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CONVERGENCE ANALYSIS OF MODULUS BASED METHODS FOR LINEAR COMPLEMENTARITY PROBLEMS

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Apstrakt

Problemi linearne komplementarnosti su savremena tema koja privlači naročitu pažnju značajnog dela naučne zajednice. Ovo nije neobično, budući da ovi problemi nastaju kao matematički odgovor tokom rešavanja realnih problema iz raznovrsnih primena, koji najčešće uključuju optimizacioni problem, poput linearnog ili kvadratnog programiranja, ali i kao instanca u rešavanju konkretnog problema iz ostalih interesantnih sfera primena, kao što su, na primer, problemi sa graničnim slojem iz mehanike fluida, problemi mrežnog ekvilibrijuma, kontaktni problemi, modeli tržišne ravnoteže, optimalna alokacija kapitala, optimalno zaustavljanje, određivanje Nešovog ekvilibrijuma u teoriji bimatričnih igara kao i mnogi drugi.

Uopšteno govoreći, rešavanju problema linearne komplementarnosti moguće je pristupiti dvojako. Prvi pristup podrazumeva upotrebu takozvanih direktnih metoda, koje su poznate i pod nazivom metode pivota. Drugoj kategoriji, koja je i sa stanovišta ove teze interesantna, pripadaju iterativni postupci. S obzirom da je ova kategorija izuzetno bogata, mi smo se opredelili za jednu od najznačajnijih varijanti, a to je modulski iterativni postupak. Međutim, ni ova odrednica nije dovoljno adekvatna, budući da modulski postupci obuhvataju nekolicinu različitih pravaca. Zato smo se odlučili da posmatramo postupke koji se zasnivaju na razlaganjima, ali i višestrukim razlaganjima matrice.

Ovu ideju o primeni najpre splitinga [8], a nekoliko godina kasnije

i multisplitinga [9], kada je reč o modulskim iterativnim postupcima prvi je uveo Bai, a postupke nazvao modulskim metodama zasnovanim na (multi)splitinzima matrice. Ideja primene multisplitinga javila se sa jasnim ciljem da se iskoriste mogućnosti paralelnog izračunavanja na višeprocesorskim računarskim jedinicama. Pored formulacije pomenutih postupaka, Bai je u svojim radovima analizirao i konvergenciju istih. Nakon toga, usledile su brojne generalizacije i novi rezultati o konvergenciji. Među njima, ističu se oni sa novim tehnikama, koje se mogu iskoristiti za dalju analizu konvergencije, [70]. To je jedan od osnovnih razloga zbog kojeg smo se opredelili za upotrebu teorije H-matrica u dokazu konvergencije pomenutih metoda, kao i dodatnih analiza, kao što je analiza greške, [26].

Uzevši sve navedeno u obzir, glavni cilj ove doktorske disertacije jeste upotreba teorije H-matrica u teoremama o konvergenciji modulskih metoda zasnovanih na multisplitinzima matrice i korišćenje ove nove tehnike, sa ciljem analize bitnih osobina, nakon što je konvergencija postupka zagarantovana. Za kraj, spomenimo i to da je rad na ovoj savremenoj temi izrodio nove i originalne rezultate [26], koji su prezentovani kao sastavni deo ove disertacije.

Abstract

Linear complementarity problems are a contemporary scientific research field which attract a lot of attention. This is not a surprise, since they arise from the need to solve a problem from real life applications, which usually includes an optimization, such as linear or quadratic programming. Then, a variety of other interesting applications appear, for example free boundary problems from fluid mechanics, network equilibrium problems, contact problems, market equilibria problems, optimal capital invariant stock, optimal stopping problems, determining Nash equilibrium in bimatrix games and many more.

Generally speaking, solving the linear complementarity problem can be approached from two essentially different perspectives. One of them includes the use of so-called direct methods, in the literature also known under the name pivoting methods. The other, and from our perspective - more interesting one, which we will actually focus on in this thesis, is the iterative approach. Among the vast collection of iterative solvers, our choice was one particular class of modulus based iterative methods. Since the subclass of modulus based-methods is again diverse in some sense, it can be specialized even further, by the introduction and the use of matrix splittings.

The idea to use splittings [8], and later multisplittings [9] in the modulus-based methods, was first proposed by Bai who named these methods as the modulus-based matrix (multi)splitting iteration meth-

ods for solving $LCP(q,A)$. The multisplitiing idea was incorporated only a couple of years later, in order to take full advantage of parallel computations on multiprocessor systems. In addition to the introduction of these methods, the first results of their convergence have been analysed by the same author. After his pioneer work, many generalizations and new convergence results have been published. Among them, there are several novel approaches, with a great potential to enable further convergence analysis, [70]. This is the main reason because of which we chose to use the theory of H-matrices for proving convergence of such methods, and to make more benefits of this kind of proof, for example, in the error analysis of proposed methods, [26].

With all this in mind, the main goal of this thesis is to use the theory of H-matrices for proving convergence of the modulus-based multisplitting methods, and to use this new technique to analyze some important properties of iterative methods once the convergence has been guaranteed. Finally, working on this conteporary topic has created new and original results [26], which are presented in this thesis.

Contents

1. Introduction

1.1 History

The history of linear complementarity problem can be traced back to the 40s of the twentieth century, when Du Val [32] formulated the problem of finding the smallest element (in the vector sense). The problem aimed to solve the system of linear inequations of the form

$$z \geq 0,$$
$$q + Az \geq 0.$$

But this problem has not always been known under this name. Its initial name was the composite problem, until it was changed to the fundamental problem and lastly - the complementarity pivoting problem. The name that is used in the contemporary science was proposed by Cottle in 1995. It is exaclty around that time that the problem attracted much more attention, especially through the work done by Lemke [47], Cottle [17] and Cottle and Dantzig [18].

1.2 Motivation

Linear complementarity problems are a contemporary scientific research field which attract a lot of attention. This is not a surprise, since they arise from the need to solve a problem from real life applications, which usually includes an optimization, such as linear or quadratic programming. Then, a variety of other interesting applications appear as well - for further reference see [19]. For example, some of these possible applications include free boundary problems from fluid mechanics, network equilibrium problems, contact problems, market equilibria problems [54, 55, 56, 57], optimal capital invariant stock [45, 46, 40, 33], optimal stopping problems [15, 16], determining Nash equilibrium in bimatrix games [59, 60] and many more.

The thesis consists of the following chapters: after some historical references, motivation and preliminaries from numerical linear algebra background, Chapter 2 is related to the LCP formulation, as well as some selected possible applications, discussed in detail. Chapter 3 represents an overview of methods for solving systems of linear equations, since it is a good starting point for understanding the formulation of iterative methods for LCP. Chapter 4 presents iterative methods for LCP, with an emphasis on modulus based splitting and multisplitting methods. Chapter 5 is related to the convergence analysis, while Chapter 6 deals with the error control, both of which contain original results from [26]. Finally, Chapter 7 gives some numerical examples, which illustrate theoretical results, their usefullness and efficiency. Thesis ends with a list of relevant literature.

1.3 Notations

- $\mathbb{N}$ — the set of natural numbers
- $\mathbb{N}_0$ — the set of natural numbers including 0
- $\mathbb{Z}$ — the set of integers
- $\mathbb{R}$ — the set of real numbers
- $\mathbb{C}$ — the set of complex numbers
- $N_n := \{1, 2, ..., n\}$ — the index set
- $\mathbb{R}^n$ — the n-dimensional real vector space
- $\mathbb{R}^n_+$ — the n-dimensional real vector space of positive vectors
- $\mathbb{C}^n$ — the n-dimensional complex vector space
- $x = [x_i] \in \mathbb{R}^n$ — n-dimensional vector with coordinates $x_i \in \mathbb{R}$
- $x = [x_i] \in \mathbb{C}^n$ — n-dimensional vector with coordinates $x_i \in \mathbb{C}$
- x_i — the i^{th} coordinate of x
- $|x| := [|x_i|] \in \mathbb{C}^n$
- x^T — the transpose of x
- x^H — the conjugate transpose of x

- $x \geq y,\ \ x, y \in \mathbb{R}^n\ -$ for all $i \in N_n,\ \ x_i \geq y_i$
- $x > y,\ \ x, y \in \mathbb{R}^n\ -$ for all $i \in N_n,\ \ x_i > y_i$
- $x \leq y,\ \ x, y \in \mathbb{R}^n\ -$ for all $i \in N_n,\ \ x_i \leq y_i$
- $x < y,\ \ x, y \in \mathbb{R}^n\ -$ for all $i \in N_n,\ \ x_i < y_i$
- $x = y \quad -$ x is equal to y
- $x \geq 0,\ x \in \mathbb{R}^n\ -$ x is a nonnegative vector
- $x > 0,\ x \in \mathbb{R}^n\ -$ x is a (strictly) positive vector
- $x \leq 0,\ x \in \mathbb{R}^n\ -$ x is a nonpositive vector
- $x < 0,\ x \in \mathbb{R}^n\ -$ x is a (strictly) negative vector
- $x = 0,\ \ x \in \mathbb{R}^n\ -$ x is a zero vector
- $e\ -$ vector with all coordinates equal to 1
- $e^i\ -$ zero vector with the i^{th} coordinate equal to 1
- $\|x\|_\infty := \max\limits_{i \in N_n} |x_i| \quad -$ infinity norm of vector x
- $\|x\|_1 := \sum\limits_{i \in N_n} |x_i| \quad -$ 1-norm of vector x
- $\|x\|_2 := \left(\sum\limits_{i \in N_n} |x_i|^2 \right)^{1/2} \quad -$ 2-norm of vector x
- $\mathbb{R}^{n,n} -$ the set of all n by n real matrices

- $\mathbb{C}^{n,n}$ – the set of all n by n complex matrices
- $A = [a_{ij}] \in \mathbb{R}^{n,n}$ – a matrix with entries $a_{ij} \in \mathbb{R}$
- $A = [a_{ij}] \in \mathbb{C}^{n,n}$ – a matrix with entries $a_{ij} \in \mathbb{C}$
- a_{ij} – the matrix entry in the i^{th} row and the j^{th} column
- a^j – the j^{th} column of matrix A
- $|A| := [|a_{ij}|] \in \mathbb{C}^{n,n}$
- A^T – the transpose of matrix A
- A^H – the conjugate transpose of matrix A
- $A \geq B,\ A, B \in \mathbb{R}^{n,n}$ – for all $i, j \in N_n,\quad a_{ij} \geq b_{ij}$
- $A > B,\ A, B \in \mathbb{R}^{n,n}$ – for all $i, j \in N_n,\quad a_{ij} > b_{ij}$
- $A \leq B,\ A, B \in \mathbb{R}^{n,n}$ – for all $i, j \in N_n,\quad a_{ij} \leq b_{ij}$
- $A < B,\ A, B \in \mathbb{R}^{n,n}$ – for all $i, j \in N_n,\quad a_{ij} < b_{ij}$
- $A \geq O$, $A \in \mathbb{R}^{n,n}$ – A is a nonnegative matrix
- $A > O$, $A \in \mathbb{R}^{n,n}$ – A is a (strictly) positive matrix
- $A \leq O$, $A \in \mathbb{R}^{n,n}$ – A is a nonpositive matrix
- $A < O$, $A \in \mathbb{R}^{n,n}$ – A is a (strictly) negative matrix
- O – zero matrix

- $\langle A \rangle$ — the comparison matrix of A
- $\det(A)$ — the determinant of A
- A^{-1} — the inverse of a nonsingular matrix A
- E — the identity matrix
- $diag(x_1, ..., x_n)$ — the diagonal matrix with diagonal entries $x_1, ..., x_n$
- $\lambda(A)$ — the eigenvalue of A
- $\sigma(A)$ — the spectrum of A
- $\rho(A)$ — the spectral radius of A
- $r_i(A) := \sum_{\substack{j \in N_n \\ j \neq i}} |a_{ij}|$ — the i^{th} deleted row sum of A
- $r(A) = [r_i(A)]$ — the column vector of deleted row sums of A
- $\|A\|_\infty = \max_{i \in N_n} \sum_{j \in N_n} |a_{ij}|$ — infinity norm of matrix A
- $\|A\|_1 = \max_{j \in N_n} \sum_{i \in N_n} |a_{ij}|$ — 1-norm of matrix A
- $\|A\|_2 = \sqrt{\rho(A^H A)}$ — 2-norm of matrix A
- $LCP(q, A)$ — linear complementarity problem with vector q and matrix A

1.4 Preliminaries

We begin this section of preliminaries with some basic overview of fundamental definitions and theorems, which we will use throughout this thesis.

In addition to vector norms included in the list of notations, there is one important, which is frequently used. For an arbitrary vector $z \in \mathbb{R}^n$ with the property $z > 0$, the weighted vector norm is defined in the following way

$$\|x\|_{(z)} = \max_{1 \le i \le n} \frac{|x_i|}{z_i}, \quad x \in \mathbb{R}^n.$$

Matrix norm induced by this vector norm is given by

$$\|A\|_{(z)} = \max_{1 \le i \le n} \sum_{j=1}^{n} \frac{|a_{ij}| z_j}{z_i}, \quad A \in \mathbb{R}^{n,n}.$$

The relationship between these vector and matrix norms is established in the following theorem.

Theorem 1.4.1. *Let $z \in \mathbb{R}^n$, $z > 0$ and $A \in \mathbb{R}^{n,n}$. It is always true that*

$$\|A\|_{(z)} \ge \|Az\|_{(z)}.$$

However, if $A \ge O$, then

$$\|A\|_{(z)} = \|Az\|_{(z)}.$$

The following theorems establish some properties of the spectral radius of a matrix, $\rho(A)$, which is the largest eigenvalue of A taken by the moduli, but also its relationship with the matrix norm.

Theorem 1.4.2. *For an arbitrary matrix norm $\|\cdot\|$ on $\mathbb{C}^{n,n}$ and every square matrix $A \in \mathbb{C}^{n,n}$, it holds that $\rho(A) \leq \|A\|$.*

Theorem 1.4.3. *For any square matrix $A \in \mathbb{C}^{n,n}$ and every $\varepsilon > 0$ there exists an induced matrix norm $\|\cdot\|$, so that $\rho(A) \leq \|A\| < \rho(A) + \varepsilon$.*

Lemma 1.4.1. *Suppose that $A \in \mathbb{R}^{n,n}$ is nonnegative, $A \geq O$. If there exists a positive vector $x \in \mathbb{R}^n$ and a positive scalar α, so that $Ax < \alpha x$, then $\rho(A) < \alpha$.*

Obviously, choosing $\alpha = 1$, based on Lemma 1.4.1, we have the following immediate result.

Lemma 1.4.2. *Suppose that $A \in \mathbb{R}^{n,n}$ is nonnegative. If there exist a positive vector $x \in \mathbb{R}^n$ satisfying $Ax < x$, then $\rho(A) < 1$.*

Since a substantial part of this thesis is in connection with the iterative methods for solving linear systems, we recall some key results in connection with the convergence of matrix sequences, as well as their norms.

Theorem 1.4.4. *Let $\left\{A^{(k)}\right\}_{k\in\mathbb{N}}$ be an arbitrary matrix sequence and $\|\cdot\|$ an arbitrary matrix norm. Then, the following equivalence is true*

$$\lim_{k\to\infty} A^{(k)} = O \quad \Longleftrightarrow \quad \lim_{k\to\infty} \|A^{(k)}\| = 0.$$

Theorem 1.4.5. *For any square matrix A it is true that*

$$\lim_{k\to\infty} A^k = O \quad \Longleftrightarrow \quad \rho(A) < 1.$$

1.5 Special classes of matrices

In this section we will introduce some special types of matrices that appear in problems dealing with linear complementarity.

Definition 1.5.1. *Given $A \in \mathbb{R}^{n,n}$, we say that it is a Z-matrix provided that all of its off-diagonal entries are nonpositive. Additionally, if all the diagonal entries are strictly positive, then we say that A is a Z^+ matrix.*

Definition 1.5.2. *Matrix $A \in \mathbb{R}^{n,n}$ is an M-matrix if it is a nonsingular Z-matrix, with a nonnegative inverse, i.e. $A^{-1} \geq O$.*

It is well-known that if A is an M-matrix, then it is a Z^+ matrix, too. Let us remark that, throughout this dissertation, the name "M-matrix" always means nonsingular M-matrix.

Lemma 1.5.1 ([11])**.** *A matrix of the form $A = sE - B$, where $B \geq O$, is an M-matrix if and only if $\rho(B) < s$.*

Lemma 1.5.2. *Every triangular Z^+ matrix is an M-matrix.*

The theory of nonnegative matrices contains about 50 different and equivalent definitions of nonsingular M-matrices (for details, we suggest [11]). Here we recall one of them, in form of a theorem, which is important for this thesis.

Theorem 1.5.1. *A Z-matrix $A \in \mathbb{R}^{n,n}$ is an M-matrix if and only if there exists a positive vector z, for which vector Az is also positive.*

Lemma 1.5.3. *Let $A, B \in \mathbb{R}^{n,n}$ be M-matrices, $D \in \mathbb{R}^{n,n}$ a positive diagonal matrix and $C \in \mathbb{R}^{n,n}$ an arbitrary matrix. Then, the following implications are true:*

1. $A \leq B \Longrightarrow B^{-1} \leq A^{-1}$,
2. $A \leq C \leq D \Longrightarrow C$ *is an M-matrix.*

Definition 1.5.3. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$. We say that A is strictly diagonally dominant (SDD) if for all $i \in \{1, 2, \ldots, n\}$*

$$|a_{ii}| > r_i(A) := \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

One very important class of matrices, especially for the topic and results of this thesis, is the class of nonsingular H-matrices.

Definition 1.5.4. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be an arbitrary matrix. Then, its comparison matrix, $\langle A \rangle := [\alpha_{ij}] \in \mathbb{R}^{n,n}$, is defined entrywise as*

$$\alpha_{ij} := \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

Now, we define the class of H-matrices.

Definition 1.5.5. *Matrix $A \in \mathbb{C}^{n,n}$ is an H-matrix if and only if its comparison matrix, $\langle A \rangle$ is an M-matrix, that is if and only if $\langle A \rangle$ is nonsingular and $\langle A \rangle^{-1} \geq O$.*

The class of nonsingular H-matrices can also be characterized as a generalization of SDD matrices, also known under the name generalized diagonally dominant (GDD) matrices.

Definition 1.5.6. *Matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix if and only if there exists a positive diagonal matrix $X = diag(x_1, x_2, \ldots, x_n) \in \mathbb{C}^{n,n}$, such that AX is SDD.*

The following properties of H-matrices are well-known and we give them without proof.

Theorem 1.5.2. *Let $A \in \mathbb{C}^{n,n}$ be an H-matrix, $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$ and $A = D - B$. Then:*

a) A is nonsingular,

b) $|A^{-1}| \leq \langle A \rangle^{-1}$,

c) $|D|$ is nonsingular and $\rho(|D|^{-1}|B|) < 1$.

Additionally, if all diagonal entries of an H-matrix are strictly positive, then the matrix is an H^+-matrix.

Lemma 1.5.4 ([66])**.** *If $e = [1, 1, \ldots, 1]^T$ and P is a strictly diagonal dominant (SDD) matrix, then for any matrix Q the inequality*

$$\|P^{-1}Q\|_\infty \leq \max_i \frac{(|Q|e)_i}{(\langle P \rangle e)_i},$$

holds.

The following lemma is an original result from [26], so we present it here along with its proof.

Lemma 1.5.5 ([26]). *Let P be an H-matrix and $w > 0$ such that $\langle P\rangle w > 0$. Then, for any matrix Q we have*

$$\|P^{-1}Q\|_{(w)} \leq \|\langle P\rangle^{-1}|Q|\|_{(w)} \leq \max_i \frac{(|Q|w)_i}{(\langle P\rangle w)_i}.$$

Proof: Since P is an H-matrix, it follows that there exists $w > 0$ such that $\langle P\rangle w > 0$. Also, according to Theorem 1.5.2 $|P^{-1}| \leq \langle P\rangle^{-1}$. Therefore, the following is straightforward

$$\|P^{-1}Q\|_{(w)} = \||P^{-1}Q|\|_{(w)} \leq \||P^{-1}||Q|\|_{(w)} \leq \|\langle P\rangle^{-1}|Q|\|_{(w)}.$$

The nonnegativity of $\langle P\rangle^{-1}|Q|$ implies that

$$\|\langle P\rangle^{-1}|Q|\|_{(w)} = \|\langle P\rangle^{-1}|Q|w\|_{(w)} = \|z\|_{(w)} = \|W^{-1}z\|_\infty = \max_i \frac{z_i}{w_i},$$

where $W = diag(w_1, \ldots, w_n)$ and $z := \langle P\rangle^{-1}|Q|w \geq 0$, or equivalently $\langle P\rangle z = |Q|w$. Componentwise, this is the same as

$$|p_{ii}|z_i - \sum_{j\neq i} |p_{ij}|z_j = \sum_j |q_{ij}|w_j.$$

Now, let us observe the row index k with the property that

$$\|z\|_{(w)} = \frac{z_k}{w_k} \geq \frac{z_j}{w_j}, \quad \forall j \in N_n,$$

for which

$$|p_{kk}|z_k - \sum_{j\neq k} |p_{kj}|\frac{z_k w_j}{w_k} \leq |p_{kk}|z_k - \sum_{j\neq k} |p_{kj}|z_j = \sum_j |q_{kj}|w_j,$$

i.e.

$$\frac{z_k}{w_k}\left\{|p_{kk}|w_k-\sum_{j\neq k}|p_{kj}|w_j\right\}\leq\sum_j|q_{kj}|w_j,$$

i.e.

$$\frac{z_k}{w_k}\left(\langle P\rangle w\right)_k\leq\left(|Q|w\right)_k.$$

Finally,

$$\|z\|_{(w)}=\frac{z_k}{w_k}\leq\frac{\left(|Q|w\right)_k}{\left(\langle P\rangle w\right)_k}\leq\max_i\frac{\left(|Q|w\right)_i}{\left(\langle P\rangle w\right)_i},$$

which completes the proof. □

For an arbitrary $A\in\mathbb{R}^{n,n}$ we say that it is symmetric if $A=A^T$.

Definition 1.5.7. *A symmetric matrix $A\in\mathbb{R}^{n,n}$ is positive definite if $x^TAx>0$ for every $x\neq 0$. If $x^TAx\geq 0$ for every x, then A is positive semi-definite.*

1.6 Matrix splittings

Definition 1.6.1. *Let $A\in\mathbb{C}^{n,n}$. If there exist matrices $M,N\in\mathbb{C}^{n,n}$, where M is nonsingular and*

$$A=M-N, \tag{1.1}$$

then (1.1) is called a splitting of A.

There are many kinds of splitting of a matrix. Some of them, used in this dissertation, we present now. For details, see [5, 11].

Definition 1.6.2. *[30] Let $A=M-N$ be a splitting of A. The splitting is:*

a) convergent if $\rho(M^{-1}N)<1$,

b) H-splitting if $\langle M\rangle - |N|$ *is an M-matrix,*

c) H-compatible if $\langle A\rangle = \langle M\rangle - |N|$.

Lemma 1.6.1. *[35] Let* $A = M - N$ *be an H-splitting. Then, matrices A and M are both H-matrices and* $\rho(M^{-1}N) \leq \rho(\langle M\rangle^{-1}|N|) < 1$, *i.e. every H-splitting is convergent.*

Lemma 1.6.2. *[35] Every H-compatible splitting of an H-matrix is also an H-splitting.*

Definition 1.6.3. *Let* $A \in \mathbb{C}^{n,n}$. *The splitting of the form*

$$A = D - L - U, \tag{1.2}$$

where $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$ *and*

$$(L)_{ij} := \begin{cases} -a_{ij} & for \quad i > j, \\ 0 & for \quad i \leq j, \end{cases} \qquad (U)_{ij} := \begin{cases} -a_{ij} & for \quad i < j, \\ 0 & for \quad i \geq j, \end{cases}$$

is called the standard splitting of A.

Definition 1.6.4. *Let* $\ell \in \mathbb{N}$ *be such that* $\ell \leq n$, *let* $A = M_p - N_p$, $p = 1, 2, \ldots, \ell$ *be splittings of the system matrix* $A \in \mathbb{R}^{n,n}$, *and let* $E_p \in \mathbb{R}^{n,n}$ *be nonnegative diagonal matrices satisfying* $\sum_{p=1}^{\ell} E_p = E$ *(called the weighting matrices). Then the collection of triples* $(M_p, N_p, E_p)(p = 1, 2, \ldots, \ell)$ *is called a multisplitting of A.*

Definition 1.6.5. *Let* $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$, *and let for* $p = 1, 2, \ldots, \ell$, *matrices* L_p *and* $U_p = D - L_p - A$ *be strictly lower triangular and zero-diagonal matrices, respectively. Let nonnegative diagonal matrices* $E_p \in$

$\mathbb{R}^{n,n}$, $p = 1, 2, \ldots, \ell$, *satisfy* $\sum_{p=1}^{\ell} E_p = E$. *Then the collection of triples* $(D - L_p, U_p, E_p)(p = 1, 2, \ldots, \ell)$ *is called a triangular multisplitting of A.*

Definition 1.6.6. *Let* $(M_p, N_p, E_p)(p = 1, 2, \ldots, \ell)$ *be a multisplitting of A. Let us observe the splittings of matrices* M_p *of the following form:* $M_p = F_p - G_p$, *for* $p = 1, 2, \ldots, \ell$. *Then, the collection of triples* $(M_p : F_p, G_p; N_p; E_p)(p = 1, 2, \ldots, \ell)$ *is called a two-stage multisplitting of A.*

Definition 1.6.7. *Let* $(M_p, N_p, E_p)(p = 1, 2, \ldots, \ell)$ *be a multisplitting of A, and let* $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$. *For each* $p = 1, 2, \ldots, \ell$, *let us define the following parts of* M_p*: diagonal,* D_p, *strictly lower triangular,* L_p *and zero-diagonal,* U_p, *so that* $M_p = D_p - L_p - U_p$. *Then, the collection of triples* $(M_p : D_p - L_p, U_p; N_p; E_p)(p = 1, 2, \ldots, \ell)$ *is called a two-stage triangular multisplitting of A.*

2. Linear complementarity problem

2.1 Problem formulation

For an arbitrary finite positive integer n, let $A \in \mathbb{R}^{n,n}$ be an arbitrary real-valued, square matrix of order n. In addition, let $q \in \mathbb{R}^n$ be a real-valued vector of dimension n. Then, we may try to solve the following problem: *find a nonnegative vector $z \in \mathbb{R}^n$ for which the vector $q + Az$ is also nonnegative*. Mathematically, we can formulate this task in the following way, as a system of linear inequations

$$z \geq 0, \tag{2.1}$$

$$q + Az \geq 0. \tag{2.2}$$

For every z satisfying both inequations (2.1) and (2.2), we say that it is a feasible vector. Therefore, the set of all feasible vectors, that is - the set of all solutions of (2.1)-(2.2) is called the feasible set, denoted by $\mathscr{F}(q,A)$. As a special case, the inequalities in (2.1)-(2.2) can become strict. Then, if there still exists a vector z satisfying both of them, we

say that z is a strictly feasible vector. The corresponding set of all such vectors is called strictly feasible set.

From the geometrical point of view, the set $\mathscr{F}(q,A)$ is the *intersection* of the first orthant (hyperoctant) - condition (2.1), and the cone given by (2.2). Clearly, this set may be empty when there is no intersection, which means that the cone defined by (2.2) lies entirely outside the first orthant. We illustrate and comment on the cases of the existence of solutions to (2.1) and (2.2) for $n = 2$, since these can be easily represented geometrically.

Example 2.1.1. *Consider the following matrix-vector pair for our problem,*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad and \quad q = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

From now on, we will write $z = [z_1, z_2]^T$, which means that (2.2) becomes:

$$z_1 + 2z_2 \geq -2, \quad 2z_1 - z_2 \geq 2. \tag{2.3}$$

Solving this system of linear inequations can be done graphically, and since part of this region belongs to the first quadrant, there exists $\mathscr{F}(q,A)$. This is illustrated in Figure 2.1.

The feasible set may not always exist. The following example shows this fact.

Example 2.1.2. *Consider the following matrix-vector pair for our problem,*

$$A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} \quad and \quad q = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

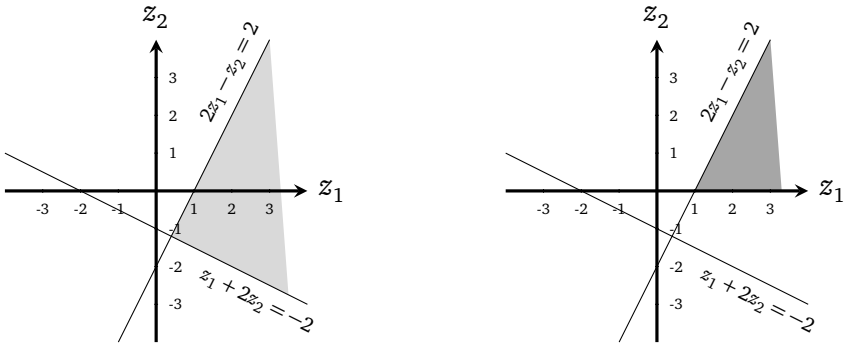


Figure 2.1: Inequality constraint cone defined by (2.3) (left) and its intersection with the first quadrant, producing the feasible set, $\mathscr{F}(q,A)$ (right).

In this case, (2.2) becomes

$$-z_1 - 2z_2 \geq 2, \quad 2z_1 - z_2 \geq 2. \tag{2.4}$$

Since the cone described by this system does not intersect with the first quadrant, there is no feasible set, $\mathscr{F}(q,A)$. This is illustrated in Figure 2.2.

The challenge of the problem becomes obvious if we introduce an additional requirement:

$$z^T(q + Az) = 0, \tag{2.5}$$

where z^T denotes the transpose of vector z.

Condition (2.5) is known as the complementarity condition. What this condition algebraically represents is the orthogonality condition of vectors z and $q + Az$.

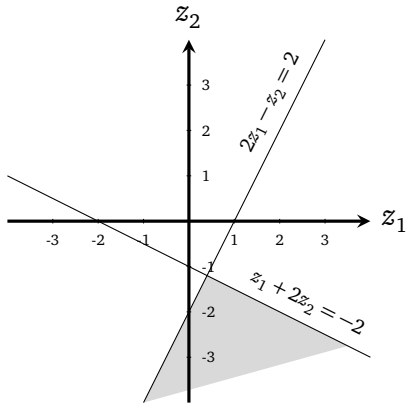


Figure 2.2: Inequality constraint (2.4) which has no intersection with the first quadrant, so there is no feasible set.

Obviously, the linear complementarity problem is about finding a vector in a finite-dimensional real-valued vector space that satisfies a certain set of constraints. Since we have analyzed its parts from different angles, we may now define the problem formally.

Definition 2.1.1. *Let $A \in \mathbb{R}^{n,n}$ be an arbitrary matrix and $q \in \mathbb{R}^n$ an arbitrary vector. The Linear Complementarity Problem (LCP), for the given pair of vector and matrix, in the notation $LCP(q,A)$, is defined as follows:*

$$\begin{aligned} z &\geq 0, \\ q + Az &\geq 0, \\ z^T(q + Az) &= 0. \end{aligned} \tag{2.6}$$

Therefore, the process of finding a solution of $LCP(q,A)$ can be divided in two stages. First, identify the feasible region, $\mathscr{F}(q,A)$, and then

choose all the vectors for which the complementarity condition (2.5) is satisfied. This means that if the feasible region is empty, as in Example 2.1.2, then $LCP(q,A)$ has no solution. Even if the feasible region is nonempty, which means that there are candidates for vector z to be chosen from, this does not guarantee that any of them will satisfy condition (2.5), in which case $LCP(q,A)$, again, has no solution.

In the literature dealing with the linear complementarity problems the usual practice is to define an auxiliary vector

$$r = q + Az,$$

which leads to a simpler notation of $LCP(q,A)$. We say that z belonging to $\mathscr{F}(q,A)$ solves $LCP(q,A)$ if and only if

$$z_i r_i = 0 \qquad \text{for every} \quad i = 1,2,\ldots,n. \tag{2.7}$$

Note that z_i and r_i are the corresponding components of vectors z and r, respectively. Since z belongs to $\mathscr{F}(q,A)$, this implies that both z and r are nonnegative. Therefore, in order the condition (2.7) to hold, it needs to be true that whenever there exists a nonzero component of one vector, the corresponding component in the other one must be equal to zero. This complementarity nature of the condition justifies the attribute in the name of the problem.

2.2 LCP in applications

Linear complementarity problems can be observed as a unifying framework for the problems of linear and quadratic programming, as well

as bimatrix games and, not to be forgotten, market equilibrium models from numerical optimization. The wide range of practical applications makes them very important. Since many problems from practice either reduce or can be written in the form of a linear complementarity problem, this motivates methods for their solution.

2.2.1 Quadratic programming

A lot of problems from practice, when expressed in the language of mathematics, become problems of numerical optimization. One important class of such problems is the quadratic programming problem (QP),

$$\begin{array}{lll} \text{minimize} & c^T x + \dfrac{1}{2} x^T Q x & \textit{(objective function)} \\ \text{subject to} & Ax \geq b & \textit{(main constraint)} \\ & x \geq 0 & \textit{(nonnegativity constraint)} \end{array} \tag{2.8}$$

The representation (2.8) is known as the *standard minimum problem* and the goal is to find vector $x \in \mathbb{R}^n$ as a solution of (2.8). The name quadratic programming comes from the quadratic term $x^T Q x$ appearing in the objective function. The main assumption is that the real square matrix Q of order n is symmetric, where $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. It is interesting here to see that in the case when matrix Q is equal to zero matrix, the quadratic term disappears, and the quadratic program becomes a linear one. For that reason, since quadratic programming represents a generalization of linear programming, we will talk about it first, and then analyze linear programming, as its special case, in detail. Both of these programs are interesting from the point of view of this

thesis, since at the equilibrium point, the optimality conditions are, in fact, in form of a linear complementarity problem.

In addition to the standard minimum problem, there also exists the *standard maximum problem*, which aims to find vector $y \in \mathbb{R}^m$ which solves

$$\begin{aligned} \text{maximize} \quad & b^T y - \frac{1}{2} u^T Q u \qquad \textit{(objective function)} \\ \text{subject to} \quad & A^T y - Qu \leq c \qquad \textit{(main constraint)} \\ & y \geq 0 \qquad \textit{(nonnegativity constraint)} \end{aligned} \tag{2.9}$$

where $u \in \mathbb{R}^n$. The constraints, main and nonnegativity, together make the *feasible set*, or the *domain* of the problem. The procedure of solving a general QP can be described as follows. First, if the domain is empty, meaning that there is no vector with the property that all inequalities hold true at the same time, the problem is *infeasible* and has no solution. However, if the set is feasible, then the problem is *feasible*. Now we try to find the extreme value of the objective within the domain. This creates two possibilities. If the objective function is unbounded on domain, then QP has no solution and it is unbounded feasible. Otherwise, it is bounded feasible. The feasible vector from domain, for which the objective function becomes maximal/minimal is called the *optimal vector*.

From the theory of numerical optimization, it is well-known that the necessary optimality condition for x to minimize the quadratic objective function is that there exists y, such that the vector pair (x, y) satisfies

the Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned} u &= c + Qx - A^T y \geq 0, \quad & x \geq 0, \quad & x^T u = 0, \\ v &= -b + Ax \geq 0, & y \geq 0, \quad & y^T v = 0. \end{aligned} \tag{2.10}$$

These conditions can be written in a compact form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix} + \begin{bmatrix} Q & -A^T \\ A & O \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \geq 0, \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

If we denote by $r = [u \;\; v]^T$ and $z = [x \;\; y]^T$, (2.10) is exactly a linear complementarity problem for the vector-matrix pair

$$\begin{bmatrix} c \\ -b \end{bmatrix} \in \mathbb{R}^{n+m} \quad \text{and} \quad \begin{bmatrix} Q & -A^T \\ A & O \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}.$$

The symmetry of Q does not imply the symmetry of the block matrix for the linear complementarity problem. However, the symmetry is true only if A is a zero matrix, which means that the inequations from QP disappear. This simplifies the problem.

2.2.2 Linear programming

In case when matrix Q from QP is a zero matrix, quadratic programming becomes simplified, that is - linear. A *linear programming* problem (LP) represents a problem of finding the maximum or minimum of a

linear function. This function is called the *objective* function. Additionally, as in the case of QP, the problem includes constraints, which are linear functions of the variables. These constraints can be equalities or inequalities.
All linear programs can be divided in two main types of standard problems:

- *standard minimum problem*, which is to find $x \in \mathbb{R}^n$ that solves

$$\begin{array}{lll} \text{minimize} & c^T x & \textit{(objective function)} \\ \text{subject to} & Ax \geq b & \textit{(main constraint)} \\ & x \geq 0 & \textit{(nonnegativity constraint)} \end{array}$$

- *standard maximum problem*, which is to find $y \in \mathbb{R}^m$ that solves

$$\begin{array}{lll} \text{maximize} & b^T y & \textit{(objective function)} \\ \text{subject to} & A^T y \leq c & \textit{(main constraint)} \\ & y \geq 0 & \textit{(nonnegativity constraint)} \end{array}$$

where c is from $\mathbb{R}^n$, vector b belongs to $\mathbb{R}^m$ and $A \in \mathbb{R}^{m,n}$.

One of the key concepts in numerical optimization in connection with linear and quadratic programming is the concept of duality. There is always more than one aspect to every problem which we need to solve. It is a tool which enables us to look at the same thing but from two different angles. So, the principle of duality makes life much easier, and is a very important concept in numerical optimization.

So far, we have seen the definitions of the standard minimum and maximum problem. Each of them can be considered as a *primal* problem. We define the concept of its *dual* as follows.

Definition 2.2.1. *The dual of the standard minimum problem is defined to be the standard maximum problem, and vice versa.*

Therefore, the usual practice is to be aware that along with the primal, the dual is also available. This is important, since we will choose the one which is more suitable for the situation we are dealing with. In case of linear programming, let us pronounce for the primal (P) the standard minimum problem, and consequently, for the dual (D) the standard maximum problem. Taken together as "two sides", the way to solve them can be described as follows. We are obviously interested to find a pair of vectors, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, that solve (P) and (D), respectively.

As in the case of QP, it is well known that the optimality conditions for the primal-dual pair are

$$\begin{aligned} u &= c - A^T y \geq 0, \qquad x \geq 0, \quad x^T u = 0, \\ v &= -b + Ax \geq 0, \qquad y \geq 0, \quad y^T v = 0. \end{aligned} \tag{2.11}$$

where $x \in \mathbb{R}^n$ is the optimal feasible solution of the primal, $y \in \mathbb{R}^m$ is the optimal feasible solution of the dual, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. But, if we take a closer look, (2.11) is equivalent to

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix} + \begin{bmatrix} O & -A^T \\ A & O \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \geq 0, \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Finally, if we denote $r = [u \quad v]^T$, $z = [x \quad y]^T$, (2.11) is exactly a linear complementarity problem for the vector-matrix pair

$$\begin{bmatrix} c \\ -b \end{bmatrix} \in \mathbb{R}^{n+m} \quad \text{and} \quad \begin{bmatrix} O & -A^T \\ A & O \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}.$$

2.2.3 Market equilibrium

Now we will consider another possible practical application of LCP that comes from linear programming model of market equilibrium, as it has been done in [19]. Imagine a company which produces goods for the consumers (for simplicity we consider only one firm, although the model can be extended to a more general case). If we take c to be the cost vector of raw goods for the supply activities, containing information of the unit production costs of all the goods that are necessary for the production, and if we denote by x the unknown vector of production activity levels, then the total cost of production can be represented as the scalar product of these two vectors, $c^T x$. Obviously, vector x is, by nature, nonnegative. The objective of the firm is

$$\text{minimize} \quad c^T x. \tag{2.12}$$

The constraints are as follows. First, it is known that the technological constraints on production can be expressed in form

$$Ax \geq b, \tag{2.13}$$

where A is a known matrix and b a known vector. Also, the demand requirement constraint is

$$Bx \geq r^\star, \tag{2.14}$$

where $r^\star$ is the threshold for demand of products at equilibrium. On the other side, observe the demand, represented by consumers. The demand vector r^* at equilibrium is given as a linear function of the vector of prices of produced goods p^*

$$r^\star = Q(p^\star) = Dp^\star + d. \tag{2.15}$$

Market equilibrium is defined to be the state of an economy at which the supplies of producers and demands of consumers are balanced. The goal is to find vectors $r^\star$ and $p^\star$ so that conditions (2.12)-(2.15) are satisfied. The last condition providing equilibrium is

$$p^\star = \pi^\star \tag{2.16}$$

where $\pi^\star$ denotes the (dual) vector of shadow prices (i.e., the market supply prices) corresponding to (2.14).

Next step is to convert the introduced model to a form of LCP. From the theory of numerical optimization and the duality, it is known that vector x^* solves (2.12)-(2.14) optimally if and only if there exists a vector $v^\star$ such that

$$\begin{aligned}
&y^\star = c - A^T v^\star - B^T \pi^\star \geq 0, && x^\star \geq 0, && (y^\star)^T x^\star = 0, \\
&u^\star = -b + Ax^\star \geq 0, && v^\star \geq 0, && (u^\star)^T v^\star = 0, \\
&\delta^\star = -r^\star + Bx^\star \geq 0, && \pi^\star \geq 0, && (\delta^\star)^T \pi^\star = 0.
\end{aligned} \tag{2.17}$$

Finally, it only remains to substitute (2.15) and (2.16) into (2.17) to obtain the LCP with

$$q = \begin{bmatrix} c \\ -b \\ -d \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & -A^T & -B^T \\ A & 0 & 0 \\ B & 0 & -D \end{bmatrix}.$$

2.2.4 Bimatrix games

Another possible application of LCP comes from the game theory, [51]. Imagine a game with two players, which we will denote $P1$ and $P2$. Suppose that $P1$ has m and $P2$ has n choices and they act independently. Next, we introduce two matrices, $A' = [a'_{ij}] \in \mathbb{R}^{m,n}$ and $B' = [b'_{ij}] \in \mathbb{R}^{m,n}$ in the following way. If $P1$ decides to choose choice i and $P2$ choice j, then the loss of $P1$ is a'_{ij} and the loss of $P2$ is b'_{ij}. For that reason, matrices A' and B' are called the *loss matrices*. If $A' + B' = O$, then the game is a *zero sum game*. Otherwise, it is a *nonzero sum game* or a *bimatrix game*.

Bimatrix games can be transformed into a LCP form by the concept of equilibrium pair of strategies. Suppose that $P1$ decides for choice i with probability x_i, for every $i = 1, 2, \ldots, m$. Then $x = [x_i] \in \mathbb{R}^m$ completely defines the strategy of $P1$. Similarly, we can introduce vector $y = [y_j] \in \mathbb{R}^n$ which determines the strategy of $P2$. Then, the *expected loss* of $P1$ is $x^T A' y$ and of $P2$ is $x^T B' y$. The *equilibrium pair* is the strategy pair $(x^\star, y^\star)$ such that

$$\begin{aligned} (x^\star)^T A' y^\star &\leq x^T A' y^\star, && \forall x \in \mathbb{R}^m \\ (x^\star)^T B' y^\star &\leq (x^\star)^T B' y, && \forall y \in \mathbb{R}^n. \end{aligned} \tag{2.18}$$

Next, we can always choose $\alpha \geq 0$ and $\beta \geq 0$ such that for every i and j $a_{ij} := a'_{ij} + \alpha > 0$ and $b_{ij} := b'_{ij} + \beta > 0$, defining two strictly positive matrices A and B. Since, for every x and y

$$x^T A' y \leq x^T A y - \alpha$$
$$x^T B' y \leq x^T B y - \beta,$$

it is clear that $(x^\star, y^\star)$ is the equilibrium pair for the game with loss matrices A' and B' if and only if it is the equilibrium pair for the game with loss matrices A and B.

Recalling that x and y are probability vectors, meaning that $0 \leq x, y \leq e$, using condition (2.18) and denoting by e_r the vector e from $\mathbb{R}^r$, we have

$$\begin{aligned} ((x^\star)^T A y^\star) e_m &\leq A y^\star, \\ ((x^\star)^T B y^\star) e_n &\leq B^T x^\star. \end{aligned} \tag{2.19}$$

But, $A, B > O$ implies that $(x^\star)^T A y^\star > 0$ and $(x^\star)^T B y^\star > 0$. Introducing

$$\xi^\star := \frac{x^\star}{(x^\star)^T B y^\star} \quad \text{and} \quad \eta^\star := \frac{y^\star}{(x^\star)^T A y^\star}$$

and substituting these vectors in (2.19), we obtain

$$\begin{aligned} e_m &\leq A \eta^\star, \\ e_n &\leq B^T \xi^\star. \end{aligned} \tag{2.20}$$

Introducing slack variables $u^\star$ and $v^\star$, (2.20) becomes equivalent to

$$\begin{bmatrix} u^* \\ v^* \end{bmatrix} - \begin{bmatrix} O & A \\ B^T & O \end{bmatrix} \cdot \begin{bmatrix} \xi^* \\ \eta^* \end{bmatrix} = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix},$$

$$\begin{bmatrix} u^* \\ v^* \end{bmatrix} \geq 0, \quad \begin{bmatrix} \xi^* \\ \eta^* \end{bmatrix} \geq 0, \quad \begin{bmatrix} u^* \\ v^* \end{bmatrix}^T \begin{bmatrix} \xi^* \\ \eta^* \end{bmatrix} = 0,$$

which is in form of a LCP for a vector-matrix pair

$$\begin{bmatrix} -e_m \\ -e_n \end{bmatrix} \in \mathbb{R}^{m+n} \quad \text{and} \quad \begin{bmatrix} O & A \\ B^T & O \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}.$$

3. Overview of methods for solving systems of linear equations

One of the key tasks of numerical linear algebra is to solve systems of linear equations, which come from many models of physical and social phenomena. By an *equation* we understand a mathematical statement made of unknowns, x_i, $i = 1, 2, \ldots, n$, which need to be found. If all the unknowns are arguments of linear functions, then the equation is said to be *linear*, for example

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n = b. \tag{3.1}$$

Once we find a solution and plug it in, the equation becomes a valid equality. Combining two or more equations containing mutual unknowns creates a *system of (linear) equations* (SLE),

$$S : \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} . \tag{3.2}$$

The usual practice is to write the system (3.2) in a matrix-vector form $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \tag{3.3}$$

As far as the solvability of (3.2) is concerned, it is well known that if the number of unknowns and number of equations coincide, menaning that the system is square, then the nonsingularity of A is a necessary and sufficient condition for the existence of a unique solution of (3.2), which we will mark as $x^\star$. However, if the system matrix is singular, then the existence and number of solutions depends on vector b.

3.1 Direct methods

By direct methods we understand all methods that compute the exact solution in a finite number of steps. Under the condition that A is nonsingular, the direct method would suggest that

$$x^\star = A^{-1}b.$$

This approach is acceptable for elementary calculations. However, from the practical point of view, working with matrices of large and super high dimensions, it is clear that the calculation of the inverse is computationally expensive and inefficient. Also, the rounding errors may affect the final solution in a way that it may become far away from the exact solution. Therefore, we will focus on the iterative methods.

3.2 Iterative methods

Iterative methods abandon the idea to find the exact solution, but are, instead, based on the idea to construct an iterative sequence of approximations which will, under certain conditions, converge to the solution of the system. The advantage of iterative methods is that they allow the restart of the iterative process at any iteration, since the convergence does not depend on the choice of the starting vector.

3.2.1 Splitting iterative methods for SLE

Solving systems of linear equations is very often treated by using splitting methods. Because of that, in this section and the next to come, we present the splitting idea and its application. Then, in the next chapter, we will use the same idea for solving the linear complementarity problem.

We consider a system of linear equations

$$Ax = b, \tag{3.4}$$

where we assume that $A \in \mathbb{C}^{n,n}$ is a nonsingular matrix with all diagonal entries nonzero, x and b are n-dimensional vectors. If we consider the splitting of A, that is

$$A = M - N,$$

system (3.4) can be rewritten as

$$Mx = Nx + b \tag{3.5}$$

and the nonsingularity of M implies

$$x = M^{-1}Nx + M^{-1}b, \tag{3.6}$$

which is known as the *fixed point equation*. This representation naturally implies an iterative rule for solving system (3.4),

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k \geq 0. \tag{3.7}$$

The idea is to start with an initial approximation $x^{(0)}$ of the solution $x^\star$ and then generate a sequence $\{x^{(k)}\}_{k=0}^{\infty}$ that converges to $x^\star$, if certain conditions are satisfied. Denoting by $T := M^{-1}N$ and $d = M^{-1}b$, (3.7) becomes

$$x^{(k+1)} = Tx^{(k)} + d, \quad k \geq 0, \tag{3.8}$$

which is known as the *general iterative method*. Its convergence is characterized by the following theorem.

Theorem 3.2.1. *The iterative sequence $\{x^{(k)}\}_{k=0}^{\infty}$, generated by (3.8), converges to the exact solution $x^\star$ of (3.4) for every starting iterate $x^{(0)}$ if and only if $\rho(T) < 1$.*

Jacobi method

The Jacobi method is the simplest iterative method for solving a square linear system $Ax = b$. It is a splitting iterative method (3.7) for the choice $M = D$ and $N = L + U$. By assumption, D is nonsingular, so the Jacobi iterative rule says

$$x^{(k+1)} = D^{-1}(L + U)x^{(0)} + D^{-1}b, \quad k \geq 0. \tag{3.9}$$

Theorem 3.2.2. *The iterative sequence* $\{x^{(k)}\}_{k=0}^{\infty}$*, generated by (3.9), converges to the exact solution* $x^{\star}$ *of (3.4) for every starting vector* $x^{(0)}$ *if and only if* $\rho(D^{-1}(L+U)) < 1$.

Theorem 3.2.3. *Provided that the coefficient matrix A of (3.4) is SDD or H matrix, the sequence* $\{x^{(k)}\}_{k=0}^{\infty}$*, generated by (3.9), converges to the unique solution of that system, for every starting vector* $x^{(0)}$.

Gauss-Seidel method

The choice $M = D - L$ and $N = U$ in (3.7) defines the Gauss-Seidel iterative method:

$$x^{(k+1)} = (D-L)^{-1}Ux^{(k)} + (D-L)^{-1}b, \quad k \geq 0. \tag{3.10}$$

Theorem 3.2.4. *The iterative sequence* $\{x^{(k)}\}_{k=0}^{\infty}$*, generated by (3.10), converges to the exact solution* $x^{\star}$ *of (3.4) for every starting vector* $x^{(0)}$ *if and only if* $\rho((D-L)^{-1}U) < 1$.

Theorem 3.2.5. *Provided that the coefficient matrix A of (3.4) is SDD or H matrix, the sequence* $\{x^{(k)}\}_{k=0}^{\infty}$*, generated by (3.10), converges to the unique solution of that system, for every starting vector* $x^{(0)}$.

3.2.2 Relaxation iterative methods

If we allow that the iteration matrix depends on one or possibly more parameters, then the convergence will obviously depend on their choice, too. These parameters are called *relaxation parameters* and the corresponding iterative methods are known as *relaxation iterative methods*.

First, we will recall one-parameter generalizations of the Jacobi and Gauss-Seidel methods, JOR and SOR respectively, and, then, the two-parameter AOR iterative method.

JOR method

The next iteration by the Jacobi method can be written as

$$x^{(k+1)} = x^{(k)} + (x^{(k+1)} - x^{(k)}),$$

where

$$Dx^{(k+1)} = (L+U)x^{(k)} + b. \tag{3.11}$$

Jacobi Over-Relaxation Method (JOR) is extrapolated Jacobi method, meaning that the current iterate, $x^{(k)}$, is corrected by the nonzero multiple of the update, where $\omega \neq 0$ is the relaxation parameter,

$$x^{(k+1)} = x^{(k)} + \omega(x^{(k+1)} - x^{(k)}). \tag{3.12}$$

Obviously, we see that JOR method reduces to Jacobi method if $\omega = 1$. Substituting (3.11) into (3.12), we can derive the iterative method

$$x^{(k+1)} = \mathscr{L}_{JOR}(A,\omega)x^{(k)} + d_{JOR}(A,b,\omega), \quad k \geq 0, \tag{3.13}$$

where the iterative matrix is

$$\mathscr{L}_{JOR}(A,\omega) := (1-\omega)E + \omega D^{-1}(L+U),$$

and

$$d_{JOR}(A,b,\omega) = \omega D^{-1}b.$$

Theorem 3.2.6. *The iterative sequence* $\{x^{(k)}\}_{k=0}^{\infty}$, *generated by (3.13), converges to the exact solution* $x^{\star}$ *of (3.4) for every starting vector* $x^{(0)}$ *if and only if* $\rho(\mathscr{L}_{JOR}(A,\omega)) < 1$.

Theorem 3.2.7 ([11])**.** *Let* $A \in \mathbb{C}^{n,n}$ *be an SDD (H) matrix and let the relaxation parameter* ω *satisfy*

$$0 < \omega < \frac{2}{1+\rho(J)},$$

where $J := D^{-1}(L+U)$ *is the Jacobi iterative matrix. Then, the iterative sequence* $\{x^{(k)}\}_{k=0}^{\infty}$, *generated by (3.13), converges to the exact solution* $x^{\star}$ *of (3.4) for every starting vector* $x^{(0)}$.

SOR method

Analogously to JOR, the SOR iterative method can be treated as an extrapolation of Gauss-Seidel method. The next iterate for the Gauss-Seidel iteration is obtained as

$$x^{(k+1)} = x^{(k)} + (x^{(k+1)} - x^{(k)}),$$

where

$$Dx^{(k+1)} = Lx^{(k+1)} + Ux^{(k)} + b. \tag{3.14}$$

In Successive Over-Relaxation (SOR) method, the current iterate, $x^{(k)}$, is obtained by

$$x^{(k+1)} = x^{(k)} + \omega(x^{(k+1)} - x^{(k)}), \tag{3.15}$$

where $\omega \neq 0$ is the relaxation parameter. Obviously, SOR method reduces to Gauss-Seidel if $\omega = 1$. Substituting (3.14) into (3.15), we are

able to derive the iterative method

$$x^{(k+1)} = \mathscr{L}_{SOR}(A,\omega)x^{(k)} + d_{SOR}(A,b,\omega), \quad k \geq 0, \tag{3.16}$$

where the iterative matrix is

$$\mathscr{L}_{SOR}(A,\omega) := (D-\omega L)^{-1}\left((1-\omega D)+\omega U\right),$$

and

$$d_{SOR}(A,b,\omega) = \omega(D-\omega L)^{-1}b.$$

Theorem 3.2.8. *The iterative sequence* $\{x^{(k)}\}_{k=0}^{\infty}$*, generated by (3.16), converges to the exact solution* $x^{\star}$ *of (3.4) for every starting vector* $x^{(0)}$ *if and only if* $\rho(\mathscr{L}_{SOR}(A,\omega)) < 1$.

Theorem 3.2.9 ([11])**.** *Let* $A \in \mathbb{C}^{n,n}$ *be an SDD (H) matrix and let the relaxation parameter* ω *satisfy*

$$0 < \omega < \frac{2}{1+\rho(|J|)},$$

where $J := D^{-1}(L+U)$ *is the Jacobi iterative matrix. Then, the iterative sequence* $\{x^{(k)}\}_{k=0}^{\infty}$*, generated by (3.16), converges to the exact solution* $x^{\star}$ *of (3.4) for every starting vector* $x^{(0)}$.

AOR method

The concept of Accelerated Over-Relaxation has been introduced by Hadjidimos, [37]. Essentially, this method represents a two-parameter generalization of SOR iterative method, where there is one additional parameter, σ, known as the *acceleration parameter*. The AOR method is formulated as

$$x^{(k+1)} = \mathscr{L}_{AOR}(A,\sigma,\omega)x^{(k)} + d_{AOR}(A,b,\sigma,\omega), \quad k \geq 0, \tag{3.17}$$

where the iterative matrix is

$$\mathscr{L}_{AOR}(A,\sigma,\omega) := (D-\sigma L)^{-1}\left((1-\omega D)+(\omega-\sigma)L+\omega U\right),$$

and

$$d_{AOR}(A,b,\sigma,\omega) = \omega(D-\sigma L)^{-1}b.$$

Remark that if $\sigma = 0$ then the AOR method reduces to the JOR method, where if $\sigma = \omega$ the AOR method becomes the SOR method.

Theorem 3.2.10. *The iterative sequence $\{x^{(k)}\}_{k=0}^{\infty}$, generated by (3.17), converges to the exact solution $x^{\star}$ of (3.4) for every starting vector $x^{(0)}$ if and only if $\rho(\mathscr{L}_{AOR}(A,\sigma,\omega)) < 1$.*

The spectral radius of the AOR iterative matrix can be bounded from above, as stated in the following theorem from [20].

Theorem 3.2.11 ([20])**.** *Let $A = D - L - U$ be the standard splitting of A with all the diagonal entries nonzero. For every $i = 1,2,\ldots,n$ define the following quantities*

$$l_i = r_i(D^{-1}L) \qquad \textit{and} \qquad u_i = r_i(D^{-1}U).$$

If $1-|\sigma|l_i > 0, \quad i = 1,2,\ldots,n$, *then*

$$\rho(\mathscr{L}_{AOR}(A,\sigma,\omega)) \leq \max_i \frac{|1-\omega| + |\omega-\sigma|l_i + |\omega|u_i}{1-|\sigma|l_i}.$$

Theorem 3.2.12 ([20]). *Let* $A = D - L - U$ *be a standard splitting of the system matrix which is an SDD matrix,* $\mathscr{L}_{AOR}(A,\sigma,\sigma)$ *be the iterative matrix of SOR method with parameter* σ *and*

$$l_i = r_i(D^{-1}L), \qquad u_i = r_i(D^{-1}U),$$

$$p = \frac{2\sigma}{1+\rho(\mathscr{L}_{AOR}(A,\sigma,\sigma))}, \qquad s = \frac{2}{1+\rho\left(|D|^{-1}(|L|+|U|)\right)},$$

$$t = \frac{2}{1+\max\limits_i(l_i+u_i)}.$$

Then, the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$, *generated by (3.17), converges for every starting vector* $x^{(0)}$, *provided that parameters* σ *and* ω *satisfy*

$$0 \leq \sigma < s, \qquad 0 < \omega < \max\{p,t\}$$

or

$$0 < \omega < t, \qquad \max_i \frac{-\omega(1-l_i-u_i) + 2\max\{0,\omega-1\}}{2l_i} < \sigma < 0$$

or

$$0 < \omega < t, \qquad t \leq \sigma < \min_i \frac{2\min\{0,1-\omega\} + \omega(1+l_i-u_i)}{2l_i}.$$

4. Iterative methods for LCP

As we have seen in the previous chapters, the linear complementarity problem has a long tradition and great practical significance. Therefore, the question of its solvability is very important. This includes a couple of questions. First question is obviously is the problem solvable. If the answer is positive, we are interested in the number of possible solutions and, finally, in a way how to determine them. One of the key theorems in connection with the existence and uniqueness of the solution of $LCP(q,A)$ is the famous result of Bai and Evans, [4]. All the results that we mention in this thesis, both known and new, revolve around the following lemma.

Lemma 4.0.1 ([4])**.** *Let $A \in \mathbb{R}^{n,n}$ be an H-matrix with positive diagonal elements. Then, the $LCP(q,A)$ has a unique solution, $z^\star \in \mathbb{R}^n$.*

The condition upon A being an H-matrix with a positive diagonal is not over-restrictive, since in many applications the problem matrix satisfies this condition.

The next step in the process is to determine the solution. The problem of solving $LCP(q,A)$ can be approached in more than one way. Just

as in the case of systems of linear equations, numerical methods for the solution of LCP can be divided in two major categories - direct and iterative. In this thesis, we will focus on iterative methods only.

As in the case of systems of linear equations, the idea behind the iterative process is to generate a sequence of solution approximations, also known as trial solutions, which converge to the exact solution, [2]. Iterative methods have been widely studied, see [41, 14, 27, 53, 1, 50].

Being fully aware that there are many possible directions of approach, most of which we will not be able to fully cover in this thesis, we will primarily focus on modulus based iterative methods, [8] and their numerous variations.

4.1 Modulus-based methods

Iterative methods for solving LCP are diverse and very well studied. Among them, the special place belongs to modulus-based methods, which have been introduced by Murty, [51]. The idea of these methods is that vectors $|x|+x$ and $|x|-x$ are simultaneously nonnegative and orthogonal. More precisely,

$$|x|+x=\begin{cases}2x, & x\geq 0,\\ 0, & x<0,\end{cases}\quad\text{and}\quad |x|-x=\begin{cases}0, & x\geq 0,\\ -2x, & x<0,\end{cases}$$

which means that $(|x|+x)_i(|x|-x)_i=0$, for all $i\in N_n$, since at least one component in the above product is equal to zero.

4.1.1 Splitting idea

The modulus-based methods have been developed even further, using the tool of matrix splitting, by Zhong-Zhi Bai, [8]. These methods provide an equivalent reformulation of $LCP(q,A)$, which led to the establishment of some well-known modulus-based matrix splitting iteration methods.

Theorem 4.1.1 ([8]). *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n,n}$, Ω_1 and Ω_2 be nonnegative diagonal matrices, and Ω and Γ be positive diagonal matrices such that $\Omega = \Omega_1 + \Omega_2$. For the LCP$(q,A)$, the following statements hold true:*

i) if (z,r) is a solution of $LCP(q,A)$, then $x = \frac{1}{2}(\Gamma^{-1}z - \Omega^{-1}r)$ satisfies the implicit fixed-point equation (IFPE)

$$(M\Gamma + \Omega_1)x = (N\Gamma - \Omega_2)x + (\Omega - A\Gamma)|x| - q, \tag{4.1}$$

ii) if x satisfies the implicit fixed-point equation (4.1), then

$$z = \Gamma(|x| + x) \quad \text{and} \quad r = \Omega(|x| - x) \tag{4.2}$$

is a solution of $LCP(q,A)$.

Obviously, Theorem 4.1.1 requires a choice of certain parameters. In the remaining part of this thesis, we will work with the following choice: $\Omega_1 = \Omega$, $\Omega_2 = O$ and $\Gamma = \gamma^{-1}E$. Then, we may formulate a special variant of the previous theorem.

Proposition 4.1.2. *Given $A \in \mathbb{R}^{n,n}$, let $A = M - N$ be a splitting of A, Ω a positive diagonal matrix, and γ a positive constant. By setting*

$$z = \frac{1}{\gamma}(|x| + x) \quad \text{and} \quad r = \frac{1}{\gamma}\Omega(|x| - x),$$

the linear complementarity problem can be equivalently transformed into an implicit fixed-point equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q. \tag{4.3}$$

Note that if (4.3) has a solution, its form is $x = \frac{1}{2}\gamma(z - \Omega^{-1}r)$.

The importance of the previous proposition is that (4.3) enables an introduction of an iterative method, based on the splitting of the matrix.

Method 1. THE MODULUS-BASED MATRIX SPLITTING ITERATION METHOD *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n,n}$, Ω a positive diagonal matrix and γ a positive constant. Given an initial vector $x^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$ until the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ is convergent, compute $x^{(k+1)} \in \mathbb{R}^n$ by solving the linear system*

$$(M + \Omega)x^{(k+1)} = Nx^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q, \tag{4.4}$$

and set

$$z^{(k+1)} = \frac{1}{2}(|x^{(k+1)}| + x^{(k+1)}). \tag{4.5}$$

method	*description*	M	N	Ω	γ
M	modulus-based	A	0	E	1
MM	modified modulus-based	A	0	αE	1
MJ	modulus-based Jacobi	D	$L+U$		2
MGS	modulus-based Gauss-Seidel	$D-L$	U		2
MSOR	modulus-based SOR	$\frac{1}{\alpha}D-L$	$\left(\frac{1}{\alpha}-1\right)D+U$		2
MAOR	modulus-based AOR	$\frac{1}{\alpha}(D-\beta L)$	$\frac{1}{\alpha}\left[(1-\alpha)D+(\alpha-\beta)L+\alpha U\right]$		2

Table 4.1: Overview of modulus-based matrix splitting iterative methods.

This method was introduced by Bai in 2010, and it provides a general framework of modulus-based matrix splitting iteration methods for solving $LCP(q,A)$. Previously considered methods, modulus and the modified modulus iteration methods, which were studied by Murty in 1988 [51] and Dong in 2009 [29], can be considered as its special cases, too. However, this method can also yield a series of other modulus-based methods, such as modulus-based Jacobi, modulus-based Gauss-Seidel and their relaxed variants. The overview of these methods is given in Table 4.1.

Now, since we have defined the modulus-based method, based on the splitting of matrix, the question is, under which conditions does the iterative sequence, generated by the method, converge to the solution $x^\star$, for every starting vector $x^{(0)}$? Analogously as for the systems of linear equations, these conditions will be connected to the matrix A itself, but also to the possible choice of relaxation parameters, if the method includes any.

Let us assume that $z^\star \in \mathbb{R}^n_+$ solves $LCP(q,A)$. Then, according to Proposition 4.1.2 and (4.4), we deduce that $x^\star = \frac{1}{2}\gamma(z^\star - \Omega^{-1}r^\star)$ satisfies

$$(M+\Omega)x^\star = Nx^\star + (\Omega - A)|x^\star| - \gamma q, \tag{4.6}$$

where $A = M - N$ is the splitting of A, $\Omega > O$ a positive diagonal matrix and γ a positive constant. Subtracting (4.6) from (4.4) we get

$$(M+\Omega)(x^{(k+1)} - x^\star) = N(x^{(k)} - x^\star) + (\Omega - A)(|x^{(k)}| - |x^\star|),$$

that is,

$$x^{(k+1)} - x^\star = (M+\Omega)^{-1}N(x^{(k)} - x^\star) + (M+\Omega)^{-1}(\Omega - A)(|x^{(k)}| - |x^\star|). \tag{4.7}$$

The expression (4.7) is known as the approximation error and is a key part for the results in connection with the convergence of modulus-based methods, based on the splittings of matrix. Note that, in order to prove the convergence of $\{z^{(k)}\}_{k=0}^{\infty}$, that is $\lim_{k\to\infty} z^{(k)} = z^{\star}$, it is sufficient to prove the convergence of $\{x^{(k)}\}_{k=0}^{\infty}$ instead.

Finally, from now on, we will only focus on a special case, which assume that the system matrix A is an H^{+}-matrix. As we have seen, this is a sufficient condition for the existence and uniqueness of the solution of $LCP(q,A)$.

Theorem 4.1.3 ([8]). *Let $A \in \mathbb{R}^{n,n}$ be an H^{+}-matrix, and $A = M - N$ be an H-compatible splitting of the matrix A, i.e., $\langle A\rangle = \langle M\rangle - |N|$. Assume that Ω is a positive diagonal matrix and γ is a positive constant. If the parameter matrix Ω satisfies $\Omega \geq diag(M)$, then the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty}$, generated by Method 1, converges to the unique solution $z^{\star} \in \mathbb{R}_{+}^{n}$ of the $LCP(q,A)$ for any initial vector $x^{(0)} \in \mathbb{R}^{n}$.*

4.1.2 Multisplitting idea

In order to take full advantage of the high-speed multiprocessor environment and parallel computing, Bai and Zhang proposed synchronous variant of the modulus-based method, [9]. There also exist asynchronous versions, but we will focus here on the first one. The idea behind the synchronous method is to distribute the tasks among all processor units at disposal, each of them solving a separate modulus-based iterative method at every iteration step k, which is then used to determine an

outer approximation of the solution as a kind of matrix convex combination of the calculated results.

MSM iterative method

This method was introduced by Bai and Zhang in 2013, [9].

Given a positive diagonal matrix Ω and a positive constant γ, from Theorem 4.1.1, we straightforwardly conclude that if x satisfies each of the implicit fixed-point equations

$$(\Omega + M_p)x = N_p x + (\Omega - A)|x| - \gamma q, \quad p = 1, 2, ...\ell, \tag{4.8}$$

then the solution of $LCP(q, A)$ is given by

$$z = \frac{1}{\gamma}(|x| + x) \quad \text{and} \quad r = \frac{1}{\gamma}\Omega(|x| - x). \tag{4.9}$$

Therefore, using the equivalent reformulations (4.8) and (4.9) for $LCP(q, A)$, Bai and Zhang were able to establish the modulus-based synchronous multisplitting (MSM) iterative method.

Method 2. THE MODULUS-BASED SYNCHRONOUS MULTISPLITTING METHOD
Let $(M_p, N_p, E_p)(p = 1, 2, \ldots, \ell)$ be a multisplitting of the matrix A.Given an initial vector $x^{(0)} \in \mathbb{R}^n$ and provided that $x^{(k)}$ has been obtained, compute $x^{(k+1,p)}$ by solving the linear subsystem

$$(\Omega + M_p)x^{(k+1,p)} = N_p x^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q, \quad p = 1, 2, \ldots, \ell, \tag{4.10}$$

then set

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p x^{(k+1,p)},$$

and

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}),$$

for $k = 0, 1, 2, \ldots$ *until the iteration sequence* $\left\{z^{(k)}\right\}_{k=0}^{\infty}$ *is convergent.*

The multisplitting idea is to solve each of the ℓ subsystems (4.10) at every iteration k on separate processors, and obtain a local update $x^{(k+1,p)}$, which is then recombined in order to produce a new iteration of MSM with the aid of weighting matrices. Finally, the approximation of the solution of $LCP(q,A)$ at step k is then generated by plugging the $x^{(k+1)}$ in (4.5). The sufficient conditions for the convergence of Method 2 are given in the following theorem.

Theorem 4.1.4 ([9]). *Let* $A \in \mathbb{R}^{n,n}$ *be an* H^+*-matrix with* $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$ *and* (M_p, N_p, E_p) $(p = 1, 2, \ldots, \ell)$ *be a multisplitting of the matrix A. Assume that* $A = M_p - N_p$, $p = 1, 2, ..., \ell$ *are H-compatible splittings,* $\gamma > 0$ *and* Ω *is a positive diagonal matrix satisfying* $\Omega \geq D$*. Then, for any initial vector* $x^{(0)} \in \mathbb{R}^n$*, the iteration sequence* $\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}_+^n$*, generated by Method 2, converges to the unique solution* $z^\star$ *of* $LCP(q,A)$ *.*

4.1.3 Triangular multisplitting idea

At every iteration step and on every processor, in the MSM iterative method, the system (4.10) needs to be solved. The system matrix does not have any special structure. However, by the use of triangular multisplittings, it is achieved that the system matrix for (4.10) becomes a lower triangular one, so the solutions can be obtained more efficiently.

There are many possibilities to choose triangular multisplittings. We will present here the "AOR-approach", i.e. the "AOR-idea", which we have described in the section about systems of linear equations.

MSMAOR iterative method

The modulus-based synchronous multisplitting accelerated overrelaxation (MSMAOR) iterative method for solving $LCP(q,A)$ has been introduced by Bai and Zhang, [9]. Let $(D-L_p, U_p, E_p)$ $(p=1,2,\ldots,\ell)$ be a triangular multisplitting of the matrix A. Taking

$$M_p = \frac{1}{\alpha}(D-\beta L_p) \quad \text{and} \quad N_p = \frac{1}{\alpha}\big((1-\alpha)D+(\alpha-\beta)L_p+\alpha U_p\big),$$

in MSM we obtain MSMAOR method.

Method 3. THE MODULUS-BASED SYNCHRONOUS MULTISPLITTING AOR
Let $(D-L_p, U_p, E_p)(p=1,2,\ldots,\ell)$ be a triangular multisplitting of $A \in \mathbb{R}^{n,n}$. Given $z^{(0)} \in \mathbb{R}^n_+$ and iterations $k \geq 0$, until the iteration sequence is convergent $\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}^n_+$, calculate $z^{(k+1)} \in \mathbb{R}^n_+$ as

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}),$$

and $x^{(k+1)} \in \mathbb{R}^n$ as

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p x^{(k,p)},$$

where each of the vectors $x^{(k,p)}$ $p=1,2,\ldots,\ell$ we find as solutions of

$$(\alpha\Omega+D-\beta L_p)x^{(k,p)} = \Big[(1-\alpha)D+(\alpha-\beta)L_p+\alpha U_p\Big]x^{(k)}+\alpha\Big[(\Omega-A)|x^{(k)}|-\gamma q\Big],$$

with $x^{(0)} = \frac{1}{2}\gamma\Omega^{-1}((\Omega-A)z^{(0)}-q)$.

In a similar way that AOR has generalized the Jacobi and Gauss-Seidel iterative methods, MSMAOR method can be looked at as a generalization of MSM variants of mentioned methods. This is illustrated in Table 4.2.

Method	Description	(α, β)
MSMJ	Modulus-based synchronous multisplitting Jacobi method	$(1,0)$
MSMGS	modulus-based synchronous multisplitting Gauss Seidel method	$(1,1)$
MSMSOR	Modulus-based synchronous multisplitting successive overrelaxation method	(α,α)

Table 4.2: The modulus-based synchronous multisplitting relaxation methods which are special cases of MSMAOR.

The convergence of MSMAOR iterative method has been a subject of study in [9].

Theorem 4.1.5 ([9]). *Let $A \in \mathbb{R}^{n,n}$ be an H^+-matrix with $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$ and $B = D - A$. Further, let $(D - L_p, U_p, E_p)$ $(p = 1,2,\ldots,\ell)$ be triangular multisplittings of matrix A. Assume that $A = D - L_p - U_p$ satisfies $\langle A \rangle = D - |L_p| - |U_p|$, for $p = 1,2,\ldots,\ell$, $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. Then, for any initial vector $x^{(0)} \in \mathbb{R}^n$, the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}^n_+$, generated by Method 3, converges to the unique solution $z^\star$ of $LCP(q,A)$, provided that*

$$0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}. \tag{4.11}$$

In 2014, this convergence area has been extended by Cvetković and Kostić, [24]. The corresponding theorem will be part of the next chapter, so we will ommit it here.

4.1.4 Two-stage multisplitting idea

The numerical benefits and computational power of the multisplitting idea has been confirmed in the work of Bai and Zhang. However, the idea of multisplitting - that is - the distribution of computational tasks among available processor units, can be advanced further, with the idea that at every iteration step k and on every processor p - for which we already have an initial splitting $A = M_p - N_p$, matrix M_p can splitted as well, thus assuming $M_p = F_p - G_p$. This naturally induced the modulus-based synchronous two-stage multisplitting (MSTM) iterative method.

MSTM iterative method

The modulus-based synchronous two-stage multisplitting (MSTM) iterative method has been developed by Bai and Zhang in 2013, [10]. The method they formulated is given as follows.

Method 4. THE MODULUS-BASED SYNCHRONOUS TWO-STAGE MULTISPLITTING

Let $(M_p : F_p, G_p; N_p; E_p)(p = 1, 2, \ldots, \ell)$ *be a two-stage multisplitting of* $A \in \mathbb{R}^{n,n}$ *and* ν_p *for* $p = 1, 2, \ldots, \ell$, *positive integers. Given an arbitrary starting vector* $z^{(0)} \in \mathbb{R}^n_+$, *for iterations* $k \geq 0$, *until the sequence*

$\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}_+^n$ *is convergent, calculate* $z^{(k+1)} \in \mathbb{R}_+^n$ *as*

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}),$$

and $x^{(k+1)} \in \mathbb{R}^n$ *as*

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p x^{(k,p,\nu_p)},$$

where vectors $x^{(k,p,\nu_p)}$ $p = 1, 2, \ldots, \ell$ *are the solutions of systems of linear equations*

$$\begin{cases} (\Omega + F_p)x^{(k,p,j+1)} = G_p x^{(k,p,j)} + b^{(k,p)} \\ p = 1, 2, \ldots, \ell \\ j = 0, 1, \ldots, \nu_p - 1 \end{cases} \tag{4.12}$$

with $b^{(k,p)} = N_p x^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q$, $x^{(k,p,0)} = x^{(k)}$ *and* $x^{(0)} = \frac{1}{2}\gamma\Omega^{-1}((\Omega - A)z^{(0)} - q)$.

The idea of the MSTM method is that initially, we assign the corresponding two-stage multisplitting of A, that is $A = F_p - G_p - N_p$ and the weight E_p to each of the ℓ available processors , as well as an arbitrary positive integer, ν_p. This integer defines the finite number of inner iterations of (4.12) which are necessary to find the local update $x^{(k,p,\nu_p)}$ after which the next iterate $x^{(k+1)}$ is processed and consequently we obtain $z^{(k+1)}$. Sufficient conditions for the convergence of MSTM method are given in the following theorem.

Theorem 4.1.6 ([10]). *Let* $A \in \mathbb{R}^{n,n}$ *be an* H^+*-matrix, where* $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$, $B = D - A$ *and suppose that* $(M_p : F_p, G_p; N_p; E_p)(p = 1, 2, \ldots, \ell)$

is a two-stage multisplitting of A. Finally, assume that $\gamma > 0$ *and that* Ω *is a positive diagonal matrix with the property* $\Omega \geq D$. *Then, if* $A = M_p - N_p$ *and* $M_p = F_p - G_p$, *for* $p = 1, 2, \ldots, \ell$, *are H-compatible splittings, the iteration sequence* $\{z^{(k)}\}_{k=0}^{\infty}$, *generated by Method 4, converges to the unique solution* $z^\star$ *of* $LCP(q,A)$ *for every initial vector* $z^{(0)} \in \mathbb{R}_+^n$ *and arbitrary positive integers* ν_p, $p = 1, 2, \ldots, \ell$.

4.1.5 Two-stage triangular multisplitting idea

It is obvious that the load on the processing units in MSTM method is higher than the one in MSM method. Depending on the number of processor units and the choice of numbers ν_p at start, the number of systems of the form (4.12), which need to be solved, can become high and this can slow down the calculations. However, if these systems have a special structure, for example, if they are lower triangular, then signifficant improvement in speed can be achieved, [10]. Therefore, the idea is to use triangular splittings on the second level. Again, we will present it in the form of the "AOR-approach".

MSTMAOR iterative method

The modulus-based synchronous two-stage multisplitting accelerated overrelaxation (MSTMAOR) iterative method was introduced by Bai and Zhang in 2013, [10]. This additional level of triangular splitting has been chosen in order to compensate for the uneven distribution of tasks among the processors, which may reduce the computational advantage that the multiprocessor system has to offer.

Method 5. THE MODULUS-BASED SYNCHRONOUS TWO-STAGE MULTISPLITTING AOR

Let $(M_p : D_p - L_p, U_p; N_p; E_p)$ $(p = 1, 2, \ldots, \ell)$ *be a two-stage triangular multisplitting of the system matrix* $A \in \mathbb{R}^{n,n}$ *and* ν_p, *for* $p = 1, 2, \ldots, \ell$, *be prescribed positive integers. Given an initial vector* $x^{(0)} \in \mathbb{R}^n$ *for* $k = 0, 1, 2, \ldots$, *until the iteration sequence* $\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}_+^n$ *is convergent, compute* $z^{(k+1)} \in \mathbb{R}_+^n$ *by*

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)})$$

and $x^{(k+1)} \in \mathbb{R}^n$ *according to*

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p x^{(k,p,\nu_p)},$$

where $x^{(k,p,\nu_p)}$, $p = 1, 2, \ldots, \ell$, *are obtained by*

$$(\alpha\Omega + D_p - \beta L_p)x^{(k,p,j+1)}$$

$$= ((1-\alpha)D_p + (\alpha - \beta)L_p + \alpha U_p)x^{(k,p,j)} + \alpha b^{(k,p)}$$

$$p = 1, 2, \ldots, \ell, \quad j = 0, 1, \ldots, \nu_p - 1,$$

respectively, with

$$b^{(k,p)} = N_p x^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q$$

and $x^{(k,p,0)} = x^{(k)}$.

The convergence of this method was studied in [10].

Theorem 4.1.7 ([10]). *Let $A \in \mathbb{R}^{n,n}$ be an H^+-matrix with $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$ and $B = D - A$ and let $(M_p : D_p - L_p, U_p; N_p; E_p)(p = 1, 2, \ldots, \ell)$ be a two-stage triangular multisplitting of the matrix A. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If, for $p = 1, 2, \ldots, \ell$, $A = M_p - N_p$ are H-compatible splittings and $M_p = D - L_p^{(M)} - U_p^{(M)}$ satisfy $\langle M_p \rangle = D - |L_p| - |U_p|$ with $diag(M_p) = D$, then the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty}$, generated by Method 5, converges to the unique solution $z^\star$ of $LCP(q, A)$ for any initial vector $x^{(0)} \in \mathbb{R}^n$ and any positive integers ν_p, $p = 1, 2, \ldots, \ell$, provided that the relaxation parameters α and β satisfy the relation*

$$0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}. \tag{4.13}$$

The condition on the relaxation parameters in the previous theorem is very specific, requiring that one relaxation parameter is greater than the other. This assumption is restrictive in nature, and can be weakened. This was done by Cvetković, Kostić and Šanca in 2016, [25].

According to these authors, due to the form of the splittings, for every $p = 1, 2, \ldots, \ell$, we can write $L_p = \Xi_p \circ (-M_p)$, where $\circ$ denotes Hadamard product and $\Xi_p = [\xi_{ij}^p]$ are such that

$$0 \leq \xi_{ij}^p \leq 1, \quad \text{for} \quad 1 \leq j < i \leq n, \quad \text{and} \quad \xi_{ij}^p = 0, \quad \text{otherwise.}$$

Also, for every $p = 1, 2, \ldots, \ell$, we can write $M_p = \Theta_p \circ A$, where $\Theta_p = [\theta_{ij}^p]$ are chosen to be such that

$$0 \leq \theta_{ij}^p \leq 1 \quad \text{for} \quad 1 \leq i,\ j \leq n.$$

Further, denote the values

$$\xi = \max_{1\le p\le \ell}\max_{1\le j<i\le n}\xi_{ij}^{p}, \quad \theta = \max_{1\le p\le \ell}\max_{1\le j<i\le n}\theta_{ij}^{p}. \tag{4.14}$$

Then, the following extension of the convergence area for the choice of relaxation parameters is possible.

Theorem 4.1.8. *[25] Theorem 4.1.7 is still valid if we replace condition (4.13) with the following one*

$$\beta \ge 0 \ \textit{and} \ (\theta max\{\alpha, \xi\beta\} + (1-\theta)\alpha)\rho(D^{-1}|B|) < min\{1,\alpha\} \tag{4.15}$$

which is equivalent to

$$0 < \alpha < \frac{1}{\rho(D^{-1}|B|)}, \ 0 \le \beta < \frac{min\{1,\alpha\} - (1-\theta)\alpha\rho(D^{-1}|B|)}{\xi\theta\rho(D^{-1}|B|)}. \tag{4.16}$$

TMSM iterative method

The two-step modulus-based synchronous multisplitting (TMSM) iteration method was interduced by Li Li Zhang in 2015, [69]. The goal was to reduce the interaction among processors and improve the computing time.

The idea is to use two multisplittings, one for each phase of the process.

Method 6. THE TWO-STEP MODULUS-BASED SYNCHRONOUS MULTISPLITTING

Let $(M_p^{'}, N_p^{'}, E_p)$ and $(M_p^{''}, N_p^{''}, E_p)$ $(p = 1, 2, \ldots, \ell)$ be two multisplittings of the matrix A. Given an initial vector $x^{(0)} \in \mathbb{R}^n$ and provided that

$x^{(k)}$ has been obtained, compute $x^{(k+1,p)}$ by solving the linear subsystems

$$\begin{cases} (\Omega + M_p^{'})x^{(k+\frac{1}{2},p)} = N_p^{'}x^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q, \\ (\Omega + M_p^{''})x^{(k+1,p)} = N_p^{''}x^{(k+\frac{1}{2},p)} + (\Omega - A)|x^{(k+\frac{1}{2},p)}| - \gamma q, \end{cases} \tag{4.17}$$

on the p^{th} processor, then set

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p x^{(k+1,p)}$$

and

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}),$$

for $k = 0, 1, 2, \ldots$, until the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ is convergent.

Theorem 4.1.9. *[69] Let $A \in \mathbb{R}^{n,n}$ be an H^+-matrix with $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$, $(M_p^{'}, N_p^{'}, E_p)$ and $(M_p^{''}, N_p^{''}, E_p)$ $(p = 1, 2, \ldots, \ell)$ be two multisplittings of the matrix A. Assume that $A = M_p^{'} - N_p^{'}$ and $A = M_p^{''} - N_p^{''}$ are H-compatible splittings, for $(p = 1, 2, \ldots, \ell)$, $\gamma > 0$ and Ω a positive diagonal matrix satisfying $\Omega \geq D$. Then, for any initial vector $x^{(0)} \in \mathbb{R}^n$, the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty} \subset \mathbb{R}_+^n$, generated by Method 6, converges to the unique solution $z^\star$ of $LCP(q, A)$.*

5. Convergence Analysis

In this chapter we will present two different approaches in the convergence analysis of modulus-based splitting and multisplitting methods, which both lead to the same conclusion, but one of them is more suitable for error analysis, which is discussed in detail in the next chapter. The same techniques work for all mentioned methods, so we restrict ourselves to one of them – MSMAOR. The main reason is that our original results are related to this method, [26].

The convergence of the modulus-based synchronous multisplitting accelerated overrelaxation (MSMAOR) iterative method (Method 3) was extensively studied in [9]. The sufficient condition for the convergence of MSMAOR in Theorem 4.1.5 includes a very particular choice of relaxation parameters, (4.11). The authors in [24] have shown that this assumption is restrictive and can be avoided, which enables the extension of the area for the choice of relaxation parameters. This extension is meaningful, since it contains the choice of parameters for which the rate of convergence can be accelerated.

If we assume that A is an H^+-matrix, then $|D| = D$, for $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$. Moreover, if $B := D - A$, then $\rho(D^{-1}|B|) < 1$. If the trian-

gular multisplittings of A additionally satisfy $\langle A\rangle = D - |L_p| - |U_p|$, for $p = 1, 2, \ldots, \ell$, the authors in [24] have used the fact that $L_p = \Xi_p \circ (-A)$, where $\circ$ denotes the Hadamard product of matrices and matrix $\Xi_p = [\xi_{ij}^p]$ is defined entrywise as

$$0 \leq \xi_{ij}^p \leq 1 \quad \text{for} \quad 1 \leq j < i \leq n \quad \text{and} \quad \xi_{ij}^p = 0 \quad \text{otherwise.}$$

Besides, let $A = D - L^\star - U^\star$ be the standard splitting of A into the diagonal D, strictly lower triangular $L^\star$ and strictly upper triangular $U^\star$ parts. Then, for all $p = 1, 2, \ldots, \ell$, L_p matrices have the form $L_p = \Xi_p \circ L^\star$ and the following inequality holds

$$|L_p| \leq \xi |L^\star|,$$

where

$$\xi = \max\{\xi_{ij}^p : \ p = 1, 2, \ldots, \ell, \quad i, j \in N_n\}.$$

We now state the extension of the area of relaxation parameters, for which the convergence is guaranteed, as presented in [24].

Theorem 5.0.1 ([24]). *Let $A \in \mathbb{R}^{n,n}$ be an H^+-matrix with $D = diag(A)$, $B = D - A$ and let $(D - L_p, U_p, E_p)$, $p = 1, 2, \ldots, \ell$, be a triangular multisplitting of matrix A. Assume that $\gamma > 0$ and Ω is a positive diagonal matrix such that $\Omega \geq D$. If $A = D - L_p - U_p$ satisfies $\langle A\rangle = D - |L_p| - |U_p|$, for $p = 1, 2, \ldots, \ell$, then the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty}$, generated by Method 3, converges to the unique solution $z^\star$ of the LCP(q, A) for any initial vector $x^{(0)} \in \mathbb{R}^n$, provided that the relaxation parameters α and β satisfy*

$$\max\{\alpha, \xi\beta\}\rho(D^{-1}|B|) < \min\{1, \alpha\}. \tag{5.1}$$

Old proof

Let $(D-L_p, U_p, E_p)(p=1,2,\ldots,\ell)$ be a triangular multisplitting of A. Since A is an H^+-matrix, the $LCP(q,A)$ has a unique solution, $z^\star$. Then, according to Theorem 4.1.1, vector $x^\star = \frac{1}{2}\gamma(z^\star - \Omega^{-1}r^\star)$ satisfies each of the ℓ implicit fixed point equations (4.8) for the choice

$$M_p = \frac{1}{\alpha}(D-\beta L_p) \quad \text{and} \quad N_p = \frac{1}{\alpha}\big((1-\alpha)D + (\alpha-\beta)L_p + \alpha U_p\big).$$

On the other hand, based on research in [9], it is known that if the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by Method 3 is convergent, then its limit is exactly x^*. Since

$$\begin{aligned} x^{(k,p)} - x^\star = (\alpha\Omega + D - \beta L_p)^{-1}[&((1-\alpha)D + (\alpha-\beta)L_p + \alpha U_p)(x^{(k)} - x^\star) \\ &+ \alpha(\Omega - A)(|x^{(k)}| - |x^\star|)], \quad p = 1,2,\ldots,\ell, \end{aligned} \tag{5.2}$$

and from Method 3 we know that

$$x^{(k+1)} = \sum_{p=1}^{\ell} E_p x^{(k,p)},$$

the approximation error is

$$\begin{aligned} x^{(k+1)} - x^\star = &\sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta L_p)^{-1}\big[(1-\alpha)D + (\alpha-\beta)L_p + \alpha U_p\big]\cdot \\ &\cdot (x^{(k)} - x^\star) + \alpha\sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta L_p)^{-1}(\Omega - A)(|x^{(k)}| - |x^\star|). \end{aligned} \tag{5.3}$$

Making use of the estimate $||x^{(k)}| - |x^\star|| \leq |x^{(k)} - x^\star|$, and arranging similar terms together, we obtain

$$|x^{(k+1)} - x^\star| \leq \mathscr{L}_{MSMAOR}(\alpha, \beta)|x^{(k)} - x^\star|,$$

where

$$\mathscr{L}_{MSMAOR}(\alpha, \beta) := \sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}\Big[|1-\alpha|D + (\alpha-\beta)|L_p| + \alpha|U_p| + \alpha|\Omega - A|\Big].$$

Also note that

$$\begin{aligned} |(1-\alpha)D + (\alpha-\beta)L_p + \alpha U_p| &= |(1-\alpha)D| + |(\alpha-\beta)L_p + \alpha U_p| \\ &= |1-\alpha|D + |\alpha(L_p + U_p) - \beta L_p| \\ &= |1-\alpha|D + |\alpha B - \beta L_p| \end{aligned}$$

and

$$\alpha|\Omega - A| = \alpha|\Omega - D + B| = \alpha(\Omega - D) + \alpha|B| = \alpha\Omega - \alpha D + \alpha|B|.$$

Therefore, the estimate of the iterative matrix becomes

$$\begin{aligned} \mathscr{L}_{MSMAOR}(\alpha, \beta) &= \sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}\Big[|1-\alpha|D + |\alpha B - \beta L_p| + \alpha\Omega \\ &= E - \sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}\Big[\big(1 + \alpha - |1-\alpha|\big)D \\ &\qquad - |\alpha B - \beta L_p| - \alpha|B| - \beta|L_p|\Big]. \end{aligned}$$

Observe that

$$1+\alpha-|1-\alpha| = 1+\alpha-\begin{cases} 1-\alpha & \text{for} \quad \alpha \le 1 \\ \alpha-1 & \text{for} \quad \alpha > 1 \end{cases}$$
$$= \begin{cases} 2\alpha & \text{for} \quad \alpha \le 1 \\ 2 & \text{for} \quad \alpha > 1 \end{cases}$$
$$= 2\min\{1,\alpha\},$$

and

$$\left(|\alpha B - \beta L_p| + \alpha|B| + \beta|L_p|\right)_{ij} = \begin{cases} 0 & \text{for} \quad i=j \\ |\alpha(-a_{ij}) - \beta\xi_{ij}^p(-a_{ij})| & \text{for} \quad i>j \\ \quad +\alpha|-a_{ij}| + \beta\xi_{ij}^p|-(a_{ij})| & \\ 2\alpha|a_{ij}| & \text{for} \quad i<j \end{cases},$$

where

$$|\alpha(-a_{ij}) - \beta\xi_{ij}^p(-a_{ij})| + \alpha|-a_{ij}| + \beta\xi_{ij}^p|-(a_{ij})| =$$
$$= \begin{cases} (\alpha - \beta\xi_{ij}^p + \alpha + \beta\xi_{ij}^p)|a_{ij}| & \text{for} \quad \alpha \ge \beta\xi_{ij}^p \\ (-\alpha + \beta\xi_{ij}^p + \alpha + \beta\xi_{ij}^p)|a_{ij}| & \text{for} \quad \alpha < \beta\xi_{ij}^p \end{cases}$$
$$= \begin{cases} 2\alpha|a_{ij}| & \text{for} \quad \alpha \ge \beta\xi_{ij}^p \\ 2\beta\xi_{ij}^p|a_{ij}| & \text{for} \quad \alpha < \beta\xi_{ij}^p \end{cases}$$
$$= 2\max\{\alpha, \beta\xi_{ij}^p\}|a_{ij}|$$
$$\le 2\max\{\alpha,\beta\xi\}|a_{ij}|, \quad \forall i,j \in N.$$

From all of the above, it is true that $\left(|\alpha B - \beta L_p| + \alpha|B| + \beta|L_p|\right)_{ij} \le 2\max\{\alpha,\beta\xi\}|a_{ij}|, \ \forall i,j \in N.$

Hence, we can conclude that

$$|x^{(k+1)} - x_\star| \leq \widetilde{\mathscr{L}}_{MSMAOR}(\alpha,\beta)|x^{(k)} - x_\star|,$$

where $\widetilde{\mathscr{L}}_{MSMAOR}(\alpha,\beta)$ is defined as

$$\begin{aligned}
&E - 2\sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}\big(\min\{1,\alpha\}D - \max\{\alpha,\beta\xi\}|B|\big) \\
&= E - 2\sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}D\big(\min\{1,\alpha\}E - \max\{\alpha,\beta\xi\}D^{-1}|B|\big).
\end{aligned}$$

In order to complete the proof, it suffices to show that

$$\rho(\widetilde{\mathscr{L}}_{MSMAOR}(\alpha,\beta)) < 1.$$

Due to the fact that $\rho(D^{-1}|B|) < 1$, we can choose $\varepsilon > 0$ sufficiently small so that $\rho_\varepsilon := \rho(J_\varepsilon) < 1$, where $J_\varepsilon := D^{-1}|B| + \varepsilon ee^T$. Then, according to Perron-Frobenius theorem for positive matrices, there exists a strictly positive vector, $v_\varepsilon > 0$, such that $J_\varepsilon v_\varepsilon = \rho_\varepsilon v_\varepsilon$. Thus,

$$\begin{aligned}
&\widetilde{\mathscr{L}}_{MSMAOR}(\alpha,\beta)v_\varepsilon < \\
&\qquad < v_\varepsilon - 2\sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}D\big(\min\{1,\alpha\}E - \max\{\alpha,\beta\xi\}J_\varepsilon\big)v_\varepsilon \\
&\qquad = v_\varepsilon - 2\sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}D\big(\min\{1,\alpha\} - \max\{\alpha,\beta\xi\}\rho_\varepsilon\big)v_\varepsilon \\
&\qquad = v_\varepsilon - 2\theta_\varepsilon\sum_{p=1}^{\ell} E_p(\alpha\Omega + D - \beta|L_p|)^{-1}Dv_\varepsilon,
\end{aligned}$$

where term θ_ε is defined in the following way

$$\theta_\varepsilon := \min\{1, \alpha\} - \max\{\alpha, \beta\xi\}\rho_\varepsilon.$$

Due to the following relationship of M-matrices (see Lemma 1.5.2),

$$\alpha\Omega + D - \beta|L_p| \leq \alpha\Omega + D,$$

Lemma 1.5.3 implies that

$$(\alpha\Omega + D - \beta|L_p|)^{-1} \geq (\alpha\Omega + D)^{-1} \geq 0,$$

and, hence,

$$\begin{aligned}\widetilde{\mathscr{L}}_{MSMAOR}(\alpha,\beta)v_\varepsilon &< v_\varepsilon - 2\theta_\varepsilon \sum_{p=1}^{\ell} E_p(\alpha\Omega + D)^{-1}Dv_\varepsilon \\ &= v_\varepsilon - 2\theta_\varepsilon(\alpha\Omega + D)^{-1}Dv_\varepsilon \\ &= (\alpha\Omega + D)^{-1}\big(\alpha\Omega + (1 - 2\theta_\varepsilon)D\big)v_\varepsilon.\end{aligned}$$

Finally, using the fact that, for ε sufficiently small, ρ_ε approaches ρ, condition (5.1) guarantees that $\theta_\varepsilon > 0$. This accumulates to

$$\widetilde{\mathscr{L}}_{MSMAOR}(\alpha,\beta)v_\varepsilon < v_\varepsilon,$$

implying $\rho(\widetilde{\mathscr{L}}_{MSMAOR}(\alpha,\beta)) < 1$. □

New proof

In contrast to the old proof, using a different approach, as it has been done in [70], based on the theory of M-matrices, we were able to offer a more elegant proof of the same theorem, [26]. But this new technique did not only have an influence to perform a simpler proof. It has also offered an adequate tool for further analysis of the convergence, in terms of the error analysis and its control. This is the topic which is covered in the next chapter of this thesis.

We have seen that the error formula for Method 3 is

$$|x^{(k+1)} - x^\star| \leq \mathscr{L}_{MSMAOR}|x^{(k)} - x^\star|,$$

where

$$\mathscr{L}_{\text{MSMAOR}} := \sum_{p=1}^{\ell} E_p \mathscr{L}(L_p, U_p, \Omega)$$

and

$$\mathscr{L}(L_p, U_p, \Omega) = (\alpha\Omega + D - \beta|L_p|)^{-1}\Big(|(1-\alpha)D + (\alpha-\beta)L_p + \alpha U_p| + \alpha|\Omega - A|\Big).$$

Using the ideas of [24], which are presented in this dissertation in the previous subsection *Old proof*, one can show that the norm of the matrix $\mathscr{L}_{\text{MSMAOR}}$ can be bounded above by

$$\widetilde{\mathscr{L}}_{\text{MSMAOR}} := E - 2(\alpha\Omega + D - \xi\beta|L^\star|)^{-1}DC,$$

where

$$C := \min\{1, \alpha\}E - \max\{\alpha, \xi\beta\}D^{-1}|B|.$$

More precisely, since $\alpha\Omega+D-\beta|L_p|$ and $\alpha\Omega+D-\xi\beta|L^\star|$, $p=1,2,\dots,\ell$ are triangular M-matrices with a positive diagonal (see Lemma 1.5.2) satisfying the inequality

$$\alpha\Omega+D-\beta|L_p|\geq\alpha\Omega+D-\xi\beta|L^\star|,$$

it implies that

$$O\leq(\alpha\Omega+D-\beta|L_p|)^{-1}\leq(\alpha\Omega+D-\xi\beta|L^\star|)^{-1}.$$

On the other hand,

$$|(1-\alpha)D+(\alpha-\beta)L_p+\alpha U_p|+\alpha|\Omega-A|=|(1-\alpha)D|+|\alpha B-\beta L_p|+\alpha(\Omega-D)+\alpha|B|,$$

hence

$$\mathscr{L}(L_p,U_p,\Omega)\leq(\alpha\Omega+D-\xi\beta|L^\star|)^{-1}\big(|(1-\alpha)D|+|\alpha B-\beta L_p|+\alpha(\Omega-D)+\alpha|B|\big),$$

and

$$\begin{aligned}&\mathscr{L}(L_p,U_p,\Omega)\leq\\&\leq E-\big(\alpha\Omega+D-\xi\beta|L^\star|\big)^{-1}\big((1+\alpha-|1-\alpha|)D-|\alpha B-\beta L_p|-\alpha|B|-\xi\beta|L^\star|\big).\end{aligned}$$

Since

$$1+\alpha-|1-\alpha|=2\min\{1,\alpha\}$$

and

$$(|\alpha B-\beta L_p|+\alpha|B|+\xi\beta|L^\star|)_{i,j}\leq 2\max\{\alpha,\xi\beta\}|a_{ij}|,$$

we obtain

$$\mathscr{L}(L_p,U_p,\Omega)\leq E-2(\alpha\Omega+D-\xi\beta|L^\star|)^{-1}DC.$$

Finally, using the equality $\sum_{p=1}^{\ell} E_p = 1$ we infer that

$$O \leq \mathscr{L}_{\text{MSMAOR}} \leq \widetilde{\mathscr{L}}_{\text{MSMAOR}}.$$

In order to prove the convergence, we need to show that $\rho(\widetilde{\mathscr{L}}_{\text{MSMAOR}}) < 1$. Let us now consider matrix C. It has the form $C = \mu I - F$ with $\mu = \min\{1, \alpha\}$ and $F = \max\{\alpha, \xi\beta\} D^{-1}|B| \geq 0$, so it is a Z-matrix. Moreover, the condition (5.1) is equivalent to the inequality $\rho(F) < \mu$ which, according to Lemma 1.4.1, implies that C is an M-matrix. But then, Theorem 1.5.1 guarantees that there exists a positive vector u such that $Cu > 0$, and since $(\alpha\Omega + D - \xi\beta|L^\star|)^{-1} DCu > 0$, we obtain

$$\begin{aligned} \widetilde{\mathscr{L}}_{\text{MSMAOR}} u &= \big(I - 2(\alpha\Omega + D - \xi\beta|L^\star|)^{-1} DC\big)u \\ &= u - 2(\alpha\Omega + D - \xi\beta|L^\star|)^{-1} DCu < u. \end{aligned}$$

Recalling Lemma 1.4.2, we conclude that $\rho(\widetilde{\mathscr{L}}_{\text{MSMAOR}}) < 1$, which completes the proof. □

6. Error control

The previous two chapters provided a series of sufficient conditions upon which the convergence of iterative methods is guaranteed, with a special attention to MSMAOR. But, once the iteration sequence is convergent, some other interesting questions usually appear. For instance, is it possible to say something in the sense of the error control? More precisely, given a norm $\|\cdot\|$, denoting the error at iteration k by $\delta_k := \|x^* - x^{(k)}\|$, one may be interested in a stopping criteria which means that the error is controlled by a given tolerance tol, i.e., $\delta_k < tol$.

Typically two kinds of error bounds are used to that end: *a priori* ($\mathscr{A}_k$) and *a posteriori* ($\mathscr{P}_k$) error estimates, which in our case provide

$$\delta_k = \|x^\star - x^{(k)}\|_\infty \leq \frac{\|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_\infty^k}{1 - \|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_\infty} \|x^{(1)} - x^{(0)}\|_\infty =: \mathscr{A}_k$$

and

$$\delta_k = \|x^\star - x^{(k)}\|_\infty \leq \frac{\|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_\infty}{1 - \|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_\infty} \|x^{(k)} - x^{(k-1)}\|_\infty =: \mathscr{P}_k.$$

Therefore, one can not neglect the fact that it is very useful to have a fairly good and computationally inexpensive norm estimation of the

iteration matrix. Understanding the need of that kind, here we present how this can be done in the case of H^+-matrices, starting from its significant subcase of SDD matrices.

6.1 SDD-matrix case

We will work with the errors estimated in the infinity norm, which is well suited to SDD matrices. To that end, let

$$\ell_i := r_i(L^\star), \quad u_i := r_i(U^\star), \quad i \in N_n.$$

Theorem 6.1.1. *Let $\Omega \geq D$ and A be a strictly diagonal dominant (SDD) matrix with positive diagonal entries and with the triangular multisplitting as in Theorem 4.1.5. If the relaxation parameters α and β satisfy the inequality*

$$\max\{\alpha, \xi\beta\}\||D^{-1}|B|\|_\infty < \min\{1, \alpha\}, \tag{6.1}$$

then

$$\|\mathscr{L}_{MSMAOR}\|_\infty \leq \|\widetilde{\mathscr{L}}_{MSMAOR}\|_\infty \leq \mathscr{E}_1(A) < 1, \tag{6.2}$$

where

$$\mathscr{E}_1(A) := \max_i \frac{(\alpha + 1 - 2\min\{1, \alpha\})\, a_{ii} + \alpha\theta_i + 2\max\{\alpha, \xi\beta\} r_i(B) - \xi\beta l_i}{(\alpha+1)a_{ii} + \alpha\theta_i - \xi\beta l_i}.$$

Proof. Since $\Omega \geq D$, we can write $\Omega = D + \Theta$, where $\Theta = diag(\theta_1, \theta_2, \ldots, \theta_n)$ is a diagonal matrix with non-negative entries θ_i, $i \in N_n$. Since the infinity norm is monotone for non-negative matrices, $O \leq \mathscr{L}_{\text{MSMAOR}} \leq$

$\widetilde{\mathscr{L}}_{\mathrm{MSMAOR}}$ implies $\|\mathscr{L}_{\mathrm{MSMAOR}}\|_\infty \le \|\widetilde{\mathscr{L}}_{\mathrm{MSMAOR}}\|_\infty$. Recalling that

$$\widetilde{\mathscr{L}}_{\mathrm{MSMAOR}} = M^{-1}N,$$

with matrices

$$\begin{aligned} M &= (\alpha+1)D + \alpha\Theta - \xi\beta|L^\star|, \\ N &= (\alpha+1-2\min\{1,\alpha\})D + \alpha\Theta - \xi\beta|L^\star| + 2\max\{\alpha,\xi\beta\}|B|, \end{aligned}$$

and the condition (6.1), we get

$$\xi\beta\frac{\ell_i}{a_{ii}} \le \xi\beta\frac{\ell_i+u_i}{a_{ii}} \le \max\{\alpha,\xi\beta\}\||D^{-1}|B|\|_\infty < \min\{1,\alpha\} \le \alpha+1.$$

Thus $(\alpha+1)D-\xi\beta|L^\star|$ is an SDD matrix and so is the matrix M. Therefore, Lemma 1.5.4 yields the estimate (6.2).

It only remains to show that $\mathscr{E}_1(A) < 1$. To do this, we observe that

$$\begin{aligned} &\max\{\alpha,\xi\beta\}\||D^{-1}|B|\|_\infty < \min\{1,\alpha\}, \\ &\max\{\alpha,\xi\beta\}r_i(B) < \min\{1,\alpha\}a_{ii}, \quad i\in N_n, \\ &-\min\{1,\alpha\}a_{ii} + \max\{\alpha,\xi\beta\}r_i(B) < 0, \quad i\in N_n, \\ &(\alpha+1)a_{ii} + \alpha\theta_i - 2\min\{1,\alpha\}a_{ii} + 2\max\{\alpha,\xi\beta\}r_i(B) - \xi\beta\ell_i \\ &\quad < (\alpha+1)a_{ii} + \alpha\theta_i - \xi\beta\ell_i, \quad i\in N_n, \end{aligned} \tag{6.3}$$

and note that M being an SDD matrix implies that $(\alpha+1)a_{ii} + \alpha\theta_i - \xi\beta\ell_i > 0$ and the inequality (6.3) is equivalent to $\mathscr{E}_1(A) < 1$. □

The condition (6.1) is sufficient for M to be an SDD matrix and is also equivalent to the inequality $\mathscr{E}_1(A) < 1$.

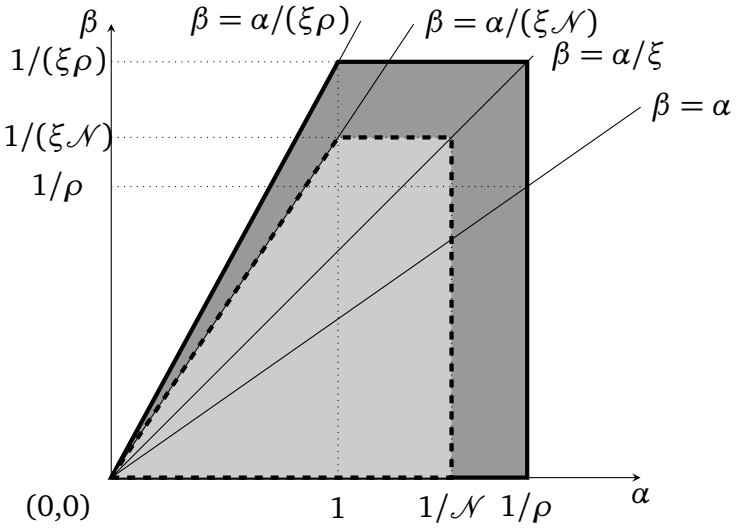


Figure 6.1: Parameter choice. Bold: the area with guarantied convergence. Dashed: Convergence area suggested by Theorem 6.1.1, $\rho(D^{-1}|B|) := \rho$, $\||D^{-1}|B|\|_\infty := \mathscr{N}$.

If A is not an SDD matrix, then there are no (α, β) which satisfy the condition (6.1), therefore the convergence area is empty and the norm $\|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_\infty$ cannot be estimated from above, using (6.2).

Provided that A is an SDD matrix with positive diagonal entries and condition (6.1) is satisfied, then the parameters (α, β) lie within the area of convergence of the MSMAOR method from [24]. This is illustrated in Fig. 6.1.

Finally, Theorem 6.1.1 provides the following a priori and a posteri-

ori stopping criteria:

$$\widehat{\mathscr{A}_k} := \frac{\mathscr{E}_1(A)^k}{1-\mathscr{E}_1(A)}\|x^{(1)}-x^{(0)}\|_\infty < tol,$$
$$\widehat{\mathscr{P}_k} := \frac{\mathscr{E}_1(A)}{1-\mathscr{E}_1(A)}\|x^{(k)}-x^{(k-1)}\|_\infty < tol.$$

6.2 H-matrix case

Switching to the weighted infinity norm,

$$\|x\|_{(w)} := \|W^{-1}x\|_\infty = \max_i \frac{|x_i|}{w_i},$$

where $W = diag(w_1,\dots,w_n)$ and $w = [w_1 \ \dots \ w_n]^T > 0$ is a vector of weights, as the following proposition shows, Theorem 6.1.1 can be generalized from SDD to the H-matrix case.

Theorem 6.2.1. *Let $\Omega \geq D$, A be an H^+-matrix with triangular multi-splitting as in Theorem 4.1.5, and $w > 0$ be a weight vector such that $\langle A\rangle w > 0$. If the relaxation parameters α and β satisfy the condition*

$$\max\{\alpha, \xi\beta\}\||D^{-1}|B|\|_{(w)} < \min\{1,\alpha\}, \tag{6.4}$$

then

$$\|\mathscr{L}_{MSMAOR}\|_{(w)} \leq \|\widetilde{\mathscr{L}}_{MSMAOR}\|_{(w)} \leq \mathscr{E}_2(A) < 1, \tag{6.5}$$

where

$$\mathscr{E}_2(A) := \max_i \frac{\Big((\alpha+1-2\min\{1,\alpha\})\,a_{ii}+\alpha\theta_i\Big)w_i + 2\max\{\alpha,\xi\beta\}\widetilde{r}_i(B) - \xi\beta\widetilde{\ell}_i}{\Big((\alpha+1)a_{ii}+\alpha\theta_i\Big)w_i - \xi\beta\widetilde{\ell}_i}$$

and

$$\widetilde{\ell}_i := \sum_{j<i} w_j|a_{ij}|, \qquad \widetilde{r}_i(B) := \sum_{j\neq i} w_j|a_{ij}|, \quad \forall i \in \mathbf{N}_n.$$

Proof: Obviously, for $W = diag(w_1, \ldots, w_n)$, the condition $\langle A\rangle w > 0$ means that AW is an SDD matrix with positive diagonal entries. Now, the proof is a direct consequence of Theorem 6.1.1 coupled with the following facts:

- $\||D^{-1}|B|\|_{(w)} = \|W^{-1}D^{-1}|B|W\|_\infty = \|(DW)^{-1}|BW|\|_\infty$,
- $\|\mathscr{L}_{\text{MSMAOR}}\|_{(w)} = \|W^{-1}\mathscr{L}_{\text{MSMAOR}}W\|_\infty = \|\sum_{p=1}^{\ell} E_p\mathscr{L}(L_pW, U_pW, \Omega W)\|_\infty$
- $\|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_{(w)} = \|W^{-1}\widetilde{\mathscr{L}}_{\text{MSMAOR}}W\|_\infty = \|(MW)^{-1}(NW)\|_\infty$, and
- $\mathscr{E}_2(A) = \mathscr{E}_1(AW)$,

which can be readily checked. □

The proof of this theorem is a natural generalization of Theorem 6.1.1, applied to matrix AW. It is obvious that this theorem offers a more general view, since it reduces to Theorem 6.1.1 for $W = E$, implying that A is an SDD matrix with a positive diagonal. The apparent benefit of this theorem is the fact that the scaling matrix can be cheaply acquired for a fair amount of well-known subclasses of H-matrices.

Finally, knowing the appropriate vector of weights $w > 0$, as before, we obtain a priori

$$\widehat{\mathscr{A}_k^w} := \frac{\mathscr{E}_2(A)^k}{1-\mathscr{E}_2(A)}\|x^{(1)} - x^{(0)}\|_{(w)} < tol$$

and a posteriori stoping criteria

$$\widehat{\mathscr{P}}_k^w := \frac{\mathscr{E}_2(A)}{1-\mathscr{E}_2(A)}\|x^{(k)}-x^{(k-1)}\|_{(w)} < tol,$$

that control the error in such weighted infinity norm $\delta_k^w = \|x^*-x^{(k)}\|_{(w)}$.

7. Numerical examples

In the final chapter of this thesis we turn to numerical examples, which illustrate the theoretical results presented in Chapters 5 and 6. These results include the extension of the area of convergence for MSMAOR, which has been presented in forms of two proofs, with the second one using the new technique. Then, using this new technique, we were able to go one step further and provide a priori and a posteriori error estimates for the MSMAOR iteration method.

7.1 Convergence area extension

First, we begin with the extension of the area. This has been presented as Theorem 5.0.1 in Chapter 5. Not only does the convergence area become wider, but also the optimal choice of relaxation parameters, for which a significant acceleration, measured by the spectral radius of the approximation of the MSMAOR iteration matrix, can be achieved. In what follows, we observe two cases for each matrix, $\xi = 0.4$ and $\xi = 1$. BZ_val presents the spectral radius of the of the approximation of the MSMAOR iteration matrix, for the choice of parameters BZ_alpha and BZ_beta. Similar reasoning applies to terms with prefix CK.

Generating matrix $A1$.

```
function A = lcp1(n)
E = eye(n); E = fliplr(E);
E(1,1) = 1; E(end,end) = 1;
S = tril(ones(n));
A = S; F = E;
for j=1:n-1
    A = blkdiag(A,S);
    F = blkdiag(F,E);
end
M = F(:,1:n);
F(:,1:n) = []; F = [F M];
A = A+F;
A = A+8*eye(n*n);
```

```
A1=lcp1(10)
```

rows/columns:	100/100
nonzero entries:	670

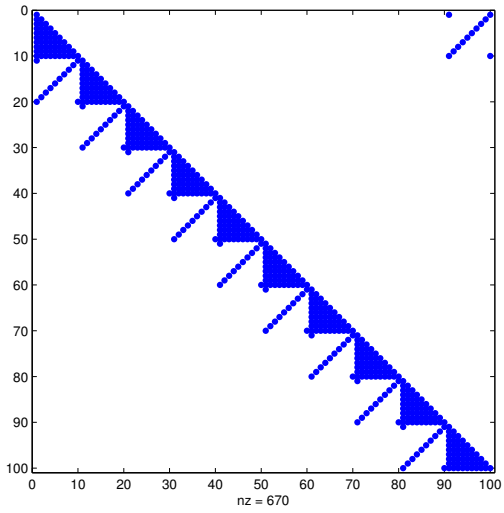
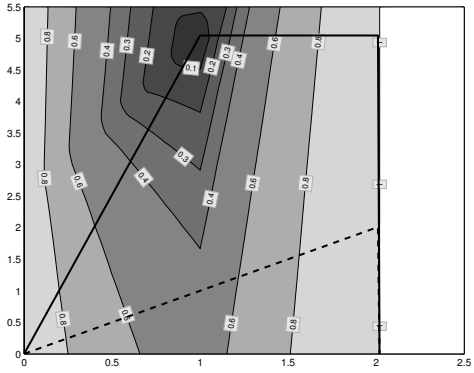
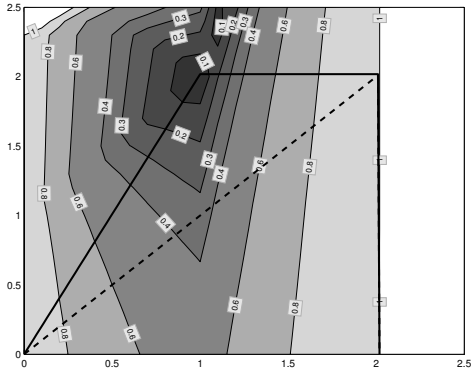


Figure 7.1: Nonzero sparsity pattern of matrix A1



(a) $\xi = 0.4$



(b) $\xi = 1$

A1	BZ_val	CK_val	BZ_alpha	BZ_beta	CK_alpha	CK_beta
$\xi = 0.4$	0.4423	$2.72 \cdot 10^{-6}$	1	1	1	5
$\xi = 1$	0.3369	$2.72 \cdot 10^{-6}$	1	1	1	2

Figure 7.2: Level curves for $\rho(\mathscr{L}_{MSMAOR}(\alpha,\beta))$ and matrix $A1$ in the (α,β)-plane and numerical comparisons of parameter choice and corresponding $\rho(\mathscr{L}_{MSMAOR}(\alpha,\beta))$ (BZ_val and CK_val) for matrix $A1$.

Generating matrix *A*2.

```
function A = lcp2(s,mi,a,b)
S = (4+mi)*eye(s)
 -a*diag(ones(s-1,1),1)
 -b*diag(ones(s-1,1),-1);
A = S;
for j=1:s-1
    A = blkdiag(A,S);
end
A = A
 -a*diag(ones(s^2-s,1),s)
 -b*diag(ones(s^2-s,1),-s);
```

```
A2=lcp2(10,4,0.5,3.5)
```

rows/columns:	100/100
nonzero entries:	460

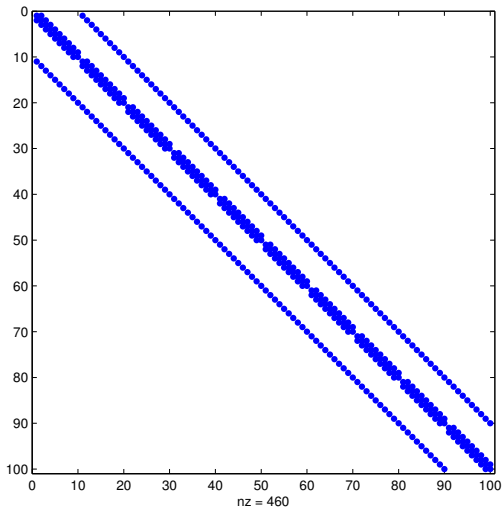
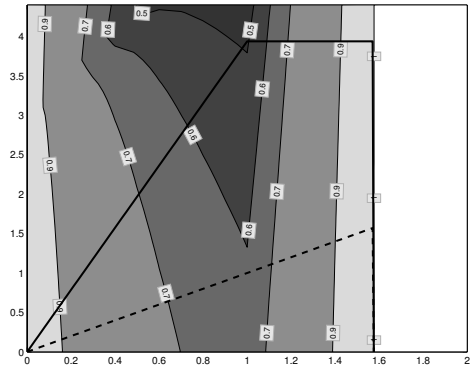
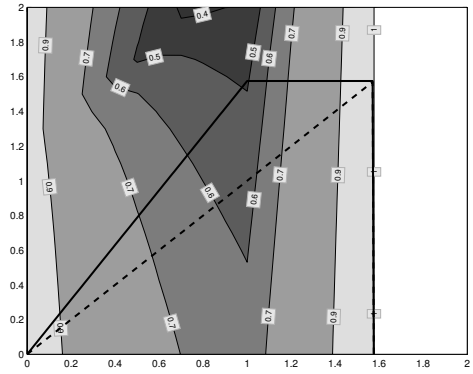


Figure 7.3: Nonzero sparsity pattern of matrix A2



(a) $\xi = 0.4$



(b) $\xi = 1$

A2	BZ_val	CK_val	BZ_alpha	BZ_beta	CK_alpha	CK_beta
$\xi = 0.4$	0.6093	0.4937	1	1	1	3.9
$\xi = 1$	0.5606	0.5025	1	1	1	1.5

Figure 7.4: Level curves for $\rho(\mathscr{L}_{MSMAOR}(\alpha,\beta))$ and matrix $A2$ in the (α,β)-plane and numerical comparisons of parameter choice and corresponding $\rho(\mathscr{L}_{MSMAOR}(\alpha,\beta))$ (BZ_val and CK_val) for matrix $A2$.

Generating matrix $A3$.

```
function A = lcp3(n)
T = zeros(n)
 -0.5*diag(ones(n-1,1),1)
 +0.5*diag(ones(n-2,1),-2);
E = eye(n); E(1,end) = -1;
E(end,1)=-1; S=10*eye(n)+T;
A = S; P = E; Q = E;
for j=1:(n-1)
    A = blkdiag(A,S);
    P = blkdiag(P,E);
    Q = blkdiag(Q,E);
end
P(:,(n^2-n+1):end)=[];
P=[zeros(n^2,n),P];
Q(:,1:n)=[];
Q = [Q,zeros(n^2,n)];
A = A+P+Q;
```

```
A3=lcp3(10)
```

rows/columns:	100/100
nonzero entries:	486

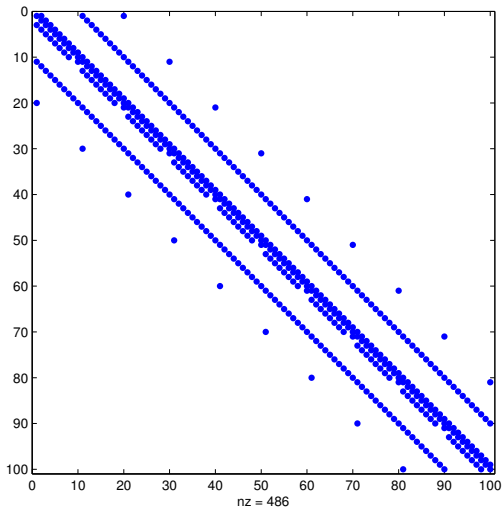
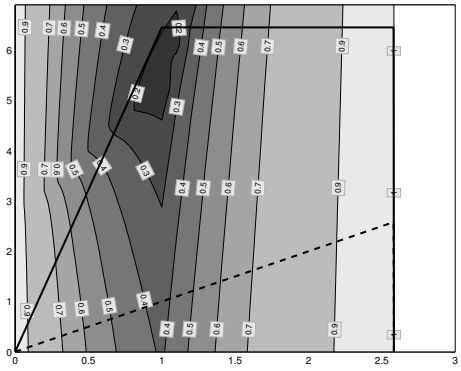
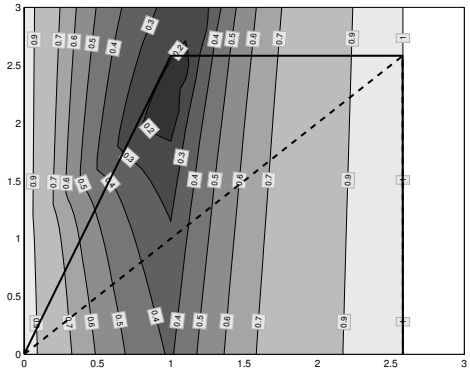


Figure 7.5: Nonzero sparsity pattern of matrix A3



(a) $\xi = 0.4$



(b) $\xi = 1$

A3	BZ_val	CK_val	BZ_alpha	BZ_beta	CK_alpha	CK_beta
$\xi = 0.4$	0.3620	0.1095	1	1	1	5.3
$\xi = 1$	0.3146	0.1083	1	1	1	2.1

Figure 7.6: Level curves for $\rho(\mathscr{L}_{MSMAOR}(\alpha,\beta))$ and matrix $A3$ in the (α,β)-plane and numerical comparisons of parameter choice and corresponding $\rho(\mathscr{L}_{MSMAOR}(\alpha,\beta))$ (BZ_val and CK_val) for matrix $A3$.

Below is the Matlab function for plotting the level curves for $\rho(\mathscr{L}_{MSMAOR}(\alpha,\beta))$ and calculating the optimal parameter choices.

```
function rho_min=srMSMAOR(A,ksi)
D=diag(diag(A));

B=D-A;
rho=abs(eigs(D\abs(B),1));
alpha_bai=1:0.01:1/rho;

tol=0.1;

clf;
hold on;

rho_min.BZ_val=NaN;
rho_min.CK_val=NaN;

[x,y]=meshgrid(0:tol:1/rho+0.5,0:tol:1/(ksi*rho)
+0.5);
rho_surf=zeros(size(x,1),size(x,2));
for k=1:size(x,1)
    for j=1:size(x,2)
        alpha=x(k,j); beta=y(k,j);
        L=((alpha+1)*D-beta*ksi*abs(-(tril(A,-1))))
        \(abs(1-alpha)*D+(alpha-beta)*ksi*
```

```
abs(-(tril(A,-1)))+alpha*abs(D-ksi*
(-(tril(A,-1)))-A)+alpha*abs(D-A));
rho_surf(k,j)=max(abs(eig(L)));


if beta <= alpha && beta > 0
    if rho_surf(k,j)==min(rho_min.BZ_val,
    rho_surf(k,j))
        rho_min.BZ_val=min(rho_min.BZ_val,
        rho_surf(k,j));
        rho_min.BZ_alpha=alpha;
        rho_min.BZ_beta=beta;
    end
end

if max(alpha,ksi*beta)*rho<min(1,alpha)
&& beta >0
    if rho_surf(k,j)==min(rho_min.CK_val,
    rho_surf(k,j))
        rho_min.CK_val=min(rho_min.CK_val,
        rho_surf(k,j));
        rho_min.CK_alpha=alpha;
        rho_min.CK_beta=beta;
    end
end
```

```
    end
end

axis equal;
[C,h]=contourf(x,y,rho_surf,[0:0.1:0.4 0.6 0.8 1]);
colormap([0.2:0.008:1; 0.2:0.008:1; 0.2:0.008:1]');
set(h,'ShowText','on');
text_handle = clabel(C,h);
set(text_handle,'BackgroundColor',[0.9 0.9 .9],
'Edgecolor',[.7 .7 .7], 'FontSize',8);
box on;
plot([0:0.01:1 alpha_bai 1/rho],
[(0:0.01:1)/(ksi*rho) 1/(ksi*rho)+0*alpha_bai 0],
'k','LineWidth',2);
plot([0:0.01:1 alpha_bai 1/rho],[0:0.01:1 alpha_bai
0],'k--','LineWidth',2);

end
```

7.2 Error control

The second part of the numerical examples is dedicated to the error control and its analysis. In order to perform numerical experiments and present them accordingly, we consider a slight modification of the example of the LCP that can be found in [8]. As it has been done before in Chapter 6, we will treat the cases of SDD and H-matrices separately.

7.2.1 SDD-matrix case

Let m be a given positive integer and $n = m^2$. Let us look at the $LCP(q,A)$, with matrix $A \in \mathbb{R}^{n\times n}$ given in its block-tridiagonal form

$$A = Tridiag(-E,S,-E) = \begin{bmatrix} S & -E & & & & \\ -E & S & -E & & & \\ & -E & S & & & \\ & & & \ddots & & \\ & & & & S & -E \\ & & & & -E & S \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{7.1}$$

where $S = \text{tridiag}\,(-1,5,-1) \in \mathbb{R}^{m\times m}$ is also tridiagonal matrix.

It is not difficult to confirm that A is an SDD matrix. We set $q := r^\star - Az^\star$ with the vectors $z^\star = [1,0,1,0,\ldots,1,0]^T$ and $r^\star = [0,1,0,1,\ldots,0,1]^T$ providing the unique solution of LCP(q,A) - i.e. $x^\star = z^\star - D^{-1}r^\star$ being the exact solution of (4.3). We consider two cases. In the first one, the initial approximation $z^{(0)}$ is $z^{(0)} = 0.05z^\star + r^\star$, while in the second case

$z^{(0)} = 5z^\star + 3r^\star$. Moreover,

$$x^{(0)} = \frac{1}{2}\gamma\Omega^{-1}\left((\Omega - A)z^{(0)} - q\right) = D^{-1}\left((D - A)z^{(0)} - q\right).$$

Working in a single processor setting, which means that $\ell = 1$, we choose the relaxation parameters $\alpha = 1$ and $\beta = 1/\xi = 1/0.4$. For simplicity, we also assume that $\Omega = D$ and $\gamma = 2$. For a priori and a posteriori error estimations we use Theorem 6.1.1 and compute a priori $(\mathscr{A}_k)$, $(\widehat{\mathscr{A}_k})$ and a posteriori $(\mathscr{P}_k)$, $(\widehat{\mathscr{P}_k})$ error estimations and $\mathscr{E}_1(A)$ bound. In Table 7.1 we present CPU times (in seconds) required for finding the norm $\|\widetilde{\mathscr{L}_{\text{MSMAOR}}}\|_\infty$ and $\mathscr{E}_1(A)$. Tables 7.2 and 7.3, for a prescribed tolerance tol, contain the number of iterations k and CPU times (CPU) for the MSMAOR which are needed to satisfy the stopping criteria for the error δ_k, a priori error estimations $\mathscr{A}_k$, $\widehat{\mathscr{A}_k}$, and a posteriori error estimations $\mathscr{P}_k$ and $\widehat{\mathscr{P}_k}$, with the preliminary calculated $\|\widetilde{\mathscr{L}_{\text{MSMAOR}}}\|_\infty$ and $\mathscr{E}_1(A)$. Note that the number of iterations required when using

n	$\|\widetilde{\mathscr{L}_{\text{MSMAOR}}}\|_\infty$	$\mathscr{E}_1(A)$
10^2	0.0029s	0.0045s
10^4	4.3538s	0.4168s
$4 \cdot 10^4$	338.4741s	2.7620s

Table 7.1: CPU times for computing $\|\widetilde{\mathscr{L}_{\text{MSMAOR}}}\|_\infty$ and $\mathscr{E}_1(A)$.

a priori and a posteriori error estimations is greater than the one with the stopping criteria, but the latter assumes that the exact solution is known. Therefore, from the practical point of view, the use of a priori or a posteriori error estimations as stopping criteria is more suitable and

	e_k	$\mathscr{A}_k$	$\mathscr{P}_k$	$\widehat{\mathscr{A}_k}$	$\widehat{\mathscr{P}_k}$	tol
k	12	48	13	57	13	10^{-6}
CPU	0.0183s	0.0873s	0.0197s	0.0784s	0.0198s	
k	24	96	26	114	26	10^{-12}
CPU	0.0414s	0.1466s	0.0460s	0.1658s	0.0415s	

Table 7.2: Number of iterations k and CPU for (7.1) with the initial approximation $z^{(0)} = 0.05z^\star + r^\star$ and $m = 100$.

	e_k	$\mathscr{A}_k$	$\mathscr{P}_k$	$\widehat{\mathscr{A}_k}$	$\widehat{\mathscr{P}_k}$	tol
k	16	54	18	64	18	10^{-6}
CPU	0.0270s	0.0866s	0.0294s	0.0925s	0.0313s	
k	29	102	31	120	31	10^{-12}
CPU	0.0439s	0.1706s	0.0469s	0.1671s	0.0481s	

Table 7.3: Number of iterations k and CPU for (7.1) with the initial approximation $z^{(0)} = 5z^\star + 3r^\star$ and $m = 100$.

convenient. Moreover, the computation time for $\|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_\infty$ is about 100 times larger than for $\mathscr{E}_1$. Comparing $\mathscr{A}_k$ to $\widehat{\mathscr{A}_k}$ and $\mathscr{P}_k$ to $\widehat{\mathscr{P}_k}$, we can conclude that the number of iterations and CPU time remain relatively close to each other, even though the terms required for their calculation exhibit significant discrepancies in terms of CPU times.

7.2.2 H-matrix case

For the results connected to the H-matrix case, we will consider the following matrix

$$A = \begin{bmatrix} F & & & & & & & O \\ -E & F & & & & & & \\ -E & -E & \widetilde{F} & & & & & \\ & -E & -E & \widetilde{F} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -E & -E & \widetilde{F} & & \\ & & & & -E & -E & \widetilde{F} & \\ O & & & & & -E & -E & \widetilde{F} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{7.2}$$

where $F = \text{tridiag}(-1,4,-1) \in \mathbb{R}^{m\times m}$ and $\widetilde{F} = \text{tridiag}(-1,5,-1) \in \mathbb{R}^{m\times m}$. Even though A is not an SDD, it is an H-matrix, more precisely an S-SDD for the choice of the index set $S = \{1,2,\ldots,2m\}$. For more information on this subclass of nonsingular H-matrices, see [21, 23]. In this case, the matrix W scaling A to an SDD matrix is known, [21]. Moreover, the diagonal of this matrix represents a possible positive vector w. We take the n-dimensional vector $w = [0.5, 0.5, \ldots, 0.5, 1, 1 \ldots, 1]^T$ with

n	$\|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_w$	$\mathscr{E}_2(A)$
10^2	0.0063s	0.0101s
10^4	4.6139s	0.2896s
$4 \cdot 10^4$	1078.1s	1.5s

Table 7.4: CPU times for computing $\|\widetilde{\mathscr{L}}_{\text{MSMAOR}}\|_w$ and $\mathscr{E}_2(A)$.

	e_k^w	$\mathscr{A}_k^w$	$\mathscr{P}_k^w$	$\widehat{\mathscr{A}_k^w}$	$\widehat{\mathscr{P}_k^w}$	tol
k	10	46	12	61	12	10^{-6}
CPU	0.0183s	0.0730s	0.0216s	0.0948s	0.0220s	
k	20	87	22	115	22	10^{-12}
CPU	0.0357s	0.1563s	0.0396s	0.1763s	0.0400s	

Table 7.5: The number of iterations k and CPU time for (7.2) with the initial approximation $z^{(0)} = 0.05z^\star + r^\star$ and $m = 100$.

	e_k^w	$\mathscr{A}_k^w$	$\mathscr{P}_k^w$	$\widehat{\mathscr{A}_k^w}$	$\widehat{\mathscr{P}_k^w}$	tol
k	14	49	15	64	15	10^{-6}
CPU	0.0275s	0.0783s	0.0306s	0.1124s	0.0313s	
k	24	90	25	118	25	10^{-12}
CPU	0.0473s	0.1495s	0.0480s	0.1979s	0.0478	

Table 7.6: The number of iterations k and CPU time for (7.2) with the initial approximation $z^{(0)} = 5z^\star + 3r^\star$ and $m = 100$.

the first $2m$ entries to be 0.5. An analysis similar to the previous example shows the advantage of the norm estimation $\mathscr{E}_2(A)$ from Theorem 6.2.1. The results are summarised in Tables 7.4, 7.5 and 7.6.

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Novi Sad, January 2019. Abear Saeed Aboglida Saeed

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Izvod: Problemi linearne komplementarnosti (LCP) se javljaju kod problema linearnog i kvadratnog programiranja i kod mnogih drugih problema iz prakse, kao što su, na primer, problemi sa graničnim slojem, problemi mrežnih ekvilibrijuma, kontaktni problemi, problemi određivanja tržišne ravnoteže, problemi bimatričnih igara i mnogi drugi. Ne tako davno, veliki broj autora se bavio razvijanjem postupaka za rešavanje LCP sa matricom koja ispunjava neko specijalno svojstvo, na primer, da pripada klasi H_+-matrica, budući da je dobro poznato da je ovaj uslov dovoljan da obezbedi egzistenciju i jedinstvenost rešenja LCP. Uopšteno govoreći, rešavanju LCP moguće je pristupiti dvojako. Prvi pristup podrazumeva upotrebu takozvanih direktnih metoda, koje su u literaturi poznate i pod nazivom metode pivota. Drugoj kategoriji, koja je i sa stanovišta ove teze interesantna, pripadaju iterativni postupci. S obzirom da je ova kategorija izuzetno bogata, mi smo se opredelili za jednu od najznačajnijih varijanti, a to je modulski iterativni postupak. Međutim, ni ova odrednica nije dovoljno adekvatna, budući da modulski postupci obuhvataju nekolicinu različitih pravaca. Zato smo se odlučili da posmatramo postupke koji se zasnivaju na razlaganjima ali i višestrukim ra-

zlaganjima matrice. Glavni cilj ove doktorske disertacije jeste upotreba teorije H-matrica u teoremama o konvergenciji modulskih metoda zasnovanih na multisplitinzima matrice i korišćenje ove nove tehnike, sa ciljem analize bitnih osobina, nakon što je konvergencija postupka zagarantovana.

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Abstract: The linear complementarity problems (LCP) arise from linear or quadratic programming, or from a variety of other particular application problems, like boundary problems, network equilibrium problems, contact problems, market equilibria problems, bimatrix games etc. Recently, many people have focused on the solver of LCP with a matrix having some kind of special property, for example, when this matrix is an H_+-matrix, since this property is a sufficient condition for the existence and uniqueness of the soluition of LCP. Generally speaking, solving LCP can be approached from two essentially different perspectives. One of them includes the use of so-called direct methods, in the literature also known under the name pivoting methods. The other, and from our perspective - more interesting one, which we actually focus on in this thesis, is the iterative approach. Among the vast collection of iterative solvers, our choice was one particular class of modulus based iterative methods. Since the subclass of modulus based-methods is again diverse in some sense, it can be specialized even further, by the introduction and the use of matrix splittings. The main goal of this thesis is to use the theory of H-matrices for proving convergence of the modulus-based multisplit-

ting methods, and to use this new technique to analyze some important properties of iterative methods once the convergence has been guaranteed.

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