University of Novi Sad
Faculty of Sciences
Department of Mathematics and Informatics

Nonlinear Control Problems With and Without Fractional Derivatives

- doctoral dissertation -

Nelinearni problemi upravljanja sa i bez frakcionih izvoda

- doktorska disertacija -

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Mathematical analysis and computer modeling are revealing to us that the shapes and processes we encounter in nature—the way that plants grow, the way that mountains erode or rivers flow, the way that snowflakes or islands achieve their shapes, the way that light plays on a surface, the way the milk folds and spins into your coffee as you stir it, the way that laughter sweeps through a crowd of people—all these things in their seemingly magical complexity can be described by the interaction of mathematical processes that are, if anything, even more magical in their simplicity.

Douglas Adams

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	be satisfied so that, for any given initial and final data, one can find a control function for which the solution of the system reaches the desired state at the end of the given interval. In order to do so, we derive several auxiliary results regarding fractional differential equations and linear time-varying fractional control problems. We define the Riemann-Liouville and the Caputo state-transition matrices, which are essential part of the solutions of linear fractional systems, and derive estimates of those matrices. Further, we consider linear time-varying fractional control problems, introduce the controllability Gramian matrix, prove the equivalence between controllability and regularity of the Gramian, introduce the associated adjoint problem and prove the equivalence between controllability of the control problem and observability of the associated adjoint problem. Moreover, we apply the Hilbert uniqueness method and techniques from the calculus of variations to obtain the optimal control function in the weighted L^2-space. Then, using properties of the solution of the linearized control problem and the Leray-Schauder fixed point theorem, we derive controllability result for one class of nonlinear control problems with unbounded dynamics. We consider the cases with fractional and with the integer-order derivative, since in the non-integer case the construction of the solution requires to take into account the memory embedded in the fractional derivative.
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	end of the given interval. In order to do so, we derive several auxiliary results regarding fractional differential equations and linear time-varying fractional control problems. We define the Riemann-Liouville and the Caputo state-transition matrices, which are essential part of the solutions of linear fractional systems, and derive estimates of those matrices. Further, we consider linear time-varying fractional control problems, introduce the controllability Gramian matrix, prove the equivalence between controllability and regularity of the Gramian, introduce the associated adjoint problem and prove the equivalence between controllability of the control problem and observability of the associated adjoint problem. Moreover, we apply the Hilbert uniqueness method and techniques from the calculus of variations to obtain the optimal control function in the weighted L^2-space. Then, using properties of the solution of the linearized control problem and the Leray-Schauder fixed point theorem, we derive controllability result for one class of nonlinear control problems with unbounded dynamics. We consider the cases with fractional and with the integer-order derivative, since in the non-integer case the construction of the solution requires to take into account the memory embedded in the fractional derivative.
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Rezime rada

U ovoj doktorskoj disertaciji bavimo se linearnim i nelinearnim problemima upravljanja opisanim sistemima običnih diferencijalnih jednačina, u kojima figuriše Kaputov izvod reda $\alpha \in (0, 1]$, koji se u slučaju $\alpha = 1$ svodi na klasičan izvod prvog reda. Bavimo se pitanjem globalne kontrolabilnosti sistema, odnosno ispitujemo pod kojim uslovima, za proizvoljno početno i krajnje stanje, postoji funkcija upravljanja takva da rešenje sistema u zadatom vremenskom intervalu dostigne željeno krajnje stanje.

Najpre ispitujemo osobine matrica prelaska stanja koje su fundamentalne matrice rešenja sistema i izvodimo ograničenja tih matrica. Zatim se bavimo osobinama rešenja određenog tipa nelinearnog sistema sa Kaputovim izvodom. Novi rezultati do kojih dolazimo publikovani su u radu [35].

Nakon toga, koristeći rezultate iz klasične teorije linearnog upravljanja, razvijamo teoriju upravljanja za opšti oblik linearnih sistema sa Kuputovim izvodima. Uvodimo Gramovu matricu kontrolabilnosti i izvodimo nekoliko potrebnih i dovoljnih uslova za globalnu kontrolabilnost sistema. Dalje, dolazimo do optimalne funkcije upravljanja u težinskim L^2 prostorima, koji se prirodno nameću zbog singulariteta koji se javlja u rešenjima adjungovanih sistema sa Riman-Ljuvilovim frakcionim izvodima. Dobijeni rezultati objavljeni su u radu [33].

Na kraju, dobijene rezultate primenjujemo na problem nelinearnog upravljanja i koristeći linearizaciju sistema i Lere-Šauderovu teoremu fiksne tačke dolazimo do dovoljnih uslova za kontrolabilnost jedne klase nelinearnih sistema. Rezultati koje predstavljamo bazirani su na radu [34].

Contents

Apstrakt

Glavni predmet istraživanja ove teze su nelinearni problemi upravljanja sledećeg oblika

$$ {}_0^C\mathrm{D}_t^\alpha y(t) = -Af(y(t))y(t) + Bu(t), \quad t \in (0,T), \quad y(0) = y_0, \tag{$*$} $$

gde je $y : [0,T] \to \mathbb{R}^d$ funkcija stanja sistema, $u : [0,T] \to \mathbb{R}^N$ funkcija upravljanja, $f : \mathbb{R}^d \to (0,\infty)$ neprekidna funkcija, $A \in \mathbb{R}^{d\times d}$ realna simetrična pozitivno semidefinitna matrica, $B \in \mathbb{R}^{d\times N}$ realna matrica i ${}_0^C\mathrm{D}_t^\alpha$ Kaputov izvod reda $\alpha \in (0,1]$, koji se u slučaju $\alpha = 1$ svodi na klasičan izvod prvog reda. Bavimo se problemom kontrolabilnosti sistema ($*$), odnosno pokazujemo da ako A i B zadovoljavaju Kalmanov uslov ranga i funkcija f ispunjava određene uslove regularnosti, onda za proizvoljna početna i krajnja stanja y_0 i y_T, možemo naći funkciju upravljanja u takvu da rešenje sistema ($*$) dostiže željeno stanje u krajnjem trenutku, tj. $y(T) = y_T$. Da bismo to ostvarili, neophodno je da izvedemo nekoliko pomoćnih rezultata vezanih za frakcione diferencijalne jednačine kao i za linearne frakcione probleme upravljanja sa nekonstantim koeficijentima.

Prvo, definišemo Riman-Ljuvilove i Kaputove matrice prelaska stanja koje se javljaju kao esencijalni delovi rešenja linearnih frakcionih sistema i izvodimo ocene tih matrica. Dalje, posmatramo nelinearni frakcioni sistem jednačina

$$ {}_0^C\mathrm{D}_t^\alpha y(t) = -Af(y(t))y(t), \quad t \in (0,T), \quad y(0) = y_0, $$

i pokazujemo postojanje, jedinstvenost i uniformnu ograničenost njegovog rešenja, što će biti od velikog značaja za konstrukciju rešenja nelinearnog problema upravljanja.

Drugo, rezultate iz linearne teorije upravljanja sa celobrojnim izvodima uopštavamo na frakcioni slučaj. Preciznije, posmatramo linearne frakcione probleme upravljanja sa nekonstantim koeficijentima, za njih definišemo Gramovu matricu kontrolabilnosti, pokazujemo ekvivalenciju između kontrolabilnosti i regularnosti Gramove matrice, uvodimo pridruženi adjungovani problem i pokazujemo ekvivalenciju između kontrolabilnosti problema upravljanja i opservabilnosti adjungovanog problema. Dalje, koristeći Hilbertov metod jedinstvenosti i tehnike varijacionog računa, dolazimo do optimalne funkcije upravljanja u težinskom L^2 prostoru. Težinski L^2 prostor se

prirodno nameće jer, u frakcionom slučaju, kada je $\alpha \in (0, \frac{1}{2}]$, rešenje adjungovanog sistema ne pripada klasičnom prostoru L^2.

Zatim, koristeći osobine rešenja linearnizovanog problema upravljanja i Lere-Šauderovu teoremu fiksne tačke, pokazujemo kontrolabilnost sistema $(*)$. Slučajeve $\alpha = 1$ i $\alpha \in (0, 1)$ analiziramo zasebno, jer, u slučaju kada red izvoda nije celobrojan, konstrukcija rešenja zahteva uzimanje u obzir memorije koja je sadržana u frakcionom izvodu.

Abstract

The main subject of research of this thesis are nonlinear control problems of the following type

$$ {}_0^C\mathrm{D}_t^\alpha y(t) = -Af(y(t))y(t) + Bu(t), \quad t \in (0,T), \quad y(0) = y_0, \tag{$*$} $$

where $y : [0,T] \to \mathbb{R}^d$ is the state of the system, $u : [0,T] \to \mathbb{R}^N$ is the control function, $f : \mathbb{R}^d \to (0,\infty)$ is a continuous function, $A \in \mathbb{R}^{d\times d}$ is a real symmetric positive semidefinite matrix, $B \in \mathbb{R}^{d\times N}$ is a real matrix and ${}_0^C\mathrm{D}_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1]$, which in the case $\alpha = 1$ reduces to the classical first order derivative. We consider the question of global controllability of ($*$), more precisely, we prove that if A and B satisfy the Kalman rank condition and function f meats certain regularity conditions, then for any given initial and final data y_0 and y_T, we can find a control function u such that the solution of the system ($*$) reaches the desired state at the end of the interval, i.e., $y(T) = y_T$. In order to do so, we derive several auxiliary results regarding fractional differential equations and linear time-varying fractional control problems.

Firstly, we define the Riemann-Liouville and the Caputo state-transition matrices, which are essential part of the solutions of linear fractional systems, and derive estimates of those matrices. Further, we consider nonlinear fractional system

$$ {}_0^C\mathrm{D}_t^\alpha y(t) = -Af(y(t))y(t), \quad t \in (0,T), \quad y(0) = y_0, $$

and prove existence, uniqueness and uniform boundedness of its solution, which will be substantial for the construction of the solution of our nonlinear control problem.

Secondly, we generalize results from the integer-order linear control theory to the fractional setting. More precisely, we consider linear time-varying fractional control problems, introduce the controllability Gramian matrix, prove the equivalence between controllability and regularity of the Gramian, introduce the associated adjoint problem and prove the equivalence between controllability of the control problem and observability of the associated adjoint problem. Moreover, we apply the Hilbert uniqueness method and techniques from the calculus of variations to obtain the optimal control function in the weighted L^2-space. The weighted L^2-spaces arise

naturally in the fractional setting, since when $\alpha \in (0, \frac{1}{2}]$, the solution of the fractional adjoint problem does not belong to the classical L^2-space.

Then, using properties of the solution of the linearized control problem and the Leray-Schauder fixed point theorem, we prove controllability of the system ($*$). We consider the cases $\alpha = 1$ and $\alpha \in (0, 1)$ separately, since in the non-integer case the construction of the solution requires to take into account the memory embedded in the fractional derivative.

Preface

From the ancient cultures of Mesopotamia to the modern age, one of the key ingredients of the development of human civilization was the need for understanding and controlling natural processes and phenomena. In order to adapt and make everyday life increasingly comfortable, we (the humanity) managed to understand and describe many physical laws and use them to our advantage. Furthermore, we have constructed machines to help us overcome our boundaries, and do the hard (or just the boring) work for us. Behind a huge number of machines and technology based systems lies a control problem that represents the interconnection of two main functions—the state function and the control function. To "solve" the control problem means to "control" the state of the system, i.e., to make it reach a desired state or perform a desired task.

The development of modern control theory started in 1960s, and it was oriented into two directions—linear and nonlinear control. Initially, the problems with ordinary differential equations were studied, and then the research extended to the systems involving partial differential equations. Recently, the systems with derivatives of noninteger order found their application in control theory. Namely, due to the memory property, they allow a greater degree of accuracy in modeling the behavior of materials and processes with hereditary effects [3, 4, 37].

In this thesis we are focused on finite-dimensional control problems, with integer and noninteger derivatives. The main subject of research are quasilinear control problems with unbounded and dispersive dynamics, which have been studied in [19, 34]. As it often occurs, in order to deal with nonlinearities, firstly, one needs to understand and examine the linear counterpart of the system. Classical linear control theory was thoroughly studied and there is plenty of literature addressing linear control problems with integer-order derivatives [1, 14, 23, 38, 60, 63, 64]. On the other hand, when it comes to linear fractional control theory, most of the focus has been on the time-invariant systems (see e.g. [8, 6, 12, 52, 55]). One of the reasons for that was the lack of a detailed analysis of the theory of fractional differential equations. Over the last two decades many contributions were made in that field (see e.g. [13, 18, 24, 25, 45, 48, 53]), also contributing to the research in fractional-order control. Since there was no general theory addressing linear fractional control problems, we tried to generalize most of the classical results from linear control theory to the fractional setting, cf. [33]. The Section 3.2 of this thesis covers the results obtained in [33], which are, beside being a contribution to the linear control theory, also significant for the construction of the solution of the linearized control problem in Section 4.2. Furthermore, when dealing with fractional derivatives which are nonlocal operators, the construction of the piecewise solution is not so simple as in the integer-derivative case, since one needs to take into account the accumulated memory. That is why the properties of the solution of the nonlinear system from [35] will also play an important role in the analysis of our nonlinear control problem.

The thesis is organized as follows:

In Chapter 1, we establish the basis for later work by introducing notation, recalling definitions and properties of several classes of function spaces, presenting the main results of the Schauder fixed point theory and by giving a short introduction to the basics of control theory. Furthermore, we give a brief historical overview of the breakthroughs in control theory.

Chapter 2 is devoted to the theory of fractional calculus. We recall definitions of fractional operators and then focus on the systems of fractional differential equations. First, we give classical results regarding the existence and uniqueness of the solution. Next, we take a closer look at the linear systems of fractional differential equations, specifically at their fundamental solution matrices. We conclude this chapter with the result related to the solution of a nonlinear fractional system, obtained in [35].

Chapter 3 deals with linear control theory and it is divided into two main sections. In Section 3.1, we present classical results from the integer-order linear control theory. Then, in Section 3.2, we present the general theory of linear time-varying fractional control problems, developed in [33].

In Chapter 4, nonlinear control problems are studied. Section 4.1 deals with the system studied in [19]. We use the same main idea as in [19], which is to consider a linearized system and apply the Leray-Schauder fixed point theorem to obtain a solution of the nonlinear problem, although we provide a different approach to construction of the solution of linearized problem. Section 4.2 contains novel results from [34], where a quasilinear fractional control problem is studied. Using the Leray-Schauder fixed point theorem, properties of linear control systems from Section 3.2, and the properties of the solutions of fractional differential equations from Chapter 2, we prove controllability of our nonlinear system.

Chapter 5 contains the summary of the obtained results and a short discussion of the possible future work and application of the presented ideas.

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I dedicate this thesis to my grandfather, Jolić Isidor, who recognized, supported and encouraged my desire for knowledge and growth.

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Novi Sad, septembar 2023.

Maja Jolić

Chapter 1

Introduction

Firstly, we introduce notation that will be used throughout the thesis, recall definitions of several classes of function spaces and present main results of the Schauder fixed point theory. Secondly, we give an introduction to control theory, with basic notions and properties of control problems. Then, we give a brief historical review of the development of control theory as a branch of applied mathematics.

1.1 Preliminaries

We begin with notations in vector spaces $\mathbb{R}^d$, $d \in \mathbb{N}$. We denote by

- $|\cdot|$ the Euclidean norm, i.e., $|x| = \left(\sum\limits_{k=1}^{d} x_k^2\right)^{\frac{1}{2}}$, for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$;
- $|\cdot|_1$ the l^1-norm, i.e., $|x|_1 = \sum\limits_{k=1}^{d} |x_k|$, for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$;
- $|\cdot|_\infty$ the maximum norm, i.e., $|x|_\infty = \max\{|x_1|, \ldots, |x_d|\}$, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$;
- $\langle \cdot, \cdot \rangle$ the inner product, i.e., $\langle x, y \rangle := \sum\limits_{k=1}^{d} x_k y_k$, for any $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

We recall the definition of the adjoint operator.

Consider a linear operator $A : H_1 \to H_2$ between Hilbert spaces H_1 and H_2, with inner products $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$, respectively. The adjoint operator of A is a linear operator $A^* : H_2 \to H_1$ such that

$$\langle Ax, y \rangle_{H_2} = \langle x, A^* y \rangle_{H_1}, \quad \text{for every } x \in H_1, y \in H_2.$$

Specifically, when A is a real $m \times d$ matrix and $H_1 = \mathbb{R}^d$, $H_2 = \mathbb{R}^m$ are Euclidian spaces, then $A^* = A^{\mathrm{T}}$, where $(\cdot)^{\mathrm{T}}$ denotes the transpose operation.

1.1.1 Function spaces

Let $-\infty < a < b < \infty$, $d \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$.

We begin by recalling definition of the L^p-spaces, i.e., spaces of p-integrable functions. For $1 \leq p < \infty$,

$$L^p([a,b];\mathbb{R}^d) := \left\{ f : [a,b] \to \mathbb{R}^d \; : \; \int_a^b |f(t)|^p \, dt < \infty \right\},$$

and it is equipped with the norm

$$\|f\|_p := \Big(\int_a^b |f(t)|^p \, dt \Big)^{\frac{1}{p}}.$$

For $p = \infty$, we have the space of measurable, almost everywhere bounded functions:

$$L^\infty([a,b];\mathbb{R}^d) := \left\{ f : [a,b] \to \mathbb{R}^d \; : \; \|f\|_\infty < \infty \right\},$$

where

$$\|f\|_\infty := \inf\{C \geq 0 : |f(t)| \leq C, \text{ for almost every } t \in [a,b]\}.$$

We recall that for $p = 2$, $L^2([a,b];\mathbb{R}^d)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{L^2} := \int_a^b \langle f(t), g(t) \rangle \, dt.$$

Further, for the analysis of fractional control problems we shall need the weighted L^2-spaces. For $\omega \in (-1,1)$, we define the space:

$$L^2_\omega([a,b];\mathbb{R}^d) := \left\{ f : [a,b) \to \mathbb{R}^d \; : \; (b-t)^{\frac{\omega}{2}} f(t) \in L^2([a,b];\mathbb{R}^d) \right\} \tag{1.1}$$

equipped with the norm

$$\|f\|_{L^2_\omega} = \Big(\int_a^b (b-t)^\omega |f(t)|^2 \, dt \Big)^{\frac{1}{2}}.$$

We have that $L^2_\omega([a,b];\mathbb{R}^d)$ is a Hilbert space with inner product

$$\langle f,g\rangle_{L^2_\omega} := \int_a^b (b-t)^\omega \langle f(t),g(t)\rangle\, dt.$$

Specifically, when $\omega=0$, $L^2_0([a,b];\mathbb{R}^d)=L^2([a,b];\mathbb{R}^d)$. Of a special interest to us will be the cases $\omega=\alpha-1$ and $\omega=1-\alpha$, with $\alpha\in(0,1)$ being the order of the fractional derivative figuring in the control system. In this case, we have the following inclusions:

$$L^\infty([a,b];\mathbb{R}^d)\subset L^2_{\alpha-1}([a,b];\mathbb{R}^d)\subset L^2([a,b];\mathbb{R}^d)\subset L^2_{1-\alpha}([a,b];\mathbb{R}^d)\subset L^1([a,b];\mathbb{R}^d).$$

Notice that if $f\in L^2_{\alpha-1}([a,b];\mathbb{R}^d)$ and $g\in L^2_{1-\alpha}([a,b];\mathbb{R}^d)$, then $\langle f,g\rangle_{L^2}$ is well-defined:

$$\begin{aligned}\langle f,g\rangle_{L^2} &= \int_a^b \langle f(t),g(t)\rangle\,dt = \int_a^b \langle (b-t)^{\frac{\alpha-1}{2}} f(t),(b-t)^{\frac{1-\alpha}{2}} g(t)\rangle\,dt\\ &= \langle (b-t)^{\frac{\alpha-1}{2}} f(t),(b-t)^{\frac{1-\alpha}{2}} g(t)\rangle_{L^2}.\end{aligned}$$

Now we move to the notion of continuity. By $C([a,b];\mathbb{R}^d)$, we denote the space of continuous functions $f:[a,b]\to\mathbb{R}^d$, and by $C^k([a,b];\mathbb{R}^d)$ the space of k times continuously differentiable functions $f:[a,b]\to\mathbb{R}^d$, equipped with norms:

$$\|f\|_\infty=\max_{t\in[a,b]}|f(t)| \quad\text{and}\quad \|f\|_{C^k}=\sum_{m=0}^k \|f^{(m)}\|_\infty=\sum_{m=0}^k \max_{t\in[a,b]}|f^{(m)}(t)|.$$

We say that the set $X\subset C([a,b];\mathbb{R}^d)$ is *equicontinuous* if for every $\varepsilon>0$, there exists $\delta>0$ (which depends only on ε) such that

$$(\forall f\in X)(\forall s,t\in[a,b])(|s-t|<\delta\Rightarrow|f(s)-f(t)|<\varepsilon).$$

Here, we recall the Arzelà-Ascoli theorem, which will be used in the thesis.

1.1.1 Theorem (Arzelà-Ascoli) *If a sequence $\{f_n\}$ in $C([a,b];\mathbb{R}^d)$ is bounded and equicontinuous, then it has a uniformly convergent subsequence.*

When dealing with fractional derivatives of order $\alpha\in(0,1)$, and regularity of solutions of fractional differential equations, several more classes of functions arise naturally. Firstly, the class of weighted continuous functions: for $\gamma\in[0,1]$, we denote by $C_\gamma([a,b];\mathbb{R}^d)$ the space of functions $f:(a,b]\to\mathbb{R}^d$ such that the function

$f_\gamma(t) = (t-a)^\gamma f(t)$ belongs to the space $C([a,b];\mathbb{R}^d)$. Naturally, $C_\gamma([a,b];\mathbb{R}^d)$ is endowed with the norm

$$\|f\|_{C_\gamma} = \|f_\gamma\|_\infty = \max_{t\in[a,b]} |(t-a)^\gamma f(t)|.$$

For $\gamma = 0$, $C_0([a,b];\mathbb{R}^d) = C([a,b];\mathbb{R}^d)$.

Secondly, the class of Hölder continuous functions.

1.1.2 Definition *A function $f : [a,b] \to \mathbb{R}^d$ is Hölder continuous of order $\alpha \in (0,1]$ if there exists a positive constant C_α such that for every $s,t \in [a,b]$ it holds that*

$$|f(s) - f(t)| \le C_\alpha |s-t|^\alpha. \tag{1.2}$$

By $H^\alpha([a,b];\mathbb{R}^d)$ we denote the space of Hölder continuous functions of order α on $[a,b]$.

For $\alpha = 1$, $H^1([a,b];\mathbb{R}^d)$ is the space of Lipschitz continuous functions on $[a,b]$. The case $\alpha > 1$ is not of an interest since it reduces to constant functions only, and for any $0 < \alpha < \beta < 1$ we have the following strict inclusions:

$$C^1([a,b];\mathbb{R}^d) \subset H^1([a,b];\mathbb{R}^d) \subset H^\beta([a,b];\mathbb{R}^d) \subset H^\alpha([a,b];\mathbb{R}^d) \subset C([a,b];\mathbb{R}^d).$$

The space $H^\alpha([a,b];\mathbb{R}^d)$ is a Banach space (see [62]), when equipped with the norm:

$$\|f\|_{H^\alpha} = \max_{t\in[a,b]} |f(t)| + \sup_{s,t\in[a,b],s\neq t} \frac{|f(s)-f(t)|}{|s-t|^\alpha},$$

where the second term coincides with the infimum of all possible constants C_α for which (1.2) holds.

Next, we recall the notion of absolute continuity.

1.1.3 Definition *A function $f : [a,b] \to \mathbb{R}^d$ is absolutely continuous on an interval $[a,b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}$ and any family of disjoint intervals $[a_k, b_k] \subset [a,b]$, $k = 1, 2, ..., n$, it holds*

$$\sum_{m=1}^{n} (b_m - a_m) < \delta \quad \Rightarrow \quad \sum_{m=1}^{n} |f(b_m) - f(a_m)| < \varepsilon.$$

The space of absolutely continuous functions on $[a,b]$ is denoted by $AC([a,b];\mathbb{R}^d)$. By $AC^k([a,b];\mathbb{R}^d)$ we denote the space of functions $f : [a,b] \to \mathbb{R}^d$ which have continuous derivatives up to order $k-1$ on $[a,b]$ and $f^{(k-1)} \in AC([a,b];\mathbb{R}^d)$.

Throughout the thesis, the notation $f(t) = O(g(t))$, $t \to a$, means that there exist $\varepsilon > 0$ and $M > 0$ such that $\left|\frac{f(t)}{g(t)}\right| \le M$, when $|t-a| < \varepsilon$.

1.1.2 Schauder fixed point theory

In the theory of nonlinear equations, a classical method for the proof of existence of a solution is based on a fixed point theorem approach. There are many variants of fixed point theorems, depending on the type of the mapping being considered. For example a well-known Banach fixed point theorem is a standard tool in theory of ordinary differential equations. In this thesis, we will use the Leray-Schauder fixed point theorem, also known as the Schaefer fixed point theorem, to prove the existence of a solution for nonlinear control problems. Compared to the Banach theorem, it gives only the existence of the fixed point, not uniqueness, but also requires relaxed conditions. First, it was proved by Schauder in 1930, for Banach spaces, and later Tychonoff proved its generalization to locally convex spaces. Here, we recall only the statements of the theorems, and for the proofs we refer to [17, Th. B.17.] and [56, Ch. 10 & 11].

Since the Schauder fixed point theory relies on convex and compact properties of sets and maps, we recall definitions of these notions.

1.1.4 Definition *Let X and Y be vector spaces. The set $K \subset X$ is convex if for every $x_1, x_2 \in K$ and $\lambda \in [0, 1]$,*

$$\lambda x_1 + (1 - \lambda)x_2 \in K.$$

The function $f : X \to Y$ is convex if for every $x_1, x_2 \in K$ and $\lambda \in [0, 1]$ it holds

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

1.1.5 Definition *Let X and Y be topological vector spaces.*

The set $K \subset X$ is compact if every open cover of K has a finite subcover, i.e., for every collection $\{U_i\}_{i \in I}$ of open subsets of X such that $K \subset \bigcup\limits_{i \in I} U_i$, there exists a finite subcollection $\{U_{i_k}\}_{k=1,\ldots,n}$ such that $K \subset \bigcup\limits_{k=1}^{n} U_{i_k}$.

The set $K \subset X$ is relatively compact if its closure $\overline{K}$ is a compact set.

The map $f : X \to Y$ is compact if for every bounded set $B \subset X$, $f(B)$ is a relatively compact set in Y.

For metric spaces, we have that the compactness (of a set) is equivalent to the sequential compactness.

1.1.6 Proposition *Let X be a metric space. The set $K \subset X$ is compact if and only if every sequence from K has a convergent subsequence, whose limit is also in K.*

Now we move to the Schauder fixed point theorem, which addresses the mappings defined on Banach spaces.

1.1.7 Theorem (Schauder) *Let X be a Banach space and $\mathcal{T} : X \to X$ a continuous map. If the image of $\mathcal{T}$ is included in a compact subset of X, then $\mathcal{T}$ has a fixed point.*

Tychonoff's generalization to locally convex spaces is given in the sequel.

1.1.8 Theorem (Schauder-Tychonoff) *Let X be a locally convex space, let $K \subset X$ be nonempty and convex, and let $K_0 \subset K$ be a compact set. If $\mathcal{T} : K \to K_0$ is a continuous map, then there exists $x^* \in K_0$ such that $\mathcal{T}(x^*) = x^*$.*

The Leray-Schauder variant of the theorem, which is more suitable for applications, states as follows.

1.1.9 Theorem (Leray-Schauder/Schaefer) *Let X be a Banach space, $\mathcal{T} : X \to X$ a continuous and compact map, and assume that the set*

$$\{x \in X \,:\, x = \lambda \mathcal{T}(x), \text{ for some } \lambda \in [0,1]\}$$

is bounded. Then $\mathcal{T}$ has a fixed point $x^* \in X$.

1.2 Basics of control theory

We begin with mathematical description of a control problem. Consider a system governed by the equation

$$D(y) = 0, \tag{1.3}$$

where $y \in Y$ is the state of the system, Y is a vector space and D is an operator on Y. The operator D represents the nature of the system, it describes a process or a phenomenon that is analyzed. Usually, D is determined by some physical laws that govern the system, i.e., by the laws that the state of the system needs to obey.

For example, Newton's law of cooling describes a heat transfer between a body and its surrounding. More precisely, it states that the rate of the heat loss from a body is directly proportional to the difference in temperature between the body and its surroundings. Simplified formulation is given by the following linear differential equation

$$\frac{dT}{dt} = r(T_{env} - T(t)), \tag{1.4}$$

where $T(t)$ is the temperature of the body at the moment t, T_{env} is the temperature of the environment and r is the coefficient of heat transfer. For a given initial temperature $T(0) = T_0 > T_{env}$ and a time interval $[0, t_1]$, we can determine the temperature at the moment t_1. By solving differential equation (1.4), we get

$$T(t_1) = T_{env} - (T_{env} - T_0)e^{-rt_1}.$$

Now, assume that the temperature of the environment is not constant, and that we are able to modify it. Naturally, changes in the temperature $T_{env} = T_{env}(t)$, will cause changes in the outcome $T(t_1)$, and by allowing these changes, we transform an ODE into a control problem. Let us rewrite the equation (1.4) in the following way

$$\frac{dT}{dt} + rT(t) = rT_{env}(t), \quad t \in [0, t_1], \quad T(0) = T_0, \tag{1.5}$$

and consider the following question:

For fixed initial and final temperatures T_0, T_1, can we find the function $T_{env}(t)$, $t \in [0, t_1]$, such that the temperature of the body will go from initial state $T(0) = T_0$ to final state $T(t_1) = T_1$ during the time interval $[0, t_1]$?

This is a question of controllability, one of the key notions in control theory, which, plainly speaking, can be formulated in this way:

Are we able to interfere in such a way that the outcome of the system will be exactly what we want it to be?

The term on the right hand side of the equation (1.5) represents our involvement in the system, and in general case, this involvement is introduced by a control function. For our general system (1.3), the associated control problem is given by

$$D(y) = h(u),$$

where, beside the natural dynamics of the system (described by the operator D), we have a term $h(u)$. Here u is a control function and h describes how the control acts on the system. Usually, the control u is chosen from an admissible set of controls $\mathcal{U}$, which is determined according to a specific problem. For example, $\mathcal{U}$ can be L^2-space, L^∞-space, space of continuous functions or some subset of those spaces.

In order to give a precise definition of controllability, let us focus on more specific type of control problems, which will be analyzed in this thesis.

Consider a system

$${}_a^C\mathrm{D}_t^\alpha y(t) = f(t, y, u), \quad t \in (a, b), \quad y(a) = y_a, \tag{1.6}$$

where $y : [a, b] \to \mathbb{R}^d$, $u : [a, b] \to \mathbb{R}^N$, $N < d$, $f : [a, b] \times \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d$ and ${}_a^C\mathrm{D}_t^\alpha$ stands for the left Caputo derivative of order $\alpha \in (0, 1]$ (for $\alpha = 1$, ${}_a^C\mathrm{D}_t^1 = \frac{d}{dt}$).

The admissible set of controls $\mathcal{U}$ we choose to be either the space $L^2_{\alpha-1}([a, b]; \mathbb{R}^N)$ (when $\alpha = 1$ it reduces to $L^2([a, b]; \mathbb{R}^N)$) or $L^\infty([a, b]; \mathbb{R}^N)$.

1.2.1 Definition *The system (1.6) is said to be controllable if for any given initial and final data $y_a, y_b \in \mathbb{R}^d$, there exists a control function $u \in \mathcal{U}$ such that the solution of the system (1.6) satisfies $y(b) = y_b$.*

Controllability defined in this way is also called global controllability or complete controllability, since we do not impose any conditions on the choice of initial and final states y_a, y_b nor on the length of the interval $[a, b]$. Contrary to global, we have local controllability, which is related to a specific state $\overline{y}$, and where instead of letting $y_a, y_b \in \mathbb{R}^d$, we consider y_a and y_b from a neighbourhood of $\overline{y}$. Furthermore, there are many other definitions of controllability. For example, in null-controllability the goal is to steer the solution to the state $y_b = 0$. In approximate controllability one does not need to derive the solution exactly to the final state y_b, but to a certain neighborhood of y_b. Further, when the function f from (1.6) depends also on some parameters, then we have averaged controllability, where the goal is to control the expected or averaged value of the system.

As we can see, depending on the type of the problem being considered, we have different criteria for controllability, but in all these problems, the control function can be chosen freely from the specified set $\mathcal{U}$. It often occurs that we have more than one control which steers the solution to a desired state. Thus, the question arises: "Which one is "the best" choice?", and it leads us to another important aspect of control theory—*optimization*. The branch of control theory which deals with problems where the goal is to find an optimal control function, and consequently an optimal solution of the system, is called *Optimal control theory*. It relies on mathematical techniques from optimization theory and variational calculus. When addressing such problems, firstly, one needs to define the criterion by which the optimal control is chosen. For example, one can consider a problem of finding control with minimal L^2-norm (energy optimization), control which steers the solution to a desired state in minimal time (time-optimal control), control which minimizes the cost or maximizes the payoff, etc. In Chapter 3, we will analyze in more details these types of problems for linear systems with both integer and non-integer derivatives.

So far, we mentioned two important concepts in control theory—controllability and optimality. Now we move to the notion of *observability*, which addresses the problem of recovering the information about the state variables from the knowledge of observations (measurements). Hence, it takes into account the constraints imposed by restrictions on measured variables (often we are not able to measure all the state variables), and deals with an inverse problem. Although it can be considered as a problem for itself, it is often associated with a problem of controllability, since there is a certain duality between these two notions. Roughly speaking, controllability indicates whether the output of the system (state function $y(t)$) can be controlled by acting on the inputs (control function), while observability indicates whether the internal behaviour of the system can be observed (detected, reconstructed) from its outputs. Specifically, in the case of linear systems, this concept of duality manifests in a form of equivalence between controllability of control problem and observability of its adjoint problem. For illustration, let us consider linear time-invariant system

$$x' = Ax, \quad x(0) = x_0 \tag{1.7}$$

where $A \in \mathbb{R}^{d\times d}$, $x : [0,T] \to \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$. Suppose that we can measure an N-dimensional output

$$z(t) = Bx(t), \tag{1.8}$$

where $B \in \mathbb{R}^{N\times d}$, $N < d$. Of a special interest is the situation where $N \ll d$, and z is interpreted as a low-dimensional observation of a high-dimensional dynamics x.

In this setting, the observability question is: Can we reconstruct $x(t)$ from the observations $z(t)$?

If yes, we say that the system is observable. Specifically, from [20] we have the following definition.

1.2.2 Definition *The pair (1.7), (1.8) is called observable if knowing the measurements $z(\cdot)$ on any time interval $[0,t]$, we are able to compute x_0.*

As we shall see in Chapter 3, there are other definitions of observability (via observability inequality), which will be more suitable for our analysis.

The following proposition (see [20, Th. 2.7]), indicates how observability and controllability of linear time-invariant systems are connected, through mutually adjoint operators.

1.2.3 Proposition *The system*

$$\begin{aligned} x'(t) &= Ax(t) \\ z(t) &= Bx(t) \end{aligned}$$

is observable if and only if the system

$$y'(t) = A^{\mathrm{T}}y(t) + B^{\mathrm{T}}u(t)$$

is controllable.

We conclude this section by mentioning one important classification of control systems. Depending on whether the choice of the input is influenced by the output, we have the open-loop and the closed-loop systems. In the open-loop systems the selection of the input (or control) is based on the a priori knowledge about the systems and the desired goal, i.e., the input is precomputed and it is not influenced by the output of the system. On the other hand, in closed-loop systems the output effects the choice of the input. More precisely, the information from the output is “fed back” to the input, and the control is calculated according to a certain *feedback law*. These feedback laws are suitable for modeling of the real systems in which random perturbations and fluctuations can occur (which can not be determined a priori). In this case, the feedback law is there to correct possible perturbations and to stabilize the output.

For more details on the above mentioned notions, as well as on the many other concepts and ideas from control theory we refer to [63, 64] (for an outline of mathematical control theory, the main concepts regarding linear systems, nonlinear systems, optimal control and infinite dimensional systems are covered), [1, 14, 23, 38, 60] (for the control theory of linear systems), [22, 30, 31, 47] (for the classical topics in optimal control) and [17, 32] (topics in nonlinear control).

1.3 A brief history of control theory

Control theory is an interdisciplinary field of research which includes several areas of mathematics and engineering. It deals with the analysis and design of control systems—a process or a phenomenon described by a set of equations. The mathematical control theory focuses on the development of a mathematical model of the system, analysis of the solution, stability, etc., while the control engineering deals with the design and practical implementation of the results.

Although the modern control theory developed in the 20th century, the applications of control systems can be found even in the ancient time. The control of the irrigation systems in Mesopotamia, the systems of regulating valves in Roman aqueducts, the water clocks in ancient Greece are one of the earliest known examples.

A significant improvement in the design of control systems came during the 17th century. The work by Christiaan Huygens and his contemporary Robert Hook on the oscillations of the pendulum resulted in a new device for a precise measurement of time—the pendulum clock. Furthermore, their analysis of circular motion and centrifugal force found an application in the construction of windmills. The flyballs that were used to regulate the velocity of windmills worked in the following way:
Two balls were attached to an axis that rotates with velocity proportional to the velocity of the windmill. When the angular velocity increases, the centrifugal force causes the balls to rise and this upward movement affects the positions of the mill's sails.

These flyball governors were adapted by James Watt in 1769 to the governors in steam engines. They were connected to several valves that regulate the pressure of the steam, and in that way keep the velocity of the engine close to a constant. At first, these control mechanisms were based on observations and empirical knowledge, without the rigor mathematical background. Around 1840, the mathematician and astronomer George Airy was the first to work on the mathematical analysis of the governors, while the complete mathematical description of their properties was published by a physicist James Clerk Maxwell in 1868. Furthermore, many 19th cen-

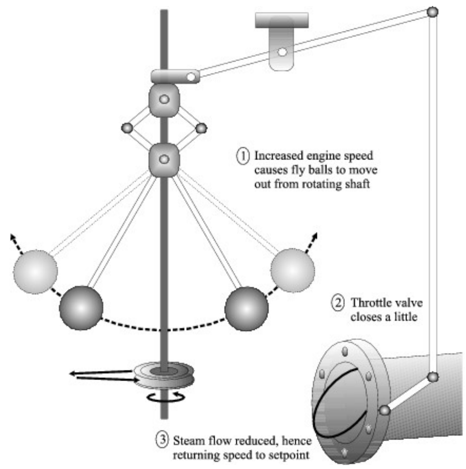

Figure 1.1: Flyball governor in a steam engine (taken from [26])

tury mathematicians (Euler, Laplace, Fourier) contributed to theoretical research in control systems by developing methods for mathematical modeling of natural phenomena.

The first three decades of the 20th century brought some fundamental developments in control theory. Harry Nyquist and Hendrik Wade Bode, together with collaborators from the Bell Telephone Laboratories, developed the theory of feedback amplifiers, which were used in telephone systems. Even today, the amplifiers are the foundations of frequency design. Another widely used concept is the proportional-integral-derivative controller (PID controller), which was introduced by a mathematician Nicolas Minorsky and applied to automatic controllers for steering ships. Around that time, development of analog computers formed the basis for the application of controllers in the chemical and petroleum industries.

Then, during the Second World War, developed techniques were used in the design of anti-aircraft batteries, the control mechanisms of plane tracking and ballistic missiles.

In 1948, mathematician Norbert Wiener published a book "Cybernetics: or Control and Communication in the Animals and the Machines", where he presented an interconnection between constructed control systems (machines) and naturally-designed control systems (which are present in living organisms, such as, the system that regulates body temperature or blood sugar level, or how our eyes perceive the world around us and how we react (the interconnection of muscles, nerves, visual information and our actions form a complex control system)). Wiener was also known for introducing randomness and noise in mathematical models and control systems.

Around 1950s, the methods and ideas developed so far were referred to as a part of classical control theory, while great improvements in the analysis of more complex control systems opened the door to modern control theory. In the classical control theory mostly the systems described by linear time-invariant ODEs were analyzed. Using the Laplace or the Fourier transform, they were transformed into the frequency domain where the analysis was easier to perform. Those were the systems with Single Input Single Output (SISO systems). The need for more accurate models required the tools for the analysis of systems which are of higher order, time-variant, linear and (often) nonlinear, as well as for the systems with Multiple Input Multiple Output (MIMO systems). In 1960s, the above mentioned problems were addressed and that was the beginning of modern control theory. The foundations of modern control theory were made by Rudolf Kalman in linear control and Richard Bellman and Lev Pontryagin in nonlinear control. Kalman introduced the filtering techniques and developed an algebraic approach for the analysis of linear systems. Bellman introduced an optimization method today well-known as the dynamic programming, while the Pontryagin's maximum principle provided a powerful tool for finding an optimal control.

From 1970s, with enormous development of new technologies and their applications in many areas of science, medicine and industry, control systems became a part of our everyday life. We single out some examples: regulation and control of the electrical power grid, control of communication systems (telephones, mobile phones, internet), transportation (cars, aircraft), space systems (rockets, satellites), home-used devices (bathroom tanks, systems of heating, ventilation and air conditioning), etc.

Today, the control theory is a part of many research areas, and a great number of control systems are used to model and explain processes in engineering, physics, medicine, biology, economy, social studies, etc. Furthermore, with the development of computer science, data science and artificial intelligence, new fields for the application of the control theory arise.

Chapter 2

Fractional Calculus

The origin of Fractional calculus, or, more precisely, integrals and derivatives of non-integer order, goes back to the 17th century and correspondence between Leibniz and L' Hospital regarding derivation of order $\frac{1}{2}$. Later, this idea intrigued many mathematicians (Euler, Lagrange, Laplace, Riemann and many more), and during the 18th and 19th century the theory of fractional calculus developed as a generalization of integer-order derivatives and integrals. Firstly, it was considered as "a rather esoteric mathematical theory without applications" [51], which in the 20th century was proved to be wrong. With its applications in many areas of engineering and industry (viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, etc.), fractional calculus became a significant domain of research.

Since, in this thesis, we study control problems with both integer and non-integer derivatives, we shall recall some basic notions of fractional calculus. The chapter is organized in the following way. In the first section we present definitions and properties of fractional operators. Then we define the Mittag-Leffler functions, which are essential part of fractional calculus since they often appear as solutions of fractional differential equations. In the third section we analyze systems of fractional differential equations. The original results, published in [35], are stated in Proposition 2.3.19, Proposition 2.3.20 and Subsection 2.3.3.

2.1 Fractional operators

In fractional calculus there are several different approaches for defining the integration and differentiation operators. In this thesis we use the Riemann-Liouville and the Caputo approach.

Let $-\infty < a < b < \infty$, and consider a function $f : [a, b] \to \mathbb{R}$. First we give definition of fractional integral.

2.1.1 Definition *The left and right Riemann-Liouville fractional integrals of order* $\alpha > 0$ *are defined by*

$$ {}_a\mathrm{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau $$

and

$$ {}_t\mathrm{I}_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau, $$

where Γ *denotes the Euler gamma function.*

In the next Theorem, which is a special case of [62, Th. 3.6], we have the boundedness property of operator ${}_a\mathrm{I}_t^\alpha$.

2.1.2 Theorem *If* $\alpha > 0$*, then the fractional integration operator* ${}_a\mathrm{I}_t^\alpha$ *is bounded from* $L^\infty([a,b];\mathbb{R}^d)$ *into* $H^\alpha([a,b];\mathbb{R}^d)$*.*

Since in our control problems we consider systems with fractional derivatives of order between 0 and 1, in this section we focus on fractional differentiation operators of order $\alpha \in (0,1)$. For more details on fractional operators and their applications we refer to [2, 18, 45, 46, 54, 57, 62].

2.1.3 Definition *The left and right Riemann-Liouville fractional derivatives of order* $\alpha \in (0,1)$*, are given by*

$$ {}_a\mathrm{D}_t^\alpha f(t) = \frac{d}{dt}\big({}_a\mathrm{I}_t^{1-\alpha} f(t)\big) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha}\, d\tau $$

and

$$ {}_t\mathrm{D}_b^\alpha f(t) = -\frac{d}{dt}\big({}_t\mathrm{I}_b^{1-\alpha} f(t)\big) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(\tau)}{(\tau-t)^\alpha}\, d\tau. $$

2.1.4 Definition *The left and right Caputo fractional derivatives of order* $\alpha \in (0,1)$*, are given by*

$$ {}_a^C\mathrm{D}_t^\alpha f(t) = {}_a\mathrm{I}_t^{1-\alpha}\big(f'(t)\big) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau $$

and

$$ {}_t^C\mathrm{D}_b^\alpha f(t) = {}_t\mathrm{I}_b^{1-\alpha}\big(-f'(t)\big) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b \frac{f'(\tau)}{(\tau-t)^\alpha}\, d\tau. $$

Let us introduce several classes of functions, which will be needed for the analysis of the solutions to differential equations involving the Riemann-Liouville and the Caputo fractional derivative. We denote by:

- $AC_a^\alpha([a,b];\mathbb{R}^d)$ the set of functions $f \in L^1([a,b];\mathbb{R}^d)$ for which the left Riemann-Liouville derivative of order α, ${}_a\mathrm{D}_t^\alpha f(t)$, is defined on (a,b);
- $AC_b^\alpha([a,b];\mathbb{R}^d)$ the set of functions $f \in L^1([a,b];\mathbb{R}^d)$ for which the right Riemann-Liouville derivative of order α, ${}_t\mathrm{D}_b^\alpha f(t)$, is defined on (a,b);
- ${}^cAC_a^\alpha([a,b];\mathbb{R}^d)$ the set of continuous functions $f \in C([a,b];\mathbb{R}^d)$ for which the left Caputo derivative of order α, ${}_a^C\mathrm{D}_t^\alpha f(t)$, is defined on (a,b);
- ${}^cC_a^\alpha([a,b];\mathbb{R}^d)$ the set of continuous functions $f \in C([a,b];\mathbb{R}^d)$ for which the left Caputo derivative of order α, ${}_a^C\mathrm{D}_t^\alpha f(t)$, is continuous on $[a,b]$.

Let us mention that an equivalent way to define the Caputo fractional derivative is via the Riemann-Liouville derivative. More precisely, starting with

$$ {}_a^C\mathrm{D}_t^\alpha f(t) = {}_a\mathrm{D}_t^\alpha[f(t)-f(a)] \quad \text{and} \quad {}_t^C\mathrm{D}_b^\alpha f(t) = {}_t\mathrm{D}_b^\alpha[f(t)-f(b)], $$

and using Definition 2.1.3 together with integration by parts, one can obtain expressions given in Definition 2.1.4 (cf. [45, Th. 2.1]).

For $\alpha \in (0,1)$, we recall the relations between fractional integral and differential operators. From [45, Lemma 2.5 & Lemma 2.22] we have the following.

2.1.5 Lemma

(i) If $f \in L^1([a,b];\mathbb{R}^d)$ and ${}_a\mathrm{I}_t^{1-\alpha} f(t) \in AC([a,b];\mathbb{R}^d)$, then the equality

$$ {}_a\mathrm{I}_t^\alpha\left({}_a\mathrm{D}_t^\alpha f(t)\right) = f(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\, {}_a\mathrm{I}_t^{1-\alpha} f(t)|_{t=a} $$

holds almost everywhere on $[a,b]$.

(ii) If $f \in C([a,b];\mathbb{R}^d)$, then it holds

$$ {}_a\mathrm{I}_t^\alpha\left({}_a^C\mathrm{D}_t^\alpha f(t)\right) = f(t) - f(a). \tag{2.1} $$

In fractional domain, integration by parts formula has several variants. For fractional integrals, from [62, Th. 3.5 & Cor.], we have the following result.

2.1.6 Proposition *Let $f \in L^p([a,b];\mathbb{R})$, $g \in L^q([a,b];\mathbb{R})$. The relation*

$$ \int_a^b g(t)\,{}_a\mathrm{I}_t^\alpha f(t)\,dt = \int_a^b {}_t\mathrm{I}_b^\alpha g(t) f(t)\,dt \tag{2.2} $$

holds for $\frac{1}{p}+\frac{1}{q} \le 1+\alpha$, $p \ge 1$, $q \ge 1$, with $p \ne 1$, $q \ne 1$ in the case $\frac{1}{p}+\frac{1}{q} = 1+\alpha$.

Using the above property, we derive fractional integration by parts, which relates the left Caputo and the right Riemann-Louville fractional derivatives.

2.1.7 Proposition *Let $\alpha \in (0,1)$, $g \in AC_b^\alpha([a,b];\mathbb{R}^d)$ and $f \in {}^cAC_a^\alpha([a,b];\mathbb{R}^d)$, such that $f' \in L^\infty([a,b];\mathbb{R}^d)$. Then*

$$\int_a^b \langle {}_a^C\mathrm{D}_t^\alpha f(t), g(t)\rangle \, dt = \langle f(t), {}_t\mathrm{I}_b^{1-\alpha} g(t)\rangle|_a^b + \int_a^b \langle f(t), {}_t\mathrm{D}_b^\alpha g(t)\rangle \, dt. \tag{2.3}$$

Proof. From the definition of the Caputo derivative and (2.2) (here with $q = 1$, $p = \infty$), we obtain

$$\int_a^b \langle {}_a^C\mathrm{D}_t^\alpha f(t), g(t)\rangle \, dt = \int_a^b \langle {}_a\mathrm{I}_t^{1-\alpha} \left(f'(t)\right), g(t)\rangle \, dt = \int_a^b \langle f'(t), {}_t\mathrm{I}_b^{1-\alpha} g(t)\rangle \, dt.$$

Then, by applying integration by parts for integer-order derivatives, we get

$$\int_a^b \langle {}_a^C\mathrm{D}_t^\alpha f(t), g(t)\rangle \, dt = \langle f(t), {}_t\mathrm{I}_b^{1-\alpha} g(t)\rangle|_a^b - \int_a^b \langle f(t), \frac{d}{dt}\Big({}_t\mathrm{I}_b^{1-\alpha} g(t)\Big)\rangle \, dt,$$

which together with the definition of the right Riemann-Liouville derivative, concludes the proof. □

For different variants of fractional integration by parts formulas, involving both the Caputo and the Riemann-Liouville derivatives, we refer to [44].

2.2 The Mittag-Leffler functions

The special functions, such as the Gamma, the Beta function, the Mittag-Leffler functions and the Wright function, are essential part of fractional calculus. They appear in definitions of fractional operators and in the solutions of fractional differential equations. In this section, we recall basic definitions and properties of the Mittag-Leffler functions. A detailed analysis of these classes of functions can be found in [27].

2.2.1 Definition *For $\alpha \in \mathbb{C}$, $\mathrm{Re}\{\alpha\} > 0$, the one-parameter Mittag-Leffler function $E_\alpha(z)$ is defined by the series*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}. \tag{2.4}$$

For $\alpha \in \mathbb{C}$, $\mathrm{Re}\{\alpha\} > 0$, and $\beta \in \mathbb{C}$, the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by the series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \tag{2.5}$$

The one-parameter Mittag-Leffler function is in fact a special case of the two-parameter Mittag-Leffler function with $\beta = 1$, i.e., $E_{\alpha,1}(z) = E_\alpha(z)$.

For $\mathrm{Re}\{\alpha\} > 0$, $\beta \in \mathbb{C}$, series (2.4) and (2.5) converge in the whole complex plane. Furthermore, $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are entire functions of the complex variable z.

Next, we recall the complete monotonicity property of the Mittag-Leffler functions with real negative argument.

2.2.2 Definition *A function $f : [0, \infty) \to \mathbb{R}$ is called completely monotonic if for every $n \in \mathbb{N}$, $f^{(n)}(x)$ exists on $(0, \infty)$ and satisfies*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in (0, \infty).$$

Clearly, a completely monotonic function is a non-increasing function on $[0, \infty)$.

2.2.3 Proposition *For $0 \leq \alpha \leq 1$ and $\beta \geq \alpha$, Mittag-Leffler functions $E_\alpha(-x)$ and $E_{\alpha,\beta}(-x)$ are completely monotonic for $x \geq 0$.*

2.3 Systems of fractional differential equations

2.3.1 Existence and uniqueness of solutions

We start with the classical results regarding existence and uniqueness of the solution for the system of fractional differential equations (FDEs). Among a broad list of literature devoted to this type of problems, we refer to [18], [29], Chapter 3 in [45], Chapter 8 in [62], and references therein.

First, we consider the Cauchy problem with the Caputo fractional derivative in general form:

$$\begin{aligned} {}_a^C\mathrm{D}_t^\alpha y(t) &= F(t, y(t)), \quad t \in [a, b], \\ y(a) &= y_a, \end{aligned} \tag{2.6}$$

where $\alpha \in (0, 1]$, $y : [a, b] \to \mathbb{R}^d$ and $F : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$.

Next theorem gives sufficient conditions for the existence of a unique solution to (2.6) and it is a special case of [45, Th. 3.25].

2.3.1 Theorem *Let G be an open set in $\mathbb{R}^d$, $y_a \in G$, and let $F : [a,b] \times G \to \mathbb{R}^d$ be a function such that:*

(i) for every $y \in G$, $F(\cdot, y) \in C([a,b];\mathbb{R}^d)$;

(ii) F is Lipschitz continuous with respect to the second variable, i.e., there exists a positive constant A, such that for every $t \in [a,b]$, $y_1, y_2 \in G$,

$$|F(t,y_1) - F(t,y_2)| \leq A|y_1 - y_2|.$$

Then, there exists a unique solution to the Cauchy problem (2.6) in ${}^cC_a^\alpha([a,b];\mathbb{R}^d)$.

2.3.2 Remark If, in the above theorem, only continuity of $F : [a,b] \times G \to \mathbb{R}^d$ is assumed, then (only) the existence of the solution is obtained.

The main idea of the proof of Theorem 2.3.1 is based on the reduction of problem (2.6) to the Volterra integral equation

$$y(t) = y_a + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{F(s, y(s))}{(t-s)^{1-\alpha}}\, ds, \quad a \leq t \leq b, \tag{2.7}$$

and application of the Banach fixed point theorem. From [45, Th. 3.24] we have the following equivalence.

2.3.3 Theorem *Let $0 < \alpha \leq 1$, let G be an open set in $\mathbb{R}^d$, $y_a \in G$, and let $F : [a,b] \times G \to \mathbb{R}^d$ be a function such that for every $y \in G$, $F(\cdot, y) \in C([a,b];\mathbb{R}^d)$. Let $r = \lfloor \alpha \rfloor$, where $\lfloor \alpha \rfloor$ is the greatest integer less than or equal to α , and $y \in C^r([a,b];\mathbb{R}^d)$. Then, y satisfies (2.6) if and only if it satisfies the Volterra integral equation (2.7).*

2.3.4 Remark Theorems 2.3.1 and 2.3.3 have an analog result involving the Riemann-Liouville fractional derivative, see [45, Th. 3.10, 3.11].

Conditions (i) and (ii) on F in Theorem 2.3.1 are sufficient for the solution y to be in the space ${}^cC^\alpha([a,b];\mathbb{R}^d)$. In addition, if we want a higher regularity of the solution, we need to impose stronger regularity conditions on F. In Section 6.4 in [18], one can find a detailed analysis of the smoothness of the solution. Here we give the result from [18, Th. 6.28], which follows from a more general theory of Fredholm integral equations studied in [15].

2.3.5 Theorem *Assume the conditions of Theorem 2.3.1 hold. Moreover, let $F \in C^1([a,b] \times \mathbb{R}^d;\mathbb{R}^d)$. Then the unique solution of (2.6) satisfies $y \in C([a,b];\mathbb{R}^d) \cap C^1((a,b];\mathbb{R}^d)$ and $y'(t) = O((t-a)^{\alpha-1})$, as $t \to a$.*

2.3.2 Linear systems of FDEs

Now we are interested in properties of solutions to linear fractional differential systems. More precisely, we are going to consider linear time-varying initial value problems with the Riemann-Liouville and Caputo derivatives of order $\alpha \in (0,1)$, given by:

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha x(t) &= A(t)x(t) + g(t), \quad t \in [a,b], \\ {}_a\mathrm{I}_t^{1-\alpha} x(t)|_{t=a} &= x_a, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} {}_a^C\mathrm{D}_t^\alpha x(t) &= A(t)x(t) + g(t), \quad t \in [a,b], \\ x(a) &= x_a, \end{aligned} \tag{2.9}$$

where $A : [a,b] \to \mathbb{R}^{d\times d}$ is a matrix function, $g : [a,b] \to \mathbb{R}^d$ is a vector function, $x : [a,b] \to \mathbb{R}^d$ is state function and $x_a \in \mathbb{R}^d$ is given initial state.

We introduce the notation

$$\Omega = \{(\tau,t) \in [a,b] \times [a,b] : \tau \leq t\} \quad \text{and} \quad \Omega_0 = \{(\tau,t) \in [a,b] \times [a,b] : \tau < t\},$$

and by $\mathbb{I}$ we denote the identity matrix.

As we shall see, essential part of the analysis of solutions to (2.8) and (2.9) are fundamental solution matrices associated to the matrix function A. In control theory, they are often referred to as *the state-transition matrices.*

2.3.6 Definition *(i) The left Riemann-Liouville state-transition matrix is the matrix function $\Phi : \Omega_0 \to \mathbb{R}^{d\times d}$, such that for every fixed $\tau \in [a,b)$, $\Phi(\tau,\cdot) : (\tau,b] \to \mathbb{R}^{d\times d}$ satisfies the initial value problem*

$$\begin{aligned} {}_\tau\mathrm{D}_t^\alpha \Phi(\tau,t) &= A(t)\Phi(\tau,t), \quad t \in (\tau,b], \\ {}_\tau\mathrm{I}_t^{1-\alpha} \Phi(\tau,t)|_{t=\tau} &= \mathbb{I}. \end{aligned} \tag{2.10}$$

(ii) The left Caputo state-transition matrix is the matrix function $\Psi : \Omega \to \mathbb{R}^{d\times d}$, such that for every fixed $\tau \in [a,b)$, $\Psi(\tau,\cdot) : [\tau,b] \to \mathbb{R}^{d\times d}$ is a solution to the matrix initial value problem

$$\begin{aligned} {}_\tau^C\mathrm{D}_t^\alpha \Psi(\tau,t) &= A(t)\Psi(\tau,t), \quad t \in [\tau,b], \\ \Psi(\tau,\tau) &= \mathbb{I}. \end{aligned} \tag{2.11}$$

From [13, Section 4] we have the following results.

2.3.7 Theorem *Suppose $A \in L^\infty([a,b];\mathbb{R}^{d\times d})$ and $g \in L^\infty([a,b];\mathbb{R}^d)$, and let $\tau \in [a,b)$.*

(i) Initial value problem (2.10) admits a unique solution $\Phi(\tau,\cdot)$ on the interval $(\tau,b]$. Moreover, $\Phi(\tau,\cdot)\in AC^{\alpha}_{\tau}([\tau,b];\mathbb{R}^{d\times d})$ and satisfies integral equation

$$\Phi(\tau,t)=\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathbb{I}+\int\limits_{\tau}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)\Phi(\tau,s)\,ds,\quad t\in(\tau,b]. \tag{2.12}$$

(ii) Initial value problem (2.11) admits a unique solution $\Psi(\tau,\cdot)$ on the interval $[\tau,b]$. Moreover, $\Psi(\tau,\cdot)\in {}^{c}AC^{\alpha}_{\tau}([\tau,b];\mathbb{R}^{d\times d})$ and satisfies integral equation

$$\Psi(\tau,t)=\mathbb{I}+\int\limits_{\tau}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)\Psi(\tau,s)\,ds,\quad t\in[\tau,b]. \tag{2.13}$$

(iii) There exists a unique solution $x\in AC^{\alpha}_{a}([a,b];\mathbb{R}^{d})$ to the Cauchy problem (2.8) on $(a,b]$, and it is given by the Duhamel formula

$$x(t)=\Phi(a,t)x_a+\int\limits_{a}^{t}\Phi(\tau,t)g(\tau)\,d\tau,\quad t\in(a,b]. \tag{2.14}$$

(iv) There exists a unique solution $x\in {}^{c}AC^{\alpha}_{a}([a,b];\mathbb{R}^{d})$ to the Cauchy problem (2.9) on $[a,b]$, and it is given by the Duhamel formula

$$x(t)=\Psi(a,t)x_a+\int\limits_{a}^{t}\Phi(\tau,t)g(\tau)\,d\tau,\quad t\in[a,b]. \tag{2.15}$$

2.3.8 Remark From [45, Th. 7.5 & Th. 7.6] we have that if $A\in C([a,b];\mathbb{R}^{d\times d})$ and $g\in C_{1-\alpha}([a,b];\mathbb{R}^{d})$, then initial value problems (2.8) and (2.9) have unique continuous solutions on $(a,b]$. Furthermore, the solution to (2.8) satisfies

$$\lim_{t\to a^{+}}(t-a)^{1-\alpha}x(t)=\frac{x_a}{\Gamma(\alpha)}.$$

The properties of Φ and Ψ, given in Theorem 2.3.7, were obtained by reduction of the initial value problems (2.10) and (2.11) to the equivalent integral equations (2.12) and (2.13), respectively. On the other hand, in [53], the state-transition matrices were introduced by the method of successive approximations.

Let $\tau\in[a,b)$ be fixed, and $t\in(\tau,b]$. By starting with the initial condition and then successively multiplying it by A and performing integration of order α, one obtains a generalized Peano-Baker series:

$$^{PB}\Phi(\tau,t):=\sum_{k=0}^{\infty}{}_{\tau}\mathrm{I}^{k\circ\alpha}_{t}A(t),\quad t\in(\tau,b], \tag{2.16}$$

where ${}_{\tau}\mathrm{I}_t^{0\circ\alpha}A(t) = \dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathbb{I}$ and for $k \geq 1$, ${}_{\tau}\mathrm{I}_t^{k\circ\alpha}A(t) = {}_{\tau}\mathrm{I}_t^{\alpha}(A(t){}_{\tau}\mathrm{I}_t^{(k-1)\circ\alpha}A(t))$.

If the series (2.16) converges uniformly on $(\tau, b]$, then it coincides with the left Riemann-Liouville state-transition matrix associated to A (cf. [53, Lemma 3.]), i.e., it solves the initial value problem (2.10). Since $\tau \in [a, b)$ was arbitrary, we have

$$\Phi(\tau, t) = {}^{PB}\Phi(\tau, t), \quad a \leq \tau < t \leq b,$$

providing that (2.16) converges uniformly on $(\tau, b]$, for every $\tau \in [a, b)$.

Similarly, for the problem with the Caputo derivative one can define

$$^{PB}\Psi(\tau, t) := \sum_{k=0}^{\infty} {}_{\tau}\mathrm{J}_t^{k\circ\alpha}A(t), \quad t \in [\tau, b], \tag{2.17}$$

where ${}_{\tau}\mathrm{J}_t^{0\circ\alpha}A(t) = \mathbb{I}$ and ${}_{\tau}\mathrm{J}_t^{k\circ\alpha}A(t) = {}_{\tau}\mathrm{I}_t^{\alpha}(A(t){}_{\tau}\mathrm{J}_t^{(k-1)\circ\alpha}A(t))$, for $k \geq 1$. Then from [53, Lemma 5.], it follows that if the series (2.17) converges uniformly on $[\tau, b]$, then it coincides with the solution to (2.11). Assuming that the above statement holds for every (fixed) $\tau \in [a, b)$, we get that the left Caputo state-transition matrix associated to A can be represented in a form of Peano-Baker series, i.e.,

$$\Psi(\tau, t) = {}^{PB}\Psi(\tau, t), \quad a \leq \tau \leq t \leq b.$$

We point out two important cases for our analysis:

- If $A(t) = A$ is a constant matrix, then the associated state-transition matrices reduce to the Mittag-Leffler functions:

$$\Phi(a, t) = (t-a)^{\alpha-1}E_{\alpha,\alpha}(A(t-a)^{\alpha}) = (t-a)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k(t-a)^{k\alpha}}{\Gamma(\alpha(k+1))} \tag{2.18}$$

  and

$$\Psi(a, t) = E_{\alpha}(A(t-a)^{\alpha}) = \sum_{k=0}^{\infty}\frac{A^k(t-a)^{k\alpha}}{\Gamma(\alpha k+1)}.$$

- If $A(t) = Af(t)$, where A is a constant matrix and $f : [a, b] \to \mathbb{R}$ is a continuous scalar function, then

$$^{PB}\Phi(a, t) = \sum_{k=0}^{\infty} A^k {}_{a}\mathrm{I}_t^{k\circ\alpha}(f(t)), \tag{2.19}$$

  and

$$^{PB}\Psi(a, t) = \sum_{k=0}^{\infty} A^k {}_{a}\mathrm{J}_t^{k\circ\alpha}(f(t)). \tag{2.20}$$

From [35, *Prop. 1*] we have the following convergence property.

2.3.9 Property *If A is a constant matrix and $f : [a,b] \to \mathbb{R}$ is a continuous function, then the series* (2.19) *converges uniformly on* $(a,b]$.

Proof. Since f is continuous on $[a,b]$, there exists $M > 0$ such that $M = \max_{t\in[a,b]} |f(t)|$. By induction with respect to $k \in \mathbb{N}$ we obtain:

$$\begin{aligned} \|A^k\| \sup_{t\in(a,b]} \left| {}_a\mathrm{I}_t^{k\circ\alpha}(f(t)) \right| &\leq \|A^k\| \sup_{t\in(a,b]} {}_a\mathrm{I}_t^{k\circ\alpha}(|f(t)|) \leq \|A\|^k \sup_{t\in(a,b]} {}_a\mathrm{I}_t^{k\circ\alpha}(M) \\ &\leq {}_a\mathrm{I}_b^{k\circ\alpha}(\|A\|M). \end{aligned}$$

Since $\sum_{k=0}^{\infty} {}_a\mathrm{I}_b^{k\circ\alpha}(\|A\|M) = (b-a)^{\alpha-1}E_{\alpha,\alpha}(M\|A\|(b-a)^{\alpha})$, it follows that ${}^{PB}\Phi(a,t)$ is uniformly convergent on $(a,b]$ and $\Phi(a,t) = {}^{PB}\Phi(a,t)$, $t \in (a,b]$. □

Following the lines of the proof of Property 2.3.9, we obtain:

2.3.10 Property *If $f : [a,b] \to \mathbb{R}$ is a continuous function, then the series* (2.20) *converges uniformly on* $[a,b]$.

Duality of Φ

The Riemann-Liouville state-transition matrix Φ has a duality property. More precisely, it can be considered as a solution to the initial value problem with the left Riemann-Liouville derivative, as well as a solution to the dual initial value problem with the right Riemann-Liouville derivative. In order to give these properties in more details, we introduce the following notation:

- By ${}_A\Phi^l$ we denote the left Riemann-Liouville state-transition matrix associated to matrix A from Definition 2.3.6 (i), meaning that, for every fixed $\tau \in [a,b)$, ${}_A\Phi^l(\tau,\cdot) : (\tau,b] \to \mathbb{R}^{d\times d}$ solves

$$ {}_\tau\mathrm{D}_t^{\alpha}\, {}_A\Phi^l(\tau,t) = A(t)\, {}_A\Phi^l(\tau,t), \quad t \in (\tau,b], \quad {}_\tau\mathrm{I}_t^{1-\alpha}\, {}_A\Phi^l(\tau,t)|_{t=\tau} = \mathbb{I}, \tag{2.21}$$

  where τ is fixed starting-point and ${}_\tau\mathrm{D}_t^{\alpha}$ denotes the left Riemann-Liouville derivative with respect to the variable t;

- By Φ_A^r we denote the right Riemann-Liouville state-transition matrix associated to matrix A, i.e., $\Phi_A^r : \Omega_0 \to \mathbb{R}^{d\times d}$ is such that, for every fixed $t \in (a,b]$, $\Phi_A^r(\cdot,t) : [a,t) \to \mathbb{R}^{d\times d}$ solves

$$ {}_\tau\mathrm{D}_t^{\alpha}\Phi_A^r(\tau,t) = \Phi_A^r(\tau,t)A(\tau), \quad \tau \in [a,t), \quad {}_\tau\mathrm{I}_t^{1-\alpha}\Phi_A^r(\tau,t)|_{\tau=t} = \mathbb{I}, \tag{2.22}$$

  where t is fixed end-point and ${}_\tau\mathrm{D}_t^{\alpha}$ represents the right Riemann-Liouville derivative with respect to the variable τ;

- By ${}_A\Phi^r$ we denote the matrix function ${}_A\Phi^r : \Omega_0 \to \mathbb{R}^{d\times d}$ such that, for every fixed $t \in (a,b]$, ${}_A\Phi^r(\tau,\cdot) : (\tau,b] \to \mathbb{R}^{d\times d}$ solves

$$
{}_\tau \mathrm{D}_t^\alpha {}_A\Phi^r(\tau,t) = A(\tau){}_A\Phi^r(\tau,t), \quad \tau \in [a,t), \quad {}_\tau \mathrm{I}_t^{1-\alpha} {}_A\Phi^r(\tau,t)|_{\tau=t} = \mathbb{I}, \quad (2.23)
$$

where t is fixed end-point and ${}_\tau \mathrm{D}_t^\alpha$ represents the right Riemann-Liouville derivative with respect to the variable τ.

Using the same arguments as in the proof of [13, Th. 7], one can derive the following duality property.

2.3.11 Proposition *Let $A \in L^\infty([a,b];\mathbb{R}^{d\times d})$.*

(i) It holds that

$$
{}_A\Phi^l = \Phi^r_A.
$$

More precisely, the left Riemann-Liouville state-transition matrix Φ from Definition 2.3.6 (i) solves both (2.21) and (2.22).

(ii) If matrix function A is such that $A(s)A(t) = A(t)A(s)$, for every $s,t \in [a,b]$, then it holds

$$
{}_A\Phi^l = \Phi^r_A = {}_A\Phi^r.
$$

Let us notice that analog properties can be derived in terms of Peano-Baker series representation of state-transition matrices. Denote by ${}^{PB}{}_A\Phi^l(\tau,t)$ Peano-Baker series given by (2.16), and for every fixed $t \in (a,b]$, define

$$
{}^{PB}\Phi^r_A(\tau,t) := \sum_{k=0}^{\infty} {}_\tau \mathrm{I}_{t,A}^{r,k\circ\alpha} A(\tau), \quad \tau \in [a,t),
$$

where ${}_\tau \mathrm{I}_{t,A}^{r,k\circ 0} A(\tau) = \dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathbb{I}$ and ${}_\tau \mathrm{I}_{t,A}^{r,k\circ\alpha} A(\tau) = {}_\tau \mathrm{I}_t^\alpha\Big(\Big({}_\tau \mathrm{I}_{t,A}^{r,(k-1)\circ\alpha} A(\tau)\Big)A(\tau)\Big)$, for $k \geq 1$, with ${}_\tau \mathrm{I}_t^\alpha$ being the right fractional integral with respect to variable τ.

Then, for every $k \in \mathbb{N}$, by changing the order of integration, we obtain

$$\begin{aligned}
{}_{\tau}\mathrm{I}_t^{k\circ\alpha} A(t) &= \frac{1}{\Gamma(\alpha)^{k+1}} \int_{\tau}^{t} \frac{A(s_1)}{(t-s_1)^{1-\alpha}} \int_{\tau}^{s_1} \frac{A(s_2)}{(s_1-s_2)^{1-\alpha}} \cdots \\
&\quad \cdots \int_{\tau}^{s_{k-2}} \frac{A(s_{k-1})}{(s_{k-2}-s_{k-1})^{1-\alpha}} \int_{\tau}^{s_{k-1}} \frac{A(s_k)(s_k-\tau)^{\alpha-1}}{(s_{k-1}-s_k)^{1-\alpha}}\, ds_k\, ds_{k-1} \cdots ds_2\, ds_1 \\
&= \frac{1}{\Gamma(\alpha)^{k+1}} \int_{\tau}^{t} \int_{s_k}^{t} \cdots \int_{s_3}^{t} \Big(\int_{s_2}^{t} \frac{A(s_1)(t-s_1)^{\alpha-1}}{(s_1-s_2)^{1-\alpha}}\, ds_1 \Big) \frac{A(s_2)}{(s_2-s_3)^{1-\alpha}}\, ds_2 \cdots \\
&\quad \cdots \frac{A(s_{k-1})}{(s_{k-1}-s_k)^{1-\alpha}}\, ds_{k-1} \frac{A(s_k)}{(s_k-\tau)^{1-\alpha}}\, ds_k \\
&= {}_{\tau}\mathrm{I}_{t,A}^{r,k\circ\alpha} A(\tau).
\end{aligned}$$

Hence, we get

$$ {}^{PB}{}_{A}\Phi^{l}(\tau,t) = {}^{PB}\Phi^{r}_{A}(\tau,t). $$

Furthermore, if, for every fixed $t \in (a,b]$, we define

$$ {}^{PB}{}_{A}\Phi^{r}(\tau,t) := \sum_{k=0}^{\infty} {}_{A,\tau}\mathrm{I}_t^{r,k\circ\alpha} A(\tau), \quad \tau \in [a,t), $$

where ${}_{A,\tau}\mathrm{I}_t^{r,k\circ 0} A(\tau) = \dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathbb{I}$ and ${}_{A,\tau}\mathrm{I}_t^{r,k\circ\alpha} A(\tau) = {}_{\tau}\mathrm{I}_t^{\alpha}\Big(A(\tau)\Big({}_{A,\tau}\mathrm{I}_t^{r,(k-1)\circ\alpha} A(\tau)\Big)\Big)$, for $k \geq 1$, with ${}_{\tau}\mathrm{I}_t^{\alpha}$ being the right fractional integral with respect to the variable τ. Then, if matrix A is such that for every $s,t \in [a,b]$, $A(s)A(t) = A(t)A(s)$, we have that, for every $k \in \mathbb{N}$, it holds

$$\begin{aligned}
{}_{\tau}\mathrm{I}_t^{k\circ\alpha} A(t) &= \frac{1}{\Gamma(\alpha)^{k+1}} \int_{\tau}^{t} \frac{A(s_1)}{(t-s_1)^{1-\alpha}} \int_{\tau}^{s_1} \frac{A(s_2)}{(s_1-s_2)^{1-\alpha}} \cdots \\
&\quad \cdots \int_{\tau}^{s_{k-1}} \frac{A(s_k)(s_k-\tau)^{\alpha-1}}{(s_{k-1}-s_k)^{1-\alpha}}\, ds_k \cdots ds_2\, ds_1 \\
&= \frac{1}{\Gamma(\alpha)^{k+1}} \int_{\tau}^{t} \frac{A(s_k)}{(s_k-\tau)^{1-\alpha}} \int_{s_k}^{t} \frac{A(s_{k-1})}{(s_{k-1}-s_k)^{1-\alpha}} \cdots \\
&\quad \cdots \int_{s_2}^{t} \frac{A(s_1)(t-s_1)^{\alpha-1}}{(s_1-s_2)^{1-\alpha}}\, ds_1 \cdots ds_{k-1}\, ds_k \\
&= {}_{A,\tau}\mathrm{I}_t^{r,k\circ\alpha} A(\tau).
\end{aligned}$$

Hence, if $A(t)$ and $A(s)$ commute, we obtain

$$ {}^{PB}{}_{A}\Phi^{l}(\tau,t) = {}^{PB}{}_{A}\Phi^{r}(\tau,t). $$

Now, we want to emphasize one property which follows from derived duality of matrix Φ and which will be substantial for the analysis of adjoint control problem and notion of observability. By transposing system (2.22) and using Proposition 2.3.11 (i), we obtain the following.

2.3.12 Corollary *It holds that*

$$ \left({}_{A}\Phi^{l}\right)^{\mathrm{T}} = {}_{A^{\mathrm{T}}}\Phi^{r}. $$

Furthermore, if Φ is the left Riemann-Liouville state-transition matrix from Definition 2.3.6 (i), then, for every fixed $t \in (a,b]$, function $\Phi(\cdot,t)^{\mathrm{T}} : [a,t) \to \mathbb{R}^{d\times d}$ is the unique solution to the initial value problem

$$ \begin{aligned} {}_{\tau}\mathrm{D}_{t}^{\alpha}\Phi(\tau,t)^{\mathrm{T}} &= A(\tau)^{\mathrm{T}}\Phi(\tau,t)^{\mathrm{T}}, \quad \tau \in [a,t), \\ {}_{\tau}\mathrm{I}_{t}^{1-\alpha}\Phi(\tau,t)^{\mathrm{T}}|_{\tau=t} &= \mathbb{I}. \end{aligned} \tag{2.24} $$

One-dimensional linear FDEs

Let us note that the results presented above still hold in the case of one-dimensional FDEs, i.e., when $d = 1$. In this case, we will use notation ϕ and ψ for the one-dimensional matrices Φ and Ψ.

2.3.13 Examples Let us recall the form of the solutions of linear fractional initial value problems with constant coefficients. Let $k \in \mathbb{R}$.

1. Solution to ${}_{a}^{C}\mathrm{D}_{t}^{\alpha}x(t) = kx(t), \quad t \in (a,b), \quad x(a) = x_a,$ is given by
$$ x(t) = E_{\alpha}(k(t-a)^{\alpha})x_a. $$
If $k = 0$, then $x(t) = x_a$.

2. Solution to ${}_{a}\mathrm{D}_{t}^{\alpha}x(t) = kx(t), \quad t \in (a,b), \quad {}_{a}\mathrm{I}_{t}^{1-\alpha}x(t)|_{t=a} = x_a,$ is given by
$$ x(t) = (t-a)^{\alpha-1}E_{\alpha,\alpha}(k(t-a)^{\alpha})x_a. $$
If $k = 0$, then $x(t) = x_a\dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}$.

3. Solution to ${}_{t}\mathrm{D}_{b}^{\alpha}x(t) = kx(t), \quad t \in (a,b), \quad {}_{t}\mathrm{I}_{b}^{1-\alpha}x(t)|_{t=b} = x_b,$ is given by
$$ x(t) = (b-t)^{\alpha-1}E_{\alpha,\alpha}(k(b-t)^{\alpha})x_b. $$
If $k = 0$, then $x(t) = x_b\dfrac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}$.

In each example, the solution is unique.

Next lemma is a result from [21, Th. 1.4, Th. 1.5. & Th. 1.6.] which states that the sign of the solution of a linear fractional differential equation is determined by the sign of the initial condition.

2.3.14 Lemma *Let $\alpha \in (0,1)$ and $k \in C([a,b];\mathbb{R})$.*

(i) If x is a solution of the initial value problem

$$ {}_a^C\mathrm{D}_t^\alpha x(t) = k(t)x(t), \quad x(a) = x_a > 0, $$

then $x(t) > 0$.

(ii) If x is a solution of the initial value problem

$$ {}_a\mathrm{D}_t^\alpha x(t) = k(t)x(t), \quad {}_a\mathrm{I}_t^{1-\alpha} x(t)|_{t=a} = x_a > 0, $$

then $x(t) > 0$.

(iii) If x is a solution of the initial value problem

$$ {}_t\mathrm{D}_b^\alpha x(t) = k(t)x(t), \quad {}_t\mathrm{I}_b^{1-\alpha} x(t)|_{t=b} = x_b > 0, $$

then $x(t) > 0$.

There are several comparison results regarding solutions of fractional differential equations. For a general type of equation, we refer to [50], and for linear equations the results can be found in [16, 24].

2.3.15 Lemma *Let $\alpha \in (0,1)$ and $a_1, a_2 \in C([a,b];\mathbb{R})$ such that $a_1(t) \leq a_2(t)$, $t \in [a,b]$.*

(i) If $x_i : [a,b] \to \mathbb{R}$, $i = 1,2$, is a solution of the initial value problem

$$ {}_a^C\mathrm{D}_t^\alpha x_i(t) = a_i(t)x_i(t), \quad x_i(a) = x_a, $$

then $|x_1(t)| \leq |x_2(t)|$.

(ii) If $x_i : [a,b] \to \mathbb{R}$, $i = 1,2$, is a solution of the initial value problem

$$ {}_a\mathrm{D}_t^\alpha x_i(t) = a_i(t)x_i(t), \quad {}_a\mathrm{I}_t^{1-\alpha} x_i(t)|_{t=a} = x_a, $$

then $|x_1(t)| \leq |x_2(t)|$.

(iii) If $x_i : [a,b] \to \mathbb{R}$, $i = 1,2$, is a solution of the initial value problem

$$ {}_t\mathrm{D}_b^\alpha x_i(t) = a_i(t)x_i(t), \quad {}_t\mathrm{I}_b^{1-\alpha} x_i(t)|_{t=b} = x_b, $$

then $|x_1(t)| \leq |x_2(t)|$.

Proof. We give the proof for (i), and (ii) and (iii) follow analogously. We analyze different cases, depending on the sign of the initial condition.

(1) If $x_a = 0$, then $x_1(t) = x_2(t) = 0$, for every $t \in [a,b]$.

(2) Suppose that $x_a > 0$. Then Lemma 2.3.14 (i) implies that both x_1 and x_2 are positive functions. Define $\varepsilon(t) := x_2(t) - x_1(t)$. We have that $\varepsilon(0) = 0$ and

$$ \begin{aligned} {}_a^C\mathrm{D}_t^\alpha \varepsilon(t) &= {}_a^C\mathrm{D}_t^\alpha x_2(t) - {}_a^C\mathrm{D}_t^\alpha x_1(t) = a_2(t)x_2(t) - a_1(t)x_1(t) \\ &= a_2(t)\varepsilon(t) + (a_2(t) - a_1(t))x_1(t). \end{aligned} $$

According to Theorem 2.3.7 (iv), $\varepsilon(t) = \int\limits_0^t \phi(\tau,t)(a_2(\tau) - a_1(\tau))x_1(\tau)\, d\tau$, where $\phi(\tau,t)$ satisfies

$$ {}_\tau\mathrm{D}_t^\alpha \phi(\tau,t) = a_2(\tau)\phi(\tau,t), \quad {}_\tau\mathrm{I}_t^{1-\alpha}\phi(\tau,t)|_{\tau=t} = 1. $$

Hence, from Lemma 2.3.14 (iii) we have that $\phi(\tau,t) > 0$, $\tau \in [a,t]$, which together with $a_2(\tau) - a_1(\tau) \geq 0$ and $x_1(\tau) > 0$ implies that $\varepsilon(t) \geq 0$. Therefore, $x_2(t) \geq x_1(t)$.

(3) If $x_a < 0$, then from Lemma 2.3.14 (i) we have that x_1 and x_2 are negative functions. By defining $\varepsilon(t) := x_2(t) - x_1(t)$, and proceeding as in (2), we obtain $\varepsilon(t) = \int\limits_0^t \phi(\tau,t)(a_2(\tau) - a_1(\tau))x_1(\tau)\, d\tau$, where $\phi(\tau,t) > 0$, $a_2(\tau) - a_1(\tau) \geq 0$ and $x_1(\tau) < 0$. Therefore, $\varepsilon(t) \leq 0$ and $x_2(t) \leq x_1(t) < 0$.

Now, (1), (2) and (3) imply $|x_1(t)| \leq |x_2(t)|$. □

Lower and upper bounds for Ψ and Φ

In order to better understand the behaviour of the solution of the system of FDEs, we need to analyze closely the state-transition matrices Φ and Ψ.

Due to the nature of the Riemann-Liouville derivative, the function $\Phi(\tau,t)$ has singularities along the line $\tau = t$ (see for example (2.18)). That is the reason why we need to be careful when examining its regularity. In [25] the author considered a modification of Φ, given by

$$ F(\tau,t) = (t-\tau)^{1-\alpha}\Phi(\tau,t), \quad (\tau,t) \in \Omega_0, $$

and

$$ F(\tau,\tau) = \lim_{t\to\tau}(t-\tau)^{1-\alpha}\Phi(\tau,t) = \frac{1}{\Gamma(\alpha)} {}_\tau\mathrm{I}_t^{1-\alpha}\Phi(\tau,t)|_{t\to\tau} = \frac{\mathbb{I}}{\Gamma(\alpha)}, \quad \tau \in [a,b]. $$

and derived the following properties.

2.3.16 Proposition *There exist $M_F > 0$ and $H_F > 0$ such that*

(i) $\|F(\tau,t)\| \leq M_F$, *for every* $(\tau,t) \in \Omega$*;*

(ii) $\|F(\tau_1,t_1)-F(\tau_2,t_2)\| \leq H_F(|\tau_1-\tau_2|^\alpha+|t_1-t_2|^\alpha)$, *for every* $(\tau_1,t_1),(\tau_2,t_2) \in \Omega$.

In particular, the function $(\tau,t) \mapsto F(\tau,t)$ *is continuous on* Ω.

From the proofs of [25, Prop. 4.1. & Prop. 4.2.], we can explicitly derive the values of constants M_F and H_F:

$$M_F = \frac{e^{(b-a)k}}{\Gamma(\alpha)(1-k^{-\alpha}M_A M_J)} \quad \text{and} \quad H_F = H_J M_A M_F E_\alpha((b-a)^\alpha M_A M_J),$$

where $M_A = \sup\limits_{t\in[a,b]} \|A(t)\|$, $M_J = 1+\frac{\sin(\alpha\pi)}{\alpha\pi}$, $H_J = \frac{2M_J}{\Gamma(\alpha+1)}$, and k is a positive number such that $\dfrac{M_A M_J}{k^\alpha} < 1$.

Let us mention that in [13, Lemma 5], similar estimates were established for matrix Φ.

2.3.17 Proposition *There exists $\Theta \geq 0$ such that*

$$|\Phi_{i,j}(\tau,t)| \leq (t-\tau)^{\alpha-1}\Theta,$$

for almost every $a \leq \tau < t \leq b$ and for every $i,j \in \{1,\ldots,d\}$.

As one can infer from the proof of [13, Lemma 5], the author proved the existence of the upper bound Θ, without its precise calculation.

In [35] sharper estimates for Φ and Ψ were obtained, for some special classes of systems (2.10) and (2.11). In the sequel, we present the results from [35, Prop. 1 & Rem. 3] in more details.

2.3.18 Remark (**Diagonalization**) If A is a real symmetric matrix it can be diagonalized, i.e., there exists a diagonal matrix

$$D = \operatorname{diag}(\lambda_1,\ldots,\lambda_d), \quad \lambda_i \in \mathbb{R}, \quad i=1,\ldots,d,$$

and an orthogonal matrix U such that

$$A = UDU^{-1} = UDU^{\mathrm{T}}.$$

Elements on the main diagonal of D are the eigenvalues of A, and if A is positive semidefinite matrix, then $\lambda_i \geq 0$, $i=1,\ldots,d$. Throughout the thesis, we will often use this property to diagonalize symmetric systems.

2.3.19 Proposition *Let A be a real symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_d$, $g \in C([a,b];\mathbb{R})$, and let $\lambda_{\max} = \max_{1\le i\le d} |\lambda_i|$ and $M \ge 0$ be such that $|g(t)| \le M$, $t \in [a,b]$.*

(i) If $\Psi(a,t)$ is a solution of the initial value problem

$$ {}_a^C\mathrm{D}_t^\alpha \Psi(a,t) = Ag(t)\Psi(a,t), \quad t \in (a,b), \quad \Psi(a,a) = \mathbb{I}, \tag{2.25}$$

then

$$E_\alpha(-\lambda_{\max} M(b-a)^\alpha) \le \|\Psi(a,t)\| \le E_\alpha(\lambda_{\max} M(b-a)^\alpha), \quad t \in [a,b]. \tag{2.26}$$

(ii) Let $t \in (a,b]$. If $\Phi(\tau,t)$ is a solution of the initial value problem

$$ {}_\tau\mathrm{D}_t^\alpha \Phi(\tau,t) = Ag(\tau)\Phi(\tau,t), \quad \tau \in (a,t), \quad {}_\tau\mathrm{I}_t^{1-\alpha}\Phi(\tau,t)|_{\tau=t} = \mathbb{I}, \tag{2.27}$$

then

$$E_{\alpha,\alpha}(-\lambda_{\max} M(t-\tau)^\alpha) \le \|(t-\tau)^{1-\alpha}\Phi(\tau,t)\| \le E_{\alpha,\alpha}(\lambda_{\max} M(t-\tau)^\alpha), \quad \tau \in (a,t). \tag{2.28}$$

Proof. (i) From diagonalization $A = UDU^{\mathrm{T}}$, it follows that the solution of (2.25) can be written as $\Psi(a,t) = UP(a,t)U^{\mathrm{T}}$, where $P(a,t)$ is a solution of

$$ {}_a^C\mathrm{D}_t^\alpha P(a,t) = Dg(t)P(a,t), \quad P(a,a) = \mathbb{I}.$$

Since D and $\mathbb{I}$ are diagonal, $P(a,t) = \mathrm{diag}(p_1(a,t), \dots, p_d(a,t))$ with $p_i(a,t)$ satisfying

$$ {}_a^C\mathrm{D}_t^\alpha p_i(a,t) = \lambda_i g(t) p_i(a,t), \quad p_i(a,a) = 1, \quad i = 1, ..., d.$$

Using $-\lambda_{\max} M \le \lambda_i g(t) \le \lambda_{\max} M$, Lemma 2.3.15 (i), and Example 2.3.13 1., we obtain $E_\alpha(-\lambda_{\max} M(t-a)^\alpha) \le |p_i(a,t)| \le E_\alpha(\lambda_{\max} M(t-a)^\alpha)$, for every $i = 1, ..., d$. Furthermore, Lemma 2.3.14 (i) and complete monotonicity of $E_\alpha(-x)$ provide the following estimates

$$E_\alpha(-\lambda_{\max} M(b-a)^\alpha) \le p_i(a,t) \le E_\alpha(\lambda_{\max} M(b-a)^\alpha). \tag{2.29}$$

Hence, $E_\alpha(-\lambda M(b-a)^\alpha) \le \|P(a,t)\| \le E_\alpha(\lambda_{\max} M(b-a)^\alpha)$ and

$$\|\Psi(a,t)\| \le \|U\|\|P(a,t)\|\|U^{\mathrm{T}}\| \le E_\alpha(\lambda_{\max} M(b-a)^\alpha). \tag{2.30}$$

Next, we consider inverse matrix of $\Psi(a,t)$. It is given by $\Psi(a,t)^{-1} = UP(a,t)^{-1}U^{\mathrm{T}}$, where $P(a,t)^{-1} = \mathrm{diag}\Big(\frac{1}{p_1(a,t)}, \dots, \frac{1}{p_d(a,t)}\Big)$. Now, (2.29) implies

$$\frac{1}{E_\alpha(\lambda_{\max} M(b-a)^\alpha)} \le \|P(a,t)^{-1}\| \le \frac{1}{E_\alpha(-\lambda_{\max} M(b-a)^\alpha)}$$

and
$$\|\Psi(a,t)^{-1}\| \le \|U\|\|P(a,t)^{-1}\|\|U^{\mathrm{T}}\| \le \frac{1}{E_\alpha(-\lambda_{\max}M(b-a)^\alpha)}.$$
By using $1 \le \|\Psi(a,t)\|\|\Psi(a,t)^{-1}\|$, we get
$$\|\Psi(a,t)\| \ge \|\Psi(a,t)^{-1}\|^{-1} \ge E_\alpha(-\lambda_{\max}M(b-a)^\alpha),$$
which together with (2.30) gives (2.26).

(ii) The proof of this part follows the same lines as the previous one. Starting with the diagonalization, we express the solution of (2.27) by $\Phi(\tau,t) = UQ(\tau,t)U^{\mathrm{T}}$, where $Q(\tau,t) = \mathrm{diag}(q_1(\tau,t),\dots,q_d(\tau,t))$ and $q_i(\tau,t)$ satisfy
$${}_\tau\mathrm{D}_t^\alpha q_i(\tau,t) = \lambda_i g(\tau) q_i(\tau,t), \quad \tau\in(a,t), \quad {}_\tau\mathrm{I}_t^{1-\alpha} q_i(\tau,t)|_{\tau=t} = 1, \quad i=1,...,d.$$
Using that $-\lambda_{\max}M \le \lambda_i g(t) \le \lambda_{\max}M$, Lemma 2.3.15 (iii), Example 2.3.13 3. and complete monotonicity of $E_{\alpha,\alpha}(-x)$, we conclude that for every $i=1,...,d$ it holds
$$E_{\alpha,\alpha}(-\lambda_{\max}M(t-\tau)^\alpha) \le (t-\tau)^{1-\alpha} q_i(\tau,t) \le E_{\alpha,\alpha}(\lambda_{\max}M(t-\tau)^\alpha), \quad \tau\in(a,t), \tag{2.31}$$
which further implies (2.28). □

When we impose stronger conditions on the definiteness of the matrix of the system, we are able to derive stronger bounds.

2.3.20 Proposition *Let A be a real, symmetric and positive semidefinite matrix with eigenvalues $\lambda_1,\dots,\lambda_d$, $g\in C([a,b];[0,\infty))$, and let $\lambda = \max\limits_{1\le i\le d}\lambda_i$ and $M\ge 0$ be such that $g(t)\le M$, $t\in[a,b]$.*

(i) If $\Psi(a,t)$ is a solution of the initial value problem
$${}_a^C\mathrm{D}_t^\alpha \Psi(a,t) = -Ag(t)\Psi(a,t), \quad t\in(a,b), \quad \Psi(a,a) = \mathbb{I}, \tag{2.32}$$
then $\quad E_\alpha(-\lambda M(b-a)^\alpha) \le \|\Psi(a,t)\| \le 1, \quad t\in[a,b].$

(ii) Let $t\in(a,b]$. If $\Phi(\tau,t)$ is a solution of the initial value problem
$${}_\tau\mathrm{D}_t^\alpha \Phi(\tau,t) = -Ag(\tau)\Phi(\tau,t), \quad \tau\in(a,t), \quad {}_\tau\mathrm{I}_t^{1-\alpha}\Phi(\tau,t)|_{\tau=t} = \mathbb{I}, \tag{2.33}$$
then $\quad E_{\alpha,\alpha}(-\lambda M(t-\tau)^\alpha) \le \|(t-\tau)^{1-\alpha}\Phi(\tau,t)\| \le 1, \quad \tau\in(a,t).$

Proof. (i) Again, by diagonalizing the system, we express the solution of (2.32) in the form $\Psi(a,t) = UR(a,t)U^{\mathrm{T}}$, where $R(a,t)$ satisfies
$${}_a^C\mathrm{D}_t^\alpha R(a,t) = -Dg(t)R(a,t), \quad R(a,a) = \mathbb{I}.$$

Furthermore, $R(a,t) = \mathrm{diag}(r_1(a,t),\dots,r_d(a,t))$ with $r_i(a,t)$ satisfying

$$ {}_a^C\mathrm{D}_t^\alpha r_i(a,t) = -\lambda_i g(t) r_i(a,t), \quad r_i(a,a) = 1, \quad i = 1,...,d. $$

Since now $-\lambda M \le -\lambda_i g(t) \le 0$, we obtain

$$ E_\alpha(-\lambda M(b-a)^\alpha) \le r_i(a,t) \le 1, \tag{2.34} $$

and $E_\alpha(-\lambda M(b-a)^\alpha) \le \|R(a,t)\| \le 1$. Therefore, using the same arguments as in the proof of Proposition 2.3.19 (i), we conclude

$$ E_\alpha(-\lambda M(b-a)^\alpha) \le \|\Psi(a,t)\| \le 1. $$

(ii) Can be proved analogously. □

2.3.3 A nonlinear system

Based on the previous results, we are able to prove existence of a solution to a nonlinear system of FDEs, which is substantial for the control problem that we are going to consider.

First, we give an auxiliary result form [35, Lemma 4].

2.3.21 Lemma *Let $x_a \in \mathbb{R}^d$ and $X \subset C([a,b];\mathbb{R}^d)$ be such that*

(i) for every $x \in X$, $x(a) = x_a$;

(ii) there exists a constant $C > 0$ such that for every $x \in X$, $\sup_{t\in[a,b]} |{}_a^C\mathrm{D}_t^\alpha x(t)| \le C$.

Then X is bounded in $H^\alpha([a,b];\mathbb{R}^d)$ and equicontinuous equicontinuous subset of $C([a,b];\mathbb{R}^d)$.

Proof. From (2.1) and assumption (i), it follows that for every $x \in X$,

$$ x(t) - x_a = {}_a\mathrm{I}_t^\alpha \left({}_a^C\mathrm{D}_t^\alpha x(t) \right), \quad t \in [a,b]. \tag{2.35} $$

Further, the assumption (ii) implies that the set $\{{}_a^C\mathrm{D}_t^\alpha x(t) : x \in X\}$ is bounded in $L^\infty([a,b];\mathbb{R}^d)$. Then, from Theorem 2.1.2 and (2.35) it follows that $\{x - x_a : x \in X\}$ is bounded in $H^\alpha([a,b];\mathbb{R}^d)$. Since x_a is a constant vector, we have that X is also bounded in $H^\alpha([a,b];\mathbb{R}^d)$. Hence, there exists a constant $K_\alpha > 0$ (not depending on x) such that for every $x \in X$, $\|x\|_{H^\alpha} \le K_\alpha$, implying that for every $x \in X$ and for every $t_1, t_2 \in [a,b]$ it holds that

$$ |x(t_1) - x(t_2)| \le K_\alpha |t_1 - t_2|^\alpha. \tag{2.36} $$

Let $\varepsilon > 0$. By taking $\delta = \left(\frac{\varepsilon}{K_\alpha}\right)^{\frac{1}{\alpha}}$ and using (2.36) we obtain

$$(\forall x \in X)(\forall t_1, t_2 \in [a,b])(|t_1 - t_2| < \delta \Rightarrow |x(t_1) - x(t_2)| < \varepsilon), \tag{2.37}$$

implying that X is an equicontinuous subset of $C([a,b];\mathbb{R}^d)$. □

Now we present the main result from [35].

2.3.22 Theorem *Let $A \in M^{d\times d}$ be a symmetric, positive semidefinite matrix, and $f : \mathbb{R}^d \to [0,\infty)$ a continuous function. Then, the Cauchy problem*

$$\begin{aligned} {}_a^C\mathrm{D}_t^\alpha z(t) &= -\,Af(z(t))\,z(t), \quad t \in [a,b], \\ z(a) &= z_a, \end{aligned} \tag{2.38}$$

has a solution $z \in C([a,b];\mathbb{R}^d)$. Moreover, if f is such that $F(z) = f(z)z$ is continuously differentiable on $\mathbb{R}^d$, then $z \in C([a,b];\mathbb{R}^d) \cap C^1((a,b];\mathbb{R}^d)$, $z'(t) = O((t-a)^{\alpha-1})$, as $t \to a$, and the solution is unique.

Proof. We use the Leray-Schauder fixed point theorem. Let us consider the mapping $\mathcal{T} : C([a,b];\mathbb{R}^d) \to C([a,b];\mathbb{R}^d)$, $\mathcal{T}(v) = z$, where z is a solution to

$$\begin{aligned} {}_a^C\mathrm{D}_t^\alpha z(t) &= -\,Af(v(t))\,z(t), \quad t \in [a,b], \\ z(a) &= z_a. \end{aligned} \tag{2.39}$$

From (2.15) it follows that $z(t) = \Psi(a,t)z_a$ and Proposition 2.3.19 (i) implies

$$|z(t)| \le |z_a|, \quad t \in [a,b]. \tag{2.40}$$

We shall show that the mapping $\mathcal{T}$ admits a fixed point. According to the Leray-Schauder fixed point theorem, we need to prove that

(i) the mapping $\mathcal{T}$ is continuous and compact;

(ii) the set of the solutions of $z = \lambda\mathcal{T}(z)$, $\lambda \in [0,1]$, is bounded.

Since the solutions of the Cauchy problem (2.39) depend continuously on the coefficients $-Af(v)$, we have that $\mathcal{T}$ is continuous. For the compactness, it suffices to show that $\mathcal{T}$ maps bounded sets into relatively compact sets. Let $V \subset C([a,b];\mathbb{R}^d)$ be a bounded set such that, for every $v \in V$, $\|v\|_{C([a,b];\mathbb{R}^d)} \le K$, i.e., $\max\limits_{t\in[a,b]} |v(t)| \le K$. Then (2.40) implies that $\mathcal{T}(V)$ is uniformly bounded by $|z_a|$. Furthermore, continuity of $f : \mathbb{R}^d \to [0,\infty)$ implies that there exists $M > 0$ such that $M = \max\limits_{y\in B_K} f(y)$, where $B_K = \{y \in \mathbb{R}^d : |y| \le K\}$. Then for every $v \in V$ we have $\max\limits_{t\in[a,b]} |f(v(t))| \le M$. Hence, if $z \in \mathcal{T}(V)$ then for the fractional derivative ${}_a^C\mathrm{D}_t^\alpha z$ it holds

$$|{}_a^C\mathrm{D}_t^\alpha z| \le \| - A\| \, |f(v(t))| \, |z(t)| \le \|A\| M |z_a|,$$

i.e., it is uniformly bounded by $\|A\|M|z_a|$, and by Lemma 2.3.21 we have that $\mathcal{T}(V)$ is equicontinuous set in $C([a,b];\mathbb{R}^d)$. Therefore, $\mathcal{T}(V)$ is uniformly bounded and equicontinuous, and by the Arzela-Ascolli theorem, $\mathcal{T}(V)$ is relatively compact. From (2.40) we see that (ii) is satisfied as well. Thus, according to the Leray-Schauder fixed point theorem, $\mathcal{T}$ has a fixed point, i.e., problem (2.38) has a solution $z \in C([a,b];\mathbb{R}^d)$.

Moreover, if we have that $F(z) = f(z)z$ is continuously differentiable on $\mathbb{R}^d$, then according to Theorem 2.3.5, the solution is unique, $z \in C([a,b];\mathbb{R}^d) \cap C^1((a,b];\mathbb{R}^d)$ and $z'(t) = O((t-a)^{\alpha-1})$, as $t \to a$. □

2.3.23 Remark From the proof of Theorem 2.3.22, it follows that the solution z satisfies $|z(t)| \le |z_a|$, $t \in [a,b]$.

2.3.24 Remark In our nonlinear system we considered the Caputo fractional derivative, in order to avoid singularity at the origin, which arises in the case of the Riemann-Liouville derivative. If we want to consider the Riemann-Liouville version of (2.38) given by:

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha z(t) &= - A f(z(t))\, z(t), \quad t \in [a,b], \\ {}_a\mathrm{I}_t^{1-\alpha} z(t)|_{t=a} &= z_a, \end{aligned}$$

then we need to look for the solution in the weighted space $C_{1-\alpha}([a,b];\mathbb{R}^d)$. In that case, the function $f(v(t))$, with $v \in C_{1-\alpha}([a,b];\mathbb{R}^d)$, will not necessarily be continuous on $[a,b]$, because of the nonlinearity of f. That will lead to a problem with the definition of the mapping $\mathcal{T} : v \mapsto z$.

Another reason for using the Caputo derivative is the nature of the initial condition. In the Caputo-type problems, the initial condition is given by the initial value of the solution, while, in the case of the Riemann-Liouville derivative, the initial condition is given by ${}_a\mathrm{I}_t^{1-\alpha} z(t)|_{t=a}$, and it does not have a natural physical interpretation.

Chapter 3

Linear Control

This chapter is devoted to the control theory of linear systems. A systematic and detailed analysis of linear control started in the 1960s with Rudolf Kalman and his papers [40, 41, 42], where necessary and sufficient conditions for controllability were established. Then followed a great expansion of the linear control in both theoretical and applied domain. Today there is a vast number of literature in this field. For a review of classical results, we refer to the following books [1, 14, 23, 38, 60, 63, 64].

In the case of linear control problems with fractional derivatives, most of the known results are related to the systems with constant coefficients. First result, regarding controllability and observability of linear time-invariant systems, was published in [52]. Later, several authors considered similar problems (see, for example, [6, 12]). Over the last 30 years, research in fractional order systems is of a great interest since for some processes they provide a more accurate models than the one with integer order derivatives. For an overview of the results in linear fractional control and their applications in engineering and industry, we refer to the books [37, 55].

This chapter contains two sections, which are organized in the following way:
In Section 3.1 we recall some of the classical results for linear control systems with integer order derivatives. In addition, we provide a proof for the equivalent condition for controllability in the case when the matrix of the system is of the form $A(t) = Ag(t)$, where A is a constant matrix and g is a scalar function.
Section 3.2 is devoted to the systems with non-integer derivatives. Using methods from control theory of linear systems with integer-order derivative, and adapting them to the fractional setting, we derive and prove new results in linear fractional control. The majority of this section is based on the original results published in [33], so we cite this paper as the source of the results and proofs unless stated otherwise.

3.1 Control theory for linear systems of ODEs

The results that will be presented in this section are based on the Chapter 1 of the book [17] and Section 2 of [65].

We start with a linear control problem

$$\begin{aligned} y'(t) &= A(t)y(t) + B(t)u(t), \quad t \in [a,b] \\ y(a) &= y_a, \end{aligned} \tag{3.1}$$

where $A : [a,b] \to \mathbb{R}^{d\times d}$ and $B : [a,b] \to \mathbb{R}^{d\times N}$ are continuous matrix functions, i.e., $A \in C([a,b];\mathbb{R}^{d\times d})$ and $B \in C([a,b];\mathbb{R}^{d\times N})$.

We recall some classical results from the theory of linear systems of ODEs. First, we define the state-transition matrix (principal solution matrix) related to the system

$$y' = A(t)y$$

as the matrix function $\Pi : [a,b] \times [a,b] \to \mathbb{R}^{d\times d}$, $(\tau,t) \mapsto \Pi(\tau,t)$, such that for any fixed $\tau \in [a,b)$, matrix $\Pi(\tau,t)$ is a solution to the Cauchy problem

$$\begin{aligned} \frac{d}{dt}\Pi(\tau,t) =& A(t)\Pi(\tau,t), \quad t \in [\tau,b], \\ \Pi(\tau,\tau) =& \mathbb{I}. \end{aligned} \tag{3.2}$$

3.1.1 Theorem *Let $A \in C([a,b];\mathbb{R}^{d\times d})$, $B \in C([a,b];\mathbb{R}^{d\times N})$, $u \in L^1([a,b];\mathbb{R}^N)$ and $y_a \in \mathbb{R}^d$.*

(i) The state-transition matrix $\Pi(\tau,t)$ is a unique solution to (3.2). Furthermore, $\Pi \in C^1([a,b]^2;\mathbb{R}^{d\times d})$ and it satisfies the following properties:

(1) $\Pi(t,t) = \mathbb{I}$, for every $t \in [a,b]$;

(2) $\Pi(\tau,s)\Pi(s,t) = \Pi(\tau,t)$, for every $\tau,s,t \in [a,b]$.

(3) $\Pi(\tau,t)$ is nonsingular and $\Pi(\tau,t)^{-1} = \Pi(t,\tau)$, for every $\tau,t \in [a,b]$.

(ii) For any fixed $t \in (a,b]$, matrix $\Pi(\tau,t)$ is a unique solution to the Cauchy problem

$$\begin{aligned} \frac{d}{d\tau}\Pi(\tau,t) =& -\Pi(\tau,t)A(\tau), \quad \tau \in [a,t], \\ \Pi(t,t) =& \mathbb{I}. \end{aligned} \tag{3.3}$$

(iii) The unique solution of the initial value problem

$$\begin{aligned} y'(t) =& A(t)y(t), \quad t \in [a,b], \\ y(a) =& y_a \end{aligned} \tag{3.4}$$

is given by $y(t) = \Pi(a,t)y_a$, and $y \in C^1([a,b];\mathbb{R}^d)$.

(iv) The unique solution of (3.1) *is given by*

$$y(t) = \Pi(a,t)y_a + \int\limits_a^t \Pi(\tau,t)B(\tau)u(\tau)d\tau \tag{3.5}$$

and $y \in C([a,b];\mathbb{R}^d)$.

If $A(t)$ is such that $A(t)A(s) = A(s)A(t)$, for every $s,t \in [a,b]$, we can express Π in terms of a matrix exponential function:

$$\Pi(\tau,t) = e^{\int\limits_\tau^t A(s)ds}.$$

3.1.1 Controllability

In order to derive equivalent conditions for controllability, we need to introduce the controllability Gramian.

3.1.2 Definition *The controllability Gramian of the control system* (3.5) *is the symmetric* $d \times d$ *matrix*

$$W(a,b) = \int\limits_a^b \Pi(t,b)B(t)B(t)^{\mathrm{T}}\Pi(t,b)^{\mathrm{T}}\,dt. \tag{3.6}$$

Let us notice that controllability Gramian is a positive semidefinite matrix, since

$$x^{\mathrm{T}}W(a,b)x = \int\limits_a^b x^{\mathrm{T}}\Pi(t,b)B(t)B(t)^{\mathrm{T}}\Pi(t,b)^{\mathrm{T}}x\,dt = \int\limits_a^b |x^{\mathrm{T}}\Pi(t,b)B(t)|^2\,dt \geq 0,$$

for any $x \in \mathbb{R}^d$. Therefore, the Gramian is nonsingular if and only if it is a positive definite matrix.

3.1.3 Theorem *The linear control system* (3.1) *is controllable if and only if its controllability Gramian is invertible. Furthermore, the control function which steers the solution from the state* y_a *to* y_b *during the time interval* $[a,b]$ *is given by*

$$\overline{u}(t) = B(t)^{\mathrm{T}}\Pi(t,b)^{\mathrm{T}}W(a,b)^{-1}(y_b - \Pi(a,b)y_a), \tag{3.7}$$

and it is the control with minimal L^2*-norm, i.e., if* $u \in L^2([a,b];\mathbb{R}^N)$ *is a control function such that the solution of* (3.1) *satisfies* $y(b) = y_b$*, then*

$$\int\limits_a^b |\overline{u}(t)|^2\,dt \leq \int\limits_a^b |u(t)|^2\,dt,$$

with equality if and only if u *coincides with* $\overline{u}$ *almost everywhere on* $[a,b]$.

Proof. Let $y_a, y_b \in \mathbb{R}^d$. If controllability Gramian $W(a,b)$ of system (3.1) is invertible, then control function (3.7) is well defined and by (3.5), the solution of the Cauchy problem

$$\begin{aligned}\overline{y}'(t) &= A(t)\overline{y}(t) + B(t)\overline{u}(t), \quad t \in [a,b]\\ \overline{y}(a) &= y_a,\end{aligned}$$

satisfies

$$\begin{aligned}\overline{y}(b) &= \Pi(a,b)y_a + \int_a^b \Pi(t,b)B(t)\overline{u}(t)\,dt\\ &= \Pi(a,b)y_a + \int_a^b \Pi(t,b)B(t)B(t)^{\mathrm{T}}\Pi(t,b)^{\mathrm{T}}W(a,b)^{-1}(y_b - \Pi(a,b)y_a)\,dt\\ &= \Pi(a,b)y_a + W(a,b)W(a,b)^{-1}(y_b - \Pi(a,b)y_a)\\ &= y_b.\end{aligned}$$

Hence, the system (3.1) is controllable. To prove other implication, assume that the system (3.1) is controllable and that $W(a,b)$ is not invertible. Then there exists $w \in \mathbb{R}^d \setminus \{0\}$ such that $w^{\mathrm{T}}W(a,b)w = 0$. Therefore

$$\int_a^b |w^{\mathrm{T}}\Pi(t,b)B(t)|^2\,dt = \int_a^b w^{\mathrm{T}}\Pi(t,b)B(t)B(t)^{\mathrm{T}}\Pi(t,b)^{\mathrm{T}}w\,dt = 0,$$

which implies

$$w^{\mathrm{T}}\Pi(t,b)B(t) = 0, \quad \text{almost everywhere on } [a,b]. \tag{3.8}$$

On the other hand, the controllability assumption implies that for initial state $y_a = 0$ and final state $y_b = w$ there exists a control function u_0 such that the solution of the system (3.1) satisfies $y(b) = w$. Hence, (3.5) and (3.8) imply

$$w^{\mathrm{T}}w = w^{\mathrm{T}}y(b) = \int_a^b w^{\mathrm{T}}\Pi(t,b)B(t)u_0(t)\,dt = 0,$$

which leads to a contradiction with $w \neq 0$.

To prove L^2-optimality of $\overline{u}$, let $u \in L^2([a,b];\mathbb{R}^N)$ be a control function such that the solution of (3.1) satisfies $y(b) = y_b$, and define $v := u - \overline{u}$. Then (3.5) implies

$$\begin{aligned}\int_a^b \Pi(t,b)B(t)v(t)\,dt &= \int_a^b \Pi(t,b)B(t)u(t)\,dt - \int_a^b \Pi(t,b)B(t)\overline{u}(t)\,dt\\ &= (y_b - \Pi(a,b)y_a) - (y_b - \Pi(a,b)y_a)\\ &= 0.\end{aligned}$$

Having in mind (3.7), we conclude

$$\begin{aligned}\int_a^b \langle \overline{u}(t), v(t)\rangle\, dt &= \int_a^b \langle B(t)^{\mathrm{T}}\Pi(t,b)^{\mathrm{T}}W(a,b)^{-1}(y_b - \Pi(a,b)y_a), v(t)\rangle\, dt \\ &= \langle W(a,b)^{-1}(y_b - \Pi(a,b)y_a), \int_a^b \Pi(t,b)B(t)v(t)\, dt\rangle \\ &= 0\end{aligned}$$

Then, using

$$\int_a^b |u(t)|^2\, dt = \int_a^b |\overline{u}(t)|^2\, dt + \int_a^b |v(t)|^2\, dt + 2\int_a^b \langle \overline{u}(t), v(t)\rangle\, dt,$$

we obtain

$$\int_a^b |\overline{u}(t)|^2\, dt = \int_a^b |u(t)|^2\, dt - \int_a^b |u(t) - \overline{u}(t)|^2\, dt,$$

which concludes the proof. □

The result from Theorem 3.1.3 tells us that in order to prove (or disprove) controllability of the system, we need to compute the matrix $W(a,b)$ and its inverse (if possible). These computations, in many cases, might be very difficult to perform. So, naturally, the question arises: Is there any simpler way to check whether the system is controllable or not? And the answer is: Yes, for some special class of the system.

Firstly, for the systems with constant coefficients we have a simple algebraic criterion, today widely known as the Kalman rank condition, given in the following theorem.

3.1.4 Theorem *Linear time-invariant system*

$$y'(t) = Ay(t) + Bu(t), \quad t \in [a,b], \tag{3.9}$$

is controllable if and only if for matrices A and B it holds

$$\mathrm{rank}[B|AB|A^2B|\cdots|A^{d-1}B] = d. \tag{3.10}$$

Proof. For system (3.9) we have

$$\Pi(\tau,t) = e^{(t-\tau)A}, \quad (\tau,t) \in [a,b]^2,$$

and

$$W(a,b) = \int_a^b e^{(b-t)A} B B^{\mathrm{T}} e^{(b-t)A^{\mathrm{T}}}\, dt. \tag{3.11}$$

First we show that the Kalman rank is a sufficient condition for controllability. Assume that system (3.9) is not controllable. Then, Theorem (3.1.3) implies that the Gramian $W(a,b)$ is singular, specifically that there exists $w \in \mathbb{R}^d \setminus \{0\}$ such that

$$\int_a^b |w^{\mathrm{T}} e^{(b-t)A} B|^2\, dt = w^{\mathrm{T}} W(a,b) w = 0.$$

Therefore, function $k : [a,b] \to \mathbb{R}^N$ given by

$$k(t) := w^{\mathrm{T}} e^{(b-t)A} B, \tag{3.12}$$

satisfies

$$k(t) = 0, \quad t \in [a,b]. \tag{3.13}$$

Differentiating k $i-$times and using (3.13), we obtain

$$k^{(i)}(b) = (-1)^i w^{\mathrm{T}} A^i B = 0, \quad i = 0, \ldots, d.$$

Hence, for some $w^{\mathrm{T}} \neq 0$ we have

$$w^{\mathrm{T}}[B|AB|\cdots|A^{d-1}B] = [w^{\mathrm{T}}B|w^{\mathrm{T}}AB|\cdots|w^{\mathrm{T}}A^{d-1}B] = 0,$$

which contradicts (3.10).

In order to prove that the Kalman rank is necessary condition for controllability, it suffices to show that if (3.10) does not hold then the Gramian (3.11) is singular. Let $w \in \mathbb{R}^d \setminus \{0\}$ be such that

$$w^{\mathrm{T}} A^i B = 0, \quad \text{for every } i = 0, \ldots, d. \tag{3.14}$$

If $P_A(x) = x^d - \alpha_d x^{d-1} \cdots - \alpha_2 x - \alpha_1$ is the characteristic polynomial of the matrix A, then the Cayley-Hamilton theorem implies $P_A(A) = 0$, i.e.,

$$A^d = \alpha_d A^{d-1} \cdots - \alpha_2 A - \alpha_1 \mathbb{I}. \tag{3.15}$$

Now, by induction, from (3.14) and (3.15) it follows

$$w^{\mathrm{T}} A^i B = 0, \quad \text{for every } i \in \mathbb{N}.$$

Therefore, function (3.12) satisfies $k^{(i)}(b) = (-1)^i w^{\mathrm{T}} A^i B = 0$, for every $i \in \mathbb{N}$, and since k is analytic on $[a,b]$, we get $k = 0$ on $[a,b]$. Hence, $w^{\mathrm{T}} W(a,b) w = \int\limits_a^b |k(t)|^2\,dt = 0$, which further implies $W(a,b)w = 0$, for $w \neq 0$, i.e., $W(a,b)$ is singular. □

Now we focus on the following type of linear systems:

$$\begin{aligned} y'(t) =& Ag(t)y(t) + Bu(t), \quad t \in [a,b] \\ y(a) =& y_a, \end{aligned} \tag{3.16}$$

where A and B are constant matrices and $g : [a,b] \to \mathbb{R}$ is a continuous function such that $g(t) \neq 0$, for almost every $t \in [a,b]$. Let us note that state-transition matrix of this system is $\Pi(\tau,t) = e^{A \int_\tau^t g(s)\,ds}$. The following proposition is a modification of the result from [19, Prop. 2.3].

3.1.5 Proposition *System (3.16) is controllable if and only if A and B satisfy the Kalman rank condition (3.10).*

Proof. ($\Rightarrow$) Suppose that the system is controllable and that

$$\operatorname{rank}[B|AB|A^2B|\cdots|A^{d-1}B] < d.$$

Then there exists a column vector $q \in \mathbb{R}^d \setminus \{0\}$ such that

$$qB = 0,\ qAB = 0,\ \ldots\ qA^{d-1}B = 0. \tag{3.17}$$

Furthermore, controllability implies that for $y_b = 0$ there exists a control function u_0 such that the solution of the system satisfies $y(b) = 0$. Then, from (3.5) we have

$$0 = y(b) = \Pi(a,b)y_a + \int\limits_a^b \Pi(\tau,b)Bu_0(\tau)d\tau.$$

Therefore,

$$-\Pi(a,b)y_a = \int\limits_a^b \Pi(\tau,b)Bu_0(\tau)\,d\tau = \int\limits_a^b \sum_{k=0}^{\infty} A^k B \frac{(\int_\tau^b g(s)\,ds)^k}{k!} u_0(\tau)\,d\tau. \tag{3.18}$$

From the Cayley-Hamilton theorem, it follows that for every $m \geq d$, $A^m = \sum\limits_{i=0}^{d-1} a_{m,i} A^i$, for some constants $a_{m,i} \in \mathbb{R}$. Then, by induction, from (3.17) we conclude that for every $k \in \mathbb{N}$, $qA^kB = 0$, and by multiplying (3.18) by q, we obtain

$$-q\Pi(a,b)y_a = \int\limits_a^b \sum_{k=0}^{\infty} qA^k B \frac{(\int_\tau^b g(s)\,ds)^k}{k!} u_0(\tau)\,d\tau = 0.$$

Since $\Pi(a,b)$ is regular matrix and y_a is arbitrary, it follows that $q = 0$—a contradiction.
($\Leftarrow$) Suppose that $\operatorname{rank}[B|AB|A^2B|\cdots|A^{d-1}B] = d$ and that the system (3.16) is not controllable. From Theorem 3.1.3 it follows that Gramian $W(a,b)$ is singular. Therefore, there exists $w \in \mathbb{R}^d \setminus \{0\}$ such that

$$\int_a^b |w^{\mathrm{T}}\Pi(t,b)B|^2\, dt = w^{\mathrm{T}}W(a,b)w = 0.$$

Hence,

$$p(t) := w^{\mathrm{T}}\Pi(t,b)B = 0, \quad \text{for almost every } t \in [a,b]. \tag{3.19}$$

Since $\Pi(\cdot,b) \in C([a,b];\mathbb{R}^d)$ and $\Pi(b,b) = \mathbb{I}$, from (3.19) we conclude $w^{\mathrm{T}}B = 0$.

Differentiating (3.19) and using Theorem 3.1.1 (ii), we get

$$p'(t) = -g(t)w^{\mathrm{T}}\Pi(t,b)AB = 0, \quad \text{for almost every } t \in [a,b].$$

Using $g \neq 0$, almost everywhere, we conclude

$$w^{\mathrm{T}}\Pi(t,b)AB = 0, \quad \text{for almost every } t \in [a,b], \tag{3.20}$$

which, as in the previous case, implies that $w^{\mathrm{T}}AB = 0$. By consecutively differentiating (3.20), using $g \neq 0$ and properties of $\Pi(t,b)$, we obtain $w^{\mathrm{T}}A^kB = 0$, for $k = 0,1,\ldots,d-1$. Since $w \neq 0$, this contradicts the Kalman rank condition. ☐

3.1.2 Hilbert uniqueness method and observability

The relation (3.7) gives us a straightforward computation of the L^2-optimal control function and it is based on the controllability approach. There are, however, other methods for obtaining the same results. In the sequel, we shall present the Hilbert uniqueness method (HUM) based on the properties of Hilbert spaces, and the variational method, where the problem of finding the L^2-optimal control function is reduced to the minimization problem of a suitably chosen functional. As we shall see, they are very closely related, and both rely on the duality between controllability and observability. Furthermore, both methods are an adaptation of techniques used for infinite-dimensional control problems to the finite-dimensional case.

First, we define the adjoint system of (3.1) by the following backward Cauchy problem:

$$\begin{aligned} z'(t) &= -\,A(t)^{\mathrm{T}}z(t), \quad t \in [a,b] \\ z(b) &= z_b. \end{aligned} \tag{3.21}$$

From (3.3) we have that the solution of the adjoint problem is given by

$$z(t) = \Pi(t,b)^{\mathrm{T}} z_b. \tag{3.22}$$

We have the following property, which justifies the choice of the adjoint system.

3.1.6 Proposition *Let $u \in L^2([a,b];\mathbb{R}^N)$ and $z_b \in \mathbb{R}^d$. Let $y : [a,b] \to \mathbb{R}^d$ be a solution to the Cauchy problem (3.1), and let $z : [a,b] \to \mathbb{R}^d$ be a solution to the adjoint problem (3.21). Then*

$$\langle y(b), z_b\rangle - \langle y_a, z(a)\rangle = \int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle \, dt. \tag{3.23}$$

Proof. Straightforward computations yield

$$\begin{aligned}
\langle y(b), z_b\rangle - \langle y_a, z(a)\rangle &= \int_a^b \frac{d}{dt}\langle y(t), z(t)\rangle \, dt \\
&= \int_a^b \langle \frac{dy}{dt}(t), z(t)\rangle + \langle y(t), \frac{dz}{dt}(t)\rangle \, dt \\
&= \int_a^b \langle A(t)y(t) + B(t)u(t), z(t)\rangle + \langle y(t), -A(t)^{\mathrm{T}} z(t)\rangle \, dt \\
&= \int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle \, dt,
\end{aligned}$$

which proves the claim. □

3.1.7 Remark From observability point of view, introduced in Section 1.2, the result from Proposition 3.1.6 can be interpreted in this way:
Knowing $y(b) = y_b$, control function u and observations $B^{\mathrm{T}} z$ on $[a,b]$, we can deduce the initial state of the system $y(a) = y_a$.

Denote by $\mathcal{R}$ the set of all reachable states, i.e., the set of all $y_b \in \mathbb{R}^d$ such that there exists $u \in L^2([a,b];\mathbb{R}^N)$ for which the solution of (3.1) satisfies $y(b) = y_b$. Further, define the mapping $\mathcal{L} : \mathbb{R}^d \to \mathbb{R}^d$, such that

$$\mathcal{L} : z_b \mapsto y(b), \tag{3.24}$$

where $y : [a,b] \to \mathbb{R}^d$ is the solution of the Cauchy problem

$$y'(t) = A(t)y(t) + B(t)u^*(t), \quad y(a) = y_a,$$

with

$$u^*(t) := B(t)^{\mathrm{T}} z(t)$$

and $z(t)$ is the solution of the adjoint problem (3.21).

Now we present the result obtained by the HUM, stated in [17, Th. 1.25.].

3.1.8 Theorem *The image of the mapping $\mathcal{L}$ given by (3.24) coincides with the set of reachable states, i.e.,*

$$\mathcal{R} = \mathcal{L}(\mathbb{R}^d).$$

Moreover, if $y_b = \mathcal{L}(z_b)$ and if $u_1 \in L^2([a,b];\mathbb{R}^N)$ is a control which steers the solution of (3.1) from y_a to y_b during the time interval $[a,b]$, then

$$\int_a^b |u^*(t)|^2 \, dt \leq \int_a^b |u_1(t)|^2 \, dt, \tag{3.25}$$

with equality if and only if $u_1(t) = u^*(t)$, for almost every $t \in [a,b]$.

Proof. The definition of $\mathcal{L}$ directly implies $\mathcal{L}(\mathbb{R}^d) \subseteq \mathcal{R}$. To prove the converse, let $y_1 \in \mathcal{R}$ and denote by u_1 the control function which steers the solution of the system (3.1) from y_a to y_1 during time interval $[a,b]$.

Let $\mathcal{U} \subset L^2([a,b];\mathbb{R}^N)$ be the set of all maps given by $t \mapsto B(t)^{\mathrm{T}} z(t)$, $t \in [a,b]$, where z is the solution of the adjoint problem and $z_b \in \mathbb{R}^d$. Using $z(t) = \Pi(t,b)^{\mathrm{T}} z_b$, we conclude that $\mathcal{U}$ is a finite-dimensional vector subspace of $L^2([a,b];\mathbb{R}^N)$ (the dimension of $\mathcal{U}$ is less than or equal to d). Hence $\mathcal{U}$ is closed. If u_1^* is the orthogonal projection of u_1 on $\mathcal{U}$, then the following holds

$$\int_a^b \langle u_1^*(t), u(t) \rangle \, dt = \int_a^b \langle u_1(t), u(t) \rangle \, dt, \quad u \in \mathcal{U}. \tag{3.26}$$

Denote by y^* the solution of the Cauchy problem

$$\frac{dy^*}{dt}(t) = A(t)y^*(t) + B(t)u_1^*(t), \quad y^*(a) = y_a. \tag{3.27}$$

Then Proposition 3.1.6 and (3.26) imply

$$\begin{aligned} \langle y^*(b), z_b\rangle - \langle y_a, z(a)\rangle &= \int_a^b \langle u_1^*(t), B(t)^{\mathrm{T}} z(t)\rangle \, dt \\ &= \int_a^b \langle u_1(t), B(t)^{\mathrm{T}} z(t)\rangle \, dt \\ &= \langle y_1, z_b\rangle - \langle y_a, z(a)\rangle, \end{aligned}$$

with $z_b \in \mathbb{R}^d$ arbitrary. Therefore

$$y^*(b) = y_1. \tag{3.28}$$

From the definition of $\mathcal{U}$ and $u_1^* \in \mathcal{U}$, it follows that there exists $z_b^* \in \mathbb{R}^d$ such that

$$u_1^*(t) = B(t)^{\mathrm{T}} z^*(t), \tag{3.29}$$

where z^* is the solution of the adjoint problem

$$\frac{dz^*}{dt}(t) = -A(t)^{\mathrm{T}} z^*(t), \quad z^*(b) = z_b^*.$$

Now, using (3.27), (3.28) and (3.29), we obtain

$$y_1 = y^*(b) = \mathcal{L}(z_b^*).$$

It remains to prove L^2-optimality of the control u^*. Assume $u_1 \in L^2([a,b];\mathbb{R}^N)$ is a control which steers the system (3.1) from y_a to y_b during the time interval $[a,b]$, where $y_b = \mathcal{L}(z_b)$ and $z_b \in \mathbb{R}^d$. Then, using Proposition 3.1.6, as in the previous part of the proof, and having in mind that $u^* \in \mathcal{U}$, we get

$$\int_a^b \langle u^*(t), u(t)\rangle \, dt = \int_a^b \langle u_1(t), u(t)\rangle \, dt, \quad u \in \mathcal{U},$$

which implies that u^* coincides with the orthogonal projection of u_1 on $\mathcal{U}$. Thus, $u_1 - u^*$ and u^* are orthogonal in $L^2([a,b];\mathbb{R}^N)$, and

$$\|u_1\|_{L^2}^2 = \|(u_1 - u^*) + u^*\|_{L^2}^2 = \|u_1 - u^*\|_{L^2}^2 + \|u^*\|_{L^2}^2.$$

Therefore, (3.25) holds. □

Now we move to the notion of observability and its relation with controllability.

3.1.9 Definition *The adjoint system* (3.21) *is observable if there exists a constant* $C > 0$ *such that for every* $z_b \in \mathbb{R}^d$ *the solution of the system satisfies*

$$|z_b|^2 \leq C \int_a^b |B(t)^{\mathrm{T}} z(t)|^2 \, dt. \tag{3.30}$$

The inequality (3.30) is called an observability inequality. It reflects the ability of the adjoint system to be observed through measurements $B^{\mathrm{T}} z$. More precisely, (3.30) assures that the N-dimensional observations $B^{\mathrm{T}} z$ provide sufficient information for the reconstruction of all components of the adjoint system. On the other hand, from controllability point of view, this means that the N-dimensional control u, through matrix B, can efficiently act on all the components of the state of the system. This illustrates the duality between controllability and observability properties, whose equivalence will be proved in the sequel. In order to do that, we need an equivalent condition for observability, in literature known as the unique continuation property, which is given in the next proposition.

3.1.10 Proposition *System* (3.21) *is observable if and only if its solution satisfies*

$$B(t)^{\mathrm{T}} z(t) = 0, \ \textit{for almost every } t \in [a, b] \quad \Rightarrow \quad z_b = 0. \tag{3.31}$$

Proof. If system (3.21) is observable, then (3.30) directly implies (3.31). To prove the converse, define the mapping $\| \cdot \|_s : \mathbb{R}^d \to [0, \infty)$ which every $z_b \in \mathbb{R}^d$ maps to

$$\|z_b\|_s := \Big(\int_a^b |B(t)^{\mathrm{T}} z(t)|^2 \, dt \Big)^{1/2},$$

where z is the unique solution of the adjoint problem (3.21). One can easily see that $\| \cdot \|_s$ is a seminorm (using properties of the L^2-norm). Furthermore, by (3.31) we have that if $\|z_b\|_s = 0$, then $z_b = 0$. Hence, $\| \cdot \|_s$ is a norm on $\mathbb{R}^d$. Now the existence of the constant C follows from the equivalence of all the norms on $\mathbb{R}^d$. □

Now we are able to prove the equivalence between controllability and observability.

3.1.11 Theorem *System* (3.1) *is controllable if and only if the adjoint system* (3.21) *is observable.*

Proof. ($\Leftarrow$) Assume that (3.1) is not controllable. Then, according to Theorem 3.1.3, Gramian $W(a, b)$ is singular. Hence, there exists $w \in \mathbb{R}^d \setminus \{0\}$ such that $w^{\mathrm{T}} W(a, b) w = 0$, which further implies

$$B(t)^{\mathrm{T}} \Pi(t, b)^{\mathrm{T}} w = 0, \ \text{for almost every } t \in [a, b].$$

If $\tilde{z}$ is the solution of the adjoint problem satisfying $\tilde{z}(b) = w$, then

$$B(t)^{\mathrm{T}}\tilde{z}(t) = B(t)^{\mathrm{T}}\Pi(t,b)^{\mathrm{T}}w = 0, \text{ almost everywhere on } [a,b],$$

and $\tilde{z}_b \neq 0$. Thus Proposition 3.1.10 implies that (3.21) is not observable.

($\Rightarrow$) Suppose that (3.1) is controllable and (3.21) is not observable. Then, by Definition 3.1.9, for every $C > 0$ there exists $z_b(C) \in \mathbb{R}^d$ such that (3.30) does not hold. Let $\{C_k\}$ be a sequence of positive numbers such that $\lim\limits_{k\to\infty} \frac{1}{C_k} = 0$ and $z_b(C_k)$, $k \in \mathbb{N}$, satisfy

$$|z_b(C_k)|^2 > C_k \int_a^b |B(t)^{\mathrm{T}} z_k(t)|^2\, dt, \tag{3.32}$$

where z_k is the solution of the adjoint system with $z_k(b) = z_b(C_k)$. Without loss of generality, we can assume $|z_b(C_k)| = 1$, so that $z_b(C_k)$ has a convergent subsequence, which we do not relabel. Let $\overline{z}_b = \lim\limits_{k\to\infty} z_b(C_k)$. Then $|\overline{z}_b| = 1$ and solution $\overline{z}$ of the adjoint system satisfying the condition $\overline{z}(b) = \overline{z}_b$ coincides with the limit of the solutions z_k. From (3.32), we have that for every k

$$0 \leq \int_a^b |B(t)^{\mathrm{T}} z_k(t)|^2\, dt < \frac{1}{C_k}|z_b(C_k)|^2,$$

and letting $k \to \infty$, we obtain

$$B(t)^{\mathrm{T}}\overline{z}(t) = 0, \quad \text{for almost every } t \in [a,b]. \tag{3.33}$$

Since, by assumption, system (3.1) is controllable, we have that for $y_a = 0$ and for any $y_b \in \mathbb{R}^d$ there exists a control function u which steers the solution y from $y(a) = 0$ to $y(b) = y_b$. Then, by Proposition 3.1.6 and (3.33) we have

$$\langle y_b, \overline{z}_b \rangle = \int_a^b \langle u(t), B(t)^{\mathrm{T}}\overline{z}(t)\rangle\, dt = 0.$$

Since y_b was arbitrary, it follows that $\overline{z}_b = 0$, which contradicts $|\overline{z}_b| = 1$. □

Variational approach for obtaining the L^2-optimal control function is based on the minimization of the functional $J : \mathbb{R}^d \to \mathbb{R}$, given by

$$J(z_b) = \frac{1}{2}\int_a^b |B(t)^{\mathrm{T}} z(t)|^2\, dt - \langle y_b, z_b\rangle + \langle y_a, z(a)\rangle, \tag{3.34}$$

where z is the solution of the adjoint problem (3.21).

3.1.12 Theorem *If the system (3.1) is controllable (or equivalently (3.21) is observable), then the functional J has a minimum, and the control function which steers the solution from y_a to y_b during the time interval $[a,b]$, is given by*

$$\hat{u}(t) = B(t)^{\mathrm{T}}\hat{z}(t) \tag{3.35}$$

where $\hat{z}$ is the solution of the adjoint problem with final state $\hat{z}(b) = \hat{z}_b$ being the minimum point of J. Furthermore, for any control function $u \in L^2([a,b];\mathbb{R}^N)$ such that the solution of the control problem (3.1) satisfies $y(b) = y_b$, it holds

$$\int_a^b |\hat{u}(t)|^2\,dt \leq \int_a^b |u(t)|^2\,dt,$$

with equality if and only if $\hat{u}$ and u coincide almost everywhere on $[a,b]$.

Proof. By definition, the functional J is strictly convex and continuous on $\mathbb{R}^d$. Thus, in order to prove that it has a minimum, it suffices to show that J is coercive, i.e., that

$$\lim_{|z_b|\to\infty} J(z_b) = \infty. \tag{3.36}$$

From observability, it follows that there exists $C > 0$ such that for every z_b and associated solution of (3.21) we have (3.30). Thus, by the definition of J and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} J(z_b) &\geq \frac{|z_b|^2}{2C} - \langle y_b, z_b\rangle + \langle y_a, z(a)\rangle \\ &\geq \frac{|z_b|^2}{2C} - |\langle y_b, z_b\rangle| + |\langle y_a, z(a)\rangle| \\ &\geq \frac{|z_b|^2}{2C} - |y_b||z_b| + |\langle y_a, z(a)\rangle|, \end{aligned}$$

which implies (3.36). Therefore, J has a unique minimum point in $\mathbb{R}^d$, which we denote by $\hat{z}_b$. Thus, the first variation of $J(\hat{z}_b)$ is equal to zero, i.e.,

$$\delta J(\hat{z}_b, z_b) = \lim_{h\to 0} \frac{J(\hat{z}_b + hz_b) - J(\hat{z}_b)}{h} = 0, \quad \text{for every } z_b \in \mathbb{R}^d.$$

By calculating the limit above, we get that for every $z_b \in \mathbb{R}^d$

$$\int_a^b \langle B(t)^{\mathrm{T}}\hat{z}(t), B(t)^{\mathrm{T}}z(t)\rangle\,dt - \langle y_b, z_b\rangle + \langle y_a, z(a)\rangle = 0,$$

which coincides with (3.23), for $u(t) = B(t)^{\mathrm{T}}\hat{z}(t)$. Therefore, the control $\hat{u}$ given by

$$\hat{u}(t) = B(t)^{\mathrm{T}}\hat{z}(t)$$

steers the solution to a desired state y_b. The second statement follows from Theorem 3.1.8. □

3.1.13 Remark Comparing the results from Theorem 3.1.3, Theorem 3.1.8 and Theorem 3.1.12 we conclude that $\overline{u} = u^* = \hat{u}$ almost everywhere on $[a, b]$. Furthermore, by calculating the first variation of the functional J, one could find that the point $\hat{z}_b$ where J reaches its minimum is given by

$$\hat{z}_b = W(a, b)^{-1}(y_b - \Pi(a, b)y_a).$$

Hence, from (3.22) and relations (3.7) and (3.35), it follows that the functions $\overline{u}$ and $\hat{u}$ coincide on $[a, b]$.

3.1.3 Bang-bang controls

In the methods presented so far, the optimization of control function was considered from energy-minimization approach, which corresponds to the L^2-setting. On the other hand, the optimization in the L^∞-setting, where the goal is to find a control with minimal L^∞-norm, provides a control of a simpler form. The L^∞-optimal control functions are called *bang-bang controls* since they turn out to be piecewise constant functions, and consist of switching from one constant state to another (often 0 and 1, on/off). This makes them more convenient for applications, since they are easier to implement than, for example, smooth controls which have fine changes in shape and magnitude.

The bang-bang controls can also be obtained via minimization problem, with certain modifications of a functional to be minimized.

For fixed initial and final data $y_a, y_b \in \mathbb{R}^d$, define a functional $J_{bb} : \mathbb{R}^d \to \mathbb{R}$, such that for $z_b \in \mathbb{R}^d$

$$J_{bb}(z_b) = \frac{1}{2}\Big(\int_a^b |B(t)^{\mathrm{T}}z(t)|_1\, dt\Big)^2 - \langle y_b, z_b\rangle + \langle y_a, z(a)\rangle, \tag{3.37}$$

where z is the solution of the adjoint system (3.21), with final state $z(b) = z_b$. Compared to the definition of J, where the square of the L^2-norm of $B^{\mathrm{T}}z$ takes part, here we have the square of the L^1-norm. Accordingly, we are going to modify observability inequality, which will provide coercivity of continuous convex functional J_{bb}.

We say that the adjoint system (3.21) is observable if there exists $C > 0$ such that for every $z_b \in \mathbb{R}^d$

$$|z_b|^2 \le C\Big(\int_a^b |B(t)^{\mathrm{T}} z(t)|_1 \, dt\Big)^2. \tag{3.38}$$

Note that observability defined via inequality (3.38) is also equivalent to the unique continuation property given in Proposition 3.1.10.

Hence, if the system (3.1) is controllable, i.e., (3.21) is observable, then J_{bb} has a minimum. Let $\tilde{z}_b$ be the point where J_{bb} reaches its minimum. Then, for every $z_b \in \mathbb{R}^d$ we have

$$\lim_{h\to 0} \frac{J_{bb}(\tilde{z}_b + hz_b) - J_{bb}(\tilde{z}_b)}{h} = 0. \tag{3.39}$$

To calculate the first variation of J_{bb}, we use the following relation:

$$\lim_{h\to 0} \frac{1}{h}\Big[\Big(\int_a^b |F+hG|_1 \, dt\Big)^2 - \Big(\int_a^b |F|_1 \, dt\Big)^2\Big] = 2\int_a^b |F|_1 \, dt \int_a^b \langle \mathrm{sign}\,(F), G\rangle \, dt, \tag{3.40}$$

which holds if $\{t \in [a,b] : F_i(t) = 0, \text{ for some } i = 1,\dots,N\}$ is the set of the zero Lebesgue measure, $F, G : [a,b] \to \mathbb{R}^N$. In (3.40) and in the sequel, the sign function of a vector is calculated componentwise.

Let $\tilde{z}$ be the solution of (3.21) with final state $\tilde{z}(b) = \tilde{z}_b$ being the minimizer of J_{bb}, $F(t) = B(t)^{\mathrm{T}}\tilde{z}(t)$ and $G(t) = B(t)^{\mathrm{T}} z(t)$. Note that if matrices A, B are continuous, then $F = B^{\mathrm{T}} z$ is also a continuous function and each component F_i changes sign on $[a,b]$ finitely many times. Hence, we can apply (3.40) in (3.39), and obtain that for every $z_b \in \mathbb{R}^d$

$$\int_a^b |B(t)^{\mathrm{T}}\tilde{z}(t)|_1 \, dt \int_a^b \langle \mathrm{sign}\,(B(t)^{\mathrm{T}}\tilde{z}(t)), B(t)^{\mathrm{T}} z(t)\rangle \, dt - \langle y_b, z_b\rangle + \langle y_a, z_a\rangle = 0.$$

The above equation is equivalent to

$$\int_a^b \langle \int_a^b |B(t)^{\mathrm{T}}\tilde{z}(t)|_1 \, dt \, \mathrm{sign}\,(B(t)^{\mathrm{T}}\tilde{z}(t)), B(t)^{\mathrm{T}} z(t)\rangle \, dt = \langle y_b, z_b\rangle - \langle y_a, z_a\rangle,$$

which coincides with (3.23) for

$$u(t) = u_{bb}(t) = \int_a^b |B(t)^{\mathrm{T}}\tilde{z}(t)|_1 \, dt \, \mathrm{sign}\,(B(t)^{\mathrm{T}}\tilde{z}(t)). \tag{3.41}$$

Hence, from Proposition 3.1.6, we have that u_{bb} is a desired control function, i.e., the one which steers the solution of (3.1) from y_a to y_b during the time interval $[a,b]$.

It remains to show that u_{bb} is a control with minimal L^∞-norm. Assume that $v \in L^\infty([a,b];\mathbb{R}^N)$ is a control which also steers the solution of (3.1) from y_a to y_b. Again, using Proposition 3.1.6 for $z_b = \tilde{z}_b$, we get that u_{bb} and v satisfy

$$\int_a^b \langle u_{bb}(t), B(t)^{\mathrm{T}}\tilde{z}(t)\rangle\, dt = \langle y_b, \tilde{z}_b\rangle - \langle y_a, \tilde{z}_a\rangle = \int_a^b \langle v(t), B(t)^{\mathrm{T}}\tilde{z}(t)\rangle\, dt.$$

Therefore,

$$\begin{aligned}
\|u_{bb}\|_\infty^2 &= \Big(\int_a^b (|B(t)^{\mathrm{T}}\tilde{z}(t)|_1\, dt\Big)^2 = \int_a^b \langle u_{bb}(t), B(t)^{\mathrm{T}}\tilde{z}(t)\rangle\, dt \\
&= \int_a^b \langle v(t), B(t)^{\mathrm{T}}\tilde{z}(t)\rangle\, dt \le \|v\|_\infty \int_a^b |B(t)^{\mathrm{T}}\tilde{z}(t)|_1\, dt = \|v\|_\infty \|u_{bb}\|_\infty.
\end{aligned}$$

After dividing by $\|u_{bb}\|_\infty$, we get

$$\|u_{bb}\|_\infty \le \|v\|_\infty.$$

Since $v \in L^\infty([a,b];\mathbb{R}^N)$ was arbitrary, we get that among all control functions the bang-bang control (3.41) has minimal L^∞-norm.

Note that each component $u_{bb,j}$, $j = 1,\dots,N$, of the vector-valued function u_{bb}, takes only two values $\pm \int_a^b |B(t)^{\mathrm{T}}\tilde{z}(t)|_1\, dt$, depending on the sign of the $j-$th component of $B(t)^{\mathrm{T}}\tilde{z}(t)$.

3.2 Control theory for linear systems of FDEs

In this section we consider a fractional analog of problem (3.1) with the Caputo fractional derivative of order $\alpha \in (0,1)$:

$$\begin{aligned}
{}_a^C\mathrm{D}_t^\alpha y(t) =& A(t)y(t) + B(t)u(t), \quad t \in [a,b] \\
y(a) =& y_a,
\end{aligned} \tag{3.42}$$

where $A \in L^\infty([a,b];\mathbb{R}^{d\times d})$ and $B \in L^\infty([a,b];\mathbb{R}^{d\times N})$ are given matrix functions.

From Theorem 2.3.7 (iv) we have that the solution of (3.42) is given by

$$y(t) = \Psi(a,t)y_a + \int_a^t \Phi(\tau,t)B(\tau)u(\tau)\, d\tau. \tag{3.43}$$

3.2.1 Controllability

Let us start by introducing the controllability Gramian matrix.

3.2.1 Definition *The controllability Gramian of the control system* (3.42) *is the symmetric $d \times d$ matrix*

$$W_\alpha(a,b) = \int_a^b (b-t)^{1-\alpha}\Phi(t,b)B(t)B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}\,dt. \tag{3.44}$$

3.2.2 Remark When comparing (3.44) with the definition (3.6), one can notice the extra term $(b-t)^{1-\alpha}$. This addition is necessary in order to assure the convergence of the integral (3.44) since, due to the nature of the Riemann-Liouville fractional derivative, the function $\Phi(\cdot,b) : [a,b) \to \mathbb{R}^{d\times d}$ has a singularity at the endpoint $t = b$.

For illustration, let us consider a simple example of one-dimensional linear time-invariant control problem

$${}_a^C\mathrm{D}_t^\alpha y(t) = a_1 y(t) + b_1 u(t), \quad y(a) = y_a,$$

where $a_1, b_1 \in \mathbb{R}^+$ and $y, u : [a,b] \to \mathbb{R}$. In this case, the matrix $\Phi(t,b)$ reduces to $\phi(t,b) = (b-t)^{\alpha-1}E_{\alpha,\alpha}(a_1(b-t)^\alpha)$. Hence, if we define the Gramian in the same way as in (3.6), we have

$$W(a,b) = \int_a^b \phi(t,b)^2 b_1^2\,dt = b_1^2 \int_a^b (b-t)^{2(\alpha-1)}E_{\alpha,\alpha}^2(a_1(b-t)^\alpha)\,dt$$

Since $a_1 > 0$ and $t \in [a,b]$, we have $E_{\alpha,\alpha}(a_1(b-t)^\alpha) \geq 1$. Therefore

$$W(a,b) \geq b^2 \int_a^b (b-t)^{2(\alpha-1)}\,dt. \tag{3.45}$$

Since the integral on the right-hand side of (3.45) is divergent for $\alpha \leq \frac{1}{2}$, we do not have a well defined Gramian for $\alpha \in (0, \frac{1}{2}]$. Hence, some adjustments need to be made. A natural one is the addition of the term $(b-t)^{1-\alpha}$, since it is a minimal modification which assures convergence of the integral and reduces to the integer-order case for $\alpha = 1$.

To show the convergence of (3.44), we use Proposition 2.3.16, and obtain that for some $M_F > 0$

$$\begin{aligned}\|W_\alpha(a,b)\| &\leq \int_a^b (b-t)^{\alpha-1}\|(b-t)^{1-\alpha}\Phi(t,b)\|\|B(t)B(t)^{\mathrm{T}}\|\|(b-t)^{1-\alpha}\Phi(t,b)^{\mathrm{T}}\|\,dt \\ &\leq M_F^2\|B\|^2\int_a^b (b-t)^{\alpha-1}dt = \frac{M_F^2\|B\|^2(b-a)^\alpha}{\alpha},\end{aligned}$$

where $\|B\| = \sup\limits_{t\in[a,b]}\|B(t)\|$.

3.2.3 Remark Let us notice that in the fractional case we also have

$$\begin{aligned}x^{\mathrm{T}}W_\alpha(a,b)x &= \int_a^b (b-t)^{1-\alpha}x^{\mathrm{T}}\Phi(t,b)B(t)B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}x\,dt \\ &= \int_a^b (b-t)^{1-\alpha}|x^{\mathrm{T}}\Phi(t,b)B(t)|^2\,dt \geq 0,\end{aligned}$$

for any $x \in \mathbb{R}^d$. Hence, the regularity of the Gramian is also equivalent to its positive definiteness.

3.2.4 Remark In the sequel, we shall use the definition of the Gramian given by (3.44), since it is well-defined for every $\alpha \in (0,1)$ and it reduces to (3.6), for $\alpha = 1$. However, we want to emphasize that for $\alpha \in (\frac{1}{2}, 1)$, one can define the Gramian without the extra term $(b-t)^{1-\alpha}$, i.e., for $\alpha = \overline{\alpha} \in (\frac{1}{2}, 1)$, definition

$$W_{\overline{\alpha}}(a,b) = \int_a^b \Phi(t,b)B(t)B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}\,dt$$

is also valid.

The first result is the equivalence between controllability and invertibility of the Gramian.

3.2.5 Theorem *System (3.42) is controllable if and only if the controllability Gramian matrix $W_\alpha(a,b)$ is nonsingular.*

Proof. ($\Leftarrow$) If $W_\alpha(a,b)$ is nonsingular, then the control function

$$\overline{u}(t) = (b-t)^{1-\alpha} B(t)^{\mathrm{T}} \Phi(t,b)^{\mathrm{T}} W_\alpha(a,b)^{-1}[y_b - \Psi(a,b)y_a] \tag{3.46}$$

is well defined and it steers the solution of the system to a desired value y_b. Indeed, inserting $u = \overline{u}$ in (3.43) we obtain

$$\begin{aligned} y(b) &= \Psi(a,b)y_a + \int_a^b \Phi(t,b)B(t)\overline{u}(t)\,dt \\ &= \Psi(a,b)y_a \\ &\quad + \int_a^b (b-t)^{1-\alpha}\Phi(t,b)B(t)B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}W_\alpha(a,b)^{-1}[y_b - \Psi(a,b)y_a]\,dt \\ &= \Psi(a,b)y_a + W_\alpha(a,b)W_\alpha(a,b)^{-1}[y_b - \Psi(a,b)y_a] \\ &= y_b. \end{aligned}$$

($\Rightarrow$) Suppose that the system is controllable and that $W_\alpha(a,b)$ is singular. Then there exists a column vector $w \in \mathbb{R}^d \setminus \{0\}$ such that $w^{\mathrm{T}} W_\alpha(a,b) w = 0$, which yields

$$\int_a^b w^{\mathrm{T}} (b-t)^{1-\alpha}\Phi(t,b)B(t)B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}} w\,dt = 0. \tag{3.47}$$

If

$$\phi(t) = w^{\mathrm{T}}\Phi(t,b)B(t)$$

then (3.47) implies that $\phi(t) = 0$, almost everywhere on $[a,b]$. Since the system is controllable, there exists a control u such that the solution of the system satisfies $y(b) = w$, when $y(a) = 0$. Then we have

$$w = y(b) = \int_a^b \Phi(t,b)B(t)u(t)\,dt,$$

and

$$\|w\|^2 = w^{\mathrm{T}} w = \int_a^b w^{\mathrm{T}}\Phi(t,b)B(t)u(t)\,dt = \int_a^b \phi(t)u(t)\,dt = 0.$$

This leads to a contradiction with $w \neq 0$. □

If the system is controllable, the control function given by (3.46) is an optimal control in the weighted space defined by (1.1).

3.2.6 Proposition *If $u \in L^2_{\alpha-1}([a,b];\mathbb{R}^N)$ is a control function such that the solution of (3.42) satisfies $y(b) = y_b$, then*

$$\int_a^b (b-t)^{\alpha-1}|\overline{u}(t)|^2\,dt \le \int_a^b (b-t)^{\alpha-1}|u(t)|^2\,dt,$$

with equality if and only if u and $\overline{u}$ coincide almost everywhere on $[a,b]$.

Proof. Let $v := u - \overline{u}$. Since u and $\overline{u}$ both steer the solution from y_a to y_b, from (3.43) we have

$$\begin{aligned}\int_a^b \Phi(t,b)B(t)v(t)\,dt &= \int_a^b \Phi(t,b)B(t)u(t)\,dt - \int_a^b \Phi(t,b)B(t)\overline{u}(t)\,dt \\ &= (y_b - \Psi(a,b)y_a) - (y_b - \Psi(a,b)y_a) = 0.\end{aligned}$$

This together with (3.46) implies

$$\begin{aligned}\langle \overline{u}, v\rangle_{L^2_{\alpha-1}} &= \int_a^b (b-t)^{\alpha-1}\langle \overline{u}(t), v(t)\rangle\,dt = \int_a^b \langle B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}w_\alpha, v(t)\rangle\,dt \\ &= \int_a^b \langle w_\alpha, \Phi(t,b)B(t)v(t)\rangle\,dt = \langle w_\alpha, \int_a^b \Phi(t,b)B(t)v(t)\,dt\rangle = 0,\end{aligned}$$

where $w_\alpha = W_\alpha(a,b)^{-1}(y_b - \Psi(a,b)y_a)$. On the other hand

$$\|u\|^2_{L^2_{\alpha-1}} = \|\overline{u}+v\|^2_{L^2_{\alpha-1}} = \|\overline{u}\|^2_{L^2_{\alpha-1}} + \|v\|^2_{L^2_{\alpha-1}} + 2\langle \overline{u}, v\rangle_{L^2_{\alpha-1}}.$$

Therefore,

$$\int_a^b (b-t)^{\alpha-1}|u(t)|^2\,dt = \int_a^b (b-t)^{\alpha-1}|\overline{u}(t)|^2\,dt + \int_a^b (b-t)^{\alpha-1}|v(t)|^2\,dt,$$

which concludes the proof. □

3.2.7 Remark Having in mind Remark 3.2.4, and taking $W_\alpha = W_{\overline{\alpha}}$, for $\alpha = \overline{\alpha} \in (\frac{1}{2}, 1)$, the control function from the proof of Theorem 3.2.5 reduces to

$$\overline{u}(t) = B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}W_{\overline{\alpha}}(a,b)^{-1}[y_b - \Psi(a,b)y_a],$$

and it is an optimal control in the space $L^2([a,b];\mathbb{R}^N)$.

The next proposition states that for linear time-invariant systems with fractional derivatives, as well as in the case with integer derivatives, the Kalman rank is a necessary and sufficient condition for controllability. Using the sketch of the proof given in [52, Th. 3], we obtain the following.

3.2.8 Proposition *Linear time-invariant system*

$$\begin{aligned} {}_a^C\mathrm{D}_t^\alpha y(t) &= Ay(t) + Bu(t), \quad t \in [a,b] \\ y(a) &= y_a, \end{aligned} \tag{3.48}$$

is controllable if and only if matrices A and B satisfy

$$\mathrm{rank}[B|AB|A^2B|\cdots|A^{d-1}B] = d.$$

Proof. First, recall that for the system (3.48), the associated state-transition matrices are given by the Mittag-Leffler functions:

$$\Psi(a,t) = E_\alpha(A(t-a)^\alpha) \quad \text{and} \quad \Phi(\tau,t) = (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha). \tag{3.49}$$

Furthermore, the controllability Gramian reduces to

$$W_\alpha(a,b) = \int_a^b (b-t)^{\alpha-1}E_{\alpha,\alpha}(A(b-t)^\alpha)BB^{\mathrm{T}}E_{\alpha,\alpha}(A^{\mathrm{T}}(b-t)^\alpha)\,dt. \tag{3.50}$$

From Theorem 3.2.5, we have that it suffices to prove that the Kalman rank condition is satisfied if and only if the Gramian (3.50) has inverse. We use the same arguments as in the integer-derivative case.

($\Leftarrow$) Suppose that (3.50) is singular matrix. Then, there exists $w \in \mathbb{R}^d \setminus \{0\}$ such that

$$\int_a^b (b-t)^{\alpha-1}|w^{\mathrm{T}}E_{\alpha,\alpha}(A(b-t)^\alpha)B|^2\,dt = w^{\mathrm{T}}W_\alpha(a,b)w = 0. \tag{3.51}$$

Denote by $\phi : [a,b] \to \mathbb{R}^N$ the function given by $\phi(t) = w^{\mathrm{T}}E_{\alpha,\alpha}(A(b-t)^\alpha)B$. Now (3.51), and the fact that ϕ is continuous, imply $\phi(t) = 0$, $t \in [a,b]$. Let us introduce the change of variables $z = (b-t)^\alpha$. Then

$$\tilde{\phi}(z) = w^{\mathrm{T}}E_{\alpha,\alpha}(Az)B = \sum_{k=0}^{\infty} \frac{w^{\mathrm{T}}A^kBz^k}{\Gamma(\alpha k+\alpha)} = 0, \quad z \in [0,(b-a)^\alpha]. \tag{3.52}$$

Taking derivative of (3.52) $i-$times, $i \in \{0,1,\dots,d-1\}$, we obtain

$$w^{\mathrm{T}}A^iB = 0, \quad i = 0,\dots,d-1. \tag{3.53}$$

Hence, the Kalman rank condition is not satisfied.

($\Rightarrow$) We show that if the Kalman rank is not satisfied, then (3.50) is singular. Let $w \in \mathbb{R}^d \setminus \{0\}$ be such that (3.53) holds. This, together with the Cayley-Hamilton theorem implies

$$w^{\mathrm{T}} A^i B = 0, \quad \text{for every } i \in \mathbb{N}. \tag{3.54}$$

Therefore, the function $\phi(t) = w^{\mathrm{T}} E_{\alpha,\alpha}(A(b-t)^\alpha)B = \sum\limits_{k=0}^{\infty} \frac{w^{\mathrm{T}} A^k B (b-t)^{\alpha k}}{\Gamma(\alpha k + \alpha)}$ satisfies

$$\phi(t) = 0, \quad t \in [a, b].$$

Hence $w^{\mathrm{T}} W_\alpha(a,b) w = \int_a^b (b-t)^{\alpha-1} |\phi(t)|^2 \, dt = 0$, which implies singularity of the Gramian. □

We conclude this section with the analysis of the following system:

$$\begin{aligned} {}_a^C\mathrm{D}_t^\alpha y(t) =& A g(t) y(t) + B u(t), \quad t \in [a, b] \\ y(a) =& y_a, \end{aligned} \tag{3.55}$$

where $A \in \mathbb{R}^{d \times d}$ is a symmetric real matrix, $B \in \mathbb{R}^{d \times N}$ is a real matrix and $g : [a, b] \to \mathbb{R}$ is a continuous function and $g(t) \neq 0$, for almost every $t \in [a, b]$.

First, we prove a fractional analog of Proposition 3.1.5.

3.2.9 Proposition *System (3.55) is controllable if and only if the Kalman rank condition is satisfied, i.e.,* $\operatorname{rank}[B|AB|A^2B|\cdots|A^{d-1}B] = d$.

Proof. ($\Rightarrow$) Suppose that the system is controllable and that A and B do not satisfy the Kalman rank condition. Then, we can find a vector $q \in \mathbb{R}^d \setminus \{0\}$ such that

$$qB = 0, \, qAB = 0, \ldots, \, qA^{d-1}B = 0. \tag{3.56}$$

Using the Cayley-Hamilton theorem and (3.56) we conclude that

$$qA^k B = 0, \quad \text{for every } k \in \mathbb{N}. \tag{3.57}$$

From controllability, it follows that for $y_b = 0$ there exists a control function u_0 such that the solution of the system satisfies $y(b) = 0$. Then, from the representation of the solution (3.43), we have

$$0 = y(b) = \Psi(a,b) y_a + \int\limits_a^b \Phi(\tau, b) B u_0(\tau) d\tau.$$

Therefore,

$$-\Psi(a,b) y_a = \int\limits_a^b \Phi(\tau,b) B u_0(\tau)\, d\tau = \int\limits_a^b \sum_{k=0}^{\infty} A^k B ({}_\tau\mathrm{I}_b^{k\circ\alpha} g(\tau)) u_0(\tau)\, d\tau. \tag{3.58}$$

Then, by multiplying (3.58) by q and using (3.57), we obtain

$$-q\Psi(a,b)y_a = 0. \tag{3.59}$$

From the proof of Proposition 2.3.19 (i) we have that $\Psi(a,b)$ is a regular matrix and since y_a was arbitrary, (3.59) implies $q=0$—a contradiction.
$(\Leftarrow)$ Suppose that $\operatorname{rank}[B|AB|A^2B|\cdots|A^{d-1}B] = d$ and that the system (3.55) is not controllable. Then Theorem 3.2.5 implies that $W_\alpha(a,b)$ is singular. Hence, there exists $w \in \mathbb{R}^d \setminus \{0\}$ such that $w^{\mathrm{T}}W_\alpha(a,b)w = 0$, which will lead to

$$w^{\mathrm{T}}\Phi(t,b)B = 0, \quad \text{for almost every } t \in [a,b]. \tag{3.60}$$

Applying ${}_t\mathrm{I}_b^{1-\alpha}$ to (3.60), and using $\lim_{t\to b} {}_t\mathrm{I}_b^{1-\alpha}\Phi(t,b) = {}_t\mathrm{I}_b^{1-\alpha}\Phi(t,b)|_{t=b} = \mathbb{I}$, we obtain $w^{\mathrm{T}}B = 0$. Moreover, from ${}_t\mathrm{D}_b^{\alpha}\Phi(t,b) = Ag(t)\Phi(t,b)$ and (3.60) it follows that

$$w^{\mathrm{T}}Ag(t)\Phi(t,b)B = 0, \qquad \text{for almost every } t \in [a,b].$$

Since $g(t)$ is continuous and $g \neq 0$ almost everywhere, we obtain $w^{\mathrm{T}}A\Phi(t,b)B = 0$, for almost every $t \in [a,b]$. Again, using ${}_t\mathrm{I}_b^{1-\alpha}\Phi(t,b)|_{t=b} = \mathbb{I}$, we get $w^{\mathrm{T}}AB = 0$.

By taking derivatives ${}_t\mathrm{D}_b^{\alpha}$ of (3.60) subsequently k-times, and then consecutively using $g \neq 0$, almost everywhere on $[a,b]$, we get $w^{\mathrm{T}}A^k\Phi(t,b)B = 0$, which together with ${}_t\mathrm{I}_b^{1-\alpha}\Phi(t,b)|_{t=b} = \mathbb{I}$ implies that $w^{\mathrm{T}}A^kB = 0$, for every $k = 0,1,2,...,d-1$. This contradicts the Kalman rank condition. □

3.2.10 Remark (**Diagonalization of Gramian matrix**) Let us notice that using Remark 2.3.18, and introducing notation $\tilde{B} = U^TB$ and $x(t) = U^{\mathrm{T}}y(t)$, we can reduce (3.55) to an equivalent system:

$$\begin{aligned} {}_a^C\mathrm{D}_t^{\alpha}x(t) &= Dg(t)x(t) + \tilde{B}u(t), \quad t \in [a,b] \\ x(a) &= U^{\mathrm{T}}y_a. \end{aligned} \tag{3.61}$$

Moreover, from the proof of Proposition 2.3.19 (ii), we have that $\Phi(\tau,t) = UQ(\tau,t)U^{\mathrm{T}}$ and $Q(\tau,t) = \Phi_D(\tau,t)$, with $\Phi_D(\tau,t)$ being the solution to

$${}_\tau\mathrm{D}_t^{\alpha}\Phi_D(\tau,t) = Dg(\tau)\Phi_D(\tau,t), \quad \tau \in (a,t), \quad {}_\tau\mathrm{I}_t^{1-\alpha}\Phi_D(\tau,t)|_{\tau=t} = \mathbb{I}.$$

Hence, the controllability Gramians of (3.55) and (3.61), denoted by $W_{\alpha,A}(a,b)$ and $W_{\alpha,D}(a,b)$, respectively, satisfy

$$\begin{aligned} W_{\alpha,A}(a,b) &= \int_a^b (b-t)^{1-\alpha}\Phi(t,b)BB^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}\,dt \\ &= \int_a^b (b-t)^{1-\alpha}U\Phi_D(t,b)U^{\mathrm{T}}BB^{\mathrm{T}}U\Phi_D(t,b)^{\mathrm{T}}U^T\,dt \\ &= UW_{\alpha,D}(a,b)U^{\mathrm{T}}. \end{aligned}$$

3.2.2 Adjoint system and observability

In this section we consider the adjoint problem of (3.42) and the notion of observability. For fractional linear systems with constant coefficients, observability was introduced in [52], where it was shown to be equivalent to the positive definiteness of the observability Gramian matrix. Here we are going to take different approach and introduce observability via the observability inequality. Furthermore, we are going to adapt the results from Section 3.1.2 to the fractional setting.

We define the adjoint system of (3.42) by the following backward problem:

$$\begin{aligned} {}_t\mathrm{D}_b^\alpha z(t) &= A(t)^{\mathrm{T}} z(t), \quad t \in (a,b) \\ {}_t\mathrm{I}_b^{1-\alpha} z(t)|_{t=b} &= z_b. \end{aligned} \tag{3.62}$$

Having in mind Corollary 2.3.12, it follows that the solution of (3.62) is given by $z(t) = \Phi(t,b)^{\mathrm{T}} z_b$. Moreover, z satisfies the following properties:

- $z \in AC_b^\alpha([a,b];\mathbb{R}^d)$ and

$$\lim_{t\to b}(b-t)^{1-\alpha} z(t) = \frac{1}{\Gamma(\alpha)} {}_t\mathrm{I}_b^{1-\alpha} z(t)|_{t=b} = \frac{z_b}{\Gamma(\alpha)}.$$

  In particular, $z \in L^1([a,b];\mathbb{R}^d)$.

- $z \in L^2_{1-\alpha}([a,b];\mathbb{R}^d)$. Using Proposition 2.3.16 we get

$$\begin{aligned} \int_a^b (b-t)^{1-\alpha} |z(t)|^2\, dt &= \int_a^b (b-t)^{\alpha-1} |(b-t)^{1-\alpha}\Phi(t,b)^{\mathrm{T}} z_b|^2\, dt \\ &\leq M_F^2 |z_b|^2 \int_a^b (b-t)^{\alpha-1}\, dt = \frac{M_F^2 |z_b|^2 (b-a)^\alpha}{\alpha} < \infty. \end{aligned}$$

- For $\alpha \in (\frac{1}{2}, 1)$, $z \in L^2([a,b];\mathbb{R}^d)$. Again, Proposition 2.3.16 implies

$$\begin{aligned} \int_a^b |z(t)|^2\, dt &= \int_a^b (b-t)^{2(\alpha-1)} |(b-t)^{1-\alpha}\Phi(t,b)^{\mathrm{T}} z_b|^2\, dt \\ &\leq M_F^2 |z_b|^2 \int_a^b (b-t)^{2(\alpha-1)}\, dt = \frac{M_F^2 |z_b|^2 (b-a)^{2\alpha-1}}{2\alpha-1} < \infty. \end{aligned}$$

In the next lemma, we present an auxiliary result.

3.2.11 Lemma *If $w(t)$ is the solution of the Cauchy problem*

$$ {}_a^C\mathrm{D}_t^\alpha w(t) = A(t)w(t), \quad w(a) = w_a, $$

and $z(t)$ the solution of the adjoint problem (3.62), then

$$ \langle w(b), z_b\rangle - \langle w_a, z_a\rangle = 0, \tag{3.63} $$

where $z_a = {}_t\mathrm{I}_b^{1-\alpha} z(t)|_{t=a}$.

Proof. For $w(t)$ and $z(t)$ we have that for every $t \in [a,b]$

$$ \langle {}_a^C\mathrm{D}_t^\alpha w(t), z(t)\rangle = \langle A(t)w(t), z(t)\rangle = \langle w(t), A(t)^{\mathrm{T}} z(t)\rangle = \langle w(t), {}_t\mathrm{D}_b^\alpha z(t)\rangle. $$

Hence $\int_a^b \langle {}_a^C\mathrm{D}_t^\alpha w(t), z(t)\rangle\, dt = \int_a^b \langle w(t), {}_t\mathrm{D}_b^\alpha z(t)\rangle\, dt$, and (3.63) follows from (2.3). $\square$

The next proposition explains why the definition of the adjoint problem given by (3.62) is the natural one.

3.2.12 Proposition *Let $u \in L^2_{\alpha-1}([a,b];\mathbb{R}^N)$, or $u \in L^\infty([a,b];\mathbb{R}^N)$.*

(i) If $y : [a,b] \to \mathbb{R}^d$ is the solution of the system (3.42), and $z : [a,b] \to \mathbb{R}^d$ is the solution of (3.62), with $z_b \in \mathbb{R}^d$ arbitrary, then

$$ \int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt = \langle y(b), z_b\rangle - \langle y_a, z_a\rangle, \tag{3.64} $$

where $z_a = {}_t\mathrm{I}_b^{1-\alpha} z(t)|_{t=a}$.

(ii) Let $y_a, y_b \in \mathbb{R}^d$. If u is such that for every $z_b \in \mathbb{R}^d$ it holds

$$ \int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt = \langle y_b, z_b\rangle - \langle y_a, z_a\rangle, \tag{3.65} $$

where $z(t)$ is solution of (3.62) and $z_a = {}_t\mathrm{I}_b^{1-\alpha} z(t)|_{t=a}$, then the solution of (3.42) satisfies $y(b) = y_b$.

Proof. (i) Since y satisfies (3.42), we have

$$\begin{aligned}\int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt &= \int_a^b \langle B(t)u(t), z(t)\rangle\, dt\\ &= \int_a^b \langle {}_a^C\mathrm{D}_t^\alpha y(t), z(t)\rangle\, dt - \int_a^b \langle A(t)y(t), z(t)\rangle\, dt\\ &= \int_a^b \langle {}_a^C\mathrm{D}_t^\alpha y(t), z(t)\rangle\, dt - \int_a^b \langle y(t), A(t)^{\mathrm{T}} z(t)\rangle\, dt.\end{aligned}$$

Applying fractional integration by parts, given in Proposition 2.3, we obtain

$$\int_a^b \langle {}_a^C\mathrm{D}_t^\alpha y(t), z(t)\rangle\, dt = \int_a^b \langle y(t), {}_t\mathrm{D}_b^\alpha z(t)\rangle\, dt + \langle y(t), {}_t\mathrm{I}_b^{1-\alpha} z(t)\rangle\Big|_{t=a}^{t=b}.$$

Hence

$$\int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt = \int_a^b \langle y(t), {}_t\mathrm{D}_b^\alpha z(t) - A(t)^{\mathrm{T}} z(t)\rangle\, dt + \langle y(b), z_b\rangle - \langle y_a, z_a\rangle, \quad (3.66)$$

for $z_a = {}_t\mathrm{I}_b^{1-\alpha} z(t)|_{t=a}$. Since, by assumption, z is a solution of the adjoint problem, (3.66) reduces to (3.64).

(ii) From (3.43), it follows that the value of the solution $y(t)$ at the end point is given by

$$y(b) = \Psi(a,b)y_a + \int_a^b \Phi(t,b)B(t)u(t)\, dt.$$

Hence, using $z(t) = \Phi(t,b)^{\mathrm{T}} z_b$, we obtain

$$\int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt = \int_a^b \langle \Phi(t,b)B(t)u(t), z_b\rangle\, dt = \langle y(b) - \Psi(a,b)y_a, z_b\rangle.$$

Furthermore, Lemma 3.2.11 implies $\langle \Psi(a,b)y_a, z_b\rangle = \langle y_a, z_a\rangle$, and we have

$$\int_a^b \langle u(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt = \langle y(b), z_b\rangle - \langle y_a, z_a\rangle.$$

Thus, if (3.65) holds for every $z_b \in \mathbb{R}^d$, then

$$\langle y_b, z_b\rangle = \langle y(b), z_b\rangle, \quad \text{for every } z_b \in \mathbb{R}^d,$$

implying $y(b) = y_b$. □

Hilbert uniqueness method for linear fractional control

Let $\mathcal{R}_\alpha$ be the set of all reachable states for problem (3.42), i.e., $\mathcal{R}_\alpha$ is a set of all $y_b \in \mathbb{R}^d$ such that there exists a control function $u \in L^2_{\alpha-1}([a,b];\mathbb{R}^N)$ for which the solution of (3.42) satisfies $y(b) = y_b$.

Define the mapping $\mathcal{L}_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathcal{L}_\alpha : z_b \mapsto y(b)$, where $y : [a,b] \to \mathbb{R}^d$ is the solution of the Cauchy problem

$$ {}_a^C\mathrm{D}_t^\alpha y(t) = A(t)y(t) + B(t)u^*(t), \quad y(a) = y_a, \tag{3.67}$$

with control function

$$u^*(t) := (b-t)^{1-\alpha} B(t)^{\mathrm{T}} z(t), \tag{3.68}$$

and $z(t)$ being the solution of the adjoint problem (3.62).

3.2.13 Theorem *The mapping $\mathcal{L}_\alpha$ and the set of reachable states are related by*

$$\mathcal{R}_\alpha = \mathcal{L}_\alpha(\mathbb{R}^d). \tag{3.69}$$

Moreover, if $y_b = \mathcal{L}_\alpha(z_b)$ and if $u_1 \in L^2_{\alpha-1}([a,b];\mathbb{R}^N)$ is a control which steers the solution of (3.42) from y_a to y_b during the time interval $[a,b]$, then

$$\int_a^b (b-t)^{\alpha-1} |u^*(t)|^2 \, dt \le \int_a^b (b-t)^{\alpha-1} |u_1(t)|^2 \, dt,$$

with equality if and only if $u^*(t) = u_1(t)$, for almost every $t \in [a,b]$.

Proof. The inclusion $\mathcal{L}_\alpha(\mathbb{R}^d) \subseteq \mathcal{R}_\alpha$ follows from the definition of $\mathcal{L}_\alpha$. For the other direction, assume that $y_1 \in \mathcal{R}_\alpha$, and let u_1 be the control which steers the solution of (3.42) from y_a to y_1 during the time interval $[a,b]$.

Define the set $\mathcal{U}$ of all the maps $u \in L^2_{\alpha-1}([a,b];\mathbb{R}^N)$ given by

$$u : t \mapsto (b-t)^{1-\alpha} B(t)^{\mathrm{T}} z(t), \quad t \in [a,b],$$

where $z(t)$ is the solution of (3.62) for some $z_b \in \mathbb{R}^d$. Since $z(t) = \Phi(t,b)^{\mathrm{T}} z_b$, we have

$$\mathcal{U} = \{u : [a,b] \to \mathbb{R}^N \, : \, u(t) = (b-t)^{1-\alpha} B(t)^{\mathrm{T}} \Phi(t,b)^{\mathrm{T}} z_b, \quad z_b \in \mathbb{R}^d\},$$

which implies that $\mathcal{U}$ is of finite dimension. Hence, $\mathcal{U}$ is a closed subspace of $L^2_{\alpha-1}([a,b];\mathbb{R}^N)$. Denote by u_1^* the orthogonal projection of u_1 on $\mathcal{U}$. Then, for every $u \in \mathcal{U}$ it holds

$$\int_a^b (b-t)^{\alpha-1}\langle u_1^*(t), u(t)\rangle\, dt = \int_a^b (b-t)^{\alpha-1}\langle u_1(t), u(t)\rangle\, dt. \tag{3.70}$$

Let $y^* : [a,b] \to \mathbb{R}^d$ be the solution of the Cauchy problem

$${}_a^C\mathrm{D}_t^\alpha y^*(t) = A(t)y^*(t) + B(t)u_1^*(t), \quad y^*(a) = y_a. \tag{3.71}$$

Then, from Proposition 3.2.12 (i) and (3.70), it follows that for every $z_b \in \mathbb{R}^d$

$$\begin{aligned}
\langle y^*(b), z_b\rangle - \langle y_a, z_a\rangle &= \int_a^b \langle u_1^*(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt \\
&= \int_a^b (b-t)^{\alpha-1}\langle u_1^*(t), (b-t)^{1-\alpha}B(t)^{\mathrm{T}} z(t)\rangle\, dt \\
&= \int_a^b (b-t)^{\alpha-1}\langle u_1(t), (b-t)^{1-\alpha}B(t)^{\mathrm{T}} z(t)\rangle\, dt \\
&= \int_a^b \langle u_1(t), B(t)^{\mathrm{T}} z(t)\rangle\, dt \\
&= \langle y_1, z_b\rangle - \langle y_a, z_a\rangle.
\end{aligned}$$

Hence $\langle y^*(b), z_b\rangle = \langle y_1, z_b\rangle$, for every $z_b \in \mathbb{R}^d$, implying that

$$y^*(b) = y_1. \tag{3.72}$$

Therefore, u_1^* also steers the solution from y_a to y_1, and since $u_1^* \in \mathcal{U}$, there exists $z_b^* \in \mathbb{R}^d$ such that $u_1^*(t) = (b-t)^{1-\alpha}B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}} z_b^* = (b-t)^{1-\alpha}B(t)^{\mathrm{T}} z^*(t)$. Then, by the definition of $\mathcal{L}_\alpha$, (3.71) and (3.72) we have

$$\mathcal{L}_\alpha(z_b^*) = y^*(b) = y_1,$$

which concludes the proof of (3.69).

Finally, let $y_b = \mathcal{L}_\alpha(z_b)$ and let u^* be given by (3.67) and (3.68). Assume that $u_1 \in L^2_{\alpha-1}([a,b];\mathbb{R}^N)$ is a control function which steers the system (3.42) from y_a to y_b during the time interval $[a,b]$. Again, using Proposition 3.2.12 (i), we obtain

$$\int_a^b (b-t)^{\alpha-1}\langle u^*(t), u(t)\rangle\, dt = \int_a^b (b-t)^{\alpha-1}\langle u_1(t), u(t)\rangle\, dt, \text{ for every } u \in \mathcal{U}.$$

Since, by definition, $u^* \in \mathcal{U}$, it follows that u^* is the orthogonal projection of u_1 on $\mathcal{U}$, which implies

$$\int_a^b (b-t)^{\alpha-1}|u_1(t)|^2\,dt = \int_a^b (b-t)^{\alpha-1}|u^*(t)|^2\,dt + \int_a^b (b-t)^{\alpha-1}|u_1(t)-u^*(t)|^2\,dt,$$

and we have proved the second statement. □

3.2.14 Remark Since for $\frac{1}{2} < \alpha < 1$ the solution z of the adjoint problem (3.62) belongs to $L^2([a,b];\mathbb{R}^d)$, in that case, the HUM can be applied in the L^2-setting (without weighted L^2-spaces), to obtain the L^2-optimal control function of the form

$$u^*(t) = B(t)^{\mathrm{T}} z(t).$$

Variational approach to linear fractional control

We start by introducing the notion of observability.

3.2.15 Definition *System (3.62) is observable if there exists $C > 0$ such that for every $z_b \in \mathbb{R}^d$, solution of (3.62) satisfies*

$$|z_b|^2 \le C \int_a^b (b-t)^{1-\alpha}|B(t)^{\mathrm{T}} z(t)|^2\,dt. \tag{3.73}$$

3.2.16 Proposition *Observability inequality (3.73) is equivalent to the following unique continuation property*

$$B(t)^{\mathrm{T}} z(t) = 0, \quad \textit{for almost every } t \in [a,b] \quad \Rightarrow \quad z_b = 0. \tag{3.74}$$

Proof. The implication (3.73) $\Rightarrow$ (3.74) follows directly. To prove (3.74) $\Rightarrow$ (3.73), we define a mapping $\|\cdot\|_r : \mathbb{R}^d \to [0,\infty)$ such that for $z_b \in \mathbb{R}^d$,

$$\|z_b\|_r := \Big(\int_a^b (b-t)^{1-\alpha}|B(t)^{\mathrm{T}} z(t)|^2\,dt \Big)^{1/2},$$

with z being the solution of the adjoint system (3.62). Since $B \in L^\infty([a,b];\mathbb{R}^d)$ and $z \in L^2_{1-\alpha}([a,b];\mathbb{R}^d)$ we have that $\|\cdot\|_r$ is well defined. Using $z(t) = \Phi(t,b)^{\mathrm{T}} z_b$, we get that for every $\mu \in \mathbb{R}$, $\|\mu z_b\|_r = |\mu|\|z_b\|_r$. Furthermore, the Minkowski inequality for $L^2([a,b];\mathbb{R}^d)$ provides the triangle inequality for $\|\cdot\|_r$. Hence, $\|\cdot\|_r$ is a seminorm on $\mathbb{R}^d$, and the assumption (3.74) implies that it is a norm. From the equivalence of all norms on $\mathbb{R}^d$ it follows that there exists a constant $C > 0$ such that (3.73) holds. □

The next step is to show that controllability and observability conditions are equivalent.

3.2.17 Theorem *System* (3.42) *is controllable if and only if the adjoint system* (3.62) *is observable.*

Proof. ($\Leftarrow$) Suppose (3.62) is observable and (3.42) is not controllable. Then the controllability Gramian $W_\alpha(a,b)$ is singular and there exists $w \in \mathbb{R}^d \setminus \{0\}$ such that $B(t)^{\mathrm{T}}\Phi(t,b)^{\mathrm{T}}w = 0$, for almost every $t \in [a,b]$.

Let $\tilde{z} = \Phi(t,b)^{\mathrm{T}}w$ be the solution to the adjoint system with final condition $z_b = w$. Then we have $B(t)^{\mathrm{T}}\tilde{z}(t) = 0$, for almost every $t \in [a,b]$ and $z_b = w \neq 0$—a contradiction with the observability.

($\Rightarrow$) Suppose that (3.42) is controllable and that for every $C > 0$ there exists $z_b \in \mathbb{R}^d$ such that (3.73) does not hold. Then we can define a sequence of numbers $\frac{1}{C_k} \to 0$, $k \to \infty$, and vectors z_{kb} such that, without loss of generality, $|z_{kb}| = 1$ and

$$|z_{kb}|^2 > C_k \int_a^b (b-t)^{1-\alpha}|B(t)^{\mathrm{T}}z_k(t)|^2\,dt, \quad k \in \mathbb{N}.$$

Since z_{kb} is bounded sequence it has a convergent subsequence, which we do not relabel. Then for $z_b := \lim\limits_{k\to\infty} z_{kb}$ we have $|z_b| = 1$, and the solution $z(t)$ of the adjoint system with final condition z_b can be obtained as a limit of the solutions $z_k(t)$.

Now for solutions z_k we have $0 \le \int_a^b (b-t)^{1-\alpha}|B(t)^{\mathrm{T}}z_k(t)|^2\,dt < \frac{1}{C_k}|z_{kb}|^2$. Letting $k \to \infty$, we conclude that $B(t)^{\mathrm{T}}z(t) = 0$, almost everywhere on $[a,b]$.

Controllability assumption and (3.64) imply that for $y_a = 0$ and for every $y_b \in \mathbb{R}^d$ there exists a control u such that

$$\langle y_b, z_b\rangle = \int_a^b \langle u(t), B(t)^{\mathrm{T}}z(t)\rangle\,dt.$$

Since $B(t)^{\mathrm{T}}z(t) = 0$ almost everywhere, it follows that $\langle y_b, z_b\rangle = 0$, for every y_b. Therefore, $z_b = 0$, leading to a contradiction with $|z_b| = 1$. □

In the sequel we shall consider observability as a minimization problem. More precisely, we shall show that observability inequality implies the coercivity of the suitable quadratic functional, and that by minimizing the functional we obtain the $L^2_{\alpha-1}$-optimal control.

For fixed initial and final states $y_a, y_b \in \mathbb{R}^d$, we define the functional $J : \mathbb{R}^d \to \mathbb{R}$ by

$$J(z_b) = \frac{1}{2}\int_a^b (b-t)^{1-\alpha}|B(t)^{\mathrm{T}}z(t)|^2\,dt - \langle y_b, z_b\rangle + \langle y_a, z_a\rangle, \tag{3.75}$$

where $z(t)$ is the solution of (3.62), and $z_a = {}_t\mathrm{I}_b^{1-\alpha}z(t)|_{t=a}$.

3.2.18 Theorem *If system (3.42) is controllable (or equivalently (3.62) is observable), then the functional J has a minimum and the control function which steers the solution of (3.42) to the state $y(b) = y_b$ is given by*

$$\hat{u}(t) = (b-t)^{1-\alpha} B(t)^{\mathrm{T}} \hat{z}(t), \tag{3.76}$$

where $\hat{z}(t)$ is the solution of the adjoint problem with final state ${}_t\mathrm{I}_b^{1-\alpha}\hat{z}(t)|_{t=b} = \hat{z}_b$ being the minimum point of J. Furthermore, the control given by (3.76) is a control with minimum $L^2_{\alpha-1}$-norm, i.e., if $u \in L^2_{\alpha-1}([a,b];\mathbb{R}^N)$ is a control such that the solution of the system (3.42) satisfies $y(b) = y_b$, then

$$\int_a^b (b-t)^{\alpha-1} |\hat{u}(t)|^2 \, dt \leq \int_a^b (b-t)^{\alpha-1} |u(t)|^2 \, dt,$$

with equality if and only if u and $\hat{u}$ coincide almost everywhere on $[a,b]$.

Proof. Since J is continuous and convex, it suffices to show that it is coercive, i.e., that $\lim\limits_{|z_b|\to\infty} J(z_b) = \infty$. From (3.73) it follows that

$$|z_b|^2 \leq C \int_a^b (b-t)^{1-\alpha} |B(t)^{\mathrm{T}} z(t)|^2 \, dt.$$

Hence,

$$J(z_b) \geq \frac{|z_b|^2}{2C} - \langle y_b, z_b \rangle + \langle y_a, z_a \rangle \geq \frac{|z_b|^2}{2C} - |\langle y_b, z_b \rangle| + |\langle y_a, z_a \rangle|,$$

and, applying the Cauchy-Schwartz inequality,

$$J(z_b) \geq \frac{|z_b|^2}{2C} - |y_b||z_b| + |\langle y_a, z_a \rangle|.$$

Here the right hand side tends to $+\infty$ when $|z_b| \to \infty$, implying that

$$\lim_{|z_b|\to\infty} J(z_b) = \infty.$$

Let $\hat{z}_b$ be the point where J reaches its minimum. Then for every $z_b \in \mathbb{R}^d$ it holds

$$\lim_{h\to 0} \frac{J(\hat{z}_b + h z_b) - J(\hat{z}_b)}{h} = 0. \tag{3.77}$$

Denote by $\hat{z}(t)$ and $z(t)$ solutions of the adjoint problem (3.62) with final conditions ${}_t\mathrm{I}_b^{1-\alpha}\hat{z}|_{t=b} = \hat{z}_b$ and ${}_t\mathrm{I}_b^{1-\alpha}z|_{t=b} = z_b$, respectively. Since $\hat{z}(t) = \Phi(t,b)^{\mathrm{T}}\hat{z}_b$ and $z(t) =$

$\Phi(t,b)^{\mathrm{T}} z_b$, it follows that the solution z_h of equation (3.62) that satisfies condition ${}_t\mathrm{I}_b^{1-\alpha} z_h|_{t=b} = \hat{z}_b + h z_b$ is given by $z_h(t) = \hat{z}(t) + h z(t)$. Hence, using (3.75) we have

$$\begin{aligned}
J(\hat{z}_b + h z_b) &= \frac{1}{2}\int_a^b (b-t)^{1-\alpha} |B^{\mathrm{T}}(\hat{z}(t) + h z(t))|^2\, dt - \langle y_b, \hat{z}_b + h z_b\rangle + \langle y_a, \hat{z}_a + h z_a\rangle \\
&= \frac{1}{2}\int_a^b \langle B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}}(\hat{z}(t) + h z(t)), B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}}(\hat{z}(t) + h z(t))\rangle\, dt \\
&\quad - \langle y_b, \hat{z}_b\rangle - h\langle y_b, z_b\rangle + \langle y_a, \hat{z}_a\rangle + h\langle y_a, z_a\rangle \\
&= \frac{1}{2}\int_a^b |B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}}\hat{z}(t)|^2\, dt + + \frac{h^2}{2}\int_a^b |B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}} z(t)|^2\, dt \\
&\quad + h\int_a^b \langle B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}}\hat{z}(t), B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}} z(t)\rangle\, dt \\
&\quad - \langle y_b, \hat{z}_b\rangle - h\langle y_b, z_b\rangle + \langle y_a, \hat{z}_a\rangle + h\langle y_a, z_a\rangle \\
&= J(\hat{z}_b) + \frac{h^2}{2}\int_a^b |B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}} z(t)|^2\, dt - h\langle y_b, z_b\rangle + h\langle y_a, z_a\rangle \\
&\quad + h\int_a^b \langle B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}}\hat{z}(t), B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}} z(t)\rangle\, dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{J(\hat{z}_b + h z_b) - J(\hat{z}_b)}{h} &= \frac{h}{2}\int_a^b |B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}} z(t)|^2\, dt - \langle y_b, z_b\rangle + \langle y_a, z_a\rangle \\
&\quad + \int_a^b \langle B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}}\hat{z}(t), B^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}} z(t)\rangle\, dt.
\end{aligned}$$

Letting $h \to 0$ and using (3.77) we obtain

$$\int_a^b \langle B(t)^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}}\hat{z}(t), B(t)^{\mathrm{T}}(b-t)^{\frac{1-\alpha}{2}} z(t)\rangle\, dt + \langle y_a, z_a\rangle - \langle y_b, z_b\rangle = 0. \tag{3.78}$$

Since $z_b \in \mathbb{R}^d$ was arbitrary, we have that the function $\hat{u}(t) = (b-t)^{1-\alpha} B(t)^{\mathrm{T}}\hat{z}(t)$ satisfies (3.65), for every z_b. Hence, Proposition 3.2.12 (ii) implies that the control $\hat{u}$

steers the solution to $y(b) = y_b$. The proof of $L^2_{\alpha-1}$-optimality of $\hat{u}$ follows the same lines as the proof of the second statement from Theorem 3.2.13 □

3.2.19 Remark Comparing the results from Proposition 3.2.6, Theorem 3.2.13 and Theorem 3.2.18, we conclude that

$$\overline{u}(t) = u^*(t) = \hat{u}(t), \quad \text{for almost every } t \in [a, b].$$

Furthermore, from (3.78), we have that for every $z_b \in \mathbb{R}^d$

$$\int_a^b (b-t)^{1-\alpha} \langle B(t)^{\mathrm{T}} \Phi(t,b)^{\mathrm{T}} \hat{z}_b, B(t)^{\mathrm{T}} \Phi(t,b)^{\mathrm{T}} z_b \rangle \, dt + \langle y_a, z_a \rangle - \langle y_b, z_b \rangle = 0.$$

Using the property $\langle y_a, z_a \rangle = \langle \Psi(a,b) y_a, z_b \rangle$, from Lemma 3.2.11, we obtain

$$\langle \int_a^b (b-t)^{1-\alpha} \Phi(t,b) B(t) B(t)^{\mathrm{T}} \Phi(t,b)^{\mathrm{T}} dt \, \hat{z}_b, z_b \rangle + \langle \Psi(a,b) y_a, z_b \rangle - \langle y_b, z_b \rangle = 0.$$

By noticing that the integral in the above equation is the controllability Gramian matrix (3.44), we get

$$\langle W_\alpha(a,b) \hat{z}_b + \Psi(a,b) y_a - y_b, z_b \rangle = 0, \quad \text{for every } z_b \in \mathbb{R}^d.$$

Hence $W_\alpha(a,b)\hat{z}_b + \Psi(a,b) y_a - y_b = 0$, and we obtain the value of the minimum point of $J(z_b)$:

$$\hat{z}_b = W_\alpha(a,b)^{-1} (y_b - \Psi(a,b) y_a). \tag{3.79}$$

Therefore, the optimal control $\hat{u}$ from Theorem 3.2.18 reduces precisely to the control $\overline{u}$ given by (3.46).

Theorem 3.2.18 provides several estimates of the control function $\hat{u}$.

3.2.20 Proposition *The control function $\hat{u}$ given by (3.76) satisfies*

(i) $\|\hat{u}\|_{L^2_{\alpha-1}} \leq \sqrt{C} |y_b - \Psi(a,b) y_a|$, *where C is the observability constant from Definition 3.2.15.*

(ii) $\|\hat{u}\|_\infty \leq M_F \|B\|_\infty |\hat{z}_b|$, *where M_F is the constant from Proposition 2.3.16.*

Proof. (i) Theorem 3.2.18 and Proposition 3.2.12 imply

$$\|\hat{u}\|^2_{L^2_{\alpha-1}} = \int_a^b (b-t)^{1-\alpha} |B(t)^{\mathrm{T}} \hat{z}(t)|^2 \, dt = \langle y_b, \hat{z}_b \rangle - \langle y_a, \hat{z}_a \rangle. \tag{3.80}$$

Furthermore, from Lemma 3.2.11 and (3.73) it follows

$$\|\hat{u}\|^2_{L^2_{\alpha-1}} = \langle y_b - \Psi(a,b)y_a, \hat{z}_b\rangle \leq |y_b - \Psi(a,b)y_a||\hat{z}_b| \leq |y_b - \Psi(a,b)y_a|\sqrt{C}\|\hat{u}\|_{L^2_{\alpha-1}}.$$

After dividing by $\|\hat{u}\|_{L^2_{\alpha-1}}$, we obtain the desired estimate.

(ii) Follows directly from $\hat{z}(t) = \Phi(t,b)^{\mathrm{T}}\hat{z}_b$ and Proposition 2.3.16. □

Let us illustrate the obtained results with one example.

3.2.21 Example Consider a control problem governed by the following system:

$$\begin{aligned} {}^C_0\mathrm{D}^\alpha_t y_1(t) &= t y_2(t), \\ {}^C_0\mathrm{D}^\alpha_t y_2(t) &= u(t). \end{aligned} \tag{3.81}$$

Assume that we have initial state $y_0 = (1,1)$ and that for given time interval $t \in [0,T]$ we need to find control function $u : [0,T] \to \mathbb{R}$ which will steer the solution to $y(T) = (0,0)$. In (3.81) matrices A and B are the following

$$A(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the associated state-transition matrices are given by

$$\Psi(0,t) = \begin{bmatrix} 1 & \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Phi(\tau,t) = \begin{bmatrix} \phi_1(\tau,t) & \phi_2(\tau,t) \\ 0 & \phi_1(\tau,t) \end{bmatrix},$$

where $\phi_1(\tau,t) = \dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$ and $\phi_2(\tau,t) = \dfrac{\alpha}{\Gamma(2\alpha+1)}(t-\tau)^{2\alpha-1}(t+\tau)$.

Now, the controllability Gramian is equal to

$$\begin{aligned} W_\alpha(0,T) &= \int_0^T (T-t)^{1-\alpha} \begin{bmatrix} \phi_2(t,T)^2 & \phi_1(t,T)\phi_2(t,T) \\ \phi_1(t,T)\phi_2(t,T) & \phi_1(t,T)^2 \end{bmatrix} dt \\ &= \begin{bmatrix} c_1 T^{3\alpha+2} & c_{12}T^{2\alpha+1} \\ c_{12}T^{2\alpha+1} & c_2 T^\alpha \end{bmatrix}, \end{aligned}$$

with constants

$$\begin{aligned} c_1 &= \frac{\alpha}{3\Gamma(2\alpha+1)^2}\Big(1 + \frac{2}{3\alpha+1} + \frac{2}{(3\alpha+1)(3\alpha+2)}\Big), \\ c_{12} &= \frac{\alpha+1}{\Gamma(\alpha)\Gamma(2\alpha+2)}, \\ c_2 &= \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}. \end{aligned}$$

It can be shown that for $\alpha \in (0,1)$,

$$D_\alpha = \det(W_\alpha(0,T)) = T^{4\alpha+2}(c_1c_2 - c_{12}^2) > 0$$

hence, system (3.81) is controllable. Using representation of the $L^2_{\alpha-1}$-optimal control function (3.46), we obtain

$$\overline{u}(t) = a_1(T-t)^\alpha(t+T) + a_2,$$

where

$$\begin{aligned} a_1 &= \frac{\alpha T^\alpha}{D_\alpha \Gamma(2\alpha+1)}\Big(c_{12}T^{\alpha+1} - c_2 - \frac{c_2 T^{\alpha+1}}{\Gamma(\alpha+2)}\Big) \\ a_2 &= \frac{T^{2\alpha+1}}{D_\alpha \Gamma(\alpha)}\Big(c_{12} + \frac{c_{12}T^{\alpha+1}}{\Gamma(\alpha+2)} - c_1 T^{\alpha+1}\Big). \end{aligned}$$

3.2.22 Remark Note that for a special case of problem (3.55), given by

$$\begin{aligned} {}_a^C\mathrm{D}_t^\alpha y(t) &= -Ag(t)y(t) + Bu(t), \quad t \in [a,b], \\ y(a) &= y_a, \end{aligned}$$

where $A \in \mathbb{R}^{d\times d}$ is a symmetric positive semidefinite matrix and $g : [a,b] \to [0,\infty)$ is a continuous function, we have stronger estimates for matrices Φ and Ψ. More precisely, from Proposition 2.3.20 it follows that

$$\|\Psi(a,t)\| \le 1 \quad \text{and} \quad \|(t-\tau)^{1-\alpha}\Phi(\tau,t)\| \le 1,$$

which will give us the following bounds for the solution of the adjoint problem and the control function constructed as in the Theorem 3.2.18:

$$|(b-t)^{1-\alpha}z(t)| \le |z_b| \quad \text{and} \quad |\hat{u}(t)| \le \|B\|_\infty|\hat{z}_b|, \quad t \in [a,b].$$

3.2.23 Remark Here also we want to emphasize that in the case $\frac{1}{2} < \alpha < 1$, the extra term $(b-t)^{1-\alpha}$ can be omitted from (3.73), (3.75) and (3.76). In that case, the results from Theorem 3.2.17 and Theorem 3.2.18 remain valid, and we are able to obtain the L^2-optimal control function $\hat{u}(t) = B(t)^{\mathrm{T}}\hat{z}(t)$.

3.2.24 Remark Let us mention that the results presented in Section 3.1.3 can also be applied for the system (3.42). Since the solution of the adjoint system (3.62) is in the space $L^1([a,b];\mathbb{R}^d)$, we can replace observability inequality (3.30) by (3.38), define the functional J_{bb} as in (3.37) and proceed in the same way as in the integer-derivative case, to obtain the bang-bang control (3.41).

Chapter 4

Nonlinear Control

Nonlinear control theory is a research area of a great interest since many processes modeled by dynamical systems have nonlinear nature. When dealing with nonlinearities, one does not have a general method for finding a solution or for proving its existence, and in the most cases, is not able to find an analytical representation of the solution or control function. That is one of the reasons why nonlinear control problems are very challenging and require various mathematical skills.

This chapter is devoted to the analysis of nonlinear control problems given by

$$\begin{aligned} {}_0^C\mathrm{D}_t^\alpha y(t) &= -Af(y(t))y(t) + Bu(t), \quad t \in [0,T] \\ y(0) &= y_0, \quad y(T) = y_T, \end{aligned} \tag{4.1}$$

where ${}_0^C\mathrm{D}_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1]$, which for $\alpha = 1$ reduces to the classical first order derivative. We assume that, for system (4.1), the following conditions hold

(a1) $f : \mathbb{R}^d \to (0,\infty)$ is a continuous function;

(a2) $A \in \mathbb{R}^{d\times d}$ is a real symmetric positive semidefinite matrix;

(a3) $B \in \mathbb{R}^{d\times N}$ is a real matrix;

(a4) A and B satisfy the Kalman rank condition: $\operatorname{rank}[B|AB|A^2B|\cdots|A^{d-1}B] = d$.

The motivation for the research of model (4.1) comes from a porous media equation, as well as from the models of population dynamics that describe the interactions between the species which tend to avoid crowding. The PDE models of such interactions were studied in [11, 59]. In [11] the space-time distribution of the species was considered, and here we have a simpler case where we are interested in a global density of species (it depends only on time). An example of an ODE model of this type of system can be found in [19]. Non-positive definiteness of the matrix $-A$

represents the dispersion tendency, while the function f determines the intensity of dispersion (large $f(y)$ implies faster dispersion). The matrix B limits our possibility to influence the population densities, while u represents controlled "birth" or "death" rate. We stress that a fundamental issue here is how to choose a source term (usually called the control) in modeling equations which would govern the system from the given initial density to the prescribed final density. Finally, we note that the fractional derivative models memory effect, which in this case could be influence of the accumulated species.

The results presented in Section 4.1 are based on the ideas developed in [19], while here slightly different methods are used, which are more suitable for systems with fractional derivatives. The Section 4.2 contains original results from [34].

4.1 Controllability of a nonlinear system of ODEs

The year 1960 is considered to be the beginning of a "new era" or "modern era" of control theory. That year the first IFAC (International Federation of Automatic Control) Congress was held in Moscow. Furthermore, many crucial methods and results in control theory were developed during the 1960s. Particularly, nonlinear control became an important subject of research. The analysis of nonlinear control problems started with works on stability: R. Kalman and J. Bertram reintroduced the Lyapunov methods on stability into control theory context [39], E. Lorenz established the basis of modern chaos theory, V. Popov presented new techniques for absolute stability [58], etc. At the same time, from the aspect of optimal control, two historical contributions were made by mathematicians L. Pontryagin (Pontryagin's maximum principle) and R. Bellman (dynamic programming, the Hamilton-Jacobi-Bellman equation).

Over the last 60 years nonlinear control problems have been extensively studied and there is a great amount of literature addressing these problems. Here we single out some of the results, relevant to our problems.

For nonlinear control systems, the question of controllability has two distinct aspects—local and global controllability. Local controllabilty is related to a fixed state, more precisely, to an equilibrium point.

Consider an autonomous system

$$y' = f(y, u), \tag{4.2}$$

with $y : [0, T] \to \mathbb{R}^d$, $u : [0, T] \to \mathbb{R}^N$ and $f : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d$. The point (y_e, u_e) is called an equilibrium point of f if $f(y_e, u_e) = 0$. Analyzing linearized system around (y_e, u_e)

$$y' = \frac{\partial f}{\partial y}(y_e, u_e)y + \frac{\partial f}{\partial u}(y_e, u_e)u,$$

and conditions for its controllability, one can obtain small-time local controllability of (4.2) at (y_e, u_e). For more details see [17, Sec. 3.1.].

On the other hand, the question of global controllability is more complex and there are no general criteria (necessary and sufficient conditions for controllability) that can be applied for any nonlinear system. There are, however, many results for some special classes of nonlinear systems. Here we mention two main approaches in nonlinear control.

- *Geometric approach.* Using theory of Lie algebras, Lie groups and Lie brackets, one can derive necessary and sufficient conditions for global controllability of control systems given by

$$y' = f_0(x) + \sum_{i=1}^{N} u_i f_i(y),$$

where $f_i \in C^\infty(\Omega; \mathbb{R}^d)$, $i = 0, \ldots, N$, and Ω is a connected, nonempty open subset of $\mathbb{R}^d$, see [36] and [17, Sec. 3.2. & 3.3.].

- *Fixed point approach.* In the survey [5], the authors gave an overview of the results related to applications of fixed point theorems to controllability of nonlinear systems. For an overview of more recent results we refer to [17, Sec. 3.5.], where, using fixed point theorem approach, the author proved global controllability for the system

$$y' = A(t, y)y + B(t, y)u + f_0(t, y),$$

under assumptions: $A \in L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R}^{d\times d})$, $B \in L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R}^{N\times d})$ and $f \in L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ (cf. [17, Th. 3.40.]). Furthermore, an interesting result based on degree theory and homogeneity is given in [17, Th. 3.46.], where the author proved global controllability of the system

$$y' = Ay + F(y) + Bu,$$

under a certain homogeneity condition of nonlinear term $F \in C^1(\mathbb{R}^d; \mathbb{R}^d)$.

In this section we consider nonlinear control problem (4.1) with $\alpha = 1$, i.e.,

$$\begin{aligned} y'(t) &= -\,Af(y)y + Bu, \quad t \in [0, T] \\ y(0) &= y_0, \quad y(T) = y_T \end{aligned} \tag{4.3}$$

and assume that conditions (a1)–(a4) are satisfied. Under these conditions, we shall prove global controllability of nonlinear system (4.3) with unbounded dynamics (f does not need to be uniformly bounded). Global controllability of this type of systems (in both deterministic and stochastic case) was studied in [19]. Here, we give a different version of the proof. The main idea is the same—to consider linearization of (4.3) given by

$$\begin{aligned} y'(t) &= -Af(v)y + Bu, \quad t \in [0,T] \\ y(0) &= y_0, \end{aligned} \tag{4.4}$$

where $v \in C([0,T];\mathbb{R}^d)$, and prove the existence of a fixed point $y = v$, using the Schauder fixed point theorem. The difference is in the construction of the solution of a linearized problem.

To begin with, we introduce notation and derive a few auxiliary results. For every $v \in C([0,T];\mathbb{R}^d)$, define constants

$$M_v := \max_{t\in[0,T]} |f(v(t))|, \quad K_v := \max\{1, M_v\}, \quad \text{and} \quad T_v := T - \frac{T}{K_v}. \tag{4.5}$$

4.1.1 Lemma *Let $f \in C(\mathbb{R}^d;(0,\infty))$ and let $\{v_n\}$ be a sequence in $C([0,T];\mathbb{R}^d)$ which converges uniformly on $[0,T]$ to a function $v \in C([0,T];\mathbb{R}^d)$. Then $\{f \circ v_n\}$ converges uniformly on $[0,T]$ to $f \circ v$. Furthermore, M_{v_n}, K_{v_n} and T_{v_n} converge to M_v, K_v and T_v, respectively.*

Proof. Since $\{v_n\}$ is a convergent sequence, it follows that it is bounded too, and there exists $K > 0$ such that

$$\max_{t\in[0,T]} |v(t)| \le K \quad \text{and} \quad \max_{t\in[0,T]} |v_n(t)| \le K, \text{ for every } n \in \mathbb{N}.$$

Let $B_K = \{x \in \mathbb{R}^d : |x| \le K\}$. Since f is continuous on $\mathbb{R}^d$ and B_K is a compact set, we have that f is uniformly continuous on B_K. For a given $\varepsilon > 0$, let $\delta > 0$ be such that

$$(\forall x, y \in B_K)(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon).$$

By choosing n_0 such that for $n > n_0$, $\max_{t\in[0,T]} |v_n(t) - v(t)| < \delta$, we get

$$n > n_0 \Rightarrow \max_{t\in[0,T]} |f(v_n(t)) - f(v(t))| < \varepsilon.$$

Since ε was arbitrary, it follows that $f \circ v_n$ converges uniformly to $f \circ v$ on $[0,T]$. Hence, $M_{v_n} = \|f \circ v_n\|_C \to \|f \circ v\|_C = M_v$, $n \to \infty$, which further implies $K_{v_n} \to K_v$ and $T_{v_n} \to T_v$, $n \to \infty$. □

Now, for every v, we construct the solution of (4.4) in a form

$$y(t) = \begin{cases} y_1(t), & 0 \le t \le T_v \\ y_2(t), & T_v < t \le T \end{cases}, \qquad u(t) = \begin{cases} 0, & 0 \le t \le T_v \\ u_2(t), & T_v < t \le T \end{cases}, \tag{4.6}$$

where $y_1(t)$ is a restriction on $[0,T_v]$ of the solution to initial value problem

$$\begin{aligned} y'(t) =& -Af(v)y, \quad t\in[0,T] \\ y(0) =& y_0, \end{aligned} \tag{4.7}$$

and $y_2(t)$ is a solution of the control problem

$$\begin{aligned} y'(t) =& -Af(v)y + Bu, \quad t\in(T_v,T] \\ y(T_v) =& y_1(T_v), \quad y(T) = y_T, \end{aligned} \tag{4.8}$$

obtained with control function $u(t) = u_2(t)$ given by (3.7), i.e., control with minimal L^2-norm. From Proposition 3.1.5 and assumptions (a1)–(a4), it follows that the system (4.8) is controllable for every $v \in C([0,T];\mathbb{R}^d)$.

Denote by Π_v the state-transition matrix of the system (4.7). We have

$$\Pi_v(\tau,t) = e^{-A\int\limits_{\tau}^{t} f(v(s))\,ds}, \quad (\tau,t)\in[0,T]^2,$$

and solution y_1 is given by

$$y_1(t) = \Pi_v(0,t)y_0, \quad t\in[0,T]. \tag{4.9}$$

Further, denote by W_v the controllability Gramian associated to control problem (4.8), i.e.,

$$W_v = W(T_v,T) = \int\limits_{T_v}^{T} \Pi_v(t,T)BB^{\mathrm{T}}\Pi_v(t,T)^{\mathrm{T}}\,dt. \tag{4.10}$$

Then

$$u_2(t) = B^{\mathrm{T}}\Pi_v(t,T)^{\mathrm{T}}W_v^{-1}(y_T - \Pi_v(T_v,T)y_1(T_v)),$$

and solution y_2 is given by

$$y_2(t) = \Pi_v(T_v,t)y_1(T_v) + \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau.$$

By noticing that $y_1(T_v) = \Pi_v(0,T_v)y_0$ and $\Pi_v(T_v,t)\Pi_v(0,T_v) = \Pi_v(0,t)$, for any $t\in[T_v,T]$, we can express u_2 and y_2 as given below:

$$u_2(t) = B^{\mathrm{T}}\Pi_v(t,T)^{\mathrm{T}}W_v^{-1}(y_T - \Pi_v(0,T)y_0), \tag{4.11}$$

$$y_2(t) = \Pi_v(0,t)y_0 + \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau. \tag{4.12}$$

Furthermore, from (4.6), (4.9) and (4.12), we have that

$$y(t) = \Pi_v(0,t)y_0 + y_p(t), \quad t \in [0,T], \tag{4.13}$$

where

$$y_p(t) = \begin{cases} 0, & 0 \le t \le T_v \\ \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau, & T_v < t \le T \end{cases}.$$

4.1.2 Lemma *Let $v \in C([0,T];\mathbb{R}^d)$. There exist constants $\lambda \ge 0$, $c_w > 0$ and $C_u > 0$, not depending on v such that*

(i) $e^{-\lambda M_v T} \le \|\Pi_v(\tau,t)\| \le 1$, *for every* $0 \le \tau \le t \le T$;

(ii) $\|W_v^{-1}\| \le \frac{K_v}{c_w}$;

(iii) $|u_2(t)| \le K_v C_u$, *for every* $t \in [T_v, T]$.

Proof. (i) Since A is symmetric and positive semidefinite, from Remark 2.3.18 we have $A = UDU^{\mathrm{T}}$, where $D = \operatorname{diag}(\lambda_1,\dots,\lambda_d)$, $\lambda_i \ge 0$, and U is an orthogonal matrix. Hence,

$$\Pi_v(\tau,t) = UD_v(\tau,t)U^{\mathrm{T}},$$

where

$$D_v(\tau,t) = \operatorname{diag}\Big(e^{-\lambda_1 \int\limits_{\tau}^{t} f(v(s))\,ds}, \dots, e^{-\lambda_d \int\limits_{\tau}^{t} f(v(s))\,ds}\Big).$$

Using $0 < f(v(s)) \le M_v$, for every $s \in [0,T]$, and taking $\lambda = \max\limits_{1\le i\le d} \lambda_i$, we get that for every $i = 1,\dots,d$ it holds

$$e^{-\lambda M_v T} \le e^{-\lambda_i \int\limits_{\tau}^{t} f(v(s))\,ds} \le 1,$$

which in turn implies desired estimate for $\Pi_v(\tau,t)$.

(ii) Since W_v is nonsingular, we can determine the norm of its inverse in the form $\|W_v^{-1}\| = \dfrac{1}{s}$, where $s = \min\limits_{x\in S^{d-1}} |W_v x|$ and $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. For any $x \in S^{d-1}$ we have

$$\begin{aligned} |W_v x| &\ge |x^{\mathrm{T}} W_v x| = \Big| \int\limits_{T_v}^{T} x^{\mathrm{T}} \Pi_v(t,T)BB^{\mathrm{T}}\Pi_v(t,T)^{\mathrm{T}} x\,dt \Big| \\ &= \int\limits_{T_v}^{T} |x^{\mathrm{T}}\Pi_v(t,T)B|^2\,dt = \int\limits_{T_v}^{T} |x^{\mathrm{T}} e^{-A\int\limits_{t}^{T} f(v(s))\,ds} B|^2\,dt. \end{aligned}$$

Introducing change of variables $\tau = K_v(T-t)$ and $\xi = K_v(T-s)$, and denoting $\tilde{\Pi}_v(\tau,T) = e^{-A\int_0^\tau \frac{f(v(T-\xi/K_v))}{K_v}\,d\xi}$, the above inequality reduces to

$$|W_v x| \geq \frac{1}{K_v}\int_0^T |x^{\mathrm{T}}\tilde{\Pi}_v(\tau,T)B|^2\,d\tau. \tag{4.14}$$

From the proof of part (i), we have $\tilde{\Pi}_v(\tau,T) = U\tilde{D}_v(\tau,T)U^{\mathrm{T}}$, where $\tilde{D}_v(\tau,T) = \operatorname{diag}(\tilde{p}_1(\tau),\ldots,\tilde{p}_d(\tau))$ and $\tilde{p}_i(\tau) = e^{-\lambda_i\int_0^\tau \frac{f(v(T-\xi/K_v))}{K_v}\,d\xi}$, $i = 1,\ldots,d$.

From the definition of K_v and assumption (a1), it follows

$$0 < \frac{f(v(\cdot))}{K_v} \leq 1, \text{ on } [0,T],$$

and we have uniform boundedness (with respect to both v and τ) of $\tilde{p}_i(\tau)$:

$$0 < e^{-\lambda T} \leq \tilde{p}_i(\tau) \leq 1, \quad \tau \in [0,T].$$

Therefore, there exists a constant $c_w > 0$ (independent of v), such that

$$\int_0^T |x^{\mathrm{T}}\tilde{\Pi}_v(\tau,T)B|^2\,d\tau \geq c_w,$$

and (4.14) implies

$$\min_{x\in S^{d-1}} |W_v x| \geq \frac{c_w}{K_v}.$$

Hence, $\|W_v^{-1}\| \leq \frac{K_v}{c_w}$.

(iii) Let $b = \max\{|b_{ij}| : i = 1,\ldots,d,\ j = 1,\ldots,N\}$. From (i) and (ii), we have that control function given by (4.11) satisfies

$$|u_2(t)| \leq b\frac{K_v}{c_w}(|y_T| + |y_0|) = K_v C_u, \quad t \in [T_v, T],$$

with $C_u = \frac{b(|y_T|+|y_0|)}{c_w}$. □

Now, we are able to prove controllability of our nonlinear problem.

4.1.3 Theorem *Assume that (a1)–(a4) hold. Then, for any $T > 0$ and $y_0, y_T \in \mathbb{R}^d$, there exists $u \in L^2([0,T];\mathbb{R}^N)$ such that the solution of (4.3) satisfies $y(T) = y_T$.*

Proof. Define the mapping $\mathcal{T} : C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$ which every $v \in C([0,T];\mathbb{R}^d)$ maps to the solution $y(t)$ of linear control problem (4.4), given by (4.13) and constructed as described above.

$\mathcal{T}$ **is continuous.** Assume that $\{v_n\}$ is a sequence in $C([0,T];\mathbb{R}^d)$ which converges uniformly to a function $v \in C([0,T];\mathbb{R}^d)$. Then $\{v_n\}$ is bounded in $C([0,T];\mathbb{R}^d)$, and continuity of f implies the existence of a constant $K > 1$ such that

$$\max_{t\in[0,T]} |f(v(t))| \le K \quad \text{and} \quad \max_{t\in[0,T]} |f(v_n(t))| \le K, \quad n \in \mathbb{N}. \tag{4.15}$$

Let $y = \mathcal{T}(v)$ and $y_n = \mathcal{T}(v_n)$, $n \in \mathbb{N}$. Let us show that y_n converges uniformly to y on $[0,T]$. From (4.13) we have

$$\max_{t\in[0,T]} |y_n(t) - y(t)| \le \max_{t\in[0,T]} |\Pi_{v_n}(0,t)y_0 - \Pi_v(0,t)y_0| + \max_{t\in[0,T]} |y_{p,n}(t) - y_p(t)|. \tag{4.16}$$

From continuous dependence of the solution of initial value problem (4.7), with respect to the coefficients v_n, we get that the first term on the right hand side of (4.16) tends to 0, when $n \to \infty$. More precisely, for any $\varepsilon_1 > 0$, there exists $n_1 \in \mathbb{N}$ such that for $n \ge n_1$ it holds

$$\max_{t\in[0,T]} |\Pi_{v_n}(0,t)y_0 - \Pi_v(0,t)y_0| < \varepsilon_1. \tag{4.17}$$

Now we consider second term in (4.16). Since $T_{v_n} \to T_v$, $n \to \infty$, for every $\delta > 0$, we can find $n_0 \in \mathbb{N}$ such that $T_{v_n} \in (T_v - \delta, T_v + \delta)$, for $n \ge n_0$. Then, for $n \ge n_0$ we have

$$y_{p,n}(t) - y_p(t) = \begin{cases} 0, & t \in [0, T_v - \delta] \\ R_n(t), & t \in (T_v - \delta, T_v + \delta) \\ \int\limits_{T_{v_n}}^{t} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau - \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau, & t \in [T_v + \delta, T] \end{cases}$$

where, if $T_{v_n} \le T_v$, the function $R_n(t)$ is given by

$$R_n(t) = \begin{cases} 0, & T_v - \delta < t \le T_{v_n} \\ \int\limits_{T_{v_n}}^{t} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau, & T_{v_n} < t \le T_v \\ \int\limits_{T_{v_n}}^{t} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau - \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau, & T_v < t \le T_v + \delta \end{cases}$$

and similarly, if $T_v < T_{v_n}$,

$$R_n(t) = \begin{cases} 0, & T_v - \delta < t \leq T_v \\ \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_{2,}(\tau)\,d\tau, & T_v < t \leq T_{v_n} \\ \int\limits_{T_{v_n}}^{t} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau - \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau, & T_{v_n} < t \leq T_v + \delta \end{cases}.$$

In both cases, for integrals $\int\limits_{T_{v_n}}^{t} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau$ and $\int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau$, from Lemma 4.1.2 and (4.15) we get

$$\begin{aligned} \Big|\int\limits_{T_{v_n}}^{t} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau\Big| &\leq \int\limits_{T_{v_n}}^{t} \|\Pi_{v_n}(\tau,t)\|\|B\||u_{2,n}(\tau)|\,d\tau \\ &\leq \|B\|C_uK(t - T_{v_n}) \\ &< \|B\|C_uK\delta \end{aligned}$$

Thus, on $[T_v - \delta, T_v + \delta]$ we have

$$\max_{[T_v-\delta,T_v+\delta]} |y_{p,n}(t) - y_p(t)| = \max_{[T_v-\delta,T_v+\delta]} |R_n(t)| < 2\|B\|C_uK\delta. \tag{4.18}$$

For $t \in [T_v + \delta, T]$

$$\begin{aligned} y_{p,n}(t) - y_p(t) &= \int\limits_{T_{v_n}}^{t} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau - \int\limits_{T_v}^{t} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau \\ &= \int\limits_{T_{v_n}}^{T_v+\delta} \Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)\,d\tau - \int\limits_{T_v}^{T_v+\delta} \Pi_v(\tau,t)Bu_2(\tau)\,d\tau \\ &\quad + \int\limits_{T_v+\delta}^{t} \Big(\Pi_{v_n}(\tau,t)Bu_{2,n}(\tau) - \Pi_v(\tau,t)Bu_2(\tau)\Big)\,d\tau. \end{aligned}$$

Hence, using uniform boundedness of Π_{v_n} and $u_{2,n}$ we get

$$|y_{p,n}(t) - y_p(t)| \leq 2\|B\|C_uK\delta + \int\limits_{T_v+\delta}^{t} \Big|\Pi_{v_n}(\tau,t)Bu_{2,n}(\tau) - \Pi_v(\tau,t)Bu_2(\tau)\Big|\,d\tau. \tag{4.19}$$

Now, from continuous dependence on coefficients, we have that $\Pi_{v_n}(\tau,t)$ converges uniformly to $\Pi_v(\tau,t)$ on $[0,T]\times[0,T]$. Moreover, from definition of the Gramian W_{v_n}, (4.10), and Lemma 4.1.1, it follows $W_{v_n}\to W_v$, $n\to\infty$.

Let us denote by $\overline{u}_{2,n}$ and $\overline{u}_2$ the extensions of $u_{2,n}$ and u_2, on $[0,T]$. More precisely, $\overline{u}_{2,n}$ and $\overline{u}_2$ are given by (4.11), with $t\in[0,T]$. Then $\Pi_{v_n}(\tau,t)B\overline{u}_{2,n}(\tau)$ converges uniformly to $\Pi_v(\tau,t)B\overline{u}_2(\tau)$ on $[0,T]^2$. Hence, for every $\varepsilon_2>0$ there exists $n_2\in\mathbb{N}$ such that for $n\geq n_2$ it holds

$$\max_{[0,T]^2}\Big|\Pi_{v_n}(\tau,t)B\overline{u}_{2,n}(\tau)-\Pi_v(\tau,t)B\overline{u}_2(\tau)\Big|<\varepsilon_2.$$

Therefore, for $n\geq\max\{n_0,n_2\}$ we have

$$\max_{[T_v+\delta,T]^2}\Big|\Pi_{v_n}(\tau,t)Bu_{2,n}(\tau)-\Pi_v(\tau,t)Bu_2(\tau)\Big|<\varepsilon_2.$$

and (4.19) implies

$$\max_{t\in[T_v+\delta,T]}|y_{p,n}(t)-y_p(t)|\leq 2\|B\|C_uK\delta+T\varepsilon_2. \tag{4.20}$$

Summarizing all the results, (4.16), (4.17), (4.18) and (4.20), we obtain that for any $\varepsilon>0$, by taking $\varepsilon_1,\varepsilon_2$ and δ sufficiently small, there exists $n^*=\max\{n_0,n_1,n_2\}$ such that

$$n\geq n^*\quad\Rightarrow\quad\max_{t\in[0,T]}|y_n(t)-y(t)|<\varepsilon,$$

which concludes the proof of continuity for $\mathcal{T}$.

Now we move to the compactness. First, let us derive estimates of the solutions. From Lemma 4.1.2, we get

$$|y_1(t)|=|\Pi_v(0,t)y_0|\leq|y_0|,\quad t\in[0,T_v), \tag{4.21}$$

and for $t\in[T_v,T]$ we have

$$\begin{aligned}|y_2(t)| &= |\Pi_v(0,t)y_0+\int_{T_v}^{t}\Pi_v(\tau,t)Bu(\tau)\,d\tau|\\ &\leq |y_0|+\int_{T_v}^{t}\|\Pi_v(\tau,t)\||Bu(\tau)|\,d\tau\leq|y_0|+bC_uK_v(T-T_v)\\ &= |y_0|+bC_uT\\ &=: C_y.\end{aligned}$$

Hence, for every $t\in[0,T]$ it holds

$$|y(t)|\leq C_y. \tag{4.22}$$

$\mathcal{T}$ **is compact.** Let V be a bounded set in $C([0,T];\mathbb{R}^d)$. Since f is continuous, there exists $K > 1$ such that, for every $v \in V$, $\max_{t\in[0,T]} |f(v(t))| \leq K$, further implying $K_v \leq K$. We need to prove that $Y := \mathcal{T}(V)$ is relatively compact set. To that end, let $\{y_n\}$ be a sequence in Y. For every $n \in \mathbb{N}$, denote by v_n the function from V such that $\mathcal{T}(v_n) = y_n$. From (4.22) we have that $\{y_n\}$ is uniformly bounded. Furthermore, from (4.7) and (4.21), it follows

$$|y'_{n,1}(t)| \leq \|A\|K|y_0|, \quad t \in [0,T_v]. \tag{4.23}$$

Similarly, from (4.8), (4.22) and Lemma 4.1.2 (iii), we get

$$|y'_{n,2}(t)| \leq \|A\|KC_y + \|B\|C_uK_v \leq (\|A\|C_y + \|B\|C_u)K, \quad t \in (T_v, T). \tag{4.24}$$

Therefore, the derivatives $y'_n(t)$ are also uniformly bounded, implying that $\{y_n\}$ is equicontinuous sequence. Hence, the Arzela-Ascoli theorem implies that $\{y_n\}$ has a convergent subsequence, which concludes the proof of compactness.

Uniform boundedness of the solution (4.22) also implies that the set

$$\{v \in C([0,T];\mathbb{R}^d) : v = \omega\mathcal{T}(v),\ \omega \in [0,1]\}$$

is bounded.

Therefore, $\mathcal{T}$ satisfies conditions of the Leray-Schauder fixed point theorem, and we have the existence of the fixed point $y^* = v^*$, which is the desired solution of nonlinear control problem (4.3). □

Let us note that analog results can be obtained for more general class of nonlinear control problems, given by

$$\begin{aligned} y'(t) &= -A(t,y)y + B(t,y)u, \quad t \in [0,T] \\ y(0) &= y_0, \quad y(T) = y_T, \end{aligned} \tag{4.25}$$

if they satisfy the following conditions:

(A1) $A : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a symmetric, positive semidefinite and continuous matrix function;

(A2) $B : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times N}$ is an essentially bounded matrix function, i.e., $B \in L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^{d\times N})$;

(A3) for every $v \in C([0,T];\mathbb{R}^d)$, $T_1 \in (0,T)$ and $y_1 \in \mathbb{R}^d$, controllability Gramian of a linearized problem

$$\begin{aligned} y'(t) &= -A(t,v)y + B(t,v)u, \quad t \in [T_1,T] \\ y(T_1) &= y_1, \quad y(T) = y_T, \end{aligned}$$

is a regular matrix, i.e.,

$$W(T_1,T)=\int_{T_1}^{T}\Pi_v(t,T)B(t,v)B(t,v)^{\mathrm{T}}\Pi_v(t,T)^{\mathrm{T}}\,dt$$

has an inverse.

4.1.4 Remark Let us mention that the construction of the solution of linearized problem (4.4) can also be performed in the following way:

- choose $K_v=\max\{2,M_v\}$ and let $T_v^1=\frac{T}{K_v}$ and $T_v^2=T-\frac{T}{K_v}$;
- find the control $u_{1,v}$ which will steer the solution of (4.4) from y_0 to $y(T_v^1)=0$, during the time interval $[0,T_v^1]$;
- let $y=u=0$ on $[T_v^1,T_v^2]$;
- find the control $u_{2,v}$ which will steer the solution of (4.4) from 0 to $y(T)=y_T$, during the time interval $[T_v^2,T]$.

Then, proceeding as in Theorem 4.1.3, we can obtain a fixed point, $y^*=v^*$, and desired control function and solution will be in the form:

$$u(t)=\begin{cases}u_{1,v^*}(t), & 0\le t\le T_{v^*}^1\\ 0, & T_{v^*}^1<t\le T_{v^*}^2\\ u_{2,v^*}(t), & T_{v^*}^2\le t\le T\end{cases},\qquad y(t)=\begin{cases}y_1(t), & 0\le t\le T_{v^*}^1\\ 0, & T_{v^*}^1<t\le T_{v^*}^2\\ y_2(t), & T_{v^*}^2\le t\le T\end{cases}.$$

4.2 Controllability of a nonlinear system of FDEs

The analysis of nonlinear control problems with fractional derivatives started relatively recently, and, in the beginning, it was motivated by specific application-oriented problems. Over the last 20 years there have been an increase in the research of fractional nonlinear control problems, from both application and theoretical aspect. Several authors considered problems governed by fractional semi-linear systems of the form

$${}_0^C\mathrm{D}_t^\alpha y(t)=Ay+Bu+f(t,y,u),\quad y(0)=y_0,\quad t\in[0,T],\tag{4.26}$$

and, in the sequel, we single out some of the most relevant results related to the question of controllability.

- In [9], the authors proved global controllability of (4.26), under assumptions that $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^N\to\mathbb{R}^d$ is continuous and $\lim\limits_{|(y,u)|\to0}\dfrac{|f(t,y,u)|}{|(y,u)|}=0$.

- In [61], the question of the existence of a mild solution and approximate controllability was considered. Further, under assumptions: $f(t, y, u) = f(t, y)$ is continuous on $[0, T] \times \mathbb{R}^d$ and uniformly bounded, the results on approximate controllability of (4.26) were obtained.

- In [28], the authors proved global controllability of (4.26), with nonlinear term $f(t, y, u) = f(t, y)$ satisfying growth condition $|f(t, y)| \leq d(t) + \eta|y|$, and monotonicity condition $\langle f(t, x) - f(t, y), x - y\rangle \leq 0$.

Furthermore, in [7], the authors considered fractional nonlinear control problems with the Caputo derivative of order $1 < \alpha < 2$.

Alongside the question of controllability, the question of optimization for nonlinear fractional systems was analyzed, as well. In the papers [10, 43] one can find an overview of the results in fractional optimal control, in which the fractional analog of the Pontryagin maximum principle is stated and applied.

In this section our goal is to prove controllability of nonlinear control problem (4.1), with the Caputo fractional derivative of order $\alpha \in (0, 1)$:

$$\begin{aligned} {}_0^C\mathrm{D}_t^\alpha y(t) &= -Af(y)y + Bu, \quad t \in [0, T] \\ y(0) &= y_0, \quad y(T) = y_T. \end{aligned} \tag{4.27}$$

Mainly, the idea is to use the same procedure as in the integer-derivative case, i.e., to consider linearized problem and prove the existence of a fixed point. As we shall see, for the systems with fractional derivatives, the construction of the piecewise solution is more complex (than in the integer-derivative case) since, it requires to take into consideration the memory imposed by the fractional derivative.

Assume that the matrices A and B satisfy assumptions (a2)–(a4), and for the function f suppose that the following modification of (a1) holds

(a1)' $f : \mathbb{R}^d \to (0, \infty)$ is a continuous function and $F : \mathbb{R}^d \to \mathbb{R}^d$, given by $F(y) = f(y)y$, is continuously differentiable on $\mathbb{R}^d$.

First, let us denote by z the solution of the nonlinear initial value problem

$${}_0^C\mathrm{D}_t^\alpha z(t) = -Af(z)z, \quad t \in [0, T], \quad z(0) = y_0. \tag{4.28}$$

From Theorem 2.3.22 we have: $z \in C([0, T]; \mathbb{R}^d) \cap C^1((0, T]; \mathbb{R}^d)$,

$$|z(t)| \leq |y_0| \tag{4.29}$$

and there exists $K_z > 0$ such that for every $t \in (0, T]$ it holds

$$|t^{1-\alpha} z'(t)| \leq K_z. \tag{4.30}$$

Next, for every $v \in C([0,T];\mathbb{R}^d)$ we define constants

$$M_v := \max_{t\in[0,T]} |f(v(t))|, \quad K_v := \Big(\max\{1, M_v\}\Big)^{\frac{1}{\alpha}} \quad \text{and} \quad T_v := T - \frac{T}{K_v}. \tag{4.31}$$

Note that we have K_v such that

$$\max_{t\in[0,T]} |f(v(t))| \le K_v^{\alpha}. \tag{4.32}$$

Now, we construct the solution y in the following way: first, we let the system (4.27) to be "uncontrolled", i.e., let $u = 0$, up to the time $T_v \in (0,T)$, and then consider linearized control problem

$$\begin{aligned} {}_0^C\mathrm{D}_t^{\alpha} y(t) = &\ -Af(v)y + Bu, \quad t \in [T_v, T] \\ y(T_v) = &\ z(T_v), \quad y(T) = y_T, \end{aligned} \tag{4.33}$$

and find its solution on $[T_v, T]$. More precisely, for every $v \in C([0,T];\mathbb{R}^d)$ we define functions

$$y(t) = \begin{cases} z(t), & 0 \le t \le T_v \\ y_2(t), & T_v < t \le T \end{cases} \quad \text{and} \quad u(t) = \begin{cases} 0, & 0 \le t \le T_v \\ u_2(t), & T_v < t \le T \end{cases}, \tag{4.34}$$

where z is a restriction on $[0, T_v]$ of the solution of (4.28), and y_2 and u_2 satisfy linear control problem (4.33). Let us notice that, although in (4.33) we look for the solution on the interval $[T_v, T]$, the derivative ${}_0^C\mathrm{D}_t^{\alpha} y$ takes into account the values of y from the starting point $t = 0$. More precisely, having in mind (4.34), we have that for $t \in (T_v, T]$

$${}_0^C\mathrm{D}_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^{\alpha}} ds = \frac{1}{\Gamma(1-\alpha)} \int_0^{T_v} \frac{z'(s)}{(t-s)^{\alpha}} ds + \frac{1}{\Gamma(1-\alpha)} \int_{T_v}^t \frac{y_2'(s)}{(t-s)^{\alpha}} ds.$$

By introducing notation

$$h_v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^{T_v} \frac{z'(s)}{(t-s)^{\alpha}} ds, \quad t \in (T_v, T], \tag{4.35}$$

we get

$${}_0^C\mathrm{D}_t^{\alpha} y(t) = h_v(t) + {}_{T_v}^C\mathrm{D}_t^{\alpha} y_2(t), \tag{4.36}$$

and transform equation (4.33) into

$$\begin{aligned} {}_{T_v}^C\mathrm{D}_t^{\alpha} y(t) = &\ -Af(v(t))y(t) + Bu(t) - h_v(t), \quad t \in (T_v, T], \\ y(T_v) = &\ z(T_v), \quad y(T) = y_T. \end{aligned} \tag{4.37}$$

Then, we divide (4.37) into two problems and find solution y_2 in the form $y_2 = y_p + y_c$, where y_p solves

$$\begin{aligned} {}_{T_v}^{C}\mathrm{D}_t^{\alpha} y_p(t) &= \ - Af(v(t))y_p(t) - h_v(t), \quad t \in (T_v, T], \\ y_p(T_v) &= 0, \end{aligned} \tag{4.38}$$

and y_c and u_2 are solutions to

$$\begin{aligned} {}_{T_v}^{C}\mathrm{D}_t^{\alpha} y_c(t) &= \ - Af(v(t))y_c(t) + Bu_2(t), \quad t \in (T_v, T], \\ y_c(T_v) &= z(T_v) = y_{c,0}, \quad y_c(T) = y_T - y_p(T) = y_{c,T}. \end{aligned} \tag{4.39}$$

Notice that from (4.29) we have

$$|y_{c,0}| \leq |y_0|. \tag{4.40}$$

Denote by Ψ_v and Φ_v the state-transition matrices associated with system (4.39). According to (2.15), the solution of (4.38) is given by

$$y_p(t) = -\int\limits_{T_v}^{t} \Phi_v(\tau, t) h_v(\tau) d\tau. \tag{4.41}$$

Using (4.30), we get that the function h_v given by (4.35) satisfies:

$$\begin{aligned} |h_v(t)| \ &= \ \Big|\frac{1}{\Gamma(1-\alpha)} \int\limits_0^{T_v} \frac{z'(s)}{(t-s)^{\alpha}} ds\Big| \leq \frac{1}{\Gamma(1-\alpha)} \int\limits_0^{T_v} \frac{|s^{1-\alpha} z'(s)|}{s^{1-\alpha}(t-s)^{\alpha}} ds \\ &\leq \ \frac{K_z}{\Gamma(1-\alpha)} \int\limits_0^{T_v} s^{\alpha-1}(t-s)^{-\alpha} ds \leq \frac{K_z}{\Gamma(1-\alpha)} \int\limits_0^{t} s^{\alpha-1}(t-s)^{-\alpha} ds. \end{aligned}$$

Introducing the change of variables $\xi = \dfrac{s}{t}$ in the above integral, we obtain

$$|h_v(t)| \leq \frac{K_z}{\Gamma(1-\alpha)} \int\limits_0^1 \xi^{\alpha-1}(1-\xi)^{-\alpha} d\xi = \frac{K_z}{\Gamma(1-\alpha)} B(\alpha, 1-\alpha) = K_z \Gamma(\alpha). \tag{4.42}$$

Now, (4.41), (4.42) and Proposition 2.3.20 (ii) imply

$$|y_p(t)| \leq \int\limits_{T_v}^{t} (t-\tau)^{\alpha-1} |h_v(\tau)| d\tau \leq K_z \Gamma(\alpha) \int\limits_{T_v}^{t} (t-\tau)^{\alpha-1} d\tau \leq K_z \Gamma(\alpha) \frac{T^{\alpha}}{\alpha}. \tag{4.43}$$

Hence, for the final state $y_{c,T}$ figuring in (4.39), we have

$$|y_{c,T}| \leq |y_T| + \frac{K_z \Gamma(\alpha) T^\alpha}{\alpha} =: C_T \tag{4.44}$$

For the solution y_c, we use properties from Section 3.2, and define y_c as the solution corresponding to the $L^2_{\alpha-1}$-optimal control u_2 given by (3.46), i.e.,

$$u_2(t) = (T-t)^{1-\alpha} B^{\mathrm{T}} \Phi_v(t,T)^{\mathrm{T}} W_{\alpha,v}^{-1}(y_T - \Psi_v(T_v,T) z(T_v)), \tag{4.45}$$

where $W_{\alpha,v}$ is the controllability Gramian associated with control problem (4.39):

$$W_{\alpha,v} = W_{\alpha,v}(T_v,T) = \int_{T_v}^{T} (T-t)^{1-\alpha} \Phi_v(t,T) B B^{\mathrm{T}} \Phi_v(t,T)^{\mathrm{T}}\, dt.$$

Further, the solution y_c is given by

$$y_c(t) = \Psi_v(T_v,t) y_{c,0} + \int_{T_v}^{t} \Phi_v(\tau,t) B u_2(\tau)\, d\tau. \tag{4.46}$$

Let us prove some auxiliary results related to Gramian and state-transition matrices of linearized problem.

4.2.1 Lemma *Let $v \in C([0,T];\mathbb{R}^d)$. There exist constants $\lambda \geq 0$, $c_w > 0$ and $C_u > 0$, not depending on v such that:*

(i) $\|\Psi_v(T_v,t)\| \leq 1$, *for every* $t \in [T_v,T]$*;*

(ii) $E_{\alpha,\alpha}(-\lambda M_v (t-\tau)^\alpha) \leq \|\Phi_v(\tau,t)\| \leq 1$, *for every* $T_v \leq \tau \leq t \leq T$*;*

(iii) $\|W_{\alpha,v}^{-1}\| \leq \dfrac{K_v^\alpha}{c_w}$*;*

(iv) the control function given by (4.45) *satisfies* $|u_2(t)| \leq C_u K_v^\alpha$, *for every* $t \in [T_v,T]$.

Proof. Let $\lambda = \max\{\lambda_1, \dots, \lambda_d\}$. Properties (i) and (ii) follow from Proposition 2.3.20.

(iii) Since we know that $W_{\alpha,v}$ is nonsingular, we can define $\|W_{\alpha,v}^{-1}\| = \frac{1}{s}$, where $s = \min\limits_{x \in S^{d-1}} |W_{\alpha,v} x|$. Using Remark 3.2.10, we get that for every $x \in S^{d-1}$

$$|W_{\alpha,v} x| \geq |x^{\mathrm{T}} W_{\alpha,v} x| = |x^{\mathrm{T}} U W_{\alpha,v,D} U^{\mathrm{T}} x|.$$

Since U is orthogonal matrix, we have $\{x^{\mathrm{T}}U : x \in S^{d-1}\} = S^{d-1}$. Therefore,

$$\begin{aligned}\min_{x\in S^{d-1}} |W_{\alpha,v}x| &\geq \min_{x\in S^{d-1}} |x^{\mathrm{T}}W_{\alpha,v,D}x| \\ &= \min_{x\in S^{d-1}} \Big|\int_{T_v}^{T} (T-t)^{1-\alpha} x^{\mathrm{T}} Q_v(t,T)\tilde{B}\tilde{B}^{\mathrm{T}} Q_v(t,T)^{\mathrm{T}} x\, dt\Big| \\ &= \min_{x\in S^{d-1}} \int_{T_v}^{T} (T-t)^{1-\alpha} |x^{\mathrm{T}} Q_v(t,T)\tilde{B}|^2\, dt,\end{aligned}$$

where $Q_v(t,T) = \mathrm{diag}(q_1(t),\ldots,q_d(t))$ (cf. proof of Proposition 2.3.19 and Remark 3.2.10), with element $q_i(t)$ being the solution of

$${}_t\mathrm{D}_T^{\alpha} q_i(t) = -\lambda_i f(v(t)) q_i(t), \quad t \in [T_v, T], \quad {}_t\mathrm{I}_T^{1-\alpha} q_i(t)|_{t=T} = 1.$$

Using comparison principle, as in the proof of Proposition 2.3.19, we derive

$$E_{\alpha,\alpha}(-\lambda_i M_v (T-t)^{\alpha}) \leq (T-t)^{1-\alpha} q_i(t) \leq 1, \quad t \in [T_v, T].$$

Furthermore, from (4.31), we have

$$M_v(T-t)^{\alpha} \leq M_v(T-T_v)^{\alpha} = M_v \frac{T^{\alpha}}{K_v^{\alpha}} \leq T^{\alpha}.$$

Hence, $e_\alpha := E_{\alpha,\alpha}(-\lambda T^{\alpha}) \leq E_{\alpha,\alpha}(-\lambda_i M_v(T-t)^{\alpha})$, and we obtain uniform boundedness (with respect to both v and t) of $q_i(t)$:

$$0 < e_\alpha \leq (T-t)^{1-\alpha} q_i(t) \leq 1. \tag{4.47}$$

Now, we have

$$\min_{x\in S^{d-1}} |W_{\alpha,v}x| \geq \min_{x\in S^{d-1}} \int_{T_v}^{T} (T-t)^{\alpha-1} |x^{\mathrm{T}} (T-t)^{1-\alpha} Q_v(t,T)\tilde{B}|^2\, dt,$$

and from (4.47), it follows that there exists a constant $c_1 \geq 0$, $c_1 = c(U, e_\alpha, B)$ (independent on v), such that

$$\min_{x\in S^{d-1}} \int_{T_v}^{T} (T-t)^{\alpha-1} |x^{\mathrm{T}} (T-t)^{1-\alpha} Q_v(t,T)\tilde{B}|^2\, dt \geq c_1 \int_{T_v}^{T} (T-t)^{\alpha-1}\, dt = \frac{c_1 T^{\alpha}}{\alpha K_v^{\alpha}}.$$

Furthermore, the constant c_1 is strictly greater than 0 since the assumptions (a1)', (a2)–(a4) imply positive definiteness of both $W_{\alpha,v}$ and $W_{\alpha,v,D}$ (cf. Proposition 3.2.9 and Remark 3.2.3).

Denoting by $c_w = \frac{c_1 T^\alpha}{\alpha}$, we obtain

$$\min_{x\in S^{d-1}} |W_v x| \geq \frac{c_w}{K_v^\alpha}.$$

Hence, $\|W_v^{-1}\| \leq \frac{K_v^\alpha}{c_w}$.

(iv) From (4.45), $(i)-(iii)$ and (4.29) we get

$$\begin{aligned}
|u_2(t)| &\leq \|B^{\mathrm{T}}\| \|(T-t)^{1-\alpha}\Phi_v(t,T)^{\mathrm{T}}\| \|W_{\alpha,v}^{-1}\|(|y_T| + \|\Psi_v(T_v,T)\||z(T_v)|) \\
&\leq \frac{\|B^{\mathrm{T}}\| K_v^\alpha (|y_T| + |y_0|)}{c_w} \\
&= C_u K_v^\alpha,
\end{aligned}$$

with $C_u = \frac{\|B^{\mathrm{T}}\|(|y_T|+|y_0|)}{c_w}$. □

Now, we move to controllability of our nonlinear problem.

4.2.2 Theorem *Assume that (a1)' and (a2)–(a4) hold. Then, for any $T>0$ and $y_0, y_T \in \mathbb{R}^d$, there exists $u \in L^2_{\alpha-1}([0,T];\mathbb{R}^N)$ such that the solution of (4.27) satisfies $y(T) = y_T$.*

Proof. Define the mapping $\mathcal{T}_\alpha : C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$ which every $v \in C([0,T];\mathbb{R}^d)$ maps to the solution y given by (4.34), and constructed as described above.

First, let us prove that the solutions y are uniformly bounded, independently on v.

From (4.46), Lemma 4.2.1 and (4.40) we have that for every $t \in [T_v, T]$

$$\begin{aligned}
|y_c(t)| &= |\Psi_v(T_v,t)y_{c,0} + \int_{T_v}^{t} \Phi_v(\tau,t)Bu(\tau)\,d\tau| \\
&\leq |y_0| + \int_{T_v}^{t} (t-\tau)^{\alpha-1}\|(t-\tau)^{1-\alpha}\Phi_v(\tau,t)\||Bu(\tau)|\,d\tau \\
&\leq |y_0| + \|B\|C_u K_v^\alpha \int_{T_v}^{t} (t-\tau)^{\alpha-1}\,d\tau \\
&\leq |y_0| + \|B\|C_u K_v^\alpha \frac{(T-T_v)^\alpha}{\alpha} \\
&= |y_0| + \frac{\|B\|C_u K_v^\alpha}{\alpha}\left(\frac{T}{K_v}\right)^\alpha \\
&= |y_0| + \frac{\|B\|C_u T^\alpha}{\alpha}.
\end{aligned}$$

Furthermore, using (4.43), we get

$$|y_2(t)| \le |y_c(t)| + |y_p(t)| \le |y_0| + \frac{\|B\|C_u T^\alpha}{\alpha} + \frac{K_z \Gamma(\alpha) T^\alpha}{\alpha} =: C_y.$$

Since on $[0, T_v]$ we have $|y(t)| = |z(t)| \le |y_0| < C_y$, we conclude that

$$|y(t)| \le C_y, \quad t \in [0, T]. \tag{4.48}$$

Now we are able to show that $\mathcal{T}_\alpha$ satisfies conditions of the Leray-Schauder fixed point theorem.

$\mathcal{T}_\alpha$ **is compact.** Let V be a bounded set in $C([0,T];\mathbb{R}^d)$. Since f is continuous, there exists $K > 1$ such that, for every $v \in V$,

$$\max_{t\in[0,T]} |f(v(t))| \le K^\alpha \quad \text{and} \quad \max_{|z|\le y_0} f(z) \le K^\alpha. \tag{4.49}$$

Then, (4.31) implies $K_v \le K$. For the compactness of $\mathcal{T}_\alpha$, it suffices to prove that $Y := \mathcal{T}_\alpha(V)$ is relatively compact set. To that end, let $\{y_n\}$ be a sequence in Y. For every $n \in \mathbb{N}$, denote by v_n the function from V such that $\mathcal{T}_\alpha(v_n) = y_n$. From (4.48) we have that $\{y_n\}$ is uniformly bounded. Furthermore, for every n, the solution y_n is equal to z on $[0, T_v]$, i.e., y_n satisfies (4.28) on $[0, T_v]$. Hence, from (4.49) and (4.29) we obtain

$$|{}_0^C\mathrm{D}_t^\alpha y_n(t)| = |-Af(z)z| \le \|A\|K^\alpha |y_0|, \quad t \in [0, T_v]. \tag{4.50}$$

On the interval $(T_v, T]$ we have that $y_n(t)$ satisfies

$${}_0^C\mathrm{D}_t^\alpha y_n(t) = -Af(v_n)y_n + Bu_{2,n}.$$

Then, using (4.49), (4.48) and Lemma 4.2.1 (iv), we get

$$|{}_0^C\mathrm{D}_t^\alpha y_n(t)| \le \|A\|K^\alpha C_y + \|B\|C_u K_v^\alpha \le (\|A\|C_y + \|B\|C_u)K^\alpha =: C_\alpha, \quad t \in (T_v, T]. \tag{4.51}$$

Therefore, (4.50) and (4.51) imply that the sequence of derivatives ${}_0^C\mathrm{D}_t^\alpha y_n(t)$, $n \in \mathbb{N}$, is bounded on $(0,T)$, independently on v. Note here that ${}_0^C\mathrm{D}_t^\alpha y_n(t)$ may not be continuous at T_v. Nonetheless, $\sup_{t\in(0,T)} |{}_0^C\mathrm{D}_t^\alpha y_n(t)| \le C_\alpha$, for every $n \in \mathbb{N}$, and conditions of Lemma 2.3.21 are satisfied. Hence, $\{y_n\}$ is uniformly bounded and equicontinuous sequence in $C([0,T];\mathbb{R}^d)$, and by the Arzela-Ascoli theorem it follows that $\{y_n\}$ has a convergent subsequence. This concludes the proof of compactness.

$\mathcal{T}_\alpha$ **is continuous.** Assume that $\{v_n\}$ is a sequence in $C([0,T];\mathbb{R}^d)$ which converges uniformly to a function $\overline{v} \in C([0,T];\mathbb{R}^d)$. Then $\{v_n\}$ is bounded in $C([0,T];\mathbb{R}^d)$, and compactness of $\mathcal{T}_\alpha$ implies that the sequence $y_n = \mathcal{T}_\alpha(v_n)$ has a convergent subsequence y_{n_k}. Let $y = \lim_{k\to\infty} y_{n_k}$. Now, from the construction of the solutions y_{n_k}, assumption that $v_n \to \overline{v}$ uniformly on $[0,T]$, and Lemma 4.1.1, it follows that y coincides with the solution $\overline{y}$, obtained for $v = \overline{v}$. Hence, y_n converges to $y = \mathcal{T}_\alpha(\overline{v})$.

Furthermore, uniform boundedness of the solutions (4.48) implies that the set

$$\{v \in C([0,T];\mathbb{R}^d) : v = \omega\mathcal{T}_\alpha(v),\ \omega \in [0,1]\}$$

is bounded.

Thus, by the Leray-Schauder theorem, it follows that $\mathcal{T}_\alpha$ has a fixed point $y^* = v^*$, which is a desired solution of nonlinear control problem (4.27). □

4.2.3 Remark Let us mention that the construction given in Remark 4.1.4 can not be applied in the fractional setting. Namely, in the fractional case, when the system is driven to zero and there is no action on the system (the control is equal to 0), the state of the system does not rest. This occurs due to the memory embedded in the fractional derivative. For example, consider a linear time-invariant system

$$ {}_0^C\mathrm{D}_t^\alpha y(t) = Ay(t) + Bu(t), \quad t \geq 0, \tag{4.52}$$

and suppose that u is the control which steers the solution of the system from initial state $y(0) = y_0$ to the final state $y(T) = 0$, $T > 0$. Now, if we let $y(t) = u(t) = 0$, $t > T$, the equation (4.52) is no longer satisfied, since the derivative at $t > T$ depends on all the accumulated values of y from $[0,t]$. That is why we can not simply insert a zero-interval in the solution, as it is possible in the integer derivative case. This property of the fractional in time systems, that they can not achieve null-equilibrium controllability, was studied in [49].

Chapter 5

Conclusion

The main contributions of this thesis are divided into three parts. The first part concerns theory of fractional differential equations (FDEs), or, more precisely, systems of FDEs. In Section 2.3 an overview of the recently obtained results regarding the existence, uniqueness and analytical representation of the solution to the system of FDEs was presented, with a special attention paid to the properties of the state-transition matrices.

The second contribution is related to the theory developed in [33], where linear fractional time-varying control problems with the Caputo derivative were examined and the classical linear control theory was adapted and applied to the fractional setting. Specifically, the equivalent conditions for controllability were established, and the methods for finding an optimal control in the weighted L^2 space (systems of order $\alpha \in (0, \frac{1}{2}]$), or classical L^2 space (systems of order $\alpha \in (\frac{1}{2}, 1)$) were derived. These results are given in Section 3.2. Since the linear time-varying control problems with fractional derivatives have been studied marginally, the obtained results present one approach to general analysis of this class of systems.

The third part concerns nonlinear control problems (4.3) and (4.27), for which, in Chapter 4, the sufficient conditions for controllability were stated. The main results, given in Theorem 4.1.3 and Theorem 4.2.2, are based on the linearization of the problem and application of the Leray-Schauder fixed point theorem. Furthermore, in order to obtain uniform boundedness of the solution of the associated linearized problem, a novel method was used, which consists of the compression of the interval in which the control acts on the state of the system. In future, it would be interesting to consider application of this idea to different types of control problems with dissipative dynamics.

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Biography

Maja Jolić was born on October 18th, 1991 in Kikinda. In 2013, she obtained a Bachelor's degree in Mathematics at the Faculty of Sciences, University of Novi Sad, and enrolled in the Master program of pure mathematics at the same faculty. During her Bachelor and Master studies, Maja received a scholarship of the Fund for Young Talents of Serbia "Dositeja" in the academic year 2012/2013 and 2014/2015. In 2016 she earned a Master's degree and enrolled in the program of PhD School of Mathematics, module Analysis, at the Faculty of Sciences, University of Novi Sad. During her PhD studies, Maja attended several international workshops and conferences. In 2020 she completed a 3-month study-research visit at the Linnaeus University in Växjö, Sweden, as a part of the Erasmus+ International Credit Mobility, under supervisor Prof. Joachim Toft. She passed all the exams of her PhD program with a GPA 10.00, and she is a co-author of two scientific articles.

From 2015 to 2017 she was employed as a teaching assistant at the Chair of Mathematics, Faculty of Technical Sciences, University of Novi Sad. From 2017 she has been working as a teaching assistant at the Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad.

Овај Образац чини саставни део докторске дисертације, односно докторског уметничког пројекта који се брани на Универзитету у Новом Саду. Попуњен Образац укоричити иза текста докторске дисертације, односно докторског уметничког пројекта.

План третмана података

Назив пројекта/истраживања
Нелинеарни проблеми управљања са и без фракционих извода
Назив институције/институција у оквиру којих се спроводи истраживање
Природно-математички факултет, Универзитет у Новом Саду
Назив програма у оквиру ког се реализује истраживање
1. Опис података
1.1 Врста студије *У овој студији нису прикупљани подаци.*
2. Прикупљање података
3. Третман података и пратећа документација
4. Безбедност података и заштита поверљивих информација
5. Доступност података
6. Улоге и одговорност