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Ω-Algebraic Systems

by

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Apstrakt

Tema ovog rada je fazifikovanje algebarskih i relacijskih struktura u okviru Ω-skupova, gde je Ω (omega) kompletna mreža. U radu se bavimo sintezom oblasti univerzalne algebre i teorije rasplinutih (fazi) skupova. Naša istraživanja Ω-algebarskih struktura bazirana su na omega-vrednosnoj jednakosti, zadovoljivosti identiteta i tehnici rada sa nivoima. U radu uvodimo omega-algebre, Ω-vrednosne kongruencije, odgovarajuće omega-strukture, i Ω-vrednosne homomorfizme i istražujemo veze izmedju ovih pojmova. Dokazujemo da postoji Ω-vrednosni homomorfizam iz Ω-algebre na odgovarajuću količničku Ω-algebru. Jezgro Ω-vrednosnog homomorfizma je Ω-vrednosna kongruencija. U vezi sa nivoima struktura, dokazujemo da Ω-vrednosni homomorfizam odredjuje klasične homomorfizme na odgovarajućim količničkim strukturama preko nivoa podalgebri. Osim toga, Ω-vrednosna kongruencija odredjuje sistem zatvaranja klasične kongruencije na nivo podalgebrama. Dalje, identiteti su očuvani u Ω-vrednosnim homomorfnim slikama. U nastavku smo u okviru Ω-skupova uveli Ω-mreže kao uredjene skupove i kao algebre i dokazali ekvivalenciju ovih pojmova. Ω-poset je definisan kao Ω-relacija koja je antisimetrična i tranzitivna u odnosu na odgovarajuću Ω-vrednosnu jednakost. Definisani su pojmovi pseudo-infimuma i pseudo-supremuma i tako smo dobili definiciju Ω-mreže kao uredjene strukture. Takodje je definisana Ω-mreža kao algebra, u ovim kontekstu nosač te strukture je bi-grupoid koji je saglasan sa Ω-vrednosnom jednakošću i ispunjava neke mrežno-teorijske formule. Koristeći aksiom izbora dokazali smo da su dva pristupa ekvivalentna. Dalje smo uveli i pojam potpune Ω-mreže kao uopštenje klasične potpune mreže. Dokazali smo još neke rezultate koji karakterišu Ω-strukture. Data je i veza izmedju Ω-algebre i pojma slabih

kongruencija. Na kraju je dat prikaz pravaca daljih istraživanja.

Abstract

The research work carried out in this thesis is aimed at fuzzifying algebraic and relational structures in the framework of Ω-sets, where Ω is a complete lattice. Therefore we attempt to synthesis universal algebra and fuzzy set theory. Our investigations of Ω-algebraic structures are based on Ω-valued equality, satisfiability of identities and cut techniques.

We introduce Ω-algebras, Ω-valued congruences, corresponding quotient Ω-valued-algebras and Ω-valued homomorphisms and we investigate connections among these notions. We prove that there is an Ω-valued homomorphism from an Ω-algebra to the corresponding quotient Ω-algebra. The kernel of an Ω-valued homomorphism is an Ω-valued congruence. When dealing with cut structures, we prove that an Ω-valued homomorphism determines classical homomorphisms among the corresponding quotient structures over cut subalgebras. In addition, an Ω-valued congruence determines a closure system of classical congruences on cut subalgebras. In addition, identities are preserved under Ω-valued homomorphisms.

Therefore in the framework of Ω-sets we were able to introduce Ω-lattice both as an ordered and algebraic structures. By this Ω-poset is defined as an Ω-set equipped with Ω-valued order which is antisymmetric with respect to the corresponding Ω-valued equality. Thus defining the notion of pseudo-infimum and pseudo-supremum we obtained the definition of Ω-lattice as an ordered structure. It is also defined that the an Ω-lattice as an algebra is a bi-groupoid equipped with an Ω-valued equality fulfilling some particular lattice-theoretical formulas. Thus using axiom of choice we proved that the two approaches are equivalent.

Then we also introduced the notion of complete Ω-lattice based on Ω-lattice. It was defined as a generalization of the classical complete lattice.

We proved results that characterizes Ω-structures and many other interesting results.

Also the connection between Ω-algebra and the notion of weak congruences

is presented.

We conclude with what we feel are the most interesting areas for future work.

Declaration

I, Elijah Eghosa EDEGHAGBA, declare that this thesis titled, Ω-Algebraic Systems and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

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Dedication

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Contents

Chapter 1

Introduction

In this chapter we briefly give the historical and chronological development of the fields from which our research originates. These are fuzzy sets and structures, omega sets and some topics from Universal algebra (in particular weak congruences). Then the summary of the various chapters of the research work carried out in this thesis are discussed.

1.1 Historical and chronological development

A paradigm shift began in mid sixties. This shift heralded a new area in Mathematics, Fuzzy Sets (sets whose elements have degrees of membership) and fuzzy logic (logic in which the truth values of variables may be any real number between 0 and 1), generally referred to as Fuzzy Mathematics. This was a shift since most of our old traditional methodology for formal modeling, reasoning, and computing are based on the principle of bivalence, as well on the idea of precision. The principle of bivalence says that any statement is either true or false. In conventional crisp logic, for example, a statement can be true or false and nothing else. In crisp set theory, an element can either belong to a set or not; and in optimization, a solution is either feasible or not, etc. On the other hand, the idea of precision requires that statements about parameters of a model must be exact and lacking no knowledge about the real system that it models. Otherwise a model is considered imprecise and as such are said to be unreliable.
The real world is complex; this complexity generally arises from uncertainty, i.e, deficiency in information and knowledge needed to reach a perfect reliable conclusion.
In 1927, Heisenberg's enunciated the principles of uncertainty (which says that we cannot measure the position x and the momentum p of a parti-

cle with absolute precision) in quantum theory, which placed an absolute theoretical limit on the accuracy of certain measurements; as a result, the assumption by earlier scientists that the physical state of a system could be measured exactly and used to predict future states had to be reconsidered or even abandoned. Therefore, this influenced science in its considerations and thoughts about the idea of precision.

Two distinctive forms of uncertainty considered are; statistical (randomness) and non-statistical uncertainty, The former is the uncertainty in the occurrence of an event, i.e, the randomness inherent in the system under investigation. This type of uncertainty can be handled by probabilistic approach. While the latter is subdivided into vagueness and ambiguity. Philosophically, vagueness is the characteristic of words whose meaning is not determined with precision. Therefore, vagueness is the uncertainty due to the lack of sharpness of relevant distinction among different entities. Ambiguity need not be confused with vagueness. Ambiguity is the presence of two or more distinct meanings for a single word, i.e, it is the uncertainty that is due to lack of certain distinctions characterizing an entity. For example "I will *ring* you later today" this could signify giving a gift of jewelry or could be to give a call across. In this case what the person intend to do later today is not really clear. Unlike the case "He is a tall man", this is difficult to evaluate since there is no borderline between tall and not tall. Therefore, through seeking to quantify the imprecision that characterizes our linguistic description of perception and comprehension, fuzzy set theory provides a formal framework for handling vagueness. In general, the reasoning process that handles uncertainty is called approximate or imprecise reasoning.

In the real world there are vaguely specified data values in many applications. This situation had already been recognized by thinkers in the past. In 1923 Bertrand Russell ([81]), who perhaps is the first thinker who gave a definition to vagueness, in his work wrote: " All traditional logic habitually assumes that precise symbols are being employed. It is, therefore, not applicable to this terrestrial life but only to an imagined celestial existence ". And then in 1937, Max Black ([10]) in his work considered vagueness as one of the fundamental issues in science and then devised a formalism for dealing with vague terms. Hence the final turn in recognizing vagueness as inherent to human description of the world and its utilization came with L. Zadeh, who introduced a natural formalization, so-called ***fuzzy sets***, for dealing with vagueness.

Therefore, fuzzy set was introduced in mid sixties by Zadeh ([109]) and then the development of fuzzy logic ([105]) in 1975. Of course before Zadeh's work, there has been expressions of concern over the need for intermediary truth-values and modalities. Surprisingly, the first classical logician to express this concern was Aristotle (who, ironically, is also generally considered to be the first classical logician and the "father of logic"([73])). Where he admitted that the **law of excluded middle** did not all apply to future events, but he did not create a system of multi-valued logic to explain his thought. It was only in the twentieth century that the Polish logician and philosopher Jan Łukasiewicz brought back this idea of multi-valued logic- a logic where propositions may assume more than two truth values- where he developed three-valued logics, and other many-valued systems in 1920, using a third value, "possible". Also in 1921, Emil L. Post ([76]) introduced the formulation of additional truth degrees with $n \geq 2$, where n is a truth value. It is widely accepted that multi-valued logic and fuzzy logic among others- supervaluationism and contextualism- are expressions of vagueness. Fuzzy logic differs from conventional logical systems in that it aims at providing a model for approximate rather than precise reasoning. Since this work is in the direction of fuzzy set theory and its application to other areas of mathematics we will then focus more on it and if necessary make reference to multi-valued logic.

Sets and set theory are inherent in mathematics, for example, one approach to the foundation of mathematics is based on the idea that sets are the most fundamental object in mathematics. Set can be roughly considered as any collection which is well defined. According to the classical view point, given an object a and a set A, then $a \in A$ or $a \notin A$. By relaxing this condition, we obtain the generalization of the concept of a set. One way to generalizing a set is to allow element belong to a set to some degree, which is the idea behind **fuzzy set**.

The first publication in fuzzy set theory by Zadeh ([109]) showed the intention of the author to generalize the crisp notion of a set, its characteristic function and a proposition to accommodate fuzziness (mathematical model of vagueness). Zadeh stated in his seminal paper: "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables".

Fuzzy sets are defined on any universal set of concern by **membership functions**. Each membership function defines a fuzzy set on a given universal set by assigning to each element of the universal set its membership grade in the fuzzy set. Because the common understanding is that fuzzy sets are completely characterized by their membership degree behavior, they may not only be characterized by, but identified with suitable membership functions. Moreover it is rather natural to consider the membership degrees of fuzzy sets as truth degrees of a generalized membership predicate. This approach offers the additional possibility to consider fuzzy sets as extensions of predicates in interpretations of suitable first-order many-valued logics.

These membership grades are represented by real numbers ranging in the unit interval, $[0, 1]$. This membership function is usually denoted by $\mu_A(x)$. Membership functions are mathematical tools for indicating flexible membership to a set, modeling and quantifying the meaning of symbols. Vague notions like "beauty, tallness, size, etc." can be represented by membership functions. We must note that precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group. Moreover, they are context-dependent. Fuzzy sets are commonly denoted in this way: If U is some universal set (set of men), then a given fuzzy set A ("tall"), can be written as a set of ordered pairs $A = \{(u, \mu_A(u))\}$. Where the first component u is an element from U and the second component $\mu_A(u)$ is the degree to which u is tall, i.e, the degree to which u belongs to A.

Following quickly in 1967 Goguen, who was a student of Zadeh in ([40]) replaced the unit interval with a lattice and Brown in ([17]), replaced the unit interval with a Boolean lattice. These were proposed and introduced as a more general variant of the notion of fuzzy set, with membership functions taking values in suitable partially ordered set (poset or lattice). These are usually termed L-valued fuzzy set to distinguish them from those of the unit interval. In ([83]), Sanchez began the study of fuzzy relations and their composition. He defined a fuzzy relation from a set A to a set B as a map from the Cartesian product $A \times B$ into a complete lattice L.
The concept of fuzzy was first applied to various aspects of universal algebra - theory of groupoids and groups - in 1971 by Rosenfeld ([80]), his work pioneered the investigations of the concept of fuzzy algebras. In 1987 Di Nola et. al ([32]) proposed a general approach to the theory of fuzzy in universal algebras. The theory of fuzzy algebras has continue to grow since this concept was introduced by Rosenfeld ([80]) with several authors also proposing other general approaches, for example ([3, 24, 30, 70, 82, 56, 57, 18]). For these works we note that they all have similar purpose, to unify and generalize the earlier results concerning particular algebras. In paper ([18]) Bošnjak et. al. extended the notion of the so-called algebras of fuzzy sets induced by an algebra which where first defined in ([71]) to a more general case, where the structure of the truth values is a complete lattice or a complete residuated lattice. They investigated two kinds of algebras of fuzzy sets induced by an algebra, which they obtained using Zadehs extension principle. They gave

conditions under which a homomorphism between two algebras induces a homomorphism between corresponding algebras of fuzzy sets.
The concept of fuzzy group was latter redefined by Anthony and Sherwood ([2]) and since then there have been several works in this regard ([103, 7, 22]). The notion of fuzzy subring and fuzzy ideal was introduced by Liu ([60]) and since then many researchers have studied theories of fuzzy subrings and ideals ([6]). The developments of algebra in fuzzy setting are very much evident in the books of D. S. Malik and J. N. Mordeson ([67, 68]).

The study of basic type of fuzzy relations started with Zadeh, ([108]), where he defined the notion of fuzzy equivalence and gave the concept of fuzzy ordering. Since then more investigations on fuzzy order has been carried out by ([5, 35, 50, 58]). Fan in ([35]), proposed a fuzzy poset (X, R), where R is a reflexive, antisymmetric and transitive fuzzy relation over set X. Fuzzy poset, also called L-ordered set, was originally introduced by Bělohlávek ([4]) in order to fuzzify the fundamental theorem of concept lattices. Considerable contribution to fuzzy orderings and applications is done by Bodenhofer, see ([13, 14, 11]) and his work with De Baets, and Fodor, ([12]). Namely, a fuzzy order in this approach is introduced with respect to a fuzzy equivalence and a t-norm, hence membership values belong to the unit interval. A connection to our work is the mentioned dependence of a fuzzy order on a fuzzy equivalence. What makes our approach different is the co-domain lattice (a complete lattice) and a reflexivity which, in our case determines fuzzy ordered substructures.

Wu and Yuan ([104]) introduced the concepts of fuzzy lattices and fuzzy sublattices. Ajmal and Thomas ([1]) presented a general development of the theory of fuzzy lattices.

Tepavčević and Trajkovski ([100]) studied L-fuzzy lattices (fuzzy lattices valued by lattices) where two types of L-fuzzy lattices where defined and their connection was observed. More recently, L-fuzzy complete lattices, was introduced by Zhang et al ([111]), based on complete Heyting algebras and fuzzy L-order relation.
Particularly, a fuzzy poset (X, R) is called a fuzzy complete lattice, if every fuzzy subset has both join and meet that belong to X. In ([63]), Martinek discussed completely lattice L-ordered sets with and without L-valued equality.
The concepts of filter and ideal, in posets, are dual to each other. Both concepts are very useful when studying problems concerning ordered structures. Liu in his work ([59]) introduced and investigated the notion of a fuzzy ideal in a ring. Mukherjee and Sen ([69]) introduced the notion of fuzzy prime ideal of a ring. Since then several researchers have obtained interesting results on fuzzy ideals of different algebraic structures ([64, 110, 62]).

1.2 Motivation and problem statement

Among the most basic objects of study in all of mathematics are algebras. An algebra $\mathcal{M} = (M, \mathbb{F})$ consists of a nonempty set M and a collection $\mathbb{F}$ of operations. Most important examples are lattices, groups, rings and modules. Understanding a particular algebra $\mathcal{M}$, is usually studied via its subalgebras, congruences, and lattices representing them. It happens that weak congruences on a particular algebra $\mathcal{M}$ (which can be referred to congruences on subalgebras, say $\mathcal{N}$ of $\mathcal{M}$) turns out to be a tool for investigating the notion of congruences and subalgebras of an algebra commonly.

This thesis is devoted to the study of L-algebraic and relational structures in a general way. These will be called Ω-Algebraic systems, Ω being a complete lattice.

The present investigation is in the framework of Ω-sets. Ω-sets were introduced by Fourman and Scott ([38]), whose intention was to model intuitionistic logic. An Ω-set is a nonempty set A equipped with an Ω-valued equality $E : A \times A \to \Omega$, where Ω is a complete Heyting algebra such that the map sends a pair of elements $x, y \in A$ to an element $E(x, y)$ in Ω. In addition $E(x, y) = E(y, x)$ and $E(x, y) \wedge E(y, z) \leq E(x, z)$ holds for every $x, y, z \in A$, these conditions implies that E is a symmetric and transitive map from A^2 to Ω. In which case the fuzziness consist of identifying elements of a given set A only to some certain extent and in particular single elements $x \in A$ are discern with certain accuracy, $E(x, x) \in \Omega$. This notion has been further applied to non-classical predicate logics, and also to foundations of fuzzy set theory ([42, 46, 102]). Ω-posets were further developed by Borceaux and Cruciani ([16], see also [15]), still keeping the Heyting lattice as a co-domain. As elaborated in the sequel, this makes an essential difference to our approach, due to cut properties of Ω-structures based on the use of infimum instead of multiplication.

In this work Ω is necessarily not a Heyting algebra but a complete lattice. The main reason for this co-domain is that it allows main algebraic and set-theoretic properties to be generalized from classical structures to L-structures, in the framework of cut sets ([55]). In the recent times a complete lattice is often replaced by a complete residuated lattice ([5]). The notion presented in this work is new, where we introduced Ω-valued equality in the study of fuzzy algebraic systems. Although the notion of fuzzy(lattice)-valued equality as a generalization of its classical counterpart has been introduced by Höhle in ([48]), and then it was used in investigations of fuzzy functions and fuzzy algebraic structures by many authors, in particular by Aleš Pultr ([77]), Novák ([72]), Demirci ([31, 29, 28]), and Bělohlávek and V. Vychodil in ([3]) initiated a new fuzzy approach to universal algebra, where they studied the so called algebras with fuzzy equalities and developed fuzzy equational logic. Šešelja, et. al. ([19]), introduced and developed L-valued identities as a generalization of the classical counterpart.

Our main task in this thesis is to use Ω-valued equalities E in order to deal with identities $E(u, v)$, where u and v are terms in the language of the basic algebra $\mathcal{M} = (M, \mathbb{F})$, which hold on Ω-algebras $(\mathcal{M}, E)$ and not necessarily on the basic algebra.
The notion of Ω-structures was introduced by Šešelja, and Tepavčević. One of the basic tools employed in the investigations of these structures is an Ω-valued equality which is transitive and symmetric as a function and as well compatible with the operations. In view of fuzzy identities introduced in ([19]) (also see([22, 93])), we have another tool, the definition of satisfiability of identities. A classical identity is transferred into lattice theoretic formula. In this way an Ω-algebra fulfills a set of identities which the basic classical algebra, on which our Ω-algebra is defined, does not necessary satisfy.

Still a very important approach that will be adopted in this work is the approach of cut sets and structures which has been extensively studied by Šešelja and Tepavčević. These cuts link very well the L-structures (in our case Ω-structures) with the classical structures. For example the cut of the diagonal of an Ω-valued equality is a classical substructure of the supporting structure. A cut of an omega valued equality is a congruence on a substructure of the support. A basic property of Ω-algebraic systems is that the quotient structures of diagonal cuts i.e. Ω-set μ determined by an Ω-valued equality E that is $\mu(x) = E(x, x), \forall x \in M$ over the corresponding cuts of the Ω-valued equality E are classical algebras fulfilling all identities which are satisfied by this Ω-algebraic system in Ω-valued framework.

1.3 Weak Congruences

The notion of weak congruences on an algebra $\mathcal{M}$, i.e. symmetric and transitive subalgebras of $\mathcal{M} \times \mathcal{M}$, are considered in this research because they link very well with Ω-structures. Weak congruences are congruences on subalgebras of $\mathcal{M}$. Therefore, we investigate the existing connection between cuts of Ω-valued equalities on Ω-algebras and weak congruences on our basic algebras. This connection is due to the fact that for an Ω-valued equality E, each cut relation $E_p, p \in \Omega$ is a weak congruence relation on the (basic) algebra $\mathcal{M}$. Consequently, the diagonals determine the corresponding sub-

algebras. Then also, these cuts of congruences (or equivalently the quotient structures) form a closure system, hence a complete lattice, a subposet in the weak congruence lattice, closed under infima.
Weak congruences have been extensively researched by Šešelja and his collaborators ([93, 88]), where their application to lattice theory and in general to universal algebra have been well investigated.

1.4 Thesis Chapters Outline

This thesis is outlined as follows. It is divided into seven chapters.
In chapter one titled: Introduction, we present a brief historical overview and development of fuzzy set, Ω-sets, and weak congruences. The chapter also contains a brief summary of the research work carried out in this thesis.
In chapter two, titled : Preliminaries, we presented basic notions such as ordinary set theory, ordered sets and lattices, closure systems, and closure operators, notions in universal algebra and basic fuzzy set theory.
In chapter three, titled : L-structures, we presented an overview of existing notions and their relation to our work. As earlier mentioned in the introduction L-structures are generalization of the basic fuzzy set notion.
In chapter four, titled : Ω-algebras in universal algebra, we introduce and develop basic structural notions: Ω-subalgebras, morphisms between Ω-algebras, congruences on Ω-algebras and then direct products between Ω-algebras. Furthermore, we investigated the connecting properties that exist between Ω-algebras and their quotient algebras, which of course are classical structures.
In chapter five, titled : Ω-relational systems, we introduced and investigated Ω-lattices both as algebras and as ordered structures Then as in the case of ordinary mathematics we proved that under the assumption of axiom of choice, the two notions are equivalent. Meaning that it is possible to define operations on an Ω-lattice as an ordered structure and obtain an Ω-lattice as an algebra, and vice versa. Furthermore, the notion of complete Ω-lattices was investigated with its special elements.
In chapter 6, titled : Weak congruence relations and Ω-algebras. We investigated the notion of weak congruences and their relationship with Ω-algebras. This relationship is investigated via the cut relations of the Ω-valued equalities defined on the basic algebras. Of course the cut relations are congruence relations on subalgebras of the basic algebras, which turns out to be weak congruences on the basic algebras.
The last chapter, chapter 7 encapsulates all that we did in the work and possible future research.

Chapter 2

Preliminaries

This chapter contains an overview of some important fundamental notions and results that are needed for the development of the remainder of this work.

2.1 Classical theory

2.1.1 Classical Set Theory

Let $\mathcal{U}$ be a universe, then an object x in $\mathcal{U}$ is said to belong to a set S, denoted $x \in S$, if x satisfies the defining rule of membership in S. For example $S = \{y : 0 \leq y < 1\}$. In the case x is not in S, it is written as $x \notin S$.
Similarly, a set can be defined by a function, usually referred to as the **characteristic function**, which declares, which elements of the universe, $\mathcal{U}$ are members of a given set, S or not. Therefore set S is defined by its characteristic function

$$\chi_S(x) = \begin{cases} 1 & if\ x \in S \\ 0 & if\ x \notin S \end{cases}$$

This is formally the map is expressed as $\chi : \mathcal{U} \to \{0, 1\}$.
The empty set or null set, denoted $\emptyset$, is the set containing no element.

For **subsets** A and B of a set X,

$$A = B \text{ if and only if } \chi_A(x) = \chi_B(x), \forall x \in X$$

$$A \subseteq B \text{ if and only if } \chi_A(x) \leq \chi_B(x), \forall x \in X$$

$$(A \cup B)(x) = \chi_A(x) \vee \chi_B(x), \forall x \in X$$

$$(A \cap B)(x) = \chi_A(x) \wedge \chi_B(x), \forall x \in X$$
$$\chi_{A^c(x)} = 1 - \chi_A(x), \forall x \in X$$

Definition 2.1.1. An **equivalence relation** on a set X gives rise to a partition of X. We say $x, y \in X$ are equivalent if and only if they belong to the same block of partition. We call a block an **equivalence class** of the equivalence relation.

If the symbol $\equiv$ denotes the equivalence relation, then we write $x \equiv y$ to indicate that x and y are equivalent (in the same block) and $x \not\equiv y$ to denote that they are not equivalent.
Here's a trivial equivalence relation that is used all the time. For a set X, let all the blocks of the partition have one element. Two elements of X are equivalent if and only if they are the same. This rather trivial equivalence relation is, of course, denoted by " $=$ ".

2.1.2 Ordered sets and Lattices

Partial orders and Preorders

Definition 2.1.2. A **preorder** is a binary relation $\leq$ on a set X which is reflexive and transitive:

$$x \leq x;\ x \leq y\ \wedge\ y \leq z \Rightarrow x \leq z,\ x, y, z \in X.$$

A **partial order** is a preorder that is antisymmetric:

$$x \leq y\ \wedge\ y \leq x \Rightarrow x = y,\ x, y \in X.$$

Let $(X, \preccurlyeq)$ be a preordered set, then a relation $\sim$ defined on X such that

$$x \sim y \text{ iff } x \preccurlyeq y \text{ and } y \preccurlyeq x$$

is an equivalence relation on X, and $\preccurlyeq$ induces a partial order $\leq$ on the set $X/\sim$ of equivalence classes of $\sim$ defined by

$$[x] \leq [y] \text{ iff } x \preccurlyeq y.$$

In particular $\leq$ satisfies antisymmetry even though $\preccurlyeq$ does not.
The above notion of partial order induced by a pre-order is well known, in what follows we present a less known but more general notion;

Definition 2.1.3. Let $X \neq \emptyset$, then a binary relation ρ on X is said to be **strict (or weakly reflexive)** if,

$$x\rho y \Rightarrow \ x\rho x \ \wedge \ y\rho y.$$

Then a strict (or weakly reflexive) and transitive binary relation $\preccurlyeq$ on X is a **strict preorder** on the set.
In this case, let $\preccurlyeq$ be a strict preorder on X. Then the relation $\sim$ defined on $Y \subseteq X$, such that

$$Y = \{x \in X : x \preccurlyeq x\}, \text{ and } x \sim y \text{ iff } x \preccurlyeq y \text{ and } y \preccurlyeq x$$

is an equivalence relation on Y and $\preccurlyeq$ induces a partial order $\leq$ on the set $Y/\sim$ of equivalence classes of $\sim$ in the subset Y of X. In particular the relation $\sim$ might not be an equivalence relation on X, since $\preccurlyeq$ is not necessarily reflexive. $\leq$ is a partial order even though $\preccurlyeq$ is not.

Definition 2.1.4. Let $\leq$ be an order relation then the inverse relation $\geq$ is also an order relation. It is called the dual order of $\leq$. Let A be an ordered set, we can form a new ordered set A^∂, the dual of A, equipped with the dual order. Two ordered sets A and B are called dually isomorphic when $A \cong B^\partial$.

Note that each statement about an ordered set A corresponds to a statement about A^∂.

Let $(X, \leq)$ be an ordered set then for a subset $A \subseteq X$, an element $b \in A$ is said to be a **maximal** element of A if for all $a \in A$, $b \leq a$ implies $b = a$; the **greatest** element of A if for all $a \in A$, $a \leq b$. The **minimal** and **least** elements are dually defined. The **upper bound** of A is an element $u \in X$ such that for all $a \in A$, $a \leq u$. An upper bound u of A is called a **supremum** (or least upper bound, or join) of A, if for all other upper bounds z of A, $u \leq z$ (i.e u is the least among the upper bounds of A). The definition for **lower bound** and **infimum** (or greatest lower bound, or meet) are obtained dually.

For a subset $S \subseteq X$, the **down-set** of S denoted by $\downarrow S$ and the **up-set** of S denoted by $\uparrow S$ are defined by

$$\downarrow S := \{x \in X : x \leq y \text{ for some } y \in S\}$$

and

$$\uparrow S := \{x \in X : x \geq y \text{ for some } y \in S\},$$

respectively. Similarly, for $x \in X$ the down-set of $\{x\}$ denoted by $\downarrow x$ and the up-set of $\{x\}$ denoted by $\uparrow x$ are defined by

$$\downarrow x := \{y \in X : y \leq x \}$$

and

$$\uparrow x := \{x \in X : x \leq y \},$$

respectively. Down-set (up-set) $\downarrow x$ ($\uparrow x$) are called principal ideal (filter).
By [27], lemma 1.30, let X be an ordered set and $x, y \in X$, then the following are equivalent;

(i) $x \leq y$

(ii) $\downarrow x \subseteq \downarrow y$

(i) $\forall S \subseteq X,$ if $y \in \downarrow S$ implies $x \in \downarrow S$.

Definition 2.1.5. A surjective map $\varphi : A \to B$ between two ordered sets $(A; \leq)$ and $(B; \leq)$ is an order-isomorphism if:

$$\forall x, y \in A,\ x \leq y \Rightarrow \varphi x \leq \varphi y,$$

that is φ is order-preserving, and

$$\forall x, y \in A,\ x \leq y \Leftarrow \varphi x \leq \varphi y,$$

that is φ is order-embedding. In this case φ is necessary injective: using antisymmetry of $\leq$

$$\varphi x = \varphi y \Longleftrightarrow \varphi x \leq \varphi y \text{ and } \varphi x \geq \varphi y \Longleftrightarrow x \leq y \text{ and } x \geq y \Longleftrightarrow x = y.$$

As a result, every order isomorphism φ has a well-defined inverse φ^{-1}. It can easily be seen that φ^{-1} is also an order isomorphism.

Remark 2.1.6. A bijective order-preserving map is not necessarily an order isomorphism.

Definition 2.1.7. A map $\varphi : A \to B$ between two ordered sets is called order-reversing if φ defines an order-preserving map between A and B^{∂}. Formally, $\varphi : (A; \leq) \to (B; \leq)$ is order-reversing if for all $x, y \in A$, $x \leq y \Rightarrow \varphi x \geq \varphi y$.

Lattices

Definition 2.1.8. A partially ordered set $(X, \leq)$ is to be a **lattice ordered** (or lattice), if join and meet exist for any two elements of X. It is said to be **completely lattice ordered** (or a complete lattice) if join and meet exist for all subsets of X.

Alternatively, a lattice can be defined as an algebra. Or:

Definition 2.1.9. A lattice as an algebra $\mathcal{L} = (L, \vee, \wedge)$, is a set $L \neq \emptyset$ with two binary operations which satisfy the identities

$x \vee (y \vee z) = (x \vee y) \vee z$ $\quad$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associativity)
$x \vee y = y \vee x$ $\quad$ $x \wedge y = y \wedge x$ (commutativity)
$x \vee (x \wedge y) = x$ $\quad$ $x \wedge (x \vee y) = x$ (absorption)

Clearly the idempotence axiom follows from these other axioms.

The two definitions of a lattice are connected in the following way:

Proposition 2.1.10. *If the algebra $(L, \vee, \wedge)$ is a lattice, we define the relation $x \leq y$ iff $x \wedge y = x$. The relation $\leq$ is a partial order in which every pair of elements has an infimum and a supremum.*

On the other hand, let the partially ordered set $(L, \leq)$ be a lattice. Define $x \vee y$ as the supremum of $\{x, y\}$ and $x \wedge y$ as the infimum of $\{x, y\}$. In the algebra $(L, \vee, \wedge)$, the two binary operations satisfy the above identities for a lattice as an algebra, so it is a lattice.

If a lattice $\mathcal{L}$ contains both the least and the greatest element 0 and 1, respectively, it is called a **bounded** lattice, written as $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ or $\mathcal{L} = (L, \leq, 0, 1)$.

Proposition 2.1.11. *An ordered set in which infimum exists for all subsets is a complete lattice.*

Definition 2.1.12. Let $\mathcal{L}$ be a complete lattice. An element $a \in L$ is said to be **compact** if and only if for all $X \subseteq L$, if $a \leq \bigvee X$, then $a \leq \bigvee Y$ for some finite subset Y of X. Furthermore, $\mathcal{L}$ is said to be **algebraic** if and only if every element of L is the join of some compact elements in L.

Definition 2.1.13. Let a be an element of a lattice L, then a is said to be **distributive** if for every $x, y \in L$ the following holds

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \tag{2.1}$$

Dually, a is said to be **codistributive** if for every $x, y \in L$ the following holds

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y). \tag{2.2}$$

Definition 2.1.14. A lattice is said to be distributive if every element is both distributive and codistributive.

Proposition 2.1.15. *For an element a of a lattice $\mathcal{L}$, the following conditions are equivalent;*

1. *a is codistributive*
2. *the mapping $m_a : L \to \downarrow a$ defined by $m_a(x) := a \wedge x$ is a lattice homomorphism;*
3. *a binary relation θ_a defined by, $(x, y) \in \theta_a$ if and only if $a \wedge x = a \wedge y$, is a congruence relation on $\mathcal{L}$.*

Proof. ($1 \Longrightarrow 2$) Suppose a is codistributive. Let $x, y \in L$ and $m_a(x) = a \wedge x$, then $m_a(x \vee y) = a \wedge (x \vee y)$. Since a is codistributive it follows that

$$m_a(x \vee y) = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = m_a(x) \vee m_a(y)$$

and so $m_a(x \vee y) = m_a(x) \vee m_a(y)$ (join-homomorphism) . Dually,

$$m_a(x \wedge y) = a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) = m_a(x) \wedge m_a(y)$$

(meet-homomorphism) holds. Therefore the mapping $m_a : \mathcal{L} \to \downarrow a$ is a lattice homomorphism.
($2 \Longrightarrow 3$) Suppose the mapping $m_a : \mathcal{L} \to \downarrow a$ is a lattice homomorphism. Let $x, y \in L$, $(x, y) \in \theta_a$, and $m_a(x) = a \wedge x$ and $m_a(y) = a \wedge y$. Therefore $m_a(x) \wedge m_a(y) = a \wedge x$ since $a \wedge x = a \wedge y$ and $m_a(x) \vee m_a(y) = a \wedge x$ since $a \wedge x = a \wedge y$, hence

$$\begin{aligned}
m_a(x) \wedge m_a(y) &= m_a(x) \vee m_a(y) \Rightarrow m_a(x \wedge y) \\
&= m_a(x \vee y) \text{ since } m_a \text{ is a homomorpism} \\
&\Rightarrow a \wedge (x \wedge y) = a \wedge (x \vee y) \\
&\text{then } \quad (x, y) \in \theta_a \\
&\Rightarrow ((x \wedge y), (x \vee y)) \in \theta_a
\end{aligned}$$

($3 \Longrightarrow 1$) From the definition of congruence θ_a, it follows that $(x, (a \wedge x)) \in \theta_a$ and $(y, (a \wedge y)) \in \theta_a$ and following the relationship between the congruence and the operation $\vee$

$$(x \vee y)\theta_a((a \wedge x) \vee (a \wedge y))$$

and by the definition of θ_a, we have

$$a \wedge (x \vee y) = a \wedge ((a \wedge x) \vee (a \wedge y)) = (a \wedge x) \vee (a \wedge y).$$

Hence a is codistributive. $\square$

Dual statement holds for distributive element a where the homomorphism is given by $n_a : L \to \uparrow a$ defined by $n_a(x) := a \vee x$ and the proof is analogous.

Proposition 2.1.16. *Let a be a distributive element in a lattice $\mathcal{L}$. The lattice $\downarrow a$ is distributive if and only if each of its elements are codistributive in $\mathcal{L}$.*

Proof. ("*if part*") Let $b, c, d \in \downarrow a$. Then

$$\begin{aligned} (b \vee c) \wedge (b \vee d) = \quad & (((b \vee c) \wedge b) \vee ((b \vee c) \wedge d)) \text{ (since } b \vee c \in \downarrow a \text{ is codistributive)} \\ = \quad & ((b \vee c) \wedge b) \vee (b \wedge d) \vee (c \wedge d) \text{ (since } d \text{ is codistributive)} \\ = \quad & b \vee (c \wedge d) \text{ (using the absorption law twice) .} \end{aligned}$$

Therefore we have shown that each element of the ideal $\downarrow a$ is distributive.
("*only if part*") Let $b \in \downarrow a$ and $c, d \in L$. Then

$$\begin{aligned} b \wedge (c \vee d) = \quad & (b \wedge (c \vee d)) \wedge a \text{ (since } b \wedge (c \vee d) \leq b \leq a) \\ = \quad & (a \wedge (c \vee d)) \wedge b \\ = \quad & ((a \wedge c) \vee (a \wedge d)) \wedge b \text{ (since a is codistributive)} \\ = \quad & ((a \wedge c) \wedge b) \vee ((a \wedge d) \wedge b) \text{ (since } \downarrow a \text{ is distributive)} \\ = \quad & ((b \wedge c) \wedge a) \vee ((b \wedge d) \wedge a) \\ = \quad & (b \wedge c) \vee (b \wedge d). \end{aligned}$$

Therefore, b is a codistributive element in $\mathcal{L}$. $\square$

Lemma 2.1.17. *Let a be a distributive element in a lattice $\mathcal{L}$. Then for all $x, y \in L$, $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ if and only if*

$$x \leq y \Rightarrow x \vee (a \wedge y) = (x \vee a) \wedge y. \tag{2.3}$$

Proof. ("*if part*") Since $x \wedge y \leq x$ and by our hypothesis, it follows that

$$\begin{aligned}(x \wedge a) \vee (x \wedge y) &= ((x \wedge y) \vee a) \wedge x \quad \text{(by (2.3))}\\ &= (x \vee a) \wedge (y \vee a) \wedge x \quad \text{(by distributivity of } a)\\ &= x \wedge (y \vee a).\end{aligned}$$

("*only if part*") Suppose $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ and $x \leq y$, then

$$x \vee (a \wedge y) = (x \wedge y) \vee (a \wedge y) = (x \vee a) \wedge y.$$

□

For a codistributive element a in a lattice $\mathcal{L}$ and for a congruence block $[x]_{\theta_a}$ of an element $x \in L$, if the congruence block has a top element then we denote it by $\bar{x}$. Then for a distributive element a in a lattice $\mathcal{L}$ and for a congruence block $[x]_{\theta_a}$ of an element $x \in L$, if the congruence block has a bottom element then denote by $\underline{x}$

Lemma 2.1.18. *If a is a codistributive element of a lattice $\mathcal{L}$. Then the following conditions are equivalent for all $x, y \in L$:*

1. *if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ then $x = y$;*
2. *if $x \leq y$, then $x \vee (a \wedge y) = (x \vee a) \wedge y$;*
3. *if $x \leq \bar{y}$, then $x \vee (a \wedge \bar{y}) = (x \vee a) \wedge \bar{y}$;*
4. *$x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y)$.*

Proof. Proved in ([98]) and ([97]). □

Lemma 2.1.19. *If a is a codistributive element of a lattice $\mathcal{L}$ satisfying any of the properties listed in lemma (2.1.18) . Then for all $x, y \in L$,*

$$(x \wedge a) \vee (x \wedge y) = ((x \wedge a) \vee y) \wedge x \tag{2.4}$$

if and only if a is a distributive element of lattice $\mathcal{L}$.

Proof. (”*if part*”) We shall use condition (2) of lemma (2.1.18).

$$\begin{aligned}
a \vee (x \wedge y) &= a \vee (x \wedge a) \vee (x \wedge a) \\
&= a \vee (((x \wedge a) \vee y) \wedge x) \quad \text{(by equation (2.4))} \\
&= a \vee (((x \wedge a) \vee (y \vee a) \vee y) \wedge x) \\
&= a \vee ((((x \vee y) \wedge a) \vee y) \wedge x) \\
&= a \vee ((y \vee a) \wedge (x \vee y) \wedge x) \quad \text{(by (2) of lemma (2.1.18))} \\
&= a \vee (x \wedge (a \vee y)) = ((a \vee y) \wedge a) \vee (x \wedge (a \vee y)) \\
&= (a \vee x) \wedge (a \vee y) \quad \text{(by equation (2.4))}.
\end{aligned}$$

(”*only if part*”)

$$\begin{aligned}
(x \wedge a) \vee (x \wedge y) \vee a = \quad & a \vee (x \wedge y); \\
(((x \wedge a) \vee y) \wedge x) \vee a &= (a \vee x) \wedge (a \vee y \vee (a \wedge x)) \\
&= (a \vee x) \wedge (a \vee y) = a \vee (x \wedge y); \\
((x \wedge a) \vee y) \wedge x \wedge a &= x \wedge a. \\
\text{since:} \quad ((x \wedge a) \vee (x \wedge y) \vee a) &= (((x \wedge a) \vee y) \wedge x) \vee a) \\
\text{and} \quad ((x \wedge a) \vee (x \wedge y) \wedge a) &= ((x \wedge a) \vee y) \wedge x \wedge a, \\
\text{it follows that:} \quad (x \wedge a) \vee (x \wedge y) &= (x \wedge a) \vee y) \wedge x \qquad .
\end{aligned}$$

□

2.1.3 Closure systems and Closure operators

Definition 2.1.20. Let X be a set. A closure system $\mathcal{F}$ on X is a set of subsets such that

$$\forall Y \subseteq \mathcal{F} \text{ then } \bigcap Y \in \mathcal{F}.$$

Since we consider intersection of the empty family then it follows that the whole set X, is an element of $\mathcal{F}$.
Some examples of closure systems are the power set of sets, the system of substructures of algebraic structures like groups, and the set of all down-sets, as well as the set of all up-sets of ordered sets.

Definition 2.1.21. A closure operator on a set X is a map $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ that assigns to each subset of Y of X it closure $\varphi Y \subseteq X$ such that the following conditions hold

i . Monotonicity. $Y \subseteq Z \Rightarrow \varphi Y \subseteq \varphi Z$

ii . Extensivity. $Y \subseteq \varphi Y$

iii . Idempotence. $\varphi Y = \varphi\varphi Y$.

The next theorem shows the close relationship that exist between a closure system and a closure operator.

Theorem 2.1.22. *If $\mathcal{F}$ is a closure system on X then*

$$\varphi_{\mathcal{F}} Y := \bigcap \{A \in \mathcal{F} : Y \subseteq A\} \tag{2.5}$$

defines a closure operator on X. Conversely, the set

$$\mathcal{F}_{\varphi} := \{\varphi Y = Y : Y \subseteq X\} \tag{2.6}$$

of all closed sets under a closure operator φ is always a closure system.

Since every closure system $\mathcal{F}$ can be understood as the set of all closed sets of a closure operator, the elements of $\mathcal{F}$ are called closed sets as well.

Proposition 2.1.23. *Let $\mathcal{F}$ be a closure system on X, then $(\mathcal{F}, \subseteq)$ is a complete lattice, with*

$$\bigwedge \mathcal{C} = \bigcap \mathcal{C} \text{ and } \bigvee \mathcal{C} = \varphi_{\mathcal{F}}(\bigcup \mathcal{C}).$$

for all $\mathcal{C} \subseteq \mathcal{F}$. Conversely every complete lattice is isomorphic to the lattice of some closure system.

Proof. Obviously, $\bigcap \mathcal{C}$ is the infimum of $\mathcal{C}$. By proposition (2.1.11) we can find the supremum of a set as a result of taking the infimum of the upper bounds. Therefore,

$$\begin{aligned}
\bigvee \mathcal{C} &= \bigwedge \{B \in \mathcal{F} : B \text{ is an upper bound of } \mathcal{C}\} \\
&= \bigwedge \{B \in \mathcal{F} : \bigcup \mathcal{C} \subseteq B\} \\
&= \bigcap \{B \in \mathcal{F} : \bigcup \mathcal{C} \subseteq B\} \\
&= \varphi_{\mathcal{F}}(\bigcup \mathcal{C}).
\end{aligned}$$

Let L be a complete lattice then the set $\{\downarrow x : x \in L\}$ is a closure system, since $\bigcap_{x \in T} \downarrow x = \downarrow(\bigwedge T)$ for all $T \subseteq L$. Also, $L = \downarrow(\bigvee L)$. The order-isomorphism map between $(L; \leq)$ and $\{\downarrow x : x \in L\}$ is defined by $x \mapsto \downarrow x$. $\square$

2.1.4 Universal Algebras

Subalgebras, homomorphisms and direct products

Formation of new algebras from existing ones is an integral part of universal algebra. Most fundamental are the formation of subalgebras, homomorphic images, and direct products.

Definition 2.1.24. A type or an algebraic language is a pair $\tau = (\mathbb{F}, \alpha)$, where $\mathbb{F}$ is a set, and $\alpha : \mathbb{F} \to \mathbb{N}$ is a function from $\mathbb{F}$ to the set of natural numbers. Each $f \in \mathbb{F}$ is said to be a (basic) operation symbol and $\alpha(f)$ is its arity.

The set of all operation symbols of arity n will be denoted by $\mathbb{F}_n$. If $\alpha(f) = \{1, 2, 3, ...\}$, then f is said to be a unary, binary, ternary and so on, operation symbol. If $\alpha(f) = 0$, then f is said to be a constant symbol, which is defined as a 0-ary (nullary) operation.

Definition 2.1.25. An algebra of type τ is a pair $\mathcal{A} = (A, \mathbb{F})$, where $A \neq \emptyset$ is called the universe of $\mathcal{A}$ and $\mathbb{F}$ is a family of finitary operation on A indexed by the language τ, such that each $f \in \mathbb{F}$ corresponds to an n-ary operation A. In the case when there is a need to distinguish between algebras, $f^{\mathcal{A}}$ is used instead.

Remark 2.1.26. If $\mathbb{F} = \{f_0, f_1, ..., f_m\}$, then $\mathbf{A} = (A, f_0, f_1, ..., f_m)$ is called an algebraic system (or simply an algebra). The arity of each operation is understood from context.

An algebra is said to be trivial if its universe is a singleton set and it is said to be finite or infinite if its universe is finite or infinite.

Definition 2.1.27. Let f be an n-ary operation on a nonempty set A and let $B \subseteq A$. Then B is said to be closed with respect to f (or preserve f) if and only if $f(a_1, ..., a_n) \in B$ for all $a_1, ..., a_n \in B$.
If f is a constant operation, B is closed with respect to f if and only if $f \in B$.

Definition 2.1.28. Let $\mathcal{A}$ be an algebra. A **subuniverse** of $\mathcal{A}$ is a subset of the universe A of $\mathcal{A}$ which is closed with respect to every fundamental operations f on $\mathcal{A}$. Therefore, an algebra $\mathcal{B}$ is a **subalgebra** of an algebra $\mathcal{A}$, if $\mathcal{B}$ is of the same type as $\mathcal{A}$, B the universe of $\mathcal{B}$ is a nonempty subuniverse of $\mathcal{A}$ and for every n-ary operation $f \in \mathbb{F}$ of $\mathcal{A}$

$$f^{\mathcal{B}}(a_1, ..., a_n) = f^{\mathcal{A}}(a_1, ..., a_n)$$

for all $a_1, ..., a_n \in B$. Then $f^{\mathcal{B}}$ is a called a restriction of $f^{\mathcal{A}}$ to B.

Theorem 2.1.29. *Let $\mathcal{A}$ be an algebra and X a nonempty collection of subuniverses of $\mathcal{A}$. Then $\bigcap X$ is a subuniverse of $\mathcal{A}$.*

A homomorphism of two algebras which are of the same type can be understood as a mapping which is compatible with all the corresponding operations of the algebras.

Definition 2.1.30. Let $\mathcal{A} = (A; f_0^{\mathcal{A}}, f_1^{\mathcal{A}}, ..., f_m^{\mathcal{A}})$ and $\mathcal{B} = (B; f_0^{\mathcal{B}}, f_1^{\mathcal{B}}, ..., f_m^{\mathcal{B}})$ be two algebraic systems of the same type τ, where both $f_i^{\mathcal{A}}$ and $g_i^{\mathcal{A}}$ are n_i-ary operations $(i = 0, 1, ..., n)$. If there exists a mapping $\phi : A \to B$ such that $\forall a_1, ..., a_{n_i} \in A$,

$$\phi(f_i^{\mathcal{A}}(a_1, ..., a_{n_i})) = f_i^{\mathcal{B}}(\phi(a_1), ..., \phi(a_{n_i}))$$

then the algebraic systems $\mathbf{A} = (A; f_0^{\mathcal{A}}, f_1^{\mathcal{A}}, ..., f_m^{\mathcal{A}})$ and $\mathbf{B} = (B; f_0^{\mathcal{B}}, f_1^{\mathcal{B}}, ..., f_m^{\mathcal{B}})$ are **homomorphic** and ϕ is called a **homomorphism**.
If ϕ is injective, then ϕ is called an **embedding** of $\mathbf{A} = (A; f_0^{\mathcal{A}}, f_1^{\mathcal{A}}, ..., f_m^{\mathcal{A}})$ into $\mathbf{B} = (B; f_0^{\mathcal{B}}, f_1^{\mathcal{B}}, ..., f_m^{\mathcal{B}})$, and if ϕ is surjective, then $\mathcal{B}$ is called a **homomorphic image** of $\mathcal{A}$ under ϕ. If ϕ is both injective and surjective, then ϕ is a **isomorphism** from $\mathcal{A} = (A; f_0^{\mathcal{A}}, f_1^{\mathcal{A}}, ..., f_m^{\mathcal{A}})$ onto $\mathcal{B} = (B; f_0^{\mathcal{B}}, f_1^{\mathcal{B}}, ..., f_m^{\mathcal{B}})$ and the algebras are said to be **isomorphic**.

It is easy to see that the image $B_i = \phi(A_i)$ of a subuniverse of A_i of $\mathcal{A}$ is a subuniverse of $\mathcal{B}$ and the preimage $\phi^{-1}(B_i) = A_i$ of a subuniverse B_i of the homomorphic image $\phi(\mathcal{A})$ of $\mathcal{A}$ is a subuniverse of $\mathcal{A}$ under the homomorphism $\phi : A \to B$.

Definition 2.1.31. Let $\{\mathcal{A}_i : i \in \mathcal{I}\}$ be a family of algebras of the same type. Its **direct product** is the algebra $\mathcal{A}$ whose universe is the cartesian product $\Pi_{i\in\mathcal{I}}A_i$ of the universes of the algebras $\mathcal{A}_i$, while if f_i is an n_i-ary operation symbol in $\mathbb{F}$ and $a_1, a_2, ..., a_{n_i} \in \mathcal{A}$ then

$$f_i^{\mathcal{A}}(a_1, ..., a_{n_i})(j) = f_i^{\mathcal{A}}(a_1(j), ..., a_{n_i}(j)).$$

The operations on $\mathcal{A}$ are done coordinatewise and this algebra, $\mathcal{A}$ is usually denoted by $\Pi_{i\in\mathcal{I}}\mathcal{A}_i$. The empty product $\Pi\emptyset$ is the trivial algebra with the universe $\{\emptyset\} = A^0$. For each $j \in \mathcal{I}$ there is an associated homormophism

$$p_j : \Pi_{i\in\mathcal{I}}\mathcal{A}_i \to \mathcal{A}_j$$

given by $p_j(a) = a(j)$ and called the **projection map** on the j^{th} component

Congruences and quotient algebras

We recall that an equivalence relation on a set is any binary relation which is reflexive, symmetric and transitive, and as such that every equivalence relation on a set determines a partition of the set into mutually exclusive and jointly exhaustive subsets, called equivalence classes of the equivalence relation.

Definition 2.1.32. A binary relation θ on an algebra $\mathcal{A}$ (i.e $\theta \subseteq \mathcal{A}^2$) is called a **congruence** relation on $\mathcal{A}$ if θ is an equivalence relation on $\mathcal{A}$ which is **compatible** with the basic operations in $\mathcal{A}$. That is, for any $f_i \in \mathbb{F}$, $a_j, b_j \in \mathcal{A}$,

$$a_j \theta b_j \ , \ j = 1, ..., n_i \Longrightarrow (f_i^{\mathcal{A}}(a_1, ..., a_{n_i}), f_i^{\mathcal{B}}(b_1, ..., b_{n_i})) \in \theta.$$

Hence every kernel of a homomorphism from $\mathcal{A}$ to some other algebra is a congruence on $\mathcal{A}$

For example the relation of equality $\Delta = \{(a, a) : a \in A\}$ and the universal relation $\nabla = \{(a, b) : a, b \in A\} = A^2$ are both congruences in $\mathcal{A}$. The set of all congruence on $\mathcal{A}$ will be denoted by $Con\mathcal{A}$ and this set forms an algebraic lattice under inclusion.

Definition 2.1.33. Let $\theta \in ConA$. For $a \in \mathcal{A}$, let $a/\theta = \{b \in A : a\theta b\}$ be the congruence class of θ containing a. Then A/θ is the quotient set of all such classes. On A/θ is defined as the natural structure $\mathcal{A}/\theta$ of the algebra of type τ in the following way; for $f_i \in \mathbb{F}$, $a_1, .., a_{n_i} \in \mathcal{A}$

$$f_i^{\mathcal{A}/\theta}(a_1/\theta, ..., a_{n_i}/\theta) = f_i^{\mathcal{A}}(a_1, ..., a_{n_i})/\theta.$$

This is well defined since the congruence class $f_i^{\mathcal{A}/\theta}(a_1/\theta, ..., a_{n_i}/\theta)$ is independent of the choice of representatives $a_1, ..., a_{n_i}$ of the classes $(a_1/\theta, ..., a_{n_i}/\theta)$. Hence, we have a new algebraic system $\mathcal{A}/\theta$= $(A/\theta, f_1^{\mathcal{A}/\theta}, ..., f_m^{\mathcal{A}/\theta})$ called the quotient algebraic system of $\mathcal{A} = (A; f_0^{\mathcal{A}}, f_1^{\mathcal{A}}, ..., f_m^{\mathcal{A}})$ factored by the congruence θ.

Definition 2.1.34. Let $\phi : A \to B$ be a homomorphism from the algebra $\mathcal{A}$ into $\mathcal{B}$. Then the kernel of ϕ, denoted by $\ker(\phi)$ is defined by

$$\ker(\phi) = \{(a, b) \in A^2 : \phi(a) = \phi(a)\}$$

is the equivalence relation. The fact that ϕ is compatible with the basic operations of $\mathcal{A}$ and $\mathcal{B}$ carries over to the kernel. That is, if $(a_j, b_j) \in ker(\phi)$ for $j \leq n_i$ and f_i an n_i-ary operation on $\mathcal{A}$, then

$$(f_i^{\mathcal{A}}(a_1, ..., a_{n_i}); f_i^{\mathcal{B}}(b_1, ..., b_{n_i})) \in ker(\phi).$$

Thus, every homomorphism on an algebra determines a congruence on the algebra. On the other hand, every congruence relation θ on an algebra, $\mathcal{A}$ determines a homomorphism

$$\pi : A \to A/\theta$$

called the **canonical homomorphism** which maps algebra $\mathcal{A}$ into its quotient algebra $\mathcal{A}/\theta$. This is defined by $\pi(a) = a/\theta$.

The following theorem is a consequence of the above observation.

Theorem 2.1.35. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type. Let ϕ be a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ and θ a congruence relation on $\mathcal{A}$. Then*

1. *the mapping $\pi : A \to A/\theta$ defined by $\pi(a) = a/\theta$ is an epimorphism from $\mathcal{A}$ onto $\mathcal{A}/\theta$ having its kernel to be θ*

2. *let $\theta = ker\pi$, then there a unique isomorphism $\hat{\phi}$ between the quotient algebra $\mathcal{A}/\theta$ and the homomorphic image $\mathcal{B}$ of ϕ, such that $\phi = \hat{\phi} \circ \pi$.*

Definition 2.1.36. A relation θ on an algebra, $\mathcal{A}$ is a **weak congruence relation** on $\mathcal{A}$ if θ is **symmetric**, **transitive** and **compatible** with all the operations defined on $\mathcal{A}$ including nullary operations.
We denote the set of all weak congruence relations on $\mathcal{A}$ by $\mathcal{C}w\mathcal{A}$.

The empty set $\emptyset$ by definition is also a weak congruence if and only if there are no nullary operations in F.

Definition 2.1.37. A **subdirect product** of an indexed family $\{\mathcal{A}_i : i \in \mathcal{I}\}$ of algebras is a subalgebra $\mathcal{B}$ of the direct product $\Pi_{i\in\mathcal{I}}A_i$ such that for all $j \in \mathcal{I}$, $p_j(\mathcal{B}) = \mathcal{A}_j$.

A subdirect representation of $\mathcal{B}$ by the family $\{\mathcal{A}_i : i \in \mathcal{I}\}$ of algebras is an embedding $\phi : \mathcal{B} \to \Pi_{i\in\mathcal{I}}\mathcal{A}_i$ such that $\phi(\mathcal{B})$ is a subdirect product of the $\mathcal{A}_i's$.

An algebra is **subdirectly irreducible** if $\mid A \mid > 1$ and for every subdirect embedding $f : \mathcal{A} \to \Pi_{i\in\mathcal{I}}\mathcal{A}_i$ there is some $i \in \mathcal{I}$ such that $p_i \circ f : \mathcal{A} \to \mathcal{A}_i$ is an isomorphism.

Free Algebras, Varieties and Birkhoff's Theorem

Here we introduce notions of equations, algebras of terms and free algebras, which are essential tools of universal algebra and its applications.

Definition 2.1.38. The set of terms of type τ in the variable X is the smallest collection $T(X)$ of the finite strings such that

1. $X \cup \mathbb{F}_0 \subseteq T(X)$
2. if $t_1, ..., t_n \in T(X)$ and f an n-ary operation then the string $f(t_1, ..., t_n) \in T(X)$.

Definition 2.1.39. A **term algebra** of type τ in the variable X is the algebra $\mathcal{T}(X)$ of type τ with universe $T(X)$, such that

$$f^{\mathcal{T}(X)}(t_1, ..., t_n) = f(t_1, t_2, ..., t_n)$$

for all $f(t_1, ..., t_n) \in T(X)$ and n-ary operations f of type τ.

Definition 2.1.40. Let $t = t(x_1, ..., x_n)$ be a term of type τ over some set X and $\mathcal{A}$ be any algebra of the same type. $t^{\mathcal{A}}$ is an n-ary **term operation** defined on $\mathcal{A}$ by the mapping $t^{\mathcal{A}} : A^n \to A$ and for $a_1, ..., a_n \in A$, as follows;

1. if t is the variable x_i then $x_i^{\mathcal{A}}(a_1, ..., a_n) = a_i$,
2. if t is of the form $f(t_1, ..., t_n), f \in \mathbb{F}_n$ then

$$f^{\mathcal{A}}(t_1^{\mathcal{A}}, ..., t_n^{\mathcal{A}})(a_1, ...a_n) = f^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, ...a_n), t_2^{\mathcal{A}}(a_1, ...a_n), ..., t_n^{\mathcal{A}}(a_1, ...a_n)).$$

A formal equation of type τ is an ordered pair of terms both of which are from the same free algebra. Formal equations are written in the form

$$s(x_1, ..., x_n) \approx t(x_1, ..., x_n).$$

Such an equation is called an identity of an algebra $\mathcal{A}$ of type τ iff

$$s^{\mathcal{A}}(a_1, ..., a_n) = t^{\mathcal{A}}(a_1, ..., a_n), \quad \text{for all } a_1, ..., a_n \in A$$

This is usually denoted by

$$\mathcal{A} \models s \approx t.$$

If Σ is any set of equations of type τ, the class of all algebras of type τ satisfying all the equations in Σ is denoted by $Mod(\Sigma)$. This class is called an **equational class**, axiomatized by Σ. On the other hand, for any class $\mathcal{K}$ of the same type of algebras, we define $\Theta(\mathcal{K})$ to be the set of all equations that hold true in every algebra in $\mathcal{K}$ over a fixed countably infinite set X of variables. A set of equations of the form $\Theta(\mathcal{K})$ for some $\mathcal{K}$ is called an **equational theory**.

A very important part of universal algebra is the study of classes of algebras of the same type.

Definition 2.1.41. A nonempty class of algebras $\mathcal{K}$ of the same type τ is called a **variety** if it is closed under subalgebras, homomorphic image, and direct products.

The next theorem by Birkhoff (1935) shows the equivalence of the definitions.

Theorem 2.1.42. *(Birkhoff's theorem) Let $\mathcal{K}$ be a class of algebras of the same type. Then $\mathcal{K}$ is a variety if and only if $\mathcal{K}$ is an equational class.*

2.2 Fuzzy Sets

In this section, we focus on the introduction of fundamentals in fuzzy set theory, including some set-theoretic operations and their extensions.

A fuzzy set is a concept that forms in essence the generalization of the crisp set. In the previous section we presented the notion of a characteristic function which assigns value of either 0 or 1 to individual elements of a universe denoting their membership and non-membership of the (crisp) set under consideration. The generalization of this function will be presented here, where values assigned to elements of a universe run through specified range and indicate the membership of these elements in the set under consideration. Functions of this type are referred to as *membership functions*, and the set they define are called *fuzzy sets*. Formally, fuzzy notion as defined by Zadeh ([109]) is given as;

Definition 2.2.1. Let X be a classical set of objects, called the universe, whose generic elements are denoted x. A **fuzzy set** A in X is characterized by a membership function

$$\mu_A : X \to [0,1]$$

which associates with each point in X a real number in the unit interval $[0,1]$, with the values of $\mu_A(x)$ at x representing the **degree of membership** of x in A.

Therefore, the closer the value of μ_A is to 1, the more x belongs to A. Clearly, A is a subset of X that has no sharp boundary. Hence a fuzzy set A of the set X is completely characterized by the set of pairs:

$$A = \{(x, \mu_A(x)) : x \in X\} \tag{2.7}$$

It is clear that if $\mu(A) = \{0,1\} \subset [0,1]$, then μ is the (classical) characteristic function on A (or equivalently, a classical subset of X).

Another notation widely employed in literature to denote membership functions is given by

$$A : X \to [0,1].$$

No confusion arises in the use of both notations. The first notation simply made a distinction between the symbol for the fuzzy set A and its membership function μ_A, while the second notation did not make such distinction and there is no ambiguity in the use of same notation for both the set and the function.

Definition 2.2.2. Let A and B be $[0,1]$-valued functions defined on a fixed set X (i.e., fuzzy sets in X).Then

$$A \leq B \text{ means that } A(x) \leq B(x), \forall x \in X \tag{2.8}$$

and

$$A = B \text{ means that } A(x) = B(x), \forall x \in X \tag{2.9}$$

The maximum function $A \vee B$, the minimum function $A \wedge B$ and the complement function $1 - A$ defined on the set X by the rules

$$(A \vee B)(x) = max\{A(x), B(x)\}$$
$$(A \wedge B)(x) = min\{A(x), B(x)\}$$
$$(1 - A)(x) = 1 - A(x),$$

respectively are $[0,1]$-valued functions.
Thus $A \vee B$, $A \wedge B$ and $1 - A$ are fuzzy sets on X if A and B are.

Example 2.2.3.

1. Let $X = \mathbb{R}$, the set of real numbers. Let A be the set of all real numbers close to 10. Then this notion can be modeled by the fuzzy set defined by

$$A(x) = \frac{1}{1 + [\frac{1}{5}(x - 10)]^2}$$

2. Let X be the set of all books, and let A be the set of thick books. Then a fuzzy set can be defined by giving the following membership grade function: if x is the number of pages in the book, then

$$A(x) = \begin{cases} 0 & if\ x < 350 \\ (x-350)/150 & if\ 350 \le x \le 500 \\ 1 & if\ x > 500 \end{cases} .$$

Definition 2.2.4. The support of a fuzzy set A, $SuppA$, is the crisp set of all $x \in X$ such that $A(x) > 0$.
The height of A is $hgt(A) = sup_{x \in X} A(x)$, i.e., the least upper bound of $A(x)$.
A fuzzy subset A of a classical set X is called normal if there exists an $x \in X$ such that $A(x) = 1$. Otherwise, A is subnormal.

Definition 2.2.5. The (crisp) set of elements that belong to the fuzzy set A, at least to the degree α, is called the α-cut set: $A_\alpha = \{x \in X : A(x) \ge \alpha\}$.

Definition 2.2.6. A fuzzy singleton, is a fuzzy set with a membership function that is unity at a single particular point on the universe of discourse and zero everywhere else.

Definition 2.2.7. A fuzzy relation R on a set X is a fuzzy set on the direct product

$$X \times X = \{((x, y) \mid x, y \in X\} \tag{2.10}$$

and characterized by the membership function

$$R : X \times X \to [0, 1].$$

Therefore, R is said to be

- Reflexive: If $\forall x \in X,\ \ R(x, x) = 1$,
- Strict (weakly reflexive): If $\forall x, y \in X,\ \ R(x, x) \ge R(x, y)$ and $R(x, x) \ge R(y, x)$,
- Symmetric: If $\forall x, y \in X,\ \ R(x, y) = R(y, x)$,
- Transitive: If $\forall x, y, z \in X,\ \ min(R(x, y), R(y, z)) \le R(x, z)$, or equivalently, if $\underset{y \in X}{max}\ min(R(x, y), R(y, z)) \le R(x, z)$,
- Antisymmetric: If $\forall x, y \in X,\ \ min(R(x, y), R(y, x)) = 0,\ \ x \neq y$.

We will sometimes be using the operation "$\wedge$" in place of the operation, "min" to denote the notion of infimum on the $[0, 1]$ interval (similarly "$\vee$" in place of "max" to denote the notion of supremum on $[0, 1]$).

Strictness and weak reflexivity are equivalent notions, both appearing in the literature. Hence in the sequel we shall mostly use the term "strictness" or "strict fuzzy relation."

A reflexive and transitive fuzzy relation, R is called a fuzzy preorder, thus the pair (X, R) is called a fuzzy-preordered set. Moreover, a fuzzy preordered relation R which is antisymmetric, is called a fuzzy order, thus the pair (X, R) is called a fuzzy-ordered set. A fuzzy relation R on X is a weak fuzzy ordering relation on X if it is strict, antisymmetric and transitive. A fuzzy relation R is said to be a fuzzy equivalence relation if it is reflexive, symmetric and transitive. A fuzzy equivalence relation is called fuzzy equality if $\forall x, y \in X$,

$$R(x, y) = 1, \text{ implies } x = y.$$

Definition 2.2.8. Let $\mu : X \to [0, 1]$ be a fuzzy set on X. Then μ is said to be extensional w.r.t. the fuzzy equality relation R on X if, and only if

$$\mu(x) \wedge R(x, y) \leq \mu(y).$$

Obviously, definition (2.2.8) generalizes a property of an ordinary equality relation, for which equal elements can be substituted into any formula without affection the truth value of the formula. Therefore a fuzzy equality relation as intended is to model gradual indistinguishability between elements.
The notion of extensionality was first defined by F. Klawonn and R Kruse in ([53])

The Extension Principle

The extension principle as introduced by Zadeh ([106]), turns out to be one of the most basic concept in fuzzy set theory for generalizing crisp mathematical concept to fuzzy concepts. Now we define the extension principle, which provides a natural way for extending the domain of a mapping.

Let X and Y be two non-empty sets and let f be a mapping from X to Y. Then f extends to a mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, where $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are the sets of all fuzzy subsets of X and Y respectively, in the following way: For each $\mu \in \mathcal{F}(X)$, then $f(\mu) \in \mathcal{F}(Y)$ is defined by

$$f(\mu)(y) = \begin{cases} \displaystyle\max_{y=f(x)} \mu(x) & if\ y \in f(X),\ y \in Y \\ 0 & if\ y \notin f(X) \end{cases}$$

$f(\mu)$ is the image of the fuzzy set μ under f. Conversely, given $\nu \in \mathcal{F}(Y)$, then $f^{-1}(\nu) \in \mathcal{F}(X)$ is defined by the equation

$$f^{-1}(\nu)(x) = \nu(f(x)) \text{ for } x \in X.$$

$f^{-1}(\nu)$ is the inverse image of the fuzzy set ν under f.
The above notion of an extension of a fuzzy set can be generalized. Let $X = X_1 \times \cdots \times X_n$ and μ_i be a fuzzy set on X_i for each $i = 1, 2, \cdots, n$. Then let $\mu = \mu_1 \times \cdots \times \mu_n$, defined by

$$\mu(x) := min\{\mu_1(x_1), \cdots, \mu_n(x_n)\},$$

where $x = (x_1, \cdots, x_n)$. Let f be a mapping from X into Y, then for each $\mu \in \mathcal{F}(X)$, $f(\mu) \in \mathcal{F}(Y)$ is defined by

$$f(\mu)(y) = \begin{cases} \max\limits_{y=f(x_1,\cdots,x_n)} [min\ (\mu_1(x_1), \cdots \mu_n(x_n))] & if\ y \in f(x_1, \cdots, x_n),\ y \in Y \\ 0 & if\ y \notin f(X). \end{cases} \tag{2.11}$$

The extension principle can also be defined through a fuzzy relation. Let μ be a fuzzy set on a crisp set X, Y a crisp set also and R a fuzzy relation on $X \times Y$. Then for a function $f : X \to Y$, we define a fuzzy set $f(\mu)$ on Y as

$$f(\mu)(y) := \max_{y=f(x)} [min(\mu(x), R(x, y))],$$

for $y \in Y$.

Fuzzy subalgebras and congruences

In this section we introduce the notion of fuzzy subalgebras of an algebra of a given type.

Definition 2.2.9. Let $\mathcal{A} = (A, \mathbb{F})$ be an algebra. Then a fuzzy subset $\mu : A \to [0, 1]$ is a fuzzy subalgebra of $\mathcal{A}$ if the following are satisfied

1. $\mu(e) = 1$ for every nullary operation in $\mathbb{F}$
2. for every n-ary operation $f^{\mathcal{A}} \in \mathbb{F}$ for $n > 0$ and for every $a_1, ..., a_n \in A$ then

$$\mu(a_1) \wedge ... \wedge \mu(a_n) \leq \mu(f^{\mathcal{A}}(a_1, ..., a_n))$$

Theorem 2.2.10. *Let $\mathcal{A}$ be an algebra. For any subset of B of A, B is a subuniverse in $\mathcal{A}$ if and only if the characteristic function χ_B is a fuzzy subalgebra of $\mathcal{A}$.*

Proof. Suppose that χ_B is a fuzzy subalgebra of $\mathcal{A}$, then for each nullary operation e on $\mathcal{A}$, it is clear that $e \in B$ since $\chi_B(e) = 1$. Next we show that for each n-ary operation $f^{\mathcal{A}}$ on $\mathcal{A}$, for $n > 0$ and $b_1, ..., b_n \in B$ then $f^{\mathcal{A}}(b_1, ..., b_n) \in B$. Therefore, $\chi_B(b_i) = 1,$ for each $i = 1, 2, ..., n$, hence

$$\begin{aligned} 1 = \bigwedge_{i=1}^{n} \chi_B(b_i) &\leq \chi_B(f^{\mathcal{A}}(b_1, ..., b_n)) \\ &\Rightarrow 1 = \chi_B(f^{\mathcal{A}}(b_1, ..., b_n)) \\ &\Rightarrow f^{\mathcal{A}}(b_1, ..., b_n) \in B. \end{aligned}$$

Therefore, B is a subuniverse of A.
Conversely, suppose B is a (classical) subalgebra of $\mathcal{A}$ and χ_B is a characteristic function, then $b \in B \Rightarrow \chi_B(b) = 1$ and $b \notin B \Rightarrow \chi_B(b) = 0$. By assumption for each nullary operation e on $\mathcal{A}$, $e \in B$ and $\chi_B(e) = 1$. Next for each n-ary operation $f^{\mathcal{A}}$ on $\mathcal{A}$, for $n > 0$, let $a_1, ..., a_n \in A$. Now for $a_i \notin B$ for some i then $\chi_B(a_i) = 0$ thus

$$0 = \bigwedge_{i=1}^{n} \chi_B(a_i) \leq \chi_B(f^{\mathcal{A}}(a_1, ..., a_n)).$$

For the case where $a_i \in B, \forall i$ then $\chi_B(a_i) = 1$ thus

$$\bigwedge_{i=1}^{n} \chi_B(a_i) = 1 = \chi_B(f^{\mathcal{A}}(a_1, ..., a_n)).$$

Hence χ_B is a fuzzy subalgebra of $\mathcal{A}$. □

Next we characterize fuzzy subalgebras by their α-cuts.

Theorem 2.2.11. *Let $\mathcal{A}$ be an algebra and $\mu : A \to [0, 1]$ be a fuzzy subset of A. Then μ is a fuzzy subalgebra of $\mathcal{A}$ if and only if for each $\alpha \in [0, 1]$ each α-cut μ_α is a crisp subalgebra of $\mathcal{A}$.*

Proof. Suppose μ is a fuzzy subalgebra of $\mathcal{A}$, and α be any arbitrary element in $[0, 1]$, then for each nullary operation e, $\mu(e) = 1 \geq \alpha$ and by definition (2.2.5) $e \in \mu_\alpha$. Next for each n-ary operation $f^{\mathcal{A}}$ on $\mathcal{A}$, for $n > 0$ and

$a_1, ..., a_n \in \mu_\alpha$, then $\mu(a_i) \geq \alpha$ for each $i = 1, ..., n$, therefore

$$\begin{aligned}\alpha \leq \mu(a_1) \wedge ... \wedge \mu(a_n) &\leq \mu(f^{\mathcal{A}}(a_1, ..., a_n)) \\ &\Rightarrow \alpha \leq \mu(f^{\mathcal{A}}(a_1, ..., a_n)) \Rightarrow f^{\mathcal{A}}(a_1, ..., a_n) \in \mu_\alpha.\end{aligned}$$

Then the α-cut μ_α is a crisp subalgebra of $\mathcal{A}$ for each $\alpha \in [0, 1]$.
conversely, assume that μ_α is a crisp subalgebra of $\mathcal{A}$ for each $\alpha \in [0, 1]$. Indeed for $\alpha = 1$ and for all nullary operations e on $\mathcal{A}$, μ_1 is a crisp subalgebra of $\mathcal{A}$ with $e \in \mu_1$ for all nullary operations. Hence it follows that $\mu(e) = 1$ for all nullary operations. Next for each n-ary operation $f^{\mathcal{A}}$ on $\mathcal{A}$, for $n > 0$ and $a_1, ..., a_n \in M$, let $\alpha = \mu(a_1) \wedge ... \wedge \mu(a_n)$ hence $\alpha \leq \mu(a_i)$ and then $a_i \in \mu_\alpha$ for each $i = 1, ..., n$. Since μ_α is a crisp subalgebra of $\mathcal{A}$ then $f^{\mathcal{A}}(a_1, ..., a_n) \in \mu_\alpha$. Then it show that

$$\mu(a_1) \wedge \cdots \wedge \mu(a_n) = \alpha \leq \mu(f^{\mathcal{A}}(a_1, ..., a_n)).$$

Hence μ is a fuzzy subalgebra of $\mathcal{A}$. □

Definition 2.2.12. Let $\mathcal{A} = (A, F)$ be an algebra and R be a fuzzy equivalence relation on A. Then R is said to be a fuzzy congruence on $\mathcal{A}$ if for $a_1, ..., a_n, b_1, ..., b_n \in A$ and for each operation symbol $f \in F$

$$R(a_1, b_1) \wedge ... \wedge R(a_n, b_n) \leq R(f^{\mathcal{A}}(a_1, ..., a_n), f^{\mathcal{A}}(b_1, ..., b_n))$$

Intuitively, fuzzy congruence relations are fuzzy equivalence relations that preserve algebraic structures. In another way, fuzzy congruence relations are fuzzy subalgebras of the product algebra $\mathcal{A} \times \mathcal{A}$ which are also fuzzy equivalence relations.

Theorem 2.2.13. *Let $\mathcal{A} = (A, F)$ be an algebra and R is a fuzzy congruence on $\mathcal{A}$ if and only if for each $\alpha \in [0, 1]$ the cut relation R_α is a classical congruence relation on $\mathcal{A}$.*

The proof of theorem (2.2.13) is analogous to the proof of theorem (2.2.11).

Chapter 3

L-valued structures

The generalizations of the fuzzy concept from Zadeh's notion of taking membership values from the unit interval $[0, 1]$ to an arbitrary set L that is at least an ordered set, are generally referred to as L-valued sets and their membership functions are given by

$$\mu : X \to L.$$

This generalization was introduced by Goguen ([40]). Generally, it is obvious that the unit interval is really not sufficient to have the truth values of general fuzzy statements. For example "University of Novi Sad is a good school", the value of this statement may not be an element in the unit interval since "being a good school" involves several components like; competency of staff, facilities presence, research works, graduates performance, etc., then the truth values corresponding to each component may be an element of the unit interval. Hence the truth value of our fuzzy statement "University of Novi Sad is a good school" is considered to be n-tuple and so an element from the n^{th} power of the unit interval. This then can be considered as a function "good school" from the set of schools in the world into the n^{th} power of the unit interval (i.e. $[0, 1]^n$). Of course $[0, 1]^n$ is not a linearly ordered set but a nonlinearly ordered set indeed a complete lattice. In view of the extensive work carried out by Šešelja and Tepavčević on cut sets ([89], [91], [92]) , which turns out to be a very important tool in our investigation can not be fully studied using the unit interval, since some very important notions concerning fuzziness (such as cut functions and their structure) depend only on the fact that the codomain of the membership function is a poset. Moreover, some algebraic methods, particularly from the lattice theory, can be successfully used in applications of fuzzy structures, provided that this codomain is some special poset or a lattice.

Main properties of fuzzy sets and structures are consequences of two basic facts: (a) they are functions, and (b) they can be uniquely represented by collections of cut subsets and substructures.

It is well known classically that the set 2^X which is the set of all mappings from a nonempty set X to the Boolean lattice of two elements is a Boolean lattice induced by the operations on the Boolean lattice $\{0,1\}$. Therefore, for a fuzzy mapping $\mu : X \to L$,

$$L^X = \{\mu \mid \mu : X \to L\},$$

is a collection of all fuzzy sets of X. L^X is a lattice induced by the lattice L under the ordering defined by:

$$\nu \leq \mu \text{ if and only if for each } \ x \in X \ \nu(x) \leq \mu(x).$$

3.1 Cut sets of an L-valued set

In this section L will be a complete lattice with bottom element 0 and top element 1.
For $p \in L$, a cut function of μ, is a mapping

$$\mu_p : X \to \{0,1\},$$

such that for $x \in X$, $\mu_p(x) = 1$ if and only if $\mu(x) \geq p$. Formally, this is defined below as the p-cut of μ.

Definition 3.1.1. Let $\mu : X \to L$ be an L-valued set on X, then for $p \in L$ the set defined by

$$\mu_p = \{x \in X : \mu(x) \geq p\}$$

is called the p-cut of μ.

Remark 3.1.2. A p-cut of μ is the inverse image of the principal filter of the lattice L induced by p,

$$\mu_p = \mu^{-1}(\uparrow p).$$

For an L-valued set $\mu : X \to L$, the set of images (functional values) denoted as $\mu(X)$ is defined by

$$\mu(X) := \{p \in L : \mu(x) = p, \text{ for some } x \in X\}.$$

Therefore, for an L-valued set $\mu : X \to L$, the following propositions highlights some of the properties of μ.

Proposition 3.1.3. *Let $\mu : X \to L$ be an L-valued set on X. Then for all $x \in X$ it holds that,*

$$\mu(x) = \bigvee \{p \in L : \mu_p(x) = 1\}$$

Proof. Let $x \in X$, such that $\mu(x) = q \in L$, then $\mu_q(x) = 1$. Thus if for any $p \in L$, $\mu_p(x) = 1$ then $\mu(x) \geq p$, which implies that $p \leq q$. Giving by definition that $q \in \{p \in L : \mu_p(x) = 1\}$, then q is the supremum of the set. That is $\mu(x) = q = \bigvee \{p \in L : \mu_p(x) = 1\}$. $\square$

The above proposition (3.1.3) also holds true when L is not necessarily a (complete) lattice, for a poset this supremum is always defined ([91]). Therefore the next proposition which is a direct consequence of proposition (3.1.3) formulates the above property in terms of lattice theoretic operation. In that way, the synthesis of an L-valued set is obtained.

Proposition 3.1.4. *Let $\mu : X \to L$ be an L-valued set on X. Then for all $x \in X$ it holds that,*

$$\mu(x) = \bigvee_{x \in \mu_p} p \circ \mu_p(x), \; p \in L,$$

where

$$p \circ \mu_p(x) = \begin{cases} p & if \; \mu_p(x) = 1 \\ 0 & otherwise. \end{cases}$$

The necessity for L to be complete stems from the fact that most of the operations on fuzzy algebraic structures require arbitrary meets.

One of the best known properties of cut subsets is their connection with the order in L, precisely the fact that smaller cuts (under the set inclusion) correspond to greater elements in L. The following proposition can be easily proved.

Proposition 3.1.5. *Let $\mu : X \to L$ be an L-valued set on X. For $p, q \in L$, if $p \leq q$ then $\mu_p \supseteq \mu_q$.*

Generally, the converse of proposition (3.1.5) does not hold. The example below is a counter example.

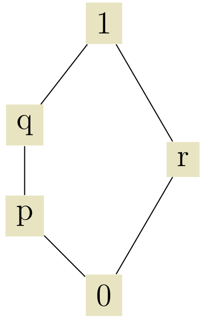

Figure 3.1: L

Example 3.1.6. Let $\mu : X = \{a, b, c, d\} \to L = \{0, p, q, r, 1\}$ be an L-valued set on X. Where L is

where $\mu(a) = 1, \mu(b) = r = \mu(d),$ and $\mu(c) = p$. Clearly, $\mu_r \supseteq \mu_q$ and not $r \leq q$

Proposition (3.1.7) below gives conditions under which the converse of proposition (3.1.5) is satisfied. This was proved in ([91]).

Proposition 3.1.7. *Let $\mu : X \to L$ be an L-valued set on X. Then*

1. *for $p \in L$ and $a \in X$, $p \leq \mu(a)$ if and only if $\mu_p \supseteq \mu_{\mu(a)}$.*

2. *for $a, b \in X$, $\mu(a) \neq \mu(b)$ if and only if $\mu_{\mu(a)} \neq \mu_{\mu(b)}$.*

Clearly, (2) follows from (1) above.

Proposition 3.1.8. *(Šešelja and Tepavčević ([99])) Let $\mu : X \to L$ be an L-valued set on X. Then,*

1. *if $L_1 \subseteq L$, then $\bigcap\{\mu_p : p \in L_1\} = \mu_{\bigvee\{\mu_p : p \in L_1\}}$.*

2. *$\bigcup\{\mu_p : p \in L\} = X$.*

3. *$\forall\ x \in X,\ \bigcap\{\mu_p : x \in \mu_p\} \in \mu_L$.*

Proof. See ([91]) $\square$

Moreover, the collection of cuts forms a (complete) lattice under set inclusion, where μ_0 is the top and μ_1 is the bottom element of this lattice. So we have the next theorem.

Theorem 3.1.9. *(Šešelja et.al ([95])) Let $\mu : X \to L$ be an L-valued set on X. Then the collection $\mu_L = \{\mu_P : p \in L\}$ of all cuts of μ forms a complete lattice under inclusion, which is a Moore family of subsets of X.*

Therefore, for every (complete) lattice there exists an L-valued set such that the collection of cut subsets of the L-valued set under set inclusion is a (complete) lattice anti-isomorphic with that (complete) lattice.

Theorem 3.1.10. *(Representation Theorem)(Šešelja and Tepavčević ([99])) Let $X \neq \emptyset$ and $\mathcal{F}$ a collection of subsets of X closed under arbitrary intersection. Let $(L, \leq)$ be a lattice dual to $(\mathcal{F}, \subseteq)$ and $\mu : X \to L$ be defined by,*

$$\mu(x) := \bigcap\{p \in \mathcal{F} : x \in p\}.$$

Then μ is an L-valued set on X, and each $p \in \mathcal{F}$ coincides with the corresponding cut μ_p.

Proof. Let $p, q \in L$, where $(L, \leq)$ is a lattice ordered by the binary relation $\leq$, therefore $p \leq q$ if and only if $p \supseteq q \in \mathcal{F}$. Clearly, μ is well defined. Indeed, since for every $x \in X$, $\bigcap\{p \in \mathcal{F} : x \in p\} \in \mathcal{F}$, therefore the family $\{p \in \mathcal{F} : x \in p\}$ is uniquely determined.
Obviously μ is an L-valued set of X. Now we prove that for every $p \in \mathcal{F}$ we have that $\mu_p = p$. We recall that $\mu_p = \{a \in X : \mu(a) \geq p\}$. Therefore, for $x \in X$

$$\begin{aligned} x \in \mu_p &\iff \mu(x) \geq p \\ &\iff \bigcap\{q \in \mathcal{F} : x \in q\} \geq p \quad \text{(by definition of } \mu) \\ &\iff \bigcap\{q \in \mathcal{F} : x \in q\} \subseteq p \quad \text{(by definition of } \geq) \\ &\iff x \in p. \end{aligned}$$

□

Observe that the above theorem gives a sufficient condition for a collection of subsets to be a collection of cuts of an L-valued set. Nevertheless, this condition is also necessary, since it is a well-known fact that the collection of cut sets of an L-valued set $\mu : X \to L$ is closed under intersections.

Let $\mu : X \to L$ be an L-valued set on X, then μ induces a partition on L. If $\approx$ is a binary relation on L, such that for $p, q \in L$, $p \approx q$ if and only if $\mu_p = \mu_q$. Clearly, $\approx$ is an equivalence relation on L and for any $p \in L$, $[p]_\approx := \{q \in L : p \approx q\}$.

Proposition 3.1.11. *(Šešelja and Tepavčević ([99])) Let $\mu : X \to L$ be an L-valued set on X and $p, q \in L$, then*

$$p \approx q \text{ if and only if } \uparrow p \cap \ \mu(X) = \uparrow q \cap \ \mu(X).$$

Proof. Let $p, q \in L$, then the binary relation $p \approx q$ holds if and only if $\mu_p = \mu_q \iff \forall x \in X \ \mu(x) \geq p \Leftrightarrow \mu(x) \geq q \iff \{x \in X : \mu(x) \in \uparrow p\} = \{x \in X : \mu(x) \in \uparrow q\} \iff \uparrow p \cap \ \mu(X) = \uparrow q \cap \ \mu(X)$. □

For any $p \in L$, $[p]_{\approx} := \{q \in L : p \approx q\}$.

Lemma 3.1.12. *Let $\mu : X \to L$ be an L-valued set on X, and let $X_1 \subseteq X$ if for all $x \in X_1$, $p = \mu(x)$, then p is the largest element of the $\approx$-class to which it belongs.*

By extension the order relation $\leq$ defined on L induces an ordering on the set of $\approx$-classes, i.e. on $L/\approx$ by:

$$[p]_{\approx} \leq [q]_{\approx} \text{ if and only if } \uparrow q \cap \ \mu(X) \subseteq \uparrow p \cap \ \mu(X).$$

Clearly $\leq$ indeed is well defined, since $p_1 \in [p]_{\approx}$ and $q_1 \in [q]_{\approx}$ implies $p_1 \approx p$ and $q_1 \approx q$ implies $\mu_{p_1} = \mu_p$ and $\mu_{q_1} = \mu_q$, hence by proposition (3.1.11) $\uparrow p_1 \cap \ \mu(X) = \uparrow p \cap \ \mu(X) \supseteq \uparrow q \cap \ \mu(X) = \uparrow q_1 \cap \ \mu(X)$, thus $\uparrow q_1 \cap \ \mu(X) \subseteq \uparrow p_1 \cap \ \mu(X)$.

Proposition 3.1.13. *Let $\mu : X \to L$ be an L-valued set on X, then;*

$$[p]_{\approx} \leq [q]_{\approx} \text{ if and only if } \mu_p \supseteq \mu_q.$$

This order is anti-isomorphic to the set inclusion among cut sets of μ, as it can be seen in the above proposition. That is the quotient lattice $(L/\approx, \leq)$ is isomorphic with the dual lattice $(\mu_L, \subseteq)$ of cuts of μ, where $\mu_L = \{\mu_p : p \in L\}$.

For the lattice $(L/\approx, \leq)$ the supremum of each $\approx$-class $[p]_{\approx}$ denoted by $\bigvee [p]_{\approx}$ exists and defined by

$$\bigvee [p]_{\approx} := \bigvee \{q \in L : q \in [p]_{\approx}\}.$$

Remark 3.1.14. For the quotient lattice $(L/\approx, \leq)$ if the $\approx$-classes are one element sets, obviously $L \cong L/\approx$. But this does not generally hold in the case where L is a poset. For a counter example see ([91])

For L a complete lattice and $X \neq \emptyset$ and $\mu \in L^X$, let

$$L_\mu := (\{\uparrow p \cap \ \mu(X) : p \in L\}, \subseteq).$$

By the definition, L_μ consists of particular collections of images of μ in L and ordered by set inclusion.

Proposition 3.1.15. *For an L-valued set μ the lattice L_μ is isomorphic to the lattice μ_L of cuts under the mapping $f : \mu_P \mapsto \uparrow\! p \cap \ \mu(X)$.*

Let L^X be the set of all L-valued sets on X, then for $\mu, \nu \in L^X$, the binary relation $\sim$ on L^X is defined by

$$\mu \sim \nu \text{ if and only if the correspondence } f : \mu(X) \mapsto \nu(X),\ x \in X$$

is a bijection from $\mu(X)$ onto $\nu(X)$ which by extension is an isomorphism from the lattice L_μ onto the lattice L_ν, given by

$$\hat{f}(\uparrow\! p \cap \mu(X)) := \uparrow \bigwedge \{\nu(x) : \mu(x) \geq p\} \cap \nu(X), p \in L.$$

Hence, if $\mu \sim \nu$, the L-valued sets μ and ν on X are said to be equivalent.

Proposition 3.1.16. *Let $\mu, \nu \in L^X$. Then $\mu \sim \nu$ if and only if the L-valued sets μ and ν have equal collections of cut sets.*

L-valued relation

For a nonempty set X and a complete lattice L, a mapping $R : X^2 \to L$ is called an L-**valued relation** on X.

Let μ be an L-valued set of X and R an L-valued relation on X. We say that R is an L-**valued relation on** μ if for every $x, y \in X$

$$R(x, y) \leq \mu(x) \wedge \mu(y). \tag{3.1}$$

Obviously, this condition is a generalization of the following crisp relational property: If ρ is a binary relation on a subset Y of X, then from $x\rho y$ it follows that $x, y \in Y$.
By condition (3.1) we modify the definition of L-valued reflexivity of L-valued relations on L-valued subsets.
An L-valued relation R on an L-valued set μ is **reflexive** if for all $x, y \in X$

$$R(x, x) = \mu(x). \tag{3.2}$$

This notion of reflexivity as stated above is known e.g. ([37]). Observe that in the classical case if μ is the characteristic function on X, i.e, if for each $x \in X$, $\mu(x) = 1$, then an L-valued reflexive relation on μ is reflexive in the

sense of $R(x,x) = 1$ for each $x \in X$.
The following lemma clearly follows.

Lemma 3.1.17. *If $R : X^2 \to L$ is reflexive on $\mu : X \to L$, then for all $x, y \in X$*
$$R(x,x) \geq R(x,y) \text{ and } R(x,x) \geq R(y,x). \quad \square$$

For any $p \in L$, a p-cut (cut relation) of R is a mapping

$$R_p : X^2 \to \{0,1\},$$

such that $R_p(x,y) = 1$ if and only if $R(x,y) \geq p$. The corresponding cut subset:
$$R_p = \{(x,y) \in X^2 : R(x,y) \geq p\} = R^{-1}(\uparrow p),$$
is a crisp relation on X.

Thus if R is an L-valued relation satisfying some of the properties as mentioned in definition (2.2.7), then the cut relation of R satisfies the analogous crisp properties.

Cutworthy approach - enables ordinary properties which are generalized in the L-valued sense to be preserved by all cuts of the L-structure (relation). This approach is supported by the classical lattice operations - join and meet, and also by lattice identities (e.g., distributivity in some cases). In such cases, the co-domain of L-valued relations is chosen to be a complete lattice (with no additional operations).

The notion of L-valued subalgebras and L-valued congruences are defined analogously as in definitions (2.2.9) and (2.2.12) respectively. In this case the unit interval is replaced by L a complete lattice.

Although several notions of L-valued sets and structures have been presented in literature, in which L was considered to be a complete lattice with additional properties for example *Šešelja* considered L to be a complete Boolean lattice, *Höhle*, considered L to be a complete Heyting lattice, *Bělohlávek* considered L to be a complete residuated lattice, etc. In [[51]] cuts and some properties were presented and in the presentation L was replaced with $\bar{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ where $\mathbb{R}$ is the set of real numbers. *Šešelja* and *Tepavčević* have also considered the use of a poset instead of a (complete) lattice. But our discussion of these topics is extensively based on the presentations given by *Šešelja* and *Tepavčević*, where L is a complete lattice without necessarily assuming additional properties.
Therefore, in view of this generalization we introduce some basic notions of the topic of our research.

3.2 Ω-valued functions, Ω-valued relations and Ω-sets

In sequel let Ω be a fixed complete lattice and in place of "L-valued..." we use "Ω-valued..." throughout the remaining part of this work. Let $E : M^2 \to \Omega$ be an Ω-valued relation over a nonempty set M.

E is said to fulfill the **strictness property** if for all $x, y \in M$

$$E(x,y) \leq E(x,x) \wedge E(y,y) \quad (\textit{strictness}) \ . \tag{3.3}$$

If for all $x, y \in M$

$$E(x,y) = E(x,x) = E(y,x) \neq 0 \ \text{ implies } \ x = y \quad (\textit{separation}), \tag{3.4}$$

then E is said to be a **separated** Ω-valued relation on M.

An **Ω-set** is a pair (M, E), where M is a nonempty set, and E is a symmetric and transitive Ω-valued relation on M, fulfilling the separation property.

Remark 3.2.1. In some particular situations which are explicitly indicated in the text we consider Ω-sets in which E does necessarily fulfill the separation property.

The following is straightforward.

Proposition 3.2.2. *An Ω-valued equality E on a set M fulfills the strictness property.*

For an Ω-set (M, E), we denote by μ the Ω-valued function on M, defined by

$$\mu(x) := E(x,x). \tag{3.5}$$

We say that μ is *determined by E*. Clearly, by strictness property, E is an Ω-valued relation on μ, namely, it is an Ω-valued equality on μ.

Corollary 3.2.3. *Let (M, E) be an Ω-set and μ be an Ω-valued function on X as given by equation* (3.5)*, then μ is extensional w.r.t the Ω-valued equality E.*

Proof. From definition (2.2.8), for $x, y \in M$ we have by equation (3.5) and the strictness property $\mu(x) \wedge E(x,y) = E(x,y) \leq \mu(y)$. $\square$

Lemma 3.2.4. *If (M, E) is an Ω-set and $p \in \Omega$, then the cut E_p is an equivalence relation on the corresponding cut μ_p of μ.*

Proof. Reflexivity of E_p over μ_p: $(x,x) \in E_p$ if and only if $E(x,x) = \mu(x) \geq p$, if and only if $x \in \mu_p$. Symmetry and transitivity are proved straightforwardly. □

Corollary 3.2.5. *Let (M,E) be an Ω-set, and $E^1 : M^2 \to \Omega$ be an Ω-valued relation on M fulfilling $E^1 \leq E$, so that the following holds:*
For all $x, y \in M$

$$E^1(x,y) = E(x,y) \wedge E^1(x,x) \wedge E^1(y,y). \tag{3.6}$$

Then E^1 a symmetric and transitive Ω-valued relation on M.

Proof. Let (M,E) be an Ω-set, and $E^1 : M^2 \to \Omega$ be an Ω-valued relation on M. Then for $x, y, z \in M$, we have by equation (3.6)

$$\begin{aligned} E^1(x,y) \wedge E^1(y,z) &= E(x,y) \wedge E(y,z) \wedge E^1(x,x) \wedge E^1(y,y) \wedge E^1(z,z) \\ &\leq E(x,z) \wedge E^1(x,x) \wedge E^1(z,z) \quad \text{(by transitivity of } E\text{)} \\ &\leq E^1(x,z) \quad \text{(by strictness property).} \end{aligned}$$

Therefore, E^1 is a transitive relation on M. The symmetry of E^1 follows easily. □

now we define the notion of an Ω-subset.

Definition 3.2.6. Let (M,E) be an Ω-set, and $E^1 : M^2 \to \Omega$ be an Ω-valued relation on M given by $E^1(x,y) = E(x,y) \wedge E^1(x,x) \wedge E^1(y,y)$. Then (M,E^1) is an Ω-set and we say that it is an **Ω-subset** of (M,E).

Clearly E^1 is the restriction of E to a nonempty Ω-subset μ^1 of μ.

Now we turn our attention to defining a mappings between Ω-sets. Let (M,E) and (N,F) be Ω-sets, where $\mu : M \to \Omega$ and $\nu : N \to \Omega$ are Ω-valued maps determined by E and F respectively, such that the mapping satisfies the properties of the Ω-valued equality relations defined on the sets.

Definition 3.2.7. Let (M,E) and (N,F) be two Ω-sets. Then the mapping $\phi : M \to N$ from the set M into the set N, such that $\forall a, b \in M$

$$E(a,b) \leq F(\phi(a), \phi(b)), \tag{3.7}$$

is called an Ω-valued **map**. Symbolically, this mapping is denoted as $\phi : (M,E) \to (N,F)$.

Definition 3.2.8. If the ordinary mapping $\phi : M \to N$ from the set M into the set N is bijective then (3.7) becomes

$$E(a,b) = F(\phi(a), \phi(b)), \tag{3.8}$$

Corollary 3.2.9. *If ϕ is a bijection, then ϕ^{-1} is an Ω-valued map if, and only if ϕ is an Ω-valued map.*

Proof. "$only\ \ if\ \ part$" : Let ϕ be a bijection and ϕ^{-1} an Ω-valued map. Then,
$E(a,b) \leq F(\phi(a), \phi(b)) \leq E(\phi^{-1}(\phi(a)), \phi^{-1}(\phi(b))) = E(a,b)$.
"$if\ \ part$" : Clearly, this part trivially follows since ϕ is a bijective and for all $a, b \in M$, $E(a,b) = F(\phi(a), \phi(b))$. □

Definition 3.2.10. Let $\phi : (M,E) \to (N,F)$ be an Ω-valued map from the Ω-set (M,E) into the Ω-set (N,F). Then for the Ω-valued image $\varphi(M)$ of M under ϕ, let the binary relation $E^{\varphi(\mathcal{M})}$ be defined by

$$E^{\varphi(\mathcal{M})}(x,y) = \begin{cases} F(x,y) & if\ x, y \in \varphi(M) \\ 0 & if\ x \text{ or } y \notin \varphi(M). \end{cases} \tag{3.9}$$

Proposition 3.2.11. *Let $\phi : (M,E) \to (N,F)$ be an Ω-valued map from the Ω-set (M,E) into the Ω-set (N,F). Then $(N, E^{\phi(\mathcal{M})})$, where $E^{\phi(\mathcal{M})}$ is as defined in (3.9) is an Ω-set which is an Ω-subset of (N,F)*

Proof. The first part of the proposition is obvious by the definition of $E^{\phi(M)}$. That is $E^{\phi(M)}$, is ν-reflexive, symmetric and transitive.
Second part: For all $x \in \varphi(M)$ and by equation (3.9), $E^{\phi(M)}(x,x) = F(x,x) = \nu(x)$. Thus $\nu(x) = \nu^{\phi(M)}(x)$.
Let $x, y \in N$, be that for $a, b \in M$ such that $\phi(a) = x$ and $\phi(b) = y$:

$$\begin{aligned} E^{\phi(M)}(x,y) &= F(x,y) \qquad \text{(by equation (3.9))} \\ &= F(x,y) \wedge \nu(x) \wedge \nu(y) \qquad \text{(since } F \text{ is } \nu\text{-relational)} \\ &= F(x,y) \wedge \nu^{\phi(M)}(x) \wedge \nu^{\phi(M)}(y). \end{aligned}$$

Therefore, $E^{\phi(M)}(x,y) = F(x,y) \wedge \nu^{\phi(M)}(x) \wedge \nu^{\phi(M)}(y)$.
Hence, $(N, E^{\phi M})$ is an Ω-set and indeed an Ω-subset of (N,F). □

Analogously to the classical notion we define the kernel of a mapping between Ω-sets.

Definition 3.2.12. Let ϕ be a mapping between two Ω-sets (M, E) and (N, F) as defined in (3.2.7). Then a binary Ω-valued relation denoted by $ker_\Omega\phi$ and given as

$$ker_\Omega\phi(a, b) = F(\phi(a), \phi(b)) \wedge \mu(a) \wedge \mu(b), \quad \forall a, b \in M, \tag{3.10}$$

is called an Ω-**valued kernel** of ϕ.

For the sake of convenience, we denote $ker_\Omega\phi$ by R_Ω. Clearly, R_Ω is an Ω-valued map on M^2. We have the following theorem:

Theorem 3.2.13. *Let $\phi : (M, E) \to (N, F)$ be an Ω-valued map. Then the Ω-valued kernel of ϕ, R_Ω as defined in equation (3.10) is a μ-reflexive, symmetric and transitive Ω-valued relation on (M, E).*

Proof. Let R_Ω be the Ω-valued kernel of ϕ as defined in (3.10). Therefore, for each $a \in M$,

$$\begin{aligned} R_\Omega(a, a) &= F(\phi(a), \phi(a)) \wedge \mu(a) \\ &= F(\phi(a), \phi(a)) \wedge E(a, a) \quad \text{(by } \mu\text{-reflexivity of } E) \\ &= E(a, a) \qquad \text{(by equation (3.10))} \\ &= \mu(a) \qquad \text{(again by } \mu\text{-reflexivity of } E). \end{aligned}$$

Therefore, R_Ω is μ-reflexive.
Next, for each $a, b \in M$,

$$\begin{aligned} R_\Omega(a, b) &= F(\phi(a), \phi(b)) \wedge \mu(a) \wedge \mu(b) \\ &= F(\phi(b), \phi(a)) \wedge \mu(b) \wedge \mu(a) \quad \text{(by symmetry of } F) \\ &= R_\Omega(b, a) \qquad \text{(by equation (3.10))}. \end{aligned}$$

Therefore, R_Ω is symmetric.
Also, for each $a, b, c \in M$,

$$\begin{aligned} R_\Omega(a, b) \wedge R_\Omega(b, c) &= (F(\phi(a), \phi(b)) \wedge \mu(a) \wedge \mu(b)) \wedge (F(\phi(b), \phi(c)) \wedge \mu(b) \wedge \mu(c)) \\ &= F(\phi(a), \phi(b)) \wedge F(\phi(b), \phi(c)) \wedge \mu(a) \wedge \mu(b) \wedge \mu(c) \\ &\leq F(\phi(a), \phi(c)) \wedge \mu(a) \wedge \mu(c) \qquad \text{(by transitivity of } F) \\ &= R_\Omega(a, c) \qquad \text{(by equation (3.10))}. \end{aligned}$$

Therefore, R_Ω is transitive. Hence R_Ω is an equivalence relation on μ. $\square$

Therefore, by theorem (3.2.13), R_Ω is an Ω-valued equivalence relation on (M, E).

Lemma 3.2.14. *Let $\phi : M \to N$ be an ordinary map from the set M into the set N, and (M, E), (N, E) be Ω-sets , and R_Ω as defined in equation* (3.10)*. Then the mapping $\phi : (M, E) \to (N, F)$ is an Ω-valued map if, and only if for all $a, b \in M$ $R_\Omega(a, b) \geq E(a, b)$.*

Proof. $(if\ \ part)$ Assuming $R_\Omega(a, b) \geq E(a, b)$, for each $a, b \in M$. Then

$$\begin{aligned} E(a, b) &\leq R_\Omega(a, b) = F(\phi(a), \phi(b)) \wedge \mu(a) \wedge \mu(b) \\ &\Longrightarrow E(a, b) \leq F(\phi(a), \phi(b)). \end{aligned}$$

$(only\ \ if\ \ part)$ Assuming ϕ is an Ω-valued map then for each $a, b \in M$

$$\begin{aligned} E(a, b) &\leq F(\phi(a), \phi(b)) \wedge E(a, a) \wedge E(b, b) \qquad \text{(by strictness property)} \\ &\leq F(\phi(a), \phi(b)) \wedge \mu(a) \wedge \mu(b) \qquad \text{(by μ-reflexivity)} \\ &= R_\Omega(a, b). \end{aligned}$$

□

The following corollary is an immediate consequence of lemma (3.2.14).

Corollary 3.2.15. *Let $\phi : (M, E) \to (N, F)$ be an Ω-valued map from the Ω-set (M, E) into the Ω-set (N, E) and R_Ω a kernel of ϕ as defined in equation* (3.10)*. Then for all $a, b \in M$, $R_\Omega(a, b) = E(a, b)$, if the ordinary mapping $\phi : M \to N$ from the set M into the set N is bijective.*

Proof. Let the ordinary mapping $\phi : M \to N$ be bijective, then for $a, b \in M$

$$\begin{aligned} R_\Omega(a, b) &= F(\phi(a), \phi(b)) \wedge \mu(a) \wedge \mu(b) \quad \text{(by equation (3.10))} \\ &\leq F(\phi(a), \phi(b)) \\ &= E(a, b) \quad \text{(by equation (3.8)).} \end{aligned}$$

Since $\phi : (M, E) \to (N, F)$ is an Ω-valued map then $R_\Omega(a, b) = E(a, b)$. This completes the proof. □

Generally, sets can be constructed from a family of sets. A well known construction is the direct product of sets. Therefore, we introduce the direct product of an indexed family of Ω-sets.

Definition 3.2.16. let I be an indexing set and $\{(M_i, E^i) \mid i \in I$ and $E^i : M_i^2 \to \Omega\}$ a family of Ω-sets . A **direct product**, $\Pi\{(M_i, E^i) \mid i \in I\}$ of the family is an Ω-set, such that for $a, b \in M$

$$E(a, b) = \bigwedge\{E^i(a^i, b^i) \mid i \in I\}. \tag{3.11}$$

where $a = (a^i)_{i\in I}$, $b = (b^i)_{i\in I}$ and $M = \Pi\{M_i \mid i \in I\}$. We denote the product by $(M, E) = \Pi\{(M_i, E^i) \mid i \in I\}$

Remark 3.2.17. Now if the function $\mu : M \to \Omega$ is an Ω-valued function determined by $E : M^2 \to \Omega$, then $\forall a \in M$,

$$\mu(a) = E(a, a) = \bigwedge\{E^i(a^i, a^i) \mid i \in I\} = \bigwedge\{\mu^i(a^i) \mid i \in I\}$$

.

$(\mathcal{M}, E)$ as defined in (3.2.16) above satisfies the properties of an Ω-set: From remark (3.2.17) it is clear that E is μ-reflexive. Now for any $a, b \in M$

$$\begin{aligned} E(a,b) &= \bigwedge_{i\in I} E^i(a^i, b^i) \leq \bigwedge_{i\in I}(\mu^i(a^i) \wedge \mu^i(b^i)) \\ &= (\bigwedge_{i\in I} \mu^i(a^i)) \wedge (\bigwedge_{i\in I} \mu^i(b^i)) = \mu(a) \wedge \mu(b) \\ &= E(a,a) \wedge E(b,b). \end{aligned}$$

Thus E satisfies the strictness property. Evidently the symmetric property follows easily. Furthermore, for any $a, b, c \in M$

$$\begin{aligned} E(a,b) \wedge E(b,c) &= \bigwedge_{i\in I} E^i(a^i, b^i) \wedge \bigwedge_{i\in I} E^i(b^i, c^i) \\ &= \bigwedge_{i\in I}(E^i(a^i, b^i) \wedge E^i(b^i, c^i)) \leq \bigwedge_{i\in I} E^i(a^i, c^i) \\ &= E(a,c). \end{aligned}$$

Thus E is transitive.

Lemma 3.2.18. *The mapping $P_j : (M, E) \to (M_j, E^j)$, $a \mapsto P_j(a)$, for each $j \in I$, is an Ω-valued map.*

Proof. Let the map for any $a \in M$ be defined by $P_j(a) = a^j$ for an arbitrary $a^j \in M_j$. Therefore, for $a, b \in M$

$$\begin{aligned} E(a,b) &= \bigwedge_{i\in I} E^i(a^i, b^i) \leq E^j(a^j, b^j) \quad \text{(for each } j \in I) \\ &= E^j(P_j(a), P_j(b)) \quad \text{(by definition of } P_j). \end{aligned}$$

So P_j is an Ω-valued map. Clearly, P_j is surjective. $\square$

The map P_j as above is referred to as the **projection map**.

Chapter 4

Ω-Algebras in Universal Algebra

In this chapter we attempt to apply fuzzy approach to universal algebra. Fuzzy approach been applied to various universal algebraic concepts started with Rosenfeld's fuzzy groups ([80]). Rosenfeld's approach has been applied to several algebraic structures and concepts, see e.g. ([24]), and also generalized to arbitrary universal algebras, as presented by Di Nola ([32]). Another attempt started by Murali ([70, 71]), where he introduced fuzzy subalgebras and the fuzzy congruences of a universal algebra as a way of uniting the existing concepts of fuzzy algebraic structures. This was further developed by Samhan ([82]), who presented a factorization of crisp algebras by fuzzy congruence. Most recently in ([3]) Bělohlávek and Vychodil presented a new approach in which algebras with fuzzy equalities were introduced. In this case, an algebra with fuzzy equality is a set with operations on it that is equipped with a fuzzy equality relation $\approx$, such that each functional operation, f on the set is compatible with $\approx$. The approach of our work is closely related to these attempts. However, there are important differences which we comment on as the chapter progresses.

Therefore, we will be dealing with Ω-algebras as introduced by Šešelja and Tepavčević. Thus, universal-algebraic properties of Ω-structures are investigated, using cut sets and relations, satisfiability of fuzzy identities as introduced by Šešelja, et. al. ([19]) , in dealing with special kinds of identities and an Ω-valued equality. Then results about homomorphisms, subalgebras and direct products of Ω-algebras are investigated.

4.1 Ω-Algebras

Let $\mathcal{M} = (M, \mathbb{F})$ be an algebra as defined in chapter two, and $\mu : M \to \Omega$, a mapping which is not constantly equal to 0. We consider μ to be compatible with operations in $\mathcal{M}$ if for any functional symbol f defined on the algebra $\mathcal{M}$, with arity greater than 0, the following property holds;

$$\forall a_1, ..., a_n \in M, \ \bigwedge_{i=1}^{n} \mu(a_i) \leq \mu(f(a_1, ..., a_n)), \tag{4.1}$$

and for each nullary operational symbol $c \in \mathbb{F}$, $\mu(c) = 1$.

Classically, terms and term operations, play very significant role in universal algebra. Therefore, in ([22]) the following L-valued version of a known property of term operation in universal algebra was formulated. In our own case we shall refer to the "L-valued..." as "Ω-valued ..."

Proposition 4.1.1. *Let $\mu : M \to \Omega$ be a compatible mapping on an algebra $\mathcal{M}$ and let $t(x_1, \ldots, x_n)$ be a term in the language of $\mathcal{M}$. If $a_1, \ldots, a_n \in M$, then the following holds:*

$$\bigwedge_{i=1}^{n} \mu(a_i) \leq \mu(t^{\mathcal{M}}(a_1, \ldots, a_n)).$$

□

For an algebra $\mathcal{M} = (M, \mathbb{F})$, an Ω-valued relation $\rho : M^2 \to \Omega$ is said to be compatible with the operational symbols defined on the algebra $\mathcal{M}$ if for each n-ary operation $f \in \mathbb{F}$ and for any $a_1, ..., a_n, b_1, ..., b_n \in M$ the following holds

$$\bigwedge_{i=1}^{n} \rho(a_i, b_i) \leq \rho(f(a_1, ..., a_n), f(b_1, ..., b_n)), \tag{4.2}$$

and for each nullary operational symbol $c \in \mathbb{F}$, $\rho(c, c) = 1$.

Obviously, the compatibility property expresses a natural constraint on the operations defined on the algebra.

Definition 4.1.2. Let $\mathcal{M} = (M, \mathbb{F})$ be an algebra and $E : M^2 \to \Omega$ an Ω-valued equality on M, which is compatible with the operations in $\mathbb{F}$. Then we say that the pair $(\mathcal{M}, E)$ is an **Ω-algebra**. Algebra $\mathcal{M}$ is the **underlying algebra** of $(\mathcal{M}, E)$.

Lemma 4.1.3. *Let $(\mathcal{M}, E)$ be an Ω-algebra and $t(x_1, \ldots, x_n)$ a term in the language of the algebra $\mathcal{M}$. Then for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$*

$$\bigwedge_{i=1}^{n} E(a_i, b_i) \leq E(t(a_1, \ldots, a_n), t(b_1, \ldots, b_n)).$$

Usually algebras are defined by equations which can be referred to as identities that holds in the algebras. So adopting the approach introduced in ([86]), we define how identities hold on Ω-algebras.

Observe that, as for the Ω-set, we denote by μ the function $\mu : M \to \Omega$, such that for every $x \in M$,

$$\mu(x) := E(x, x).$$

Obviously, μ is a compatible function over $\mathcal{M}$.

Let $t_1(x_1, \ldots, x_n) \approx t_2(x_1, \ldots, x_n)$ (briefly $t_1 \approx t_2$) be an identity in the type of an Ω-algebra $(\mathcal{M}, E)$. We assume, as usual, that variables appearing in terms t_1 and t_2 are from $x_1, \ldots, x_n$. Then, $(\mathcal{M}, E)$ **satisfies identity** $t_1 \approx t_2$ (i.e., this identity **holds** on $(\mathcal{M}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leq E(t_1(a_1, ..., a_n), t_2(a_1, ..., a_n)) \;\; \forall a_1, ..., a_n \in M \tag{4.3}$$

Proposition 4.1.4. *([22]) If an identity $u \approx v$ holds on an algebra $\mathcal{M}$, then it also holds on an Ω-algebra $(\mathcal{M}, E)$.*

Remark 4.1.5. It has been proved in ([22]) that an identity holding in an Ω-algebra $(\mathcal{M}, E)$ does not necessary holds in the (basic) algebra $\mathcal{M}$.

Next we introduce and define Ω-subalgebras.

Let $(\mathcal{M}, E)$ be an Ω-algebra, and (M, E^1) an Ω-subset of (M, E). By (3.9), E^1 is a symmetric and transitive Ω-valued relation on M, fulfilling for all $x, y \in M$

$$E^1(x, y) = E(x, y) \wedge E^1(x, x) \wedge E^1(y, y).$$

Let also E^1 be compatible with the operations in $\mathcal{M}$. Obviously, $(\mathcal{M}, E^1)$ is an Ω-algebra and we say that it is an **Ω-subalgebra** of $(\mathcal{M}, E)$.

Proposition 4.1.6. *If $(\mathcal{M}, E^1)$ is an Ω-subalgebra of an Ω-algebra $(\mathcal{M}, E)$, and $\mu^1 : M \to \Omega$ is the Ω-valued function on M defined by $\mu^1(x) := E^1(x, x)$,* *then* μ^1 *is compatible over* $\mathcal{M}$*, i.e., it fulfills* (4.1) *and* $\mu^1(c) = 1$ *for any nullary operation* c *on* $\mathcal{M}$*.*

Proof. Assuming that $(\mathcal{M}, E^1)$ is an Ω-subalgebra of the Ω-algebra $(\mathcal{M}, E)$. Therefore for $x_1, \dots, x_n \in M$, it is clear that

$$\bigwedge_{i=1}^{n} \mu^1(x_i) = \bigwedge_{i=1}^{n} E^1(x_i, x_i) \leq E^1(f(x_1, ..., x_n), f(x_1, ..., x_n)) = \mu^1(f(x_1, ..., x_n)),$$

fulfilling (4.1). Next for a nullary operation c it follows that

$$\mu^1(c) = E^1(c, c) = 1.$$

This completes the proof. $\square$

An Ω-subalgebra $(\mathcal{M}, E^1)$ of $(\mathcal{M}, E)$ fulfills all the identities that the latter does:

Theorem 4.1.7. *Let* $(\mathcal{M}, E^1)$ *be an* Ω*-subalgebra of an* Ω*-algebra* $(\mathcal{M}, E)$*. If* $(\mathcal{M}, E)$ *satisfies the set* Σ *of identities, then also* $(\mathcal{M}, E^1)$ *satisfies all identities in* Σ*.*

Proof. Let $(\mathcal{M}, E^1)$ be an Ω-subalgebra of $(\mathcal{M}, E)$, $E(u(x_i, ..., x_n), v(x_i, ..., x_n)) \in \Sigma$, where u, v are terms of the same type as $\mathcal{M}$ and the variables occurring in u, v are among $x_i, ..., x_n$. Since by assumption $(\mathcal{M}, E)$ satisfies all identities in Σ hence for $a_1, ..., a_n \in M$

$$\bigwedge_{i=1}^{n} \mu^1(a_i) \leq E(u^{\mathcal{M}}(a_1, ..., a_n), v^{\mathcal{M}}(a_1, ..., a_n)).$$

Therefore by definition of Ω-subalgebra we have

$$\begin{gathered} E^1(u^{\mathcal{M}}(a_1, ..., a_n), v^{\mathcal{M}}(a_1, ..., a_n)) \\ = E(u^{\mathcal{M}}(a_1, ..., a_n), v^{\mathcal{M}}(a_1, ..., a_n)) \wedge \mu^1(u^{\mathcal{M}}(a_1, ..., a_n)) \wedge \mu^1(v^{\mathcal{M}}(a_1, ..., a_n)) \geq \\ \bigwedge_{i=1}^{n} \mu(a_i) \wedge \bigwedge_{i=1}^{n} \mu^1(a_i) \wedge \bigwedge_{i=1}^{n} \mu^1(a_i) = \bigwedge_{i=1}^{n} \mu^1(a_i). \end{gathered}$$

Hence $E(u(x_i, ..., x_n), v(x_i, ..., x_n)) \in \Sigma$ holds in $(\mathcal{M}, E^1)$. $\square$

4.2 Ω-valued Morphisms and Congruences on Ω-Algebras

Classically when considering mathematical constructions (or classes of these constructions) sharing some general structural properties, it is a usual mathematical practice to consider mappings between them preserving these properties. These mappings are often called morphisms. In this section we will be considering such mappings in the context of Ω-structures.
Maps play very important role in the study of the structural relationship between algebraic objects.

Before we introduce these maps between Ω-structures, we define a congruence (i.e. an Ω-**valued congruence**) on an Ω-structure. Classically, a congruence relation on an algebra $\mathcal{M}$ is an equivalence relation ρ, which is a subalgebra of $\mathcal{M} \times \mathcal{M}$, whose operational structure is defined componentwise. As it is known a congruence relation on an algebra gives rise to another algebra of the same type as $\mathcal{M}$ called the quotient algebra by defining the operations through representatives. Hence we endeavor to present analogous results with an Ω-algebra. Up to now, the notion of congruences in fuzzy settings has been studied by several authors: Murali ([70]), Samhan ([82]), Bělohlávek and Vychodil ([3]) etc. The first two authors considered this notion using the unit interval (which is a particular kind of a complete residuated lattice) as the set of truth degrees. Their basic algebra is an ordinary algebra, as usual equipped with the crisp equality, and their definition of a fuzzy congruence is a fuzzy equivalence relation compatible with the set of operations on the algebra. But third and fourth authors considered this notion in somewhat different way: Firstly, the crisp equality is been replaced by a fuzzy equality, such that instead of dealing with the usual ordinary algebra, their consideration was on an algebra with fuzzy equality. Secondly, the set of truth degrees was replaced by an arbitrary complete residuated lattice. Thirdly, congruence relations are defined on these algebras as fuzzy equivalence relations, compatible with the set of operations on the algebra and compatible with the fuzzy equality relations defined on the algebra. Therefore, we note that the former definition is a special case of the latter.

Our consideration of this notion of congruences in fuzzy settings as presented in this work differs in some aspects and as a result the above mentioned considerations can be regarded as special cases of our consideration.

Let $\mathcal{M} = (M, \mathbb{F}^{\mathcal{M}})$ and $\mathcal{N} = (N, \mathbb{F}^{\mathcal{N}})$ be algebras of the same type.

Definition 4.2.1. Let θ be an Ω-valued relation. Then θ is a congruence (Ω-**valued congruence**) on $(\mathcal{M}, E)$ if:

i. θ is μ-reflexive

ii. θ is an Ω-valued equivalence relation

iii. $\theta(a,b) \geq E(a,b), \ \ \forall a,b \in M$

iv. θ is compatible with all the operations from $\mathbb{F}^{\mathcal{M}}$.

Proposition 4.2.2. *Let $(\mathcal{M}, E)$ be an Ω-algebra, where E is an Ω-valued equality on the algebra $\mathcal{M} = (M, \mathbb{F}^{\mathcal{M}})$. Define an Ω-valued relation $\bar{\theta} : (\mathcal{M}, E)^2 \to \Omega$ by*

$$\bar{\theta}(a,b) := E(a,a) \wedge E(b,b).$$

Then $\bar{\theta}$ is an Ω-valued congruence relation on $(\mathcal{M}, E)$ and the greatest Ω-valued congruence on $(\mathcal{M}, E)$.

Proof. Clearly, $\bar{\theta}$ is μ-reflexive, symmetric and transitive and so an Ω-valued equivalence relation on $(\mathcal{M}, E)$. Therefore, let $a_1, ..., a_n, b_1, ..., b_n \in M$ and $f^{\mathcal{M}}$ be any n-ary operation on $\mathcal{M}$, then
$\bigwedge_{i=1}^{n} \bar{\theta}(a_i, b_i) = \bigwedge_{i=1}^{n} E((a_i, a_i)) \wedge \bigwedge_{i=1}^{n} E((b_i, b_i)) \leq$
$\leq E(f^{\mathcal{M}}(a_1, ..., a_n), f^{\mathcal{M}}(a_1, ..., a_n)) \wedge E(f^{\mathcal{M}}(b_1, ..., b_n), f^{\mathcal{M}}(b_1, ..., b_n)) =$
$= \bar{\theta}(f^{\mathcal{M}}(a_1, ..., a_n), f^{\mathcal{M}}(b_1, ..., b_n)).$
Thus the relation $\bar{\theta}$ is compatible with the operations defined on $\mathcal{M}$. Hence, $\bar{\theta}$ is an Ω-valued congruence relation on $(\mathcal{M}, E)$.
Next, let ρ be any other Ω-valued congruence relation on $(\mathcal{M}, E)$ and $x, y \in M$. Then by strictness and μ-reflexivity;

$$\rho(x,y) \leq \rho(x,x) \wedge \rho(y,y) = E(x,x) \wedge E(y,y) = \bar{\theta}(x,y).$$

Therefore, this shows that $\bar{\theta}$ is the greatest Ω-valued congruence relation on $(\mathcal{M}, E)$. $\square$

Now we define lattice operations on the set of all Ω-valued congruences on an Ω-algebra $(\mathcal{M}, E)$. We denote this set by $C_\Omega((\mathcal{M}, E))$.
Let $\theta_1, \theta_2 \in C_\Omega((\mathcal{M}, E))$, define a binary relation $\leq$ on $C_\Omega((\mathcal{M}, E))$ such that

$$\theta_1 \leq \theta_2 \Longleftrightarrow (\forall x, y \in M)(\theta_1(x,y) \leq \theta_2(x,y)).$$

Clearly, $\leq$ is order on $C_\Omega((\mathcal{M}, E))$. Then $inf\{\theta_1, \theta_2\} \in C_\Omega((\mathcal{M}, E))$ is defined by

$$inf\{\theta_1, \theta_2\}(x,y) = \theta_1(x,y) \wedge \theta_2(x,y)$$

and generally,

$$inf\{\theta_i \mid i \in I\}(x, y) = \bigwedge_{i \in I} \theta_i(x, y).$$

Theorem 4.2.3. *For any index set I, let $\{\theta_i \mid i \in I\} \subseteq C_\Omega((\mathcal{M}, E))$ be an arbitrary family of Ω-valued congruences on $(\mathcal{M}, E)$. Then $inf\{\theta_i \mid i \in I\} \in C_\Omega((\mathcal{M}, E))$.*

Proof. For $a \in M$,

$$inf\{\theta_i \mid i \in I\}(a, a) = \bigwedge_{i \in I} \theta_i(a, a) = \mu(a).$$

Thus $inf\{\theta_i \mid i \in I\}$ is μ-reflexive. Next, for any $a, b \in M$,

$$inf\{\theta_i \mid i \in I\}(a, b) = \bigwedge_{i \in I} \theta_i(a, b) = \bigwedge_{i \in I} \theta_i(b, a) = inf\{\theta_i \mid i \in I\}(b, a).$$

Thus $inf\{\theta_i \mid i \in I\}$ is symmetric. Now for any $a, b, c \in M$,

$$\begin{aligned} inf\{\theta_i \mid i \in I\}(a, b) \wedge inf\{\theta_i \mid i \in I\}(b, c) &= \bigwedge_{i \in I} \theta_i(a, b) \wedge \bigwedge_{i \in I} \theta_i(b, c) \\ &\leq \bigwedge_{i \in I} \theta_i(a, c) \\ &= inf\{\theta_i \mid i \in I\}(a, c). \end{aligned}$$

Thus $inf\{\theta_i \mid i \in I\}$ is transitive. Hence $inf\{\theta_i \mid i \in I\}$ is an Ω-valued equivalence relation on $(\mathcal{M}, E)$. Also, for any $a, b \in M$,

$$inf\{\theta_i \mid i \in I\}(a, b) = \bigwedge_{i \in I} \theta_i(a, b) \geq E(a, b).$$

Thus $inf\{\theta_i \mid i \in I\}(a, b) \geq E(a, b)$. Now for any operation $f^{\mathcal{M}} \in \mathbb{F}^{\mathcal{M}}$ and arbitrary $a_1, ..., a_n, b_1, ..., b_n \in M$, we have,

$$\begin{aligned} \bigwedge_{j=1}^{n} (inf\{\theta_i \mid i \in I\}(a_j, b_j)) &= inf\{(\bigwedge_{j=1}^{n} \theta_i(a_j, b_j)) \mid i \in I\} \\ &\leq inf\{\theta_i(f^{\mathcal{M}}(a_1, ..., a_n), f^{\mathcal{M}}(b_1, ..., b_n)) \mid i \in I\}. \end{aligned}$$

Thus $inf\{\theta_i \mid i \in I\}$ is compatible with every $f^{\mathcal{M}} \in \mathbb{F}^{\mathcal{M}}$, hence $inf\{\theta_i \mid i \in I\}$ is Ω-valued congruence on $(\mathcal{M}, E)$. Therefore, $C_\Omega((\mathcal{M}, E))$ is closed under taking an arbitrary intersection of any family of Ω-valued congruences on $(\mathcal{M}, E)$. $\square$

Theorem 4.2.4. *The collection $C_\Omega((\mathcal{M}, E))$ of all Ω-valued congruences on an Ω-algebra $(\mathcal{M}, E)$ under inclusion forms a complete lattice.*

Proof. The proof follows from theorem (4.2.3), and propositions (4.2.2, 2.1.23). □

Remark 4.2.5. (1)In the crisp set, the greatest congruence relation on an algebra, $\mathcal{M}$ is the square $\mathcal{M}^2$, but for an Ω-algebra $(\mathcal{M}, E)$ the greatest congruence relation is in general not the square as proved in proposition (4.2.2). (2) Obviously, E is also a congruence relation on $(\mathcal{M}, E)$ and the smallest one w.r.t. componentwise order in (iii) of definition (4.2.1).

A very natural way of obtaining classical congruences is in the context of mappings (morphisms), by which elements of the domain mapped into the same element of the codomain are congruent. These congruences are usually called kernels of the morphisms. In the same way we deal with this notion in Ω-algebras.

Definition 4.2.6. A mapping $\varphi : (M, E) \to (N, F)$ from an Ω-algebra $(\mathcal{M}, E)$ into another Ω-algebra $(\mathcal{N}, F)$ is an Ω-valued **homomorphism** if φ is an Ω-valued map and for all $a, a_1, ..., a_n, b_1, ..., b_n \in M$, for every constant operation $c \in M$ and the corresponding constant operation $c^1 \in N$, and all n-ary functional symbols $f^{\mathcal{M}} \in \mathbb{F}^{\mathcal{M}}$, and $f^{\mathcal{N}} \in \mathbb{F}^{\mathcal{N}}$, the following conditions hold:

$$E(a, a) = F(\varphi(a), \varphi(a)) \tag{4.4}$$

$$\mu(a_1) \wedge ... \wedge \mu(a_n) \leq F(\varphi(f^{\mathcal{M}}(a_1, ..., a_n)), f^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \tag{4.5}$$

$$\mu(c) \leq F(\varphi(c), c^1), \tag{4.6}$$

and

$$\varphi(f^{\mathcal{M}}(a_1, \ldots, a_n)) \in \varphi(M). \tag{4.7}$$

It is clear that by the separation property condition (4.6) is equivalent with $\varphi(c) = c^1$, and condition (4.7) means that the set $\varphi(M)$ of images under φ is closed with respect to operations, i.e., $\varphi(M)$ is required to be a subalgebra of $\mathcal{N}$.
The following lemma is an immediate consequence of definition (4.2.6).

Lemma 4.2.7. *Let the mapping $\varphi : (M, E) \to (N, F)$ from an Ω-algebra $(\mathcal{M}, E)$ into an Ω-algebra $(\mathcal{N}, F)$ be an Ω-valued homomorphism. Whenever $\varphi(a) = \varphi(b)$, for $a \neq b \in M$, then $E(a, a) = E(b, b)$*

Proof. Suppose the assumption of the lemma holds: Then

$$\begin{aligned} E(a,a) &= F(\varphi(a),\varphi(a)) \qquad \text{(by definition (4.2.6))} \\ &= F(\varphi(b),\varphi(b)) \\ &= E(b,b). \end{aligned}$$

□

Theorem 4.2.8. *Let φ be an Ω-valued homomorphism from the Ω-algebra $(\mathcal{M},E)$ into the Ω-algebra $(\mathcal{N},F)$. Then the Ω-valued kernel of φ, R_Ω as defined in equation* (3.10) *is a congruence relation on $(\mathcal{M},E)$.*

Proof. From theorem (3.2.13), R_Ω is an equivalence relation on $(\mathcal{M},E)$. Hence we only need to show that all functional symbols $f^{\mathcal{M}} \in \mathbb{F}^{\mathcal{M}}$ are compatible with R_Ω. Let $(a_1,b_1),...,(a_n,b_n) \in M^2$ and $f^{\mathcal{N}} \in \mathbb{F}^{\mathcal{N}}$ be functional symbols on $\mathcal{N}$, then by (3.10)

$$\begin{aligned} &R_\Omega(a_1,b_1)\wedge\cdots\wedge R_\Omega(a_n,b_n) = (F(\varphi(a_1),\phi(b_1))\wedge\mu(a_1)\wedge\mu(b_1))\wedge\cdots\wedge \\ &\quad\wedge\cdots\wedge(F(\varphi(a_n),\varphi(b_n))\wedge\mu(a_n)\wedge\mu(b_n)) = \\ &\quad= (F(\varphi(a_1),\varphi(b_1))\wedge\cdots\wedge F(\varphi(a_n),\varphi(b_n)))\wedge \\ &\quad\wedge(\mu(a_1)\wedge\cdots\wedge\mu(a_n))\wedge(\mu(b_1)\wedge\cdots\wedge\mu(b_n)) \le \\ &\quad\le F(f^{\mathcal{N}}(\varphi(a_1),...,\varphi(b_n)),f^{\mathcal{N}}(\varphi(a_1),...,\varphi(b_n)))\wedge \\ &\quad\wedge\mu(f^{\mathcal{M}}(a_1,...,a_n))\wedge\mu(f^{\mathcal{M}}(b_1,...,b_n)) \\ &\quad\text{(compatibilty of } F,\ \mu \text{ an } \Omega\text{-subalgebra)} \\ &\quad= R_\Omega(f^{\mathcal{M}}(a_1,...,a_n),f^{\mathcal{M}}(b_1,...,b_n)) \quad \text{(by equation (3.10)).} \end{aligned}$$

Hence R_Ω is a congruence relation on $(\mathcal{M},E)$. □

Obviously, Ω-valued homomorphism on Ω-algebras as defined is a mapping that preserves the functional symbols and Ω-valued equality on the algebras.
As it is known in the classical case if $h: M \to N$ is a homomorphism then $h(M)$ is a homomorphic image of $\mathcal{M}$ under h, which of course is a subalgebra of the $\mathcal{N}$. Analogously, a homomorphic image of $(\mathcal{M},E)$ under φ should be an Ω-subalgebra of $(\mathcal{N},F)$.

Proposition 4.2.9. *Let the mapping $\varphi : (M,E) \to (N,F)$ from the Ω-algebra $(\mathcal{M},E)$ into the Ω-algebra $(\mathcal{N},F)$ be an Ω-valued homomorphism. Then $(\mathcal{N},E^{\varphi(\mathcal{M})})$ is an Ω-algebra which is also an Ω-subalgebra of $(\mathcal{N},F)$.*

Proof. For the first part of the lemma, clearly the relation $E^{\varphi(\mathcal{M})}$ being ν-reflexive, symmetric and transitive follows from proposition (3.2.11). Now for all $x_1, ..., x_n \in N$ such that $\varphi(a_i) = x_i$ for $a_1, ..., a_n \in M$, then

$$\begin{aligned}
\bigwedge_{i=1}^{n} \nu^{\varphi(\mathcal{M})}(x_i) &= \bigwedge_{i=1}^{n} \nu(x_i) \qquad \text{(by equation (3.9))} \\
&\leq \nu(f^{\mathcal{N}}(x_1, ..., x_n)) \\
&\leq \nu^{\varphi(\mathcal{M})}(f^{\mathcal{N}}(x_1, ..., x_n)).
\end{aligned}$$

Trivially, in the case where for some i, $1 \leq i \leq n$ there exists $x_i \in N$, such that $\nu^{\varphi(\mathcal{M})}(x_i) = 0$, hence

$$\nu^{\varphi(\mathcal{M})}(f^{\mathcal{N}}(x_1, ..., x_n)) \geq \bigwedge_{i=1}^{n} \nu^{\varphi(\mathcal{M})}(x_i) = 0.$$

Second part: Clearly, $(N, E^{\varphi(M)})$ being an Ω-subset of (N, F) follows from proposition (3.2.11)

Hence we are only left to show that $E^{\varphi(M)}$ is compatible with the operations defined on $\mathcal{N}$. Now, for all $x_1, ..., x_n, y_1, ..., y_n \in N$, such that $\varphi(a_i) = x_i$ and $\varphi(b_i) = y_i$ for $a_1, ..., a_n, b_1, ..., b_n \in M$, then

$$\begin{aligned}
&E^{\varphi(M)}(x_1, y_1) \wedge ... \wedge E^{\varphi(M)}(x_n, y_n) = F(x_1, y_1) \wedge ... \wedge F(x_n, y_n) \\
&\quad \leq F(f^{\mathcal{N}}(x_1, ..., x_n), f^{\mathcal{N}}(y_1, ..., y_n)) \;\; \text{(by compatibility of } F \text{)} \\
&\quad = E^{\varphi(M)}(f^{\mathcal{N}}(x_1, ..., x_n), f^{\mathcal{N}}(y_1, ..., y_n)) \quad \text{(by equation (3.9))} \\
&\quad = E^{\varphi(M)}(f^{\varphi(\mathcal{M})}(x_1, ..., x_n), f^{\varphi(\mathcal{M})}(y_1, ..., y_n)).
\end{aligned}$$

Next, in the case where for some i, $1 \leq i \leq n$, $x_i, y_i \in N$, such that $E^{\varphi(M)}(x_i, y_i) = 0$, hence

$$E^{\varphi(M)}(f^{\varphi(\mathcal{M})}(x_1, ..., x_n), f^{\varphi(\mathcal{M})}(y_1, ..., y_n)) \geq \bigwedge_{i=1}^{n} E^{\varphi(M)}(x_i, y_i) = 0.$$

Therefore, $E^{\varphi(M)}$ is compatible with the operations defined on $\mathcal{N}$. Hence $(\mathcal{N}, E^{\varphi(M)})$ is an Ω-algebra and indeed an Ω-subalgebra of $(\mathcal{N}, F)$. $\square$

Now, we show that identities are preserved under the Ω-valued homomorphism, i.e if an Ω-algebra satisfies an identity then its Ω-valued image satisfies the same identity.

Lemma 4.2.10. *Let u be an n-ary term over the set $\{x_1, ..., x_n\}$ of variables and of the same type as the algebras $\mathcal{M}$ and $\mathcal{N}$. If $\varphi : (M, E) \to (N, F)$ is an Ω-valued homomorphism, then for all $a_1, ..., a_n \in M$*

$$\bigwedge_{i=1}^{n} \mu(a_i) \leq F(\varphi(u^{\mathcal{M}}(a_1, ..., a_n)), u^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))).$$

Proof. We give a proof by induction on the complexity (i.e. the number of occurrences of n-ary operation symbols) of u. Let $l(u)$ be the this length. If $l(u) = 1$, then $u = f$, for a fundamental operation symbol f, then by the definition of Ω-valued homomorphism we have

$$\begin{aligned}\bigwedge_{i=1}^{n} \mu(a_i) &\leq F(\varphi(f^{\mathcal{M}}(a_1, ..., a_n)), f^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \\ &= F(\varphi(u^{\mathcal{M}}(a_1, ..., a_n)), u^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))).\end{aligned}$$

Inductively, assume that $l(u) > 1$ and that for every term v, $l(v) < l(u)$ the assumption holds. Therefore, $u(x_1, ..., x_n) = f(u_1(x_1, ..., x_n), ..., u_k(x_1, ..., x_n))$ and since $l(u_i) < l(u)$, for $i \in \{1, ..., n\}$ we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leq F(\varphi(u_i^{\mathcal{M}}(a_1, ..., a_n)), u_i^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))),$$

then

$$\begin{aligned}&\bigwedge_{i=1}^{n} \mu(a_i) \leq \bigwedge_{i=1}^{n} F(\varphi(u_i^{\mathcal{M}}(a_1, ..., a_n)), u_i^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \\ &\leq F(f^{\mathcal{N}}(\varphi(u_1^{\mathcal{M}}(a_1, ..., a_n)), ..., \varphi(u_n^{\mathcal{M}}(a_1, ..., a_n))), \\ &f^{\mathcal{N}}(u_1^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n)), ..., u_k^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \quad \text{(By compatibility of } f).\end{aligned}$$

Hence,

$$\begin{aligned}\bigwedge_{i=1}^{n} \mu(a_i) &\leq \bigwedge_{i=1}^{n} \mu(u_i^{\mathcal{M}}(a_1, ..., a_n)) \quad \text{(By proposition (4.1.1))} \\ &\leq F(\varphi(f^{\mathcal{M}}(u_1^{\mathcal{M}}(a_1, ..., a_n), ..., u_n^{\mathcal{M}}(a_1, ..., a_n))), \\ &\quad , f^{\mathcal{N}}(\varphi(u_1^{\mathcal{M}}(a_1, ..., a_n), ..., \varphi(u_k^{\mathcal{M}}(a_1, ..., a_n)))).\end{aligned}$$

Therefore by transitivity,

$$\begin{aligned}\bigwedge_{i=1}^{n}\mu(a_i) &\leq F(\varphi(f^{\mathcal{M}}(u_1^{\mathcal{M}}(a_1,...,a_n),...,u_n^{\mathcal{M}}(a_1,...,a_n))),\\ &\quad ,f^{\mathcal{N}}(u_1^{\mathcal{N}}(\varphi(a_1),...,\varphi(a_n)),...,u_k^{\mathcal{N}}(\varphi(a_1),...,\varphi(a_n)))\\ &= F(\varphi(u^{\mathcal{M}}(a_1,...,a_n)),u^{\mathcal{N}}(\varphi(a_1),...,\varphi(a_n))).\end{aligned}$$

□

Theorem 4.2.11. *Let $(\mathcal{M},E)$, $(\mathcal{N},F)$ be Ω-algebras of the same type such that $\varphi:(M,E)\to(N,F)$ is an Ω-valued homomorphism and $(\mathcal{N},E^{\varphi(M)})$ an Ω-valued homomorphic image of $(\mathcal{M},E)$ under φ. Let u,v be n-ary terms over the set X of variables of the same type as the algebras $\mathcal{M}$ and $\mathcal{N}$. If $(\mathcal{M},E)$ fulfills an identity $u(x_1,...,x_n)\approx v(x_1,...,x_n)$, for variables among $x_1,...,x_n$, then $(\mathcal{N},E^{\varphi(M)})$ satisfies the same identity.*

Proof. We show that for $a_1,\ldots,a_n\in M$,

$$\bigwedge_{i=1}^{n}\nu^{\varphi(M)}(\varphi(a_i))\leq E^{\varphi(M)}(u^{\mathcal{N}}(\varphi(a_1),...,\varphi(a_n)),v^{\mathcal{N}}(\varphi(a_1),...,\varphi(a_n))),$$

where $u^{\mathcal{N}}$ and $v^{\mathcal{N}}$ are term operations on $\mathcal{N}$. Likewise let $u^{\mathcal{M}}$ and $v^{\mathcal{M}}$ be term operations on $\mathcal{M}$ corresponding to the terms u and v. Therefore, since the identity $u(x_1,\ldots,x_n)\approx v(x_1,\ldots,x_n)$ holds in $(\mathcal{M},E)$ it follows from the definition of Ω-valued map that

$$\begin{aligned}\bigwedge_{i=1}^{n}\mu(a_i) &\leq E(u^{\mathcal{M}}(a_1,...,a_n),v^{\mathcal{M}}(a_1,...,a_n))\\ &\leq F(\varphi(u^{\mathcal{N}}(a_1,...,a_n)),\varphi(v^{\mathcal{N}}(a_1,...,a_n))),\end{aligned}$$

thus

$$\bigwedge_{i=1}^{n}\mu(a_i)\leq F(\varphi(u^{\mathcal{N}}(a_1,...,a_n)),\varphi(v^{\mathcal{N}}(a_1,...,a_n))) \tag{4.8}$$

By lemma (4.2.10) it follows that

$$\bigwedge_{i=1}^{n}\mu(a_i)\leq F(\varphi(u^{\mathcal{M}}(a_1,...,a_n)),u^{\mathcal{N}}(\varphi(a_1),...,\varphi(a_n))) \tag{4.9}$$

and

$$\bigwedge_{i=1}^{n} \mu(a_i) \leq F(\varphi(v^{\mathcal{M}}(a_1, ..., a_n)), v^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))). \tag{4.10}$$

Hence by equations (4.8), (4.9) and (4.10) we have

$$\begin{aligned}
\bigwedge_{i=1}^{n} \mu(a_i) \;\; &\leq \;\; F(\varphi(u^{\mathcal{M}}(a_1, ..., a_n)), u^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \\
&\wedge \;\; F(\varphi(v^{\mathcal{M}}(a_1, ..., a_n)), v^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \\
&\wedge \;\; F(\varphi(u^{\mathcal{M}}(a_1, ..., a_n)), \varphi(v^{\mathcal{M}}(a_1, ..., a_n))) \\
&\leq \;\; F(u^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n)), \varphi(v^{\mathcal{M}}(a_1, ..., a_n))) \\
&\wedge \;\; F(\varphi(v^{\mathcal{M}}(a_1, ..., a_n)), v^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \quad \text{(By Transitivity)} \\
&\leq \;\; F(u^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n)), v^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))) \quad \text{(By Transitivity).}
\end{aligned}$$

Therefore by the definition of Ω-valued homomorphism, $\bigwedge_{i=1}^{n} \mu(a_i) = \bigwedge_{i=1}^{n} \nu(\varphi(a_i))$. Thus

$$\bigwedge_{i=1}^{n} \nu(\varphi(a_i)) \leq F(u^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n)), v^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))).$$

Hence by theorem (4.1.7) and proposition (4.2.9) we have that

$$\bigwedge_{i=1}^{n} \nu^{\varphi(M)}(\varphi(a_i)) \leq E^{\varphi(M)}(u^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n)), v^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n)).$$

□

Definition 4.2.12. Let $\varphi : (M, E) \to (N, F)$ be an Ω-valued homomorphism. Then if φ is injective (surjective) φ is named Ω-valued **monomorphism** (Ω-valued **epimorphism**). Hence if φ is an Ω-valued monomorphism and an Ω-valued epimorphism, then φ is an Ω-valued **isomorphism** and the relative Ω-algebras are called isomorphic.

Remark 4.2.13. From definition (3.2.8) and corollary (3.2.9) an Ω-valued isomorphism from an Ω-algebra $(\mathcal{M}, E)$ onto an Ω-algebra $(\mathcal{N}, F)$ as defined in (4.2.12) it follows that for all $a, b \in M$ $E(a, b) \leq F(\varphi(a), \varphi(b))$ and the inverse Ω-valued map φ^{-1} is an Ω-valued homomorphism such that for all

$\varphi(a), \varphi(b) \in N$, $F(\varphi(a), \varphi(b)) \leq E(\varphi^{-1}(\varphi(a)),$
$\varphi^{-1}(\varphi(b))) = E(a, b)$.

Corollary 4.2.14. *If the Ω-algebra $(\mathcal{M}, E)$ is isomorphic with the Ω-algebra $(\mathcal{N}, F)$ under the map φ, then for all $a, b \in M$, $E(a, b) = F(\varphi(a), \varphi(b))$.*

4.3 Quotient Ω-Algebras

Now we turn our attention to defining Ω-quotient algebras which are derived Ω-algebras. Of course, these algebras are induced by Ω-valued congruence relations. As in the classical case quotient algebras play fundamental role in dealing with the structure of algebras.
In ([82]), Samhan presented a notion of "fuzzy factorization" in which a crisp algebra is factorized by a fuzzy congruence. Recently in ([3]), Bělohlávek and Vychodil, presented a new approach in the generalization of fuzzy quotient algebras in which case the base algebras are equipped with fuzzy equalities, in a way generalizing the notion of a fuzzy factorization.
Here we present a different approach to existing works as mentioned. In our approach the base algebra equipped with an equality, is an algebra not necessarily fulfilling any particular axiom. And our equality (i.e.,Ω-valued equality) which is a weak fuzzy equality in the sense that reflexivity property is weakened , it is strict.

Definition 4.3.1. Let θ be an Ω-valued congruence on the Ω-algebra $(\mathcal{M}, E)$. For an element $a \in M$, the **block** of an Ω-valued congruence θ on $(\mathcal{M}, E)$ is a mapping $[a]^\theta : M \to \Omega$, such that for $x \in M$, $[a]^\theta(x) := \theta(a, x)$. The collection of all blocks, the **quotient set** of M over θ, is denoted by M/θ. Operations on M/θ are defined as induced by the operations on $\mathcal{M}$: for an n-ary operation f,

$$f([a_1]^\theta, \ldots, [a_n]^\theta) := [f(a_1, \ldots, a_n)]^\theta. \tag{4.11}$$

Remark 4.3.2. We observe that these blocks are functions and they can be equal. In our approach, we consider them to be different, since each of these functions is denoted (indexed) by the element to which it is associated: $[a]^\theta$ is a function denoted by a, and suppose that for $b \in M$, $b \neq a$, $[a]^\theta(x) = [b]^\theta(x)$, for all $x \in M$. In our approach these functions are denoted by different elements (a and b), hence they are distinct elements (blocks, functions) in M/θ.

By definition (4.3.1), it is clear that $\mathcal{M}/\theta = (M/\theta, F)$ is an algebra isomorphic with $\mathcal{M}$ under $[x]^\theta \mapsto x$. It can be endowed with an Ω-valued equality E^θ as follows:

$$E^\theta([a]^\theta, [b]^\theta) := \theta(a, b). \tag{4.12}$$

It is clear by definition 4.12 that E^θ is an Ω-valued compatible equivalence relation on $\mathcal{M}/\theta$. In this way we obtain an Ω-algebra $(\mathcal{M}/\theta, E^\theta)$ in which E^θ is not separated. In addition:

$$\mu^\theta : (\mathcal{M}/\theta, E^\theta) \to \Omega \text{ is defined by } \mu^\theta([x]^\theta) := E^\theta([x]^\theta, [x]^\theta). \tag{4.13}$$

Obviously, each $[a]^\theta \in (\mathcal{M}/\theta, E^\theta)$ for $a \in M$, is a special Ω-set.

Proposition 4.3.3. *If θ is an Ω-valued congruence on $(\mathcal{M}, E)$ and $a, b \in M$. Then for every $c \in M$, $[a]^\theta(c) = [b]^\theta(c)$ if, and only if $\theta(a, a) = \theta(a, b) = \theta(b, b)$.*

Proof. ($if \quad part$): Assuming $\theta(a, a) = \theta(a, b) = \theta(b, b)$. Then we need show that $\forall c \in M$, $[a]^\theta(c) = [b]^\theta(c)$.
Firstly, $[a]^\theta(c) = \theta(a, c)$ then

$$\begin{aligned} \theta(a, c) &= \theta(a, a) \wedge \theta(a, c) \\ &= \theta(b, a) \wedge \theta(a, c) \qquad \text{(by assumption)} \\ &\leq \theta(b, c) \qquad \text{(by transitivity of } \theta \text{).} \end{aligned}$$

The other inequality is analogous.
Hence, $\theta(a, c) = \theta(b, c)$,
($only \quad if \quad part$): Let us assume that for every $c \in M$, $[a]^\theta(c) = [b]^\theta(c)$. Then $\theta(a, c) = \theta(b, c)$ for each $c \in M$. In particular

$$\theta(a, b) = \theta(b, b)$$

and

$$\theta(b, a) = \theta(a, a).$$

Hence

$$\begin{aligned} \theta(a, a) &= \theta(b, a) = \theta(a, b) \qquad \text{(by symmetry of } \theta) \\ &= \theta(b, b). \end{aligned}$$

This completes the proof. □

Proposition 4.3.4. *Let θ be an Ω-valued congruence on $(\mathcal{M}, E)$, e a constant symbol in the language of $\mathcal{M}$. Then, for an $a \in M$ and for every $c \in M$, $[e]^\theta(c) = [a]^\theta(c)$, if, and only if $\theta(e, a) = 1$.*

Proof. (*only if part*): Suppose for every $c \in M$, $[e]^\theta(c) = [a]^\theta(c)$, then by proposition (4.3.3) $\theta(e,e) = \theta(e,a) = \theta(a,a)$. Since $\theta(e,e) = E(e,e) = 1$, then $\theta(e,a) = \theta(e,e) = 1$ implies $\theta(e,a) = 1$.
(*if part*): Assuming that $\theta(e,a) = 1$, then by the strictness property $1 = \theta(e,a) \leq \theta(e,e) \wedge \theta(a,a)$ implies that $1 = \theta(e,e) \wedge \theta(a,a)$. Thus we have $\theta(e,e) = \theta(a,a) = 1$ and so $\theta(e,e) = \theta(e,a) = \theta(a,a)$. Hence for every $c \in M$, $[e]^\theta(c) = [a]^\theta(c)$. □

Corollary 4.3.5. *Let θ be an Ω-valued congruence on $(\mathcal{M}, E)$, **e** a constant symbol in the language of $\mathcal{M}$. Therefore, for $a \in M$, if $[e]^\theta(c) = [a]^\theta(c)$ for every $c \in M$, then $E(a,a) = 1$.*

Remark 4.3.6. The conditions as presented in propositions (4.3.3) and (4.3.4) by which two blocks in $(\mathcal{M}/\theta, E^\theta)$ are equal is a special case of the same notion as presented in ([82]) and ([3]). That is for $[a]^\theta, [b]^\theta \in (\mathcal{M}/\theta, E^\theta)$ and for every $c \in M$, $[a]^\theta(c) = [b]^\theta(c) \iff \theta(a,b) = 1$. Clearly, if we take the Ω-valued equivalence relation $\theta : M^2 \to \Omega$ to be the characteristic function on the algebra, $\mathbf{M}^2$, i.e. the diagonal entries are all 1, then our condition for every $c \in M$, $[a]^\theta(c) = [b]^\theta(c)$ if, and only if $\theta(a,a) = \theta(a,b) = \theta(b,b)$ is equivalent to $[a]^\theta(c) = [b]^\theta(c)$ if, and only if $\theta(a,b) = 1$ for every $c \in M$.

In what follows we turn our attention to present a theorem analogous to a known theorem in universal algebra, usually called **the homomorphism theorem**.

Theorem 4.3.7. *Let $(\mathcal{M}, E)$ be an Ω-algebra and θ an Ω-valued congruence relation on $(\mathcal{M}, E)$. Then the mapping $\pi_\theta : (\mathcal{M}, E) \to (\mathcal{M}, E)/\theta$ where $\pi_\theta(a) = [a]^\theta$ is an Ω-valued homomorphism. This is called the **canonical quotient map**.*

Proof. π_θ is an Ω-valued map from $(\mathcal{M}, E)$ to $(M/\theta, E^\theta)$, since $E^\theta([x]^\theta, [y]^\theta) = \theta(x,y)$, and $E \leq \theta$. In addition, π_θ satisfies properties (4.4) to (4.7): for $a, a_1, \ldots, a_n \in M$, and an n-ary operational symbol f from the language,

$$E(a,a) = \theta(a,a) = E^\theta([a]^\theta, [a]^\theta) = E^\theta(\varphi(a), \varphi(a));$$

$$\begin{aligned}
&\mu(a_1) \wedge \cdots \wedge \mu(a_n) = E(a_1,a_1) \wedge \cdots \wedge E(a_n,a_n) = \\
&\theta(a_1,a_1) \wedge \cdots \wedge \theta(a_n,a_n) = \\
&E^\theta([a_1]^\theta, [a_1]^\theta) \wedge \cdots \wedge E^\theta([a_n]^\theta, [a_n]^\theta) = \\
&\theta(\pi_\theta(a_1), \pi_\theta(a_1)) \wedge \cdots \wedge \theta(\pi_\theta(a_n), \pi_\theta(a_n)) \leq \\
&\theta(f(\pi_\theta(a_1), \ldots, \pi_\theta(a_n)), f(\pi_\theta(a_1), \ldots, \pi_\theta(a_n))) = \\
&\theta(\pi_\theta(f(a_1, \ldots, a_n)), f(\pi_\theta(a_1), \ldots, \pi_\theta(a_n))),
\end{aligned}$$

by the definition (4.11) of operations on classes. If c is a constant in $\mathcal{M}$, then $\pi_\theta(c) = [c]^\theta = 1$, hence

$$E^\theta(\pi_\theta([c]^\theta, [c]^\theta)) = \theta(c, c) = 1 = \mu(c) = E(c, c),$$

and (4.6) holds. Finally, (4.7) holds, since π_θ is a bijection. □

Theorem 4.3.8. *Let $(\mathcal{M}, E)$, $(\mathcal{N}, F)$ be two Ω-algebras, φ an Ω-valued homomorphism from $(\mathcal{M}, E)$ to $(\mathcal{N}, F)$, and K^φ the Ω-valued kernel of φ, as defined by* (3.10). *Let also $(\mathcal{N}, E^{\varphi(\mathcal{M})})$ be the Ω-subalgebra of $(\mathcal{N}, F)$, determined by φ. Then, the map $\psi : M/K^\varphi \to N$, given by $\psi([x]^{K^\varphi}) := \varphi(x)$ is an Ω-valued homomorphism from $(\mathcal{M}, E)/K^\varphi$ onto $\varphi((\mathcal{M}, E))$.*

Proof. The map ψ is an Ω-valued function:

$$E^{K^\varphi}([a]^{K^\varphi}, [b]^{K^\varphi}) = K^\varphi(a, b) \leq F(\varphi(a), \varphi(b)) = F(\psi([a]^{K^\varphi}), \psi([b]^{K^\varphi})),$$

by (3.10). It is an Ω-valued homomorphism:

$$(\mu^{K^\varphi})([a]) = E^{K^\varphi}([a]^{K^\varphi}, [a]^{K^\varphi}) = K^\varphi(a, a) = F(\varphi(a), \varphi(a)) \wedge \mu(a) = \nu(\varphi(a)) = \nu(\psi([a]^{K^\varphi}),$$

by the definitions of an Ω-valued kernel, and of the function ψ. Next,

$$\begin{aligned} &\bigwedge_{i=1}^{n} \mu^{K^\varphi}([a_i]^{K^\varphi}) = \bigwedge_{i=1}^{n} K^\varphi(a_i, a_i) = \bigwedge_{i=1}^{n} \mu(a_i) \leq \\ &F(\varphi(f^{\mathcal{M}}(a_1, \ldots, a_n)), f^{\mathcal{N}}(\varphi(a_1), \ldots, \varphi(a_n))) = \\ &F(\psi([f^{\mathcal{M}}(a_1, \ldots, a_n)]^{K^\varphi}), f^{\mathcal{N}}(\psi([a_1]^{K^\varphi}), \ldots, \psi([a_n]^{K^\varphi}))), \end{aligned}$$

since φ is an Ω-valued homomorphism.

□

Remark 4.3.9. Let us mention that Theorem 4.3.8 shows that the quotient Ω-algebra over an Ω-valued congruence turns out to be essentially different from the classical quotient algebra. As indicated in Remark 4.3.2, equal blocks (functions) need not be compatible under operations, hence they are treated as different objects, denoted by elements of the underlying algebra. Therefore, in the case of an Ω-valued homomorphism, the quotient Ω-structure with respect to the kernel is not isomorphic with the image subalgebra, as in the classical case. Still, the cut structures do possess this property.

4.4 Direct Products

Generally, larger algebraic structures can be constructed from a set of smaller ones. A well known construction is the direct product. Therefore, we introduce the direct product of an indexed family of Ω-algebras.

Definition 4.4.1. let I be an indexing set and $\{(\mathcal{M}_i, E^i) \mid i \in I$ and $E^i : M_i^2 \to \Omega\}$ a family of Ω-algebras . Then a **direct product** of this family is an Ω-algebra $(\mathcal{M}, E) = \Pi\{(\mathcal{M}_i, E^i) \mid i \in I\}$ which is an Ω-set according to definition (3.2.16) and also for each n-ary operation $f^{\mathcal{M}}$ on the algebra $\mathcal{M}$ and each $a_1, ..., a_n \in M$ we have

$$f^{\mathcal{M}}(a_1, ..., a_n)(i) = f^{\mathcal{M}_i}(a_1^i, ..., a_n^i). \tag{4.14}$$

$(\mathcal{M}, E)$ as defined in (4.4.1) above satisfies the properties of an Ω-algebra: From remark (3.2.17) it is clear that E is μ-reflexive, symmetric, transitive and satisfies the strictness property.

Hence, for each n-ary operation $f^{\mathcal{M}}$ on the algebra $\mathcal{M}$ and for each $a_1, ..., a_n, b_1, ..., b_n \in M$;

$$\begin{aligned}
\bigwedge_{j=1}^{n} E(a_j, b_j) &= \bigwedge_{i\in I} E^i(a_1^i, b_1^i) \wedge ... \wedge \bigwedge_{i\in I} E^i(a_n^i, b_n^i) \\
&= \bigwedge_{i\in I}(\bigwedge_{j=1}^{n}(E^i(a_j^i, b_j^i))) \\
&\leq \bigwedge_{i\in I} E^i(f^{\mathcal{M}_i}(a_1^i, ..., a_n^i), f^{\mathcal{M}_i}(b_1^i, ..., b_n^i)) \\
&= E(f^{\mathcal{M}}(a_1, ..., a_n), f^{\mathcal{M}}(b_1, ..., b_n)).
\end{aligned}$$

So E is compatible with the operations on $\mathcal{M}$.

Proposition 4.4.2. *Let $\mathcal{M}_E$ be a direct product of the family $\{\mathcal{M}^i_{E^i} \mid i \in I,\ \mu^i(a^i) := E^i(a^i, a^i)\}$ of Ω-algebras and the Ω-valued function, $\mu : M \to \Omega$ determined by E (i.e., $\mu(a) := E(a, a)$). Then μ an Ω-subalgebra of $\mathcal{M}$.*

Proof. Let $a_1, ..., a_n \in M$ and for each n-ary operations $f^{\mathcal{M}^i}$ defined on

$M^i,\ i \in I$, then

$$\begin{aligned}
\bigwedge_{j=1}^{n} \mu(a_j) &= \bigwedge_{j=1}^{n} E(a_j, a_j) \\
&= \bigwedge_{i\in I} \bigwedge_{j=1}^{n} E^i(a_j^i, a_j^i) \\
&\leq \bigwedge_{i\in I} E^i(f^{\mathcal{M}^i}(a_1^i, ..., a_n^i), f^{\mathcal{M}^i}(a_1^i, ..., a_n^i)) \\
&= \bigwedge_{i\in I} \mu^i(f^{\mathcal{M}^i}(a_1^i, ..., a_n^i)) \\
&= \mu(f^{\mathcal{M}}(a_1, ..., a_n)).
\end{aligned}$$

□

Lemma 4.4.3. *The mapping $P_j : (\mathcal{M}, E) \to (\mathcal{M}_j, E^j)$ for each $j \in I$, is an Ω-valued epimorphism.*

Proof. By lemma (3.2.18), the mapping, is an Ω-valued map. Clearly, P_j is surjective. Now for each n-ary operation $f^{\mathcal{M}}$ on the algebra $\mathcal{M}$ and for each $a_1, ..., a_n \in \mathcal{M}$,

$$\begin{gathered}
E^j(P_j(f^{\mathcal{M}}(a_1, ..., a_n)), f^{\mathcal{M}_j}(P_j(a_1), ..., P_j(a_n))) \\
= E^j(P_j(f^{\mathcal{M}}(a_1, ..., a_n)), f^{\mathcal{M}_j}(a_1^j, ..., a_n^j)) \quad \text{(By definition of } P_j\text{)} \\
= E^j((f^{\mathcal{M}}(a_1, ..., a_n))(j), f^{\mathcal{M}_j}(a_1^j, ..., a_n^j)) \quad \text{(By definition of } P_j\text{)} \\
= E^j(f^{\mathcal{M}_j}(a_1^j, ..., a_n^j), f^{\mathcal{M}_j}(a_1^j, ..., a_n^j)) \quad \text{(By equation4.14)} \\
\geq \bigwedge_{k=1}^{n} E^j(a_k^j, a_k^j) \geq \bigwedge_{i\in I}(\bigwedge_{k=1}^{n} E^i(a_k^i, a_k^i)) \\
= \bigwedge_{k=1}^{n} E(a_k, a_k) = \bigwedge_{k=1}^{n} \mu(a_k).
\end{gathered}$$

Thus P_j is an Ω-valued homomorphism. □

The map P_j as above is referred to as the **projection map**.

Ordinary direct products of algebras do satisfy the same identities as satisfied by each of the algebras in the family. Analogously we prove that for an Ω-valued direct product of a family of Ω-algebras of the same type fulfill the same identities.

Theorem 4.4.4. *Let $\{(\mathcal{M}_i, E^i) \mid i \in I\}$ be a family of Ω-algebras of the same type and let the identity $u(x_1, ..., x_n) \approx v(x_1, ..., x_n)$ holds for each Ω-algebra in the family. Then $E(u(x_1, ..., x_n), v(x_1, ..., x_n))$, holds also in $(\mathcal{M}, E)$.*

Proof. Suppose the identity $u(x_1, ..., x_n) \approx v(x_1, ..., x_n)$ holds for each Ω-algebra in the family, where $u(x_1, ..., x_n), v(x_1, ..., x_n)$ are terms in the language of the Ω-algebras $(\mathcal{M}_i, E^i)$. Then for all $a_1^i, ..., a_n^i \in M_i,\ i \in I$

$$\bigwedge_{j=1}^{n} \mu^i(a_j^i) \leq E^i(u^{\mathcal{M}_i}(a_1^i, ..., a_n^i), v^{\mathcal{M}_i}(a_1^i, ..., a_n^i)).$$

Therefore, let $a_1, ..., a_n \in M$, then

$$\begin{aligned}
\bigwedge_{k=1}^{n} \mu(a_k) = & \bigwedge_{k=1}^{n}(\bigwedge_{i\in I} \mu^i(a_k)) = \bigwedge_{i\in I}(\bigwedge_{k=1}^{n} \mu^i(a_k)) \\
\leq & \bigwedge_{i\in I} E^i(u^{\mathcal{M}_i}(a_1^i, ..., a_n^i), v^{\mathcal{M}_i}(a_1^i, ..., a_n^i)) \\
= & \bigwedge_{i\in I} E^i(u^{\mathcal{M}_i}(a_1, ..., a_n)(i), v^{\mathcal{M}_i}(a_1, ..., a_n)(i)) \\
= & E(u^{\mathcal{M}}(a_1, ..., a_n), v^{\mathcal{M}}(a_1, ..., a_n)).
\end{aligned}$$

Thus we have shown that identities are preserved over taking direct products. □

4.5 Cut Properties

Next we observe that for an Ω-valued equality E the followings properties hold as a consequence of well known properties of cut sets. The following is a well known property of cut set.

Proposition 4.5.1. *Let E be an Ω-valued equality on the algebra $\mathcal{M}$, then the following holds;*

if $p, q \in \Omega$ and $p \leq q$, then $E_q \subseteq E_p$.

Let $E_\Omega = \{E_p : p \in \Omega\}$ be the collection of all cut relations of an Ω equality E, then the following proposition holds.

Proposition 4.5.2. *Let $E : M^2 \to \Omega$ be an Ω-valued equality. Then for $\Omega_1 \subset \Omega$,*

$$\bigcap\{E_p : p \in \Omega_1,\} \in E_\Omega.$$

Proof. Let $r = \bigvee_{p \in \Omega_1} p$. We claim that $E_r = \bigcap\{E_p : p \in \Omega_1,\}$. Now for $x, y \in M$ then

$$\begin{aligned}
(x, y) \in E_r \quad & \Longleftrightarrow E(x, y) \geq r \\
& \Longleftrightarrow E(x, y) \geq p, \ \forall p \in \Omega_1 \\
& \Longleftrightarrow (x, y) \in E_p, \ \forall p \in \Omega_1 \\
& \Longleftrightarrow (x, y) \in \bigcap_{p \in \Omega_1} E_p.
\end{aligned}$$

□

Obviously, for all $p \in \Omega$ the union of the collection of all cut relations E_p of E on M is the whole set, M^2.

Therefore, from theorem (3.1.9) we have the following theorem.

Theorem 4.5.3. *Let $E : M^2 \to \Omega$ be an Ω-valued equality on the algebra $\mathcal{M}$. Then the collection E_Ω of all cut relations on E forms a complete lattice under inclusion.*

Clearly, the above complete lattice turns out to be a closure system of family of subsets of M^2, since the cut relation $E_0 = M^2$. Therefore in the lattice the meet operation is the set intersection on E_Ω and the lattice join operation $\vee$ is given by $E_p \vee E_q = \bigcap\{E_r \in E_\Omega : E_p \cup E_q \subseteq E_r\}$.

Corollary 4.5.4. *Let $(\mathcal{M}, E)$ be an Ω-algebra and μ be the function on $\mathcal{M}$, determined by $\mu(x) = E(x, x)$. Then for each $p \in \Omega$ the cut relation E_P is a congruence on its corresponding cut set μ_p.*

Proof. We prove that E_p is a congruence relation on μ_p. It is an equivalence relation on μ_p by Lemma (3.2.4). In addition, E_p is compatible with the operations in μ_p. Indeed, take $f \in \mathbb{F}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mu_p$, and suppose that for $i = 1, \ldots, n$, $(x_i, y_i) \in E_p$. Then for every i, $E(x_i, y_i) \geq p$. Now, we have that

$$E(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \geq \bigwedge_{i=1}^{n} E(x_i, y_i) \geq p,$$

i.e., $(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)) \in E_p$. $\square$

Theorem 4.5.5. *Let $\mathcal{M} = (M, F)$ be an algebra. Then $(\mathcal{M}, E)$ is an Ω-algebra in the same language as $\mathcal{M}$ fulfilling a set of identities $\mathcal{I}$ if and only if for every $p \in \Omega$ the quotient structure μ_p/E_p is a classical algebra fulfilling the same set of identities.*

Proof. Let $(\mathcal{M}, E)$ be an Ω-algebra satisfying the set of identities $\mathcal{I}$ and let for every $p \in \Omega$ we consider the quotient set of μ_p over E_p denoted by μ_p/E_p. Furthermore, we denote each n-ary operation $f \in \mathbb{F}$ on congruence classes by f_p. These operations are introduced in a natural way (by class representatives) and it is easily proved that they are well defined. Hence the structure $(\mu_p/E_p, \{f_p \mid f \in \mathbb{F}\})$ is an algebra of the same type as $\mathcal{M}$. Then we have to prove that this classical algebra satisfies the same identities in $\mathcal{I}$. For $x_1, ..., x_n \in \mu_p$, let $[x_1]_{E_p}, ..., [x_n]_{E_p} \in \mu_p/E_p$, and $u(x_1, ..., x_n)$ and $v(x_1, ..., x_n)$ be terms in the language of $\mathcal{M}$. Therefore, since $(\mathcal{M}, E)$ is an Ω-algebra and for each identity $u(x_1, ..., x_n) \approx v(x_1, ..., x_n)$ that holds in $(\mathcal{M}, E)$, $E(u(x_1, ..., x_n), v(x_1, ..., x_n)) \geq \bigwedge_{i=1}^{n} \mu(x_i) \geq p$, hence $(u(x_1, ..., x_n), v(x_1, ..., x_n)) \in E_p$. Therefore, for $x_1, ..., x_n \in \mu_p$, it follows that $u([x_1]_{E_p}, ..., [x_n]_{E_p}) = [u(x_1, ..., x_n)]_{E_p} = [v(x_1, ..., x_n)]_{E_p} = v([x_1]_{E_p}, ..., [x_n]_{E_p})$. So we proved that for each operation f_p in $(\mu_p/E_p, \{f_p \mid f \in F\})$ the classical identity analogous to $E(u(x_1, ..., x_n), v(x_1, ..., x_n))$ holds in $(\mu_p/E_p, \{f_p \mid f \in \mathbb{F}\})$.
Conversely, suppose that for every $p \in \Omega$ the quotient structures $(\mu_p/E_p, \{f_p \mid f \in \mathbb{F}\})$ fulfills the statement of the theorem, then we show that $(\mathcal{M}, E)$ is an Ω-algebra in the language of $\mathcal{M}$. For $x_1, ..., x_n \in M$ let $\bigwedge_{i=1}^{n} \mu(x_i) = p$, then $x_1, ..., x_n \in \mu_p$ and since $(\mu_p/E_p, \{f_p \mid f \in \mathbb{F}\})$ fulfills the statement of the theorem by assumption and for terms $u(x_1, ..., x_n)$ and $v(x_1, ..., x_n)$ in the language of $\mathcal{M}$ we have $u([x_1]_{E_p}, ..., [x_n]_{E_p}) = v([x_1]_{E_p}, ..., [x_n]_{E_p})$, hence $[u(x_1, ..., x_n)]_{E_p} = [v(x_1, ..., x_n)]_{E_p}$, and therefore, $E(u(x_1, ..., x_n), v(x_1, ..., x_n)) \geq p = \bigwedge_{i=1}^{n} \mu(x_i)$. Thus we proved the theorem.

$\square$

Observe that an Ω-valued equality E is compatible with the operations of $\mathcal{M}$ if and only if each E_p is compatible with the operations of the corresponding (basic) algebra $\mathcal{M} = (M, F)$.
Clearly the restriction $\varphi_{\restriction \mu_p}$ of φ to the subalgebra μ_p, $p \in \Omega$ of $\mathcal{M}$ is necessar-

ily not an ordinary homomorphism from μ_p into ν_p a subalgebra of $\mathcal{N}$, since our mapping between our basic algebras is not necessarily a homomorphism.

Theorem 4.5.6. *An Ω-valued map $\varphi : M \to N$ from an Ω-algebra $(\mathcal{M}, E)$ to an Ω-algebra $(\mathcal{N}, E)$ of the same type is an Ω-valued homomorphism, if and only if for every $p \in \Omega$, the mapping $\bar{\varphi} : \mu_p/E_p \to \nu_p/F_p$, defined by*

$$\bar{\varphi}([x]_{E_p}) = [\varphi(x)]_{E_p}, \tag{4.15}$$

is an ordinary homomorphism.

Proof. Assume that φ is an Ω-valued homomorphism. Then for $a_1 \dots, a_n \in \mu_p$ we have

$$F(\varphi(f(a_1, \dots, a_n)), f(\varphi(a_1), \dots, \varphi(a_n))) \geq \mu(a_1) \wedge \dots \wedge \mu(a_n) \geq p,$$

implying

$$(\varphi(f^{\mathcal{M}}(a_1, \dots, a_n)), f^{\mathcal{N}}(\varphi(a_1), \dots, \varphi(a_n))) \in E_p^{\nu}.$$

Hence, for an n-ary fundamental operation f,

$$\begin{aligned} &f(\bar{\varphi}([a_1]_{E_p}), \dots, \bar{\varphi}([a_n]_{E_p})) = f([\varphi(a_1)]_{F_p}, \dots, [\varphi(a_n)]_{F_p}) = \\ &[f(\varphi(a_1), \dots, \varphi(a_n))]_{F_p} = [\varphi(f(a_1, \dots, a_n))]_{F_p} = \\ &[\varphi(f(a_1, \dots, a_n))]_{F_p} = \bar{\varphi}([f(a_1, \dots, a_n)]_{E_p}) = \\ &\bar{\varphi}(f([a_1]_{E_p}, \dots, [a_n]_{E_p})). \end{aligned}$$

Therefore, the mapping $\bar{\varphi}$ is a homomorphism.

For the converse, assume that for every $p \in \Omega$, $\bar{\varphi}$ is a homomorphism and let $a_1 \dots, a_n \in M$ such that $\mu(a_1) \wedge \dots \wedge \mu(a_n) = p$, hence $a_1 \dots, a_n \in \mu_p$. Now, since $\bar{\varphi}_p$ is a homomorphism, we have

$$\bar{\varphi}(f([a_1]_{E_p}, \dots, [a_n]_{E_p})) = f(\bar{\varphi}([a_1]_{E_p}), \dots, \bar{\varphi}([a_n]_{E_p})),$$

implying

$$[\varphi(f(a_1, \dots, a_n))]_{F_p} = f([\varphi(a_1)]_{F_p}, \dots, [\varphi(a_n)]_{F_p}) = [f(\varphi(a_1), \dots, \varphi(a_n))]_{F_p}.$$

Hence,

$$F(\varphi(f(a_1, \dots, a_n)), f(\varphi(a_1), \dots, \varphi(a_n))) \geq p = \mu(a_1) \wedge \dots \wedge \mu(a_n).$$

Therefore, the mapping φ is an Ω-valued homomorphism. This completes the proof. $\square$

Corollary 4.5.7. *Let $(\mathcal{M}, E)$ and $(\mathcal{N}, F)$ be Ω-algebras, and $\varphi : M \to N$ is an Ω-valued homomorphism from $(\mathcal{M}, E)$ to $(\mathcal{N}, F)$. Let θ and ϑ be Ω-valued congruence relations on $(\mathcal{M}, E)$ and $(\mathcal{N}, F)$ respectively such that for all $a, b \in M$, $\theta(a, b) \leq \vartheta(\varphi(a), \varphi(b))$. Then $\bar{\varphi}_p : \mu_p/\theta_p \to \nu_p/\vartheta_p$, $\bar{\varphi}_p([x]^{\theta_p}) := [\varphi(x)]^{\vartheta_p}$ is an ordinary homomorphism for each $p \in \Omega$.*

Proof. Let $\bar{\varphi}_p$ be defined by, $\bar{\varphi}_p([x]^{\theta_p}) = [\varphi(x)]^{\vartheta_p}$. Let $a_1, ..., a_n \in \mu_p$ and assume φ an Ω-valued homomorphism. Then we have

$$E(\varphi(f^{\mathcal{M}}(a_1, \ldots, a_n)), f^{\mathcal{N}}(\varphi(a_1), \ldots, \varphi(a_n))) \geq \mu(a_1) \wedge \ldots \wedge \mu(a_n) \geq p,$$

implying

$$(\varphi(f^{\mathcal{M}}(a_1, \ldots, a_n)), f^{\mathcal{N}}(\varphi(a_1), \ldots, \varphi(a_n))) \in E_p.$$

Hence by definition (4.2.1),

$$\theta(\varphi(f^{\mathcal{M}}(a_1, \ldots, a_n)), f^{\mathcal{N}}(\varphi(a_1), \ldots, \varphi(a_n))) \geq p$$

and so

$$(\varphi(f^{\mathcal{M}}(a_1, \ldots, a_n)), f^{\mathcal{N}}(\varphi(a_1), \ldots, \varphi(a_n))) \in \theta_p.$$

Therefore for n-ary operations f^{μ_p} and f^{ν_p},

$$\begin{aligned}
&\bar{\varphi}_p(f^{\mu_p/\theta_p}([a_1]^{\theta_p}, \ldots, [a_n]^{\theta_p})) = \bar{\varphi}_p([f^{\mu_p}(a_1, \ldots, a_n)]^{\theta_p}) = \\
&[\varphi(f^{\mu_p}(a_1, \ldots, a_n))]^{\vartheta_p} = [f^{\nu_p}(\varphi(a_1), \ldots, \varphi(a_n))]^{\vartheta_p} = \\
&f^{\nu_p/\vartheta_p}([\varphi(a_1)]^{\vartheta_p}, \ldots, [\varphi(a_n)]^{\vartheta_p}) = f^{\nu_p/\vartheta_p}(\bar{\varphi}_p([a_1]^{\theta_p}), \ldots, \bar{\varphi}_p([a_n]^{\theta_p})).
\end{aligned}$$

Therefore $\bar{\varphi}_p$ is an ordinary homomorphism for each $p \in \Omega$.

□

The following corollary is a consequence of proposition (4.3.3).

Corollary 4.5.8. *Let $[a]^\theta, [b]^\theta \in (\mathcal{M}/\theta, E^\theta)$ and $[a]^\theta = [b]^\theta$. Then for each $p \in \Omega$, $(a, a), (b, b) \in \theta_p$ if, and only if $(a, b) \in \theta_p$.*

Remark 4.5.9. Corollary (4.5.8) implies that for $[a]^\theta, [b]^\theta \in (\mathcal{M}/\theta, E^\theta)$, $[a]^\theta = [b]^\theta$ if, and only if a and b belongs to the same congruence classes for each classical quotient structures μ_p/θ_p, $[a]^{\theta_p} = [b]^{\theta_p}$, $p \in \Omega$.
It is possible that a and b do not belong to any congruence class since we deal with weak congruences (i.e., either they are in the same congruence classes or they both do not belong to any congruence class).

Theorem 4.5.10. *Let θ be an Ω-valued equivalence on an Ω-algebra $(\mathcal{M}, E)$. Then θ is an Ω-valued congruence on $(\mathcal{M}, E)$ if and only if for every $p \in \Omega$ such that $\mu_p \neq \emptyset$, the mapping $\phi_p : \mu_p \to \mu_p/\theta_p$ given by $\phi_p(x) = [x]^{\theta_p}$, is a classical homomorphism.*

Proof. Assume that θ is an Ω-valued congruence on $(\mathcal{M}, E)$ then for each $p \in \Omega$, θ_p is a classical congruence relation the corresponding cut set $\mu_p \neq \emptyset$. Therefore, for $a_1, \ldots, a_n \in \mu_p$, $[a_1]^{\theta_p}, \ldots, [a_n]^{\theta_p} \in \mu_p/\theta_p$. Furthermore since μ_p/θ_p is a quotient structure then $f([a_1]^{\theta_p}, \ldots, [a_1]^{\theta_p}) \in \mu_p/\theta_p$. Clearly, ϕ_p is a well defined function. It is a homomorphism: Let f be an n-ary fundamental operation in the language of $\mathcal{M}$. Then

$$\begin{aligned}
\phi_p(f(a_1, \ldots, a_n)) &= [f(a_1, \ldots, a_n)]^{\theta_p} \\
&= f([a_1]^{\theta_p}, \ldots, [a_1]^{\theta_p}) \\
&= f(\phi_p(a_1), \ldots, \phi_p(a_n)),
\end{aligned}$$

since μ_p/θ_p is a quotient structure. Conversely, suppose ϕ_p is a homomorphism. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mu_p$ and $(a_1, b_1), \ldots, (a_n, b_n) \in \theta_p$ such that $\bigwedge_{i=1}^{n} \theta(a_i, b_i) = p$. Thus $\theta(a_i, b_i) \geq p$, for each i, hence $(a_i, b_i) \in \theta_p$, for each i. Therefore, $[a_1]^{\theta_p} = [b_1]^{\theta_p}, \ldots, [a_n]^{\theta_p} = [b_n]^{\theta_p}$, hence for an n-ary fundamental operation, f in the language of $\mathcal{M}$ we have that $f([a_1]^{\theta_p}, \ldots, [a_n]^{\theta_p}) = f([b_1]^{\theta_p}, \ldots, [b_n]^{\theta_p})$. Thus

$$\begin{aligned}
& f([a_1]^{\theta_p}, \ldots, [a_n]^{\theta_p}) = f([b_1]^{\theta_p}, \ldots, [b_n]^{\theta_p}) \\
\Rightarrow \quad & f(\phi_p(a_1), \ldots, \phi_p(a_n)) = f(\phi_p(b_1), \ldots, \phi_p(b_n)) \ \text{(By mapping, } \phi_p) \\
\Rightarrow \quad & \phi_p(f(a_1, \ldots, a_n)) = \phi_p(f(b_1, \ldots, b_n)) \ \text{(since } \phi_p \text{ is a homomorphism)} \\
\Rightarrow \quad & [f(a_1, \ldots, a_n)]^{\theta_p} = [f(b_1, \ldots, b_n)]^{\theta_p} \ \text{(by definition of the mapping, } \phi_p).
\end{aligned}$$

Therefore, $\theta(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \geq p$. Thus $\theta(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \geq \bigwedge_{i=1}^{n} \theta(a_i, b_i)$. Hence we have proved that θ is compatible with n-ary fundamental operations in the language of $\mathcal{M}$. $\square$

Lemma 4.5.11. *Let θ be an Ω-valued congruence on an Ω-algebra $(\mathcal{M}, E)$. If θ is the greatest Ω-valued congruence on $(\mathcal{M}, E)$, then for every $p \in \Omega$, the cut relation θ_p is a (classical) full congruence relation on the corresponding cut μ_p of the Ω-subalgebra μ determined E.*

Proof. Let Ω-subalgebra μ be determined by E (i.e. $\mu(x) = E(x, x), \forall x \in M$). For $p \in \Omega$, let $x, y \in \mu_p$, then $\mu(x) \geq p$ and $\mu(y) \geq p$. Therefore since by our hypothesis θ is the greatest Ω-valued congruence on $(\mathcal{M}, E)$, then

$$\theta(x,y) = E(x,x) \wedge E(y,y) = \mu(x) \wedge \mu(y) \geq p,$$

thus $\theta(x,y) \geq p$. $\square$

Proposition 4.5.12. *Let $\mathcal{M}_E$ be a direct product of the family $\{(\mathcal{M}^i, E^i) \mid i \in I,\ E^i : (M^i)^2 \to \Omega\}$ of Ω-algebras. Then $E_p = \Pi_{i\in I} E^i_p$, for each $p \in \Omega$.*

Proof. Let $(a_i)_{i\in I}, (b_i)_{i\in I} \in M$ and $p \in \Omega$ such that $((a_i)_{i\in I}, (b_i)_{i\in I}) \in E_p$. Then

$$\begin{aligned}
((a_i)_{i\in I}, (b_i)_{i\in I}) \in E_p \iff & \quad E((a_i)_{i\in I}, (b_i)_{i\in I}) \geq p \\
\iff & \quad (\Pi_{i\in I} E^i)((a_i)_{i\in I}, (b_i)_{i\in I}) \geq p \\
\iff & \quad \bigwedge_{i\in I} E^i(a_i, b_i) \geq p \\
\iff & \quad E^i(a_i, b_i) \geq p \\
\iff & \quad (a_i, b_i) \in E^i_p \quad \text{(for each } i \in I) \\
\iff & \quad ((a_i)_{i\in I}, (b_i)_{i\in I}) \in (\Pi_{i\in I} E^i_p).
\end{aligned}$$

$\square$

Corollary 4.5.13. *Let $\mathcal{M}_E$ be a direct product of the family $\{\mathcal{M}^i_{E^i} \mid i \in I,\ \mu^i(a^i) := E^i(a^i, a^i)\}$ of Ω-algebras and $\mu : M \to \Omega$, an Ω-valued function on $\mathcal{M}$ determined by E (i.e., $\mu(a) := E(a,a)$). Then $\mu_p = \Pi_{i\in I} \mu^i_p$, for each $p \in \Omega$.*

The above corollary is an immediate consequence of proposition (4.5.12) and the proof follows easily from the proof of the lemma.

Lemma 4.5.14. *Let $\mathcal{M}_E$ be a direct product of the family $\{\mathcal{M}^i_{E^i} \mid i \in I\}$ of Ω-algebras. Then for each $p \in \Omega$, E_p is a congruence relation on μ_p if, and only if $\Pi_{i\in I} E^i_p$ is a congruence relation on $\Pi_{i\in I} \mu^i_p$.*

Proof. (*only if part*) : Let $(((a^i_1)_{i\in I}, ..., (a^i_n)_{i\in I}), ((b^i_1)_{i\in I}, ..., (b^i_n)_{i\in I})) \in \Pi_{i\in I} \mu^i_p$. Then by corollary (4.5.13), $(((a^i_1)_{i\in I}, ..., (a^i_n)_{i\in I}), ((b^i_1)_{i\in I}, ..., (b^i_n)_{i\in I})) \in \mu_p$, thus since E_p is a congruence relation on μ_p and for each n-ary operations $f^{\mathcal{M}}$ on $\mathcal{M}$, it follows that $f^{\mathcal{M}}(((a^i_1)_{i\in I}, ..., (a^i_n)_{i\in I})), f^{\mathcal{M}}(((b^i_1)_{i\in I}, ..., (b^i_n)_{i\in I})) \in E_p$ and so $p \leq E(f^{\mathcal{M}}((a^i_1)_{i\in I}, ..., (a^i_n)_{i\in I}), f^{\mathcal{M}}((b^i_1)_{i\in I}, ..., (b^i_n)_{i\in I}))$ $= \bigwedge_{i\in I} E^i(f^{\mathcal{M}^i}((a^i_1, a^i_2, ..., a^i_n), f^{\mathcal{M}^i}((b^i_1, b^i_2, ..., b^i_n))))$. Then $p \leq E^i(f^{\mathcal{M}^i}((a^i_1, a^i_2, .$

.., $a_n^i), f^{\mathcal{M}^i}((b_1^i, b_2^i, ..., b_n^i))))$ thus $(f^{\mathcal{M}^i}(a_1^i, a_2^i, ..., a_n^i), f^{\mathcal{M}^i}(b_1^i, b_2^i, ..., b_n^i))$
$\in E_p^i$, for each n-ary operations $f^{\mathcal{M}^i}$ on $\mathcal{M}^i$, $i \in I$. Hence,
$(f^{\Pi_{i\in I}\mathcal{M}^i}(((a_1^i)_{i\in I}, ..., (a_n^i)_{i\in I})), f^{\Pi_{i\in I}\mathcal{M}^i}(((b_1^i)_{i\in I}, ..., (b_n^i)_{i\in I})) \in \Pi_{i\in I}E_p^i$.
$(if\ part)$: The proof follows analogously from the above argument. □

4.6 Example

In this section we provide a concrete example to illustrate our work so-far.

Example 4.6.1. Let $\mathcal{M} = (M, \circ, ^{-1}, e)$ and $\mathcal{N} = (N, *, ^{-1}, è)$ be algebras with one binary, unary and constant functions, $M = \{e, a, b, c, d, f\}$ and $N = \{è, g, h, x, y, z\}$, We have following operation tables below;

$\circ$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	a	f	a	c
b	b	a	d	d	e	b
c	c	f	d	e	d	a
d	d	a	e	d	b	d
f	f	c	d	a	b	e

Table 1

$*$	$è$	g	h	x	y	z
$è$	$è$	g	h	x	y	z
g	g	$è$	g	g	y	g
h	h	g	z	h	y	$è$
x	x	g	h	$è$	y	z
y	y	y	y	y	$è$	y
z	z	g	$è$	z	y	h

Table 2

E	e	a	b	c	d	f
e	1	q	s	q	s	q
a	q	u	q	q	q	q
b	s	q	w	q	s	q
c	q	q	q	t	q	q
d	s	q	s	q	w	q
f	q	q	q	q	q	t

Table 3

F	$è$	g	h	x	y	z
$è$	1	t	s	s	q	s
g	t	u	q	q	q	q
h	s	q	w	s	q	s
x	s	q	s	k	q	s
y	q	q	q	q	t	q
z	s	q	s	s	q	w

Table 4

μ	e	a	b	c	d	f
	1	u	w	t	w	t

Table 5: Ω-valued function

ν	$è$	g	h	x	y	z
	1	u	w	k	t	w

Table 6: Ω-valued function

The cuts of μ and ν, i.e., subset $\mu^{-1}(\uparrow p)$ and $\nu^{-1}(\uparrow p)$, $p \in \Omega$, of M and N respectively are:

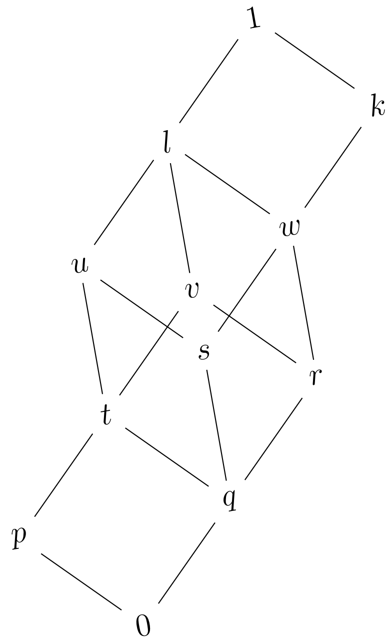

Figure 4.1: Ω

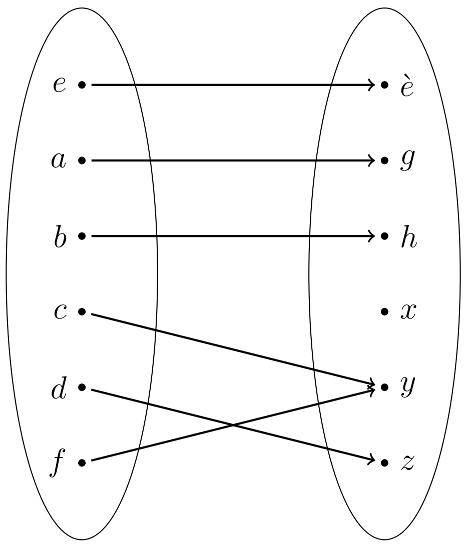

Figure 4.2: φ

$\mu_1 = \{e\} = \mu_k = \mu_l = \mu_v,$
$\mu_w = \{e, b, d\} = \mu_r,$
$\mu_u = \{e, a\},$
$\mu_t = \{e, a, c, f\} = \mu_p,$
$\mu_s = \{e, a, b, d\},$
$\mu_q = \{e, a, b, c, d, f\} = \mu_0.$

$\nu_1 = \{\grave{e}\} = \nu_l = \nu_v,$
$\nu_w = \{\grave{e}, h, x, z\} = \nu_r,$
$\nu_u = \{\grave{e}, g\}$,
$\nu_k = \{\grave{e}, x\},$
$\nu_t = \{\grave{e}, g, y\} = \nu_p,$
$\nu_s = \{\grave{e}, g, h, x, z\},$
$\nu_q = \{\grave{e}, g, h, x, y, z\} = \nu_0.$

The cuts of E and $F$are congruences on the corresponding cuts of μ and ν respectively:
$E_1 = \{(e, e)\} = E_k = E_l = E_v$; $E_u = \{(e, e), (a, a)\}$,
$E_w = \{(e, e), (b, b), (d, d)\} = E_r$; $E_t = \{(e, e), (a, a), (c, c), (f, f)\} = E_p$;
$E_s = \{(e, e), (a, a), (b, b), (d, d), (e, b), (b, e), (e, d), (d, e), (b, d), (d, d)\}$;
$E_q = \{\bigtriangledown_M\} = E_0$
$F_1 = \{(\grave{e}, \grave{e})\} = F_l = F_v$; $F_k = \{(\grave{e}, \grave{e}), (x, x)\}$;
$F_u = \{(\grave{e}, \grave{e}), (g, g)\}$;
$F_w = \{(\grave{e}, \grave{e}), (h, h), (x, x), (z, z)(\grave{e}, x), (x, \grave{e})\} = F_r$
$F_t = \{(\grave{e}, \grave{e}), (g, g), (y, y), (g, \grave{e}), (\grave{e}, g)\} = F_p$,
$F_s = \{(\grave{e}, \grave{e}), (g, g), (h, h), (x, x), (z, z), (h, x), (x, h), (h, z), (z, h), (z, x), (x, z), (\grave{e}, x), (x, \grave{e}), (\grave{e}, z), (z, \grave{e}), (\grave{e}, h), (h, \grave{e})\}$
$F_q = \{\bigtriangledown_N\} = F_0$.
Finally, the quotient structures μ_p/E_p and ν_p/F_p are as follows:

$\mu_1/E_1 = \{\{e\}\} = \mu_k/E_k$,
$\mu_u/E_u = \{\{e\},\{a\}\}$,
$\mu_w/E_w = \{\{e\},\{b\},\{d\}\}$
$\mu_t/E_t = \{\{e\},\{a\},\{c\},\{f\}\}$
$\mu_s/E_s = \{\{e,b,d\},\{a\}\}$.
$\mu_0/E_0 = \{\{e,a,b,c,d,f\}\}$,

$\nu_1/F_1 = \{\{è\}\}$,
$\nu_k/F_k = \{\{è\},\{x\}\}$,
$\nu_u/F_u = \{\{è\},\{g\}\}$,
$\nu_w/F_w = \{\{è,x\},\{h\},\{z\}\}$
$\nu_t/F_t = \{\{è,g\},\{y\}\}$,
$\nu_s/F_s = \{\{è,h,x,z\},\{g\}\}$.
$\nu_0/F_0 = \{\{è,g,h,x,y,z\}\}$.

Now for congruence relations θ and ϑ on the Ω-algebras $(\mathcal{M}, E)$ and $(\mathcal{N}, F)$ respectively:

θ^E	e	a	b	c	d	f
e	1	t	s	q	s	q
a	t	u	q	q	q	q
b	s	q	w	q	s	q
c	q	q	q	t	q	t
d	s	q	s	q	w	q
f	q	q	q	t	q	t

Table 7

ϑ^F	$è$	g	h	x	y	z
$è$	1	u	w	w	t	w
g	u	u	s	s	t	s
h	w	s	w	w	q	w
x	w	s	w	k	q	w
y	t	t	q	q	t	q
z	w	s	w	w	q	w

Table 8

we have:
The cuts of μ^θ and ν^ϑ, $p \in \Omega$, of M_E and N_F respectively are:

$\mu^\theta_1 = \{e\} = \mu_k = \mu_l = \mu_v$,
$\mu^\theta_w = \{e,b,d\} = \mu_r$,
$\mu^\theta_t = \{e,a,c,f\} = \mu_p$,
$\mu_u = \{e,a\}$, $\mu^\theta_s = \{e,a,b,d\}$,
$\mu^\theta_q = \{e,a,b,c,d,f\} = \mu_0$.

$\nu^\vartheta_1 = \{è\} = \nu_l = \nu_v$,
$\nu^\vartheta_u = \{è,g\}$,
$\nu^\vartheta_w = \{è,h,x,z\} = \nu_r$,
$\nu^\vartheta_k = \{è,x\}$,
$\nu^\vartheta_t = \{è,g,y\} = \nu_p$,
$\nu^\vartheta_s = \{è,g,h,x,z\}$,
$\nu^\vartheta_q = \{è,g,h,x,y,z\} = \nu_0$.

The cuts of E^θ and F^ϑ are congruences on the corresponding cuts of μ^θ and ν^ϑ respectively.

Therefore, the quotient structures μ^θ_p/E^θ_p and $\nu^\vartheta_p/F^\vartheta_p$ are as follows:

$\mu^\theta_1/E^\theta_1 = \{\{e\}\}$,
$\mu^\theta_u/E^\theta_u = \{\{e\},\{a\}\}$,
$\mu^\theta_w/E^\theta_w = \{\{e\},\{b\},\{d\}\}$
$\mu^\theta_t/E^\theta_t = \{\{e,a\},\{c,f\}\}$,
$\mu^\theta_s/E^\theta_s = \{\{e,b,d\},\{a\}\}$.
$\mu^\theta_0/E^\theta_0 = \{\{e,a,b,c,d,f\}\}$,

$\nu^\vartheta_1/F^\vartheta_1 = \{\{è\}\}$,
$\nu^\vartheta_k/F^\vartheta_k = \{\{è\},\{x\}\}$,
$\nu^\vartheta_w/F^\vartheta_w = \{\{è,h,x,z\}\}$,
$\nu^\vartheta_u/F^\vartheta_u = \{\{è,g\}\}$,
$\nu^\vartheta_t/F^\vartheta_t = \{\{è,g,y\}\}$,
$\nu^\vartheta_s/F^\vartheta_s = \{\{è,g,h,x,z\}\}$,
$\nu^\vartheta_0/F^\vartheta_0 = \{\{è,g,h,x,y,z\}\}$.

Obviously the algebras $\mathcal{M}$ and $\mathcal{N}$ do not satisfy any particular identity, in which they could be referred to as any particular known algebra. For example the operations $\circ$ and $\star$ by their definitions are not associative and so both algebras $\mathcal{M}$ and $\mathcal{N}$ are non-associative algebras.

Clearly, by definition (4.1.2) and equation (4.3) we can see that $(\mathcal{M}, E)$ and $(\mathcal{N}, F)$ are particular Ω-algebraic structures which satisfy the associative property and other identities not satisfied by the basic algebras.

Next we deal with cuts properties. It is well known in the classical fuzzy set theory that a fuzzy function defined on an algebra is a fuzzy subalgebra if, and only if the cuts are ordinary subalgebras of the algebra (theorem (2.2.11)). Clearly in our case, the cuts of Ω-valued functions defined on our basic algebra are ordinary subalgebras of this basic algebra. But, it is easy to see that these cuts of our Ω-valued functions defined on our basic algebra does not in general satisfy in the classical sense the identities that are satisfied in the Ω-algebras.
However as proved in theorem (4.5.5) their quotient structures with respect to the corresponding cuts of the Ω-valued equality satisfy in the classical sense the identities that are satisfied in the Ω-algebras.
Therefore, for the Ω-valued functions μ and ν determined by the Ω-valued equalities E and F defined on the algebras $\mathcal{M}$ and $\mathcal{N}$ respectively, we show that for each $p \in \Omega$ the quotient structures μ_p/E_p and ν_p/F_p are classical algebras which fulfill in the classical sense the identities, that are fulfilled in the Ω-algebras $(\mathcal{M}, E)$ and $(\mathcal{N}, F)$ respectively.

Chapter 5

Ω-Relational Systems

This chapter is an attempt to replicate the classical notion of lattices as ordered and algebraic structures in the framework of fuzzy theory. Tepavčević and Trajkovski in L-valued lattices ([100]), presented a direction into dealing with this notion in fuzzy settings, where fuzzy lattices were obtained through fuzzification of the membership of the carrier and fuzzification of the ordering relation in a (classical) lattice and then these two approaches were shown to be equal. Other approaches had been introduced as well, most dealing with the fuzzification of the carrier set. Our approach to this notion is slightly different in which we necessarily do not deal with lattices as our domain of fuzzy functions.

Therefore, in section one of this chapter we introduce Ω-lattices, both as algebraic and as order structures. An Ω-poset is an Ω-set equipped with an Ω-valued order which is antisymmetric with respect to the corresponding Ω-valued equality. Using a cut technique, we prove that the quotient cut-substructures can be naturally ordered. Introducing notions of pseudo-infimum and pseudo-supremum, we obtain a definition of an Ω-lattice as an ordering structure. An Ω-lattice as an algebra is a bi-groupoid equipped with an Ω-valued equality, fulfilling particular lattice-theoretic formulas. On an Ω-lattice we introduce an Ω-valued order, and we prove that particular quotient substructures are classical lattices.

Completing these investigations on Ω-lattices, we prove that, under the assumption of the Axiom of Choice, the two notions are equivalent. Namely, it is possible to define operations on an Ω-lattice as an ordered structure, so that it becomes an Ω-lattice as an algebra. On the other hand on an Ω-lattice as an algebra, it is possible to define an Ω-valued order, under which this structure is an Ω-lattice. A construction of Ω-posets and Ω-lattices are dealt with.

In section two of this chapter we introduce complete Ω-lattices, based on

Ω-lattices as introduced in the previous section of the chapter. Complete Ω-lattices are defined, as a generalization of the classical complete lattice. Special properties and elements of complete Ω-lattices are introduced and discussed.

5.1 Ω-lattice

5.1.1 Ω-poset; Ω-lattice as ordered structure

Let E be an Ω-valued equality on a nonempty set A. We say that an Ω-valued relation $R : A^2 \to \Omega$ on A is **E-antisymmetric**, if the following holds:

$$R(x,y) \wedge R(y,x) = E(x,y), \quad \text{for all} \quad x,y \in A. \tag{5.1}$$

Let (M,E) be an Ω-set. We say that an Ω-valued relation $R : M^2 \to \Omega$ on M is an **Ω-valued order** on (M,E), if it fulfills the strictness property (3.3), it is E-antisymmetric, and it is transitive

A structure (M,E,R) is an **Ω-poset**, if (M,E) is an Ω-set, and $R : M^2 \to \Omega$ is an Ω-valued order on (M,E).

In addition, it is clear that by (5.1), $R(x,x) = E(x,x)$, for every $x \in M$. As indicated by (3.5), we denote by μ the Ω-valued function on M, defined by $\mu(x) = E(x,x)$.

Obviously, both E and R are reflexive relations on μ, in the sense of (3.2).

By Lemma(3.2.4), every cut E_p of E is a classical equivalence relation on the cut μ_p of μ. In this context, as usual, we denote by $[x]_{E_p}$ the equivalence class of $x \in \mu_p$, and by μ_p/E_p the corresponding quotient set: for $p \in \Omega$

$$[x]_{E_p} := \{y \in \mu_p \mid xE_py\}, x \in \mu_p; \;\; \mu_p/E_p := \{[x]_{E_p} \mid x \in \mu_p\}.$$

Proposition 5.1.1. *Let (M,E,R) be an Ω-poset. Then for every $p \in \Omega$, the binary relation $\leq_p$ on μ_p/E_p, defined by*

$$[x]_{E_p} \leq_p [y]_{E_p} \;\; \textit{if and only if} \;\; (x,y) \in R_p \tag{5.2}$$

is a classic ordering relation.

Proof. First, we prove that the relation $\leq_p$ is well defined. Indeed, suppose that $[x]_{E_p} \leq_p [y]_{E_p}$, i.e., that $(x,y) \in R_p$. Now, if $u \in [x]_{E_p}, v \in [y]_{E_p}$, then $E(x,u) = R(x,u) \wedge R(u,x) \geq p$ and similarly $E(y,v) = R(y,v) \wedge R(v,y) \geq p$.

Thereby, since by assumption $R(x, y) \geq p$, using transitivity of R we obtain

$$\begin{aligned} p &\leq R(x, u) \wedge R(u, x) \wedge R(y, v) \wedge R(v, y) \wedge R(x, y) \\ &\leq R(u, v) \wedge R(x, u) \wedge R(v, y) \\ &\leq R(u, v). \end{aligned}$$

Therefore, $[u]_{E_p} \leq_p [v]_{E_p}$ and the order $\leq_p$ does not depend on class representatives.

The relation $\leq_p$ is reflexive on μ_p/E_p: for $x \in \mu_p$,

$$\begin{aligned} [x]_{E_p} \leq_p [x]_{E_p} &\iff R(x, x) \geq p \\ &\iff E(x, x) \geq p \\ &\iff \mu(x) \geq p \\ &\iff x \in \mu_p. \end{aligned}$$

Next, $\leq_p$ is antisymmetric: Suppose that for $x, y \in \mu_p$, $[x]_{E_p} \leq_p [y]_{E_p}$ and $[y]_{E_p} \leq_p [x]_{E_p}$. This is equivalent with $(x, y) \in R_p$ and $(y, x) \in R_p$, which holds if and only if $R(x, y) \wedge R(y, x) = E(x, y) \geq p$, i.e., if and only if $[x]_{E_p} = [y]_{E_p}$.

Transitivity: Let $[x]_{E_p} \leq_p [y]_{E_p}$ and $[y]_{E_p} \leq_p [z]_{E_p}$, i.e., $(x, y) \in R_p$ and $(y, z) \in R_p$, hence equivalently $R(x, y) \geq p$ and $R(y, z) \geq p$, if and only if $R(x, y) \wedge R(y, z) \geq p$. Then, by transitivity of R, $R(x, z) \geq p$, i.e., $(x, z) \in R_p$, which finally gives $[x]_{E_p} \leq_p [z]_{E_p}$. $\square$

Definition 5.1.2. Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if the following holds:
(i) $\mu(a) \wedge \mu(b) \leq R(c, a) \wedge R(c, b)$, and for every $p \leq \mu(a) \wedge \mu(b)$, for every $x \in \mu_p$, $\quad p \leq R(x, a) \wedge R(x, b) \implies p \leq R(x, c)$.
An element $d \in M$ is a **pseudo-supremum** of $a, b \in M$, if the following holds:
(ii) $\mu(a) \wedge \mu(b) \leq R(a, d) \wedge R(b, d)$, and for every $p \leq \mu(a) \wedge \mu(b)$, for every $x \in \mu_p$, $\quad p \leq R(a, x) \wedge R(b, x) \implies p \leq R(d, x)$.

Remark 5.1.3. It is straightforward that a pseudo-infimum (supremum) of a and b belongs to μ_p for every $p \leq \mu(a) \wedge \mu(b)$.

Observe that a pseudo-infimum and a pseudo-supremum for given $a, b \in M$, if they exist, are not unique in general. In the following proposition we prove that pseudo-infima (suprema) of two elements a, b, if they exist, they belong to the same equivalence class in μ_p/E_p, for $p \leq \mu(a) \wedge \mu(b)$.

Proposition 5.1.4. *Let (M, E, R) be an Ω-poset and $a, b, c, c_1, d, d_1 \in M$.*
If c is a pseudo-infimum of a and b, then

$\mu(a) \wedge \mu(b) \leq E(c, c_1)$ *if and only if c_1 is also a pseudo-infimum of a and b.*

Analogously, if d is a pseudo-supremum of a and b, then

$\mu(a) \wedge \mu(b) \leq E(d, d_1)$ *if and only if d_1 is also a pseudo-supremum of a and b.*

Proof. Suppose that c and c_1 are pseudo-infima of $a, b \in M$. Then by (i),

$$\begin{aligned} \mu(a) \wedge \mu(b) &\leq R(c,a) \wedge R(c,b) \wedge R(c_1,a) \wedge R(c_1,b) \\ &\leq R(c_1,c) \wedge R(c,c_1) \quad \text{(by transitivity)} \\ &= E(c,c_1) \quad \text{(by definition of } R\text{)}. \end{aligned}$$

Conversely, let $\mu(a) \wedge \mu(b) \leq E(c, c_1)$. Then, for every $p \leq \mu(a) \wedge \mu(b)$,

$$p \leq \mu(a) \wedge \mu(b) \leq R(c, c_1) \wedge R(c_1, c),$$

hence $p \leq R(c_1, c)$. Since $p \leq R(c, a) \wedge R(c, b)$, we have

$$p \leq R(c_1, c) \wedge R(c, a) \wedge R(c_1, c) \wedge R(c, b) \leq R(c_1, a) \wedge R(c_1, b).$$

In addition, assume $x \in \mu_p$ and $p \leq R(x, a) \wedge R(x, b)$. Then, $p \leq R(x, c)$. Since also $p \leq \mu(a) \wedge \mu(b) \leq R(c, c_1)$, by transitivity we obtain

$$p \leq R(x, c_1).$$

The proof for pseudo-suprema is analogous. □

Remark 5.1.5. Since for $p \leq q$, every equivalence class of μ_q/E_q is contained in a class of μ_p/E_p, we get that pseudo-infima (suprema) of two elements a, b, if they exist, belong to the same equivalence class in μ_p/E_p, for $p \leq \mu(a) \wedge \mu(b)$.

Remark 5.1.6. By the above definition, if E is a separated equality on M, for $p = \mu(a)$, the unique pseudo-infimum (supremum) of one element $a \in M$ (i.e., for a and b with $a = b$), is a. In terms of Proposition (5.1.4), this follows from the fact that for every $a \in M$, the class $[a]_{E_{\mu(a)}}$ consists of the single element a:

$$\begin{aligned} x \in [a]_{E_{\mu(a)}} &\iff E(a,x) \geq \mu(a) \\ &\iff E(a,x) \geq E(a,a) \\ &\iff x = a \quad \text{(by separation property of } E\text{).} \end{aligned}$$

We say that an Ω-poset (M, E, R) is an **Ω-lattice as an ordered structure**, if for every $a, b \in M$ there exist a pseudo-infimum and a pseudo-supremum.

In the following, infimum and supremum of elements a and b in an ordered set (here a lattice) are denoted by $\inf(a, b)$ and $\sup(a, b)$, respectively.

Theorem 5.1.7. *Let (M, E, R) be an Ω-poset. Then it is an Ω-lattice as an ordered structure if and only if for every $q \in \Omega$, the poset $(\mu_q/E_q, \leq_q)$ is a lattice, where the relation $\leq_q$ on the quotient set μ_q/E_q is defined by* (5.2) *and the following holds: for all $a, b \in M$, and $p = \mu(a) \wedge \mu(b)$,*

$$\inf([a]_{E_p}, [b]_{E_p}) \subseteq \inf([a]_{E_q}, [b]_{E_q}),\ \sup([a]_{E_p}, [b]_{E_p}) \subseteq \sup([a]_{E_q}, [b]_{E_q}) \quad (5.3)$$

for every q, $q \leq p$

Proof. Let (M, E, R) be an Ω-poset. Then, by Proposition (5.1.1), for every $q \in \Omega$, $(\mu_q/E_q, \leq_q)$ is a classical poset.

Assume now then in addition, (M, E, R) is an Ω-lattice. Under this assumption $(\mu_q/E_q, \leq_q)$ is a (classical) lattice. Indeed, let $a, b \in \mu_q$, and let c be a pseudo-infimum of a, b. Then, $q \leq \mu(a) \wedge \mu(b)$, and also $c \in \mu_q$ by the definition of a pseudo-infimum (see also Remark (5.1.3)). Hence by (5.2), $[c]_{E_q} \leq_q [a]_{E_q}$ and $[c]_{E_q} \leq_q [b]_{E_q}$, and for every $x \in \mu_q$, if $[x]_{E_q} \leq_q [a]_{E_q}$ and $[x]_{E_q} \leq_q [b]_{E_q}$, then also $[x]_{E_q} \leq_q [c]_{E_q}$. Therefore, $[c]_{E_q} = \inf([a]_{E_q}, [b]_{E_q})$ in μ_q/E_q.

By the above, (5.3) holds for infima, since $q \leq p$, for $p = \mu(a) \wedge \mu(b)$, and therefore

$$\inf([a]_{E_p}, [b]_{E_p}) = [c]_{E_p} \subseteq [c]_{E_q} = \inf([a]_{E_q}, [b]_{E_q}).$$

Analogously, using (ii), we can prove that the class $[d]_{E_q}$ is a supremum of $[a]_{E_q}$ and $[b]_{E_q}$ in μ_q/E_q, where d is a pseudo-supremum of a and b. Similarly as for infima, (5.3) holds for suprema.

Hence, for every $q \in \Omega$, $(\mu_q/E_q, \leq_q)$ is a lattice and (5.3) holds.

Conversely, suppose that for a given Ω-poset (M, E, R), the poset $(\mu_p/E_p, \leq_p)$ is a lattice for every $p \in \Omega$, and that (5.3) holds. We prove that then for all $a, b \in M$ there exist a pseudo-infimum and a pseudo supremum. Indeed, since for $p = \mu(a) \wedge \mu(b)$, $(\mu_p/E_p, \leq_p)$ is a lattice, there is the infimum for the classes $[a]_{E_p}$ and $[b]_{E_p}$, a class $[c]_{E_p}$, for some $c \in \mu_p$. Hence $[c]_{E_p} \leq_p [a]_{E_p}$, $[c]_{E_p} \leq_p [b]_{E_p}$, and for every $x \in \mu_p$, if $[x]_{E_p} \leq_p [a]_{E_p}$, $[x]_{E_p} \leq_p [b]_{E_p}$, then also $[x]_{E_p} \leq_p [c]_{E_p}$. By Proposition 5.1.1, this is equivalent with $(c,a),(c,b) \in R_p$, and $(x,a),(x,b) \in R_p$ implies $(x,c) \in R_p$. By (5.3), the above properties of c hold if p is replaced by q, $q \leq p$. Therefore, c is a pseudo-infimum of a and b. Analogously one can prove that there is a pseudo-supremum for a and b. □

Remark 5.1.8. According to Theorem (5.1.7) and Remark (5.1.6), if E is a separated equality in an Ω-lattice (M,E,R) and if $\mu_1 \neq \emptyset$, then (μ_1, R_1) is a lattice. Clearly, in this case E_1 is a diagonal relation and the congruence classes under E_1 are one-element sets. Then also R_1 is an order on μ_1 and the posets (μ_1, R_1) and $(\mu_1/E_1, \leq_1)$ are order isomorphic. Since the latter is a lattice by Theorem 5.1.7, the same holds for the former.

Due to the same argument, for every $p \in \Omega$, (μ_p, R_p) is a lattice whenever equivalence classes under E_p are all one-element sets.

5.1.2 An Ω-lattice as an Ω-algebra

Here we define an Ω-lattice as an Ω-algebra, according to the definition in section (4.1). This algebraic approach was first developed in [90]. Hence, basic definitions and main results from this paper are also given in the present section. However, we adopt these results to the wider framework of both algebraic and relational approach. In addition, there is a difference in the definition of the separation property. Namely, as it is defined in [90], for a separated Ω-valued equality E it is possible that for some x we have $E(x,x) = 0$. Consequently, an Ω-lattice (generally an Ω-algebra) could be considered to be a proper sublattice (subalgebra) of the basic bi-groupoid (algebra), which is not our intention here: we deal with Ω-sets, Ω-lattices, generally with Ω-algebras, not with their substructures.

Definition 5.1.9. Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, as an algebra with two binary operations, without any additional conditions and $E : M^2 \to \Omega$ is an Ω-valued equality on M such that (M,E) is an Ω-set. Then $(\mathcal{M}, E)$ is an Ω-**algebra**, if E satisfies the following additional property:

$$E(x,y) \wedge E(z,t) \leq E(x \sqcap z, y \sqcap t) \quad \text{and} \quad E(x,y) \wedge E(z,t) \leq E(x \sqcup z, y \sqcup t).$$

That is E should be compatible with operations $\sqcap$ and $\sqcup$.

The following are straightforward properties of the above notions.

Proposition 5.1.10. *If E is a compatible Ω-valued equality on a bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$, and $\mu : M \to \Omega$ is defined by $\mu(x) = E(x,x)$, then the following hold:*

(*i*) *For all $x, y \in M$,*

$$\mu(x) \wedge \mu(y) \leq \mu(x \sqcap y) \text{ and } \mu(x) \wedge \mu(y) \leq \mu(x \sqcup y). \tag{5.4}$$

(*ii*) *For every $p \in \Omega$, the cut μ_p of μ is a sub-bi-groupoid of $\mathcal{M}$.*

Proof. (*i*) for all $x, y \in M$

$$\mu(x) \wedge \mu(y) = E(x,x) \wedge E(y,y) \leq E(x \sqcap y, x \sqcap y) = \mu(x \sqcap y)$$

by the compatibility of E. Similarly we get the second formula in (5.4).

(*ii*) straightforward consequences of Proposition (6.2.1)and the part (*i*) above. □

Therefore, we say that the Ω-algebra $(\mathcal{M}, E)$ is an **Ω-lattice as an Ω-algebra** (Ω-lattice as an algebra), if it satisfies (classical) lattice identities:

$\ell 1 :\ x \sqcap y \approx y \sqcap x$
$\ell 2 :\ x \sqcup y \approx y \sqcup x$ (*commutativity*)
$\ell 3 :\ x \sqcap (y \sqcap z) \approx (x \sqcap y) \sqcap z$
$\ell 4 :\ x \sqcup (y \sqcup z) \approx (x \sqcup y) \sqcup z$ (*associativity*)
$\ell 5 :\ (x \sqcap y) \sqcup x \approx x$
$\ell 6 :\ (x \sqcup y) \sqcap x \approx x.$ (*absorption*)

As defined by (4.3), this means that for all $x, y, z \in M$, the following formulas are satisfied, where, as already indicated, the mapping $\mu : M \to \Omega$ is defined by $\mu(x) = E(x,x)$:

$L1 :\ \mu(x) \wedge \mu(y) \leq E(x \sqcap y, y \sqcap x)$
$L2 :\ \mu(x) \wedge \mu(y) \leq E(x \sqcup y, y \sqcup x)$ (commutative laws)
$L3 :\ \mu(x) \wedge \mu(y) \wedge \mu(z) \leq E((x \sqcap y) \sqcap z, x \sqcap (y \sqcap z))$
$L4 :\ \mu(x) \wedge \mu(y) \wedge \mu(z) \leq E((x \sqcup y) \sqcup z, x \sqcup (y \sqcup z))$ (associative laws)
$L5 :\ \mu(x) \wedge \mu(y) \leq E((x \sqcap y) \sqcup x, x)$
$L6 :\ \mu(x) \wedge \mu(y) \leq E((x \sqcup y) \sqcap x, x).$ (absorption laws)

Next we present some additional properties of Ω-lattices as algebras. As mentioned, these propositions (to the end of Section (5.1.2)) are versions of the results in [90].

The proof of the following lemma follows by Lemma 1 in [90].

Lemma 5.1.11. *An Ω-lattice $(\mathcal{M}, E)$ fulfills the following versions of the absorption identities:*

$$(y \sqcap x) \sqcup x \approx x \quad \textit{and} \quad (y \sqcup x) \sqcap x \approx x.$$

Proof. By (4.3) we need to show that following formulas

$$L7: \ \mu(x) \wedge \mu(y) \leq E((y \sqcap x) \sqcup x, x)$$

and

$$L8: \ \mu(x) \wedge \mu(y) \leq E((y \sqcup x) \sqcap x, x)$$

are satisfied. Therefore, by compatibility and $L2$, we have

$$\begin{aligned} \mu(x) \wedge \mu(y) &\leq E(x \sqcap y, y \sqcap x) \wedge E(x, x) \\ &\leq E((x \sqcap y) \sqcup x, (y \sqcap x) \sqcup x) \end{aligned}$$

Hence by $L5$, symmetry and transitivity of E,

$$\begin{aligned} \mu(x) \wedge \mu(y) &\leq E((x \sqcap y) \sqcup x, (y \sqcap x) \sqcup x) \wedge E((x \sqcap y) \sqcup x, x) \\ &\leq E((y \sqcap x) \sqcup x, x). \end{aligned}$$

Thus we have proved that the absorption identity $(y \sqcap x) \sqcup x \approx x$ is valid in $(\mathcal{M}, E)$. The validity of the second law is proved dually.

□

The next result is also proved in [90] as Proposition 3.

Proposition 5.1.12. *In an Ω-lattice $(\mathcal{M}, E)$ as an algebra, the idempotent identities*

$$x \sqcup x \approx x \quad \textit{and} \quad x \sqcap x \approx x$$

are valid.

Proof. Again by (4.3) we need to show that following formulas

$$\mu(x) \leq E(x \sqcap x, x)$$

and

$$\mu(x) \leq E(x \sqcup x, x)$$

are satisfied. Therefore by the absorptive law $L6$, letting $y = x$ and using the fact that E is μ reflexive, we have that

$$\begin{aligned}\mu(x) &\leq E((x \sqcup x) \sqcap x, x) \wedge E(x, x) \\ &\leq E(((x \sqcup x) \sqcap x) \sqcup x, x \sqcup x) \text{ (by compatibility of } E \text{) .}\end{aligned}$$

By $L7$ in lemma (5.1.11) and $y = x \sqcup x$, it follow that

$$\mu(x) \leq E(((x \sqcup x) \sqcap x) \sqcup x, x).$$

Therefore by the symmetric and transitive properties of E, we have that

$$\begin{aligned}\mu(x) &\leq E(((x \sqcup x) \sqcap x) \sqcup x, x) \wedge E(((x \sqcup x) \sqcap x) \sqcup x, x \sqcup x) \\ &\leq E(x \sqcup x, x).\end{aligned}$$

Hence we have proved that the idempotent law $x \sqcup x \approx x$ is valid in $(\mathcal{M}, E)$. The validity of the other law is proved analogously.

□

Under the separation property the idempotence in the Ω-valued framework implies that the same identity should be satisfied by the bi-groupoid on which an Ω-lattice is defined (Proposition 4 in [90]):

Proposition 5.1.13. *[90] Let $(\mathcal{M}, E)$ be an Ω-lattice, in which E is a separated Ω-valued equality. Then the idempotent law $x \sqcap x \approx x$ is valid in $(\mathcal{M}, E)$ if and only if the operation $\sqcap$ is idempotent in the bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$, and analogously $x \sqcup x \approx x$ holds in $(\mathcal{M}, E)$ if and only if $\sqcup$ is idempotent in $\mathcal{M}$.*

Proof. Let E be separable and assuming that the idempotent law $x \sqcap x \approx x$ is valid in $(\mathcal{M}, E)$, that is by (4.3), $\mu(x) \leq E(x \sqcap x, x)$. Since $E(x \sqcap x, x) \leq E(x, x) = \mu(x)$, we have that $E(x \sqcap x, x) = \mu(x)$. Hence by the definition of Ω-valued equality, we have that $x \sqcap x \approx x$ in $\mathcal{M}$. Conversely, suppose $x \sqcap x \approx x$ in $\mathcal{M}$ and since $E(x, x) = \mu(x)$, then $E(x \sqcap x, x) = \mu(x)$. Thus $E(x \sqcap x, x) \leq \mu(x)$. Analogously, the other part is proved. □

If a bi-groupoid $\mathcal{M}$ is a classical lattice, and E is an Ω-valued compatible equality on $\mathcal{M}$, then $(\mathcal{M}, E)$ is an Ω-lattice, as we prove in the sequel. In other words, classical lattice properties imply formulas $L1$ – $L6$ (see Proposition 5 in [90]).

Proposition 5.1.14. *If $\mathcal{M} = (M, \sqcap, \sqcup)$ is a lattice and E is a compatible Ω-valued equality on $\mathcal{M}$, then $(\mathcal{M}, E)$ is an Ω-lattice.*

Proof. Let the bi-groupoid $\mathcal{M} = (M, \sqcup, \sqcap)$ be a lattice then all the lattice identities holds. Since $x \sqcap y \approx y \sqcap x$ and E is an Ω-valued equality then $E(x \sqcap y, y \sqcap x) = \mu(y \sqcap x)$. Since μ is compatible on $\mathcal{M}$, then $\mu(y \sqcap x) \geq \mu(y) \wedge \mu(x)$. Thus $\mu(y) \wedge \mu(x) \leq E(x \sqcap y, y \sqcap x)$. Analogously we prove for the other formulas $L2 - L6$. $\square$

If $(\mathcal{M}, E)$ is an Ω-lattice with $\mathcal{M}$ being a classical lattice, then, by Proposition (5.1.10), μ is compatible on $\mathcal{M}$.
However, an Ω-lattice is a notion which is more general then an Ω-sublattice. Namely, if $\mathcal{M}$ is a bi-groupoid which is not a lattice and $(\mathcal{M}, E)$ is an Ω-lattice, then obviously μ is by Proposition (5.1.10) an Ω-sub-bi-groupoid of $\mathcal{M}$, hence it is not an Ω-sublattice of $\mathcal{M}$ in the sense of fuzzy substructures in general.

Next we describe the notion of an Ω-lattice $(\mathcal{M}, E)$ in the framework of classical lattices obtained by cuts of the bi-groupoid $\mathcal{M}$. Observe that by Proposition (5.1.10), for every $p \in \Omega$, the cut μ_p is a sub-bi-groupoid of $\mathcal{M}$ and E_p is a congruence relation on μ_p. Hence, μ_p/E_p is a quotient bi-groupoid of μ_p over E_p.

Following from theorem (4.5.5), we have theorem (5.1.15) which is analogous to theorem (5.1.7), which deals with Ω-lattices as ordered structures.The present one is concerned with Ω-lattices as algebras; it is already given in [90], but here the proof is slightly reformulated.

Theorem 5.1.15. *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, and let E be an Ω-valued compatible equality on $\mathcal{M}$. Then, $(\mathcal{M}, E)$ is an Ω-lattice if and only if for every $p \in \Omega$, the quotient structure μ_p/E_p is a lattice.*

Proof. Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and E a compatible Ω-valued equality on $\mathcal{M}$, such that $(\mathcal{M}, E)$ is an Ω-lattice. According to the above comment, for $p \in \Omega$, we consider the quotient set of μ_p over E_p, denoted as usual by μ_p/E_p. In addition, we denote operations on congruence classes by $\sqcap_p$ and $\sqcup_p$. These operations are introduced in a natural way (by class representatives) and it is easy to prove that they are well defined. Now, $(\mu_p/E_p, \sqcap_p, \sqcup_p)$ is a bi-groupoid. We prove that this bi-groupoid is a lattice. Let $[x]_{E_p}, [y]_{E_p}, [z]_{E_p}$ be elements (classes) from μ_p/E_p. We have to prove lattice axioms $\ell1 - \ell6$.

We prove $\ell1$, commutativity of $\sqcap_p$:

For $x, y \in \mu_p$, since $(\mathcal{M}, E)$ is an Ω-lattice, by $L1$ we have

$$E(x \sqcap y, y \sqcap x) \geq \mu(x) \sqcap \mu(y) \geq p,$$

hence $(x \sqcap y, y \sqcap x) \in E_p$. Therefore, for $x, y \in \mu_p$, we have that

$$[x]_{E_p} \sqcap_p [y]_{E_p} = [x \sqcap y]_{E_p} = [y \sqcap x]_{E_p} = [y]_{E_p} \sqcap_p [x]_{E_p},$$

so we proved that the operation $\sqcap_p$ in μ_p/E_p is commutative.

Similarly we can prove the remaining five lattice axioms, hence μ_p/E_p is a lattice.

Conversely, suppose that for every p from Ω, the quotient structure μ_p/E_p is a lattice. Now, we have to prove that $(\mathcal{M}, E)$ is an Ω-lattice, i.e., that formulas $L1 - L6$ hold. We prove the absorption law $L6$

$$\mu(x) \wedge \mu(y) \leq E((x \sqcup y) \sqcap x, x),$$

the others are proved analogously.

Let $\mu(x) \wedge \mu(y) = p$. Then $x, y \in \mu_p$. Since $(\mu_p/E_p, \sqcap_p, \sqcup_p)$ is a lattice by assumption, by $\ell 6$ we have $([x]_{E_p} \sqcup_p [y]_{E_p}) \sqcap_p [x]_{E_p} = [x]_{E_p}$, hence $[(x \sqcup y) \sqcap x]_{E_p} = [x]_{E_p}$, and therefore

$$E((x \sqcup y) \sqcap x, x) \geq p = \mu(x) \wedge \mu(y).$$

□

We conclude the section by describing the relationship between quotient lattices μ_p/E_p, for various $p \in \Omega$.

Proposition 5.1.16. *Let $(\mathcal{M}, E)$ be an Ω-lattice and $p, q \in \Omega$, with $p \leq q$. Then, the mapping $f : \mu_q/E_q \to \mu_p/E_p$, defined by $f([x]_{E_q}) = [x]_{E_p}$ is a lattice homomorphism.*

Proof. The function f is well defined, since by assumption $p \leq q$, hence $\mu_q \subseteq \mu_p$; therefore, if $x \in \mu_q$, then $x \in \mu_p$. For the same reason, if $[x]_{E_q} = [y]_{E_q}$, then also $[x]_{E_p} = [y]_{E_p}$. Now for $x, y \in \mu_q$, we have

$$f([x]_{E_q} \sqcap_q [y]_{E_q}) = f([x \sqcap y]_{E_q}) = [x \sqcap y]_{E_p} = [x]_{E_p} \sqcap_p [y]_{E_p},$$

and analogously for the operation $\sqcup_q$. □

5.1.3 Equivalence of two approaches

Ω-lattice as ordered structure is Ω-lattice as algebra

Here we prove that, like in the classical case, the notions of an Ω-lattice as an algebraic structure and an Ω-lattice as an ordered structure are equivalent.

In the following we assume that the Axiom of Choice (AC) holds.

Definition 5.1.17. Let (M, E, R) be an Ω-lattice as an ordered structure. We define two binary operations, $\sqcap$ and $\sqcup$ on M as follows: for every pair a, b of elements from M, $a \sqcap b$ is an arbitrary, fixed pseudo-infimum of a and b, and $a \sqcup b$ is an arbitrary, fixed pseudo-supremum of a and b.

Assuming Axiom of Choice, by which an element is chosen among all pseudo-infima (suprema) of a and b and then this element is fixed, the operations $\sqcap$ and $\sqcup$ on M are well defined. Indeed, by the definition of an Ω-lattice, for any $a, b \in M$, a pseudo-infimum and a pseudo-supremum exist; by Definition (5.1.17), they are unique.

If E is a separated Ω-valued equality, then by Remark (5.1.6), for every $a \in M$, we get $a \sqcap a = a$ and $a \sqcup a = a$, i.e., in this case these operations are idempotent.

Hence, the structure $\mathcal{M} = (M, \sqcap, \sqcup)$ is a bi-groupoid. In the following proposition we prove that μ ($\mu(x) = E(x, x)$) is an Ω-sub-bigroupoid of $\mathcal{M}$.

Proposition 5.1.18. *Let (M, E, R) be an Ω-lattice, $\mu : M \to \Omega$ defined by (3.5) ($\mu(x) = E(x, x)$) and $\mathcal{M} = (M, \sqcap, \sqcup)$ a bi-groupoid, as defined above. Then, for all $x, y \in M$*

$$\mu(x) \wedge \mu(y) \leq \mu(x \sqcap y) \quad \text{and} \quad \mu(x) \wedge \mu(y) \leq \mu(x \sqcup y). \tag{5.5}$$

Proof. Let $x, y \in M$ and $x \sqcap y = z$. Denote $\mu(x) \wedge \mu(y)$ by p. Then, by Remark (5.1.3), the class $[z]_{E_p}$ exists, i.e., $z \in \mu_p$. By the definition of a cut, we have $\mu(z) \geq \mu(x) \wedge \mu(y)$.

The proof in case $z = x \sqcup y$ is analogous. □

Next, we deal with an Ω-lattice (M, E, R) as an ordered structure, in which, by Theorem (5.1.7), for every $p \in \Omega$, the quotient structure $(\mu_p/E_p, \leq_p)$ is a lattice, where $\leq_p$ is defined by (5.2) and (5.3) holds. For $x, y \in \mu_p$, we denote infimum and supremum of $[x]_{E_p}$ and $[y]_{E_p}$ by $[x]_{E_p} \sqcap_p [y]_{E_p}$ and $[x]_{E_p} \sqcup_p [y]_{E_p}$, respectively.

Lemma 5.1.19. *Let (M, E, R) be an Ω-lattice as an ordered structure, and $p \in \Omega$. Then for all $x, y \in \mu_p$, in the lattice $(\mu_p/E_p, \leq_p)$,*

$$[x]_{E_p} \sqcap_p [y]_{E_p} = [x \sqcap y]_{E_p} \quad \text{and} \quad [x]_{E_p} \sqcup_p [y]_{E_p} = [x \sqcup y]_{E_p},$$

where $\sqcap$ and $\sqcup$ are the operations on M introduced by Definition (5.1.17).

Proof. By Proposition (5.1.1), in the lattice $(\mu_p/E_p, \leq_p)$, relation $\leq_p$ is an ordering relation given by

$$[x]_{E_p} \leq_p [y]_{E_p} \text{ if and only if } (x, y) \in R_p.$$

By the definition of a pseudo-infimum, since $x, y \in \mu_p$ and $p \leq \mu(x) \wedge \mu(y)$, we get $(x \sqcap y, x) \in R_p$, $(x \sqcap y, y) \in R_p$, hence $[x \sqcap y]_{E_p} \leq_p [x]_{E_p}$ and $[x \sqcap y]_{E_p} \leq_p [y]_{E_p}$. Therefore, $[x \sqcap y]_{E_p}$ is a lower bound for $[x]_{E_p}$ and $[y]_{E_p}$. Further, if for some $u \in \mu_p$ we have $(u, x) \in R_p$ and $(u, y) \in R_p$, then also by the definition of a pseudo-infimum, $(u, x \sqcap y) \in R_p$. Equivalently, if $[u]_{E_p} \leq_p [x]_{E_p}$ and $[u]_{E_p} \leq_p [y]_{E_p}$, then $[u]_{E_p} \leq_p [x \sqcap y]_{E_p}$. Hence $[x \sqcap y]_{E_p}$ is the greatest lower bound (infimum) for $[x]_{E_p}$ and $[y]_{E_p}$.

The proof that $[x \sqcup y]_{E_p}$ is the supremum for $[x]_{E_p}$ and $[y]_{E_p}$, i.e., that $[x]_{E_p} \sqcup_p [y]_{E_p} = [x \sqcup y]_{E_p}$ is analogous. □

As we proved in Proposition (5.1.18), the function $\mu : M \to \Omega$ ($\mu(x) = E(x, x)$) is an Ω-sub-bigroupoid of $(M, \sqcap, \sqcup)$. Therefore, for every $p \in \Omega$, μ_p is a (classical) sub-bigroupoid of $(M, \sqcap, \sqcup)$. We use this in the following proposition.

Proposition 5.1.20. *Let (M, E, R) be an Ω-lattice as an ordered structure, and $\sqcap$, $\sqcup$ the corresponding binary operations on M, introduced by Definition (5.1.17). Then, E is compatible with $\sqcap$ and $\sqcup$.*

Proof. We have to prove that for all $x, y, u, v \in M$,

$$E(x, y) \wedge E(u, v) \leq E(x \sqcap u, y \sqcap v),$$

and analogously for the operation $\sqcup$. By the definition of a cut, the above inequality is equivalent with

$$(x \sqcap u, y \sqcap v) \in E_p,$$

where $p = E(x, y) \wedge E(u, v)$. Since $E(x, y) \geq p$ and $E(u, v) \geq p$, i.e., $(x, y), (u, v) \in E_p$, our task is to prove that E_p is a congruence relation on the sub-bigroupoid μ_p of M. Indeed, if $(x, y), (u, v) \in E_p$, then $[x]_{E_p} = [y]_{E_p}$ and $[u]_{E_p} = [v]_{E_p}$ in the lattice μ_p/E_p. Then, $[x]_{E_p} \sqcap_p [u]_{E_p} = [y]_{E_p} \sqcap_p [v]_{E_p}$, implying by Lemma 5.1.19, $[x \sqcap u]_{E_p} = [y \sqcap v]_{E_p}$, which gives $(x \sqcap u, y \sqcap v) \in E_p$.

Compatibility with $\sqcup$ is proved analogously. □

Now we prove that the operations $\sqcap$ and $\sqcup$ satisfy lattice-theoretic identities $\ell 1, \ldots, \ell 6$, which means that the formulas $L1, \ldots, L6$ hold.

Proposition 5.1.21. *Let (M, E, R) be an Ω-lattice as an ordered structure, and $\sqcap$, $\sqcup$ the corresponding binary operations on M, introduced by definition (5.1.17). Then, the formulas $L1$ – $L6$ are satisfied.*

Proof. We prove that the formula $L1$, commutativity of $\sqcap$, holds: Since for every $p \in L$, μ_p/E_p is a lattice, we have that for all $x, y \in M$, with $\mu(x) \wedge \mu(y) = p$,

$$[x]_{E_p} \sqcap_p [y]_{E_p} = [y]_{E_p} \sqcap_p [x]_{E_p}.$$

Therefore

$$[x \sqcap y]_{E_p} = [y \sqcap x]_{E_p},$$

and thus, by the definition of these classes,

$$E(x \sqcap y, y \sqcap x) \geq p = \mu(x) \wedge \mu(y).$$

Using the same analogous argument, we prove that all the remaining five formulas hold. □

Theorem 5.1.22. *If (M, E, R) is an Ω-lattice as an ordered structure, and $\mathcal{M} = (M, \sqcap, \sqcup)$ the bi-groupoid in which operations $\sqcap$, $\sqcup$ are introduced in definition (5.1.17), then $(\mathcal{M}, E)$ is an Ω-lattice as an algebra.*

Proof. This is a straightforward consequence of Propositions (5.1.18), (5.1.20) and (5.1.21) and of the definition of Ω-lattice as an algebra. □

Finally, let us prove a property of Ω-lattices as ordered structures which is analogous to the following known fact about lattices: $x \leq y$ if and only if $x \wedge y = x$.

Proposition 5.1.23. *If (M, E, R) is an Ω-lattice as an ordered structure, then for all $x, y \in M$,*

$$\mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x) = R(x, y).$$

Proof. For every $p \in \Omega$, $(\mu_p/E_p, \leq_p)$ is a lattice, hence for all $x, y \in M$, by Proposition (5.1.1) and by strictness property of R, we have $R(x, y) \geq p$ if and only if $x, y \in \mu_p$ and $(x, y) \in R_p$ if and only if $\mu(x) \wedge \mu(y) \geq p$ and $[x]_{E_p} \leq_p [y]_{E_p}$ if and only if $\mu(x) \wedge \mu(y) \geq p$ and $[x]_{E_p} \sqcap_p [y]_{E_p} = [x]_{E_p}$ if and only if $\mu(x) \wedge \mu(y) \geq p$ and $[x \sqcap y]_{E_p} = [x]_{E_p}$ if and only if $\mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x) \geq p$. So, for every $p \in \Omega$ and for all $x, y \in M$

$$\mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x) \geq p \text{ if and only if } R(x, y) \geq p,$$

which proves the proposition. □

Ω-lattice as algebra is Ω-lattice as ordered structure

Here we deal with the relational aspect of Ω-lattices as algebras. First we introduce an Ω-valued order on these structures.

Theorem 5.1.24. *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $(\mathcal{M}, E)$ an Ω-lattice as an algebra, where E is a separated Ω-valued equality on $\mathcal{M}$. Then the Ω-valued relation $R : M^2 \to \Omega$, defined by*

$$R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x) \tag{5.6}$$

is an Ω-valued order on M.

Proof. We have to prove that R is E-antisymmetric and transitive.

Now we show that R is E-antisymmetric:

$$\begin{aligned}
E(x, y) &= E(x, y) \wedge E(y, x) \wedge E(x, x) \wedge E(y, y) \\
&\leq E(x \sqcap y, x \sqcap x) \wedge E(y \sqcap x, y \sqcap y) \wedge E(x, x) \wedge E(y, y) \\
&= E(x \sqcap y, x) \wedge E(y \sqcap x, y) \wedge E(x, x) \wedge E(y, y) \\
&= R(x, y) \wedge R(y, x) \\
&= E(x \sqcap y, x) \wedge E(y \sqcap x, y) \wedge E(x, x) \wedge E(y, y) \\
&\leq E(x, x \sqcap y) \wedge E(x \sqcap y, y \sqcap x) \wedge E(y \sqcap x, y) \\
&\leq E(x, y \sqcap x) \wedge E(y \sqcap x, y) \\
&\leq E(x, y),
\end{aligned}$$

by Proposition (3.2.2) (formula (3.3)), by compatibility of E; we also use idempotence of $\sqcap$ (Proposition (5.1.13), since E is separated by assumption), axiom $L1$ and transitivity of E. Hence, $R(x, y) \wedge R(y, x) = E(x, y)$, proving that R is E-antisymmetric.

Next we show that R is transitive:

$$\begin{aligned}
R(x, y) \wedge R(y, z) &= E(x, x) \wedge E(y, y) \wedge E(x \sqcap y, x) \wedge E(z, z) \wedge E(y \sqcap z, y) \\
&\leq E((x \sqcap y) \sqcap z, x \sqcap z) \wedge E(x \sqcap (y \sqcap z), x \sqcap y) \wedge E(x, x) \wedge \\
&\quad E(y, y) \wedge E(x \sqcap y, x) \wedge E(z, z) \\
&\leq E((x \sqcap y) \sqcap z, x \sqcap z) \wedge E(x \sqcap (y \sqcap z), x \sqcap y) \wedge E(x \sqcap (y \\
&\quad \sqcap z), (x \sqcap y) \sqcap z) \wedge E(x, x) \wedge E(x \sqcap y, x) \wedge E(z, z) \\
&\leq E(x \sqcap y, x \sqcap z) \wedge E(x \sqcap y, x) \wedge E(x, x) \wedge E(z, z) \\
&\leq E(x \sqcap z, x) \wedge E(x, x) \wedge E(z, z) \\
&= R(x, z).
\end{aligned}$$

□

Observe that in case E is separated, the diagonal part of R coincides with the corresponding sub-relation of E: for all $x \in M$

$$R(x,x) = E(x,x). \tag{5.7}$$

Indeed, by Proposition (5.1.12), $R(x,x) = E(x \sqcap x, x) = E(x,x)$.

We also prove that the relation R on an Ω-lattice determines the order on the cut-lattices.

Proposition 5.1.25. *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, $(\mathcal{M}, E)$ an Ω-lattice as an algebra, and $R : M^2 \to \Omega$ an Ω-valued relation on M defined by (5.6). Let $p \in \Omega$. Then, for $x, y \in \mu_p$ and $[x]_{E_p}, [y]_{E_p} \in \mu_p/E_p$,*

$$[x]_{E_p} \leq_p [y]_{E_p} \text{ if and only if } xR_py.$$

Proof. By Theorem 5.1.15, μ_p/E_p is a lattice, hence

$$\begin{aligned}
[x]_{E_p} \leq_p [y]_{E_p} &\iff x, y \in \mu_p \text{ and } [x]_{E_p} \sqcap_p [y]_{E_p} = [x]_{E_p} \\
&\iff x, y \in \mu_p \text{ and } [x \sqcap y]_{E_p} = [x]_{E_p} \\
&\iff x, y \in \mu_p \text{ and } (x \sqcap y, x) \in E_p \\
&\iff \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x) \geq p \\
&\iff R(x,y) \geq p \\
&\iff xR_py.
\end{aligned}$$

□

Theorem 5.1.26. *Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, $(\mathcal{M}, E)$ an Ω-lattice as an algebra in which E is separated, and $R : M^2 \to \Omega$ an Ω-valued relation on M defined by $R(x,y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x)$. Then, (M, E, R) is an Ω-lattice as an ordered structure.*

Proof. Under the given assumptions, by Theorem (5.1.24), (M, E, R) is an Ω-poset. Since $(\mathcal{M}, E)$ is an Ω-lattice as an algebra, for every $p \in \Omega$, the quotient sub-bi-groupoid μ_p/E_p is a lattice with respect to operations $\sqcap_p$ and $\sqcup_p$ induced on the congruence classes by the operations $\sqcap$ and $\sqcup$ respectively. By Proposition (5.1.25), the order $\leq_p$ is induced by R, i.e., it is precisely the order defined by (5.2). Finally, (5.3) holds. Indeed, for $a, b \in M$ and $p \leq \mu(a) \wedge \mu(b)$, we have

$$[a]_{E_p} \sqcap_p [b]_{E_p} = [a \sqcap b]_{E_p} \subseteq [a \sqcap b]_{E_q} = [a]_{E_q} \sqcap_q [b]_{E_q}.$$

Now by Theorem (5.1.7), (M, E, R) is an Ω-lattice as an ordered structure.

□

Next we present an example of a finite Ω-lattice. It is constructed as an ordered structure (M, E, R), and the operations are introduced by Definition (5.1.17). Of course, the two approaches (order-theoretic and algebraic) are equivalent.

Example 5.1.27. Let $M = \{a, b, c, d, e, f, g\}$, and let Ω be a membership values lattice given in Figure 5a.

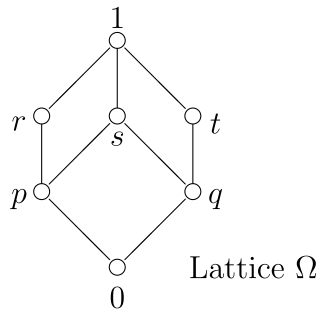

Figure 5a: Membership values lattice

An Ω-valued, separated equality $E : M^2 \to \Omega$ is given in Table 9, and in Table 10 we present an Ω-valued transitive relation $R : M^2 \to \Omega$, which, in addition, satisfies the strictness property; moreover, as it is necessary by the definition, the formula $E(x, y) = R(x, y) \wedge R(y, x)$ holds for all $x, y \in M$.

E	a	b	c	d	e	f	g
a	r	p	0	0	0	0	0
b	p	r	0	0	0	0	0
c	0	0	s	q	q	0	0
d	0	0	q	1	q	0	0
e	0	0	q	q	1	0	0
f	0	0	0	0	0	q	0
g	0	0	0	0	0	0	q

Table 9: Ω-valued equality E

R	a	b	c	d	e	f	g
a	r	r	0	0	r	0	0
b	p	r	0	0	r	0	0
c	0	0	s	q	s	q	q
d	r	r	s	1	1	q	q
e	0	0	q	q	1	q	q
f	0	0	0	0	0	q	q
g	0	0	0	0	0	0	q

Table 10: Ω-valued order R

Now (M, E, R) is an Ω-lattice as an ordered structure. This fact is shown by the following analysis.

Ω-valued function $\mu : M \to \Omega$, defined by $\mu(x) := E(x, x)$, is given by

$$\mu = \begin{pmatrix} a & b & c & d & e & f & g \\ r & r & s & 1 & 1 & q & q \end{pmatrix}.$$

The cuts of μ and the cuts of E represented by partitions are:

$$\begin{array}{ll}
\mu_0 = M \ ; & E_0 = M^2; \\
\mu_p = \{a, b, c, d, e\} \ ; & E_p = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}; \\
\mu_q = \{c, d, e, f, g\} \ ; & E_q = \{\{c, d, e\}, \{f\}, \{g\}\}; \\
\mu_r = \{a, b, d, e\} \ ; & E_r = \{\{a\}, \{b\}, \{d\}, \{e\}\}; \\
\mu_s = \{c, d, e\} \ ; & E_s = \{\{c\}, \{d\}, \{e\}\}; \\
\mu_t = \mu_1 = \{d, e\}; & E_t = E_1 = \{\{d\}, \{e\}\}.
\end{array}$$

For all $x \in \Omega$ the quotient structures μ_x/E_x are lattices with respect to the order defined by (5.2); μ_0/E_0 is obviously a one-element lattice, and the other quotient lattices are represented in Figure 5b.

Observe that by Remark (5.1.8), (μ_r, R_r), (μ_s, R_s), (μ_t, R_t) and (μ_1, R_1) are also lattices, since the corresponding congruence classes under the cuts of E are one-element sets.

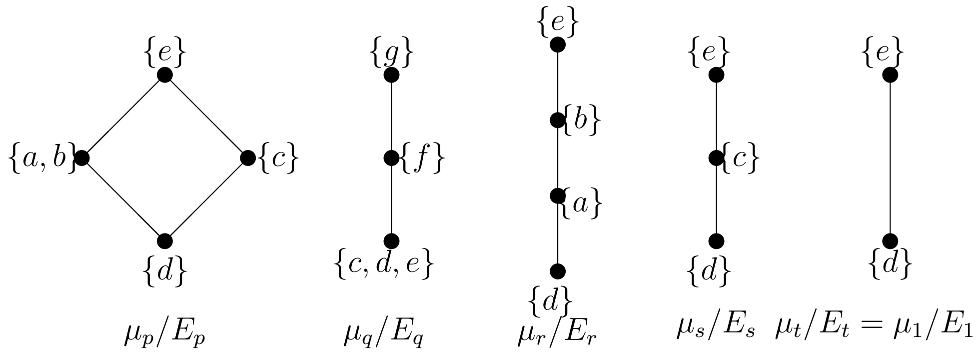

Figure 5b: Quotient lattices

Finally, two binary operations on M constructed by means of pseudo-infima and pseudo-suprema, according to Definition (5.1.17), are given in Tables 11 and 12. The operations are idempotent, since E is separated. Observe that in some fields of these tables the values could be arbitrary (the values denoted by $**$) since the corresponding class is the whole set M, or they could be chosen among several elements (c, d, e) belonging to the same class (values indicated by $*$); we give possible choices of these values. In this way, we obtain the bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$.

$\sqcap$	a	b	c	d	e	f	g
a	a	a	d	d	a	b^{**}	c^{**}
b	a	b	d	d	b	a^{**}	g^{**}
c	d	d	c	d	c	c^{*}	c^{*}
d	d	d	d	d	d	d^{*}	d^{*}
e	a	b	c	d	e	e^{*}	c^{*}
f	d^{**}	a^{**}	d^{*}	e^{*}	c^{*}	f	f
g	a^{**}	e^{**}	c^{*}	e^{*}	c^{*}	f	g

Table 11: Operation $\sqcap$

$\sqcup$	a	b	c	d	e	f	g
a	a	b	e	a	e	f^{**}	a^{**}
b	b	b	e	b	e	a^{**}	c^{**}
c	e	e	c	c	e	f	g
d	a	b	c	d	e	f	g
e	e	e	e	e	e	f	g
f	g^{**}	g^{**}	f	f	f	f	g
g	b^{**}	g^{**}	g	g	g	g	g

Table 12: Operation $\sqcup$

By Theorem (5.1.22), $(\mathcal{M}, E)$ is an Ω-lattice as an algebra.

Next we introduce the notion of complete omega lattices and investigate some special notions as in the ordinary theory of lattices and ordered sets.

5.2 Complete Ω-Lattices

Complete lattices as defined in ordinary mathematics are partially ordered set in which case every subset has a supremum and an infimum. Of course, complete lattices are special instant of lattices, which are studied as ordered structures. Therefore, in this section we present complete Ω-Lattices as special instant of Ω-Lattices.

Definition 5.2.1. Let (M, E, R) be an Ω-poset and $A \subseteq M$. An element $u \in M$ is an **upper bound** of A (under R), if for every $a \in A$

$$\bigwedge(\mu(x) \mid x \in A) \leq R(a, u).$$

An element $v \in M$ is a **lower bound** of A, if for every $a \in A$

$$\bigwedge(\mu(x) \mid x \in A) \leq R(v, a).$$

In other words, an element $u \in M$ is an upper bound of $A \subseteq M$, if

$$(\forall a)(a \in A \;\Rightarrow\; (\bigwedge(\mu(x) \mid x \in A) \leq R(a, u))). \tag{5.8}$$

Similarly, v is a lower bound of A, if

$$(\forall a)(a \in A \;\Rightarrow\; (\bigwedge(\mu(x) \mid x \in A) \leq R(v, a))). \tag{5.9}$$

Proposition 5.2.2. *If u is an upper bound of $A \subseteq M$ in an Ω-poset (M, E, R), then*

$$\bigwedge(R(x, u) \mid x \in A) = \bigwedge(\mu(x) \mid x \in A).$$

Similarly, if v is a lower bound of $A \subseteq M$ in an Ω-poset (M, E, R), then

$$\bigwedge(R(v, x) \mid x \in A) = \bigwedge(\mu(x) \mid x \in A).$$

Proof. We prove the part concerning an upper bound, the proof for a lower bound is analogous.

For $a \in A$,

$$\bigwedge(\mu(x) \mid x \in A) \leq R(a, u). \tag{5.10}$$

Taking infimum over all $x \in A$, we get

$$\bigwedge(\mu(x) \mid x \in A) \leq \bigwedge(R(x, u) \mid x \in A).$$

Further,

$$\bigwedge(R(x, u) \mid x \in A) \leq \bigwedge(R(x, x) \mid x \in A) = \bigwedge(\mu(x) \mid x \in A). \tag{5.11}$$

From (5.10) and (5.11) we get the proof. □

Definition 5.2.3. Let (M, E, R) be an Ω-poset and $A \subseteq M$. Then an element $u \in M$ is a **pseudo-supremum** of A, if for every $p \in \Omega$, such that $p \leq \bigwedge(\mu(x) \mid x \in A)$, the following hold:

- (i) u is an upper bound of A and
- (ii) if there is $u_1 \in M$ such that $p \leq R(a, u_1)$ for every $a \in A$, then $p \leq R(u, u_1)$.

Dually, an element $v \in M$ is a **pseudo-infimum** of A, if for every $p \in \Omega$, such that $p \leq \bigwedge(\mu(x) \mid x \in A)$, the following hold:

- (j) v is a lower bound of A and
- (jj) if there is $v_1 \in M$ such that $p \leq R(v_1, a)$ for every $a \in A$, then $p \leq R(v_1, v)$.

Proposition 5.2.4. *Let (M, E, R) be an Ω-poset, let $A \subseteq M$ and let $u \in M$ be a pseudo-supremum (pseudo-infimum) of $A \subseteq M$. Then $v \in M$ is also a pseudo-supremum (pseudo-infimum) of $A \subseteq M$, if and only if $\bigwedge(\mu(x) \mid x \in A) \leq E(u, v)$.*

Proof. (*only if part*) : Assume that $u, v \in M$ are pseudo-suprema of $A \subseteq M$. Then, by the definition of a pseudo-supremum, by Proposition (5.2.2) and by E-antisymmetry of R, we have

$$\begin{aligned} \bigwedge(\mu(x) \mid x \in A) &= \bigwedge(R(y,u) \mid y \in A) \wedge \bigwedge(R(y,v) \mid y \in A) \\ &\leq R(v,u) \wedge R(u,v) \\ &= E(u,v). \end{aligned}$$

(*if part*) : Suppose that u is a pseudo-supremum of A, and for $v \in M$, $\bigwedge(\mu(x) \mid x \in A) \leq E(u,v)$. Then, since R is E-antisymmetric, we have $\bigwedge(\mu(x) \mid x \in A) \leq R(u,v) \wedge R(v,u)$, which for every $a \in A$ implies

$$\bigwedge(\mu(x) \mid x \in A) \leq R(a,u) \wedge R(u,v) \wedge R(v,u) \leq R(a,u) \wedge R(u,v).$$

Since R is transitive, we have $\bigwedge(\mu(x) \mid x \in A) \leq R(a,v)$, and v is an upper bound for A. Next, let $p \in \Omega$, $p \leq \bigwedge(\mu(x) \mid x \in A)$ and suppose that for $w \in M$, $p \leq R(a,w)$ for every $a \in A$. Then $\bigwedge(\mu(x) \mid x \in A) \leq R(u,w)$, since u is a pseudo-infimum of A and $p \leq R(u,w)$. Also, since

$$p \leq \bigwedge(\mu(x) \mid x \in A) \leq R(v,u),$$

by transitivity of R we obtain $p \leq R(v,w)$.

The dual arguments enable the proof for pseudo-infima. □

Remark 5.2.5. From the above proposition it is clear that for an arbitrary subset $A \subseteq M$, if a pseudo-supremum (pseudo-infimum) exists, it is generally not unique. Two pseudo-suprema u, v of A belong to the same equivalence class μ_p/E_p for every $p \leq \bigwedge(\mu(x) \mid x \in A)$.

Next we introduce pseudo-top and bottom elements for subsets of M in an Ω-poset (M, E, R), as follows.

Definition 5.2.6. A **pseudo-top of** A**,** $A \subseteq M$, is an element $t \in A$ such that for every $y \in A$

$$\bigwedge(\mu(x) \mid x \in A) \leq R(y,t).$$

Dually, a **pseudo-bottom of** A**,** $A \subseteq M$, is an element $b \in A$, such that for every $y \in A$

$$\bigwedge(\mu(x) \mid x \in A) \leq R(b,y).$$

In particular, if $A = M$, then the above elements t and b are said to be a **pseudo-top** and a **pseudo-bottom**, respectively, of the whole Ω-poset (M, E, R).

The following can be proved analogously as Proposition (5.2.2).

Proposition 5.2.7. *Let (M, E, R) be an Ω-poset.*
An element $t \in M$ is a pseudo-top of $A \subseteq M$ if and only if

$$\bigwedge(\mu(x) \mid x \in A) = \bigwedge(R(x, t) \mid x \in A).$$

Dually, $b \in M$ is a pseudo-bottom of $A \subseteq M$ if and only if

$$\bigwedge(\mu(x) \mid x \in A) = \bigwedge(R(b, x) \mid x \in A).$$

Proposition 5.2.8. *If t is a pseudo-top element of a subset A in an Ω-poset (M, E, R), then $t_1 \in A$ is a pseudo-top element of A if and only if*

$$E(t, t_1) \geq \bigwedge(\mu(x) \mid x \in A).$$

Analogously, if b is a pseudo-bottom element of $A \subseteq M$, then an element $b_1 \in A$ is a pseudo-bottom element of A if and only if

$$E(b, b_1) \geq \bigwedge(\mu(x) \mid x \in A).$$

Proof. $(only\ if\ part)$: Let t, t_1 be two pseudo-top elements of $A \subseteq M$ in (M, E, R). Then by the definition

$$\bigwedge(\mu(x) \mid x \in A) \leq R(t_1, t), \quad \text{and} \quad \bigwedge(\mu(x) \mid x \in A) \leq R(t, t_1).$$

Hence, by the E-antisymmetry of R we get

$$E(t, t_1) = R(t_1, t) \wedge R(t, t_1) \geq \bigwedge(\mu(x) \mid x \in A). \tag{5.12}$$

$(if\ part)$: Suppose that $t \in A$ is a pseudo-top element of A, and $t_1 \in A$, such that

$$E(t, t_1) \geq \bigwedge(\mu(x) \mid x \in A).$$

Since $R(t, t_1) \wedge R(t_1, t) = E(t, t_1) \geq \bigwedge(\mu(x) \mid x \in A)$, we have that $R(t, t_1) \geq \bigwedge(\mu(x) \mid x \in A)$. By $\bigwedge(\mu(x) \mid x \in A) = \bigwedge(R(x, t) \mid x \in A)$ and transitivity, we have that t_1 is also a pseudo-top element of A.

The proof for pseudo-bottom elements is analogous. □

Observe that in the above proposition, we assume that another pseudo-top belongs to the subset A. For particular subsets of M – for cuts, we have a stronger result, namely conditions for being a pseudo-top are given for an arbitrary element of M, as follows.

Proposition 5.2.9. *Let t be a pseudo-top of a cut-subset μ_p, $p \in \Omega$, in an Ω-poset (M, E, R). Then any $t_1 \in M$ is also a pseudo-top of μ_p if and only if*

$$E(t, t_1) \geq \bigwedge(\mu(x) \mid x \in \mu_p).$$

Dually, if b is a pseudo-bottom of μ_p, then $b_1 \in M$ is also a pseudo-bottom of μ_p if and only if

$$E(b, b_1) \geq \bigwedge(\mu(x) \mid x \in \mu_p).$$

Proof. The first part of the equivalence is proved in Proposition 5.2.8.

For the converse, suppose that for a pseudo-top t and for $t_1 \in M$, we have

$$E(t, t_1) \geq \bigwedge(\mu(x) \mid x \in \mu_p).$$

Since E is a fuzzy relation on μ, we have that $E(t, t_1) \leq \mu(t) \wedge \mu(t_1) \leq \mu(t_1)$ and since $\bigwedge(\mu(x) \mid x \in \mu_p) \geq p$, we have that $\mu(t_1) \geq p$ and hence $t_1 \in \mu_p$.

The rest of the proof follows as in Proposition 5.2.8

The proof for pseudo-bottom elements is analogous. □

Corollary 5.2.10. *If t is a pseudo-top element of an Ω-poset (M, E, R), then for every $p \leq \bigwedge(\mu(x) \mid x \in M)$, the class $[t]_{E_p}$ is the top element of the poset $(\mu_p/E_p, \leq_p)$.*

Dually, if b is a pseudo-bottom element of (M, E, R), then for every $p \leq \bigwedge_{x \in M} \mu(x)$, the class $[b]_{E_p}$ is the bottom element of the poset $(\mu_p/E_p, \leq_p)$.

Proof. Let t be a pseudo-top element of (M, E, R), and let $p \leq \bigwedge(\mu(x) \mid x \in M)$. This implies $\mu_p = M$. Now, for every $x \in M$, $[x]_{E_p} \leq_p [t]_{E_p}$ if and only if $(x, t) \in R_p$ if and only if $R(x, t) \geq p$, which holds by the definition of a pseudo-top, and by the choice of p.

The proof of the second part is dual. □

Remark 5.2.11. The top and bottom elements of a classical poset if they exist are unique, but in the case of the omega poset they are generally not. Uniqueness therefore can by considered as being in the same equivalence class.

Proposition 5.2.12. *A pseudo top of a subset A of an Ω-poset (M, E, R), if it exists, is a pseudo-supremum of A.*

Dually, a pseudo bottom of A is a pseudo-infimum of this subset.

Proof. Indeed, by the definition, a pseudo top t of A is an upper bound of A. Further, if $p \leq \bigwedge(\mu(x) \mid x \in A)$, and if for some $x_0 \in M$ we have $p \leq \bigwedge(R(x, x_0) \mid x \in A)$, then obviously $p \leq R(t, x_0)$, since $t \in A$.

The proof of the second part is analogous. □

Remark 5.2.13. It is natural to deal also with pseudo-suprema (infima) of the empty subset of M, for an omega poset (M, E, R). According to formulas (5.8) and (5.9), *every element $u \in M$ is an upper (lower) bound of the empty set, as a subset of M.* Indeed, the antecedent $a \in A$ in both formulas is not fulfilled for $A = \emptyset$. Therefore, these formulas (implications) hold for every $u \in M$. A consequence is that *in an Ω-poset (M, E, R) a pseudo-infimum of the empty subset exists if and only if this Ω-poset possesses a pseudo-top element; moreover, in this case every pseudo-top element is a pseudo-infimum of the empty set. Dually, a pseudo-supremum of the empty subset exists in an Ω-poset if and only if it possesses a pseudo-bottom element, and in this case every pseudo-bottom element is a pseudo-supremum of the empty set.*

Definition 5.2.14. Let (M, E, R) be a finite Ω-poset, and $A \subseteq M$ then $a_1 \in A$ is a **maximal** element of A if

$$\bigwedge(\mu(a) \mid a \in A) \not\leq R(a_1, b) \tag{5.13}$$

for every $b \in A$, such that $b \neq a_1$. Dually, $a_0 \in A$ is a **minimal** element of A if

$$\bigwedge(\mu(a) \mid a \in A) \not\leq R(b, a_0)$$

for every $b \in A$, such that $b \neq a_0$.

In order to deal with maximal and minimal elements in Ω-posets, we need the following property of cut sets.

Lemma 5.2.15. *Let $\mu : M \to \Omega$ be an arbitrary mapping, let $p \in \Omega$ and $q = \bigwedge(\mu(x) \mid x \in \mu_p)$. Then, $\mu_p = \mu_q$. In particular, if $\mu(x_0) = p$ for some $x_0 \in \mu_p$, then also $p = q$.*

Proof. For every $x \in \mu_p$ we have $\mu(x) \geq p$, hence $\bigwedge(\mu(x) \mid x \in \mu_p) \geq p$ i.e., $q \geq p$. Therefore $\mu_q \subseteq \mu_p$.

In the other hand, if $y \in \mu_p$, then $\mu(y) \geq p$, hence $\mu(y) \geq \bigwedge(\mu(x) \mid x \in \mu_p) = q$, and therefore $y \in \mu_q$. This proves the other inclusion, $\mu_p \subseteq \mu_q$.

The second part is straightforward. □

Proposition 5.2.16. *Let a be a maximal element of a cut μ_p, $p \in \Omega$, in an Ω-poset (M, E, R), so that for some $x_0 \in \mu_p$, $\mu(x_0) = p$. Then, the class $[a]_{E_p}$ is a maximal element in the poset $(\mu_p/E_p, \leq_p)$.*

Proof. For $x \in \mu_p$, we have

$$[x]_{E_p} \not\geq_p [a]_{E_p} \text{ if and only if } R(x, a) \not\geq p.$$

By the proof of Lemma (5.2.15), $\bigwedge(\mu(x) \mid x \in \mu_p) \geq p$, and since $\mu(x_0) = p$ for some $x_0 \in \mu_p$, we have $\bigwedge(\mu(x) \mid x \in \mu_p) = p$. By assumption, a is a maximal element of $\mu_p \subseteq M$ in (M, E, R), hence by (5.13), for every $y \in \mu_p$, $R(y, a) \not\geq p$, i.e., $[y]_{E_p} \not\geq_p [a]_{E_p}$ for every $y \in \mu_p$, proving that the class $[a]_{E_p}$ is a maximal element in $(\mu_p/E_p, \leq_p)$. $\square$

Definition 5.2.17. An Ω-poset (M, E, R) is called a **complete Ω-lattice** if for every $A \subseteq M$ a pseudo-supremum and a pseudo-infimum of A exist.

By the above definition a pseudo-supremum (infimum) is required also for the empty subset of M. Therefore, by Remark (5.2.13), we have the following immediate consequence.

Proposition 5.2.18. *A complete Ω-lattice possesses a pseudo-top and a pseudo bottom element.*

Next we show that quotient cut-posets of a complete Ω-lattice are complete lattices, and that the classes represented by pseudo-suprema (infima) are classical suprema (infima) in these lattices.

Concerning notations, observe that we deal with arbitrary suprema (infima) in the lattice Ω, but also in the quotient posets $(\mu_q/E_q, \leq_q)$ of an Ω-poset (M, E, R). In the former (the lattice Ω), we denote these by $\bigwedge$ ($\bigvee$), and in the latter (quotient structures) by sup (inf). Recall also that for an Ω-poset (M, E, R), the structure $(\mu_p/E_p, \leq_p)$ is a poset for every $p \in \Omega$, where the relation $\leq_p$ on the quotient set μ_p/E_p is defined by (5.2).

Theorem 5.2.19. *Let (M, E, R) be a complete Ω-lattice. Then, for every $p \in \Omega$, the poset $(\mu_p/E_p, \leq_p)$ is a complete lattice. In addition, for $A \subseteq M$, if c is a pseudo-infimum of A in (M, E, R), then $[c]_p$ is the infimum of $\{[a]_p \mid a \in A\}$ in the lattice $(\mu_p/E_p, \leq_p)$, for every $p \in \Omega$, such that $A \subseteq \mu_p$. Analogously, if d is a pseudo-supremum of A, then $[d]_p$ is the supremum of $\{[a]_p \mid a \in A\}$ in $(\mu_p/E_p, \leq_p)$.*

Proof. Let (M, E, R) be a complete Ω-lattice.

We prove that for every $p \in \Omega$, $(\mu_p/E_p, \leq_p)$ is a (classical) complete lattice. Indeed, let $A \subseteq \mu_p$. Clearly, A determines a collection of classes, each element of A being a representative of the class. Let c be a pseudo-infimum of A. Then for $a \in A$,

$$p \leq \bigwedge(\mu(x) \mid x \in A) \leq R(c, a) \leq E(c, c) = \mu(c),$$

hence $c \in \mu_p$. In addition, in the same sequence of inequalities we have $p \leq R(c, a)$ for every $a \in \mu_p$. Therefore by (5.2), for every $a \in A$, $[c]_{E_p} \leq_p [a]_{E_p}$, and for $x \in \mu_p$, if $[x]_{E_p} \leq_p [a]_{E_p}$ for every $a \in A$, then also $[x]_{E_p} \leq_p [c]_{E_p}$. Hence, $[c]_{E_p} = \inf\{[a]_{E_p} \mid a \in A\}$ in μ_p/E_p.

The proof that for a pseudo-supremum d of A we have $[d]_{E_p} = \sup\{[a]_{E_p} \mid a \in A\}$ is analogous.

By the above, $[c]_p$ is the infimum of the corresponding collection of classes in every lattice $(\mu_p/E_p, \leq_p)$ (i.e., for every p), such that $A \subseteq \mu_p$, similarly for the supremum. □

Theorem 5.2.20. *Let (M, E, R) be an Ω-poset. Then it is a complete Ω-lattice if and only if for every $q \in \Omega$, the poset $(\mu_q/E_q, \leq_q)$ is a complete lattice, and the following holds: for all $A \subseteq M$, for $p = \bigwedge(\mu(a) \mid a \in A)$, and $q \leq p$, we have*

$$\inf\{[a]_{E_p} \mid a \in A\} \subseteq \inf\{[a]_{E_q} \mid a \in A\}, \tag{5.14}$$

$$\text{and} \quad \sup\{[a]_{E_p} \mid a \in A\} \subseteq \sup\{[a]_{E_q} \mid a \in A\}, \tag{5.15}$$

where the infima (suprema) are considered in the corresponding posets $(\mu_q/E_q, \leq_q)$ and $(\mu_p/E_p, \leq_p)$.

Proof. $(only\ if\ part)$: Let (M, E, R) be an Ω-poset, which is a complete Ω-lattice. Then, by Theorem 5.2.19, every poset $(\mu_q/E_q, \leq_q)$, $q \in \Omega$, is a complete lattice and conditions (5.14) and (5.15) clearly hold by the mentioned theorem.

$(if\ part)$: Suppose that for a given Ω-poset (M, E, R), the poset $(\mu_p/E_p, \leq_p)$ is a complete lattice for every $p \in \Omega$, and that conditions (5.14) and (5.15) hold. We prove that then for every $A \subseteq M$ there exist a pseudo-infimum and a pseudo supremum. Indeed, since for $p = \bigwedge(\mu(a) \mid a \in A)$, $(\mu_p/E_p, \leq_p)$ is a complete lattice, there is the infimum for the collection of classes $\{[a]_{E_p} \mid a \in A\}$, a class $[c]_{E_p}$, for some $c \in \mu_p$. Hence $[c]_{E_p} \leq_p [a]_{E_p}$ for every $a \in A$, and for every $x \in \mu_p$, if $[x]_{E_p} \leq_p [a]_{E_p}$ for every $a \in A$, then also $[x]_{E_p} \leq_p [c]_{E_p}$. By Proposition 5.1.1, this is equivalent with $(c, a) \in R_p$ for every $a \in A$, and $(x,a) \in R_p$ for every $a \in A$ implies $(x,c) \in R_p$. By (5.14), the above properties of c hold if p is replaced by q, $q \leq p$. Therefore, c is a pseudo-infimum of A.

Analogously, one can prove that there is a pseudo-supremum for $A \subseteq M$. $\square$

In the classical lattice theory it is well known that a poset containing infima for all subsets is a complete lattice. The following is a kind of an analogous proposition for Ω-posets.

Theorem 5.2.21. *An Ω-poset (M,E,R) is a complete Ω-lattice, if and only if the following conditions are fulfilled:*

(i) a pseudo-infimum exists for every $A \subseteq M$;

(ii) every cut μ_p, $p \in \Omega$, possesses a pseudo-top element;

(iii) for all $A \subseteq M$, $p = \bigwedge(\mu(a) \mid a \in A)$, and $q \leq p$, if $\sup\{[a]_{E_p} \mid a \in A\}$ and $\sup\{[a]_{E_q} \mid a \in A\}$ exist in the posets $(\mu_q/E_q, \leq_q)$ and $(\mu_p/E_p, \leq_p)$ respectively, then

$$\sup\{[a]_{E_p} \mid a \in A\} \subseteq \sup\{[a]_{E_q} \mid a \in A\}.$$

Proof. Let (M,E,R) be an Ω-poset, and for $p \in \Omega$, consider the poset $(\mu_p/E_p, \leq_p)$. Furthermore, let $\{[x]_{E_p} \mid x \in A\}$, $A \subseteq M$ be an arbitrary collection of classes in this quotient poset. Since, by assumption, there is a pseudo-infimum $c \in M$ of A, similarly as in the proof of Theorem (5.1.7), we get that $[c]_{E_p} = \inf\{[x]_{E_p} \mid x \in A\}$. Therefore, the poset $(\mu_p/E_p, \leq_p)$ contains the greatest element by corollary (5.2.10) and is closed under arbitrary infima, hence it is a complete lattice. Further, as in Theorem (5.2.20), we conclude that the property (5.14) holds. Obviously, (5.15) also holds by (iii). By Theorem (5.2.20), (M,E,R) is a complete Ω-lattice.

The converse holds by Theorem 5.2.20. $\square$

Finally we deal with a construction of Ω-posets and complete Ω-lattices.

In the following we use transitivity and strictness of binary relations, as presented in preliminaries. We also denote by $\Delta(f)$ the diagonal sub-relation of a binary relation f:

$$\Delta(f) := \{x \in M \mid (x,x) \in f\}. \tag{5.16}$$

Theorem 5.2.22. *Let $M \neq \emptyset$, and let $\mathcal{F} \subseteq \mathcal{P}(M^2)$ be a closure system over M^2 such that each $f \in \mathcal{F}$ is transitive and strict. Then the following hold.*

(a) There is a complete lattice Ω and a mapping $R : M^2 \longrightarrow \Omega$ such that $\mathcal{F}$ is a collection of cuts of R and (M,E,R) is an Ω-poset, where $E : M^2 \longrightarrow \Omega$ is defined by $E(x,y) = R(x,y) \wedge R(y,x)$.

(*b*) (M, E, R) *is a complete* Ω*-lattice, if in addition, for every* $f \in \mathcal{F}$ *and for every* $A \subseteq \Delta(f)$ *there is an infimum and a supremum in the relational structure* $(\Delta(f), f)$*, and for* $g \in \mathcal{F}$*, such that* $f \subseteq g$*, the following hold:*

$$\text{if } c \text{ is an inf. of } A \text{ in } \Delta(f)\text{, then } c \text{ is an inf. of } A \text{ in } \Delta(g) \quad (5.17)$$

$$\text{if } c \text{ is a sup. of } A \text{ in } \Delta(f)\text{, then } c \text{ is a sup. of } A \text{ in } \Delta(g) \quad (5.18)$$

Proof. (*a*) We take the lattice Ω to be the collection $\mathcal{F}$, ordered by the dual of the set inclusion: for $f, g \in \mathcal{F}$, $f \leq g$ if and only if $f \supseteq g$. By the assumption of closeness under intersections, $\Omega = (\mathcal{F}, \leq)$ is a complete lattice. Next, we define $R : M^2 \longrightarrow \Omega$:

$$R(x, y) := \bigcap \{f \in \mathcal{F} \mid (x, y) \in f\}. \quad (5.19)$$

We prove that the cuts of R coincide with the relations in $\mathcal{F}$. Namely, we prove that for every $f \in \mathcal{F}$ as an element of the co-domain lattice, the cut R_f coincide with f, which is now considered to be the subset of the domain. Indeed, $(x, y) \in R_f$ if and only if $R(x, y) \geq f$ if and only if $R(x, y) \subseteq f$ if and only if $\bigcap\{g \in \mathcal{F} \mid (x, y) \in g\} \subseteq f$. Therefore, if $(x, y) \in R_f$, then $(x, y) \in f$. In the other hand, if $(x, y) \in f$, then $\bigcap\{g \in \mathcal{F} \mid (x, y) \in g\} \subseteq f$, since f is one of the subsets forming the intersection. Then, by the previous series of equivalences, $(x, y) \in R_f$, proving finally $R_f = f$.

Next, we prove that R is a transitive and strict Ω-valued relation. It is transitive: For some $x, y \in M$, let $R(x, y) \wedge R(y, z) = f \in \Omega$. Then $R(x, y) \geq f$ and $R(y, z) \geq f$. Therefore, both (x, y) and (y, z) belong to R_f. Since $R_f = f$ and f is a transitive relation on M, we get $(x, z) \in R_f$, i.e., $R(x, z) \geq f$, implying $R(x, y) \wedge R(y, z) \leq R(x, z)$. Further, R is strict. Indeed, assume that for $x, y \in M$, $R(x, y) = f$. Then $(x, y) \in R_f = f$. By assumption, f is a strict relation on M, so both, (x, x) and (y, y) also belong to f, implying that $R(x, x) \geq f$ and $R(y, y) \geq f$. Therefore, $R(x, y) \leq R(x, x) \wedge R(y, y)$ and R fulfills strictness property for Ω-valued relations.

As assumed, $E : M^2 \longrightarrow \Omega$ is given by $E(x, y) = R(x, y) \wedge R(y, x)$.

By the above, we conclude that (M, E, R) is an Ω-poset.

(*b*) We prove that under the assumption in this part, the conditions of Theorem 5.2.20 are fulfilled. First, observe by the definition (5.16), for $f \in \mathcal{F}$, $\Delta(f)$ coincides with the cut μ_f (f being considered as a member of the co-domain lattice Ω), where $\mu : M \to \Omega$ has been defined by $\mu(x) = E(x, x)$. Therefore, by Proposition 5.1.1, the quotient μ_f / E_f is a poset under the order R_f. Moreover, it is a complete lattice: If c is an infimum of $A \subseteq \mu_f$, then it is straightforward that $[c]_{E_f} = \inf\{[a]_{E_f} \mid a \in A\}$. In addition, assumption

(5.17) and (5.18) coincide with conditions (5.14) and (5.15), respectively.

By Theorem 5.2.20, (M, E, R) is a complete Ω-lattice. $\square$

Chapter 6

Weak Congruence Relations and Ω-Algebras

It is well known in universal algebra that the best known ordered structures for structural investigations of algebras are lattices of subalgebras and those of congruences the former being the subsets of the universe and the latter being the subsets of the square of the universe. Due to their different nature, these two lattices are often studied separately. Weak congruences has been studied in ([88, 85, 93]) as a viable tool for studying both subalgebras and congruences of the same algebra. Therefore our notation on weak congruences is as given by [*Šešelja and Tepavčević* ([93])]

6.1 Weak Congruence Relations

In this section we introduce the notion of weak congruences. Generally, weak congruences are different from congruence by the reflexivity condition, the former is weakly reflexive compared to the latter which is reflexive. In fact weak congruences can be seen as a weakening of the concept of congruences. Weak congruence relations can be taken as particular subuniverses of squares of algebras. Furthermore, weak congruences are precisely congruence relations on the subalgebras of an algebra, which includes the subalgebras that are represented by the diagonal relations. Therefore, the lattice of weak congruences contains as sublattices collections of some different kinds of relations like the lattice of all congruence relations on the algebra. Also the lattice of all subalgebras of the algebra is embeddable in a natural way into the lattice of weak congruences.

Definition 6.1.1. Let $\mathcal{M} = (M, \mathbb{F})$ be an algebra. A relation θ on M, is a weak congruence relation on $\mathcal{M}$ if θ is a symmetric, transitive compatible relation on M^2. The lattice of all weak congruences on M will be denoted $\mathcal{C}w\mathcal{M}$.

Obviously, weak congruence relations on $\mathcal{M}$ are symmetric, transitive subalgebra of M^2, i.e. they are congruences on subalgeras of $\mathcal{M}$.

Remark 6.1.2. A weak congruence relation θ on an algebra $\mathcal{M}$ in which the compatibility property is not explicitly extented to the constants functions defined on the algebra can be defined as a compatible weak equivalence relation on $\mathcal{M}$ which is **weakly reflexive:**

$$\text{if } \ c \text{ is a constant in } \ \mathcal{M}, \ \text{ then } \ c\theta c.$$

We note that the empty set is a weak congruence relation on an algebra $\mathcal{M}$ if and only if $\mathcal{M}$ does not have fundamental constant operations.[1]

For the lattice $\mathcal{C}w\mathcal{M}$ of all weak congruences on $\mathcal{M}$, the diagonal relation $\Delta = \{(x,x) \in M^2 : x \in M\}$ on $\mathcal{M}$, plays a very significant role in determining the structure of this lattice. Namely, the filter generated by the diagonal relation Δ on $\mathcal{M}$ is the lattice $Con\mathcal{M}$ of all congruence relations on $\mathcal{M}$ and also the ideal generated by the diagonal relation Δ on $\mathcal{M}$ is isomorphic to the lattice $Sub\mathcal{M}$ of all subalgebras of $\mathcal{M}$. The meet operation defined on $\mathcal{C}w\mathcal{M}$ is the set-intersection and the join operation is the closure of the set-union.

For lattice $\mathcal{C}w\mathcal{M}$ the following lemmas are obvious

Lemma 6.1.3. *Let $\mathcal{N} \in Sub\mathcal{M}$ and $\theta \in Con\mathcal{N}$ then $\theta \in \mathcal{C}w\mathcal{M}$.* $\square$

For an arbitrary relation θ on M, let $M_\theta := \{x \in M \mid (x,x) \in \theta\}$.

Lemma 6.1.4. *If $\theta \in \mathcal{C}w\mathcal{M}$ and $\mathcal{M}_\theta = (M_\theta, \mathbb{F}_\theta)$, then $\mathcal{M}_\theta \in Sub\mathcal{M}$.* $\square$

The operations in $\mathbb{F}_\theta$ are the restriction of the operation in the algebra.

Theorem 6.1.5 (*[Vovodić and Šešelja, 1988]*)**.** *The lattice $\mathcal{C}w\mathcal{M}$ of all weak congruences on $\mathcal{M}$ is an algebraic lattice.*

Proof. Let $\mathcal{C}w\mathcal{M}$ be as defined. Then for the algebra $\mathcal{M}$, $M^2 \in \mathcal{C}w\mathcal{M}$ by (6.1.3) and since $\mathcal{C}w\mathcal{M}$ is closed under intersection then $\mathcal{C}w\mathcal{M}$ is a complete lattice. Next we show that $\mathcal{C}w\mathcal{M}$ is an algebraic lattice. Therefore for each

[1] In [Vojvodić and Šešelja, 1988], $\emptyset$ is a weak congruence relation if and only if $\mathcal{M}$ does not have the smallest algebra.

n-ary operation $f \in \mathbb{F}$ defined on $\mathcal{M}$, let the corresponding n-ary operation $\hat{f}$ on M^2 be defined by

$\hat{f}((a_1, b_1), ..., (a_n, b_n)) := (f(a_1, ..., a_n), f(b_1, ..., b_n))$, and $\hat{C} = \{(c, c) \mid c \in \mathbb{F}^{\mathcal{M}}\}$.

Let $\hat{\mathcal{M}} = (M^2, \hat{F} \cup \hat{C} \cup \{s, t\})$, where $\hat{C}$ is the set of nullary operations, s a unary operation given by $s((x, y)) = (y, x)$ and t a binary operation given by

$$t((x, y), (u, v)) = \begin{cases} (x, v) & if\ y = u \\ (x, y) & otherwise \end{cases}$$

Therefore, obviously it is clear that any subalgebra of $\hat{\mathcal{N}} \subseteq \hat{\mathcal{M}}$ is a weak congruence relation on $\mathcal{M}$. Since the lattice of all subalgebras is algebraic, then $\mathcal{C}w\mathcal{M}$ is algebraic. $\square$

Proposition 6.1.6. *If $\mathcal{N} \in Sub\mathcal{M}$, then $(Con\mathcal{N}, \leq)$ is an interval sublattice in $(\mathcal{C}w\mathcal{M}, \leq)$.* $\square$

6.2 Weak congruences and Cuts of Ω-valued equality on Ω-algebra

In this section we elaborate the relationship between an Ω-valued equality and the weak congruence relations on a (basic) algebra. This existing relationship is linked in the cut-worthy approach. Generally it is known in the fuzzy settings that the cuts of a fuzzy (set) function are some crisp subsets of the domain of the fuzzy function. Therefore, it is obvious from our definition of Ω-valued equality that the cuts of our Ω-valued equality are some weak congruences on the basic algebra.

By our definition of Ω-valued function μ, the following fact about Ω-algebras are direct consequences of some well known properties of L-structures.

Proposition 6.2.1. *Let $(\mathcal{M}, E)$ be an Ω-algebra. The function $\mu : M \to \Omega$ defined by $\mu(x) = E(x, x)$ is compatible over $\mathcal{M}$.*

Proof. For any n-ary functional symbol f defined on $\mathcal{M}$ and any $x_1, ..., x_n \in \mathcal{M}$, then by the compatibility of E

$$\bigwedge_{i=1}^{n} \mu(x_i) = \bigwedge_{i=1}^{n} E(x_i, x_i) \leq E(f(x_1, ..., x_n), f(x_1, ..., x_n)) = \mu(f(x_1, ..., x_n)).$$

$\square$

Remark 6.2.2. Observe that an analogous situation to the one presented in Proposition (6.2.1) holds in the classical case, where weak congruence relations σ on an algebra $\mathcal{M}$ determines a subalgebra $\mathcal{B}$ of $\mathcal{M}$ by the corresponding identity relation i.e. $B = \{x \in M | (x, x) \in \sigma\}$.

The following is a simple illustration of the existing relationship.

Example 6.2.3. Let $\mathcal{M} = (M, \circ, -^1, e)$ be an algebra with one binary, unary and a constant function, which does not fulfill any special particular axiom, like associativity. We see that e is the neutral element in this algebra. Let $M = \{a, b, c, d, e, f\}$, such that we have following operation tables below;

$\circ$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	a	b	c	d
b	b	c	d	a	e	f
c	c	d	a	e	a	d
d	d	a	e	a	b	f
f	f	d	f	b	f	e

Table 13

$-^1$	e	a	b	c	d	f
	e	a	d	c	b	f

Table 14

The Ω-valued equality E and $\mu(x) := E(x, x)$ are given by the tables below.

E	e	a	b	c	d	f
e	1	0	x	0	x	0
a	0	w	0	0	0	0
b	x	0	v	0	x	0
c	0	0	0	u	0	0
d	x	0	x	0	v	0
f	0	0	0	0	0	x

Table 15: Ω-valued equality

μ	e	a	b	c	d	f
	1	w	v	u	v	x

Table 16: Ω-valued function

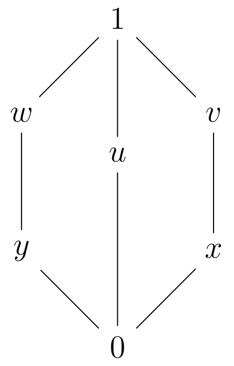

Figure 6.1: Ω

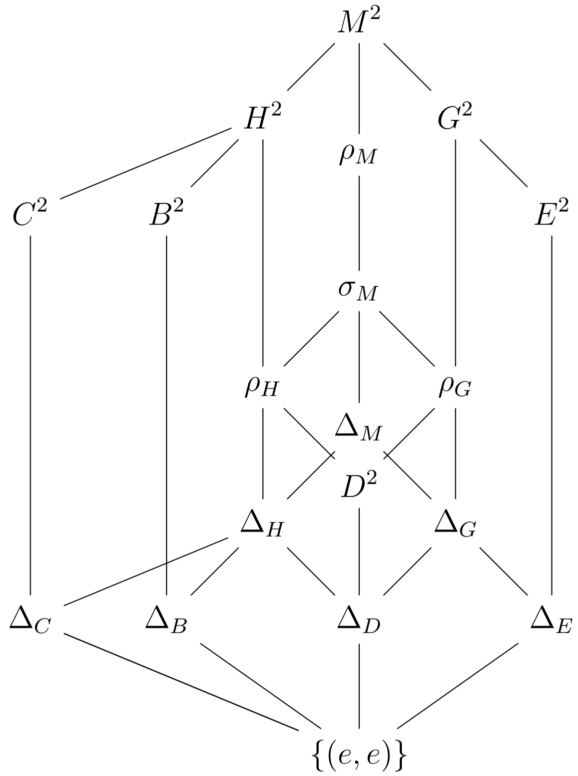

Figure 6.2: $\mathcal{C}w\mathcal{M}$

Figures $\mathcal{C}w\mathcal{M}$ and Ω are the lattices of all weak congruences on $\mathcal{M}$ and the lattice Ω respectively.

Therefore the following are the cuts of E and μ represented:
$E_0 = M^2$; $E_u = \{(e,e),(c,c)\}$
$E_x = \{(e,e),(b,b),(d,d),(e,b),(b,e),(e,d),(d,e),(b,d),(d,b),(f,f)\}$
$E_w = \{(e,e),(a,a)\} = E_y$; $E_1 = \{(e,e)\}$; $E_v = \{(e,e),(b,b),(d,d)\}$
$\mu_0 = M$; $\mu_u = \{e,c\}$; $\mu_x = \{e,b,d,f\}$
$\mu_w = \{e,a\} = \mu_y$; $\mu_1 = \{e\}$; $\mu_v = \{e,b,d\}$
Obviously, each cut relation of the Ω-valued equality E as represented above is a weak congruence relation on the (basic) algebra $\mathcal{M}$ and a congruence relation on the respective cut structure, which turns out to be a subalgebra of $\mathcal{M}$.
The following are substructures of $\mathcal{M}$,
$\mathcal{A} = (\{e\}, \circ, -^1, e)$, $\mathcal{B} = (\{e,a\}, \circ, -^1, e)$, $\mathcal{C} = (\{e,c\}, \circ, -^1, e)$,
$\mathcal{E} = (\{e,f\}, \circ, -^1, e)$ $\mathcal{D} = (\{e,b,d\}, \circ, -^1, e)$,
$\mathcal{H} = (\{e,a,b,c,d\}, \circ, -^1, e)$, $\mathcal{G} = (\{e,b,d,f\}, \circ, -^1, e)$.
Next we obtain the congruences on $\mathcal{M}$ and the congruence on substructures,

which are weak congruences on $\mathcal{M}$.
$Con\mathcal{M} = \{\Delta_{\mathcal{M}}, \{\{e,b,d\},\{a,c,f\}\}, \{\{e,b,d\},\{a,c\},\{f\}\}, \mathcal{M}^2\}$,
$Con\mathcal{A} = \{e\}$, $Con\mathcal{B} = \{\Delta_{\mathcal{B}}, \mathcal{B}^2\}$, $Con\mathcal{C} = \{\Delta_{\mathcal{C}}, \mathcal{C}^2\}$, $Con\mathcal{D} = \{\Delta_{\mathcal{D}}, \mathcal{D}^2\}$,
$Con\mathcal{E} = \{\Delta_{\mathcal{E}}, \mathcal{E}^2\}$, $Con\mathcal{H} = \{\Delta_{\mathcal{H}}, \{\{e,b,d\},\{a,c\}\}, \mathcal{H}^2\}$,
$Con\mathcal{G} = \{\Delta_{\mathcal{G}}, \{\{e,b,d\},\{f\}\}, \mathcal{G}^2\}$.
Where, $\rho_{\mathcal{M}} = \{\{e,b,d\},\{a,c,f\}\}$, $\sigma_{\mathcal{M}} = \{\{e,b,d\},\{a,c\},\{f\}\}$,
$\rho_{\mathcal{H}} = \{\{e,b,d\},\{a,c\}\}$ and $\rho_{\mathcal{G}} = \{\{e,b,d\},\{f\}\}$.

Proposition 6.2.4. *Let $\mathcal{M} = (M, \mathbb{F})$ be an algebra and Ω a complete lattice. Then $E : M^2 \to \Omega$ is an Ω-valued equality relation on $\mathcal{M}$ if and only if all the cut relations are weak congruence relations on $\mathcal{M}$.*

Proof. Let E be an Ω-valued equality relation on $\mathcal{M}$, then for all constant functional operations c on $\mathcal{M}$, $E(c,c) = 1$, hence for each $p \in \Omega$, $E(c,c) = 1 \geq p$, which means that for each $p \in \Omega$ $(c,c) \in E_p \subseteq M^2$. The transitivity and symmetry of each E_p follow directly from the transitive and symmetric properties of E. Now, since E is transitive and symmetric then for each $p \in \Omega$ and $(x,y) \in E_p$, then $(x,x),(y,y) \in E_p$. Hence by compatibility of E, then compatibility holds in E_p.
Conversely, suppose all cut relations are weak congruence relations on $\mathcal{M}$. For $x,y \in M$, let $E(x,y) = p$, $p \in \Omega$, then $(x,y) \in E_p$. By our hypothesis we have that $(y,x) \in E_p$ thus $E(y,x) \geq p$, hence $E(x,y) \leq E(y,x)$. Similarly we prove that $E(y,x) \leq E(x,y)$. Therefore, $E(x,y) = E(y,x)$ and so E is symmetric. Indeed E is transitive: For $x,y,z \in M$, let $E(x,y) \wedge E(y,z) = p$, then $(x,y),(y,z) \in E_p$. By our hypothesis E_p is transitive, therefore $(x,z) \in E_p$, thus $E(x,z) \geq p$. Hence we have that $E(x,y) \wedge E(y,z) \leq E(x,z)$. Clearly, E satisfies the strictness property. Let c be any nullary operation in the language of of $\mathcal{M}$. Therefore by our hypothesis and remark (6.2.2), since each cut relation E_p is a congruence relation on a particular subalgebra of $\mathcal{M}$, then $(c,c) \in E_p$ for each such $p \in \Omega$. Therefore, it must be that $E(c,c) = 1$. Next we show that each n-ary operation symbol f on $\mathcal{M}$ is compatible with E. For $x_1, \cdots, x_n, y_1, \cdots, y_n \in M$, let $\bigwedge_{i=1}^{n} E(x_i, y_i) = p$, then $E(x_i, y_i) \geq p$ for each $i = 1, 2, \cdots, n$, thus $(x_i, y_i) \in E_p$. Therefore by our hypothesis and remark (6.2.2) each cut relation E_p is a congruence relation on a particular subalgebra of $\mathcal{M}$, thus $(f(x_1, \cdots, x_n), f(y_1, \cdots, y_n)) \in E_p$ implying that $E(f(x_1, \cdots, x_n), f(y_1, \cdots, y_n)) \geq p$ for each such $p \in \Omega$. Hence $\bigwedge_{i=1}^{n} E(x_i, y_i) = p \leq E(f(x_1, \cdots, x_n), f(y_1, \cdots, y_n))$. Therefore we have proved that E is an Ω-valued equality on the algebra $\mathcal{M}$.

$\square$

Theorem 6.2.5. *Assume that $\mathcal{M} = (M, \mathbb{F})$ is an algebra, $\mathcal{F} = \{\rho_i \in \mathcal{C}w\mathcal{M} \mid i \in I\} \subseteq \mathcal{P}(M^2)$ is a closure system over M^2, and $\{\mathcal{N}_i : i \in I\}$ is a collection of subalgebras of $\mathcal{M}$ such that for each $i \in I$, $N_i = \{x \in M \mid x\rho_i x\}$ and $\mathcal{N}_i/\rho_i$ satisfies a collection of classical identities. Then there is complete lattice Ω and a mapping $E : M^2 \longrightarrow \Omega$ such that $(\mathcal{M}, E)$ is an Ω-algebra satisfying the same identity and $\mathcal{F}$ consists of cuts of E.*

Proof. Let $(\mathcal{F}, \subseteq)$ be a closure system as defined in the theorem and $(\Omega, \leq)$ be $\mathcal{F}$ with dual ordering of the set inclusion defined on $\mathcal{F}$ i.e. $\leq = \supseteq$. Each $\rho_i \in \mathcal{F}$ is a weak congruence relation on $\mathcal{M}$. Therefore for, $\rho_1, \rho_2 \in \mathcal{F}$, $\rho_1 \leq \rho_2 \Leftrightarrow \rho_1 \supseteq \rho_2$, hence $\Omega = (\mathcal{F}, \leq)$ is a complete lattice closed under intersection. Now we define the mapping $E : M^2 \longrightarrow \Omega$ by

$$E(x,y) := \bigcap\{\rho_i \in \mathcal{F} \mid (x,y) \in \rho_i\}.$$

Next we show that by the definition of the mapping, for each $\rho_j \in \mathcal{F}$ as an element of the co-domain lattice each cut relation E_{ρ_j} coincide with ρ_j i.e. $E_{\rho_j} = \rho_j$. Indeed $(x,y) \in E_{\rho_j}$ if and only if $E(x,y) \geq \rho_j$ if and only if $E(x,y) \subseteq \rho_j$ if and only if $\bigcap\{\rho_i \in \mathcal{F} \mid (x,y) \in \rho_i\} \subseteq \rho_j$ if and only if $(x,y) \in \rho_j$. Hence if $(x,y) \in E_{\rho_j}$ then$(x,y) \in \rho_j$. For the reverse inclusion, let $(x,y) \in \rho_j$ then $\bigcap\{\rho_i \in \mathcal{F} \mid (x,y) \in \rho_i\} \subseteq \rho_j$, since ρ_j is a one of this subsets forming the intersection, then from the above arguments we have that $(x,y) \in E_{\rho_j}$. Thus proving that $E_{\rho_j} = \rho_j$.
We next show that E is transitive, symmetric and strict. For some $x, y, z \in M$, let $E(x,y) \wedge E(y,z) = \rho_j$, then $E(x,y), E(y,z) \geq \rho_j$, but since ρ_j is transitive then $(x,z) \in \rho_j$, thus $(x,z) \in E_{\rho_j}$ implying $E(x,z) \geq \rho_j$. Therefore, $E(x,y) \wedge E(y,z) \leq E(x,z)$.
To show that E is symmetric, for some $x, y \in M$, let $E(x,y) = \rho_j$ then$(x,y) \in E_{\rho_j} = \rho_j$. Since ρ_j is symmetric, then $(y,x) \in \rho_j$. Then $(y,x) \in E_{\rho_j}$ imply $E(y,x) \geq \rho_j$ and so $E(y,x) \geq E(x,y)$. With the same argument we show that $E(y,x) \leq E(x,y)$, proving that E is symmetric.
Furthermore E is strict. Indeed, for some $x, y \in M$, let $E(x,y) = \rho_j$, then by transitivity and symmetric of E it follows that $E(x,y) \wedge E(y,x) \leq E(x,x)$ and $E(y,x) \wedge E(x,y) \leq E(y,y)$. Therefore $E(x,x), E(y,y) \geq \rho_j$, hence $E(x,y) \leq E(x,x) \wedge E(y,y)$, proving that E is strict.
Therefore by theorem (4.5.5) it follows that $(\mathcal{M}, E)$ is an Ω-algebra. $\square$

It is clear that the lattice of all cuts from E form a complete lattice, which is a subposet of the lattice of all weak congruences on the basic algebra.

Therefore, every Ω-algebra $(\mathcal{M}, E)$ uniquely determine a closure system in the lattice $\mathcal{C}w\mathcal{M}$ of all weak congruences on the algebra $\mathcal{M}$.

Corollary 6.2.6. *Let $(\mathcal{M}, E)$ be an Ω-algebra and $\{\mu_p/E_p : p \in \Omega\}$ be the set of all quotient algebras that satisfy the same identities. Then the poset*

$$(\{\mu_p/E_p \mid p \in \Omega\}, \subseteq)$$

is a closure system which is, up to an isomorphism, a subposet of the weak congruence lattice of $\mathcal{M}$.

The following example illustrates the construction described in theorem (6.2.5). Observe that the above inclusion is the classical order on quotient structures induced by the corresponding inclusion over weak congruences.

Example 6.2.7. Let $\mathcal{M}$ be as given in example (6.2.3). Therefore from Figure (6.2) we can choose $(\mathcal{F}, \subseteq)$ as given in Figure (6.3) and $(\Omega, \leq)$ as given in Figure (6.4).

E	a	b	c	d	e	f
a	Δ_B	H^2	ρ_H	H^2	B^2	ρ_M
b	H^2	Δ_D	H^2	D^2	D^2	G^2
c	ρ_H	H^2	Δ_C	H^2	C^2	ρ_M
d	H^2	D^2	H^2	Δ_D	D^2	G^2
e	B^2	D^2	C^2	D^2	$\{(e,e)\}$	E^2
f	ρ_M	G^2	ρ_M	G^2	E^2	Δ_E

table 17: Ω-valued equality induced by $\mathcal{C}w\mathcal{M}$

μ	e	a	b	c	d	f
	{(e,e)}	Δ_B	Δ_D	Δ_C	Δ_D	Δ_E

table 18: Ω-valued function

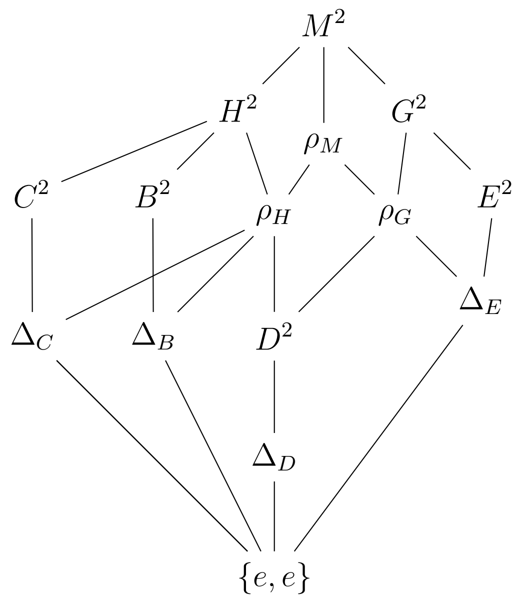

Figure 6.3: $(\mathcal{F}, \subseteq)$

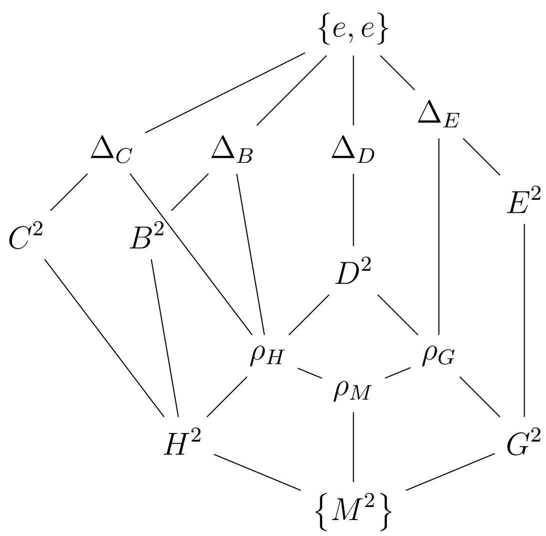

Figure 6.4: $(\Omega, \leq)$

The cut relations of E are given as follows and are congruences on the corresponding cut sets;
$E_e = \{(e,e)\}$, $E_{\Delta_B} = \{(e,e),(a,a)\}$, $E_{\Delta_C} = \{(e,e),(c,c)\}$,
$E_{\Delta_D} = \{(e,e),(b,b),(d,d)\}$, $E_{\Delta_E} = \{(e,e),(f,f)\}$,
$E_{B^2} = \{(e,e),(a,a),(e,a),(a,e)\}$, $E_{C^2} = \{(e,e),(c,c),(e,c),(c,e)\}$,
$E_{D^2} = \{(e,e),(b,b),(d,d),(e,b),(b,e),(e,d),(d,e),(d,b),(b,d)\}$,
$E_{E^2} = \{(e,e),(f,f),(e,f),(f,e)\}$, $E_{\rho_H} = \{(e,e),(a,a),(b,b),(c,c),(d,d),(e,b),(b,e),(e,d),(d,e),(d,b),(b,d),(a,c),(c,a)\}$, $E_{H^2} = \{(e,e),(a,a),(b,b),(c,c),(d,d),(e,a),(a,e),(e,b),(b,e),(e,c),(c,e),(e,d),(d,e),(a,b),(b,a),(a,c),(c,a),(a,d),(d,a),(b,c),(c,b),(b,d),(d,b),(c,d),(d,c)\}$,
$E_{G^2} = \{(e,e),(b,b),(d,d),(f,f),(e,b),(b,e),(e,d),(d,e),(e,f),(f,e),(b,d),(d,b),(b,f),(f,b),(d,f),(f,d)\}$,
$E_{\rho_G} = \{(e,e),(b,b),(d,d),(f,f),(e,b),(b,e),(e,d),(d,e),(b,d),(d,b)\}$,
$E_{\rho_M} = \{(e,e),(a,a),(b,b),(c,c),(d,d),(f,f),(e,b),(b,e),(e,d),(d,e),(b,d),(d,b),(a,c),(c,a),(a,f),(f,a),(c,f),(f,c)\}$,
$E_{M^2} = M^2$;
which turn out to be the corresponding elements of $(\mathcal{F},\subseteq)$.

Chapter 7

Conclusion and future research work

In the preceding chapters we have been able to provide a detailed foundation of this new direction of research in fuzzy set theory in universal algebra, dealing with omega-valued algebraic and relational structure from a general point of view. Therefore, below are some of the main achievements of this work and various questions that relates to them.

The notion of Ω-structures as introduced and presented in this work is dealt with by the weakening of the reflexivity property of fuzzy relations and orders. Wherefore, these functions fulfill the strictness property, whose consequence is that the cut relations are weakly reflexive. Hence, by this consequence we investigated omega structures in the frame work of weak congruences of classical algebras with its lattice weak congruences . The cuts of an Ω-valued equality are weak congruence relations on the (basic) algebra and as such they are elements in the (classical) lattice of all weak congruences on the (basic) algebra. Our investigation proved that the collection of all quotient structures, $\mu_p/E_p, \forall p \in \Omega$, where μ is an Ω-subalgebra determined by the Ω-valued equality E, forms a closure system, which is a subposet of the lattice of all weak congruence relations on the (basic) algebra.

After the definition of Ω-algebras, in chapter (4) (definition 4.1.2), basic universal algebraic constructions were introduced, Ω-subalgebras, Ω-valued congruences, Ω-valued morphisms and Ω-valued direct products. Each of these notions where appropriately presented and defined with respect to the Ω-valued equality. But it turned out that several properties are different from the classical theory, concerning e.g., quotient structures, homomorphic images etc. Still, the link between the classical homomorphisms, kernels and natural maps exists in the field of cut structures, satisfy analogous properties.

It was implicitly shown that each equational class of Ω-algebras is closed under Ω-subalgebras, Ω-valued homomorphic images and Ω-valued direct products, forming an Ω-variety. These follow from theorems (4.1.7,4.2.11,4.4.4). This investigation implicitly dealt with the Birkhoff's theorem, theorem (2.1.41) in one direction in the Ω-valued framework. But the other direction, which of course tends to be more difficult was not investigated. The question is, if we have a class of Ω-algebras of the same type which is closed under formation of Ω-subalgebras, Ω-valued homomorphic images and Ω-valued direct products, is it an equational class, , i.e., is there a set of identities fulfilled by all Ω-algebras in the class? In dealing with this question an Ω-valued free algebra in the class is needed, which was beyond the planned scope of this work. Answering this question is one of many further research tasks that can follow from this thesis. With our definition of Ω-valued homomorphism the platform for this further investigation has been set on track. Our definition of Ω-valued homomorphisms give the requirement that was missing in the definition of fuzzy homomorphisms in ([21]) where a similar notion, a fuzzy variety was also investigated. Notably, our definition of Ω-valued morphisms on Ω-structures link properly with the quotient structures, theorem (4.5.6).

Morphisms theorems of course deal with quotient structures. For this reason Ω-quotient structures were also introduced and investigated. Also future research is to investigate further topics in this framework, namely homomorphism and isomorphism theorems. In addition, applying our results to particular algebras like groups, rings etc.

Our Ω-valued equality is a special kind of an Ω-valued congruence on Ω-algebras. This is different from other Ω-valued congruences by the (strong) separation property, and it is a generalization of the ordinary equality.

It was shown that the set of all Ω-valued congruences on an Ω-algebra forms a complete lattice, theorem (4.2.4) with the greatest element of this lattice which is in general the square of the Ω-algebra, proposition (4.2.2).

Ω-posets and Ω-lattices, as were introduced in chapter (5), can be understood as a L-valued generalization of orderings, with respect to a generalized equality relation. They are based on Ω-sets, they are defined as Ω-algebras, and in addition they are extended to Ω-valued relational structures. As a follow-up, many classical notions related to complete lattices can be analyzed in this framework, like e.g., algebraic lattices and various topological structures.

Another aspect of this investigation, namely, an Ω-poset was being equivalently considered as a set M with a closure system of subsets of M^2 which are transitive and strict, theorem (5.2.22) . Therefore, there could be an interest to develop the whole theory of Ω-ordered structures in the framework of closure systems and operators.

Furthermore, we have tried presenting a detailed and consistent description of complete Ω-lattices. Most of the presented results on complete Ω-lattices are new, but some of them are generalizations of known results. Further research will be in applying the obtained results to the study of morphisms between complete Ω-lattices, special Ω-sublattices, Ω-valued congruences on complete Ω-lattices, etc.

There are strong reasons (originating in fuzzy logic) for using a (particular) residuated lattice instead of a general complete lattice, Ω (actually, Ω-sets were defined with Ω being a Heyting algebra). But in this case cut structures would not preserve classical algebraic properties. Thus, quotient structures of cuts over the corresponding congruences would not be classical groups, rings, lattices, etc. Therefore, connections to known structures should be established in a different way than in the case when the language uses only classical lattice-theoretic operations and so new techniques is needed to deal with this situation. Hence, Ω-structures where Ω is a (complete) residuated lattice is another interesting area for further research.

Furthermore, from the following research works: D. Higgs ([44, 45]), D. Ponasse ([75]), J. Goguen ([40, 41]), M. Eytan ([34]), U. Cerruti and U. Höhle ([23]), U. Höhle ([46, 49]), and J. Coulon and J. L. Coulon ([26]), an investigation into the categorical aspect of Ω-algebras: a category consisting of Ω-algebras as objects, arrows are structure preserving maps, i.e. Ω-valued morphisms, with the composition $\circ$ being the usual composition of maps, and identity arrows being the usual identity maps, will be a viable work for future research.

Our research finds its application in cluster analysis, which could be seen from our defining of quotient structures w.r.t. Ω-valued equality. This is a useful tool in data mining, which further can be used in various fields of studies, notably, machine learning, pattern recognition, image analysis, information retrieval, bioinformatics, data compression, and computer graphics, etc.

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Važna napomena VN	
Izvod: IZ	Tema ovog rada je fazifikovanje algebarskih i relacijskih struktura u okviru omega-skupova, gde je Ω kompletna mreža. U radu se bavimo sintezom oblasti univerzalne algebre i teorije rasplinutih (fazi) skupova. Naša istraživanja omega-algebarskih struktura bazirana su na omega-vrednosnoj jednakosti, zadovoljivosti identiteta i tehnici rada sa nivoima. U radu uvodimo omega-algebre, omega-vrednosne kongruencije, odgovarajuće omega-strukture, i omega-vrednosne homomorfizme i istražujemo veze izmedju ovih pojmova. Dokazujemo da postoji Ω-vrednosni homomorfizam iz Ω-algebre na odgovarajuću količničku Ω-algebru. Jezgro Ω-vrednosnog homomorfizma je Ω-vrednosna kongruencija. U vezi sa nivoima struktura, dokazujemo da Ω-vrednosni homomorfizam odredjuje klasične homomorfizme na odgovarajućim količničkim strukturama preko nivoa podalgebri. Osim toga, Ω-vrednosna kongruencija odredjuje sistem zatvaranja klasične kongruencije na nivo podalgebrama. Dalje, identiteti su očuvani u Ω-vrednosnim homomorfnim slikama. U nastavku smo u okviru Ω-skupova uveli Ω-mreže kao uredjene skupove i kao algebre i dokazali ekvivalenciju ovih pojmova. Ω-poset je definisan kao Ω-relacija koja je antisimetrična i tranzitivna u odnosu na odgovarajuću Ω-vrednosnu jednakost. Definisani su pojmovi pseudo-infimuma i pseudo-supremuma i tako smo dobili definiciju Ω-mreže kao uredjene strukture. Takodje je definisana Ω-mreža kao algebra, u ovim kontekstu nosač te strukture je bi-grupoid koji je saglasan sa Ω-vrednosnom jednakošću i ispunjava neke mrežno-teorijske formule. Koristeći aksiom izbora dokazali smo da su dva pristupa ekvivalentna. Dalje smo uveli i pojam potpune Ω-mreže kao uopštenje klasične potpune mreže. Dokazali smo još neke rezultate koji karakterišu Ω-strukture.Data je i veza izmedju Ω-algebre i pojma slabih kongruencija. Na kraju je dat prikaz pravaca daljih istraživanja.

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Abstract: AB	The research work carried out in this thesis is aimed at fuzzifying algebraic and relational structures in the framework of Ω-sets, where Ω is a complete lattice. Therefore we attempt to synthesis universal algebra and fuzzy set theory. Our investigations of Ω-algebraic structures are based on Ω-valued equality, satisfiability of identities and cut techniques. We introduce Ω-algebras, Ω-valued congruences, corresponding quotient Ω-valued-algebras and Ω-valued homomorphisms and we investigate connections among these notions. We prove that there is an Ω-valued homomorphism from an Ω-algebra to the corresponding quotient Ω-algebra. The kernel of an Ω-valued homomorphism is an Ω-valued congruence. When dealing with cut structures, we prove that an Ω-valued homomorphism determines classical homomorphisms among the corresponding quotient structures over cut subalgebras. In addition, an Ω-valued congruence determines a closure system of classical congruences on cut subalgebras. In addition, identities are preserved under Ω-valued homomorphisms. Therefore in the framework of Ω-sets we were able to introduce Ω-lattice both as an ordered and algebraic structures. By this Ω-poset is defined as an Ω-set equipped with Ω-valued order which is antisymmetric with respect to the corresponding Ω-valued equality. Thus defining the notion of pseudo-infimum and pseudo-supremum we obtained the definition of Ω-lattice as an ordered structure. It is also defined that the an Ω-lattice as an algebra is a bi-groupoid equipped with an Ω-valued equality fulfilling some particular lattice-theoretical formulas. Thus using axiom of choice we proved that the two approaches are equivalent. Then we also introduced the notion of complete Ω-lattice based on Ω-lattice. It was defined as a generalization of the classical complete lattice. We proved results that characterizes Ω-structures and many other interesting results. Also the connection between Ω-algebra and the notion of weak congruences is presented. We conclude with what we feel are most interesting areas for future work.

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