t} + (1-p)\int_{1/2}^1 \log M_1(t^a)\frac{dt}{t} \\
&= C(p,n,a) + p\int_{1/2}^1 \log M_p(t)\frac{dt}{t} + \frac{1-p}{a}\int_{(1/2)^a}^1 \log M_1(\rho)\frac{d\rho}{\rho}.
\end{aligned}$$

Now we choose $a = 1 - p^2$. If $M_1(2^{-a}) \leq 1$ then we have $|u(a)| \leq 1$. So we assume $M_1(2^{-a}) > 1$.
From the above estimates we obtain

$$\begin{aligned}
\frac{p}{1+p}\int_{1/2}^1 \log M_1(t)\frac{dt}{t} &\leq C(p,n) + \int_{1/2}^1 \log M_p^p(t)\frac{dt}{t} \\
&\leq C(p,n) + \int_{1/2}^1 M_p^p(t)\frac{dt}{t} \\
&\leq C(p,n) + 2^n\int_{1/2}^1 t^{n-1}M_p^p(t)dt
\end{aligned}$$

and we get

$$\int_{1/2}^{1} \log M_1(t)\frac{dt}{t} \leq K(u,p).$$

Since $M_1(t)$ is increasing in $0 \leq t < 1$ we get $M_1(y_2) \leq C(u,p)$, so

$$|u(0)| \leq M_1(y_2) \leq C(u,p).$$

As desired. □

Remark 5. *:*
(i) This proof shows that it is true also for complex valued u.
(ii) More generally the theorem remains true if $|u|$ is replaced by an arbitrary nonnegative subharmonic function on Ω.
(iii)When $0 < p < 1$ the volume mean

$$\frac{1}{V(B(a,r))}\int_{B(a,r)} |u(x)|^p dx$$

in the theorem 22 can not be replaced by the spherical mean

$$\frac{1}{Area(S(a,r))}\int_{S(a,r)} |u(\xi)|^p d\sigma(\xi).$$

Corollary 11. *Suppose that $0 < p < \infty$. Then there exists a constant $C = C_p$ depending only on p such that*

$$|u(0)|^p \leq C_p \int_{\mathbb{D}} |u|^p ds$$

for every harmonic function u on unit disc $\mathbb{D}$. When $p \geq 1$, the inequality holds with $C_p = 1$.

Remark 6. *More generally this inequality remains valid if we assume that $u \geq 0$ is an arbitrary subharmonic function in $\mathbb{D}$.*

1.3 Analytic definition of planar quasiconformal mapping

Definition 6. (Sense-preserving homeomorphisms). A homeomorphism $f : \Omega \to \Omega'$ is called sense-preserving if f preserves the orientation of the boundary of every Jordan domain D such that $\overline{D} \subset \Omega$.

Definition 7. (regular mapping). A map $f : \Omega \to \Omega'$ is called regular at the point z if z lies in the interior of Ω , f is differentiable at z, and $J_f(z) \neq 0$.

Lemma 6. *If a homeomorphism $f : \Omega \to \Omega'$ possesses a regular point z where $J_f(z) > 0$ then f is sense-preserving. Conversely, the Jacobian of a sense preserving homeomorphism is positive at every regular point.*

Definition 8. (The directional derivative). Let f be regular throughout Ω and the partial derivatives f_x and f_y are continuous in Ω. Then the directional derivative of f at z in the direction of θ is

$$\partial_\theta f(z) = \lim_{r\to 0} \frac{f(z+re^{i\theta}) - f(z)}{re^{i\theta}} = e^{-i\theta}(f_x(z)\cos\theta + f_y(z)\sin\theta)$$

exists at every point $z \in \Omega$.

Definition 9. (Dilatation quotient). Let f be regular at $z \in \Omega$. If f has directional derivative at z, then the dilatation quotient D, is

$$D(z) = \frac{\max\limits_{\alpha} |\partial_\alpha f(z)|}{\min\limits_{\alpha} |\partial_\alpha f(z)|},$$

is bounded in every compact subset of Ω.

Definition 10. (Absolutely continuous functions on an interval). The function $f : (a, b) \to \mathbb{R}^m$ is absolutely continuous on the interval (a, b) if for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\sum_{i=1}^{n} ||f(b_i) - f(a_i)|| < \epsilon,$$

then for every finite sequence of non-intersecting intervals $a_i \leq x \leq b_i, i = 1, ..., n$ contained in (a, b), such that

$$\sum_{i=1}^{n} |b_i - a_i| < \delta$$

Remark 7. *(Equivalent definition). The following conditions for a real-valued function f on a compact interval $[a, b]$ are equivalent*
(1) f is absolutely continuous;
(2) f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)dt \quad \forall x \in [a, b].$$

(3) there exists a Lebesgue integrable function g on $[a, b]$ such that

$$f(x) = f(a) + \int_a^x g(t)dt \quad \forall x \in [a, b].$$

Now before define ACL, we consider a closed 2-interval $I = \{z \equiv x + iy \in \mathbb{C} : x \in [a, b], y \in [c, d]\}$ with a function $f : I \to \mathbb{R}^m$, continuous, for almost any $(c_1, c_2) \in I$. Let $g : [a, b] \to \mathbb{R}^m$, and $h : [c, d] \to \mathbb{R}^m$ where $g(x) = f(x, c_2)$, $h(x) = f(c_1, x)$, are absolutely continuous.

Definition 11. (ACL on an interval). A function $f\ |_I$ is called absolutely continuous on lines (ACL) if $f\ |_I$ is continuous and it is absolutely continuous on almost every line segment in I parallel to the coordinate axes.

Definition 12. (ACL): Let $\Omega \subset \mathbb{C}$ be a domain, $I \subset \Omega$. A function $f : \Omega \to \mathbb{R}^m$, is ACL if for every 2-interval $I \subset \Omega$, the function $f\,|_I$ is ACL.

Remark 8. *A complex valued function $f \equiv u + iv$ is ACL in $\Omega \subset \mathbb{C}$ if u, v are both ACL in Ω.*

Definition 13. (Analytic definition). Let Ω, Ω' are subsets of $\mathbb{C}$. A sense-preserving homeomorphism $f : \Omega \to \Omega'$ is called K-quasiconformal mapping of the domain Ω if satisfy the following two conditions:
1. f is absolutely continuous on lines in Ω.
2. The dilatation condition

$$\frac{|f_z(z)| + |f_{\overline{z}}(z)|}{|f_z(z)| - |f_{\overline{z}}(z)|} \leq K \tag{1.27}$$

holds almost everywhere in Ω.

A map is quasiconformal if it is K-quasiconformal for some K.

Remark 9. *Let f be regular at $z \in \Omega$ and let $f_z(z) = \frac{1}{2}[f_x(z) - if_y(z)]$ and $f_{\overline{z}}(z) = \frac{1}{2}[f_x(z) + if_y(z)]$, be the complex derivatives. Then the directional derivative of f at z in the direction of θ can be expressed in terms of the complex derivatives as*

$$\partial_\theta f(z_0) = f_z(z_0) + f_{\overline{z}}(z_0)e^{-2i\theta},$$

and for the Jacobian we get

$$J(z_0) = |f_z(z_0)|^2 - |f_{\overline{z}}(z_0)|^2.$$

Since f is sense-preserving, $J(z) \geq 0$, and so $|f_z(z)| \geq |f_{\overline{z}}(z)|$. Therefore

$$\max_\theta |\partial_\theta f(z)| = |f_z(z)| + |f_{\overline{z}}(z)|, \quad \min_\theta |\partial_\theta f(z)| = |f_z(z)| - |f_{\overline{z}}(z)|.$$

Chapter 2

Harmonic Bergman space and reproducing kernels

Introduction.
A norm on a (complex) linear(vector) space X is a function $\|.\| : X \to \mathbb{R}$ satisfying the following conditions:
$(i)\|f\| \geq 0$ for all $f \in X$ (nonnegative),
$(ii)\|f\| = 0$ if and only if $f = 0$, where $f \in X$ (strictly positive),
$(iii)\|\lambda f\| = |\lambda|\|f\|$ for every $f \in X$ and scalar $\lambda \in \mathbb{C}$ (homogeneity),
$(iv)\|f + g\| \leq \|f\| + \|g\|$ for every $f, g \in X$ (triangle inequality).
The couple $(X, \|.\|)$ is called normed space.

The function $\|.\|$ which satisfying the properties: (i), (ii), and (iii) of a norm, and satisfy $\|f+g\| \leq C(\|f\|+\|g\|)$ (C-triangle inequality) where $C(\geq 1)$ is a constant independent of $f, g \in X$ is called a quasi-norm on X, and the couple $(X, \|.\|)$ is called quasi-normed space.

2.1 L^p spaces for $0 < p < \infty$.

One of the most important examples of a function space is the space of measurable functions whose absolute values are p^{th} power integrable where $1 \leq p < \infty$.

Definition 14. **(L^p space).** Suppose that $1 \leq p < \infty$. Let Ω be a measurable subset of $\mathbb{R}^n$, and μ is a Borel measure. then the space $L^p(\Omega)$ is the set of Lebesgue measurable function $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) whose p^{th} power is Lebesgue integrable ie:

$$\int_\Omega |f(x)|^p d\mu(x) < +\infty; \tag{2.1}$$

with the norm

$$\|f\|_{L^p(\Omega)} := \|f\|_p = \left(\int_\Omega |f(x)|^p d\mu(x)\right)^{1/p}. \tag{2.2}$$

When $p = 1$ the space $L^1(\Omega)$ consists of all integrable functions on Ω.

A function f, measurable on Ω is said to be essentially bounded on Ω if there exists a constant C such that $|f(x)| \leq C$ almost everywhere in Ω. The greatest lower bound of

C is called the essential supremum of $|f|$ on Ω and is denoted by $ess\sup\limits_{x\in\Omega}|f(x)|$.

$$ess\sup_{\Omega}|f| := inf\{k : \mu\{x \in \Omega : \quad f(x) > k\} = 0\} \tag{2.3}$$

When $p = \infty$, we denote by $L^{\infty}(\Omega)$ the space of all functions essentially bounded Lebesgue measurable on Ω with the essential supremum as the norm. Hence

$$\|f\|_{\infty} = ess\sup_{\Omega}|f| \tag{2.4}$$

is a norm on $L^{\infty}(\Omega)$.

Now when $p \in (0, 1)$ the functional is not norm, but satisfies $\|f+g\|_p \leq C_p(\|f\|_p+\|g\|_p)$ (with $C_p = 2^{1/p-1}$). In fact it is a quasi-norm.

2.1.0.1 Some properties for the space $L^p : p > 0$

1)$\|f\|_p \geq 0$ for any measurable f.
2)$\|f\|_p = 0$ if and only if $f = 0$ $a.e$ in Ω.
3)$\|\lambda f\|_p = |\lambda|.\|f\|_p$ for any λ is a scalar.

We call the positive real numbers $1 < p, q < \infty$ a pair of conjugate exponents if $\frac{1}{p}+\frac{1}{q} = 1$. We naturally regard 1, ∞ as a pair of conjugate exponents.

Theorem 23. ***(Holder's inequality).*** *Let $p \in [1, \infty]$ and q be its conjugate exponents.Then for any $f \in L^p(\Omega)$ and any $g \in L^q(\Omega)$, we have $fg \in L^1(\Omega)$ and*

$$\int\limits_{\Omega} |fg|d\mu \leq \Big(\int\limits_{\Omega} |f|^p d\mu\Big)^{1/p}\Big(\int\limits_{\Omega} |g|^q d\mu\Big)^{1/q} \tag{2.5}$$

or equivalently

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^P(\Omega)}\|g\|_{L^q(\Omega)}$$

Moreover, equality holds only if there exists a constant C such that

$$|f(x)|^p = C|g(x)|^q \quad for \quad a.e.x \in \Omega.$$

Proof. . The inequality is obviously true if $p = 1$ or ∞, or $\|f\|_{L^p(\Omega)} = 0$. For $p \in (1, \infty)$ and $\|f\|_{L^p(\Omega)} \neq 0$, we use the following inequality

$ab \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{1-q}b^q}{q}$, $\varepsilon > 0$, q is a conjugate of $p \in (1, \infty)$,with $a = |f|$,$b = |g|$, we obtain

$$\int\limits_{\Omega} |fg|d\mu \leq \frac{\varepsilon\|f\|^p_{L^p(\Omega)}}{p} + \frac{\varepsilon^{1-q}\|g\|^q_{L^q(\Omega)}}{q};$$

for all $\varepsilon > 0$. Choice $\varepsilon = \|g\|_{L^q(\Omega)}/\|f\|^{p-1}_{L^p(\Omega)}$. □

Corollary 12. ***(Schwarz inequality).*** *When $p = q = 2$ the inequality*

$$\int_\Omega |fg| d\mu \leq \left(\int_\Omega |f|^2 d\mu \right)^{1/2} \left(\int_\Omega |g|^2 d\mu \right)^{1/2}$$

is known as the Schwarz inequality .

Theorem 24. ***(Minkowski inequality [41]).*** *Let $f, g \in L^p(\Omega)$, for some $p \in [1, \infty]$. Then $f + g \in L^p(\Omega)$ and*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^P(\Omega)} + \|g\|_{L^p(\Omega)}$$

Moreover, equality holds only if there exists a constant C such that

$$f(x) = Cg(x) \quad for \quad a.e. x \in \Omega. \quad Or \quad g(x) = Cf(x) \quad for \quad a.e. x \in \Omega.$$

Proof. The inequality is obviously true for $p = 1$ and $p = \infty$. For $p \in (1, \infty)$, we applying Holder inequality, we obtain

$$\begin{aligned} \|f + g\|^p_{L^p(\Omega)} &\leq \int_\Omega |f + g|^{p-1} |f| d\mu + \int_\Omega |f + g|^{p-1} |g| d\mu \\ &\leq \left(\int_\Omega |f + g|^{(p-1)q} \right)^{1/q} (\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}) \\ &= \left(\int_\Omega |f + g|^p \right)^{1-1/p} (\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}). \end{aligned}$$

□

2.2 Hilbert space

Inner product space. We define $\|x\| = \langle x, x \rangle^{1/2}$. An inner product on a complex vector space X is a function, $\langle ., . \rangle : X \times X \to \mathbb{C}$, such that: $\langle x, x \rangle \geq 0$ with equality $\langle x, x \rangle = 0$ if and only if $x = 0$, $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ i.e. $x \to \langle x, z \rangle$ is linear, $\overline{\langle x, y \rangle} = \langle y, x \rangle$. The couple $(X, \langle ., . \rangle)$ is called an inner product space.

Lemma 7. ***(Parallelogram Law).*** *Let $(X, \langle ., . \rangle)$ be an inner product space, then*

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

for all $f, g \in X$.

Theorem 25. *:**(Schwarz Inequality [12]).*** *Let $(X, \langle ., . \rangle)$ be an inner product space, then*

$$|\langle f, g \rangle| \leq \|f\| \|g\|$$

for all $f, g \in X$, and equality holds if and only if f, g are linearly dependent.

Definition 15. (Hilbert space). A Hilbert space is an inner product space $(H, \langle ., . \rangle)$ such that the induced Hilbertian norm is complete.

For example the spaces $L^2(\Omega)$ with inner product

$$\langle f, g\rangle = \int\limits_{\Omega} f.\bar{g}d\mu$$

are Hilbert spaces.

Suppose that H is a Hilbert space, then for every bounded linear operator $T : H \to H$ we have $\|T\| = \sup\{\langle Tx, y\rangle : \|x\| \leq 1, \|y\| \leq 1\}$. A dual space of a normed linear space $(X, \|.\|)$ is set (namely X^*) of all continuous linear functions $T : X \to \mathbb{C}(or\mathbb{R})$. The dual norm is a function $\|.\|^* : X^* \to \mathbb{R}$ defined by $\|T\|^* := \sup\{|T(u)|/\|u\| : u \in X\backslash\{0\}\}$.

Theorem 26. ***(Riesz Theorem).*** *Let H^* be the dual space of a Hilbert space H. Then the function defined by $T(x) = \langle ., x\rangle \, \forall x \in H$ is an element of H^*. Every element of H^* can be written uniquely in this form. The map $T : H \to H^*$ is a conjugate linear isometric isomorphism.*

Theorem 27. ***(The Riesz-Fréchet theorem)[17].*** *Let H be a Hilbert space. Let T be a continuous linear function $T : H \to \mathbb{C}$. Then there exists a unique vector v in H such that $T(u) = \langle v, u\rangle$ for all u in H.*

Proof. : Define $F : H \to \mathbb{C}$ by

$$F(w) := \frac{1}{2}\|w\|^2 - ReT(w).$$

The functional T is Lipschitz because it is continuous. So there exists a constant C such that $|T(w)| \leq C\|w\|$. It follows that $F(w) \geq \frac{1}{2}\|w\|^2 - C\|w\|$. If we set $\rho := \|w\|$, and differentiate $F(\rho)$ we find that the minimum of F occurs when $\|w\| = C$, and hence when $T(w) = \|w\|^2$. So $F(w) \geq -\frac{1}{2}C^2$. Thus $F(w)$ is bounded from below. Assume that a be the infimum of the $F(w)$. Let w_m be a sequence in H such that $F(w_m) \to a$ as $m \to \infty$. By the parallelogram identity and the linearity of T, we have

$$\|w_m - w_k\|^2 = 2\|w_m\|^2 - \|w_m + w_k\|^2 + 2\|w_k\|^2 = 4F(w_m) - 8F(\frac{w_m + w_k}{2}) + 4F(w_k).$$

Hence

$$\|w_m - w_k\|^2 \leq 4(F(w_m) + F(w_k)) - 8a.$$

The right hand side tends to zero, so w_m is Cauchy sequence and hence has a limit v by the completeness of H. Since F is continuous, $F(v) = a$, so F assumes its minimum. For any vector $u \in H$ and any real number $t > 0$, we have $F(v) \leq F(v + tu)$. By a long computation we get

$$0 \leq Re\,\langle v, u\rangle - ReT(u) + \frac{1}{2}t\|u\|^2.$$

Since this is true for arbitrary $t > 0$, we have $ReT(u) \leq Re\,\langle v, u\rangle$ for all u. The same argument applied to $-u$ shows that $-ReT(u) \leq -Re\,\langle v, u\rangle$. Hence $ReT(u) = Re\,\langle v, u\rangle$. The above reasoning applied to $-iu$ shows that $ReT(u) = Re\,\langle v, u\rangle$. We conclude that $T(u) = \langle v, u\rangle$. □

2.2.1 Orthogonal projection on a closed subspace of a Hilbert space

Let $(X, \langle ., . \rangle)$ be an inner product space, we say $f, g \in X$ are orthogonal and write $f \perp g$ iff $\langle f, g \rangle = 0$. More generally we say $f \in X$ is orthogonal to a set $E \subset X$ and write $f \perp E$ iff $\langle f, g \rangle = 0$ for all $g \in E$. We denote the set of all elements in X which orthogonal to the set $E \subset X$ by $E^{\perp} = \{f \in X : f \perp E\}$. We also say that a set $E \subset X$ is orthogonal if $f \perp g$ for all $f, g \in E$ with $f \neq g$.
We say that $E \subset X$ orthonormal if it is orthogonal and satisfies $\|f\| = 1$ for all $f \in E$.

Every Hilbert space has an orthonormal basis.

Theorem 28. *([31].Theorem 2.6.) Suppose H is a Hilbert space with reproducing kernel R_Ω and that $\{e_m\}_{m=1}^{\infty}$ is an orthonormal basis for H. Then*

$$R_\Omega(x, y) = \sum_{m=1}^{\infty} e_m(x)\bar{e}_m(y)$$

Lemma 8. *: Let X be an inner product space and E be a subset of X. Then $E^{\perp}$ is a closed linear subspace of X.*

In fact $E^{\perp} = \cap_{f \in E} Ker(\langle ., f \rangle)$, where $Ker(\langle ., f \rangle) := \{g \in X : \langle g, f \rangle = 0\}$ is a closed subspace of X.

Definition 16. (Projection operators). Let X be a vector space. A map $T : X \to X$ is a projection operator if it is linear and satisfies $T^2 = T$.

Note that for any element f in X can be written as sum $a + b$, where $a \in ImT$ and $b \in Ker(T)$.

Now suppose that $X = H$ be a Hilbert space and Y be a closed subspace(linear) of H. The orthogonal projection of H onto Y is the function $T : H \to H$ such that for any $x \in H$, there is a unique point (namely $T(x)$) in Y closest to x such that $x - T(x)$ is orthogonal to Y.

Definition 17. (Adjoint). Let $T : X \to X$ be a bounded operator. The adjoint of T, denote T^*, is the unique operator $T^* : X \to X$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$. A bounded operator $T : X \to X$ is self - adjoint if $T^* = T$. It is normal operator if it commutes with its adjoint T^*, that is $TT^* = T^*T$.

Proposition 4. *([12]Proposition 14.13). Let H be a Hilbert space and $Y \subset H$ be a closed subspace. Then the orthogonal projection $T : H \to H$ satisfies:*
(i). T is linear.
(ii). T is projection.
(iii). T is self-adjoint.
(iv). $Ker(T) = Y^{\perp}$.

Proof. To prove (i). Let $x, y \in H$ and $\alpha \in \mathbb{C}$, then $Tx + \alpha Ty \in Y$ and
$Tx + \alpha Ty - (x + \alpha y) = [Tx - x + \alpha(Ty - y)] \in Y^{\perp}$. Thus $Tx + \alpha Ty = T(x + \alpha y)$ which means T is linear.
To prove (iii). Let $x, y \in H$. Since $x - Tx$ and $y - Ty$ are in $Y^{\perp}$. Then
$\langle Tx, y \rangle = \langle Tx, Ty + y - Ty \rangle = \langle Tx, Ty \rangle = \langle Tx + (x - Tx), Ty \rangle = \langle x, Ty \rangle$. □

The following corollary shows that for each element $y \in Y$ can be written uniquely as a sum $a + b$ with $a \in Im(T)$ and $b \in Ker(T)$. More precisely $a = T(y)$ and $b = y - T(y)$. The point $T(y)$ is the point in Y closest to y.

Corollary 13. *Let Y be a closed subspace of a Hilbert space H. Then for any $y \in Y$ there is a unique $a \in Y$ and $b \in Y^{\perp}$ such that $y = a + b$.*

Proof. Let $x \in X$ and let $y = T(x)$, then $x - y \in Y^{\perp}$ and hence $x = y + (x - y) \in Y + Y^{\perp}$. □

Note that for any $x \in H$ we have $|x|^2 = |Tx|^2 + |x - Tx|^2$ which implies, in particular, that T is a bounded linear map.

Proposition 5. *Let H be a Hilbert space and $T : H \to H$ be a linear map (not necessarily bounded) such that there exists $T^* : H \to H$ with $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$. Then T is bounded . Conversely, Let H, H' be Hilbert spaces and $T : H \to H'$ be a bounded operator. Then there exists a unique bounded operator $T^* : H' \to H$ such that $\langle Tx, y\rangle_{H'} = \langle x, T^*y\rangle_H$ for all $x \in H$ and all $y \in H'$*

2.2.2 Characterization of orthogonal projections

Recall, that if Y is a closed subspace of a complex Hilbert space H, then there is a bounded linear operator $T : H \to H$ such that $T(H) = Y$ with $T^2 = T$ and $T^* = T$. Conversely if $T : H \to H$ is any bounded linear operator for which $T^2 = T$ then the following are equivalent: (i) T is normal, (ii) T is self-adjoint, (iii) T is an orthogonal projection onto a closed subspace.

Lemma 9. *Suppose H be a complex vector space with a Hermitian inner-product $\langle ., .\rangle$. Let $T : H \to H$ be a bounded linear operator. Then*
(i) If $\langle Tx, x\rangle = 0$ for all $x \in H$ then $T = 0$.
(ii) the operator T is normal iff $|Tx| = |T^*x|$. In particular, if T is normal then $Ker(T) = Ker(T^*)$.

Now the following proposition gives a characterization of orthogonal projections:

Proposition 6. *Let $T : H \to H$ be a bounded linear map on the complex Hilbert space H such that $T^2 = T$. Then the following are equivalent:*
(i) T is self-adjoint
(ii) T is normal
(iii) $x - Tx$ is orthogonal to Tx for all $x \in H$.
If these conditions hold then T is the orthogonal projection onto its image (closed subspace).

2.3 Banach space

A Banach space is a normed linear space that is a complete metric space with respect to the metric d derived form its norm $d(x, y) = \|x - y\|$. For example finite-dimensional linear space $\mathbb{R}^n$(or $\mathbb{C}^n$) is a Banach space with respect to the Euclidean norm. Another important example: the spaces $L^p(\Omega)$ are Banach spaces for $1 \leq p \leq \infty$, but when $0 < p < 1$ the space L^p is not Banach.

Definition 18. (bounded linear operator on Banach space). Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator defined on X. T is called bounded if there exists $C \leq \infty$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$. If T is bounded, define

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

The number $\|T\|$ is called the operator norm of T. Clearly, if $\|Tx\| \leq C$ for every $x \in X$ with $\|x\| = 1$, then $\|T\| \leq C$. Moreover, the family of bounded linear operators is a linear space with respect to the addition and multiplication of operators by scalars. From the definition of the norm $\|T\|$, it follows that $\|Tx\| \leq \|T\|.\|x\|$ for every $x \in X$. Any bijection (one-one and onto) bounded linear mapping between two Banach spaces has a bounded inverse.

Lemma 10. *A linear functional is continuous if and only if it is continuous at the origin.*

Proposition 7. *A linear functional T on a Banach space is continuous, if and only if it is bounded. ie $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\} < +\infty$.*

Proof. Let T denotes the linear functional and let X be a Banach space. First suppose that $T : X \to \mathbb{R}$ is continuous, then for $f \in X$, and $\varepsilon = 1$ there exists $\delta > 0$ such that $|T(f)| \leq 1$ whenever $\|f\| \leq \delta$. Let $g \in X\backslash\{0\}$, then $\|\delta g/\|g\|\| = \delta$, which implies that $|T(\delta g/\|g\|)| \leq 1$ and hence $|T(g)| \leq \|g\|/\delta$. Conversely, if T is bounded it is clearly continuous at the origin, hence continuous. □

Theorem 29. ***(Banach-Steinhaus/uniform boundedness theorem).*** *Let $T_\alpha : X \to Y$ be a continuous linear map from a Banach space X to a normed space Y for each α in an index set A. Then either there is a uniform bound $M < \infty$ so that $|T_\alpha| \leq M$ for all $\alpha \in A$, or there is $x \in X$ such that $\sup_{\alpha \in A} |T_\alpha(x)| = +\infty$.*

Theorem 30. ***(Open mapping theorem).*** *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. If T is a surjective map, then T is an open mapping, ie $T(E)$ is open in Y for all open subsets $E \subset X$.*

Corollary 14. *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. If T is a a bijective, then the inverse map, T^{-1}, is bounded linear operator.*

Theorem 31. ***(Closed graph theorem).*** *Let X, Y be Banach spaces. A linear map $T : X \to Y$ is continuous iff T is closed.*

Proof. Suppose T is continuous. Let $\Gamma := \{(x, y) : T(x) = y\}$. If $(x_n, Tx_n) \to (x, y) \in X \times Y$ as $n \to \infty$ then $Tx_n \to Tx = y$ which implies $(x, y) = (x, Tx) \in \Gamma(T)$. Hence $\Gamma(T) \subset X \times Y$ is closed.
Conversely suppose T is closed. Let $\Gamma(x) = (x, Tx)$. Since the product of Banach spaces is a Banach space, thus the space $X \times Y$ with norm $|\langle u, v\rangle| = |u|.|v|$ is a Banach. And since Γ is a closed subspace of $X \times Y$, it is a Banach space itself with the restriction of this norm. The projection $F : X \times Y \to X$ is a continuous linear map and the restriction $G \mid_{\Gamma(T)}: \Gamma(T) \to X$ is continuous bijection and hence by the open mapping Theorem 30 $G \mid_{\Gamma(T)}^{-1}$ is bounded. Therefore $T = F \circ G \mid_{\Gamma(T)}^{-1}$ is bounded and expresses T as a composite of continuous functions. The proof is completed. □

Let Y be a vector subspace of X and let $T : Y \to Z$ a linear map to another vector space Z. A linear map $T' : X \to Z$ is an extension of T to X when the restriction $T' \mid_Y$ of T' to Y is T.

Theorem 32. ***(Hahn-Banach theorem)****. Let X be a normed vector space with scalars $\mathbb{R}$ (or $\mathbb{C}$), and let Y be a subspace. If T be a continuous linear functional on Y, then there is an extension T' of T to X such that $\|T'\| = \|T\|$.*

2.3.0.1 Quasi-Banach space

A vector space X equipped with a quasi-norm (resp. p-norm) is called a quasi-Banach (resp. p-Banach) space if it is complete (with resect to the invariant metric d). The most important class of quasi-Banach spaces which are not already Banach space is the class of $L^p(\Omega)$ spaces for $0 < p < 1$ with the usual quasi-norm.

Let X, Y be quasi-Banach spaces and let $T : X \to Y$ be a linear. As in the Banach space case, T is called bounded or continuous if $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\} < \infty$.

2.4 Bergman space $b^p(\Omega), \Omega \subset \mathbb{R}^n$

The Bergman spaces of harmonic functions $b^p(\Omega)$ are named in honor of Stefan Bergman (1895-1977), who studied analogous spaces of holomorphic functions belonging to L^p with respect to volume measure [6], with emphasis on the case $p = 2$.

Suppose that p denotes a number satisfying $1 \leq p < \infty$. The Bergman space $b^p(\Omega)$ is the set of all harmonic functions u on Ω that are p^{th} power integrable with respect to volume measure .ie:

$$\|u\|_{b^p(\Omega)} := \left(\int_\Omega |u|^p dV \right)^{1/p} < +\infty, \tag{2.6}$$

where dV denotes the usual n-dimensional Lebesgue measure on Ω

Several properties of Bergman spaces of harmonic functions on the unit ball in $\mathbb{R}^n$ are analogous to the Bergman spaces of analytic functions on the unit ball in $\mathbb{C}^n$.
When $p = 2$ the Bergman space $B^2(\Omega)$ is a Hilbert space which played a fundamental role in much of his work [Bergman 1970].

The following proposition shows that the point evaluation is continuous on $b^p(\Omega)$.

Proposition 8. *([4] Proposition 8.1) Suppose that $0 < p < \infty$ and $x \in \Omega$. Then there exists a positive constant C such that*

$$|u(x)| \leq \frac{C}{d(x, \partial\Omega)^{n/p} \omega_n^{1/p}} \|u\|_{b^p(\Omega)}$$

for every $u \in b^p(\Omega)$, where d denotes Euclidean distance. Moreover when $0 < p < 1$, the constant $C > 1$, when $p \geq 1$, the constant $C = 1$.

Proof. For $0 < p < 1$, recall subharmonic behavior of $|u|^p$ see Theorem 22, there exists a constant $C_{p,n} > 1$ such that

$$|u(x)|^p \leq \frac{C_{n,p}}{r^n \omega_n} \int\limits_{B(x,r)} |u(y)|^p dy$$

for every harmonic function u in $B(x,r) \subset \mathbb{R}^n$.
For each $x \in \Omega$, apply this theorem with $r = d(x, \partial\Omega)$, we get

$$|u(x)|^p \leq \frac{C_{n,p}}{d(x,\partial\Omega)^n \omega_n} \int\limits_{B(x,r)} |u(y)|^p dy.$$

Taking p^{th} roots to both sides we obtain the required inequality.

For $p \geq 1$, suppose that $x \in \Omega$ and let r be a positive number with $r < d(x, \partial\Omega)$, then by the volume version of the mean-value property (1.3) to u on the ball $B(x,r)$ we have

$$u(x) = \frac{1}{r^n \omega_n} \int\limits_{B(x,r)} u(y) dV(y).$$

Taking the absolute values to both sides, and then applying Jensen's inequality we get

$$|u(x)|^p \leq \frac{1}{r^n \omega_n} \int\limits_{B(x,r)} |u|^p dV \leq \frac{1}{r^n \omega_n} \|u\|^p_{b^p(\Omega)}.$$

Taking p^{th} roots, and letting the limit as $r \to d(x, \partial\Omega)$, we get

$$|u(x)| \leq \frac{1}{d(x,\partial\Omega)^{n/p} \omega_n^{1/p}} \|u\|_{b^p(\Omega)}.$$

□

Corollary 15. *For every multi-index α there exists a constant C_α such that*

$$|D^\alpha u(x)| \leq \frac{C_\alpha}{d(x,\partial\Omega)^{|\alpha|+n/p}} \|u\|_{b^p}$$

for all $x \in \Omega$ and every $u \in b^{p(\Omega)}$.

Corollary 16. *For each fixed $x \in \Omega$ the functional*

$$T_x : u \to u(x), \quad u \in b^2(\Omega)$$

is a continuous linear functional on $b^2(\Omega)$.

Proposition 9. *The Bergman space $b^p(\Omega)$ is a closed subspace of $L^p(\Omega)$*

Proof. Suppose u_m is a sequence in $b^p(\Omega)$ which converges to a function u in $L^p(\Omega)$. Let K be compact subset of Ω. By Proposition 8, there exists constant $C < \infty$ such that

$$|u_m(x) - u_j(x)| \leq C \|u_m - u_j\|_{b^p}$$

for all $x \in K$ and all $m.j$. Since u_m is a Cauchy sequence in $b^p(\Omega)$, thus the inequality above implies that u_m is a Cauchy sequence in $C(K)$ and therefore the sequence u_m converges uniformly on K. Hence by Theorem 14 the sequence u_m converges uniformly on compact subsets of Ω to a function v harmonic on Ω. Because $u_m \to u$ in $L^p(\Omega)$, then there exists subsequence of u_m converges to u pointwise almost everywhere on Ω. We conclude that $u = v$ $a.e$ on Ω and hence $u \in b^p(\Omega)$ which completes the proof. □

Corollary 17. *The Bergman space $b^p(\Omega)$ is a Banach space for $p \geq 1$.*

Note: it can be proved that $b^p(B)$ is a quasi-Banach space for $0 < p < 1$.

2.4.1 Harmonic Bergman kernel

Certainly the Bergman kernel construction was one of the great ideas of modern complex function theory. It not only gives rise to a useful and important canonical reproducing kernel, but also to the Bergman metric. The Bergman kernel deals with integration over the solid region.

The Bergman kernel has, in the past fifty years, become an important tool in the complex analysis of both one and several complex variables (see: [29] [18] [32]). Its reproducing properties, its bi-holomorphic invariance, and its relationship to the Bergman metric are all of fundamental importance. This it is important to obtain concrete information about the Bergman kernel. An explicit formula for the harmonic Bergman reproducing kernel has only been determined recently; see [4]. The name for Bergman kernel function of Ω (in the next theorem 33) comes from Stephan Bergman (1895-1987) who introduced its study in 1922; [6]]

Now we give a simple derivation for such a formula.

2.4.1.1 Reproducing Bergman kernel of Ω

When $p = 2$, the Proposition 9 shows that the Bergman space $b^2(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle = \int_{\Omega} u\bar{v} dV$$

for $u, v \in L^2(\Omega)$. Let Ω be a bounded domain, and let x be any fixed point in Ω . Define the linear functional T by

$$T(u) = u(x), \quad u \in b^2(\Omega).$$

Then the Proposition 8 implies that T is bounded on the Hilbert space $b^2(\Omega)$. Therefore, by the Fréchet-Riesz theorem 27 there is a unique function
$R_\Omega(x, .) \in b^2(\Omega)$ such that

$$u(x) = \langle u, R_\Omega(x, .)\rangle := \int_{\Omega} u(y)\overline{R}_\Omega(x, y)dV(y) \tag{2.7}$$

for every $u \in b^2(\Omega)$. The function R_Ω as a function on $\Omega \times \Omega$ is called the (harmonic) Bergman kernel function of Ω (or the reproducing kernel of Ω). Now we have the following theorem :

Theorem 33. ***(Reproducing kernel).*** *Let Ω be a bounded domain of finite connectivity, and let x be any fixed point in Ω. Then, the Hilbert space $b^2(\Omega)$ has a unique reproducing kernel $R_\Omega(x, .)$ such that*

$$\langle u, R_\Omega(x, .)\rangle = u(x), \quad \forall u \in b^2(\Omega)$$

Proposition 10. *([4].Proposition 8.4) The reproducing kernel of R_Ω has the following properties:*
(i) R_Ω is real valued.
(ii) If (u_m) is an orthonormal basis of $b^2(\Omega)$, then

$$R_\Omega(x,y) = \sum_{m=1}^{\infty} \overline{u}_m(x) u_m(y)$$

for all $x, y \in \Omega$
(iii) R_Ω is a symmetric function on Ω
(iv) $\|R_\Omega(x,.)\|_{b^2} = \sqrt{R_\Omega(x,x)}$ for all $x \in \Omega$

Proof. To prove (i), suppose that $u \in b^2(\Omega)$ is real valued and $x \in \Omega$. Then

$$Imu(x) = Im \int_\Omega u(y)\overline{R}_\Omega(x,y)dV(y) = -\int_\Omega u(y) Im R_\Omega(x,y)dV(y).$$

Choose $u = ImR_\Omega(x,.)$, we get

$$\int_\Omega (ImR_\Omega(x,y))^2 dV(y) = 0,$$

which implies $ImR_\Omega = 0$.
To prove (ii), suppose u_m is any orthonormal basis of $b^2(\Omega)$, By standard Hilbert space theory 28,

$$R_\Omega(x,y) = \sum_{m=1}^{\infty} \langle R_\Omega(x,.), u_m\rangle u_m = \sum_{m=1}^{\infty} \overline{u}_m(x) u_m(x)$$

for each $x \in \Omega$, where the infinite sums converge in norm in $b^2(\Omega)$. Since point evaluation is a continuous linear functional on $b^2(\Omega)$, the equation above shows that the conclusion of (ii) holds.
To prove (iii), using (ii), we obtain $\overline{R}_\Omega(x,y) = R_\Omega(y,x)$, and using (i), we obtain $\overline{R}_\Omega(x,y) = R_\Omega(x,y)$ for all $x, y \in \Omega$.
To prove (iv), let $x \in \Omega$. Then

$$\begin{aligned}\|R_\Omega(x,.)\|^2_{b^2(\Omega)} &= \langle R_\Omega(x,.), R_\Omega(x,.)\rangle = \\ &= R_\Omega(x,x).\end{aligned}$$

□

2.4.1.2 Bergman kernel for the ball

In this subsection we will derive an explicit formula for the harmonic Bergman reproducing kernel R on the ball B in Euclidean space. We will not make use of so-called zonal and spherical harmonics used in [4] to calculate the Bergman kernel R, but instead use Green's identity to relate R to the extended Poisson kernel P [3].

Let u, v be harmonic functions on $\overline{B}$, and fix $y \in B = B(0,1)$. Assume that $w(x) = (|x|^2 - 1)v(x)$, then the function

$$\Delta w(x) = \Delta(|x|^2)v(x) + 2\nabla|x|^2.\nabla v(x) + |x|^2\Delta v(x) = 2nv(x) + 4x.\nabla v(x) \tag{2.8}$$

is harmonic on B.

$$\nabla w(x) = 2xv(x) + (|x|^2 - 1)\nabla v(x).$$

So that

$$D_\nu w(x) = \nabla w(x).x/|x| = 2v(x)|x| + (|x|^2 - 1)D_\nu v(x).$$

Clearly $D_\nu w(x) = 2v(x)$ on $S \equiv \partial B(0,1)$, $w = 0$ on S, and $\Delta u = 0$ on B. Thus from Greens identity (with u, w), we obtain

$$\int\limits_B u\Delta w dV = \int\limits_S uD_\nu w ds = 2\int\limits_S uv ds = 2n\omega_n \int\limits_S uv d\sigma$$

For $x \in B$,, choose $v(x) = P(x,y)$, where $P(x,y)$ is extended Poisson kernel (1.10). Hence

$$\int\limits_B u\Delta w dV = 2n\omega_n \int\limits_S u(\xi)P(\xi,y)d\sigma(\xi) = 2n\omega_n u(y).$$

We conclude that the harmonic function $\Delta w/(2nV(B))$ is the reproducing kernel at y. using (2.8) we obtain the following formula for the Bergman kernel

$$R_B(x,y) = \frac{1}{n\omega_n}\big(nP(x,y) + 2x.\nabla_x P(x,y))\big)$$

Now by elementary calculation, we can formulate the following theorem

Theorem 34. *[3]. Let $x, y \in B$. Then*

$$R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x.y - 2n - 4)|x|^2|y|^2 + n}{n\omega_n(1 - 2x.y + |x|^2|y|^2)^{1+n/2}}$$

where $x.y$ denotes the Euclidean inner product in $\mathbb{R}^n$

Corollary 18. *Let $x, y \in B$. Then*

$$|R_B(x,y)| \leq \frac{4}{\omega_n(1 - 2x.y + |x|^2|y|^2)^{n/2}}$$

Proof. By elementary calculation, we can write R_B in the form

$$R_B(x,y) = \frac{1}{n\omega_n(1 - 2x.y + |x|^2|y|^2)^{n/2}}\left(\frac{n(1 - |x|^2|y|^2)^2}{1 - 2x.y + |x|^2|y|^2} - 4|x|^2|y|^2\right).$$

By Cauchy-Schwarz ($x.y \leq |x||y|$), we get

$$(1 - |x||y|)^2 = 1 - 2|x||y| + |x|^2|y|^2 \leq 1 - 2x.y + |x|^2|y|^2,$$

thus

$$(1-|x|^2|y|^2)^2 = (1+|x||y|)^2(1-|x||y|)^2 \le 4(1-2x.y+|x|^2|y|^2).$$

And hence, we get the result. □

2.4.2 Harmonic Bergman projection

As in the analytic case, there is a reproducing kernel and associated projection. Duality results follow once we know that the projection is L^p-bounded. Coifman and Rochberg [10] used deep results from harmonic analysis to establish L^p-boundedness of the harmonic Bergman projection.

Let Q denote the orthogonal projection of $L^2(B)$ onto $b^2(B)$. If $f \in L^2(B)$ and $x \in B$, then $Q[f](x) = \langle Qf, R_x\rangle = \langle f, R_x\rangle$ and we have the following formula:

$$Q[f](x) = \int_B R(x,y)f(y)dV(y). \tag{2.9}$$

For fixed $x \in B$ the function $R(x,.)$ is bounded, so that we can use the formula above for $Q[f]$ to extend the domain of Q to $L^p(B)$, where $1 < p < \infty$.

The following theorem 35 shows that the harmonic Bergman projection is L^p-bounded for $1 < p < \infty$. The proof of this theorem 35 is similar to Forelli and Rudin's proof of L^p-boundedness of the analytic Bergman projection [21], [40], but as in Axler's argument [1] in the context of the analytic Bergman spaces on the unit disk, we will avoid the use of the binomial theorem, the gamma function and Stirling's formula.

Before prove the following theorem we introduce the following lemma:

Lemma 11. *Let q be the conjugate index of $p \in (1,\infty)$. Then there exists a positive function h and a constant C such that*

$$\int_B h^q(x)|R(x,y)|dV(x) \le Ch^q(y) \tag{2.10}$$

for all $y \in B$, and

$$\int_B h^p(y)|R(x,y)|dV(y) \le Ch^p(x) \tag{2.11}$$

for all $x \in B$.

Proof. We claim that the function $h(x) = (1-|x|^2)^{-1/(pq)}$ works, that is, satisfies (2.10) and (2.11) . By symmetry in p and q, it will suffice to find a constant C_p for which

$$\int_B (1-|x|^2)^{-1/p}|R(x,y)|dV(y) \le C_p(1-|y|^2)^{-1/p} \tag{2.12}$$

for all $y \in B$.
Fix $y \in B\backslash\{0\}$. For $0 < r < 1$ and $\xi \in S$ it follows from Corollary 18 that

$$|R(r\xi, y)| \leq \frac{4}{\omega_n(1 - ry.\xi + r^2|y|^2)^{n/2}} = \frac{4}{\omega_n|\xi - ry|^n}.$$

By (1.2) and Proposition 1 (ii), we have

$$\begin{aligned}
\int_B \frac{|R(x,y)|}{(1-|x|^2)^{1/p}} dV(x) &= n\omega_n \int_0^1 r^{n-1}(1-r^2)^{-1/p} \int_S |R(r\xi, y)| d\sigma(\xi) dr \\
&\leq 4n \int_0^1 r^{n-1}(1-r^2)^{-1/p} \int_S \frac{1}{|\xi - ry|^n} d\sigma(\xi) dr \\
&\leq 2n \int_0^1 2r(1-r^2)^{-1/p} \frac{1}{1 - r^2|y|^2} dr \\
&= 2n \int_0^1 (1-t)^{-1/p}(1-t|y|^2)^{-1} dt.
\end{aligned}$$

Now

$$\int_0^{|y|^2} (1-t)^{-1/p}(1-t|y|^2)^{-1} dt \leq \int_0^{|y|^2} (1-t)^{-1-1/p} dt \leq p(1-|y|^2)^{-1/p}$$

and hence

$$\begin{aligned}
\int_{|y|^2}^1 (1-t)^{-1/p}(1-t|y|^2)^{-1} dt &\leq (1-|y|^2)^{-1} \int_{|y|^2}^1 (1-t)^{-1/p} dt \\
&= (1-|y|^2)^{-1} q(1-|y|^2)^{1-1/p} \\
&= q(1-|y|^2)^{-1/p}
\end{aligned}$$

Addition yields

$$\int_0^1 (1-t)^{-1/p}(1-t|y|^2)^{-1} dt \leq (p+q)(1-|y|^2)^{-1/p}.$$

Hence (2.12) is proved with $C_p = 2n(p+q)$. The proof is completed. □

The Fubini's theorem to reverse the order of integration says: if $f \geq 0$ then

$$\int_\Omega \left(\int_{\Omega'} f(x,y) d\mu'(y) \right) d\mu(x) = \int_{\Omega'} \left(\int_\Omega f(x,y) d\mu(x) \right) d\mu'(y). \tag{2.13}$$

Theorem 35. *[3]. If $1 < p < \infty$. Then Q maps $L^p(B)$ boundedly onto $b^p(B)$.*

Proof. Let f be given function in space $L^p(B, dV)$, applying Holder's inequality (2.5) and Lemma above 11 we have

$$|Q[f](y)| \leq \int\limits_B \frac{|f(x)|}{h(x)} h(x)|R(x,y)|dV(x)$$
$$\leq \Big(\int\limits_B \frac{|f(x)|^p}{h^p(x)}|R(x,y)|dV(x)\Big)^{1/p}\Big(\int\limits_B h^q(x)|R(x,y)|dV(x)\Big)^{1/q}$$
$$\leq C^{1/q}h(y)\Big(\int\limits_B \frac{|f(x)|^p}{h^p(x)}|R(x,y)|dV(x)\Big)^{1/p}$$

Thus, applying Fubini's theorem (2.13), and using Lemma above 11 we obtain

$$\int\limits_B |Q[f](y)|dV(y) \leq C^{p/q}\int\limits_B h^p(y)\Big(\int\limits_B \frac{|f(x)|^p}{h^p(x)}|R(x,y)|dV(x)\Big)dV(y)$$
$$= C^{p/q}\int\limits_B \frac{|f(x)|^p}{h^p(x)}\Big(\int\limits_B h^p(y)|R(x,y)|dV(y)\Big)dV(x)$$
$$\leq C^{p/q}\int\limits_B \frac{|f(x)|^p}{h^p(x)}\big(Ch^p(x)\big)dV(x)$$
$$= C^p\int\limits_B |f(x)|^p dV(x).$$

From the above estimates and the observation that $C_p = C_q$ we conclude the proof of the L^p-boundedness of Q. In fact, we obtain the following bound on the norm of Q: $\|Q\| \leq 2np^2/(p-1)$ as an operator from $L^p(B)$ onto $b^p(B)$. □

Remark 10. *Theorem above does not hold for $p = 1$ or $p = \infty$.*

Corollary 19. *Let q denotes the conjugate index of $p \in (1, \infty)$. Then the spaces $b^p(B)$ and $b^q(B)$ are dual to each other.*

Proof. It follows from of the L^p-boundedness of the Bergman projection: if $u \in b^p(B), v \in b^q(B)$, then the function T defined by $T(u) = \langle u, v\rangle$ defines a bounded linear functional on $b^p(B)$, and every bounded linear functional on $b^p(B)$ is of the form above. In fact if we assume that $T \in b^p(B)$, then by the Hahn-Banach theorem 32 , T extends to a bounded linear functional T' on $L^p(B)$. There exists a $g_0 \in L^q(B)$ such that $T'(f) = \langle f, g_0\rangle$ for all $f \in L^p(B)$. In particular, if $u \in b^p(B)$, then $T(u) = \langle u, g_0\rangle$. Note that $v = Q[g_0] \in Q(L^q(B)) = b^q(B)$. Using Fubini's theorem (2.13) it can then be shown that $\langle u, v\rangle = \langle u, g_0\rangle$, and we have $T(u) = \langle u, v\rangle$, for all $u \in b^p(B)$. □

2.5 Weighted Bergman spaces $b^p_\beta(B), \beta > -1$

There has been a great deal of work done in recent years on weighted Bergman spaces on the unit ball B^n. Bergman spaces with standard weights on the unit ball have been studied by numerous authors in recent years.

Assume that $-1 < \beta < \infty$ is a given real number and suppose $n \geq 2$ is a fixed integer number. A weighted Lebesgue spaces on B denotes by $L^p_\beta(B)$ is the space of all measurable functions u in B with norm weight

$$\|u\|_{p,\beta} = \left(\int_B |u(x)|^p (1-|x|^2)^\beta dV(x) \right)^{1/p} < +\infty, \quad 0 < p < \infty \tag{2.14}$$

where $dV(x)$ is the Lebesgue measure. A weighted harmonic Bergman space $b^p_\beta(B)$ is the space of all harmonic functions $h(B)$ in $L^p_\beta(B)$.

In the following we always assume $\beta > -1$.
The following proposition shows that the point evaluation is continuous on $b^p_\beta(B)$.

Proposition 11. *([37] Proposition 2) For any function $u \in b^p_\beta(B)$, $1 \le p < \infty$ and any point $x \in B$, we have*

$$|u(x)| \le \frac{2^{n/p}}{(1-|x|)^{(n-1)/p}} \left(n\omega_n \int\limits_{(1+|x|)/2}^{1} r^{n-1}(1-r^2)^\beta dr \right)^{-1/p} \|u\|_{p,\beta}. \tag{2.15}$$

Proof. For $x \in B$, $\xi \in S$, we obtain

$$P(x,\xi) := \frac{1-|x|^2}{|\xi - x|^n} \le \frac{1+|x|}{(1-|x|)^{n-1}} \le \frac{2}{(1-|x|)^{n-1}}. \tag{2.16}$$

Let $x \in B$ and $|x| \le r' \le 1$. Using the subharmonicity of the function $|u(r'x)|^p$ in the neighborhood of the ball $\overline{B}$ and (2.16), we get

$$\begin{aligned} |u(r'x)|^p &\le \int\limits_S |u(r'\xi)|^p P(x,\xi) d\sigma(\xi) \\ &\le \frac{2}{(1-|x|)^{n-1}} \int\limits_S |u(r'\xi)|^p d\sigma(\xi). \end{aligned} \tag{2.17}$$

Let $x = r\xi$, where $r = |x|$, $\xi \in S$. The integral means $M(r') = \int_S |u(r'\xi)|^p d\sigma(\xi)$ is nondecreasing in r', so by using the expression of the volume element in polar coordinates $dV(x) = n\omega_n r^{n-1} dr d\sigma(\xi)$ (see (1.2)) , we get

$$n\omega_n \int\limits_{r'}^{1} r^{n-1}(1-r^2)^\beta dr \int\limits_S |u(r'\xi)|^p d\sigma(\xi) \le$$

$$\begin{aligned} &\le n\omega_n \int\limits_{r'}^{1} \int\limits_S |u(r\xi)|^p r^{n-1}(1-r^2)^\beta dr d\sigma(\xi) \\ &= \int\limits_{r'<|x|<1} |u(x)|^p (1-|x|^2)^\beta dV(x) \\ &\le \|u\|^p_{p,\beta}. \end{aligned} \tag{2.18}$$

By (2.17) and (2.18), we have

$$|u(r'x)|^p \leq \frac{2}{(1-|x|)^{n-1}}\left(n\omega_n\int_{r'}^{1} r^{n-1}(1-r^2)^\beta dr\right)^{-1}\|u\|_{p,\beta}^p.$$

Changing $r'x$ by x, we obtain

$$|u(x)| \leq \frac{2^{1/p}}{(r'-|x|)^{(n-1)/p}}\left(n\omega_n\int_{r'}^{1} r^{n-1}(1-r^2)^\beta dr\right)^{-1/p}\|u\|_{p,\beta}.$$

Choose $r' = (1+|x|)/2$ we obtain the desired estimate. □

Corollary 20. *For any multi-index α there is a constant $C = C(\alpha)$ such that*

$$|D^\alpha u(x)| \leq \frac{C}{(1-|x|)^{|\alpha|+(n-1)/p}}\left(\int_{(3+|x|)/4}^{1} t^{n-1}n\omega_n(1-t^2)^\beta dt\right)^{-1/p}\|u\|_{p,\beta}. \tag{2.19}$$

Proof. Apply, the Cauchy inequalities 1.14 for the ball $B(x) = \{y : |y-x| < (1-|x|)/2\}$, we get

$$|D^\alpha u(x)| \leq \frac{C_\alpha}{(1-|x|)^{|\alpha|}}M(x), \tag{2.20}$$

where $M(x) = \max\limits_{y\in B(x)} u(y)$. By Proposition 11, for any $y \in B(x)$

$$|u(y)| \leq \frac{2^{n/p}}{(1-|y|)^{(n-1)/p}}\left(\int_{(1+|y|)/2}^{1} t^{n-1}n\omega_n(1-t^2)^\beta dt\right)^{-1/p}\|u\|_{p,\beta}. \tag{2.21}$$

Besides, the inequalities $1-|y| \geq (1-|x|)/2$ and $(1+|y|)/2 \leq (3+|x|)/4$ obviously follow from $y \in B(x)$, and taking the maximum over all $y \in B(x)$ we get

$$M(x) \leq \frac{2^{(2n-1)/p}}{(1-|x|)^{(n-1)/p}}\left(\int_{(3+|x|)/4}^{1} t^{n-1}n\omega_n(1-t^2)^\beta dt\right)^{-1/p}\|u\|_{p,\beta}. \tag{2.22}$$

Hence from (2.20) and the last inequalities we obtain the desired result. □

Proposition 12. *The weighted Bergman space $b^p_\beta(B)$ is closed subset of $L^p_\beta(B)$ for all $1 \leq p < \infty$.*

Proof. Suppose u_m is a sequence of functions in $b^p_\beta(B)$ such that $\|u_m - u\|_{p,\beta} \to 0$ as $m \to \infty$, where $u \in L^p_\beta(B)$. Let K be a compact subset of B. Then by Proposition 11 there is a constant $C = C(K, p, \beta)$ such that

$$\max_{x\in K}|u(x)| \leq C\|u\|_{p,\beta}$$

for any $u \in b^p_\beta(B)$. Hence $|u_m(x) - u_k(x)| \leq C\|u_m - u_k\|_{p,\beta}$ for any $x \in K$ and m, k. The sequence u_m is fundamental in $b^p_\beta(B)$, and hence u_m converges uniformly on compact subsets of B to a function harmonic on B namely v. Moreover $u_m \to u$ in

$L^p_\beta(B)$. Therefore, by Riesz theorem 26 there exists a subsequence of u_m converging to u pointwise almost everywhere in B. Thus, $u = v$ almost everywhere in B, and $u \in b^p_\beta(B)$. □

Corollary 21. *The weighted Bergman space $b^p_\beta(B)$ is a Banach space for all $1 \le p < \infty$.*

Note: it can be proved that $b^p_\beta(B)$ is a quasi-Banach space for $0 < p < 1$. Also, all of the above results can be extended to $b^p_\beta(\Omega)$ spaces, where $\Omega \subset \mathbb{R}^n$.

2.5.1 Reproducing kernel R_β for the ball

When $p = 2$, the Proposition 12 shows that the weighted Bergman space $b^2_\beta(B)$ is a Hilbert space with the inner product in the space $L^2_\beta(B)$ given by the expression

$$\langle u, v\rangle = \int_B u\overline{v}.(1 - |x|^2)^\beta dV(x).$$

For any fixed point $x \in B$, the Proposition 11 implies that the point evaluation functional $u \to u(x)$ is a bounded linear functional on the Hilbert space $b^2_\beta(B)$, so (it follows from the Riesz Theorem 26) that there exists a unique function $R_\beta(x, .) \in b^2_\beta(B)$ such that $u(x) = \langle u, R_\beta(x, .)\rangle$. As in the Proposition (10)(i) the function R_β is real valued, and hence

$$u(x) = \int_B u(y)R_\beta(x, y)(1 - |y|^2)^\beta dV(y) \tag{2.23}$$

for every $u \in b^2_\beta(B)$. The function $R_\beta(x, y)$ as a function on $B \times B$ is called the reproducing kernel of $b^2_\beta(B)$.

Recall, $\mathcal{H}_m(\mathbb{R}^n)$ is the space of all complex-valued homogeneous harmonic polynomials of degree m in $\mathbb{R}^n$. By homogeneity, a $u \in \mathcal{H}_m$ is determined by its restriction to S. The restrictions of functions from $\mathcal{H}_m(\mathbb{R}^n)$ on the sphere S are called spherical harmonics of degree m, and we denoted by $\mathcal{H}_m(S)$.

The spaces $\mathcal{H}_m$ are finite-dimensional. Hence each $\mathcal{H}_m$ is a closed subspace of L^2 and thus a Hilbert space with respect to the inner product of L^2. For $m \ne k$, $\mathcal{H}_m$ is orthogonal to $\mathcal{H}_k$ in b^2_β with respect to the inner product of L^2_β. Then the evaluation functional at each point $\eta \in S$ is a bounded linear functional on $\mathcal{H}_m$. Thus $\mathcal{H}_m$ is a reproducing kernel Hilbert space . Now by the Riesz representation theorem of Hilbert space theory 27 there exists a unique function (namely) $Z_m(\xi, .) \in \mathcal{H}_m(S)$ such that

$$p_m(\xi) = \int_S p_m(\eta)Z_m(\xi, \eta)d\eta, \quad p_m \in \mathcal{H}_m. \tag{2.24}$$

In other words, Z_m is the reproducing kernel of $\mathcal{H}_m$, it is called the zonal harmonic of degree m. It is real-valued (when $m = 0$, $Z_0 \equiv 1$) and symmetric (in its two variables) on $\overline{B}$, consequently, Z_m is harmonic in either of its variables since it lies in $\mathcal{H}_m$. So $Z_m(\xi, .) \in \mathcal{H}_m(S)$. By homogeneity zonal harmonics can be extended to functions on $\overline{B} \times \overline{B}$ as $Z_m(x, y) = r^m\rho^m Z_m(\xi, \eta)$, where $x = r\xi, y = \rho\eta \in B$, and $\xi, \eta \in S$.

We are going to use the dilatation. For a number $\lambda > 0$ and u a function on Ω, we defined a λ-dilatation of u, denoted u_λ, is the function $u_\lambda(x) = u(\lambda x)$ defined for $x \in \frac{1}{\lambda}\Omega := \{\frac{1}{\lambda}y : y \in \Omega\}$. For any function u in $b^p_\beta(B)$, we have

$$\|u_\lambda - u\|_{p,\beta} \to 0 \quad \text{as} \quad \lambda \to 1 - 0. \tag{2.25}$$

This means the continuity of λ-dilatation in $b_\beta(B)$. By using this fact (2.25) and Corollary 18, we can state the following lemma

Lemma 12. *Harmonic polynomials are dense in $b^p_\beta(B)$.*

Theorem 36. *([37] Theorem 1.) If $x, y \in B$. Then*

$$R_\beta(x,y) = \frac{2}{n\omega_n}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{n}{2}+m+\beta+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\beta+1)}Z_m(x,y), \tag{2.26}$$

where the zonal harmonics Z_m are harmonically extended on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover the series on the right-hand side of (2.26) converges absolutely and uniformly on the set $\{(x,y) \in \mathbb{R}^{2n} : |x||y| \leq \varepsilon, 0 < \varepsilon < 1\}$. In particular the series in (2.26) converges absolutely and uniformly on $K \times \overline{B}$, where K is any compact subset of B.

Proof. Let $x = r\xi$, $y = \rho\eta$, where $\xi, \eta \in S$. We consider that the zonal harmonics $Z_k(x,y)$ is homogeneous by both variables, we obtain

$$|Z_k(x,y)| = r^k\rho^k|Z_k(\xi,\eta)| \leq r^k\rho^k dim_k, \tag{2.27}$$

where dim_k is the dimension of $\mathcal{H}_k(S)$. The desired convergence follows from (2.27) in view of the estimate $dim_k \leq Ck^{n-2}$ and Stirling's formula (?). Assume $F(x,y)$ denotes the right-hand side of (2.26), then $F(x,.)$ is a bounded harmonic function on B for each $x \in B$. In particular, $F(x,y) \in b^2_\beta(B)$ for each $x \in B$. Fix $x \in B$. Since the zonal harmonics are reproducing kernels for the space $\mathcal{H}_m(\mathbb{R}^n)$. Thus for, we have

$$u(x) = \int_S u(\xi)Z_m(x,\xi)d\sigma(\xi), \quad u \in \mathcal{H}_m(\mathbb{R}^n) \tag{2.28}$$

for each $x \in \mathbb{R}^n$. We derive the analogue of (2.28) for integration over B with respect to the measure $(1-|x|^2)^\beta dV$. For every $u \in \mathcal{H}_m(\mathbb{R}^n)$ we have

$$\begin{aligned}
\int_B u(y)Z_m(x,y)(1-|x|^2)^\beta dV &= n\omega_n\int_0^1 r^{n-1}(1-r^2)^\beta\int_S u(r\xi)Z_m(x,r\xi)d\sigma(\xi)dr \\
&= n\omega_n\int_0^1 r^{n-1+2m}(1-r^2)^\beta\int_S u(\xi)Z_m(x,\xi)d\sigma(\xi)dr \\
&= \frac{n\omega_n}{2}u(x)\int_0^1 t^{\frac{n}{2}+m-1}(1-t)^\beta dt \\
&= \frac{n\omega_n}{2}\frac{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)}{\Gamma(\frac{n}{2}+m+\alpha+1)}u(x)
\end{aligned}$$

for every $x \in \mathbb{R}^n$. Taking into account the orthogonality in $b^2_\beta(B)$ of homogeneous harmonic polynomials of different degrees, we receive that $u(x) = \langle u, F(x,.)\rangle$ whenever u is harmonic polynomial. Because point evaluation is continuous in b^2_β due to Proposition 11 and the lemma 12, we have $u(x) = \langle u, F(x,.)\rangle$ for all $u \in b^2_\beta(B)$. Hence F is the reproducing kernel. □

From above Theorem 36, we conclude the series on the right-hand side of (2.26) converges absolutely on $B \times B$, and uniformly if one variable lives in a compact subset of B. Therefore the kernel R_β is symmetric in his variables and harmonic as a function of each.

2.5.2 Weighted Bergman Projection

The integral representation (2.23) is true for all $1 \le p < \infty$. i.e. for any fixed point $x \in B$, we have

$$u(x) = \int\limits_B u(y) R_\beta(x,y)(1-|y|^2)^\beta dV(y). \tag{2.29}$$

where $u \in b^p_\beta(B)$, $1 \le p < \infty$. In fact if $p_k \in \mathcal{H}_m(S)$, then by using (2.26) we get

$$\int\limits_B p_k(y) R_\beta(x,y) dV_\beta(y) =$$

$$= \sum_{m=0}^{\infty} \frac{2\Gamma(\frac{n}{2}+m+\beta+1)}{n\omega_n \Gamma(\frac{n}{2}+m)\Gamma(\beta+1)} \int\limits_B p_k(y) Z_m(x,y) dV_\beta(y), \tag{2.30}$$

where $dV_\beta(y) = (1-|y|^2)^\beta dV(y)$. Put $x = r\xi$, $y = \rho\eta$, where $\xi, \eta \in S$, then the expression of the volume element in polar coordinates is $dV(y) = \rho^{n-1} n\omega_n d\rho d\sigma(\eta)$. Thus by homogeneity of the functions $p_k(y)$ and $Z_m(x,y)$ we have

$$\int\limits_B p_k(y) Z_m(x,y) dV_\beta(y) = \int\limits_B \rho^k p_k(\eta) r^m \rho^m Z_m(\xi,\eta)(1-\rho^2)^\beta \rho^{n-1} n\omega_n d\rho d\sigma(\eta)$$

$$= r^m \int\limits_0^1 \rho^{k+m+n-1}(1-\rho^2)^\beta n\omega_n d\rho \int\limits_S p_k(\eta) Z_m(\xi,\eta) d\sigma(\eta).$$

The last integral vanishes for $m \ne k$ by orthogonality and is equal to $p_k(\xi)$ for $m = k$ in accordance to (2.24) . Hence

$$\int\limits_B p_k(y)Z_k(x,y)dV_\beta(y) = r^k p_k(\xi)\int\limits_0^1 \rho^{2k+n-1}(1-\rho^2)^\beta n\omega_n d\rho$$
$$= p_k(x)\frac{n\omega_n}{2}\int\limits_0^1 t^{n/2+k-1}(1-t)^\beta dt$$
$$= \frac{n\omega_n\Gamma(\frac{n}{2}+k)\Gamma(\beta+1)}{2\Gamma(\frac{n}{2}+k+\beta+1)}p_k(x). \tag{2.31}$$

By (2.30) and (2.31), we obtain

$$\int\limits_B p_k(y)R_\beta(x,y)dV_\beta(y) = p_k(x). \tag{2.32}$$

Suppose $u_\lambda(x) = u(\lambda x)$ with $0 < \lambda < 1$. By the uniform convergence of the expansion $u(\lambda x) = \sum p_k(\lambda x)$ in $\overline{B}$ and by (2.32), we have

$$u_\lambda(x) = \sum_{k=0}^{\infty}\lambda^k p_k(x) = \sum_{k=0}^{\infty}\lambda^k\int\limits_B p_k(y)R_\beta(x,y)dV_\beta(y)$$
$$= \int\limits_B\left(\sum_{k=0}^{\infty}\lambda^k p_k(y)\right)R_\beta(x,y)dV_\beta(y)$$
$$= \int\limits_B\left(\sum_{k=0}^{\infty}p_k(\lambda y)\right)R_\beta(x,y)dV_\beta(y)$$
$$= \int\limits_B u_\lambda(y)R_\beta(x,y)(1-|y|^2)^\beta dV(y)$$

By using the fact (2.25) for $u \in b^p_\beta(B)$, the passage $\lambda \to 0$ we get the formula (2.29).

The right-hand side integral of (2.29) defines the orthogonal projection of $L^2_\beta(B)$ onto its subspace $b^2_\beta(B)$ which implies that the following theorem

Theorem 37. *The operator*

$$Q_\beta[u](x) = \int\limits_B u(y)R_\beta(x,y)(1-|y|^2)^\beta dV(y), \quad u \in L^2_\beta(B), \quad x \in B \tag{2.33}$$

is the orthogonal projection of $L^2_\beta(B)$ *onto* $b^2_\beta(B)$.

Proof. As $L^2_\beta(B) = b^2_\beta(B) \oplus (b^2_\beta(B))^\perp$, any $u \in L^2_\beta(B)$ can be written in the form $u = v + w$, where $v \in b^2_\beta(B)$ and $w \in (b^2_\beta(B))^\perp$. Hence $Q_\beta[u] = Q_\beta[v] + Q_\beta[w]$, where $Q_\beta[v] = v$ by formula (2.29) and

$$Q_\beta[w](x) = \int\limits_B w(y)R_\beta(x,y)(1-|y|^2)^\beta dV(y) = \langle w, R_\beta(x,.)\rangle = 0 \tag{2.34}$$

since due to Theorem 36 for a fixed $x \in B$ the function $R_\beta(x,y)$ is harmonic in y on a domain containing $\overline{B}$, and w is orthogonal to $b^2_\beta(B)$, Thus $Q_\beta[u] = v$. Hence Q_β is the orthogonal projector $L^2_\beta(B) \to b^2_\beta(B)$. $\square$

Chapter 3

Harmonic Bergman spaces on the complement of a lattice

In this chapter we present results from [44] which extend previously obtained results from [5]. We begin with some preliminary results.

3.1 Classification of singularities of harmonic function

3.1.1 Laurent series expansion of harmonic function on annular region in $\mathbb{R}^n$

Suppose that $0 < r < R < \infty$, $K = \overline{B}_r \subset \mathbb{C}$, and $\Omega = B_R \subset \mathbb{C}$. Assume f is analytic on the annulus $\Omega \backslash K$, and let $\sum_{-\infty}^{\infty} a_i z^i$ be the Laurent expansion of f on $\Omega \backslash K$. Setting $g(z) = \sum_{0}^{\infty} a_i z^i$ and $h(z) = \sum_{-\infty}^{-1} a_i z^i$, we see that $f = g + h$ on $\Omega \backslash K$, that g, and h extend to be analytic on Ω, and $(\mathbb{C} \cup \{\infty\}) \backslash K$, respectively. The Laurent series expansion therefore gives us a decomposition for analytic functions. The decomposition theorem is the analogous result for harmonic functions.

Theorem 38. ***(Decomposition theorem of harmonic function).*** *Let $\Omega \subset \mathbb{R}^n$, and let K be a compact subset of Ω. If u is harmonic function on $\Omega \backslash K$, then u has a unique decomposition of the form*

$$u = v + w$$

where v is a harmonic function on Ω and w is a harmonic function on $\mathbb{R}^n \backslash K$ satisfying
$$\lim_{x \to \infty} w(x) = \begin{cases} C \log |x| & \text{when } n = 2 \\ 0 & \text{when } n > 2 \end{cases}$$
for some constant C.

By Theorem 38 we obtain an analogous Laurent series expansion development for harmonic functions on annular domain in $\mathbb{R}^n$.

Before we prove next theorem we introduce Kelvin transform and its properties. Let u be a given function defined on a set $\Omega \subset \mathbb{R}^n \backslash \{0\}$. For such set Ω, we defined $\Omega^* = \{x^* : x \in \Omega\}$, where $x^* = x|x|^{-2}, x \neq \infty$. Then the function $\mathcal{K}[u]$ on Ω^* define by $\mathcal{K}[u](x) = |x|^{n-2} u(x^*), n > 2$ is called the Kelvin transform of u. When $n = 2$ the Kelvin transform of u is defined by $\mathcal{K}[u](x) = u(x^*)$.
The Kelvin transform $\mathcal{K}$ is its own inverse, ie $\mathcal{K}[\mathcal{K}[u]] = u$ for every function u defined on $\Omega \subset \mathbb{R}^n \backslash \{0\}$, the Kelvin transform $\mathcal{K}$ is linear, the Kelvin transform preserves uniform

convergence on compact sets, the Kelvin transform of every harmonic function is harmonic, if p_m is homogeneous polynomial on $\mathbb{R}^n$ of degree m, then $\mathcal{K}[p_m] = |x|^{2-n-2m}p_m$.

Theorem 39. ***(Laurent series theorem of harmonic function in** $\mathbb{R}^2$ **[4]).*** *Assume that $\mathcal{A}$ is an annular region in $\mathbb{R}^2$. If u is harmonic function on $\mathcal{A}$, then there exist unique homogeneous harmonic polynomials p_m and q_m of degree m such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + q_0 \log|x| + \sum_{m=1}^{\infty} \frac{q_m(x)}{|x|^{2m}}$$

on $\mathcal{A}$. The convergence is absolute and uniform on compact subsets of $\mathcal{A}$.

Proof. Let $\mathcal{A} =: \{x \in \mathbb{R}^2 : r < |x| < R\}$, with $0 \le r < \infty$ and $0 < R \le \infty$. By the decomposition theorem u has unique written in the form $u = v + w$, where v is harmonic on RB and w is harmonic on $(\mathbb{R}^2 \cup \{\infty\}) \backslash r\overline{B}$. Since v is harmonic on the ball RB, then by Corollary 9 u has a unique homogeneous expansion of the form

$$v(x) = \sum_{m=0}^{\infty} p_m(x) \tag{3.1}$$

on RB, where p_m are homogeneous harmonic polynomials of degree m, and this series converges absolutely and uniformly on compact subsets of $\mathcal{A}$. Since w is harmonic, then The Kelvin transform of $\mathcal{K}[w]$ is harmonic on the ball $r^{-1}B$, and hence then by Corollary (9) there are homogeneous harmonic polynomials q_m of degree m such that

$$\mathcal{K}[w](x) = \sum_{m=0}^{\infty} q_m(x)$$

on $r^{-1}B$. Taking the Kelvin transform to both sides of this equation, and then applying properties of the Kelvin transform we get

$$w(x) = q_0 \log|x| + \sum_{m=1}^{\infty} \frac{q_m(x)}{|x|^{2m}} \tag{3.2}$$

on $\mathbb{R}^2 \backslash r\overline{B}$, and this series converges absolutely and uniformly on compact subsets of $\mathcal{A}$. Now by combining the series expansions (3.1) and (3.2) we obtain that the desired Laurent series unique expansion for u on $\mathcal{A}$, and this expansion converges absolutely and uniformly on compact subsets of $\mathcal{A}$. □

Corollary 22. *Assume that $\Omega \subset \mathbb{R}^2$, and $a \in \Omega$. If u is harmonic function on $\Omega \backslash \{a\}$, then there exist unique homogeneous harmonic polynomials p_m and q_m of degree m such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + q_0 \log|x-a| + \sum_{m=1}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m}} \tag{3.3}$$

for x in a deleted neighborhood of a.

Since the Theorem 38 takes a different form in case $n > 2$, thus the Laurent series expansion for harmonic function in that case takes another form which is given by the next theorem.

Theorem 40. ***(Laurent series theorem of harmonic function in*** $\mathbb{R}^n, n > 2$***).*** *Assume that $\mathcal{A}$ is an annular region in $\mathbb{R}^n, n > 2$. If u is harmonic function on $\mathcal{A}$, then there exist unique homogeneous harmonic polynomials p_m and q_m of degree m such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

on $\mathcal{A}$. The convergence is absolute and uniform on compact subsets of $\mathcal{A}$.

Corollary 23. *Assume that $\Omega \subset \mathbb{R}^n, n > 2$, and $a \in \Omega$. If u is harmonic function on $\Omega \backslash \{a\}$, then there exist unique homogeneous harmonic polynomials p_m and q_m of degree m such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}} \tag{3.4}$$

for x in a deleted neighborhood of a.

The above expression (3.4) is called Laurent series expansion of u at a, and the function

$$w(x) := \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}} \tag{3.5}$$

is called the principal part of Laurent series of u at a. Also (see (3.3)) the principal part of Laurent series of u at $a \in \mathbb{R}^2$ is

$$w(x) := q_0 \log|x-a| + \sum_{m=1}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m}} \tag{3.6}$$

3.1.2 Isolated singularities of harmonic functions in $\Omega \subset \mathbb{R}^n$

A set $\Omega \subset \mathbb{R}^n \cup \{\infty\}$ is open if it is an open subset of $\mathbb{R}^n$ or $\Omega = \{\infty\} \cup (\mathbb{R}^n \backslash K)$, where K is a compact subset of $\mathbb{R}^n$. We call a point $a \in \Omega$ an isolated singularity of any function u defined on $\Omega \backslash \{a\}$. When u is harmonic on $\Omega \backslash \{a\}$, the isolated singularity a is said to be removable if u has a harmonic extension to Ω.

We say that u has a removable singularity at a if each term in the principal part (3.5) is zero; u has a pole at a if the principal part is a finite sum of nonzero terms; u has an essential singularity at a if the principal part has infinitely many nonzero terms. If $a \in \mathbb{R}^n, n > 2$ is a pole of u, then we say that the pole a of order $k + n - 2$ if there is a largest positive integer k such that $q_k \neq 0$. We call a pole of order $n - 2$ a fundamental pole (because the principal part is then a multiple of the fundamental solution). when $n = 2$, we say that the pole $a \in \mathbb{R}^2$ has order k if $q_k \neq 0$,we say that u has a fundamental pole at a if the principal part is a nonzero multiple of $\log|x|$.

Theorem 41. *([4]Theorem 10.5.) If u is harmonic with an isolated singularity at $a \in \mathbb{R}^n, n > 2$, then u has*
(i)a removable singularity at a if and only if

$$\lim_{x \to a} |x-a|^{n-2} |u(x)| = 0;$$

(ii)a pole at a of order $k+n-2$ if and only if

$$0 < \lim_{x\to a} \sup |x-a|^{k+n-2}|u(x)| < \infty,$$

(iii)an essential singularity at a if and only if

$$\lim_{x\to a} \sup |x-a|^{N}|u(x)| = \infty,$$

for every positive integer N.

Proof. To prove (ii), first suppose that u has a pole at a of order $k+n-2$. since q_m is homogeneous for every m, then we have

$$\lim_{x\to a} \sup |x-a|^{k+n-2}|u(x)| = \sup_S |q_m|.$$

But $0 < \sup\limits_S |q_m| < \infty$ which completes the proof one direction of (ii). Conversely, suppose $0 < \lim\limits_{x\to a} \sup |x-a|^{k+n-2}|u(x)| < \infty$, then for small $r>0$ and $\xi \in S$, we get

$$|w(a+r\xi)| = \left| \sum_{m=0}^{\infty} r^{2-n-2m} q_m(r\xi) \right| \le r^{2-n-k} C,$$

where $C < \infty$ is a constant. Let j be an integer with $j > k$, then by the orthogonality of spherical harmonics of different degree we have

$$\begin{aligned} r^{4-2n-2j} \int\limits_S |q_j(\xi)|^2 d\sigma(\xi) &\le \sum_{m=0}^{\infty} r^{4-2n-2m} \int\limits_S |q_m(\xi)|^2 d\sigma(\xi) \\ &= \int\limits_S |w(a+r\xi)|^2 d\sigma(\xi) \le r^{4-2n-2k} C^2. \end{aligned}$$

Letting $r \to 0$, we get $\int_S |q_j|^2 d\sigma = 0$, which implies that $q_j = 0$. Thus u has a pole at a of order at most $k+n-2$. Since $\lim_{x\to a} \sup |x-a|^{k+n-2}|u(x)| > 0$, then the order of the pole is at least $k+n-2$.

To prove (iii), first suppose that $\lim_{x\to a} \sup |x-a|^N |u(x)| = \infty$ for every positive integer N. By (i), and (ii), u can have neither a removable singularity nor a pole at a, and thus u has an essential singularity at a. Conversely, suppose there is a positive integer N such that $\lim_{x\to a} \sup |x-a|^N |u(x)| < \infty$. By the argument used in proving (ii), this implies that $q_j = 0$ for all sufficiently large j. Thus u does not have an essential singularity at a. □

Theorem 42. *If u is harmonic with an isolated singularity at $a \in \mathbb{R}^2$, then u has (i)a removable singularity at a if and only if*

$$\lim_{x\to a} \frac{|u(x)|}{\log|x-a|} = 0;$$

(ii)a fundamental pole at a if and only if

$$0 < \lim_{x\to a} \left| \frac{u(x)}{\log|x-a|} \right| < \infty;$$

(iii)a pole at a of order k if and only if

$$0 < \limsup_{x \to a} |x-a|^k |u(x)| < \infty,$$

(iv)an essential singularity at a if and only if

$$\limsup_{x \to a} |x-a|^N |u(x)| = \infty,$$

for every positive integer N.

Remark 11. *The real-valued harmonic function may have an isolated(non-removable) singularity; for example, $u(x) = |x|^{2-n}, n > 2$ has an isolated singularity at 0. But a real-valued harmonic function cannot have isolated zeros. In fact if u is a real-valued harmonic function defined on a domain Ω, and $u(a) = 0$ where $a \in \Omega$, then the mean value property over $S(a,r)$ and continuity of u imply that u must vanish at least once on $S(a,r)$ whenever $B(a,r) \subseteq \Omega$. consequently, the zeros of real-valued harmonic function are never isolated. It is also not true that analytic functions of several complex variables have isolated zeros. For a number of years, these difficulties thwarted efforts to extend these ideas to higher dimensions.*

3.1.3 Harmonic conjugate of harmonic functions in finitely connected domains $\Omega \subset \mathbb{C}$

We say that a domain (open connected) $\Omega \subset \mathbb{R}^2$ is finitely connected if $\mathbb{R}^2 \backslash \Omega$ has finitely many bounded components. The domain Ω is simply connected if $\mathbb{R}^2 \backslash \Omega$ has no bounded components. Recall that a real-valued harmonic function on simply connected Ω always has a harmonic conjugate.

The following theorem shows that a real-valued harmonic function on a finitely connected domain has a harmonic conjugate provided that some logarithmic terms are subtracted.

Theorem 43. ***(Logarithmic Conjugation Theorem).*** *Suppose that a domain $\Omega \subset \mathbb{R}^2$ is a finitely connected domain. Let Ω_m, $m = 1, ..., k$ be the bounded components of $\mathbb{R}^2 \backslash \Omega$, and let $a_m \in \Omega_m$, for $m = 1, ..., k$. If u is real valued and harmonic on Ω, then there exist function f holomorphic on Ω and $b_m \in \mathbb{R}$, $m = 1, ..., k$ such that*

$$u(z) = Ref(z) + \sum_{m=1}^{k} b_m \log |z - a_m|$$

for all $z \in \Omega$.

Any function harmonic in an annulus $r_0 < |z| < r_1$ can be written in the form

$$u(z) = a + b \log |z| + \sum_{m=-\infty}^{\infty} (c_m z^m + c_{-m} \overline{z}^m). \tag{3.7}$$

Theorem 44. ***(Harmonic Classification Theorem).*** *If u is real valued and harmonic on the annulus $\mathcal{A} =: \{z \in \mathbb{R}^2 : r_0 < |z| < r_1\}$, then u has a series development of* *the form*

$$u(re^{i\theta}) = b\log r + \sum_{m=-\infty}^{\infty} (c_m r^m + \overline{c_{-m}} r^{-m}) e^{im\theta}.$$

The series converges absolutely for each $re^{i\theta} \in \mathcal{A}$ and uniformly on compact subsets of $\mathcal{A}$.

Proof. By the logarithmic conjugation Theorem 43 with $\Omega = \mathcal{A}$, $\Omega_1 = \{z \in \mathbb{R}^2 : |z| \leq r_0\}$, and $a_1 = 0$, there is function f holomorphic on $\mathcal{A}$ such that

$$u(z) = b\log|z| + Ref(z).$$

The function f has a Laurent series expansion on $\mathcal{A}$

$$f(z) = \sum_{m=-\infty}^{\infty} c_m z^m,$$

This series converges absolutely and uniformly on compact subsets of $\mathcal{A}$. Since

$$Ref(z) = \frac{f(z) + \overline{f(z)}}{2}.$$

Replacing f with its Laurent series, we obtain

$$u(z) = b\log|z| + \sum_{m=-\infty}^{\infty} c_m z^m + \sum_{m=-\infty}^{\infty} \overline{c_m z^m}.$$

Setting $z = re^{i\theta}$ we get desired. □

3.2 Bergman space $b^p(\Omega)$, $\Omega = \mathbb{R}^n \backslash \mathbb{Z}^n$

3.2.1 Introduction

In this section we investigate harmonic Bergman spaces $b^p(\Omega)$, where $\Omega = \mathbb{R}^n \backslash \mathbb{Z}^n$. In the planar case the analytic Bergman spaces on $\Omega = \mathbb{C}\backslash(\mathbb{Z} + i\mathbb{Z})$, namely $B^p(\Omega)$ were studied in [5].
The presence of the logarithmic factor in Laurent series expansion of harmonic function in $\mathbb{R}^2$ makes a difference between analytic and harmonic case, see for example Proposition 16 below.

We set, for $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_{1\leq j\leq n} |x_j|$. Also, $Q(x,l) = \{w : \|w - x\|_\infty < l/2\}$ denotes an open cube centered at $x \in \mathbb{R}^n$ of side length $l > 0$ and $\dot{Q}(x,l) = Q(x,l) \setminus \{x\}$.

We start from some known facts.

Proposition 13. *If $f : \mathbb{R}^n \to \mathbb{C}$ is a harmonic function, not identically equal to zero, then $f \notin b^p(\mathbb{R}^n)$, $p > 0$. Moreover:*

$$\left(\int_{B(x,R)} |f(y)|^p dy\right)^{1/p} \geq C_{p,n} R^{n/p} |f(x)|, \qquad x \in \mathbb{R}^n. \tag{3.8}$$

Proof. Indeed, (3.8) follows from subharmonic behavior of $|f|^p$ for $0 < p < \infty$, see [43]. Therefore

$$\left(\int_{\mathbb{R}^n} |f(y)|^p dy \right)^{1/p} \geq \lim_{R\to+\infty} C_{p,n} |f(x)| R^{n/p} = +\infty$$

whenever $f(x) \neq 0$ for some $x \in \mathbb{R}^n$. □

It is a standard fact that for $f \in b^p(V)$, $V \subset \mathbb{R}^n$, $0 < p < +\infty$ we have

$$|f(x)| \leq C_{p,n} \frac{\|f\|_p}{r^{n/p}}, \qquad \text{where} \quad r = d(x, V^c). \tag{3.9}$$

In fact, using (3.8), we get

$$|f(x)|^p \leq \frac{C_{n,p}}{r^n \omega_n} \int_{B(x,r)} |f|^p dm \leq C_{n,p} r^{-n} \|f\|_p^p,$$

and (3.9) easily follows. Note that the this allows one to conclude that convergence in $b^p(V)$ implies locally uniform convergence on V.

There is an alternative, but equivalent way to expand $u \in h(V \setminus \{a\})$, $V \subset \mathbb{C}$, namely to use analytic and conjugate analytic functions. We assume, for simplicity, that $a = 0$. Then we have

$$u(z) = a_0 + b_0 \log|z| + \sum_{n\neq 0} (c_n z^n + d_n \overline{z}^n), \qquad 0 < |z| < r. \tag{3.10}$$

Note that $a_0 = a_0(u)$, $b_0 = b_0(u)$, $c_n = c_n(u)$ and $d_n = d_n(u)$.

Proposition 14. *The functionals a_0, b_0, c_n and d_n, $n \neq 0$, are continuous on the Frechet space $h(V')$, $V' = V \setminus \{0\}$.*

Proof. Using

$$b_0(u) = \frac{1}{2\pi} \int_{C_\rho} \frac{\partial u}{\partial n} ds, \qquad 0 < \rho < \operatorname{dist}(0, \partial V), \tag{3.11}$$

where C_ρ is the circle centered at 0 of radius ρ, we conclude, using continuity of derivatives on the space $h(V')$ that b_0 is continuous on $h(V')$. Now we fix $0 < \rho_1 < \rho_2 < \operatorname{dist}(0, \partial V)$. For any $k \neq 0$ we have

$$\phi_k(u) = \frac{1}{2\pi\rho_1} \int_{C_{\rho_1}} u(z) z^{-k} ds = c_k(u) + \rho_1^{-2k} d_{-k}(u) \tag{3.12}$$

and

$$\psi_k(u) = \frac{1}{2\pi\rho_2} \int_{C_{\rho_2}} u(z) \overline{z^k} ds = \rho_2^{2k} c_k(u) + d_{-k}(u). \tag{3.13}$$

Both ϕ_k and ψ_k are continuous on $h(V')$, since (3.12) and (3.13) represent a system of linear equations with determinant $1 - (\rho_2/\rho_1)^k \neq 0$ it follows immediately that c_k and d_k are continuous. The case of a_0 is left to the reader. □

3.2.2 Inclusions between b^p spaces on $\mathbb{R}^n \backslash \mathbb{Z}^n$

We start with an auxiliary proposition.

Proposition 15. *Assume $f \in b^p(V')$, where $V' = V\backslash\{a\}$ for some $a \in V \subset \mathbb{R}^n$. Then*

$$|f(x)| = o(|x-a|^{-n/p}), \qquad x \to a. \tag{3.14}$$

In particular, a is either a removable singularity of f or a pole of order $k < n/p$. If $n \geq 3$ and $p \geq \frac{n}{n-2}$, then a is a removable singularity.

Proof. Applying (3.9) to $V = B(x, |x-a|)$ one gets (3.14) and that suffices in view of the above classification of isolated singularities. □

Combining the last proposition and Proposition 13 we obtain the following:

Corollary 24. *If $f \in b^p(\Omega)$, $p \geq \frac{n}{n-2}$ and $n \geq 3$, then f is identically zero.*

Our first result demonstrates a basic difference between harmonic and analytic Bergman spaces on Ω in the planar case $B^p(\Omega) = \{0\}$ for $p \geq 2$, see [5]. However we have Proposition 16 which shows difference between harmonic and analytic case.

We are going to use Lagrange's theorem. Let a, b be distinct points in $\mathbb{R}^n$, and we defined a (closed) segment as $S[a,b] = \{x(t) = (1-t)a + tb : 0 \leq t \leq 1\}$. The Lagrange mean value theorem stated as follows: let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be defined and continuous at any point of $S[a,b]$, and differentiable at any point of $S[a,b]$ with the (possible) exception of the endpoints a and b. Then there exists a $\xi \in S[a,b]$ different from a, b such that

$$u(b) - u(a) = \nabla u(\xi).(b-a). \tag{3.15}$$

Proposition 16. *If $n = 2$, then $b^p(\Omega) \neq \{0\}$ for $0 < p < \infty$.*

Proof. For $0 < p < 2$, the analytic Bergman space $B^p(\Omega) \neq \{0\}$, where $\Omega = \mathbb{C}\backslash(\mathbb{Z} + i\mathbb{Z})$ see [5]. In fact if $P(z), Q(z)$ are relatively prime polynomials, $z \in \mathbb{C}$, and $R(z) = P(z)/Q(z)$ is a (nontrivial) rational function, then by asymptotic relations $R(z) \sim |z|^{degP - degQ}$, $|z| \to +\infty$ and $R(z) \sim |z-a|^{-k}$, $z \to a \in (\mathbb{Z} + i\mathbb{Z})$, where k is the order of zero a of Q. Hence we conclude the zeroes of Q belong to the lattice $(\mathbb{Z} + i\mathbb{Z})$, $degQ - degP > 2/p$ and each zero of Q has order $k < 2/p$ which implies that $R(z) \in B^p(\Omega)$. Finally, using the fact $B^p(\Omega) \subset b^p(\Omega)$.

For $p = 2$, the function $u(z) = \log|z-1| - 2\log|z| + \log|z+1|$ is harmonic in Ω and, by Lagrange's theorem 3.15, $|u(z)| = O(|z|^{-2})$ as $z \to \infty$. Therefore $u \in b^2(\Omega)$.

For $2 < p < \infty$, similarly, $u(z) = \log|z+1| - \log|z|$ is harmonic in Ω and, by Lagrange's theorem 3.15, $|u(z)| = O(|z|^{-1})$. Therefore $u \in b^p(\Omega)$ for $2 < p < \infty$. □

Lemma 13. *Let $k \in \mathbb{N}$ and $n/(k+1) \leq q < p < n/k$. Then there is a constant $C = C_{p,q,n}$ such that*

$$\|u\|_{b^p(\dot{Q}(a,1))} \leq C\|u\|_{b^q(\dot{Q}(a,3/2))} \qquad \textit{for every} \quad u \in b^q(\dot{Q}(a,3/2)), \quad a \in \mathbb{Z}^n.$$

Proof. This lemma states that the restriction operator $R : b^q(\dot{Q}(a,3/2)) \to b^p(\dot{Q}(a,1))$ given by $Ru = u|_{\dot{Q}(a,1)}$ is continuous. Since both spaces $b^q(\dot{Q}(a,3/2))$ and $b^p(\dot{Q}(a,1))$ are complete it suffices, by the closed graph theorem, to prove that R maps $b^q(\dot{Q}(a,3/2))$ into $b^p(\dot{Q}(a,1))$. Let $u \in b^q(\dot{Q}(a,3/2))$. Since $q \geq n/(k+1)$ Proposition 15 implies that the order of pole of u at a is at most k. Therefore, $|u(z)|^p = O(|a-z|^{-kp})$ where $kp < n$. Hence $|u|^p$ is integrable in a neighborhood of a and that implies $u \in b^p(\dot{Q}(a,1))$. □

The main result of this section is the following result.

Theorem 45. *If $n/(k+1) \leq q < p < n/k$ $(k = 1, 2, \ldots)$, then $b^q(\Omega) \subset b^p(\Omega)$.*

Proof. Set $Q_\omega = Q(\omega, 1)$ for $\omega \in \mathbb{Z}^n$. Let $u \in b^q(\Omega)$. The poles of u have orders at most k hence $u(z) = O(|z-\omega|^{-k})$ as $z \to \omega$. Therefore $u|_{Q_\omega} \in L^p(Q_\omega)$. Using Lemma 13 we get

$$\begin{aligned}
\|u\|_p^p = \int_\Omega |u|^p dm &= \sum_{\omega\in\Gamma} \int_{\dot{Q}_\omega} |u|^p dm \leq C \sum_{\omega\in\Gamma} \left(\int_{\dot{Q}(\omega,3/2)} |u|^q dm \right)^{p/q} \\
&\leq C \left(\sum_{\omega\in\Gamma} \int_{\dot{Q}(\omega,3/2)} |u|^q dm \right)^{p/q} \\
&\leq 4^{p/q} C \left(\sum_{\omega\in\Gamma} \int_{\dot{Q}_\omega} |u|^q dm \right)^{p/q} = 4^{p/q} C \|u\|_q^p
\end{aligned}$$

because $p/q \geq 1$ and almost every point in $\mathbb{C}$ lies in precisely 4 squares $Q(\omega, 3/2)$. □

We note that the above proof can be used to prove Theorem 1 from [5], in fact it presents a simplification of the proof given in [5].

3.2.3 Asymptotics at infinity of functions in $b^p(\Omega)$

One might conjecture that on the set $\Omega_\epsilon = \{z \in \mathbb{C} : d(z, \mathbb{Z}^n) > \epsilon\}$ we can control the size of functions $f \in b^p(\Omega)$, for example that we can prove $f(z) = O(|z|^{-2/p})$, $|z| \to \infty$, $z \in \Omega_\epsilon$. However, this is never true in general. The following theorem was proved in the case $0 < p < 2$ for analytic Bergman spaces $B^p(\Omega)$ in [5], and the same method of proof works in the present situation. We present this proof for reader's convenience.

Theorem 46. *Implication $f \in b^p(\Omega) \Rightarrow f(z) = O(|z|^{-\alpha})$ as $|z| \to \infty$, $z \in \Omega_\epsilon$ does not hold for any $0 < p < \infty$, $\alpha > 0$, $0 < \epsilon < 1/\sqrt{2}$.*

Proof. Assume this implication holds for some $0 < p < \infty$, $\alpha > 0$ and $0 < \epsilon < 1/\sqrt{2}$. One easily proves that

$$h_{\epsilon,\alpha} = \{f \in h(\Omega_\epsilon) : \|f\|_{\epsilon,\alpha} = \sup_{z\in\Omega_\epsilon} |z|^\alpha |f(z)| < +\infty\}$$

is a Banach space. The restriction operator $R : b^p(\Omega) \to h_{\epsilon,\alpha}$ has closed graph because convergence in both (quasi)-norms $\|\cdot\|_p$ and $\|\cdot\|_{\epsilon,\alpha}$ implies pointwise convergence. Hence R is bounded, that is $\|f\|_{\epsilon,\alpha} \leq C\|f\|_p$ for all $f \in b^p(\Omega)$. Let us pick a non-trivial $f \in b^p(\Omega)$. Then

$$\begin{aligned}
|f(z_0)| &= |f_n(z_0 - n)| \leq |z_0 - n|^{-\alpha} \|f_n\|_{\epsilon,\alpha} \leq C|z_0 - n|^{-\alpha} \|f_n\|_p \\
&= C|z_0 - n|^{-\alpha} \|f\|_p
\end{aligned}$$

for all $n \in \mathbb{N}$, $z_0 \in \Omega_\epsilon$ (f_n denotes a function $f_n(z) = f(z+n)$). This gives, as $n \to \infty$, $f(z_0) = 0$, hence $f(z) = 0$ on Ω_ϵ and therefore on Ω as well. Contradiction. □

Remark 12. *The same proof works for a function $\phi(|z|)$ instead of $|z|^{-\alpha}$, where $\phi(r)$ is strictly positive and $\lim_{r\to+\infty} \phi(r) = 0$.*

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