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Singular Sylvester equation and its applications

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Singularna Silvesterova jednačina i njene primene

DOKTORSKA DISERTACIJA

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Data on Doctoral Dissertation

Doctoral supervisor:	Nebojša Dinčić, full professor at Faculty of Sciences and mathematics, University of Niš
Title:	Singular Sylvester equation and its applications
Abstract:	This thesis concerns singular Sylvester operator equations, that is, equations of the form $AX-XB=C$, under the premise that they are either unsolvable or have infinitely many solutions. The equation is studied in different cases, first in the matrix case, then in the case when A, B and C are bounded linear operators on Banach spaces, and finally in the case when A and B are closed linear operators defined on Banach or Hilbert spaces. In each of these cases, solvability conditions are derived and then, under those conditions, the initial equation is solved. Exact solutions are obtained in their closed forms, and their classification is conducted. It is shown that all solutions are obtained in the manner illustrated in this thesis. Special attention is dedicated to approximation schemes of the solutions. Obtained results are illustrated on some contemporary problems from operator theory, among which are spectral problems of bounded and unbounded linear operators, Sturm-Liouville inverse problems and some operator equations from quantum mechanics.
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Подаци о докторској дисертацији

Ментор:	Небојша Динчић, редовни професор Природно-математичког факултета Универзитета у Нишу
Наслов:	Сингуларна Силвестерова једначина и њене примене
Резиме:	У овој дисертацији се изучава сингуларна Силвестерова операторска једначина, односно, операторска једначина облика AX-XB=C, под претпоставком да је она или нерешива или да има бесконачно много решења. Једначина се посматра у више разчличитих случаја, најпре у матричном случају, затим у случају када су у питању ограничени линеарни оператори на Банаховим просторима и коначно у случају када су у питању затворени линеарни оператори на Банаховим или Хилбертовим просторима. У сваком од поменутих сценарија се прво изводе довољни услови решивости полазне једначине, а онда се под тим претпоставкама прелази на њено решавање. Долази се до егзактних решења у затвореној форми, те се прелази на њихову класификацију и карактеризацију, односно, показује се да су изведеним посупцима обухваћена сва могућа решења сингуларне Силвестерове једначине. Посебна пажња је посвећена апроксимацијама решења. Добијени резултати су илустровани на неким савременим проблемима из теорије оператора, као што су спектрални проблеми ограничених и неограничених линеарних оператора, инверзни проблеми Штурм-Лиувилове теорије и операторске једначине које се јављају у квантној механици.
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Abstract

The main goal of this doctoral dissertation is to investigate behavior of singular Sylvester equations, i. e. behavior of operator equations

$$AX - XB = C,$$

under the assumption that they are either unsolvable or have infinitely many solutions. Once solvability conditions are derived, characterization and classification of the solutions is conducted and an explicit general formula for those solutions is provided, thus forming the general solution of the given Sylvester equation.

Standard techniques, such as the generalized inverses, are omitted, because the assumption that A and B have closed ranges which are complemented in the corresponding Banach spaces is dropped. Instead, new and original methods are developed for solving this problem, and they are the original scientific contribution of the author, published in papers [24]–[29].

The dissertation is broken down into several chapters. Chapter 1 is the introductory chapter, where regular Sylvester equation (which has a unique solution) is introduced and solved. Some important applications of the equation are mentioned.

Chapter 2 concerns the singular case where A, B and C are matrices. The results are obtained by the shared-eigenvalue discussion for matrices A and B, and by the analysis of the corresponding eigenspaces. Generalized commutators of matrices A and B are characterized, and the solutions are approximated when possible. Perturbation analysis is conducted, using majorization theory for matrices. The main results in this chapter were obtained by the author and his PhD advisor in their joint works [28] and partially [29], and by the author in his individual paper [27].

Chapter 3 concerns the singular case when A, B and C are bounded linear operators on Banach spaces. Since the spectra of A and B do not necessarily consist of eigenvalues only, an alternative approach is required. First, a special operator algebra is introduced, which is not a Banach algebra *per se*, but still allows a functional calculus of its elements. This algebra gives a different form of the general solution to the given Sylvester equation, and solves every basic operator equation

$$AX - XB = C, \qquad AXB = C, \qquad X - AXB = C,$$

in the same manner, discarding regularity of the equations (only their solvability is required). The advantage of this method compared to the generalized inverses techniques (which are commonly used in singular equations) is that it does not require complementedness of the appropriate ranges and null-spaces, but rather solves each equation directly. This algebra is introduced and studied in detail by the author in [25]. Afterwards, this algebra is used to solve the initial Sylvester equation, with help from Fredholm theory. The author obtained these results in [26]. Applications to some contemporary problems in Banach spaces are illustrated as well.

Chapter 4 concerns the singular case when A and B are densely defined closed operators on Banach spaces, and C is a densely defined linear operator. The initial premise is that the point spectra of A and B intersect, and in that case, *weak* solutions X are obtained, which are defined on appropriate eigenspaces of B. Techniques used involve decompositions of the given operators and spaces. Further, the results are extended to Schauder bases when possible, and are applied to Sturm-Liouville operators. These results were achieved by the author and his PhD advisor in the joint work [29]. Afterwards, a special case is analyzed, where A and B are densely defined self-adjoint operators on Hilbert spaces, while the point-spectrum-intersection assumption is dropped. In that case, the weak solutions X obtained in [29] are extended to the largest domains possible, which are constructed by the Spectral mapping theorem for self-adjoint operators and by the Berberian-Buoni-Harte-Wickstead construction. These results were obtained by the author in [24] and they are illustrated on an example which stems from quantum mechanics.

Abstrakt

Glavni cilj ove doktorske disertacije je ispitivanje prirode singularne Silvesterove jednačine, odnosno, operatorske jednačine oblika

$$AX - XB = C,$$

pod pretpostavkom da je ona ili nerešiva, ili da ima beskonačno mnogo rešenja. Najpre bi se obezbedili dovoljni uslovi rešivosti jednačine, a potom bi se sprovela karakterizacija i klasifikacija rešenja. Ta rešenja bi se zatim izvela analitčkim i egzaktnim metodima, u zatvorenom obliku, čime bi formirala opšte rešenje polazne jednačine.

Za razliku od standarndih metoda korišćenih za rešavanje singularnih operatorskih jednačina, poput uopštenih inverza, u ovoj disertaciji se, između ostalog, posmatraju i slučajevi u kojima dati operatori nisu uopšteno invertibilni, odnonso, njihova jezgra i njihove slike ne moraju biti zatvoreni sa topološkim komplementima u odgovarajućim prostorima. Stoga se dati problem analizira na nov i originalan način, što je ujedno i naučni doprinos autora ovoj temi. Originalni rezultati autora, na kojima se i bazira ova disertacija, publikovani su u radovima [24]–[29].

Sama disertacija je podeljena u nekoliko glava. Glava 1 je uvodnog karaktera, u kojoj se uvodi i rešava regularna Silvesterova jednačina (polazna jendačina koja ima jedinstveno rešenje). U ovoj glavi su pomenute neke od najbitnijih primena ove jednačine.

U Glavi 2 se posmatra singularna Silvesterova jednačina, pod pretpostavkom da su A, B i C skalarne matrice. Rezultati prikazani u ovoj glavi su izvedeni diskusijom po zajedničkim sopstvenim vrednostima matrica A i B, kao i pomoću analize sprovedene na odgovarajućim sopstvenim prostorima tih matrica. Okarakterisani su uopšteni komutatori matrica A i B. Sama rešenja su aproksimirana u slučajevima kada je to bilo moguće. Sprovedena je perturbaciona analiza pomoću teorije majorizacija za matrice. Glavni rezultati ovog poglavlja izvedeni su u zajedničkim radovima autora i njegovog mentora [28] i delimično [29], kao i u samostalnom radu autora [27].

U Glavi 3 se proučava singularna Silvesterova jednačina, pod pretpostavkom da su A, B i C ograničeni linearni operatori zadati na Banahovim prostorima. S obzirom da spektri operatora A i B ne moraju da se sastoje isključivo od sopstvenih vrednosti tih operatora, neophodan je drugačiji pristup u odnosu na matrični slučaj. Za početak, uvedena je specijalna algebra operatora,

koja nije Banahova algebra kao takva, ali dopušta funkcionalni račun svojih elemenata. Ova algebra pruža drugačiji oblik opšteg rešenja singularne Silvesterove jednačine, i direktno rešava svaku od elementarnih operatorskih jednačina oblika

$$AX - XB = C, \qquad AXB = C, \qquad X - AXB = C,$$

na isti način, zanemarujući regularnost samih jednačina (zahteva se samo njihova rešivost). Prednost ovog metoda, u odnosu na upštene inverze, je ta da se ne zahteva zatvorenost i komlementarnost odgovarajućih slika i jezgara datih operatora. Ova algebra operatora je uvedena i analizirana od strane autora u samostalnom radu [25]. Nakon toga, ta algebra operatora je iskorišćena za rešavanje singularne Silvesterove jednačine pomoću Fredholmove teorije. Ove rezultate je autor izveo u samostalnom radu [26]. Prikazane su primene dobijenih rezultata na neke savremene probleme koji se javljaju u teoriji operatora.

U Glavi 4 se izučava singularna Silvesterova jednačina pod pretpostavkom da su A i B zatvoreni operatori sa gustim domenima u Banahovim prostorima, dok je C proizvoljan gusto definisan linearan operator. Polazna pretpostavka je ta da se tačkasti spektri operatora A i B seku, i u tom slučaju su izvedena *slaba* rešenja X, koja su definisana na odgovarajućim sopstvenim prostorima operatora B. Ova rešenja su dobjena pomoću raznih dekompozicija operatora i prostora. Dobijena slaba rešenja X su potom proširena na Šauderove baze kada je to bilo moguće, i ilustrovana su na spektralnim problemima iz Šturm-Liuvilove teorije. Ove rezultate je autor izveo u koautorstvu sa svojim mentorom u radu [29]. Nakon toga, specijalan slučaj je analiziran, u kome su A i B samokonjugovani neograničeni operatori, definisani na separabilnim Hilbertovim prostorima, dok je pretpostavka o preseku njihovih tačkastih spektara izbačena. U tom slučaju, slaba rešenja X izvedena u radu [29] su proširena na najveće moguće domene. Ona su proširena pomoću spektralne teorije samokonjugovanih operatora i pomoću konstrukcije uvedene od strane Berberijana, Buonija, Hartea i Vikstida. Autor je ove rezultate izveo u samostalnom radu [24], a potom ih je ilustrovao na primeru iz kvantne mehanike.

A word from the author

Other than being a fifth grader's worst nightmare, mathematical equations are the most recognizable "things" from the math world, a claim stated by an innocent bystander.

,,*Imagine trying to compare two variables A and B, which can simultaneously take any value as they please. Rarely enough, they sometimes do take the same number, and in those moments you can say that A and B are equal. Other times, one is always larger or smaller than the other. That is, of course, when the variables A and B are real-valued entities, which is usually not the case"*, is how I tried to explain the dissertation to my middle school students. Those bright, curious minds, who demanded an explanation to why their algebra teacher was sometimes distracted and frustrated. Luckily, math majors were a bit more sympathetic to their teaching assistant. ,,*Is it difficult?*" some of them would ask me, with the usual existential follow-ups (as one mathematician tends to ask another) ,,*What's the point of such results? Are there any real-world applications?*" At the time being, I spectacularly failed to answer these questions. With a long overdue, I am finally stating my reply:

Beauty and poetry of mathematics hide in its irregularities. Because, fact that a butterfly will flap its wings and simply fly away will not send shivers down your spine; but the idea that the flapping might start a series of events which could lead to a tsunami will surely get you there. The world does not function in a regular manner, but rather experiences chaotic behavior in every possible situation, therefore it cannot be modeled with regular equations. It demands to be studied and analyzed with the equations which behave in an irregular fashion, which are called "singular" equations. I can only hope that one day humanity will benefit from the results obtained in this dissertation, in its attempt to understand the world around us.

An endless "thank you" goes to all my students, current and former, who

managed to keep me on my toes all these years, pulling me out of my comfort zone in every way imaginable.

I was fortunate enough to get such an amazing PhD advisor, professor Nebojša Dinčić, who introduced me to this wonderful area of mathematics. Not only he'd been an extraordinary mentor during the process, but became a close friend as well. He always knew the perfect balance between pushing me forward and letting me figure it out on my own. Thank you for all the patience, knowledge and wisdom you shared with me.

Academician Vladimir Rakočević and professors Dragan Djordjević, Snežana Živković-Zlatanović, Dijana Mosić, Milica Kolundžija and Marko Djikić are just few of the many who taught me everything I know about functional analysis and operator theory. I am forever in their debt. Academician Stevan Pilipović and professor Marko Nedeljkov showed me beauty and necessity of distributions and weak partial differential equations. Academician Miodrag Mateljević helped me bridge the gap between spectral theory and complex analysis. I am grateful to them for helping me understand importance of mathematical analysis at a deeper level. Professor Peter Šemrl carefully read the manuscript and gave me constructive tips which improved quality of the dissertation. For doing so, I am very thankful to him.

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Last, but not the least, I would like to thank my entire Đorđević family, parents Dragan and Olivera, brother Dušan, sister-in-law Katarina, aunt Snežana and grandparents Gospava and Sreten, for their endless love and support during the process. Believe me, the pleasure was all mine.

Contents

Chapter 1

Introduction: Significance of the Sylvester equation

1.1 Notation

Throughout the dissertation, notation used for various mathematical entities is a standard one; by V, W, V_1, V_2, K, H, $\ldots$ we denote vector spaces (often Banach spaces), unless stated differently. Elements of such spaces are vectors, usually denoted by u, v, w and so on. The symbol $\|\cdot\|$ stands for a (given) norm, and $\langle\cdot,\cdot\rangle$ stands for the scalar product. By $\cdot^{\perp}$ we denote the orthogonal complement of the given entity. Two spaces W and V, form a direct sum denoted by $W \dot{+} V$, while $W \oplus V$ stands for the orthogonal sum of W and V. An open disc in the complex plane, with the center $a \in \mathbb{C}$ and the radius $r > 0$ is denoted by $D(a;r)$. Letters A, B, C, X, Y, L, S and so on denote linear operators. I stands for the identical operator, $Iu = u$, for every $u \in V$. $L(V,W)$ denotes the set of all linear operators S, with domains (denoted by $\mathcal{D}_S$) being subsets of V, while their images (denoted by $\mathcal{R}(S)$) are subsets of W. If $V = W$, we simply write $L(V)$. For normed spaces V_1 and V_2, the set of bounded (continuous) linear operators from V_1 to V_2 is denoted by $\mathcal{B}(V_1, V_2)$, where it is understood that the operators are defined on the entire space V_1. If such exists, the inverse of an operator $S \in L(V)$ is denoted by S^{-1} (unless stated differently, we require both S and S^{-1} to be bounded and defined on the entire V). The set of all $\lambda \in \mathbb{C}$ such that $S - \lambda I$ is not an invertible operator in $L(V)$ is denoted by $\sigma(S)$, while $\rho(S) := \mathbb{C} \setminus \sigma(S)$ denotes the resolvent set, i.e. the set of all $\lambda \in \mathbb{C}$ such that $S - \lambda I$ is an invertible operator[1]. The set of all vectors u such that $Su = 0$

[1] If S is a bounded linear operator, then $\sigma(S)$ is a non-empty compact set and $\rho(S)$ is a non-empty unbounded set.

is the null-space of S, denoted by $\mathcal{N}(S)$. A value $\lambda \in \sigma(S)$ is called an eigenvalue for S if there exists $u \neq 0$ such that $Su = \lambda u$, i.e. $u \in \mathcal{N}(S - \lambda I)$; the vector u is then called an eigenvector for S which corresponds to λ. For given Hilbert spaces V and W, for arbitrary $L \in L(V, W)$, the unique closed (if such exists) $L^* \in L(W, V)$ which satisfies

$$\langle Lu, v\rangle = \langle u, L^* v\rangle,$$

for every $u \in \mathcal{D}_L$ and every $v \in \mathcal{D}_{L^*}$, denotes the Hilbert-conjugate (or adjoint) operator of the operator L.

In this chapter we introduce Sylvester equations. We state existing results on this topic, which mostly concern *regular* Sylvester equations (this will be explained shortly). Some of the original proof are modified by the author, in order to make them more mathematically accurate and more applicable for the rest of the dissertation. These alterations are clearly pointed out in the text. Afterwards, singular Sylvester equations are introduced and motivation for their analysis (and motivation for writing this thesis) is explained.

1.2 The regular equation

Let V_1 and V_2 be given Banach spaces. Equations of the form

$$AX - XB = C \tag{1.1}$$

are called Sylvester equations, where, in general, $A \in L(V_2)$, $B \in L(V_1)$ and $C \in L(V_1, V_2)$, are given linear operators. Such equations were firstly introduced by Sylvester himself in 1884, in the matrix case, when he proved a fundamental result, which today serves as a starting point for contemporary results in matrix analysis.

Theorem 1.2.1. [96] Let A, B and C be matrices of appropriate dimensions. The equation (1.1) has a unique solution X if and only if $\sigma(A) \cap \sigma(B) = \emptyset$.

It wasn't until the mid 1900s when Rosenblum generalized the result to bounded linear operators.

Theorem 1.2.2. [89] Let V_1 and V_2 be Banach spaces and let A, B and C be bounded linear operators defined on the appropriate spaces. The equation (1.1) has a unique solution if $\sigma(A) \cap \sigma(B) = \emptyset$.

Remark. The converse statement does not hold for bounded linear operators, that is, there can be a unique solution to (1.1) even though $\sigma(A) \cap \sigma(B) \neq \emptyset$. This is because in matrices, being invertible is equivalent to being injective.

For given Banach spaces V_1 and V_2, and bounded linear operators $A \in \mathcal{B}(V_2)$ and $B \in \mathcal{B}(V_1)$, condition $\sigma(A) \cap \sigma(B) = \emptyset$ implies that for any given $C \in \mathcal{B}(V_1, V_2)$ there always exists a unique bounded solution $X \in \mathcal{B}(V_1, V_2)$ to (1.1), which means that the problem is regular. More precisely, if one defines the Sylvester operator,

$$S(L) := AL - LB, \qquad S \in \mathcal{B}(\mathcal{B}(V_1, V_2)), \quad L \in \mathcal{B}(V_1, V_2),$$

then $\sigma(A) \cap \sigma(B) = \emptyset$ implies S to be invertible in $\mathcal{B}(\mathcal{B}(V_1, V_2))$, with a bounded inverse also belonging to $\mathcal{B}(\mathcal{B}(V_1, V_2))$. In that sense, the sought solution X to (1.1) is always $X = S^{-1}(C)$, for any afore-given $C \in \mathcal{B}(V_1, V_2)$. For this reason, the equation (1.1) is said to be *regular* whenever A, B and C are bounded linear operators on the corresponding Banach spaces and $\sigma(A) \cap \sigma(B) = \emptyset$. Regular equations have been studied extensively so far, with various applications in theoretical and applied mathematics, physics and engineering, see [1], [3], [4], [10], [14], [35], [40], [42], [47], [50], [52], [55], [56], [60], [69], [70], [83], [88], [89], [90], [91], [93] and numerous references therein. In addition, there are several results regarding a unique bounded solution to (1.1), while the operators are unbounded, consult [60], [75] and [83]. These results have a huge impact on mathematical physics and quantum mechanics. However, these results impose certain solvability conditions for operator C (this will be discussed in Chapter 4).

Proofs for Theorems 1.2.1 and 1.2.2 rely on Lemma 1.2.1 below. Notice that proof of the lemma, as well as proof of Theorem 1.2.2, have a direct generalization to unital Banach algebras, see [30], [34], [37], [39], [41], [43], [72], [80], [86], [92], [99], [100], [102].

Lemma 1.2.1. If $\mathbb{A}$ and $\mathbb{B}$ are commuting bounded linear operators on a Banach space V, then

$$\sigma(\mathbb{A} - \mathbb{B}) \subset \sigma(\mathbb{A}) - \sigma(\mathbb{B}).$$

Proof. Proof provided in [10] follows from Gelfand theory of commutative Banach algebras. Imbed $\mathbb{A}$ and $\mathbb{B}$ in a maximal commutative subalgebra of the algebra of operators. Then the spectrum of an operator is equal to its spectrum, relative to a maximal commutative subalgebra. The spectrum of an element of a commutative Banach algebra with identity is the range of its Gelfand transform. This gives

$$\sigma(\mathbb{A} - \mathbb{B}) = \{\varphi(\mathbb{A} - \mathbb{B}) : \varphi \text{ is a nonzero complex homomorphism}\} =$$

$$\{\varphi(\mathbb{A}) - \varphi(\mathbb{B}) : \varphi \text{ is a nonzero complex homomorphism}\} \subset \sigma(\mathbb{A}) - \sigma(\mathbb{B}).$$

□

Proof of Theorem 1.2.1 and Theorem 1.2.2. Define operators $\mathbb{A}(X) := AX$ and $\mathbb{B}(X) := XB$. Then $\mathbb{A}$, $\mathbb{B} \in \mathcal{B}(\mathcal{B}(V_1, V_2))$ and they commute. Trivially, the Sylvester operator S satisfies $S = \mathbb{A} - \mathbb{B}$, therefore, Lemma 1.2.1 applies and $\sigma(S) \subset \sigma(A) - \sigma(B) \not\ni 0$. This proves the statements.
Conversely, if A and B are matrices and for every matrix C there exists a unique solution X to (1.1), then $\sigma(A) \cap \sigma(B) = \emptyset$. This is verified by a direct computation. Assume that, in addition with the previous assumptions, there exists $\lambda \in \sigma(A) \cap \sigma(B)$. Then $\bar{\lambda} \in \sigma(A^*)$ and there exist (non-zero) eigenvectors u and v for B and A^*, respectively, which correspond to λ and $\bar{\lambda}$, respectively. Define $Cu := v$ and let X be a unique solution to the appropriate Sylvester equation. Then

$$\begin{aligned} 0 &= \lambda\langle Xu, v\rangle - \lambda\langle Xu, v\rangle = \langle Xu, \overline{\lambda} v\rangle - \lambda\langle Xu, v\rangle \\ &= \langle Xu, A^* v\rangle - \langle \lambda Xu, v\rangle = \langle AXu, v\rangle - \langle XBu, v\rangle \\ &= \langle (AX - XB)u, v\rangle = \langle Cu, v\rangle = \langle v, v\rangle = \|v\|^2 > 0, \end{aligned}$$

which is impossible. □

That the converse does not hold for bounded linear operators is illustrated in the next example obtained by the author and his PhD mentor:

Example 1.2.1. [29, Example 1.1.] Let $V_1 = V_2$ be infinite dimensional Banach spaces and let $A = C = 0$. Assume that B is onto but is not injective. Then $\sigma(A) \cap \sigma(B) = \{0\}$, while the only solution to the equation $AX - XB = C \Leftrightarrow XB = 0$ is $X = 0$. ♣

The author is here generalizing the statement to noncommutative unital Banach algebras.

Lemma 1.2.2. Let $\mathcal{A}$ be a noncommutative unital Banach algebra that is infinite dimensional. Let a, b and $c \in \mathcal{A}$ such that $a = c = 0_{\mathcal{A}}$ and let b be a left zero divisor, which is not simultaneously a right zero divisor. Then $ax - xb = c$ has only one solution and that is $x = 0_{\mathcal{A}}$.

Proof. Obviously $\sigma(a) = \{0\}$ while $0 \in \sigma(b)$, since b is a left zero divisor. Furthermore,

$$ax - xb = c \Leftrightarrow xb = 0_{\mathcal{A}} \Leftrightarrow x = 0_{\mathcal{A}}.$$

□

Conversely, if there are no solutions, or there exist infinitely many solutions, the equation is said to be *singular*. More generally, by singular equations we can (and we will in Chapter 4) consider equations of the form (1.1), where we discard boundedness of the given operators and the equation itself does not have a unique bounded solution. It is quite trivial to give an example of a Sylvester equation which is unsolvable:

Proposition 1.2.1. Let V be a Banach space and let $A \in \mathcal{B}(V)$. Then equation $AX - XA = I$ does not have a bounded solution.

This statement can directly be generalized to unital Banach algebras as well. Since the proof is identical, we only prove the statement below.

Proposition 1.2.2. Let $\mathcal{A}$ be a unital Banach algebra, with 1 as its unity. Then 1 is not a commutator in $\mathcal{A}$, meaning that, there are no a, $b \in \mathcal{A}$ such that $ab - ba = 1$.

Remark. Note that proof can be found in numerous books on functional analysis and operator theory, to name a few, see [30], [34], [37], [39], [41], [43], [72], [80], [86], [92], [99], [100], [102]. Here, the author states his proof.

Proof. Let a, $b \in \mathcal{A}$ and let $\sigma(a)$, $\sigma(b)$ denote the spectra of a and b, respectively, in $\mathcal{A}$. Then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$. On the other hand, if $ab = 1 + ba$, then $\sigma(ab) = \sigma(1 + ba) = 1 + \sigma(ba) = \{1 + \lambda : \lambda \in \sigma(ba)\}$. Consequently,

$$\sigma(ba) \cup \{0\} = \sigma(ab) \cup \{0\} = \{1 + \lambda : \lambda \in \sigma(ba)\} \cup \{0\}.$$

Denote by $K = \sigma(ba)$, which is a compact subset of $\mathbb{C}$. We are going to prove that it is impossible to have the set equality

$$K \cup \{0\} = 1 + K \cup \{0\}.$$

Since K is a compact set, it has a finite diameter, therefore, there exists an $m \in \mathbb{Z}_0^+$ such that $m > \operatorname{diam}(K)$.

Case 1. Assume that $K \cap \mathbb{Z} = \emptyset$. Then for every $k \in K$ we have $k \in K + 1$, that is, $k - 1 \in K$. Consecutively, it follows that $k \in K$ and $k - m \in K$, which is impossible.

Case 2. Now assume that there exists $k \in K \cap \mathbb{Z}$ such that $k < 0$. Then $k \in K \subset K + 1 \cup \{0\}$, ergo $k \in K + 1$ and $k - 1 \in K$. Similarly to the previous case, it follows that $k - m \in K$ as well as $k \in K$, which is impossible

by the choice for m.
Case 3. Assume that there exists $k \in K \cap \mathbb{Z}$ such that $k > 0$. Then by the same argument, $k + 1 \in K + 1 \subset K \cup \{0\}$, that is, $k + 1 \in K$. Therefore, $k + m \in K$ as well as $k \in K$, which is not possible.
Case 4. What remains is that $K \cap \mathbb{Z} = \{0\}$. But then, $K = \{0\} \cup K_1$, where $K_1 \cap \mathbb{Z} = \emptyset$. Then $\operatorname{diam}(K_1) \leq \operatorname{diam}(K) < m$ and for any $k \in K_1$, Case 1 applies which leads to $k + m \in K_1$, which is not possible. Therefore, $K_1 = \emptyset$ and $K = \{0\}$. But then $\sigma(ab) = \sigma(ba) = \{0\}$ and $\sigma(ab) = 1 + \sigma(ba) = \{1\}$, which is not possible. □

It is important to state that the equation $AX - XA = I$ can be solved (under certain conditions) if the operator A is unbounded and in that case, the solutions X are unbounded as well. This example is the pillar for the theory of closed operators, as such equations stem quite naturally in quantum mechanics, consult [70], [99] and [101]. For this reason, it is very important to study singular Sylvester equations with unbounded operators. This will further be discussed in Chapter 4.

The previous proposition illustrates that, when the equation is singular, solvability of the equation is not automatically achieved, but rather requires a special attention. This will be emphasized in appropriate places of the dissertation. So far, singular Sylvester equations have not been studied that extensively. This dissertation is a collection of original results on that topic, published by the author in individual papers ([24]–[27]) and in joint work with his PhD mentor, professor Nebojša Dinčić ([28] and [29]).

1.2.1 Solution to the regular equation

There are numerous ways to construct the solution X in the regular case, both analytically and numerically. In what follows, we enlist some of the most common methods for obtaining the solution, as these expressions come in handy throughout the dissertation. Unless stated differently, we assume V_1 and V_2 to be Banach spaces and operators A, B, C and X are bounded linear operators, defined on the appropriate Banach spaces. Note that, if $\sigma(A) \cap \sigma(B) = \emptyset$, then A or B is invertible. Similarly to the previous remarks, note that most of the functional calculus can be directly transferred to unital Banach algebras.

Theorem 1.2.3. [10] Suppose that A is an invertible operator, such that there exist $\delta_1 > \delta_2 > 0$, $\sigma(B) \subset D(0, \delta_2)$ and $\sigma(A) \subset \left(\overline{D(0, \delta_1)}\right)^c$. Then the solution X to eq. (1.1) can be provided as

$$X = \sum_{n=0}^{+\infty} A^{-n-1} C B^n. \tag{1.2}$$

Similarly, if B is invertible with $\sigma(B) \subset \left(\overline{D(0,\delta_1)}\right)^c$ and $\sigma(A) \subset D(0,\delta_2)$, then

$$X = \sum_{n=0}^{+\infty} A^n C B^{-n-1}. \tag{1.3}$$

Proof. If A is invertible such that $\sigma(B) \subset D(0,\delta_2)$ and $\sigma(A) \subset \left(\overline{D(0,\delta_1)}\right)^c$, then the spectral radius theorem implies that $\|A^{-1}CB\| \leq \frac{\delta_2}{\delta_1}\|C\|$. Therefore, the sum in (1.2) converges and defines a bounded linear operator. Direct verification shows that

$$A\left(\sum_{n=0}^{+\infty} A^{-n-1} C B^n\right) - \left(\sum_{n=0}^{+\infty} A^{-n-1} C B^n\right) B = C.$$

Analogous procedure holds for (1.3). □

Theorem 1.2.4. [89] Let Γ be a union of closed contours in the complex plane, with total winding number around $\sigma(A)$ equal to 1 and total winding number around $\sigma(B)$ equal to zero. Then the solution to (1.1) can be expressed as

$$X = \frac{1}{2\pi i}\int_\Gamma (A-\xi)^{-1} C (B-\xi)^{-1}\,\mathrm{d}\,\xi. \tag{1.4}$$

Proof. Assume that (1.1) holds. Then for every $\xi \in \mathbb{C}$,

$$(A-\xi)X - X(B-\xi) = C.$$

Take ξ such that both $A-\xi$ and $B-\xi$ are invertible. This gives

$$X(B-\xi)^{-1} - (A-\xi)^{-1}X = (A-\xi)^{-1}C(B-\xi)^{-1}.$$

Integrating over Γ and noting that

$$\int_\Gamma (B-\xi)^{-1}\,\mathrm{d}\,\xi = 0, \qquad \int_\Gamma (A-\xi)^{-1}\,\mathrm{d}\,\xi = 2\pi i$$

finishes the proof. □

In the case when A, B and C are matrices, a polynomial construction is the most fruitful. If $\dim A = m$ and $\dim B = n$, let a and b the characteristic polynomials of A and B respectively:

$$a(s) = \sum_{i=0}^{m} \alpha_i s^i, \tag{1.5}$$

$$b(s) = \sum_{i=0}^{n} \beta_i s^i. \tag{1.6}$$

Similarly to the previously stated, for given $k \in \mathbb{N}_0$, define

$$\eta(k, A, C, B) = \sum_{i=0}^{k} A^{k-i} C B^i. \tag{1.7}$$

The following polynomial equations hold.

Lemma 1.2.3. [50, Lemma 2.1.] With respect to the previous notation, if X is the solution to (1.1), then for every $k \in \mathbb{N}$ the following equality holds

$$A^k X - X B^k = \sum_{i=0}^{k-1} A^{k-i-1} C B^i = \eta(k-1, A, C, B). \tag{1.8}$$

Proof. When $k = 1$, the eq. (1.8) holds by assumption. When $k = 2$, we have

$$A^2 X - X B^2 = A(AX) - (XB)B = A(AX - XB) - (XB - AX)B = AC + CB.$$

The rest is proved analogously, by mathematical induction. $\square$

Combining $\eta(k, A, C, B)$ with the characteristic polynomial b of B, we define

$$\phi(A, C, B) = \sum_{k=1}^{n} \beta_k \eta(k-1, A, C, B). \tag{1.9}$$

Theorem 1.2.5. [50, Theorem 2.2.] If matrices A and B have no common eigenvalues, then (1.1) is equivalent to

$$b(A) X = \phi(A, C, B). \tag{1.10}$$

Spectral mapping theorem implies that $b(A)$ is an invertible matrix, so

$$X = b(A)^{-1}\phi(A, C, B). \tag{1.11}$$

Remark. Formulas (1.2)–(1.11) motivate further inspection of expressions of the form A^nCB^n and A^nXB^n. These expressions have been studied in detail by the author in [25], and Chapter 3 relies heavily on those results.

For further results on analytical and numerical solutions to the regular Sylvester equations, the reader is referred to [7], [14], [39], [42], [52], [68], [69], [71], [73], [74], [75], [88], [89], [90], [91], [93], [94], [95] and references therein.

1.3 Some applications

1.3.1 Diagonalization of operator matrices

Simply knowing when the Sylvester equation is solvable (discarding uniqueness of the solution), gives quite interesting information about the operators A, B and C. One of the most basic consequences is the diagonalization of an operator matrix. Consider the 2×2 bounded operator matrices $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, defined on $V_2 \times V_1$. When are these two matrices similar? Note that every operator of the form $\begin{bmatrix} I_2 & X \\ 0 & I_1 \end{bmatrix}$ is invertible in $\mathcal{B}(V_2 \times V_1)$, and its inverse is $\begin{bmatrix} I_2 & -X \\ 0 & I_1 \end{bmatrix}$. Thus the given matrices are similar if there exists an X satisfying

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Multiplying out the matrices and equating the corresponding entries gives four operator equations, of which only one is not automatically satisfied. That equation is $AX + C = XB$, or $AX - XB = -C$. Ergo, if the later equation is solvable, then the afore given matrices are similar. Simple application of mathematical induction generalizes this statement to n-dimensional upper triangular operator matrices, consult [10]. The diagonalization problem is essential in applied operator theory and matrix analysis, as it drastically simplifies computational procedures, such as computation of the matrix (or operator) sign function, linear model reductions etc. consult [7], [9], [10], [14], [35], [46], [47], [54], [60], [70], [75], [83], [88], [91], [94] [95] and [100].

1.3.2 Lyapunov stability theory

Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a bounded linear operator on H. Observe the abstract differential equation

$$\frac{\mathrm{d}\, Z}{\mathrm{d}\, t} = AZ(t), \quad t \in [0, +\infty), \quad Z : [0, \infty) \to H. \tag{1.12}$$

Translating it to 0 and homogenizing the initial conditions, it follows that either all solutions to (1.12) are stable or unstable in the Lyapunov sense, where the solution $Z(t)$ is stable if and only if

$$\|Z(t)\| \to 0, \quad t \to +\infty.$$

Theorem 1.3.1. [10, Theorem 7.1.] With respect to the previous notation, if the spectrum of A is contained in the open left half plane, then there exists a unique (strictly) positive operator X satisfying $AX + XA^* = -I$.

Proof. It immediately follows that $\sigma(A) \cap \sigma(-A^*) = \emptyset$, therefore operator equation $AX + XA^* = -I$ has a unique solution. Taking the Hilbert conjugate of the operators, it follows that X^* is a solution to the equation as well, indicating that $X = X^*$ so X is self-adjoint. In order to prove that X is positive, it suffices to show that $\sigma(X) \subset \mathbb{R}^+$. Without loss of generality, we can assume that the numerical range of A is contained in the open left half plane as well.

If λ is an eigenvalue of X then there exists a normed u such that $Xu = \lambda u$ and

$$\langle -u, u\rangle = \langle (AX + X^*A)u, u\rangle = \langle AXu, u\rangle + \langle Au, Xu\rangle = 2\lambda\langle Au, u\rangle.$$

Since $\mathrm{Re}\langle Au, u\rangle$ and $\langle -u, u\rangle$ are both negative, λ must be positive. For the same reason, it follows that $\lambda \neq 0$.

$\square$

If the operator X does not have an eigenvalue, the proof provided in [10] is completed by Weyl–von Neumann Theroem (see e. g. [57]):

Theorem (Weyl–von Neumann): *Any self-adjoint operator differs from a pure point spectrum operator by an operator of arbitrarily small Hilbert-Schmidt norm.*

However, this does not imply that all approximate eigenvalues of the operator X are positive. Here the author provides his own proof, which transfers the problem to the elegant construction introduced by Berberian, see [8]. What is already proved, is that X is a self-adjoint operator, therefore $\sigma(X) = \sigma_{app}(X)$. If X has no eigenvalues, then every point λ from the spectrum of X is its approximate eigenvalue. The problem of transferring the approximate point spectrum to the set of eigenvalues was firstly solved by Berberian in [8], which was further applied to bounded Fredholm operators by Wickstead, Buoni and Harte in [12] and [44]. To start, assume that L is a bounded normal operator on a Hilbert space V. Then for fixed μ and $\lambda \in \sigma_{app}(L)$, there exist two normed sequences (x_n) and (y_n), such that $\|(L - \lambda I)x_n\|$ and $\|(L - \mu I)y_n\|$ simultaneously tend to zero as n approaches infinity. Then for every n:

$$\begin{aligned}|(\mu - \lambda)\langle x_n, y_n\rangle| &= |\langle \lambda x_n - Lx_n, y_n\rangle + \langle x_n, L^* y_n - \bar{\mu} y_n\rangle| \\ &\leq \|\lambda x_n - Lx_n\| + \|Ly_n - \nu y_n\|,\end{aligned}$$

which tends to zero as $n \to +\infty$. This implies that approximate eigenvectors corresponding to different approximate eigenvalues tend to behave in an orthogonal manner, similarly to the exact eigenvectors corresponding to the actual different eigenvalues. This motivates the characterization of the approximate point spectrum of all bounded linear operators $L \in \mathcal{B}(V)$, which goes as the following (see [8]). Denote by $\ell_\infty(V)$ the space of all bounded sequences with values in V, equipped with the sup $-$norm. The set of all sequences which converge to zero is denoted by $c_0(V)$. It follows that c_0 is, with respect to the relative topology inherited from $\ell_\infty(V)$, a proper closed subspace, and defines a quotient space $\ell_\infty(V)/c_0(V)$ in a natural way. What is left is to enclose this space, in a manner that $\overline{\ell_\infty(V)/c_0(V)}$ forms a complete inner product space, with inner product defined via the generalized limits (called Banach limits) in $\ell_\infty(V)$ (see [8] for a more detailed construction). For a sequence $(x_n)_n \in \ell_\infty(V)$, a bounded linear operator $L \in \mathcal{B}(V)$ defines a bounded linear map on $\ell_\infty(V)$ as

$$L'((x_n)_n) := (Lx_n)_n \in \ell_\infty(V).$$

Furthermore, it follows that $L'(x_n) \in c_0(V)$, whenever $(x_n) \in c_0(V)$. Hence, $L'_0 : \ell_\infty(V)/c_0(V) \to \ell_\infty(V)/c_0(V)$ defines a bounded linear operator, such that $L'_0\left((x)_n/c_0(V)\right) := \left(L'(x_n)\right)/c_0(V)$, for every $(x_n) \in \ell_\infty(V)$. This implies that $\|L\| = \|L'_0\|$, and that L'_0 extends continuously to the entire space $\overline{\ell_\infty(V)/c_0(V)}$, and that extension is denoted again by L'_0.

Theorem 1.3.2. [8, Theorem 1] For every $L \in \mathcal{B}(V)$, $\sigma_{app}(L) = \sigma_{app}(L'_0) = \sigma_p(L'_0)$.

Now applying this construction to the operator X defined on the Hilbert space H from Theorem 1.3.1, where $AX + XA^* = -I$, it follows that X_0' is a self-adjoint operator as well, but with the pure point spectrum. In that case, we conclude that X_0' has strictly positive eigenvalues (we simply analyze the equation $A_0'X_0' + X_0'{A^*}_0' = -I_0'$ in the same manner we analyzed the initial equation $AX + XA^* = -I$ in the first part of the proof of Theorem 1.3.1), therefore, all approximate eigenvalues of operator X are strictly positive, thus completing the proof.

Theorem 1.3.3. [10, Theorem 7.2.] With respect to the previous notation, if the spectrum of A is contained in the open left half plane, then every solution to the abstract differential equation (1.12) is stable in the Lyapunov sense.

Proof. Let X be the positive solution of the operator equation $A^*X + XA = -I$. Define the real-valued non-negative function $f : [0, +\infty) \to \mathbb{R}$ as $f(t) = \langle XZ(t), Z(t)\rangle$. Then

$$f'(t) = \langle XZ'(t), Z(t)\rangle + \langle XZ(t), Z'(t)\rangle.$$

However, $Z'(t) = AZ(t)$ so

$$f'(t) = \langle XAZ(t), Z(t)\rangle + \langle XZ(t), AZ(t)\rangle = -\|Z(t)\|^2.$$

Choose $d > 0$ such that $X \geq dI$. Then

$$f(t) \geq d\|Z(t)\|^2$$

and

$$\frac{f'(t)}{f(t)} \leq \frac{-\|Z(t)\|^2}{d\|Z(t)\|^2} = -\frac{1}{d}.$$

Therefore

$$\ln f(t) \leq -(t/d) + C,$$

for some constant C, that is,

$$d\|Z(t)\|^2 \leq f(t) \leq \mathrm{e}^{C-t/d}.$$

Taking $t \to +\infty$ finishes the proof. □

1.4 The homogeneous equation: generalized commutators

Specially, when $C = 0$, the equation (1.1) is said to be *homogeneous*. If the homogenous equation is regular, then the only solution is the zero operator. However, in the singular case, we characterize the set of generalized commutators of A and B:

$$\{X : \quad AX = XB\}.$$

It is important to emphasize that if X is a generalized commutator of A and B, then every restriction of X is a generalized commutator for A and B as well. Therefore, we are always interested in characterizing those X which have the largest possible supports, w. r. t. the inclusion. Furthermore, it is quite common to characterize the set of solutions to the inhomogeneous equation (1.1) as $X = X_p + X_h$, where X_p is one particular solution to the inhomogeneous equation (1.1), while X_h is an arbitrary solution to the appropriate homogeneous equation. Therefore, homogeneous equations play an important role when it comes to singular Sylvester equations, and special attention will be dedicated to them in the appropriate sections.

Chapter 2

The matrix case

In this chapter we consider the case where $\dim V_1$, $\dim V_2 < \infty$, that is, the case where A and B are square matrices of appropriate dimensions, which share s eigenvalues and C is a rectangular matrix of appropriate dimensions. Results presented in this chapter were obtained by the author, in joint work with his PhD mentor (papers [28] and partially [29]), and in author's individual paper [27].

Denote by σ the spectral intersection of matrices A and B:

$$\{\lambda_1, \dots, \lambda_s\} =: \sigma = \sigma(A) \cap \sigma(B).$$

For more elegant notation, we introduce $E_B^k = \mathcal{N}(B - \lambda_k I)$ and $E_A^k = \mathcal{N}(A - \lambda_k I)$ whenever $\lambda_k \in \sigma$. Different eigenvalues generate linearly independent eigenvectors, so the spaces E_B^k form a direct sum. Put $E_B := \sum_{k=1}^{s} E_B^k$. It is a closed subspace of V_1 and there exists $E_B^\perp$ such that $V_1 = E_B \oplus E_B^\perp$. With respect to that decomposition, denote $B_E := BP_{E_B}$, $B_1 := BP_{E_B^\perp}$ and $C_1 = CP_{E_B^\perp}$.

2.1 Solvability of the equation

We begin with the following proposition (see any functional analysis textbook).

Proposition 2.1.1. Let V be a Hilbert space and $L \in \mathcal{B}(V)$. If W is $L-$invariant subspace of V, then $W^\perp$ is L^*-invariant subspace of V.

Proof. Let $w \in W$. Then $Lw \in W$. For any $u \in W^\perp$ we have

$$0 = \langle Lw, u \rangle = \langle w, L^* u \rangle,$$

thus $L^*u \in W^\perp$ for any $u \in W^\perp$. $\square$

Theorem 2.1.1. [28, Theorem 2.1.] *(Existence of solutions)* For every $k \in \{1, \dots, s\}$, let λ_k, E_A^k and E_B^k be provided as in the previous paragraph. If

$$\mathcal{N}(C_1)^\perp = \mathcal{R}(B_1) \qquad \text{and} \qquad C\left(E_B^k\right) \subset \mathcal{R}(A - \lambda_k I), \tag{2.1}$$

then there exist infinitely many solutions X to the matrix equation

$$AX - XB = C. \tag{2.2}$$

Proof. For every $1 \le k \le s$, let E_B^k, E_B, $E_B^\perp$, B_E and B_1 be provided as in the previous paragraph. Note that $\mathcal{N}(C_1)^\perp = \mathcal{R}(C_1^*)$, where $C_1^* \in \mathcal{B}(V_2, V_1)$, with $\mathcal{R}(C_1^*) \subset E_B^\perp$.

Step 1: solutions on $E_B^\perp$.

We first analyze $E_B^\perp$. The space E_B is $BP_{E_B^\perp}-$invariant subspace of V_1 and Proposition 2.1.1 yields $E_B^\perp$ to be $(BP_{E_B^\perp})^*-$ invariant subspace of V_1, so without loss of generality we can observe restriction of B_1^* as $B_1^* : E_B^\perp \to E_B^\perp$. Since $\sigma(B_E) = \{\lambda_1, \dots, \lambda_s\}$, it follows that

$$\sigma(B_1^*) \subseteq \{0\} \cup \sigma(B^*) \setminus \{\bar{\lambda}_1, \dots, \bar{\lambda}_s\}.$$

Case 1. Assume that $\sigma(B_1^*) \cap \sigma(A^*) = \emptyset$. Then there exists a unique $X_1^* \in \mathcal{B}(V_2, E_B^\perp)$ such that

$$X_1^* A^* - B_1^* X_1^* = C_1^*,$$

that is, there exists a unique $X_1 \in \mathcal{B}(E_B^\perp, V_2)$ such that

$$AX_1 - X_1 B_1 = C_1$$

holds.

Case 2. Assume that $\sigma(A^*) \cap \sigma(B_1^*) \neq \emptyset$. It follows that $\sigma(A^*) \cap \sigma(B_1^*) = \{0\}$. A^* cannot be nilpotent. Truly, if $\sigma(A^*) = \{0\} = \sigma(A)$, then by assumption, $\sigma(B) \cap \sigma(A) \neq \emptyset$, therefore, $0 \in \sigma(B)$, that is, $0 \in \sigma$. If $u \in \mathcal{N}(B_1)$, it follows that $B_1 u = 0$ and $u \in E_B^\perp$, but then $Bu = B_1 u = 0$, so $u \in \mathcal{N}(B) \subset E_B$, therefore $u \in E_B \cap E_B^\perp = \{0\}$. Hence contradiction, implying that A^* is not nilpotent, but rather has finite ascend, $\operatorname{asc}(A^*) = m \ge 1$, where $\mathcal{N}((A^*)^m)$ is a proper subspace of V_2.

Now observe $B_1^* : E_B^\perp \to E_B^\perp$, which is not invertible by the assumption. Take arbitrary operator $Z_0^* \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{N}(B_1^*))$. Then for every $d \in \mathcal{N}(A^*)$, there exists (by (2.1)) a unique $u \in \mathcal{N}(B_1^*)^\perp$ such that

$$B_1^* u = C_1^* d.$$

Define ${X_1}^*_{Z_0^*}$ on $\mathcal{N}(A^*)$ as ${X_1}^*_{Z_0^*} d := Z_0^* d + u$. Since $\operatorname{asc}(A^*) = m$, the following recursive formula applies.
Assume that $m = 1$. Precisely, decompose $V_2 = \mathcal{N}(A^*) \oplus \mathcal{N}(A^*)^\perp$ and $A^* = 0 \oplus A_1^*$. Then A_1^* is injective from $\mathcal{N}(A^*)^\perp$ to $\mathcal{N}(A^*)^\perp$ and X_1^* can be defined on $\mathcal{N}(A^*)^\perp$ as restriction of X_1^* from Case 1.

Assume that $m > 1$. Then A_1^* is a restriction of A^* to $\mathcal{N}(A^*)^\perp$ and proceed to decompose $\mathcal{N}(A^*)^\perp = \mathcal{N}(A_1^*) \oplus \mathcal{N}(A_1^*)^\perp$ and and define X_1^* on $\mathcal{N}(A_1^*)$ as ${X_1^*}_{N_1^*} u := N_1^* u + d$, where $Z_1^* \in \mathcal{B}(\mathcal{N}(A_1^*), \mathcal{N}(B_1^*))$ is an arbitrary operator and

$$B_1^* u = C_1^* d.$$

If A_1^* is injective on $\mathcal{N}(A_1^*)^\perp$, i.e. if $m = 2$, then X_1 can be defined on $\mathcal{N}(A_1^*)^\perp$ as restriction of X_1 from Case 1. If not, then proceed to decompose $\mathcal{N}(A_1^*)^\perp = \mathcal{N}(A_2^*) \oplus \mathcal{N}(A_2^*)^\perp$ and so on. Eventually, one would get to iteration no. m, in a manner that

$$V_2 = \mathcal{N}(A^*) \oplus \mathcal{N}(A_1^*) \oplus \mathcal{N}(A_2^*) \oplus \ldots \oplus \mathcal{N}(A_m^*) \oplus \mathcal{N}(A_m^*)^\perp,$$

where $A_m^* : \mathcal{N}(A_m^*)^\perp \to \mathcal{N}(A_m^*)^\perp$ is injective. Then $\sigma(B_1^*) \cap \sigma(A_m^*) = \emptyset$, ergo define X_1^* on $\mathcal{N}(A_m^*)^\perp$ as restriction of X_1^* from Case 1 to $\mathcal{N}(A_m^*)^\perp$. Further, for $0 \leq n \leq m$, let $Z_n^* \in \mathcal{B}(\mathcal{N}(A_n^*), \mathcal{N}(B_1^*))$ be arbitrary matrices. Then define X_1^* on $\mathcal{N}(A_n^*)$ as

$${X_1^*}_{Z_n^*} d := Z_n^* d + u,$$

where once again $u \in \mathcal{N}(B_1^*)^\perp$ is the unique element such that $B_1^* u = C_1^* d$. Equivalently, there exists $X_1 \in \mathcal{B}(E_B^\perp, V_2)$ such that

$$AX_1 - X_1 B_1 = C_1, \tag{2.3}$$

where

$$X_1 = X_{1(Z_0, Z_1, \ldots, Z_m)}.$$

The condition $\mathcal{R}(C_1^*) = \mathcal{N}(B_1^*)^\perp = \mathcal{R}(B_1)$ implies X_1 to be well defined on the entire $E_B^\perp$.

Step 2: solutions on E_B.

We now conduct our analysis on E_B. Define $E_A = \sum_{k=1}^{s} E_A^k$ and split V_2 into an orthogonal sum $V_2 = E_A \oplus E_A^\perp$. Decompose $A = A_E \oplus A_1$ with respect to that sum. Then A_1 is injective on $E_A^\perp$ and $A_1 v = Av$, for every $v \in E_A^\perp$. For every $k \in \{1, \ldots, s\}$ let $N_k \in \mathcal{B}(E_B^k, E_A^k)$ be arbitrary. For every $u \in E_B^k$, by the assumption (2.1), there exists a unique $d(u) \in \left(E_A^k\right)^\perp$ such that

$$(A - \lambda_k I)d(u) = Cu.$$

Define

$$X_E^k : u \mapsto N_k u + d(u), \qquad u \in E_B^k.$$

Then $X_E^k : E_B^k \to E_A^k \oplus \left(P_{E_A^k{}^\perp}(A_1 - \lambda_k I)^{-1} C E_B^k\right)$ defines a linear map. What is left is to check whether $X_E := \sum_{k=1}^{s} X_E^k$ is a solution to the equation

$$AX_E - X_E B_E = CP_{E_B}$$

restricted to E_B. However, this is directly verifiable. For any $u \in E_B$ there exist unique $\alpha_1, \ldots, \alpha_s \in \mathbb{C}$ (or $\mathbb{R}$) and unique $u_k \in E_B^k$, $1 \leq k \leq s$, such that $u = \sum \alpha_k u_k$. Then

$$\begin{aligned}
(AX_E - X_E B)u &= A \sum_{k=1}^{s} \alpha_k X_E^k u_k - \sum_{k=1}^{s} \lambda_k \alpha_k X_E^k u_k \\
&= \sum_{k=1}^{s} \left(\alpha_k (A - \lambda_k I)\right) \left(N_k u_k + d(u_k)\right) \\
&= \sum_{k=1}^{s} \alpha_k C u_k = Cu.
\end{aligned}$$

It follows that

$$X = \begin{bmatrix} X_E & 0 \\ 0 & X_1 \end{bmatrix}. \tag{2.4}$$

is a solution to the eq. (1.1). $\square$

Theorem 2.1.1 naturally inquires answers to the following questions:

Question 1. Is every solution to the equation (2.2) of the form (2.4)?

Question 2. Under which conditions is the solution to (2.2) unique?

Both of these questions have affirmative answers, which is justified by the analysis of the following *eigen-problem associated with the given Sylvester equation*:

Assume that $0 \in \sigma = \sigma(A) \cap \sigma(B)$ *and let* $N_\lambda \in \mathcal{B}(E_B^\lambda, E_A^\lambda)$ *be arbitrary, for every* $\lambda \in \sigma$. *Define* $N_\sigma := \sum_{\lambda\in\sigma} N_\lambda$. *Find a solution* X *to the Sylvester equation such that the following eigen-problem is uniquely solved:*

$$\begin{cases} AX - XB = C \\ Xu_\lambda := P_{(E_A^\lambda)^\perp}(A-\lambda I)^{-1}Cu_\lambda + N_\lambda u_\lambda, \qquad u_\lambda \in E_B^\lambda, \qquad \lambda\in\sigma \quad (\sigma \ni 0). \end{cases} \tag{2.5}$$

Theorem 2.1.2. [28, Theorem 2.2.] *(Uniqueness of the solution to the eigen-problem)* With respect to the previous notation, assume that $0 \in \sigma$.

1. If the condition (2.1) holds for every shared eigenvalue $\lambda \in \sigma$, then the solution X depends only on the choice of operator N_σ, that is, for fixed N_σ, there exists a unique solution X such that (2.5) holds.

2. Conversely, for every solution X to (2.2) and for every shared eigenvalue λ for matrices A and B, there exists a unique quotient class $(A-\lambda I)^{-1}C(\mathcal{N}(B-\lambda I)) \oplus \mathcal{N}(A-\lambda I)$ such that X is the unique solution to the quotient eigen-problem (2.5).

Proof. Recall notation from proof of Theorem 2.1.1.

1. The first statement of the theorem is proved directly. Namely, take $V_1 = E_B \oplus E_B^\perp$, $B = B_E \oplus B_1$, $V_2 = E_A \oplus E_A^\perp$, $A = A_E \oplus A_1$ like in Theorem 2.1.1. Then there exists $X = X_E \oplus X_1$, which is a solution to (2.2). By construction, since $\sigma(B_1)\cap\sigma(A) = \emptyset$, Case 1. applies and X_1 is uniquely determined in $\mathcal{B}(E^\perp, V_2)$ while X_E^λ is uniquely determined in the class $\mathcal{B}(E_B/E_B^\lambda, V_2/E_A^\lambda)$ for every $\lambda \in \sigma$. Varying λ in σ completes the proof.

2. Conversely, let X be a solution to the eq. (2.2). Let λ be one of the shared eigenvalues for A and B and fix u as a corresponding eigenvector for B. Then $XBu = \lambda Xu$. Hence

$$AXu - XBu = (A-\lambda I)Xu = Cu.$$

Split Xu into the orthogonal sum $Xu = v_1 + v_2$, where $v_1 \in \mathcal{N}(A-\lambda I)$ and $v_2 \in (\mathcal{N}(A-\lambda I))^{\perp}$. Then v_2 is the sought expression $P_{\mathcal{N}(A-\lambda I)^{\perp}}(A-\lambda I)^{-1}Cu$ and $Xu \in v_2 + \mathcal{N}(A-\lambda I)$. Condition (2.1) follows immediately. Repeating the same procedure for every shared eigenvalue for A and B completes the proof.

□

Corollary 2.1.1. [28, Corollary 2.1.] (*Number of solutions*) let Σ be the set of all N_σ introduced in the eigen-problem associated with given Sylvester equation (2.5), that is

$$\Sigma = \big\{N_\sigma : \quad N_\sigma = \sum_{\lambda\in\sigma} N_\lambda, \quad N_\lambda \in \mathcal{B}(E_B^\lambda, E_A^\lambda), \quad \lambda \in \sigma(A)\cap\sigma(B) = \sigma \ni 0\big\}.$$

Let S be the set of all solutions to (2.2) which satisfy condition (2.1). Then $|\Sigma| = |S|$.

Proof. For arbitrary $N_\sigma \in \Sigma$, there exits a unique $X \in S$ such that (2.5) holds. Further, for arbitrary $X \in S$ and arbitrary $\lambda \in \sigma$ there exist quotient classes E_A^λ and E_B^λ such that (2.5) holds. Define $N_\lambda : E_B^\lambda \to E_A^\lambda$ to be bounded. Then $N_\sigma = \sum\limits_{\lambda\in\sigma} N_\lambda$. It follows that $N_\sigma \in \Sigma$. There is a one-to-one surjective correspondence $S \leftrightarrow \Sigma$. □

Remark. Due to Corollary 4.4.1, for fixed $N_\sigma \in \Sigma$, the solution $X_{N_\sigma} \in S$ can be referred to as a *particular solution.*

Corollary 2.1.2. [28, Corollary 2.2.] (*Size of a particular solution*) With the assumptions and notation from Theorem 2.1.1, Theorem 2.1.2 and Corollary 4.4.1, norm of X_{N_σ} is given as

$$\|X_{N_\sigma}\|^2 = \|X_E\|^2 + \|X_1\|^2 \leq \|N_\sigma\|^2 + \sum_{k=1}^{s} \|P_{(E_A^k)^\perp}(A-\lambda_k I)^{-1}CP_{E_B^k}\|^2 + \|X_1\|^2, \tag{2.6}$$

where equality holds if and only if the sum $\sum\limits_{k=0}^{s} E_B^k$ is orthogonal.

Proof. Taking the same decomposition as in Theorem 2.1.1, let $X_{N_\sigma} = X_E + X_1$. Since X_E annihilates $E_B^\perp$ and X_1 annihilates E_B, it follows that

$$\|X_{N_\sigma}\|^2 = \|X_E + X_1\|^2 = \|X_E\|^2 + \|X_1\|^2.$$

By the same argument, taking

$$\|X_E\|^2 \leq \|N_\sigma\|^2 + \sum_{k=1}^{s} \|P_{(E_A^k)^\perp}(A-\lambda_k I)^{-1}CP_{E_B^k}\|^2,$$

where the equality holds if and only if the sum $\sum_{k=1}^{s} E_B^k$ is orthogonal.

$\square$

Corollary 2.1.3. [28, Corollary 2.3.] (*Singularities on* $E_B^\perp$) Assume that $0 \notin \sigma$ but $0 \in \sigma(A) \cap \sigma(B_1)$ and let $\operatorname{dsc}(A) = m \geq 1$. For every $0 \leq n \leq m$, define $Z_n \in \mathcal{B}(R(B_1)^\perp, \mathcal{R}(A^{n+1})^\perp \cap \mathcal{R}(A^n))$ and let $Z = \sum_{n=0}^{m} Z_n$. If $\mathcal{N}(C_1)^\perp = \mathcal{R}(B_1)$, then there are infinitely many solutions to (2.2) on $E_B^\perp$. Those solutions depend only on choice for Z, that is, if Z is fixed then there exists a unique solution $X_1(Z)$ on $E_B^\perp$.

Proof. Proof is the same as part 1) in Theorem 2.1.2. Note that $\operatorname{dsc}(A) = \operatorname{asc}(A^*) = m$ and $\mathcal{R}(A^{n+1})^\perp \cap \mathcal{R}(A^n) = \mathcal{N}((A^*)^{n+1}) \cap \mathcal{N}((A^*)^n)^\perp$. Then proceed to Case 2. of proof of Theorem 2.1.1. $\square$

2.1.1 Homogeneous equation

Recall that the equation (2.2) is said to be homogeneous when $C = 0$. In that case, X_1 from Theorem 2.1.1 and Theorem 2.1.2 is always the zero matrix, and $X = 0 + X_E$. This brings our attention to the set of all X, such that $AX = XB$. The following corollary speaks of the cardinality of such set.

Corollary 2.1.4. [29, Corollary 2.4.] Let $\lambda_1, \ldots, \lambda_s$ be the s different common non-zero eigenvalues for square matrices A and B. For every $k = \overline{1, s}$, let E_B^k be the eigenspace for B which corresponds to λ_k and let E_A^k be the eigenspace for A which corresponds to λ_k. For every $k = \overline{1, s}$, put

$$q_B^k := \dim E_B^k \quad \text{and} \quad q_A^k := \dim E_A^k.$$

There are at least

$$\prod_{k=1}^{s} (q_A^k)^{q_B^k}$$

different non-zero solutions to the homogeneous equation (2.2), acting from $\sum_{k=1}^{s} E_B^k$ to $\sum_{k=1}^{s} E_A^k$, which are non-zero on every eigenspace $E_B^k, \quad k = \overline{1, s}$.

2.2 Perturbation analysis: majorization theory

It is not difficult to show that if A and B are altered, then their eigenvectors (and consequently, the corresponding eigenspaces) are changed drastically.

This makes perturbation analysis quite difficult, because the solutions X_E, defined on the said eigenspaces, are in that case incomparable. This naturally inquires the question: *how far are the solutions, if A and B are changed?*

If one observes the Sylvester operator, $S(X) = AX - XB$, and a perturbed Sylvester operator $S'(X) = A'X - XB'$, where $\|A - A'\|$, $\|B - B'\| < \delta$, then $\|S - S'\|$ gives an upper bound for the perturbation analysis. For simpler calculations, we can restrict our observation to Hermitian matrices only, as any square matrix can be presented as a combination of two Hermitian matrices. Similarly, instead of the sup $-$norm, $\|\cdot\|$, we observe the Frobenius norm, $\|\cdot\|_2$, of the given (daigonal) Hermitian matrix A,

$$\|A\|_2^2 = \sum_{i=1}^{n} |a_{ii}|^2.$$

For any two real numbers a and b, recall the parallelogram law

$$|a+b|^2 + |a-b|^2 = 2|a+ib|^2.$$

A similar statement holds in the matrix setting: if A and X are square Hermitian matrices of the same dimensions, then (see [11])

$$\|AX + XA\|_2^2 + \|AX - XA\|_2^2 = 2\|AX + iXA\|_2^2.$$

Consequently, it follows that for any square matrix X, and a Hermitian A, we have

$$\|AX + XA\|_2 \leq \sqrt{2}\|AX + iXA\|_2.$$

This implies that $\|AX + XB\|_2 \leq \sqrt{2}\|AX + iXB\|_2$, for any square X and any square Hermitian A and B. Under certain conditions, this estimate can be extended to a much broader class of matrix norms.

A norm $|||\cdot|||$ is said to be unitarily invariant (u. i. for short), if $|||A||| = |||UAV|||$, for every matrix A and every unitary U and V. It is not difficult to see that u. i. norms depend on the singular values of matrix A, see [9] and [11]. Classic examples of u. i. norms are the trace norm, the Frobenius $\|\cdot\|_2-$norm, the Ky-Fan $k-$norm, and the Schatten $p-$norm. In what follows, we state the results obtained in [11] and [27] which concern the u. i. norms and basic majorizations that involve the Sylvester operator.

For a square $n-$dimensional matrix A, the matrix A^* represents the complex Hilbert conjugate matrix of A, and $|A| = (A^*A)^{1/2}$. Notation $p\ .d.$ denotes a positive definite, and *n. d.* denotes a negative definite matrix (operator) for shorthand. Analogously, *p. s. d.* denotes a positive semi definite matrix, and *n. s. d.* denotes a negative semi definite matrix. The value $\lambda_j(A)$ represents an eigenvalue of the matrix A. If A is Hermitian, then we require

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|.$$

Similarly, $s_j(A)$ represents a singular value of the matrix A, i. e. $s_j(A) = \sqrt{\lambda_j(A^*A)}$. Hence, we always assume that singular values are ordered in a non-ascending manner:

$$\text{(if } A \text{ is Hermitian, then) } \|A\| = s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0.$$

Note that for arbitrary square complex matrices A and B, the following chain of implications holds:

$$\begin{aligned}
&(|A| - |B|) \qquad \text{is} \qquad p.\ s.\ d. && (2.7)\\
&\Rightarrow \text{ for every } 1 \leq k \leq n, \qquad s_k(A) \geq s_k(B) && (2.8)\\
&\Rightarrow \text{ for every } 1 \leq k \leq n, \qquad \prod_{j=1}^{k} s_j(A) \geq \prod_{j=1}^{k} s_j(B) && (2.9)\\
&\Rightarrow \text{ for every } 1 \leq k \leq n, \qquad \sum_{j=1}^{k} s_j(A) \geq \sum_{j=1}^{k} s_j(B) && (2.10)
\end{aligned}$$

Relation (2.10) is called the *weak majorization of the singular values of B by the singular values of A*, and it is denoted as $\{s_j(B)\} \prec_w \{s_j(A)\}$. The relation (2.9) is called the *logarithmic weak majorization of the singular values of B by the singular values of A*, and it is denoted as $\{s_j(B)\} \prec_{log(w)} \{A\}$.

Relations (2.10) and (2.9) are important, because they state that, if (2.10) holds, then $|||A||| \geq |||B|||$, for any unitarily invariant norm. This property can be extended to the trace class operators, and the corresponding $s-$numbers, consult [55], [56], [66], [67], [80] and references therein. This connection with compact and trace-class operators will be mentioned in Chapter 3.

Basic estimates regarding relations (2.7)–(2.10) were obtained by Bhatia and Kittaneh in [11].

Theorem 2.2.1. [11, Theorem 1.1] Let A and B be $n \times n$ Hermitian matrices. Then

$$\{s_j(A+B)\}_j \prec_w \sqrt{2}\{s_j(A+iB)\}_j. \tag{2.11}$$

If A is positive semidefinite, then we have the stronger inequality

$$\{s_j(A+B)\}_j \prec_{log(w)} \sqrt{2}\{s_j(A+iB)\}_j \tag{2.12}$$

If both A and B are positive semidefinite, then this can be strengthened further to

$$s_j(A+B) \le \sqrt{2} s_j(A+iB). \tag{2.13}$$

There exist 2×2 Hermitian matrices A and B for which (2.12) is not true. There exist a 2×2 *p. s. d.* matrix A and Hermitian B for which (2.13) is not true. There exist 2×2 *p. s. d.* matrices A and B for which the matrix inequality

$$A + B \le \sqrt{2}|A+iB|$$

is not true.

The proof uses a minimax principle derived in [9]. If A is a Hermitian matrix, then

$$\lambda_j(A) = \max_{\substack{M<\mathbb{C}^n \\ dimM=j}} \min_{\substack{x\in M \\ ||x||=1}} \langle x, Ax\rangle. \tag{2.14}$$

Moreover, if A is an arbitrary linear operator on $\mathbb{C}^n$, then

$$s_j(A) = \max_{\substack{M<\mathbb{C}^n \\ dimM=j}} \min_{\substack{x\in M \\ ||x||=1}} ||Ax||. \tag{2.15}$$

Further, we have:

$$k \in \{1,\dots,n\} \implies \sum_{j=1}^{k} s_j(A) = \max|\sum_{j=1}^{k}\langle y_j, Ax_j\rangle|, \tag{2.16}$$

where the maximum is taken over all k-tuples of orthonormal vectors $x_1, ..., x_k$ and $y_1, ..., y_k$. Finally, we have:

$$k \in \{1,2,\dots,n\} \implies \prod_{j=1}^{k} s_j(A) = \max|\det W^*AW| \tag{2.17}$$

where the maximum is taken over all $n \times k$ matrices W with the property $W^*W = I$.

We now formulate and prove results obtained by the author in [27]. Theorem 2.2.1 is proved using the same technique.

Lemma 2.2.1. [27, Lemma 2.1.] Let a and b be real numbers, and let $\alpha = \mu + i\nu$, $\beta = \lambda + i\eta$ be complex numbers, which satisfy the conditions (2.18) and (2.19) below. The following hold:
(1) If $ab > 0$ and $\lambda\mu + \nu\eta \geq 1$, then $|a+b| \leq |\alpha a + \beta b|$;
(2) If $ab < 0$ and $\lambda\mu + \nu\eta \leq 1$, then $|a+b| \leq |\alpha a + \beta b|$;
(3) If $ab = 0$ then $|a+b| \leq |\alpha a + \beta b|$.

Theorem 2.2.2. [27, Theorem 2.1.] Let A and B be Hermitian matrices. Let $\mu, \nu, \lambda, \eta \in \mathbb{R}$,

$$\alpha := \mu + i\nu, \beta := \lambda + i\eta \tag{2.18}$$

be provided in a way that

$$|\alpha|, |\beta| \geq 1, \tag{2.19}$$
$$\mu\lambda + \nu\eta \geq 1. \tag{2.20}$$

Then:
(1) If B and $A+B$ are *p. d.* and all singular values of $A+B$ are greater than $||B||$, then $\{s_j(A+B)\} \prec_w \{s_j(\alpha A + \beta B)\}$;
(2) If A and B are *n. d.*, then $\{s_j(A+B)\} \prec_{log(w)} \{s_j(\alpha A + \beta B)\}$;
(3) If A and B are *p. d.*, then $s_j(A+B) \leq s_j(\alpha A + \beta B)$ for all $j \in \{1, \ldots, n\}$.

Proof. (1) Under the assumptions, there exist orthonormal eigenvectors $e_1, ..., e_n$ of $A+B$, arranged in such a way that the following holds:

$$1 \leq j \leq n: \quad s_j(A+B) = |\langle e_j, (A+B)e_j\rangle| = |\langle e_j, Ae_j\rangle + \langle e_j, Be_j\rangle| \tag{2.21}$$

Note that $A+B$ and B are positive definite matrices, so it is safe to say that $\langle e_j, Ae_j\rangle$ and $\langle e_j, Be_j\rangle$ are real numbers. Denote $a := \langle e_j, Ae_j\rangle$ and $b := \langle e_j, Be_j\rangle$ for given j. Then

$$\begin{aligned} ab &= \langle e_j, Ae_j\rangle\langle e_j, Be_j\rangle \\ &= (\langle e_j, (A+B)e_j\rangle - \langle e_j, Be_j\rangle)\langle e_j, Be_j\rangle \\ &= (s_j(A+B) - \langle e_j, Be_j\rangle))\langle e_j, Be_j\rangle. \end{aligned}$$

Since, $\langle e_j, Be_j\rangle > 0$, applying the Cauchy-Schwarz inequality, we have:

$$ab \geq (\langle e_j, Be_j\rangle)(s_j(A+B) - ||Be_j|| \cdot ||e_j||)$$

Since $||e_j|| = 1$ and $||Be_j|| \leq ||B||||e_j|| = ||B||$, we get:

$$ab \geq (\langle e_j, Be_j\rangle)(s_j(A+B) - ||B||) > 0. \tag{2.22}$$

Since $ab > 0$, we can now apply Lemma 2.1 in (2.21):

$$|\langle e_j, Ae_j\rangle + \langle e_j, Be_j\rangle| \leq |\alpha\langle e_j, Ae_j\rangle + \beta\langle e_j, Be_j\rangle| = |\langle e_j, (\alpha A + \beta B)e_j\rangle|.$$

Combining the last inequality with (2.16), it follows that

$$\sum_{j=1}^{k} s_j(A+B) \leq \sum_{j=1}^{k} |\langle e_j, (\alpha A + \beta B)e_j\rangle| \leq \sum_{j=1}^{k} s_j(\alpha A + \beta B)$$

(2) Let A and B be Hermitian *n.d.* matrices. Let $\{\lambda_1, ..., \lambda_n\}$ denote the spectrum of $(-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}}$. Now we have:

$$\begin{aligned}
|\det(A+B)| &= |\det[(-B)^{\frac{1}{2}}((-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}} - I)(-B)^{\frac{1}{2}}]| \\
&= |\det(-B)\det((-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}} - I)| \\
&= |\det(-B)|\prod_{j=1}^{n}|\lambda_j - 1|.
\end{aligned}$$

Note that λ_j are negative real numbers, for all j, due to A and B being *n. d.* Therefore, we can apply Lemma 2.1. Let $a := \lambda_j$ and $b := -1$. Since the condition $\lambda\mu + \nu\eta \geq 1$ is satisfied, due to the assumption of the theorem, the following inequality holds:

$$\begin{aligned}
|\det(-B)|\prod_{j=1}^{n}|\lambda_j - 1| &\leq |\det(-B)|\prod_{j=1}^{n}|-\beta + \alpha\lambda_j| \\
&= |\det(-B)||\det(-\beta I + \alpha(-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}})| \\
&= |\det(\alpha A + \beta B)|.
\end{aligned}$$

From (2.17), we know that there exists $W \in \mathbb{C}^{n\times k}$, $W^*W = I$, such that

$$\begin{aligned}
\prod_{j=1}^{k} s_j(A+B) &= |\det(W^*(A+B)W)| \leq |\det(W^*(\alpha A + \beta B)W)| \\
&\leq \prod_{j=1}^{k} s_j(W^*(\alpha A + \beta B)W),
\end{aligned}$$

$$\prod_{j=1}^{k} s_j(W^*(\alpha A+\beta B)W) \leq \prod_{j=1}^{k} s_j(\alpha A+\beta B) \Rightarrow$$
$$\prod_{j=1}^{k} s_j(A+B) \leq \prod_{j=1}^{k} s_j(\alpha A+\beta B).$$

(3) Let A and B be Hermitian *p.d.* It follows that the singular values of $A+B$ are also its eigenvalues, because $A+B$ is also Hermitian *p. d.* Let $j \in \{1, ..., n\}$ be fixed, $e_1, ..., e_j$ be the eigenvectors of $A+B$ that correspond to the eigenvalues (singular values) $\lambda_1, ..., \lambda_j$ of $A+B$, in that order, and let M be the span over $\{e_1, \dots, e_j\}$. Now we have:

$$s_j(A+B) = \min_{\substack{x\in M\\ ||x||=1}} \langle x, (A+B)x\rangle.$$

Note that $\langle x, (A+B)x\rangle = \langle x, Ax\rangle + \langle x, Bx\rangle$, for an arbitrary normed $x \in M$, for an arbitrary $j-$dimensional subspace M of $\mathbb{C}^n$. Since A and B are *p. d.*, it follows that $\langle x, Ax\rangle > 0$ and $\langle x, Bx\rangle > 0$. Thus we have:

$$\begin{aligned}
\langle x, Ax\rangle + \langle x, Bx\rangle &= |\langle x, Ax\rangle + \langle x, Bx\rangle| \leq |\alpha\langle x, Ax\rangle + \beta\langle x, Bx\rangle| \\
&= |\langle x, (\alpha A+\beta B)x\rangle|.
\end{aligned}$$

Using the condition of the theorem: $\mu\lambda + \nu\eta \geq 1$, we can apply Lemma 2.1. Since the majorization holds in all $M < \mathbb{C}^n, x \in M, ||x|| = 1$, we have:

$$\begin{aligned}
s_j(A+B) &\leq |\langle x, (\alpha A+\beta B)x\rangle| \leq ||(\alpha A+\beta B)x|| \cdot ||x|| \\
&= ||(\alpha A+\beta B)x|| \\
&\Rightarrow \\
s_j(A+B) &\leq \min_{\substack{x\in M\\ ||x||=1}} ||(\alpha A+\beta B)x|| \leq \max_{\substack{M\langle\mathbb{C}^n\\ dim M=j}} \min_{\substack{x\in M\\ ||x||=1}} ||(\alpha A+\beta B)x|| \\
&= s_j(\alpha A+\beta B).
\end{aligned}$$

□

Corollary 2.2.1. [27, Corollary 2.2.] Let A and B be Hermitian matrices, and $\mu, \nu, \lambda, \eta \in \mathbb{R}$, $\alpha := \mu + i\nu, \beta := \lambda + i\eta$ denoted in a way that $|\alpha|, |\beta| \geq 1$, and $\lambda\mu + \nu\eta \leq 1$. Then:

(1) If B is negative semi-definite, $A+B$ is positive semi-definite, and none of the singular values of $A+B$ are smaller than $||B||$, then $\{s_j(A+B)\} \prec_w \{s_j(\alpha A+\beta B)\}$;
(2) If A is positive semi-definite and B is $n.\ d.$, then $\{s_j(A+B)\} \prec_{log(w)} \{s_j(\alpha A+\beta B)\}$;
(3) If A is positive semi-definite, B is negative semi-definite and $A+B$ is positive semi-definite, then $s_j\{A+B\} \leq s_j\{\alpha A+\beta B\}$.

2.3 Approximation schemes

As seen from the proof of Theorem 2.1.1, every solution to (2.2) has the form

$$X = \begin{bmatrix} X_E & 0 \\ 0 & X_1 \end{bmatrix},$$

where X_1 solves the ,,regular problem" (2.3), while X_E solves the eigenproblem

$$X_E u = N_k u + P_{(E_A^k)^\perp}(A-\lambda_k I)^{-1}Cu, \tag{2.23}$$

for every $u \in E_B^k$, for every shared eigenvalue λ_k and every given $N_k \in \mathcal{B}(E_B^k, E_A^k)$. It is known that, even with given eigenvalues, numerical procedures for computing the corresponding eigenvectors are highly unstable. Therefore, numerical methods for solving the singular equation (2.2) are numerically unstable in general. However, if we restrict our attention to solving only (2.5), that is, if we assume that $\lambda_1, \ldots, \lambda_s$ are provided, and the corresponding $N_k \in \mathcal{B}(E_B^k, E_A^k)$, are provided as well, for $1 \leq k \leq s$, then solving the (2.5) reduces to two numerically solvable problems: one is solving (2.3), which has been done in [7], [14], [23], [28], [50], [52], [69], [71], [88], [94], [95] and references therein. The other problem is solving (2.23), which is merely the standard problem

$$Lx = y,$$

with x and y given and L the unknown, which has been solved in [6], [18], [4], [54], [73], [74] and rich references therein.

Chapter 3

The bounded operator case

Unlike the matrix case, when it comes to bounded linear operators, the corresponding spectra of A and B can contain values which are not eigenvalues (see e. g. [17], [37], [43], [59], [86], [99] and [102]). Therefore, the standard eigenanalysis approach fails, but an alternative way solves the problem. Main results in this chapter were obtained by the author in his individual papers [25] and [26].

The main goal is to solve the Sylvester equation given in its vector form

$$AX - XB = S(X) = C, \tag{3.1}$$

where S is the Sylvester operator while X and C are treated as vectors from the space $\mathcal{B}(V_1, V_2)$. Problems of the form

$$Lx = y$$

require access to the fact whether $y \in \mathcal{R}(L)$. Then and only then, the equation $Lx = y$ is solvable. Recall that, if $0 \notin \sigma(S)$, then S is invertible on the entire space $\mathcal{B}(V_1, V_2)$, and for every C there exists a unique X such that (3.1) holds. However, if $0 \in \sigma(S)$, then the operator S is singular (which is the main premise of this dissertation), but this still does not answer the question whether $C \in \mathcal{R}(S)$ (recall condition 2.1 from Theorem 2.1.1). Therefore, sufficient conditions for solvability of the equation (3.1) are required.

In this chapter, D represents the open unit disc in the complex plane and $\overline{D}$ represents its closure. $H(D)$ denotes the set of all holomorphic functions on D, continuous on its closure. $P[\mathbb{C}]$ denotes the set of all polynomials with complex coefficients. Finally, let $AbCon$ be the set of all $f \in H(D)$ such that the power-series for f is absolutely convergent on D.

3.1 The algebra $\mathcal{A}_{AXB}$

In what follows, we assume V_1 and V_2 to be Banach spaces, $A \in \mathcal{B}(V_2)$, $B \in \mathcal{B}(V_1)$ and $X \in \mathcal{B}(V_1, V_2)$ such that $X \neq 0$. Define $n-$th power of AXB in $\mathcal{B}(V_1, V_2)$ by

$$(AXB)^n := A^n XB^n, \quad n \in \mathbb{N}_0.$$

Put

$$\mathcal{A}_{AXB} := \overline{\{p(AXB) : p \in P[\mathbb{C}]\}}. \tag{3.2}$$

These expressions raise particular interest, as they appear in numerous papers, among which are [15], [16], [22], [34], [58], [72], [76], [78], [80], [87].

3.1.1 Invertibility in $\mathcal{A}_{AXB}$

Let $\varepsilon > 0$ and $M := \max\{\|A\|, \|B\|\} + \varepsilon$. Then $\|A\|$, $\|B\| < M$ and $\|\frac{1}{M}A\|$, $\|\frac{1}{M}B\| < 1$. Put $A_1 := \frac{1}{M}A$ and $B_1 = \frac{1}{M}B$. It is not difficult to see that $\|A_1\|$, $\|B_1\| < 1$ and

$$\mathcal{A}_{AXB} = \mathcal{A}_{A_1XB_1}.$$

Theorem 3.1.1. [25, Theorem 2.1.] Assume $\|A\|$ and $\|B\|$ to be smaller than one. Let n, $m \in \mathbb{N}_0$ such that $0 \leq n < m$ and let $\mathcal{A}_{AXB}$ be provided as in (3.2). Then

1. The ordered triple $(\mathcal{A}_{AXB}, \|\cdot\|, +)$ is a separable Banach subspace of $\mathcal{B}(V_1, V_2)$. The ordered triple $(\mathcal{A}_{AXB}, +, \cdot)$ is a commutative algebra with the unity X. The ordered quadruple $(\mathcal{A}_{AXB}, \|\cdot\|, +, \cdot)$ is not necessarily a normed algebra.

2. The inequality $\|(AXB)^m\| \leq \|(AXB)^n\|$ holds, where the equality is obtained iff $(AXB)^k = 0$, for some $k \in \{0, \ldots, n\}$.

3. The series
$$\sum_{j=0}^{+\infty} (AXB)^{m \cdot j} \tag{3.3}$$
converges in $\mathcal{A}_{AXB}$. The operator $X - (AXB)^m$ is invertible in $\mathcal{A}_{AXB}$ and its inverse is given as (3.3).

Proof. Let $\mathcal{A}_{AXB}$, A, B, m and n be provided as stated in the theorem.

1. In order to prove that $\mathcal{A}_{AXB}$ is indeed a separable Banach subspace of $\mathcal{B}(V_1, V_2)$, it suffices to prove that $\mathcal{A}_{AXB}$ is a closed and separable

subspace of the Banach space $\mathcal{B}(V_1, V_2)$. The closedness follows directly from (3.2). Since the powers

$$x \mapsto 1, \quad x \mapsto x, \quad x \mapsto x^2, \quad \dots$$

form a Schauder basis for the space of polynomials $P[\mathbb{C}]$, it follows that

$$X, \quad AXB, \quad (AXB)^2, \quad \dots$$

form a Schauder basis for $\mathcal{A}_{AXB}$ (or even a Hamel basis, in the case that AXB is nilpotent). Either way, this proves that $\mathcal{A}_{AXB}$ must be separable.

The associative law $A^p \cdot A^q = A^q \cdot A^p$, for every p, $q \in \mathbb{N}_0$, and $B^p \cdot B^q = B^q \cdot B^p$ implies that

$$(AXB)^p \cdot (AXB)^q = (AXB)^q \cdot (AXB)^p,$$

so the multiplication in (3.2) is commutative. Trivially,

$$X \cdot (AXB)^n = (A^0XB^0) \cdot A^nXB^n = A^{0+n}XB^{0+n} = A^nXB^n,$$

for every $n \in \mathbb{N}_0$, so X is indeed the unity in $\mathcal{A}_{AXB}$. Fact that $\mathcal{A}_{AXB}$ is not necessarily a normed algebra is illustrated in the next example. Assume that $B = \frac{1}{\sqrt{2}} I_{V_1}$, $A = \frac{1}{\sqrt{3}} I_{V_2}$ and $\|X\| < 1$. Then $\|AXB\| = \frac{1}{\sqrt{6}}\|X\| < 1$. Now observe

$$\|(AXB)^2\| = \|AAXBB\| = \frac{1}{6}\|X\|$$

and

$$\|AXB\|^2 = \|AXB\| \cdot \|AXB\| = \frac{1}{6}\|X\|^2.$$

Since $\|X\| < 1$, it follows that

$$\|(AXB)^2\| > \|AXB\|^2,$$

therefore $\mathcal{A}_{AXB}$ in this particular case is not a normed algebra.

2. With respect to the assumption $\|A\|$, $\|B\| < 1$, we have

$$\begin{aligned}\|(AXB)^m\| &= \|A^mXB^m\| = \|A^{m-n}A^nXB^nB^{n-m}\| \\ &\leq \|A^{m-n}\| \cdot \|(AXB)^n\| \cdot \|B^{m-n}\| < \|(AXB)^n\|.\end{aligned}$$

The equality is obtained iff $(AXB)^k = 0$, for some $k \in \{0, \dots, n\}$. Another way to prove the later is with help from Banach fixed point theorem. Observe the operator $T(X) := (AXB)^{m-n}$. From $\|T\| \leq \|A\| \cdot \|B\| < 1$ it follows that T is a contraction. Hence there is only one fixed point for T and that is 0.

3. If $(AXB)^m = 0$ then $X = X - 0$ is invertible in $\mathcal{A}_{AXB}$ and its inverse is $X = (AXB)^0$. Now assume that $(AXB)^m \neq 0$. Note that $X^k - (AXB)^{m\cdot k} = X - (AXB)^{m\cdot k}$, for arbitrary $k \in \mathbb{N}$. Having proved that $\|(AXB)^m\| < \|X\|$, it follows that

$$X - (AXB)^{m\cdot k} = (X - (AXB)^m) \cdot \sum_{j=0}^{k-1} (AXB)^{m\cdot j}. \tag{3.4}$$

When $k \to +\infty$, we get

$$\|(AXB)^{m\cdot k}\| \leq \|A\|^{m\cdot k} \cdot \|X\| \cdot \|B\|^{m\cdot k} \to 0, \quad k \to +\infty$$

and consequently

$$X - (AXB)^{m\cdot k} \to X, \quad k \to +\infty.$$

On the other hand, the numerical series $\sum\limits_{j=0}^{+\infty} \|(AXB)^{m\cdot j}\|$ converges, due to the comparison criterion

$$\|(AXB)^{m\cdot j}\| \leq \|X\| \cdot (\|A\| \cdot \|B\|)^{m\cdot j},$$

where the sequence $(\|A\| \cdot \|B\|)^{m\cdot j}, \quad j \in \mathbb{N}_0$ forms a geometric progression. Therefore (3.3) converges absolutely and $\mathcal{A}_{AXB}$ is a Banach space, thus

$$\sum_{j=0}^{+\infty} (AXB)^{m\cdot j}$$

converges in $\mathcal{A}_{AXB}$. Further, $\mathcal{A}_{AXB}$ is a commutative algebra, so

$$\begin{aligned} X &= (X - (AXB)^m) \cdot \sum_{j=0}^{+\infty} (AXB)^{m\cdot j} \\ &= \left(\sum_{j=0}^{+\infty} (AXB)^{m\cdot j} \right) \cdot (X - (AXB)^m), \end{aligned}$$

which yields $X - (AXB)^m$ to be invertible in $\mathcal{A}_{AXB}$, having $\sum\limits_{j=0}^{+\infty} (AXB)^{m\cdot j}$ as its inverse.

□

Corollary 3.1.1. [25, Corollary 2.1.] If $V_1 = V_2 = V$, let A, B and $X \in \mathcal{B}(V)$, such that they all commute and $X = X^2$. Then $\mathcal{A}_{AXB}$ is a Banach algebra.

Proof. If A, B and X commute, where $X = X^2$, a direct verification shows that the multiplication defined in (3.2) coincides with the standard multiplication defined on $\mathcal{B}(V)$ (i.e. composition of the operators at hand). $B(V)$ is a Banach algebra and $\mathcal{A}_{AXB}$ is a closed subspace of $\mathcal{B}(V)$ with the same multiplication, therefore $\mathcal{A}_{AXB}$ is a Banach algebra as well. $\square$

Remark. In order to reduce the algebra $\mathcal{A}_{AXB}$ to a Banach algebra of, say, bounded linear operators over a Banach space V, existence of idempotents is a necessary condition. However, bounded projectors always exist in every maximal commutative unital subalgebra of $\mathcal{B}(V)$. This proves that our algebra $\mathcal{A}_{AXB}$ generalizes the standard notion of a Banach algebra contained in $\mathcal{B}(V)$.

The previous theorem suggests further investigation of invertible elements in $\mathcal{A}_{AXB}$. For future reference, the said set will be labeled as $\mathcal{A}^{-1}_{AXB}$. Note that invertibility in $\mathcal{A}_{AXB}$ is not correlated with actual (left or right) invertibility of operators.

Suppose $A \in \mathcal{B}(V_2)$, $B \in \mathcal{B}(V_1)$ such that $\|A\|, \|B\| < 1$. Let $\eta > 0$ and let $g : [-\eta, 1] \to \mathbb{R}$ be a real non-negative function, non-decreasing on $[0, 1]$, from the class $\mathcal{C}^\infty(-\eta, 1)$. Define the set $\mathcal{F}(g)$ as

$$\mathcal{F}(g) := \{f \in AbCon \;\; : \;\; f(z) = \sum_{k=0}^{+\infty} c_k z^k, \;\; g(|z|) = \sum_{k=0}^{+\infty} |c_k||z|^k, \;\; |z| < 1\}.$$

The value $\|g\|$ will represent the sup-norm of g on $[0, 1]$, that is

$$\|g\| = \sup\{g(|z|) : \;\; |z| \leq 1\} = \sup\{\sum_{k=0}^{+\infty} |c_k||z|^k : |z| < 1\} = \sum_{k=0}^{+\infty} |c_k| = g(1).$$

Remark. From the maximum modulus principle one can see that the aforementioned function g must be non-descending on $[0, 1]$. Therefore $\|g\| = g(1)$.

Remark. The value η plays no role in the further text. Its main purpose is to ensure differentiability of g at point zero.

Theorem 3.1.2. [25, Theorem 2.2.] Let operators A and B, function g and the set $\mathcal{F}(g)$ be provided as described in the previous paragraph. Let $X \in \mathcal{B}(V_1, V_2)$, $f \in \mathcal{F}(g)$ and $\lambda \in \mathbb{C}$ such that $AXB \neq 0$ and $\|X\| \cdot \|g\| < |\lambda|$. Then

1. Operators
$$\left(\lambda + g(0)\,\mathrm{e}^{i\varphi}\right) X - f(AXB)$$
are invertible in $\mathcal{A}_{AXB}$, for every $\varphi \in [0, 2\pi)$.

2. If $\lambda + g(0)\,\mathrm{e}^{i\varphi} \neq 0$ for every $\varphi \in [0, 2\pi)$ then
$$\left(\left(\lambda + g(0)\,\mathrm{e}^{i\varphi}\right) X - f(AXB)\right)^{-1} = \sum_{k=0}^{+\infty} \left(\lambda + g(0)\,\mathrm{e}^{i\varphi}\right)^{-k+1} \left(f(AXB)\right)^{k}. \tag{3.5}$$

3. If $\lambda + g(0)\,\mathrm{e}^{i\varphi_0} = 0$ for some $\varphi_0 \in [0, 2\pi)$ then $f(AXB)$ is invertible in $\mathcal{A}_{AXB}$. If
$$\|f(AXB) - g(0)\,\mathrm{e}^{i\varphi_0} \cdot X\| < |\lambda|$$
then its inverse is
$$\left(f(AXB)\right)^{-1} = (-\lambda)^{-1} \sum_{k=0}^{+\infty} \left(\frac{f(AXB) - g(0)\,\mathrm{e}^{i\varphi_0} \cdot X}{\lambda}\right)^{k}. \tag{3.6}$$

Proof. Let all the assumptions from the theorem hold. Put $Y := f(AXB)$. We are going to prove all three statements by conducting the following cases:

Case 1. Assume that $g(0) = 0$ and $|\lambda| = 1$. Put
$$f(z) = \sum_{k=0}^{+\infty} c_k z^k, \quad |z| < 1. \tag{3.7}$$

Observe
$$\begin{aligned}\|f(AXB)^n\| = \left\|\sum_{k=0}^{+\infty} c_k (AXB)^{k \cdot n}\right\| &\leq \sum_{k=0}^{+\infty} |c_k| \cdot \|X\| \cdot (\|A\| \cdot \|B\|)^{k \cdot n} \\ &= \|X\| \sum_{k=0}^{+\infty} |c_k| \left((\|A\| \cdot \|B\|)^n\right)^k = \|X\| \cdot g\left((\|A\| \cdot \|B\|)^n\right).\end{aligned}$$

Now $(\|A\| \cdot \|B\|)^n \to 0$, so $Y^n = f(AXB)^n \to 0$ when $n \to +\infty$. This yields that
$$X - Y^n \to X, \quad n \to +\infty.$$

Further,

$$\begin{aligned}
\left\| \sum_{k=0}^{+\infty} f(AXB)^k \right\| &= \left\| \sum_{k=0}^{+\infty} \left(\sum_{n=0}^{+\infty} c_n (AXB)^n \right)^k \right\| \\
&\leq \sum_{k=0}^{+\infty} \left\| \left(\sum_{n=0}^{+\infty} c_n (AXB)^n \right)^k \right\| \\
&\leq \sum_{k=0}^{+\infty} \left[\|X\|^k \cdot \left(\sum_{n=0}^{+\infty} |c_n| \cdot (\|A\| \cdot \|B\|)^n \right)^k \right] \\
&\leq \sum_{k=0}^{+\infty} \|X\|^k \cdot g(\|A\| \cdot \|B\|)^k,
\end{aligned} \tag{3.8}$$

which is a convergent sum, as a geometric progression

$$(\|X\| \cdot \|g\|)^k < 1, \quad k \in \mathbb{N}_0.$$

Therefore, the decomposition

$$X - Y^n = (X - Y) \sum_{k=0}^{n-1} Y^k$$

holds when $n \to +\infty$ and

$$(X - Y)^{-1} = \sum_{k=0}^{+\infty} Y^k.$$

Case 2. Assume that $g(0) = 0$ and $|\lambda| \neq 1$. Put $g_1(x) := \frac{g(x)}{|\lambda|}$. Then $g_1(0) = 0$ and for every $f_1 \in \mathcal{F}(g_1)$ it follows that

$$\|f_1(AXB)\| \leq \frac{\|g(AXB)\|}{|\lambda|} \leq \frac{\|g\| \cdot \|X\|}{|\lambda|} < 1.$$

Now apply Case 1. of this theorem on $f_1(AXB) = \frac{1}{\lambda} f(AXB)$. It follows that

$$\lambda X - f(AXB)$$

is invertible in $\mathcal{A}_{AXB}$ and its inverse is given as

$$(\lambda X - f(AXB))^{-1} = \sum_{k=0}^{+\infty} \lambda^{-k+1} \left(f(AXB) \right)^k.$$

Case 3. Assume that $g(0) = \omega \neq 0$ and let λ be arbitrary. Define

$$g_1(x) := g(x) - \omega, \quad x \in [0,1].$$

Then $g_1(0) = 0$ and since $w = |c_0| > 0$ we have $\|g_1\| = \|g\| - \omega < \|g\|$, so

$$\|g_1\| \cdot \|X\| < |\lambda|,$$

thus Case 2. of this theorem applies. In other words, $X - \frac{1}{\lambda}(f_1(AXB))$ is invertible in $\mathcal{A}_{AXB}$, for every $f_1 \in F(g_1)$, that is

$$X - \frac{f(AXB) - \omega\,\mathrm{e}^{i\varphi} \cdot X}{\lambda} = \frac{\lambda + \omega\,\mathrm{e}^{i\varphi}}{\lambda} X - \frac{f(AXB)}{\lambda}$$

is invertible in $\mathcal{A}_{AXB}$, for every $\varphi \in [0, 2\pi)$, and so are the operators

$$\left(\lambda + g(0)\,\mathrm{e}^{i\varphi}\right) X - f(AXB), \quad \varphi \in [0, 2\pi).$$

This proves the first statement of the theorem. To prove the other two, we conduct an auxiliary discussion as presented below.

If $\lambda + g(0)\,\mathrm{e}^{i\varphi} \neq 0$, for every $\varphi \in [0, 2\pi)$, then the sought inverse is (for a given φ)

$$\left(\left(\lambda + g(0)\,\mathrm{e}^{i\varphi}\right) X - f(AXB)\right)^{-1} = \sum_{k=0}^{+\infty} \left(\lambda + g(0)\,\mathrm{e}^{i\varphi}\right)^{-k+1} (f(AXB))^k .$$

If $\lambda + g(0)\,\mathrm{e}^{i\varphi_0} = 0$, for some $\varphi_0 \in [0, 2\pi)$, then

$$\begin{aligned} & f(AXB) = \frac{-\lambda}{-\lambda} f(AXB) = -\lambda \frac{-f(AXB)}{\lambda} = \\ & (-\lambda)\left(\frac{\lambda + g(0)\,\mathrm{e}^{i\varphi_0}}{\lambda} X - \frac{f(AXB)}{\lambda}\right) = \\ & (-\lambda)\left(X - \frac{f(AXB) - g(0)\,\mathrm{e}^{i\varphi_0} \cdot X}{\lambda}\right) \end{aligned} \tag{3.9}$$

is invertible in $\mathcal{A}_{AXB}$. If $\|f(AXB) - g(0)\,\mathrm{e}^{i\varphi_0} \cdot X\| < |\lambda|$, then its inverse is

$$(f(AXB))^{-1} = (-\lambda)^{-1} \sum_{k=0}^{+\infty} \left(\frac{f(AXB) - g(0)\,\mathrm{e}^{i\varphi_0} \cdot X}{\lambda}\right)^k .$$

$\square$

Corollary 3.1.2. [25, Corollary 2.2.] Let A, B, X, λ, g and $\digamma(g)$ be given as in the previous theorem.
Then for every g_1 such that $0 < \|g_1\| \leq \|g\|$ the operator

$$\|g_1\| \frac{\lambda + g(0)\, \mathrm{e}^{i\varphi}}{\|g\|} X - f_1(AXB)$$

is invertible in $\mathcal{A}_{AXB}$, for every $f_1 \in \digamma(g_1)$, for every $\varphi \in [0, 2\pi)$.

Theorem 3.1.3. [25, Theorem 2.3.] Let $Y \in \mathcal{A}_{AXB}$ be given as $Y = f(AXB)$, for some function $f \in AbCon$. There exists $\omega \in \mathbb{C}$ such that $\omega X - Y$ is in $\mathcal{A}^{-1}_{AXB}$. Furthermore, there exists $\omega_0 \in \mathbb{C}$ such that for every $\lambda \in \mathbb{C}$ with $|\lambda| > |\omega_0|$ the operator $\lambda X - Y$ is in $\mathcal{A}^{-1}_{AXB}$.

Proof. Assume that

$$f(z) = \sum_{k=0}^{+\infty} \alpha_k z^k, \quad |z| < 1$$

and that the series

$$\sum_{k=0}^{+\infty} |\alpha_k|$$

converges. Put

$$g_\alpha(|z|) := \sum_{k=0}^{+\infty} |\alpha_k| (|z|)^k, \quad |z| < 1.$$

Then $\|g_\alpha\| = \Gamma < +\infty$. There exists a numerical series

$$\sum_{k=0}^{+\infty} |\beta_k|, \quad |\beta_0| = 0$$

such that its sum is $\Gamma + \varepsilon$, for some $\varepsilon > 0$. Then there exists a function g_β such that

$$g_\beta(|z|) = \sum_{k=0}^{+\infty} |\beta_k| |z|^k, \quad |z| < 1, \quad \|g_\beta\| = \Gamma + \varepsilon, \quad g_\beta(0) = 0.$$

Now there exists a $\zeta \in \mathbb{C} \setminus \{0\}$ such that

$$\|g_\beta\| \cdot \|X\| < |\zeta|.$$

Applying Corollary 3.1.2 we see that

$$\|g_\alpha\| \frac{\zeta}{\|g_\beta\|} X - f(AXB)$$

is in $\mathcal{A}_{AXB}^{-1}$. Now take $\omega := \zeta\frac{\|g_\alpha\|}{\|g_\beta\|}$. Since there is not upper bond for $|\zeta|$ it follows that $|\omega|$ can be arbitrary large. Therefore the statement is true for every $\lambda \in \mathbb{C}$ such that $|\lambda| \geq |\omega|$.
To complete the proof, note that $|\zeta|$ does have a lower bound. Let $\varepsilon \to 0+0$ and take a family of holomorphic functions g_β^ε, such that

$$g_\beta^\varepsilon(0) = 0$$

and

$$|\|g_\beta^\varepsilon\| - \Gamma| = o(\varepsilon), \quad \varepsilon \to 0+0.$$

Then $\frac{\|g_\beta^\varepsilon\|}{\|g_\alpha\|} - 1 = o(\varepsilon), \quad \varepsilon \to 0+0$. Now take ζ_0 such that

$$|\zeta_0| = \inf\{|\zeta| : \|g_\beta^\varepsilon\| \cdot \|X\| < |\zeta|, \quad \varepsilon \to 0+0\} = \|X\| \cdot \Gamma.$$

The sought number is $|\omega_0| = |\zeta_0| \cdot \frac{\|g_\alpha\|}{\|g_\alpha\|} = \|X\| \cdot \Gamma$. □

Definition 3.1.1. Let $Y \in \mathcal{A}_{AXB}$, $Y = f(AXB)$, $f \in AbCon$. The set of all complex numbers λ such that $\lambda X - Y \in \mathcal{A}_{AXB}^{-1}$ is called the resolvent set of Y and is denoted as $\rho(Y)$. Its complement (in the complex plane), denoted as $\sigma(Y)$ is called the spectrum of Y.
The number

$$r(Y) = \inf\{r \geq 0 : \lambda \in \mathbb{C}, \quad |\lambda| > r \Rightarrow \lambda \in \rho(Y)\}$$

is called the spectral radius of Y in $\mathcal{A}_{AXB}$, denoted as $r(Y)$.
The resolvent function $R_Y : \rho(Y) \to \mathcal{A}_{AXB}$ is defined as $R_Y(\lambda) := (\lambda X - Y)^{-1}$, for every $\lambda \in \rho(Y)$.

Even though $(\mathcal{A}_{AXB}, \|\cdot\|)$ is not a Banach algebra, a simple verification shows that the following lemma holds.

Lemma 3.1.1. Let Y, $Z \in \mathcal{A}_{AXB}$ and let λ, $\theta \in \rho(Y)$, $\lambda \in \rho(Z)$.

1. The resolvent equations hold

$$R_Y(\lambda) - R_Y(\theta) = R_Y(\theta)(\theta - \lambda)XR_Y(\lambda),$$

$$R_Y(\lambda) - R_Z(\lambda) = R_Y(\lambda)(Y - Z)R_Z(\lambda).$$

2. The resolvent function is differentiable on $\rho(Y)$, in the sense that

$$\lim_{\lambda \to \theta} \frac{R_Y(\lambda) - R_Y(\theta)}{\lambda - \theta} = (-1)(\theta X - Y)^{-2}.$$

3. The resolvent function vanishes at infinity, that is

$$\lim_{|\lambda|\to+\infty} R_Y(\lambda) = 0.$$

Corollary 3.1.3. [25, Corollary 2.3.] Let assumptions from Theorem 3.1.3 hold. Then $f(AXB)$ has a non-empty bounded spectrum.

Proof. Put $Y = f(AXB) \in \mathcal{A}_{AXB}$. There exists $r(Y)$, such that, for every complex number λ with its modulus greater than $r(Y)$, it follows that $\lambda \in \rho(Y)$. Lemma 3.1.1 yields that the resolvent function R_Y is differentiable on $\rho(Y)$.
Assume $\rho(Y) = \mathbb{C}$. Then R_Y is analytic on the entire complex plane. Further, $R_Y(\lambda) \to 0, \quad |\lambda| \to +\infty$, so R_Y is bounded. But then Liouville theorem yields that R_Y must be constant on $\mathbb{C}$. Hence contradiction. Therefore there exists a $\mu \in \mathbb{C}$ such that $|\mu| \leq r(Y)$ and $\mu X - Y$ is not invertible in $\mathcal{A}_{AXB}$. $\square$

In what follows, we prove that under the same conditions, the spectrum is not necessarily compact.

Theorem 3.1.4. [25, Theorem 2.4.] Let $f \in AbCon$ and let r be an arbitrary positive number. There exists a function g holomorphic on the open unit disc and continuous on its closure, represented by the power series which is not absolutely convergent on the boundary of the unit disc, with the property

$$\|f - g\| < r.$$

Proof. Let f be represented as

$$f(z) = \sum_{k=0}^{+\infty} \alpha_k z^k, \quad |z| < 1, \quad \sum_{k=0}^{+\infty} |\alpha_k| < +\infty.$$

There exists a power series

$$\sum_{k=0}^{+\infty} \varepsilon_k \, \mathrm{e}^{i\varphi_k} \, z^k, \quad |z| < 1,$$

with $\varepsilon_k \geq 0, \quad \varphi_k \in [0, 2\pi), \quad k \in \mathbb{N}_0$, such that

$$\sum_{k=0}^{+\infty} \varepsilon_k \, \mathrm{e}^{i\varphi_k} \text{ converges and } \quad \sum_{k=0}^{+\infty} \varepsilon_k = +\infty.$$

Put $\beta_k := \alpha_k + \varepsilon_k \,\mathrm{e}^{i\varphi_k}$ and define $g_1(z) := \sum\limits_{k=0}^{+\infty} \beta_k z^k$. Since

$$|\beta_k| = |-\beta_k| = |-\alpha_k - \varepsilon_k \,\mathrm{e}^{i\varphi_k}| \geq ||\alpha_k| - |\varepsilon_k \,\mathrm{e}^{i\varphi_k}|| = |\varepsilon_k - |\alpha_k||,$$

for every $k \in \mathbb{N}_0$, it follows from the comparison criterion that $\sum\limits_{k=0}^{+\infty} |\beta_k| = +\infty$. For every $z \in \overline{D}$ let $\varepsilon : z \mapsto \varepsilon(z) \in (0, r)$ be a continuous real function. Riemann conditional convergence theorem implies existence of a bijection $j : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\left| \sum_{k=0}^{+\infty} \varepsilon_{j(k)} \,\mathrm{e}^{i\varphi_{j(k)}}\, z^{j(k)} \right| = r - \varepsilon(z).$$

Put $\beta_{j(k)} := \alpha_{j(k)} + \varepsilon_{j(k)} \,\mathrm{e}^{i\varphi_{j(k)}}$ and define $g_j(z) := \sum\limits_{k=0}^{+\infty} \beta_{j(k)} z^{j(k)}$. Note that $\sum\limits_{k=0}^{+\infty} |\beta_{j(k)}| = +\infty$. By the assumption, the numerical series $\sum\limits_{k=0}^{+\infty} |\alpha_k|$ is (absolutely) convergent, therefore, for any bijection $p : \mathbb{N}_0 \to \mathbb{N}_0$ the following two equalities hold

$$f(z) = \sum_{k=0}^{+\infty} \alpha_k z^k = \sum_{k=0}^{+\infty} \alpha_{p(k)} z^{p(k)}, \quad |z| < 1$$

and

$$\sum_{k=0}^{+\infty} |\alpha_k| = \sum_{k=0}^{+\infty} |\alpha_{p(k)}|.$$

But then

$$|g_j(z) - f(z)| = |\sum_{k=0}^{+\infty} \varepsilon_{j(k)} \,\mathrm{e}^{i\varphi_{j(k)}}\, z^{j(k)}| = r - \varepsilon(z), \quad |z| < 1,$$

and consequently

$$\|g_j - f\| < r.$$

The sought function g is g_j. □

Theorem 3.1.5. [25, Theorem 2.5.] Assume that $Y \in \mathcal{A}^{-1}_{AXB}$ if and only if

$$X - Y \in \{f(AXB) : f \in AbCon\}.$$

Then $\sigma(Y)$ is not compact.

Proof. Corollary 3.1.3 yields $\sigma(Y)$ to be bounded. Thus it suffices to prove that $\rho(Y)$ is not an open set. Observe $\varphi : \mathbb{C} \to \mathcal{A}_{AXB}$, defined as

$$\varphi(\lambda) := \lambda X - Y.$$

Then

$$\mathcal{A}_{AXB}^{-1} \cap \mathcal{R}(\varphi) = \{\lambda X - Y : \quad \lambda X - Y \in \mathcal{A}_{AXB}^{-1}\}$$

and consequently

$$\rho(Y) = \varphi^{-1}(\mathcal{A}_{AXB}^{-1} \cap \mathcal{R}(\varphi)).$$

Assume $\rho(Y)$ to be an open set. Then $\mathcal{A}_{AXB}^{-1} \cap \mathcal{R}(\varphi)$ is an open set as well. Consequently, $\mathcal{R}(\varphi)$ is an open set because the mapping φ^{-1} is continuous and $\mathcal{R}(\varphi)$ is the inverse image of the open set $\mathbb{C}$. Therefore, $\mathcal{A}_{AXB}^{-1}$ is an open set. Now take $\chi(Z) := X - Z$, for any $Z \in \mathcal{A}_{AXB}$. Obviously, χ is a continuous mapping and $\chi(\chi(Z)) = Z$. Consequently, $\{f(AXB) : f \in AbCon\}$ is the inverse image of $\mathcal{A}_{AXB}^{-1}$ via the continuous mapping χ, hence it is an open set. However, the mapping $f \mapsto f(AXB)$ is continuous on $AbCon$, and therefore $AbCon$ is an open set, which contradicts Theorem 3.1.4, concluding that $\rho(Y)$ cannot be an open set and that $\sigma(Y)$ cannot be a closed set. $\square$

3.1.2 Algebraic Representations and Extensions

Let $\mathbb{A}, \mathbb{B} \in \mathcal{B}(\mathcal{B}(V_1, V_2))$ be defined as

$$\mathbb{A}(X) := AX, \quad \mathbb{B}(X) = XB, \quad X \in \mathcal{B}(V_1, V_2).$$

Trivially, $\mathbb{A}$ and $\mathbb{B}$ commute and $AXB = (\mathbb{A} \circ \mathbb{B})(X) = (\mathbb{B} \circ \mathbb{A})(X)$. The following lemma obviously holds:

Lemma 3.1.2. [25, Lemma 2.2.] With respect to the previous notation, algebra $\mathcal{A}_{AXB}$ is isometrically isomorphic to

$$\overline{\{p(\mathbb{A} \circ \mathbb{B})(X), \quad p \in P[\mathbb{C}]\}}.$$

For the given set $\mathcal{S}$ of operators, let $\mathcal{S}^{-1}$ be the set of all invertible operators in $\mathcal{S}$. Recall that

$$\mathcal{A}_{AXB}^{-1} := \{Y \in \mathcal{A}_{AXB} : \quad Y \text{ is invertible in } \mathcal{A}_{AXB}\}$$

and for a given $L \in \mathcal{B}(V)$, the set $[L]$ represents the set of all operators from $\mathcal{B}(V)$ which commute with L. Consequently, $L^n \in [L]$, for every $n \in \mathbb{N}_0$. Define

$$[\mathcal{A}_{AXB}] := [A] \cdot \mathcal{A}_{AXB} \cdot [B] = \{CDE : C \in [A], D \in \mathcal{A}_{AXB}, E \in [B]\}$$

and

$$\mathcal{B}_{AXB} := \mathcal{B}(V_2)\cdot\mathcal{A}_{AXB}\cdot\mathcal{B}(V_1) = \{FGH : F \in \mathcal{B}(V_2), G \in \mathcal{A}_{AXB}, H \in \mathcal{B}(V_1)\}.$$

Now one takes natural extension of the multiplication from $\mathcal{A}_{AXB}$ to $[\mathcal{A}_{AXB}]$ and $\mathcal{B}_{AXB}$. More precisely, let $C_1D_1E_1$, $C_2D_2E_2 \in [\mathcal{A}_{AXB}]$ and let $F_1G_1H_1$, $F_2G_2H_2 \in \mathcal{B}_{AXB}$. Then

$$(C_1D_1E_1)\cdot(C_2D_2E_2) := (C_1\cdot C_2)\cdot(D_1\cdot D_2)\cdot(E_1\cdot E_2)$$

and

$$(F_1G_1H_1)\cdot(F_2G_2H_2) := (F_1\cdot F_2)\cdot(G_1\cdot G_2)\cdot(H_1\cdot H_2).$$

It is now possible to observe

$$[\mathcal{A}_{AXB}]^{-1} := [A]^{-1}\cdot\mathcal{A}_{AXB}^{-1}\cdot[B]^{-1},$$

One should note that $I_{V_2} \in [A]^{-1}$ and $I_{V_1} \in [B]^{-1}$, so $[A]^{-1}$ and $[B]^{-1}$ are non-empty and so is $[\mathcal{A}_{AXB}]^{-1}$. However if A and B are invertible, then A^n and B^n are invertible, and so are A^{-n} and B^{-n}, for every $n \in \mathbb{N}$. In other words, operators of the form $A^kXB^k \in [\mathcal{A}_{AXB}]^{-1}$, for every $k \in \mathbb{Z}$. The set $[\mathcal{A}_{AXB}]^{-1}$ has some important properties, as illustrated below:

Theorem 3.1.6. [25, Theorem 2.7.] With respect to the previous notation, the following statements hold

1. The ordered pair $(\mathcal{A}_{AXB}^{-1}, \cdot)$ is an abelian group.
2. The ordered pair $(\mathcal{B}_{AXB}^{-1}, \cdot)$ is a group.
3. $(\mathcal{A}_{AXB}^{-1}, \cdot)$ is a subgroup of $(\mathcal{B}_{AXB}^{-1}, \cdot)$.
4. The centralizer and the normalizer for $(\mathcal{A}_{AXB}^{-1}, \cdot)$ in $(\mathcal{B}_{AXB}^{-1}, \cdot)$ are the same set $[\mathcal{A}_{AXB}]^{-1}$.

Proof.

1. Since $(\mathcal{A}_{AXB}, +, \cdot)$ is an algebra (Theorem 3.1.1), it follows that $(\mathcal{A}_{AXB}\setminus\{0\}, \cdot)$ is a semi-group. Taking the set of all invertible elements from $\mathcal{A}_{AXB}$, that is, taking the set $\mathcal{A}_{AXB}^{-1}$, we see that $(\mathcal{A}_{AXB}^{-1}, \cdot)$ is indeed a group. Commutation follows from Theorem 3.1.1.
2. Let FGH be an element from $\mathcal{B}_{AXB}^{-1}$, where $F \in \mathcal{B}(V_2)^{-1}$, $G \in \mathcal{A}_{AXB}^{-1}$ and $H \in \mathcal{B}(V_1)^{-1}$. Its inverse is $F^{-1}G^{-1}H^{-1}$, where each inverse belongs to the corresponding operator space. It follows that $(\mathcal{B}_{AXB}^{-1}, \cdot)$ is also a group. It is not abelian, since invertible operators in general do not need to commute.

3. Specially, $I_{V_k} \in \mathcal{B}(V_k)^{-1}, \quad k = \overline{1,2}, \quad$ it follows that $\mathcal{A}_{AXB}^{-1} = I_{V_2} \cdot \mathcal{A}_{AXB}^{-1} \cdot I_{V_1} \subset \mathcal{B}_{AXB}^{-1}$. Therefore, $\mathcal{A}_{AXB}^{-1}$ is a commutative subgroup of $\mathcal{B}_{AXB}^{-1}$.

4. Let $CDE \in [\mathcal{A}_{AXB}]^{-1}$. Then

$$\begin{aligned} & CDE \cdot \mathcal{A}_{AXB}^{-1} \\ =& \{CDE \cdot f(AXB) : f(AXB) \in \mathcal{A}_{AXB}^{-1}\} \\ =& \{CDE \cdot \lim_{n\to\infty} p_n(AXB) : \quad \lim_{n\to+\infty} p_n = f, \quad f(AXB) \in \mathcal{A}_{AXB}^{-1}\} \\ =& \{\lim_{n\to+\infty} (CDE) \cdot p_n(AXB) : \quad \lim_{n\to+\infty} p_n = f, \quad f(AXB) \in \mathcal{A}_{AXB}^{-1}\}. \end{aligned} \tag{3.10}$$

Let $q_m(AXB)$ be given as

$$\begin{aligned} & q_m(AXB) = \\ & \alpha_m A^m X B^m + \alpha_{m-1} A^{m-1} X B^{m-1} + \ldots + \alpha_1 AXB + \alpha_0 X. \end{aligned}$$

Then $CDE \cdot q_m(AXB)$ is given as

$$\alpha_m (C \cdot A^m) \cdot (D \cdot X) \cdot (E \cdot B^m) + \ldots + \alpha_0 C(D \cdot X)E.$$

Since A, A^2, …, A^m, $C \in [A]$, and the same goes (respectively) for B and E, it follows that (applying $X \cdot D = D \cdot X$)

$$CDE \cdot q_m(AXB) = q_m(AXB) \cdot CDE,$$

for any polynomial q_m. This is also true for the polynomials p_n that occur in (3.10). Therefore,

$$\begin{aligned} \{CDE \cdot f(AXB) \quad &: \quad f(AXB) \in \mathcal{A}_{AXB}^{-1}\} = \\ \{f(AXB) \cdot CDE \quad &: \quad f(AXB) \in \mathcal{A}_{AXB}^{-1}\}, \end{aligned}$$

so $CDE \cdot \mathcal{A}_{AXB}^{-1} = \mathcal{A}_{AXB}^{-1} \cdot CDE$. This proves that $[\mathcal{A}_{AXB}]^{-1}$ is contained in the normalizer of $\mathcal{A}_{AXB}^{-1}$ in $\mathcal{B}_{AXB}^{-1}$.

Now let $PQR \in \mathcal{B}_{AXB}^{-1}$ such that $PQR \cdot \mathcal{A}_{AXB}^{-1} = \mathcal{A}_{AXB}^{-1} \cdot PQR$. Then $P \in \mathcal{B}(V_2)^{-1}$, $Q \in \mathcal{A}_{AXB}^{-1}$ and $R \in \mathcal{B}(V_1)^{-1}$. Let $f(AXB) \in \mathcal{A}_{AXB}^{-1}$ be arbitrary. It follows that there exists $g(AXB) \in \mathcal{A}_{AXB}^{-1}$ such that

$$PQR \cdot f(AXB) = g(AXB) \cdot PQR. \tag{3.11}$$

Assume that

$$f(AXB) = \lim_{n\to+\infty} p_n(AXB)$$

and

$$g(AXB) = \lim_{m \to +\infty} q_m(AXB).$$

For given n and m, we have

$$PQR \cdot p_n(AXB) =$$
$$c_n(P \cdot A^n) \cdot (Q \cdot X) \cdot (R \cdot B^n) + \ldots + c_0 P(Q \cdot X)R$$

and

$$q_m(AXB) \cdot PQR =$$
$$d_m(A^n \cdot P) \cdot (Q \cdot X) \cdot (B^m \cdot R) + \ldots + d_0 P(X \cdot Q)R.$$

Now taking m and $n \to +\infty$, from (3.11) we have that

$$\|PQR \cdot p_n(AXB) - q_m(AXB) \cdot PQR\| \to 0.$$

But then for large enough n and m we have $\|A^k P - PA^k\| \to 0$, for every $k \in \{1, \ldots, \min\{n, m\}\}$. This yields that $P \in [A]^{-1}$ and $R \in [B]^{-1}$, so $PQR \in [\mathcal{A}_{AXB}]^{-1}$, that is, $[\mathcal{A}_{AXB}]^{-1}$ is the normalizer for $\mathcal{A}_{AXB}^{-1}$ in $\mathcal{B}_{AXB}^{-1}$. The centralizer part goes completely analogously.

□

We summarize our algebraic representations with a brief discussion when A and B are nilpotent and finite-dimensional operators.

Lemma 3.1.3. [25, Lemma 2.3.] Let $n \in \mathbb{N}$ such that $A^n X B^n = 0$ and $A^{n-1} X B^{n-1} \neq 0$. Then $\mathcal{A}_{AXB}$ is isomorphic to

$$(\{rest_{x^n}(p_m(x)) : p_m \in P[\mathbb{C}], \quad m \in \mathbb{N}_0\}, +, \cdot),$$

where addition and multiplication are standard operations in the space of polynomials.

Proof. For every $k \in \{0, \ldots, n-1\}$ put $\varphi((AXB)^k) := x^k$, where x^k is the polynomial $x \mapsto x^k$, for some independent variable x. □

For shorter notation, the set $\{rest_{x^n}(p_m(x)) : p_m \in P[\mathbb{C}], \quad m \in \mathbb{N}_0\}$ will simply be denoted as $rest_{x^n}$. Note that

$$\{0, \quad 1, \quad x, \quad x^2, \quad \ldots, \quad x^{n-1}\} = \{rest_{x^n}(x^k), \quad k \in \mathbb{N}_0\}.$$

The corresponding multiplication forms a structure

$$(\{0, \quad 1, \quad x, \quad x^2, \quad \ldots, \quad x^{n-1}\}, \cdot) = (\langle x \rangle_{rest_{x^n}}, \cdot),$$

where the multiplication is standard and is given as

$$x^p \cdot x^q = \begin{cases} x^{p+q}, & p+q<n \\ 0, & p+q \geq 0. \end{cases}$$

Theorem 3.1.7. [25, Theorem 2.8.] Let A and B be matrices. Then there exist $n, m \in \mathbb{N}_0$ such that the semi-group $(\{(AXB)^k\}_{k\in\mathbb{N}_0}, \cdot)$ is isomorphic to

$$(\mathbb{Z}_0^+, +) \oplus (\langle x\rangle_{rest_{x^n}}, \cdot) \oplus (\langle x\rangle_{rest_{x^m}}, \cdot) \oplus (\langle x\rangle_{rest_{x^{\min\{m,n\}}}}, \cdot). \tag{3.12}$$

Proof. Since A and B are matrices, then so are the operators $(AXB)^k$, for every $k \in \mathbb{N}$. From Jordan-Chevalley decompositions of A and B (see [9]), we see that

$$A = A_1 \dot{+} A_2,$$

where A_1 is invertible on $\mathcal{R}(A^{\mathrm{ind}(A)})$, and A_2 is nilpotent on $\mathcal{N}(A^{\mathrm{ind}(A)})$. The same decomposition holds for the matrix B

$$B = B_1 \dot{+} B_2,$$

where B_1 is invertible on $\mathcal{R}(B^{\mathrm{ind}(B)})$ and B_2 is nilpotent on $\mathcal{N}(B^{\mathrm{ind}(B)})$. Then for every $k \in \mathbb{N}$,

$$A^k = A_1^k \dot{+} A_2^k, \quad B^k = B_1^k \dot{+} B_2^k,$$

and consequently,

$$(AXB)^k = (A_1^k \dot{+} A_2^k)X(B_1^k \dot{+} B_2^k) = A_1^kXB_1^k \dot{+} A_1^kXB_2^k \dot{+} A_2^kXB_1^k \dot{+} A_2^kXB_2^k.$$

Let the nilpotency index of A_2 be m and let the nilpotency index of B_2 be n, for some m, $n \in \mathbb{N}_0$. Lemma 3.1.3 yields that $(\mathcal{A}_{A_1XB_2}, +, \cdot)$ is isomorphic to $rest_{x^n}$. Analogously, $(\mathcal{A}_{A_2XB_1}, +, \cdot)$ is isomorphic to $rest_{x^m}$ and $(\mathcal{A}_{A_2XB_2}, +, \cdot)$ is isomorphic to $rest_{x^{\min\{m,n\}}}$. But then the bases of the prior spaces are isomorphic to the bases of later, respectively. So $A_2^kXB_2^k$ maps to x^k, for $k < \min\{m, n\}$, and $A_2^kXB_2^k$ maps to zero otherwise. Analogous procedure goes for $A_2^kXB_1^k$ and $A_1^kXB_2^k$. Finally, observe the invertible part $(A_1^kXB_1^k)$, $k \in \mathbb{N}$. Put $\varphi((A_1XB_1)^k) := k$, $\varphi(X) := 0$, and its inverse, $\varphi((A_1XB_1)^{-k}) := -k$. It is now obvious that $(\mathbb{Z}, +)$ is isomorphic to $\{(A_1XB_1)^k, \cdot : k \in \mathbb{Z}\}$. From all of the above, we see that the representation (3.12) holds. $\square$

3.1.3 Applications of $\mathcal{A}_{AXB}$ to some operator equations

In this section we will illustrate how to apply the previous results to some basic operator equations. Let V_1 and V_2 be Banach spaces, $B \in \mathcal{B}(V_1)$, $A \in \mathcal{B}(V_2)$ and $C \in \mathcal{B}(V_1, V_2)$. The sets $\mathcal{A}_{ACB}$, $[\mathcal{A}_{ACB}]$ and $\mathcal{B}_{ACB}$ are defined completely analogously as $\mathcal{A}_{AXB}$, $[\mathcal{A}_{AXB}]$ and $\mathcal{B}_{AXB}$, respectively, where instead of X we write C. The problem at hand is the same: for given A, B and C, find $X \in \mathcal{B}(V_1, V_2)$ such that the desired equality holds.

1. Operator equation $X - AXB = C$

Operator equations of the form

$$X - AXB = C \tag{3.13}$$

are called Stein operator equations. From Lemma 3.1.2 we see that $AXB = (\mathbb{A} \circ \mathbb{B})(X)$. If we assume $\|A\|$ and $\|B\| < 1$, then $\|\mathbb{A} \circ \mathbb{B}\| < 1$ and consequently $I - (\mathbb{A} \circ \mathbb{B})$ is invertible in $\mathcal{B}(\mathcal{B}(V_1, V_2))$. Since $X = I_{\mathcal{B}(V_1,V_2)}(X)$, we have that (3.13) is equivalent to

$$(I - (\mathbb{A} \circ \mathbb{B}))(X) = C, \tag{3.14}$$

which implies

$$X = (I - (\mathbb{A} \circ \mathbb{B}))^{-1}(C) = \sum_{k=0}^{+\infty} (\mathbb{A} \circ \mathbb{B})^k(C) = \sum_{k=0}^{+\infty} A^k C B^k. \tag{3.15}$$

Remark. From (3.15) we see that $X \in \mathcal{A}_{ACB}$. In addition, operator $I - \mathbb{A} \circ \mathbb{B}$ is invertible, so the equation is solvable for every C.

When $C = 0$, eq. (3.13) is called homogeneous Stein equation. From discussion set in Theorem 3.1.1, statement 2, we see that the only solution is $X = 0$, which agrees with the calculation in (3.14) and (3.15):

$$X = (I - (\mathbb{A} \circ \mathbb{B}))^{-1}(0) = 0.$$

One can say that for given holomorphic function f, generalized Stein equation is every operator equation of the form

$$X - f(AXB) = C. \tag{3.16}$$

Applying Lemma 3.1.2, equation (3.16) transforms into

$$(I - f(\mathbb{A} \circ \mathbb{B}))(X) = C, \tag{3.17}$$

that is

$$X = (I - f(\mathbb{A} \circ \mathbb{B}))^{-1}(C) = \sum_{k=0}^{+\infty} (f(\mathbb{A} \circ \mathbb{B}))^k(C), \tag{3.18}$$

provided that $\|f(\mathbb{A} \circ \mathbb{B})\| < 1$, which supports Theorem 4.42. The same methodology goes for $\lambda X - f(AXB)$, when $\lambda \in \mathbb{C}$ and the holomorphic function f are provided in a way that they satisfy conditions of Theorem 4.42.

Corollary 3.1.4. [25, Corollary 3.1] Let f be a holomorphic function, $\lambda \in \mathbb{C} \setminus \{0\}$, A, B and X be provided as in Theorem 4.42.
If $C \in \mathcal{B}(V_1, V_2)$ is given such that

$$\lambda X - f(AXB) = C$$

holds, then $\mathcal{A}_{AXB} = \mathcal{A}_{ACB}$.

Proof. Obviously, $C \in \mathcal{A}_{AXB}$ so $\mathcal{A}_{ACB} \subset \mathcal{A}_{AXB}$. Conversely, from the previous discussion $X \in \mathcal{A}_{ACB}$ and consequently $\mathcal{A}_{AXB} \subset \mathcal{A}_{ACB}$. □

For more results on this particular operator equation, an interested reader is referred to [32], [34], [53], [84], [85].

2. Operator equations $AX = C$ and $AXB = C$

In this section we recall how to solve the simplest operator equation,

$$AX = C, \tag{3.19}$$

for given $C \in \mathcal{B}(V_1, V_2)$ and $A \in \mathcal{B}(V_2)$. Solvability conditions require that $\mathcal{R}(A) \supset \mathcal{R}(C)$. If $A = 0$, then the eq. (3.19) is solvable if and only if $C = 0$. Otherwise, if $A \neq 0$, then A is outer regular. Consequently, there exists $A^{(2)} \in \mathcal{B}(V_2)$ such that

$$A^{(2)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R \\ \mathcal{N}(A^{(2)}) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A^{(2)}) \\ T \end{pmatrix},$$

where $V_2 = R \dotplus \mathcal{N}(A^{(2)}) = \mathcal{R}(A^{(2)}) \dotplus T$, and in that case, A can be represented via the matrix

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & N_A \end{bmatrix} : \begin{pmatrix} \mathcal{R}(A^{(2)}) \\ T \end{pmatrix} \to \begin{pmatrix} R \\ \mathcal{N}(A^{(2)}) \end{pmatrix}.$$

It follows that $A^{(2)}A = P_{\mathcal{R}(A^{(2)})}$.

Proposition 3.1.1. Assume that A is outer regular. If there exists an outer inverse for A, $A^{(2)}$, such that $\mathcal{R}(C) \subset \mathcal{R}(A^{(2)})$, then the equation (3.19) is solvable, and one of its solutions is $X = A^{(2)}C$.

For a more general problem, define operators $\mathbb{A}(L) := AL$ and $\mathbb{B}(L) := LB$, for every $L \in \mathcal{B}(V_1, V_2)$, where $B \in \mathcal{B}(V_1)$ and $A \in \mathcal{B}(V_2)$. Then solving operator equation

$$AXB = C \tag{3.20}$$

reduces to two consecutive applications of Proposition 3.1.1. Consult [18], [32], [34], [38], [45], [61], [62] and references therein.

Proposition 3.1.2. If equation (3.20) is solvable, then $C \in \mathcal{A}_{AXB}$.

3.2 Singular Sylvester equation: Algebra $\mathcal{A}_{AXB}$ meets Fredholm theory

In order to solve the initial Sylvester equation, it suffices to find one particular solution X_p, and all solutions X_h to the homogeneous equation

$$AX - XB = 0. \tag{3.21}$$

Then every solution X to (3.1) can be obtained as $X = X_p + X_h$. The following Theorem and Corollary concern the homogenized problem (3.21).

Theorem 3.2.1. [25, Theorem 3.4] Let A, X and B be provided such that $AX = XB$. Then for every $Y \in \mathcal{A}_{AXB}$ it follows that $AY = YB$. In other words, every element from $\mathcal{A}_{AXB}$ is a solution to the homogeneous Sylvester equation (3.21).

Proof. First observe the basis of $\mathcal{A}_{AXB}$:

$$X, \quad AXB, \quad A^2XB^2, \quad \dots \quad .$$

Given the way A, B and X are provided, it follows that

$$A(A^nXB^n) = A(A^{n-1}XB^{n+1}) = (A^nXB^n)B, \quad n \in \mathbb{N},$$

so $(AXB)^n$ is a solution to (3.21), for every $n \in \mathbb{N}$. Further, every finite linear combination of the basis elements is a solution to (3.21). This proves that $p_n(AXB)$ is a solution to the homogeneous Sylvester equation, for every $p_n \in P[\mathbb{C}]$. One should note that, in the bounded-operator case, the set of solutions to the equation $AX - XB = 0$ is closed. This is directly verifiable.

Nevertheless, let f be a holomorphic function on some Cauchy domain Ω, $\sigma(A)$, $\sigma(B) \subset \Omega$, given as the limit of some complex polynomials

$$f(z) = \lim_{n \to +\infty} p_n(z), \quad z \in \Omega.$$

Then Lemma 3.1.2 applies and

$$f(AXB) = \lim_{n \to +\infty} p_n(AXB)$$

and

$$\begin{aligned} &Af(AXB) = A(\lim_{n \to \infty} p_n(AXB)) = \\ &\lim_{n \to \infty} Ap_n(AXB) = \lim_{n \to \infty} (p_n(AXB)B) = f(AXB)B, \end{aligned}$$

so $\mathcal{A}_{AXB}$ is contained in the set of solutions to the homogeneous Sylvester equation. □

Corollary 3.2.1. [25, Corollary 3.3.] Let A, B and X be provided such that $AX = XB$, and let $\mathbb{A}$ and $\mathbb{B}$ be provided as in Lemma 3.1.2.
Then $\mathcal{A}_{AXB}$ is isomorphic to

$$\overline{\{p(\mathbb{A}^2)(X) : p \in P[\mathbb{C}]\}} \tag{3.22}$$

and to

$$\overline{\{p(\mathbb{B}^2)(X) : p \in P[\mathbb{C}]\}}. \tag{3.23}$$

In order for $AX = XB$ to be solvable, it is required for (3.22) and (3.23) to be isomorphic to each other.

For a moment, assume that at least one solution to (3.1) is found. Then it can be further exploited, via the algebra $\mathcal{A}_{AXB}$ introduced in this chapter. Recall Lemma 1.2.3 from Chapter 1:

Lemma 3.2.1. [50, Lemma 2.1.] Assume X is a solution to (3.1). Then for any $k \geq 1$

$$A^k X - XB^k = \sum_{i=0}^{k-1} A^{k-1-i} C B^i. \tag{3.24}$$

In the following, we briefly describe algebraic properties (w. r. t. $\mathcal{A}_{AXB}$), of one solution to the Sylvester equation (3.1).

Corollary 3.2.2. [25, Corollary 3.2.] Let A, B, C and X be provided such that (3.1) holds. Then $C \in [\mathcal{A}_{AXB}]$ and for every $k \in \mathbb{N}_0$,

$$A^k X - XB^k \in [\mathcal{A}_{ACB}].$$

Proof. The first claim follows directly $C = AX - XB \in [\mathcal{A}_{AXB}]$.

When $k = 0$ then $X - X = 0 \in [\mathcal{A}_{ACB}]$. When $k \geq 1$, then (3.24) applies, and

$$A^k X - XB^k = A^{k-1}C + \ldots + CB^{k-1}.$$

Since $A^\ell \in [A]$ and $B^s \in [B]$, for every $s, \ell \in \{0, \ldots, k-1\}$, it follows that every addend on the right-hand-side is in $[\mathcal{A}_{ACB}]$, and so is $A^k X - XB^k$. □

Now all what is left to do, is to find one particular solution to (3.1), and analogously, one particular non-trivial solution to (3.21). This is obtained in the following section, with help from Fredholm theory.

3.2.1 Finding particular solutions: Fredholm theory approach

Notation and results from [19], [20], [33] and [103]–[106] come in handy at this point, and we briefly mention those which are relevant for this section. For convenience, we denote the ideal of compact operators by $\mathcal{C}(V_1, V_2)$.

Recall that, for a bounded linear operator $L \in \mathcal{B}(V)$, the hyper range is given by $\mathcal{R}^\infty(L) = \cap_n \mathcal{R}(L^n)$ and the hyper null space is given by $\mathcal{N}^\infty(L) = \cup_n \mathcal{N}(L^n)$. With $\operatorname{asc}(L)$ and $\operatorname{dsc}(L)$ we denote, respectively, the ascend and the descend of the operator L. If $\operatorname{asc}(L)$ and $\operatorname{dsc}(L)$ are both finite, then they are equal (to, say, p) and

$$V = \mathcal{R}(L^p) \dot{+} \mathcal{N}(L^p).$$

Conversely, if

$$V = \mathcal{R}(L^m) \dot{+} \mathcal{N}(L^m),$$

for some m, then $\operatorname{asc}(L), \operatorname{dsc}(L) \leq m$. It is now clear that, if $\operatorname{asc}(L) < \infty$ and $\operatorname{dsc}(L) < \infty$, then

$$\mathcal{R}^\infty(L) \cap \mathcal{N}^\infty(L) = \{0\}, \qquad V = \mathcal{R}^\infty(L) \dot{+} \mathcal{N}^\infty(L).$$

We introduce some standard definitions from Fredholm theory.

Definition 3.2.1. A (bounded) linear operator $L \in \mathcal{B}(V_1, V_2)$ is upper semi-Fredholm if $\alpha(L) = \dim \mathcal{N}(L) < \infty$ and $\mathcal{R}(L)$ is closed in V_2. The set of upper semi-Fredholm operators is denoted as $\Phi_+(V_1, V_2)$.

Definition 3.2.2. An upper semi-Fredholm operator L is a left upper semi-Fredholm operator if there exists a bounded projection from V_2 onto $\mathcal{R}(L)$. The set of all left upper semi-Fredholm operators is denoted by $\Phi^\ell(V_1, V_2)$.

Equivalently (see [106]), an upper semi-Fredholm operator L is a left upper semi-Fredholm operator iff there exist $L_1 \in \mathcal{B}(V_2, V_1)$ and a finite rank operator $F \in \mathcal{B}(V_1)$ (or equivalently, a compact operator $K \in \mathcal{B}(V_1)$), such that $L_1 L = I + F$ (respectively, $L_1 L = I + K$). The term *left* refers to the left invertibility in the Calkin algebra, and therefore left upper semi-Fredholm operators are sometimes called *essentially left invertible operators*, see [12] and [44].

Definition 3.2.3. [106] Let $L \in \mathcal{B}(V)$. The point $\lambda \in \sigma(L)$ is a Riesz point of L if V is a direct sum of a closed subspace $E_L(\lambda)$ and a finite dimensional subspace $F_L(\lambda)$, which are invariant for L and the reduction of $L - \lambda$ to $E_L(\lambda)$ is invertible while the reduction of $L - \lambda$ to $F_L(\lambda)$ is nilpotent.

We now return to the general case, where V_1 and V_2 are Banach spaces and A, B and C are accordingly provided bounded linear operators, such that $\sigma(A) \cap \sigma(B) \neq \emptyset$.

Theorem 3.2.2. [26, Theorem 3.1.] Assume that there exists a bounded embedding $J : V_1 \to V_2$ with a closed range, such that $\mathcal{R}(J)$ is complemented in V_2. Define operators

$$\widehat{C} \in \mathcal{B}\left(\mathcal{B}(V_1, V_2)\right), \quad \widehat{C}(L) := C J^{-1} P_{\mathcal{R}(J)} L$$

and

$$S \in \mathcal{B}\left(\mathcal{B}(V_1, V_2)\right), \quad S(L) := AL - LB.$$

There exists a solution to (3.1) if and only if

$$S \cdot \widehat{X} = \widehat{C} \tag{3.25}$$

is solvable in $\mathcal{B}\left(\mathcal{B}(V_1, V_2)\right)$.

Proof. Denote by Q the bounded projection from V_2 onto $\mathcal{R}(J)$. In addition to S and $\widehat{C}$, define the following operators as previously,

$$\mathbb{A} \in \mathcal{B}\left(\mathcal{B}(V_1, V_2)\right), \qquad \mathbb{A}(L) = AL, \quad L \in \mathcal{B}(V_1, V_2),$$

$$\mathbb{B} \in \mathcal{B}\left(\mathcal{B}(V_1, V_2)\right), \qquad \mathbb{B}(L) = LB, \quad L \in \mathcal{B}(V_1, V_2).$$

If (3.25) is solved for $\widehat{X} \in \mathcal{B}(\mathcal{B}(V_1, V_2))$, then (1.1) is solved by the operator $\widehat{X}(J)$. On the other hand, for every solution X to (1.1), it follows that

$$AX - XB = C \Leftrightarrow S(X) = C \Leftrightarrow S \cdot \widehat{X}(J) = \widehat{C}(J),$$

giving one bounded solution to (3.25) at point J, where $\widehat{X} \in \mathcal{B}(\mathcal{B}(V_1, V_2))$ is given by $\widehat{X}(L) = XJ^{-1}QL$, for every $L \in \mathcal{B}(V_1, V_2)$.

We now proceed to solve (3.25). Since $S = \mathbb{A} - \mathbb{B}$ and $[\mathbb{A}, \mathbb{B}] = 0$, it follows that

$$\sigma(S) \subset \sigma(A) - \sigma(B) = \{\mu - \lambda : \quad \mu \in \sigma(A), \ \ \lambda \in \sigma(B)\}.$$

Trivially, if $0 \notin \sigma(S)$, then S is invertible and $\widehat{X} = S^{-1} \cdot \widehat{C}$. Otherwise, the eq. (3.25) can be solved by Proposition 3.1.1. $\square$

Corollary 3.2.3. [26, Corollary 3.1.] If V_1 is a closed, complemented subspace of V_2, then there exists a solution to (1.1) if and only if (3.25) is solvable, where $J = P_{V_1}$.

In what follows, we extend the statement from Theorem 3.2.2 to a more general case. For convenience, we define the following property for Riesz points of a given operator.

Definition 3.2.4. [26, Definition 3.1.] Let $B \in \mathcal{B}(V_1)$ and let $\lambda \in \sigma(B)$ be a Riesz point of B. Let operator $L \in \mathcal{B}(V_1, V)$, for some Banach space V, be given such that $\alpha(L) = \dim \mathcal{N}(L) < +\infty$. Then operator L decomposes operator B at point λ in the Riesz sense if $B_N := B \restriction_{\mathcal{N}(L)}$ has the property that

$$F_B(\lambda) = \mathcal{N}^\infty(B_N) + \mathcal{R}^\infty(B_N). \tag{3.26}$$

Remark. Note that such L always exists: fact that $F_B(\lambda)$ is a finite dimensional $B-$invariant subspace of V_1 implies that $B \restriction_{F_B(\lambda)}: F_B(\lambda) \to F_B(\lambda)$ is a square matrix. Every square matrix can be further decomposed into a sum of an invertible and a nilpotent matrix, which naturally define the hyper range and the hyper null space of $B \restriction_{F_B(\lambda)}$.

Proposition 3.2.1. [26, Proposition 3.1.] Let $B \in \mathcal{B}(V_1)$, $C \in \mathcal{B}(V_1, V_2)$ and $A \in \mathcal{B}(V_2)$ be given bounded linear operators on Banach spaces V_1 and V_2 and let F_B be a finite dimensional $B-$invariant subspace of V_1. Then there exists a finite dimensional $A-$invariant subspace of V_2, denoted by F_A, such that $C(F_B) \subset F_A$.

Proof. Since F_B is finite dimensional, it follows that $C(F_B)$ is finite dimensional as well. Then $A \restriction_{C(F_B)}$ is a finite rank operator, which has finite ascend and descend, therefore, there exists

$$F_A = \mathcal{N}^\infty(A \restriction_{C(F_B)}) + \mathcal{R}^\infty(A \restriction_{C(F_B)}),$$

which is $A-$invariant finite dimensional subspace of V_2. $\square$

Theorem 3.2.3. [26, Theorem 3.2.] Let $B \in \mathcal{B}(V_1)$ such that $\lambda \in \sigma(B)$ is a Riesz point of B. Assume there exists $J \in \Phi^\ell(V_1, V_2)$ such that it decomposes B at point λ in the Riesz sense. Let W and U be finite dimensional subspaces of V_1 and V_2, respectively, defined as

$$W = \mathcal{R}^\infty(B \restriction_{\mathcal{N}(J)}) + \mathcal{N}^\infty(B \restriction_{\mathcal{N}(J)}) = F_B(\lambda) \text{ (by (3.26))}, \tag{3.27}$$

$$U = \mathcal{R}^\infty(A \restriction_{\mathcal{R}(C\restriction_W)}) + \mathcal{N}^\infty(A \restriction_{\mathcal{R}(C\restriction_W)}). \tag{3.28}$$

If matrices $B \restriction_W$, $C \restriction_W$ and $A \restriction_U$ satisfy conditions (2.1), then there exist infinitely many solutions to (3.1) if and only if

$$AX_1 - X_1 B_1 = C_1 \tag{3.29}$$

is solvable on V_{12}, where $V_1 = W \dot{+} V_{12}$ and $B_1 = B \restriction_{V_{12}}$, $C_1 = C \restriction_{V_{12}}$.

Proof. Let $J \in \Phi^\ell(V_1, V_2)$ be a left upper semi-Fredholm operator. Then $\alpha(J) < \infty$ and $B \restriction_{\mathcal{N}(J)}$ is a finite dimensional operator. Let W be the finite dimensional space introduced in (3.27), which is the finite dimensional space $F_B(\lambda)$ on which $B - \lambda$ is nilpotent. Define $B_W := B \restriction_W$ and $C_W := C \restriction_W$. Then $\mathcal{R}(C_W)$ is a finite dimensional space as well. In that sense, let U be provided as in (3.28) and similarly $A_U := A \restriction_U$. It follows that $A_U : U \to U$. Observe the finite dimensional spaces W and U, and operators defined on them, that is,

$$B_W \in \mathcal{B}(W), \qquad C_W \in \mathcal{B}(W, U), \qquad A_U \in \mathcal{B}(U).$$

They are all scalar matrices, so if conditions (2.1) hold, there exist infinitely many solutions X_W to

$$A_U X_W - X_W B_W = C_W.$$

To complete the proof, note that $V_1 = \mathcal{N}(J) + V_{11} = \mathcal{N}(J) + (W \cap V_{11}) + V_{12} = W + V_{12}$, and each subspace is closed. Let $J_1 = J \restriction_{V_{11}}$ and $J_2 = J_1 \restriction_{V_{12}}$. Since $\mathcal{R}(J)$ is closed and J_1 is injective, with $\mathcal{R}(J) = \mathcal{R}(J_1)$, it follows that

$$\mathcal{R}(J_1) = J_1(W \cap V_{11}) + \mathcal{R}(J_2),$$

thus $\mathcal{R}(J_2)$ is closed as well and because J_2 is injective, J_2 has a bounded inverse from $\mathcal{R}(J_2)$ to V_{12}. By assumption, J is a left upper semi-Fredholm operator, so there exists a bounded projection Q_1 from V_2 onto $\mathcal{R}(J) = \mathcal{R}(J_1)$. However, since $\mathcal{R}(J_1 \restriction_{W \cap V_{11}})$ is finite dimensional, it follows that there exists a bounded projection Q_2 from V_2 onto $\mathcal{R}(J_2)$, so J_2 is a bounded embedding of V_{12} into V_2, with a closed range, which is complemented in

V_2. Further, since $V_1 = W \dot{+} V_{12}$ and λ is a Riesz point for the operator B, it follows that V_{12} is a closed, $B-$invariant subspace of V_1. Since J_2 is a bounded embedding from V_{12} to V_2, with a closed and complemented range in V_2, the equation (3.29) is solved via Theorem 3.2.2, if and only if the initial equation (3.1) is solved in $\mathcal{B}(V_1, V_2)$. Finally, every solution X to (3.1) can be expressed in the form $X = X_W + X_1$, with respect to the decomposition $V_1 = W \dot{+} V_{12}$. □

If B does not have a Riesz point in its spectrum, then Theorem 3.2.3 fails, but Theorem 3.2.2 can still be applied. In that case, we proceed with the construction firstly introduced by Berberian, which was applied to Fredholm theory by Buoni, Harte and Wickstead (see [8], [12] and [44]). By $\ell_\infty(V_1)$ we denote the Banach space of bounded sequences in V_1, equipped with the supremum norm. By $m(V_1)$ we denote the subspace of $\ell_\infty(V_1)$ which consists of those bounded sequences in V_1 such that each sequence has a subsequence which has a convergent subsequence, or, equivalently, every element of the space $m(V_1)$ is totally bounded. Now introduce $\mathcal{P}(V_1) = \ell_\infty(V_1)/m(V_1)$, equipped with the supremum norm. It follows that $\|(x)\| = q((x))$, for every $(x) \in \mathcal{P}(V_1)$, where q is the measure of noncompactness

$$q((x)) = \inf\{\delta \geq 0 : \quad (x) \text{ has a finite } \delta\text{-net}\}.$$

This defines a Banach space, and every bounded operator $L \in \mathcal{B}(V_1)$ induces $\mathcal{P}(L) \in \mathcal{B}(\mathcal{P}(V_1))$, defined entry-wise for each sequence $(x) \in \mathcal{P}(V_1)$. We state some fundamental results obtained in [12] and [44].

Theorem 3.2.4. [12, Theorem 2] If $T : V_1 \to V_2$ is a bounded linear operator between Banach spaces V_1 and V_2, then the following are equivalent:
(a) $\mathcal{P}(T) : \mathcal{P}(V_1) \to \mathcal{P}(V_2)$ is one-one;
(b) $T : V_1 \to V_2$ is upper semi-Fredholm;
(c) $\mathcal{P}(T) : \mathcal{P}(V_1) \to \mathcal{P}(V_2)$ is bounded below.

Recall that every upper semi-Fredholm operator maps bounded but not totally bounded sequences into bounded but not totally bounded sequences. Further, if T sends every $(x) \in \ell_\infty(V_1)$ to $m(V_1)$, then T must be a compact operator, so $\mathcal{B}(\mathcal{P}(V_1), \mathcal{P}(V_2)) = \mathcal{B}(V_1, V_2)/\mathcal{C}(V_1, V_2)$. In analogy to

$$Lx = 0 \Rightarrow x = 0,$$

whenever L is injective, the implication

$$TU \text{ is compact} \Rightarrow U \text{ is compact}$$

defines T as an *essentially one-one* operator. In analogy to the reverse-order-law in dual spaces, the implication

$$UT \text{ is compact} \Rightarrow U \text{ is compact}$$

defines T as an *essentially dense* operator.

Theorem 3.2.5. [12, Theorem 4] Let T be a bounded operator between two Banach spaces. Then the following implications hold:
(a) T is left upper semi-Fredholm$\Rightarrow$ T is upper semi-Fredholm $\Rightarrow$ T is essentially one-one;
(b) T is right lower semi-Fredholm $\Rightarrow$ T is lower semi-Fredholm $\Rightarrow$ T is essentially dense.

At this point we can generalize the statement from Theorem 3.2.3.

Theorem 3.2.6. [26, Theorem 3.5.] Define $\mathcal{P}(V_1)$, $\mathcal{P}(V_2)$, $\mathcal{P}(B)$, $\mathcal{P}(C)$ and $\mathcal{P}(A)$ as described above.
(a) If there exists $J \in \Phi^\ell(V_1, V_2)$, then there exists a solution to the quotient equation

$$\mathcal{P}(A)\mathcal{P}(X) - \mathcal{P}(X)\mathcal{P}(B) = \mathcal{P}(C) \tag{3.30}$$

if and only if

$$\mathcal{P}(S) \cdot \widehat{\mathcal{P}(X)} = \widehat{\mathcal{P}(C)} \tag{3.31}$$

is solvable, where

$$\mathcal{P}(S)(\mathcal{P}(L)) := \mathcal{P}(A)\mathcal{P}(L) - \mathcal{P}(L)\mathcal{P}(B)$$

$$\widehat{\mathcal{P}(C)}(\mathcal{P}(L)) := \mathcal{P}(C)\mathcal{P}(J)^{-1}P_{\mathcal{R}(\mathcal{P}(J))}\mathcal{P}(L).$$

(b) Let $\lambda \in \sigma_{app}(B)$ such that the eigenspace $F_{\mathcal{P}(B)}(\lambda)$ for $\mathcal{P}(B)$ which corresponds to λ is a finite dimensional subspace of $\mathcal{P}(V_1)$. Assume there exists an upper semi-Fredholm operator $\mathcal{P}(\varphi) \in \Phi^\ell(\mathcal{P}(V_1), \mathcal{P}(V_2))$, which decomposes $\mathcal{P}(B)$ at point λ in the Riesz sense and let $\mathcal{P}(W)$ and $\mathcal{P}(U)$ be defined as in (3.27) and (3.28), with respect to operators $\mathcal{P}(A)$, $\mathcal{P}(B)$, $\mathcal{P}(C)$ and $\mathcal{P}(\varphi)$. If operators $\mathcal{P}(B) \restriction_{\mathcal{P}(W)}$, $\mathcal{P}(C) \restriction_{\mathcal{P}(W)}$ and $\mathcal{P}(A) \restriction_{\mathcal{P}(U)}$ satisfy conditions (2.1), then there exist infinitely many solutions to (3.30) iff $\mathcal{P}(A)\mathcal{P}(X_1) - \mathcal{P}(X_1)\mathcal{P}(B_1) = \mathcal{P}(C_1)$, where $\mathcal{P}(X_1)$, $\mathcal{P}(B_1)$ and $\mathcal{P}(C_1)$ are defined as in (3.29) on V_{12}, $V_1 = W \dot{+} V_{12}$.

Proof. (a) From the discussion above, operator J defines an injective $\mathcal{P}(J)$, with closed and complemented range in $\mathcal{P}(V_2)$, so Theorem 3.2.2 applies to (3.30).
(b) Similarly, all conditions of Theorem 3.2.3 hold, so (3.30) has infinitely many solutions. □

Corollary 3.2.4. [26, Corollary 3.2.] Let $\lambda \in \sigma_{app}(B)$ such that λ is a Riesz point of $\mathcal{P}(B)$ and assume that $\mathcal{P}(\varphi) \in \Phi_+(\mathcal{P}(V_1), \mathcal{P}(V_2))$ is an upper semi-Fredholm operator which decomposes $\mathcal{P}(B)$ at point λ in the Riesz sense. Then $\varphi \notin \Phi_+(V_1, V_2)$.

Proof. Assume that $\mathcal{P}(\varphi)$ is an upper semi-Fredholm operator, which decomposes $\mathcal{P}(B)$ at point λ in the Riesz sense. If $\varphi \in \Phi_+(V_1, V_2)$, then (by [12]) $\mathcal{P}(\varphi)$ is one-one, that is, $\mathcal{N}(\mathcal{P}(\varphi)) = \{0\}$. But by assumption, $\mathcal{P}(\varphi)$ decomposes $\mathcal{P}(B)$ at point λ in the Riesz sense, so the finite dimensional part (as in (3.26)) is equal to zero:

$$F_{\mathcal{P}(B)}(\lambda) = \{0\}.$$

Then $\mathcal{P}(B) - \lambda$ is invertible in $\mathcal{P}(V_1)$, which contradicts the fact that $\lambda \in \sigma_p(\mathcal{P}(B))$. □

Corollary 3.2.5. [26, Corollary 3.3.] If there exists $J \in \Phi^{\ell}(V_1, V_2)$, then there exists $X \in \mathcal{B}(V_1, V_2)$ such that

$$AX - XB \in \mathcal{C}(V_1, V_2). \tag{3.32}$$

Proof. Immediately from Theorem 3.2.6 (a), we have that the equation (3.30) is solvable. Taking $C = 0$ completes the proof. □

3.3 Some applications

As previously mentioned, Sylvester equations have numerous applications in both theoretical and applied mathematics, physics, engineering and computer science. Simply knowing when a Sylvester equation is solvable, gives sufficient conditions for some quite important results, such as operator matrix diagonalization, perturbation analysis, commutator problems, etc. consult [10] and numerous references therein. In this section we illustrate how our results contribute to such applications.

3.3.1 Fréchet derivatives and commutators

Expressions $AX - XB$ are known as generalized derivations or weighted commutators, and are in close relations to the Fréchet derivatives (see [31], [97] and [98]). When $V_1 = V_2 = V$, let $f(A) = A^2$. Then the Fréchet derivative of $f(A)$ at point B is the expression (which is a bounded linear operator on V),

$$Df_A(B) = AB + BA.$$

Observe the abstract ODE

$$Df_A(B) = C.$$

Question 1. *At which points B does the Fréchet derivative of f at point A take the value C?*
Assuming that $\sigma(A) \cap \sigma(-A) \neq \emptyset$ it follows from Corollary 3.2.3 that there exists a B, which is the solution to the abstract ODE if and only if (3.25) is solvable.
Question 2. *When can an operator be expressed as a commutator of two idempotents?*
The problem of commuting idempotents has been characterized in the following

Theorem 3.3.1. [36, Theorem 1.] An element t in a ring R is a commutator of a pair of idempotents if and only if there exist $u \in R$ and $s \in R$ such that $u^2 = 1$, $ut + tu = 0$ $su - us = 0$, $st - ts = 0$ and $s^2 = t^2 + \frac{1}{4}$.

Although we cannot simplify the statement of the Theorem 3.3.1, our results can enable solvability of the commutator equations that appear in the paper [36]. Let $R = \mathcal{B}(V)$ and let $C \in \mathcal{B}(V)$ be given such that $\sigma(C) \cap \sigma(-C) \neq \emptyset$. Define $g(C) = C^2 + \frac{1}{4}$ and let $f(L) = L^2$, as before. Then finding U such that $U^2 = I$ and $CU + UC = 0$ reduces to

$$Dg_C(U) = Df_U(C) = UC + CU = 0, \qquad U^2 = I.$$

Notice that, in order for U to be non trivial, we require $\sigma(U) = \{-1, 1\}$. In order to apply Theorem 3.2.2 and Corollary 3.2.3, assume that we can solve the abstract Cauchy problem (if the equation were regular-which it isn't, one could simply apply results from [83])

$$Df_U(C) = UC + CU = 0, \qquad U^2 = I. \tag{3.33}$$

It follows that there exists $L \in \mathcal{B}(V)$ which solves the following system of homogeneous Sylvester equations

$$CL - LC = 0, \qquad UL - LU = 0, \qquad f(L) = g(C)$$

if and only if C is a commutator of two idempotents.

3.3.2 Connections to compact operators

Question 3. *For a bounded linear operator $A \in \mathcal{B}(V)$ given on a Banach space V, is there a bounded linear operator $B \in \mathcal{B}(V)$, such that $AB + BA$ is a compact operator? More generally, what conditions must hold for A and B, such that there exists an $X \in \mathcal{B}(V)$, making $AX - XB$ a compact operator, as in formula* (3.32)*?*

If $\sigma(A) \cap \sigma(-A) = \emptyset$, then for every compact operator $C \in \mathcal{C}(V)$, there exists a unique $B \in \mathcal{B}(V)$ such that $AB + BA = C$. However, if $\sigma(A) \cap \sigma(-A) \neq \emptyset$, then Corollary 3.2.5 gives an affirmative answer.

In general, for given operators A, B and X in $\mathcal{B}(V)$, when can we claim that they form a compact derivation, i.e. when is $AX - XB$ a compact operator? Formula (3.32) from Corollary 3.2.5 gives an answer to this question. Furthermore, let $\varphi : \Omega \to \mathbb{C}$ be an analytical function, defined in a region $\Omega \subset \mathbb{C}$ such that $\sigma(A)$ and $\sigma(B)$ are both contained in that region. Then the Spectral mapping theorem yields that

$$\sigma(A) \cap \sigma(B) \neq \emptyset \Rightarrow \sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset.$$

Thus Corollary 3.2.5 gives sufficient conditions for A, B and φ, in order for $\varphi(A)X - X\varphi(B)$ to be a compact operator, for some $X \in \mathcal{B}(V)$. This is very important for majorization theory and its applications, because there are numerous problems which concern comparing expressions $AX - XB$ and $\varphi(A)X - X\varphi(B)$ in various norms, see [9], [27], [55], [56] and [80]. Similarly to perturbation analysis conducted on matrices A and B in Chapter 2, it is very convenient to know when the said expressions are trace-class operators, Ky-Fan-k-class operators, Schatten-p-class operators etc. consult [9], [55], [56], [66], [67], [72], [80], [105] and rich references therein. Recall that each of the afore-mentioned classes of operators consists of operators which are necessarily compact operators, and every class has its own unitarily invariant norm (trace-norm, Ky-Fan-k-norm, Schatten-$p-$norm and so on). Ergo it is suitable to know under which conditions expressions $AX - XB$ and $\varphi(A)X - X\varphi(B)$ are compact, trace-class, Ky-Fan-$k-$class, Schatten $p-$class and so on.

Recall that two bounded operators T and L, defined on two different Banach spaces V_1 and V_2 respectively, are said to be *equivalent after extension*, if they can both be extended to $V_1 \dot{+} V_2$, $\widetilde{T} := T + I_{V_2}$, $\widetilde{L} = I_{V_1} + L$, and in addition satisfy $\widetilde{T} = U\widetilde{L}V$, for some bounded and invertible linear operators U and V on $V_1 \dot{+} V_2$. Specially, if $V_1 = \{0\}$ or $V_2 = \{0\}$, then T and L, which are equivalent after extension, are said to be equivalent after one-sided extension. Note that if T and L are compact operators which are equivalent after extension, then $\widetilde{T}$ and $\widetilde{L}$ are Fredholm operators, which means that they have a Riesz point in their spectra.

Question 4. *When are compact operators L and T equivalent after extension?*

It suffices to find an invertible U such that $\widetilde{L}U = U\widetilde{T}$. This is now solvable by Theorem 3.2.1, Theorem 3.2.3 or Theorem 3.2.6, Corollary 3.2.4 or Corollary 3.2.1. A necessary condition was obtained in [48], where operator ideals are constructed, which are similar to the operator algebra $\mathcal{A}_{AXB}$ introduced by the author in [25].

Definition 3.3.1. [48, Defitinion 2.1.] Let $T \in \mathcal{B}(V_1, V_2)$ be a Banach space operator. For any Banach spaces Z_1 and Z_2, we define

$$\mathcal{I}_T(Z_1, Z_2) := \bigcup_{n\in\mathbb{N}} \left\{ \sum_{j=1}^{n} R_j T R'_j : \qquad R_j \in \mathcal{B}(V_2, Z_2), \;\; R'_j \in \mathcal{B}(Z_1, V_1) \right\}.$$

Denote by $\mathcal{I}_T$ the (proper) class $\bigcup_{Z_1,Z_2} \mathcal{I}_T(Z_1, Z_2)$, and refer to $\mathcal{I}_T$ as the operator ideal generated by T.

Theorem 3.3.2. [48, Theorem 2.5.] Let $T \in \mathcal{B}(V_1)$ and $L \in \mathcal{B}(V_2)$ be two compact operators defined on Banach spaces V_1 and V_2, resp. If T and L are equivalent after extension, then $\mathcal{I}_T = \mathcal{I}_L$.

Remark. Necessary condition from Theorem 3.3.2, $\mathcal{I}_L = \mathcal{I}_T$, agrees with Corollary 3.2.1.

3.3.3 LTI systems and Schur coupling for operators in Banach spaces

One of the main applications of Sylvester equations is in systems engineering and modeling of linear time-invariant (LTI) systems (see [5], [7], [40] and

[88]). For now, we restrict our attention to finite scalar matrices, as that is the most exploited case in the systems engineering. Every linear time-invariant system, continuous in time, can be represented by a generalized state-space model of the form

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \geq 0, \\ y(t) = Cx(t) + Du(t), & t \geq 0, \\ x(0) = 0, \end{cases} \tag{3.34}$$

where t represents the time parameter, while variables u, x, y, x' and matrices A, B, C and D have a direct physical interpretation:

- $x(\cdot)$ is the state vector with n dimensions, $x(\cdot) \in \mathbb{R}^n$;
- $y(\cdot)$ is the output vector with q dimensions, $y(\cdot) \in \mathbb{R}^q$;
- $u(\cdot)$ is the input vector with p dimensions, $u(\cdot) \in \mathbb{R}^p$;
- A is the state (or system) matrix, $A \in \mathbb{R}^{n\times n}$;
- B is the input matrix, $B \in \mathbb{R}^{n\times p}$;
- C is the output matrix, $C \in \mathbb{R}^{q\times n}$;
- D is the feedthrough (or feedforward) matrix $D \in \mathbb{R}^{q\times p}$. If the system does not have a direct feedthrough, then $D = 0$;
- $x'(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}x(t)$.

Matrices A, B, C and D are constant if the system is time-invariant. Otherwise, they are time dependent. The matrix $G_1(\lambda) := D + C(\lambda - A)^{-1}B$, where $\lambda \in \rho(A)$, resembles a state space realization of the transfer function $G_1(\cdot)$ of the given system.

Often, matrices A, B, C and D are sparse and in those cases the initial system (3.34) is replaced by a so-called minimal system (see e. g. [7]), which is:

- Controllable[1], that is,

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \ldots, & A^{n-1}B \end{bmatrix} = n;$$

[1] The state controllability condition implies that it is possible by admissible inputs to steer the states from any initial value to any final value within some finite time window.

- Observable[2], that is,

$$\operatorname{rank}\begin{bmatrix} C & CA & CA^2 & \dots & CA^{n-1} \end{bmatrix}^T = n;$$

- Has exactly the same transfer function $G_1(\cdot)$ as the starting system (3.34).

Case 1. If $D = 0$, then the system can be solved via Luenenberg's scheme (see [5], [7], [40] and [88]): namely, if we introduce another dynamical system

$$z'(t) = Hz(t) + Fy(t) + Gu(t), \quad z(0) = z^0, \tag{3.35}$$

such that (A, C) is observable, and (H, F) is controllable, then there exists a unique full-rank solution to

$$HX - XA = -FC,$$

as this reduces to a regular matrix Sylvester equation. In addition, if G from (3.35) allows the decomposition $G = XB$, then $z : [0, +\infty) \to \mathbb{R}^n$, which is a solution to (3.35), is the state observer for (3.34), meaning that, for some nonsingular $Z \in \mathbb{R}^{n\times n}$ and the state vector $x(t)$ for (3.34), we have

$$\|z(t) - Zx(t)\| \longrightarrow 0, \qquad t \to +\infty.$$

Case 2. On the other hand, if $D \neq 0$, then LTI system (3.34) can be analyzed via Schur coupling. At this point, we can generalize the system (3.34) in a way that A, B, C and D are bounded linear operators on appropriate Banach spaces, as this scenario is also covered by Schur coupling and the arising singular Sylvester equations with bounded linear operators as their entries. In general, Schur complements for Banach space operators, Schur coupling, and their applications to LTI systems have been studied, among others, in [2], [5], [13], [21], [48], [49] and [63]. Introduce an operator matrix M,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

and assume that A is invertible. Then $G_1 := D - CA^{-1}B$ is in fact the first Schur complement of operator A in matrix M, often denoted as $A_{/M}$ or $W_1(M)$. Similarly, if D is invertible, then $G_2 := A - BD^{-1}C$ is the second Schur complement for operator D in matrix M, often denoted by $W_2(M)$.

[2] Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs.

Schur complements are important, because they are involved in the Schur decomposition of a given operator matrix:

$$M = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \cdot \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}.$$

Conversely, given bounded linear operators L and T are *Schur coupled* if there exists an operator matrix M, with bounded and invertible A and D, such that $L = W_1(M)$ and $T = W_2(M)$. Hence it is very important to answer the following question:

Question 5. *When are two given operators L and T Schur coupled?*

In the following, we formulate a result from [49], which answers Question 5. Notice that the answer is surprisingly similar to the one for Question 4, and is once again solved by the results obtained in this chapter, particularly, by Theorem 3.2.1, Theorem 3.2.3, Theorem 3.2.6, Corollary 3.2.4 or Corollary 3.2.1. Recall that an operator $L \in \mathcal{B}(V)$ is said to be inessential, if LT is a Riesz operator (quasi-nilpotent in the Calkin algebra), for every $T \in \mathcal{B}(V)$, see [106].

Theorem 3.3.3. [49, Theorem 1.1.] Let $L \in \mathcal{B}(V_1)$ and $T \in \mathcal{B}(V_2)$ be inessential operators. The following statements are equivalent:

(a) L and T are Schur coupled;

(b) L and T are equivalent after extension;

(c) L and T are equivalent after one-sided extension.

Chapter 4

The closed operator case

4.1 The „regular” unbounded equation

When the operators A, B and C are not bounded, a different analysis is required. To start, their domains are compromised because the operators are not automatically defined on the entire spaces. Consistency conditions always yield the operators A and B to be densely defined on the corresponding Banach spaces, and that $\mathcal{D}_B \subset \mathcal{D}_C$. In that sense, the Sylvester equation in the unbounded setting has the form

$$AXu - XBu = Cu, \qquad u \in \mathcal{D}_B. \tag{4.1}$$

The problem when A and $-B$ are given as generators of analytical semigroups and $\mathcal{C}_0-$semigroups, has been studied in [60] and [83]. These results provide a nice way to extend solvability of the equation (4.1) to quantum mechanics (see [70], [99] and [101]) and abstract differential equations, see [79]. For example, it is very convenient to note that the abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\,t} X(t) = AX(t) - X(t)B \\ X(0) = C \end{cases} \tag{4.2}$$

is uniquely solved by

$$X = \mathrm{e}^{tA}\, C\, \mathrm{e}^{-tB}, \tag{4.3}$$

and that solution is uniformly exponentially stable when the semigroup generated by $-B$ and the semigroup generated by A have negative growth limits (see below). Furthermore, the abstract inhomogeneous problems

$$u'(t) = Au(t) + f(t), \quad u(0) = x_0 \tag{4.4}$$

can be rewritten as

$$U'(t) = \begin{bmatrix} A & \delta_0 \\ 0 & \frac{d}{dt} \end{bmatrix} U(t), \quad U(0) = (x_0, f), \tag{4.5}$$

observed on the space $V_2 \times \mathcal{F}$, where $\mathcal{F}$ is the space of V_2-valued functions, defined on $\mathbb{R}$, with $\delta_0(f) = f(0)$ being an unbounded operator in $\mathcal{F}$. Recall from Chapter 1, that if the operator matrix in (4.5) is diagonalizable, then Cauchy problem (4.4) can be drastically simplified. Because of these, and several other reasons, it is important to study the Sylvester equation (4.1) when A, B and C are unbounded operators. Throughout this chapter, a strong background in closed operators and spectral theory is required. An interested reader is referred to books [37], [43], [59], [86], [99], [100] and [102].

Theorem 4.1.1. [60] Let A and $-B$ be generators of $\mathcal{C}_0-$semigroups $(T(t))$ and $(S(t))$, $t \geq 0$, on Banach spaces V_2 and V_1, respectively and let C be an operator from V_1 to V_2. Let

$$Q(t) : \mathcal{D}_B \subset V_1 \to V_2 : \quad Q(t)(f) := T(t)CS(t)(f), \quad t \geq 0,$$

$$R(t) : \mathcal{D}_B \subset V_1 \to V_2 : \quad R(t)(f) := -\int_0^t Q(s)f ds, \quad t \geq 0.$$

Assume that:

1. The weak topology closure of $\{Q(t)f\}_{t\geq 0}$ contains zero, for every $f \in \mathcal{D}_B$;
2. $R(t)$ has a continuous extension to a bounded linear operator, for every $t \geq 0$ and the family $\{R(t)\}_{t\geq 0}$ is relatively compact with respect to the weak topology.

Then the equation (4.1) has a bounded solution. Contrary, if (4.1) has a bounded solution then $R(t)$ is bounded, for every $t \geq 0$. Furthermore, if for every bounded linear operator Y from V_1 to V_2 the operator $T(t)YS(t)$ converges towards zero when $t \to +\infty$ in the weak (resp. strong, uniform) operator topology, then the solution X to the equation (4.1) is unique and $R(t)$ converges to X in the weak (resp. strong, uniform) topology.

Definition 4.1.1. [60] For the semigroup $(T(t))_{t\geq 0}$ generated by an operator A, the value $w(A)$ represents the semigroups growth limit, and is provided as

$$w(A) = \inf\{\lambda \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\| \leq M\, \mathrm{e}^{\lambda t}, \quad \forall t \geq 0\}.$$

If $w(A) < 0$, then the semigroup $(T(t))_{t\geq 0}$ is called uniformly exponentially stable.

Theorem 4.1.2. [60] Let $w(A)+w(-B) < 0$ and assume the family $(R(t))_{t\geq 0}$ from the Theorem 4.1.1 to be uniformly exponentially stable. Then the equation (4.1) has a unique bounded solution.

4.2 The singular equation

Contrary to the regular equation, singular eigenproblems that stem from Sturm-Liouville theory, partial differential equations, quantum mechanics and mathematical physics often yield the corresponding operator equations to be singular. In what follows, we solve the unbounded singular Sylvester equation (4.1) in detail. Afterwards, we illustrate our results on explicit examples where such equations emerge. The author obtained these results in his individual paper [24] and in joint work with his PhD mentor [29].

In this section we assume the spaces V_1 and V_2 to be Banach spaces and $A \in L(V_2)$ and $B \in L(V_1)$ to be closed linear operators with non-empty point spectra. We introduce a weak solution to the given inhomogeneous and homogeneous equation.

Definition 4.2.1. [29] Linear operator X is a weak solution to the equation (4.1) if

1. $\mathcal{D}_C \cap \mathcal{D}_B \neq \emptyset$.
2. $\mathcal{D}_X \subset \mathcal{D}_B \cap \mathcal{D}_C$, $\mathcal{R}(X) \subset \mathcal{D}_A$ and $\mathcal{D}_X$ is $B-$invariant subspace of V_1.
3. For every $u \in \mathcal{D}_X$ $(AX - XB)u = Cu$.

Definition 4.2.2. [29] Linear operator X is a weak solution to the homogeneous equation (4.1) if

1. $\mathcal{D}_X \subset \mathcal{D}_B$, $\mathcal{R}(X) \subset \mathcal{D}_A$ and $\mathcal{D}_X$ is $B-$invariant subspace of V_1.
2. for every $u \in \mathcal{D}_X$ $AX(u) = XB(u)$.

Remark. If A, B, C and X are bounded linear operators then $\mathcal{D}_B = \mathcal{D}_C = \mathcal{D}_X = V_1$ and $\mathcal{R}(X) \subset \mathcal{D}_A = V_2$ and V_1 is a $B-$invariant subspace of V_1. In other words, the previous definitions of a weak solution extend the definitions of a solution in the bounded operator case.

For convenience and simpler calculations, the results regarding solvability of the unbounded Sylvester equation are broken down into two cases, one regarding the homogeneous equation, and the other regarding the inhomogeneous equation.

4.2.1 The homogeneous equation

Let V be an arbitrary vector space over the field F and let I be an arbitrary index set. The set of different vectors $\{a_i\}_{i\in I}$ from V is said to be a Hamel or an algebraic basis for V, if every vector $a \in V$ can be represented as a unique finite linear combination of vectors from the family $\{a_i\}_{i\in I}$:

$$(\forall a \in V)\ (\exists! n \in \mathbb{N})\ (\exists! a_1, \dots, a_n \in \{a_i\}_{i\in I})\ (\exists! \alpha_1, \dots, \alpha_n \in F)\ a = \sum_{k=1}^{n} \alpha_k a_k.$$

It is known that every vector space has a Hamel basis. Unique representation of every vector from V, in terms of the Hamel basis $\{a_i\}_{i\in I}$ of V, implies that $\{a_i\}_{i\in I}$ are linearly independent vectors.
All Hamel bases of the same vector space have the same cardinality. Hence the term "dimension" of the given space can be extended to infinitely dimensional vector spaces. $Lin(S)$ or $span(S)$ stands for a lineal (finite linear span) over the set of vectors S.

At this point we assume V_1 and V_2 to be linear (vector) spaces and $A \in L(V_2)$, $B \in L(V_1)$ to be both one-to-one (injective). We will return to the case of closed operators in Banach spaces later. We also assume that there exists $W < \mathcal{D}_B < V_1$ which is a $B-$invariant subspace of V_1. Let $\mathcal{U} = \{u_i\}_{i\in I}$ be an algebraic basis of W. Further, since $\{u_i\}_{i\in I}$ is a basis for W, it follows that $\{B(u_i)\}_{i\in I}$ is a basis for $B(W)$. Operator B is injective, so $\mathrm{card}(\{u_i\}_{i\in I}) = \mathrm{card}(\{B(u_i)\}_{i\in I})$. Therefore, there exists a linear bijection $T_W : \{B(u_i)\}_{i\in I} \to \{u_i\}_{i\in I}$, such that for each $i \in I$ there exists unique $j \in I$ so that $T_W B(u_j) = u_i$.

For every $u \in \mathcal{U}$, we define the class of u as

$$[u] = \{(T_W B)^n(u) : n \in \mathbb{Z}\}.$$

Now $\{[u_i] : i \in I\}$ forms a partition of $\mathcal{U}$. We define a binary operation $\cdot_B$ on every $[u]$, $u \in \mathcal{U}$. Put $[\mathcal{U}] := \{[u_i] : i \in I\}$. For every $[u] \in [\mathcal{U}]$, fix one $\widehat{u} \in [u]$. It follows that

$$[\widehat{u}] = \{(T_W B)^n(\widehat{u}) : n \in \mathbb{Z}\} = [u],$$

so $\widehat{u}$ can be treated as the generating element of its equivalence class $[u]$. Define $\cdot_B : [u] \times [u] \to [u]$:

$$(\forall n, m \in \mathbb{Z})\ (T_W B)^n(\widehat{u}) \cdot_B (T_W B)^m(\widehat{u}) := (T_W B)^{n+m}(\widehat{u}).$$

Lemma 4.2.1. [29, Lemma 2.1.] Let $u \in \mathcal{U}$.

1. If $[u]$ has a finite number of different elements, say k of them, then $([u], \cdot_B)$ is isomorphic to $(\mathbb{Z}_k, +_k)$;
2. If $[u]$ has infinitely many different elements, then $([u], \cdot_B)$ is isomorphic to $(\mathbb{Z}, +)$.

Proof.

1. Let $T_W B(x) \in [u]$ be the generating element for $[u]$. Define $h(x) := 0$. For every $n \in \mathbb{N}$, define $h((T_W B)^n(x)) := n \mod k$. If $[u]$ has $k \in \mathbb{N}$ different elements, then $h([u]) = \{0, 1, \ldots, k-1\}$, and the isomorphism $([u], \cdot_B) \mapsto (\mathbb{Z}_k, +_k)$ is now obvious.
2. Assume that $[u]$ has infinitely many different elements and let x be its generating element. Define $h(x) := 0$, and for every $m \in \mathbb{Z}$, $h((T_W B)^m(x) := m$. Now $h([u]) = \{\ldots, -|m|, \ldots, -1, 0, 1, \ldots, |m|, \ldots\}$, for every $m \in \mathbb{Z}$, and the isomorphism $([u], \cdot_B) \mapsto (\mathbb{Z}, +)$ is now obvious.

□

Let $Z < \mathcal{D}_A < V_2$ be an $A-$invariant subspace of V_2 and let $\mathcal{V} = \{v_j\}_{j \in J}$ be an algebraic basis for Z. Let $S_Z \in L(A(Z), Z)$ be a bijective linear operator, such that $S_Z(\mathcal{V}) \subset (\mathcal{V})$. For every $v \in \mathcal{V}$, define $[v]$ using $S_Z A$, in the analogous way we defined $[u]$, using $T_W B$, when $u \in \mathcal{U}$. For every $[v]$ define $\cdot_A$ using $S_Z A$ in the analogous way we defined $\cdot_B$ using $T_W B$ for every $[u]$.

Corollary 4.2.1. [29, Corollary 2.1.] For every $v \in \mathcal{V}$, $([v], \cdot_A)$ is isomorphic to exactly one of the elements in $\{(\mathbb{Z}_k, +_k) : k \in \mathbb{N}\} \cup \{(\mathbb{Z}, +)\}$.

Remark. The aforementioned isomorphisms between elements of $\{([u], \cdot_B), ([v], \cdot_A)\}$ and elements of $\{(\mathbb{Z}, +), (\mathbb{Z}_k, +_k) : k \in \mathbb{N}\}$ will be denoted as "$\cong$".

Theorem 4.2.1. [29, Theorem 2.1.] (*The shifted injective homogeneous equation*) Let V_1 and V_2 be vector spaces and let $B \in L(\mathcal{D}_B, V_1)$, $A \in L(\mathcal{D}_A, V_2)$ be one-to-one linear operators, where $\mathcal{D}_B \subset V_1$ and $\mathcal{D}_A \subset V_2$, and let $W \subset \mathcal{D}_B$ be a $B-$invariant subspace of V_1 and let $Z \subset \mathcal{D}_A$ be an $A-$ invariant subspace of V_2. Let T_W and S_Z be provided as in the previous discussion. Then there exists a linear operator $X \in L(W, Z)$ which is a weak solution to the equation

$$XT_W B = S_Z AX, \tag{4.6}$$

defined on W.

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be the algebraic bases for W and Z, respectively, on which T_W and S_Z are respectively defined.

Step 1. For $u \in \mathcal{U}$, we define $X(u)$ as described below.
Assume that $([u], \cdot_B) \cong (\mathbb{Z}, +)$. If there is some $v \in \mathcal{V}$ such that $([v], \cdot_A) \cong (\mathbb{Z}, +)$, then

$$X(u) := v.$$

Further, for every $m \in \mathbb{Z}$, we define

$$X((T_W B)^m(u)) := (S_Z A)^m(v).$$

Notice that X is a correctly defined map on u because $T_W B$ and $S_Z A$ are injective. Therefore, if $u_1 = u$ then

$$X(u) = X(u_1) = v$$

and

$$XT_W B(u_1) = XT_W B(u) = S_Z A(v).$$

But then for every $m \in \mathbb{Z}$ we have

$$\begin{aligned} X((T_W B)^m(u)) &= (S_Z A)^m(v) = (S_Z A)^m(X(u_1)) \\ &= (S_Z A)^{m-1}(S_Z A)X(u_1) \\ &= (S_Z A)^{m-1}X(T_W B(u_1)) = \ldots = X((T_W B)^m(u_1)), \end{aligned}$$

so X is correctly defined on the entire class $[u]$.

If there is no $v \in \mathcal{V}$ such that $([v], \cdot_A) \cong (\mathbb{Z}, +)$, then $X([u]) := 0_{V_2}$.

Either way, we verify that $S_Z A(X(p)) = X(T_W B(p))$, $\forall p \in [u]$.
Now assume $([u], \cdot_B) \cong (\mathbb{Z}_k, +_k)$, for some $k \in \mathbb{N}$. If there exists $v \in \mathcal{V}$ such that $([v], \cdot_A) \cong (\mathbb{Z}_k, +_k)$, then

$$X(u') := v',$$

where u' is the generating element of $[u]$, and v' is the generating element of $[v]$. Further, for every $m \in \{1, \ldots, k-1\}$, define

$$X((T_W B)^m(u')) := (S_Z A)^m(v').$$

X is a correctly defined map on u' because $T_W B$ and $S_Z A$ are injective. Therefore, if $u_1' = u'$ then

$$X(u') = X(u_1') = v'$$

and

$$X(T_W B(u_1')) = X(T_W B(u')) = S_Z A(v').$$

But then for every $m = \overline{1, k-1}$ we have

$$\begin{aligned} X((T_W B)^m(u')) &= (S_Z A)^m(v') = \\ &= (S_Z A)^m(X(u_1')) = (S_Z A)^{m-1}(S_Z A)X(u_1') = \\ &= (S_Z A)^{m-1}X(T_W B(u_1')) = \ldots = X((T_W B)^m(u_1')), \end{aligned}$$

so X is correctly defined on the entire class $[u]$.
If there is no $v \in \mathcal{V}$ such that $([v], \cdot_A) \cong (\mathbb{Z}_k, +_k)$, then $X([u]) := 0_{V_2}$.

Either way, we verify that $S_Z A(X(p)) = X(T_W B(p))$, $\forall p \in [u]$.

Step 2. For any given $u \in W$, with the unique algebraic representation in $\mathcal{U}$

$$u = \sum_{k=1}^{n} \alpha_k u_k, \quad u_k \in \mathcal{U}, \quad \alpha_k \in \mathbb{C}, \quad k = \overline{1, n}, \quad n \in \mathbb{N},$$

define

$$X(u) := \sum_{k=1}^{n} \alpha_k X(u_k).$$

X is correctly defined for each u_k, $k = \overline{1, n}$, and u is uniquely represented via $\{u_1, \ldots, u_n\}$, hence X is well-defined in u. Now part 1. of the proof implies that

$$\begin{aligned} S_Z AX(u) &= S_Z AX(\sum_{k=1}^{n} \alpha_k u_k) = \sum_{k=1}^{n} \alpha_k (S_Z AX(u_k)) \\ &= \sum_{k=1}^{n} \alpha_k (XT_W B(u_k)) = XT_W B(\sum_{k=1}^{n} \alpha_k u_k) = XT_W B(u). \end{aligned}$$

Combining observations from Steps 1. and 2. we conclude that X is a well defined linear operator from W to Z and is a solution to (4.6). Also we have that $\mathcal{D}_X = W \subset \mathcal{D}_{T_W B}$, $\mathcal{R}(X) = Z \subset \mathcal{D}_{S_Z A}$ and W is $T_W B-$invariant, so X is indeed a weak solution. $\square$

Remark. The previous theorem provides a solution to the shifted injective equation (4.6). Notice that the proof only required existence of invariant subspaces and the operators to be one-to-one. So Theorem 4.2.1 holds in general linear spaces, for given one-to-one operators and the corresponding invariant subspaces.

Remark. The solution X is not uniquely determined in the sense that it depends on the choice of algebraic bases and it maps one equivalence class onto another equivalence class, where the former is isomorphic to the latter. Theoretically, there could be an infinite number of different equivalence classes, which are all isomorphic to the fixed one. Therefore, there could be infinitely many different solutions to the equation (4.6).

We now return to our observation in Banach spaces and closed operators.

Theorem 4.2.2. [29, Theorem 2.3.] (*The homogeneous equation*) Let V_1 and V_2 be given Banach spaces, $B \in L(V_1)$ and $A \in L(V_2)$ closed operators, such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ have topological complements in V_1 and V_2, respectively. If $(\sigma_p(B) \cap \sigma_p(A)) \setminus \{0\} \neq \emptyset$ then the homogeneous equation

$$AX - XB = 0 \tag{4.7}$$

has a non-trivial weak solution.

Proof. Since B is a closed operator on V_1, then $\mathcal{N}(B)$ is a closed subspace of V_1. Since $\mathcal{N}(B)$ has a topological complement in V_1, V_1 can be split into a direct sum:

$$V_1 = \mathcal{N}(B) \dot{+} V_1'.$$

In other words, $(\forall u \in \mathcal{D}_B)$ $(\exists! u_1 \in \mathcal{N}(B), u_2 \in V_1' \cap \mathcal{D}_B)$ such that $u = u_1 + u_2$. Put $V_1(B) := V_1' \cap \mathcal{D}_B$. In that sense, define $B_1 : V_1(B) \to V_1$ as: $B_1(u_2) := B(u)$. This way, B_1 is one-to-one, so $0 \notin \sigma_p(B_1)$. Note that $\sigma_p(B) \setminus \{0\} \equiv \sigma_p(B_1)$.
Assume the same thing is done with A and the Banach space V_2: $V_2 = \mathcal{N}(A) \dot{+} V_2'$, put $V_2(A) := \mathcal{D}_A \cap V_2'$ and $A_1 : V_2(A) \to V_2$ defined as $A_1(v_2) := A(v)$, whenever $v \in \mathcal{D}_A$ and $v = v_1 + v_2$, $v_1 \in \mathcal{N}(A)$ and $v_2 \in V_2(A)$. Now A_1 is one-to-one and $0 \notin \sigma_p(A_1)$. Also note that $\sigma_p(A) \setminus \{0\} = \sigma_p(A_1)$. Now conditions of the theorem yield that $\sigma_p(A_1) \cap \sigma_p(B_1) = \sigma \neq \emptyset$.
Let $\{\lambda_i\}_{i \in I} = \sigma$, for some index set I, where $\lambda_i = \lambda_j \Rightarrow i = j$. Let $u_i \in \mathcal{D}_{B_1}$ and $v_i \in \mathcal{D}_{A_1}$ such that $B_1 u_i = \lambda_i u_i$ and $A_1 v_i = \lambda_i v_i$, whenever $i \in I$. It follows that $\{u_i\}_{i \in I}$ and $\{v_i\}_{i \in I}$ are families of linearly independent vectors. Now put $\mathcal{U} := \{u_i\}_{i \in I}$ and $\mathcal{V} = \{v_i\}_{i \in I}$. It is now obvious that $W := Lin(\mathcal{U})$ is a B_1-invariant subspace of V_1 and $Z := Lin(\mathcal{V})$ is an A_1- invariant subspace of V_2.
For each $i \in I$ define bounded linear operators T_i and S_i on $Lin(u_i)$ and $Lin(v_i)$ respectively as

$$(\forall u \in Lin(u_i)) \;\; T_i(u) := \lambda_i^{-1} u, \text{ and } (\forall v \in Lin(v_i)) \;\; S_i(v) := \lambda_i^{-1} v.$$

Finally put

$$T_W(u_i) := T_i(u_i),\ S_Z(v_i) := S_i(v_i).$$

Since $Lin(u_i) \cap Lin(u_j) = \{0\}$ whenever $i \neq j$, it follows that T_W is a correctly defined operator on $\sum_{i\in I} Lin(u_i)$ (which is an eigenspace for B_1 and therefore for B). Analogously, S_Z is a correctly defined operator on $\sum_{i\in I} Lin(v_i)$. Now all conditions of Theorem 4.2.1 are satisfied, so there exists a linear operator X_1 from W to Z such that

$$X_1 T_W B_1 = S_Z A_1 X_1$$

holds.

Further, we see that $(\overline{[u_i]}, \cdot_{B_1}) \cong (\overline{[v_i]}, \cdot_{A_1}) \cong (\mathbb{Z}_2, +_2)$ for every $i \in I$. For $u \in W$, we have:

$$\begin{aligned}
&S_Z A_1 X_1(u) = S_Z A_1 X_1(\sum_{k=1}^{n} \alpha_k u_k) = \sum_{k=1}^{n} \alpha_k S_Z A_1 X_1(u_k) \\
&= \sum_{k=1}^{n} \alpha_k S_Z A_1(v_k) = \sum_{k=1}^{n} \alpha_k S_Z(\lambda_k v_k) = \sum_{k=1}^{n} \alpha_k v_k = \sum_{k=1}^{n} \alpha_k X_1(u_k) = \\
&= \sum_{k=1}^{n} \alpha_k \lambda_k X_1(\frac{1}{\lambda_k} u_k) = \sum_{k=1}^{n} \alpha_k \lambda_k X_1(T_W(u_k)) = \sum_{k=1}^{n} \alpha_k X_1 T_W(\lambda_k u_k) = \\
&= \sum_{k=1}^{n} \alpha_k X_1 T_W B_1(u_k) = X_1 T_W B_1(\sum_{k=1}^{n} \alpha_k u_k) = X_1 T_W B_1(u).
\end{aligned} \tag{4.8}$$

Since S_Z and T_W act in the same way on the corresponding spaces, it directly follows that:

$$\begin{aligned}
&A_1 X_1(u) = A_1 X_1(\sum_{k=1}^{n} \alpha_k u_k) = \sum_{k=1}^{n} \alpha_k A_1 X_1(u_k) = \sum_{k=1}^{n} \alpha_k A_1(v_k) \\
&= \sum_{k=1}^{n} \alpha_k \lambda_k v_k = \sum_{k=1}^{n} \alpha_k \lambda_k X_1(u_k) = \sum_{k=1}^{n} \alpha_k X_1(\lambda_k u_k) \\
&= \sum_{k=1}^{n} \alpha_k X_1(B_1(u_k)) = X_1 B_1(\sum_{k=1}^{n} \alpha_k u_k) = X_1 B_1(u).
\end{aligned} \tag{4.9}$$

Therefore, $X_1 \in L(W, Z)$ where $\mathcal{D}_X = W < \mathcal{D}_B$ and $\mathcal{R}(X) = Z < \mathcal{R}(A)$, so X is a weak solution to the equation:

$$A_1 X_1 = X_1 B_1.$$

Let $N \in L(\mathcal{N}(B), \mathcal{N}(A))$ be arbitrary. Put

$$X = N \dotplus X_1 = \begin{bmatrix} N & 0 \\ 0 & X_1 \end{bmatrix}, \, X \in L\left(\mathcal{D}_X, (\mathcal{N}(A) \dotplus Z)\right), \mathcal{D}_X = \mathcal{N}(B) \dotplus W.$$

We see that X is a weak solution to (4.7) (domains of X and B intersect, as the images of X and A do. Further, $\mathcal{N}(B) \dotplus W$ is a $B-$invariant subspace of V_1). □

Remark. When constructing the injective operators A_1 and B_1, one encounters the problem of losing information about the null-spaces of A and B. However, this property is not as restrictive as it may seem at the first sight. In particular, suppose that

$$\{0\} = \sigma_p(A) \cap \sigma_p(B).$$

Then $\mathcal{N}(B)$ and $\mathcal{N}(A)$ are the corresponding eigenspaces of B an A, respectively, which correspond to the shared eigenvalue $\lambda = 0$. But then an arbitrary operator $N \in L(\mathcal{N}(B), \mathcal{N}(A))$ (provided in the proof of Theorem 4.2.2) is the desired map that maps the 0-eigenspace of B into the corresponding 0-eigenspace of A. In other words, one could simply put

$$X := \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix}.$$

However, this case is somewhat irrelevant because both XB and AX vanish on $\mathcal{N}(B)$. Nevertheless, Theorem 4.2.2 holds even if

$$\sigma_p(A) \cap \sigma_p(B) = \{0\}.$$

This assertion agrees with the classification of solutions conducted in the matrix case, in particular, the eigenproblem (2.5) in Chapter 2.

Remark. Theorem 4.2.2 provides results which concern Banach spaces and closed operators defined on them (defined on their subsets, to be precise), whose point spectra intersect. However, the only reason why we required A and B to be closed operators was to ensure closedness of the null spaces $\mathcal{N}(A)$ and $\mathcal{N}(B)$. Note that this could be weakened in the following sense: let V_1 and V_2 be Banach spaces and let $B \in L(V_1)$ and $A \in L(V_2)$ be (arbitrary) linear operators such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ are closed subsets of V_1 and V_2, respectively, and have topological complements in the corresponding vector spaces. If $\sigma_p(A) \cap \sigma_p(B) \neq \emptyset$, then the same statements from Theorem 4.2.2 hold.

Recall that every closed subspace M of a given Hilbert space $\mathcal{H}$ has a topological complement N. Furthermore, N can be provided such that M and N form an orthogonal sum, i. e. $\mathcal{H} = M \oplus N$.

Corollary 4.2.2. [29, Corollary 2.2] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $A \in L(\mathcal{H}_2)$ and $B \in L(\mathcal{H}_1)$ closed operators. If $\sigma_p(A) \cap \sigma_p(B) \neq \emptyset$ then the homogeneous equation (4.7) has a non-trivial weak solution.

If we restrict the previous analysis to finite dimensional spaces, we obtain the same results as in Chapter 2:

Corollary 4.2.3. [29, Corollary 2.3.] Let $K \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be vector spaces over the field K, $\dim(\mathcal{H}_1) = m$, $\dim(\mathcal{H}_2) = n$ and let $A \in M_n(K)$ and $B \in M_m(K)$ be square matrices with entries and spectra in K. If A and B share some eigenvalues, then the matrix equation $AX = XB$ has a non-trivial solution. Further, that solution (as proved in the Chapter 2) must be in such a form that it maps the appropriate eigenspaces of B into the appropriate eigenspaces of A, where the generating eigenvectors correspond to the shared eigenvalues.

The next example concerns the case where A, B and C are matrices, but it is solved with Theorem 4.2.2.

Example 4.2.1. [29, Example 2.1.] Let A, B be some linear operators such that, according to the standard basis $\{e_1, e_2, e_3\}$ for V_1 and $\{E_1, E_2\}$ for V_2, appropriate matrices are $A = \lambda I_2$ and $B = \lambda I_3$ for some $\lambda \neq 0$ (the case where $\lambda = 0$ is trivial). We have $\sigma(A) \cap \sigma(B) = \{\lambda\}$ and the eigenvectors corresponding to the eigenvalue λ are precisely $\{e_1, e_2, e_3\}$ with respect to B and $\{E_1, E_2\}$ with respect to A.
Therefore, $\mathcal{U} = \{e_1, e_2, e_3\}$, $W = Lin(\mathcal{U}) = V_1$ and $\mathcal{V} = \{E_1, E_2\}$, $Z = Lin(\mathcal{V}) = V_2$. Now, $T_W := 1/\lambda \cdot \pi_3$ and $S_Z := 1/\lambda \cdot \pi_2$, where π_n denotes some $n \times n$ permutation matrix (recall that there are $n!$ of them), for $n \in \{2, 3\}$.
Indeed, $\mathcal{V} = \{E_1, E_2\}$, $A(E_1) = \lambda E_1$, $A(E_2) = \lambda E_2$, and S_Z is a matrix which maps either $\lambda E_1 \mapsto E_1$, $\lambda E_1 \mapsto E_2$, or $\lambda E_1 \mapsto E_2$, $\lambda E_2 \mapsto E_1$, so there are two different choices for the matrix S_Z :

$$S_Z^{(1)} = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; S_Z^{(2)} = \begin{bmatrix} 0 & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 0 \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Similar can be done for $\mathcal{U}$ and B, but there are now 6 possibilities for T_W :

$$\frac{1}{\lambda}\left\{I_3, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right\}.$$

Therefore, $T_W B = \pi_3,\ S_Z A = \pi_2$.
In order to find the appropriate equivalence classes and construct some solutions, we note the following:

$$\pi_{132}^2 = \pi_{213}^2 = \pi_{321}^2 = \pi_{123},\ \pi_{231}^2 = \pi_{312},\ \pi_{231}^3 = \pi_{123} = \pi_{312}^3,\ \pi_{312}^2 = \pi_{213},$$

where π_{ijk} denotes the permutation $\binom{1\ 2\ 3}{i\ j\ k}$, i.e. $\pi(1) = i,\ \pi(2) = j,\ \pi(3) = k$.
The equivalence classes $[e_1]_{ijk}$ of the vector e_1 when the permutation π_{ijk} is chosen are:

- $[e_1]_{123} = \{\pi_{123}^n(e_1) : n \in \mathbb{Z}\} = \{\pi_{123}(e_1)\} = \{e_1\}$,
- $[e_1]_{132} = \{\pi_{132}(e_1), \pi_{132}^2(e_1)\} = \{\pi_{132}(e_1), \pi_{123}(e_1)\} = \{e_1\}$,
- $[e_1]_{213} = \{\pi_{213}(e_1), \pi_{213}^2(e_1)\} = \{\pi_{213}(e_1), \pi_{123}(e_1)\} = \{e_2, e_1\}$,
- $[e_1]_{231} = \{\pi_{231}(e_1), \pi_{231}^2(e_1), \pi_{231}^3(e_1)\} = \{\pi_{231}(e_1), \pi_{312}(e_1), \pi_{123}\} = \{e_2, e_3, e_1\}$,
- $[e_1]_{312} = \{\pi_{312}(e_1), \pi_{312}^2(e_1), \pi_{312}^3(e_1)\} = \{\pi_{312}(e_1), \pi_{231}(e_1), \pi_{123}\} = \{e_3, e_2, e_1\}$,
- $[e_1]_{321} = \{\pi_{321}(e_1), \pi_{321}^2(e_1)\} = \{\pi_{321}(e_1), \pi_{123}(e_1)\} = \{e_3, e_1\}$.

The same should be done for both e_2, e_3 and E_1, E_2. All possible combinations (actually, the partitions of $\mathcal{U}$ and $\mathcal{V}$) are:

- $\{e_1\},\ \{e_2\},\ \{e_3\}$,
- $\{e_1\},\ \{e_2, e_3\};\ \{e_1, e_2\},\ \{e_3\};\ \{e_1, e_3\},\ \{e_2\}$,
- $\{e_1, e_2, e_3\}$,
- $\{E_1\},\ \{E_2\}$,
- $\{E_1, E_2\}$.

Now we can construct the solution X in each of the cases:

- $[e_i] = \{e_i\},\ i = \overline{1,3},\ [E_j] = \{E_j\},\ j = \overline{1,2}$: Each class is isomorphic to $(\mathbb{Z}_1, +_1)$ so $X(e_i)$ can be any E_j, and there are precisely 8 possibilities:

$$[E_1\ E_1\ E_1] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[E_1\ E_1\ E_2] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\vdots$$

$$[E_2\ E_2\ E_2] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

- $[e_i] = \{e_i\},\ i = \overline{1,3},\ [E_1] = [E_2] = \{E_1, E_2\}$: since $([e_i], \cdot_B) \equiv (\mathbb{Z}_1, +_1)$, but $([E_1], \cdot_A) \equiv (\mathbb{Z}_2, +_2)$, it follows that $X = 0$. The same is true when $[e_1] = \{e_1, e_2, e_3\}$ and $[E_1] = \{E_1, E_2\}$.

- $[e_1] = \{e_1\},\ [e_2] = [e_3] = \{e_2, e_3\},\ [E_1] = \{E_1\},\ E_2 = \{E_2\}$: In this case $X(e_1)$ is either E_1 or E_2, while $X(e_2) = X(e_3) = 0$. Therefore, solutions in this case are:

$$[E_1\ 0\ 0] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ [E_2\ 0\ 0] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

  The analogous is true for $\{e_1, e_2\},\ \{e_3\}$; and $\{e_1, e_3\},\ \{e_2\}$, respectively.

- $[e_1] = \{e_1\},\ [e_2] = [e_3] = \{e_2, e_3\},\ [E_1] = [E_2] = \{E_1, E_2\}$: In this case $X(e_1) = 0$ and $X(e_2) = X(e3)$ is either E_1 or E_2, so the solutions are

$$[0\ E_1\ E_1] = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ [0\ E_2\ E_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

♣

4.2.2 The inhomogeneous equation

We now return to the inhomogeneous equation (4.1), where A and B are closed operators on the corresponding Banach spaces V_2 and V_1, respectively, and $C \in L(V_1, V_2)$ is an arbitrary linear operator. Our main result on this topic, Theorem 4.2.3 below, concerns the case where the point spectra of A and B intersect. This theorem is proved by reducing the equation (4.1) to the homogeneous equation.

Lemma 4.2.2. [29, Lemma 2.2.] Let V_1 and V_2 be Banach spaces, B, $\Psi_1 \in L(V_1)$, A, $\Psi_2 \in L(V_2)$ closed operators and $C \in L(V_1, V_2)$, such that for every $u \in \mathcal{D}_{\Psi_1} \cap \mathcal{R}(\Psi_1) \cap \mathcal{D}_C$ we have $C(u) \in \mathcal{D}_{\Psi_2}$ and

$$\Psi_2 C(u) - C\Psi_1(u) = C(u). \tag{4.10}$$

Suppose $\mathcal{D}_{\Psi_2} \cap \mathcal{D}_A \neq \emptyset$ and $\mathcal{D}_{\Psi_1} \cap \mathcal{D}_B \neq \emptyset$. Finally, we require that $\mathcal{N}(A - \Psi_2)$ and $\mathcal{N}(B - \Psi_1)$ have topological complements and

$$(\sigma_p(A - \Psi_2) \cap \sigma_p(B - \Psi_1)) \setminus \{0\} \neq \emptyset. \tag{4.11}$$

Then for every $Y \in L(\mathcal{D}_Y, \mathcal{R}(Y))$, $\mathcal{D}_Y = \mathcal{D}_{\Psi_1} \cap \mathcal{R}(\Psi_1) \cap \mathcal{D}_C$, $\mathcal{R}(Y) \subset \mathcal{D}_{\Psi_2} \cap \mathcal{D}_A$, which is a weak solution to

$$\Psi_2 Y - Y\Psi_1 = 0, \tag{4.12}$$

the operator $X := Y + C$ is a weak solution to the inhomogeneous Sylvester equation (4.1) iff it is a weak solution to the homogeneous equation

$$(A - \Psi_2)X - X(B - \Psi_1) = 0. \tag{4.13}$$

Proof. Assume there exists Y such that the equation (4.12) is satisfied. Put $X := Y + C$. By applying Theorem 4.2.2, we see that (4.11) yields that there exists a non-trivial weak solution X to the equation (4.13). Finally, we verify that

$$\begin{aligned} (A - \Psi_2)X - X(B - \Psi_1) = 0 \Leftrightarrow \\ AX - XB = \Psi_2 X - X\Psi_1 \Leftrightarrow \\ AX - XB = \Psi_2 Y + \Psi_2 C - Y\Psi_1 - C\Psi_1 = C. \end{aligned}$$

□

Remark. Such Ψ_1 and Ψ_2 always exist, e.g. $\Psi_2 = (\alpha + 1)I$, $\Psi_1 = \alpha I$ for any $\alpha \in \mathbb{C}$.

Before we formulate Theorem 4.2.3, we give some preliminaries.

Let V_1 and V_2 be Banach spaces, $B \in L(V_1)$, $A \in L(V_2)$ closed operators such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ have topological complements (denoted respectively by V_1' and V_2') in V_1 and V_2, respectively. The projector from V_2 to V_2' will be denoted as $P_{V_2'}$. Let $C \in L(V_1, V_2)$ be such that $\mathcal{D}_C \cap \mathcal{D}_B \neq \emptyset$ and $C(\mathcal{D}_C \cap \mathcal{D}_B) \subset \mathcal{R}(A)$.
We assume that $(\sigma_p(B) \cap \sigma_p(A)) \setminus \{0\} \neq \emptyset$ and label such intersection as

$$\sigma \equiv (\sigma_p(B) \cap \sigma_p(A)) \setminus \{0\}.$$

Theorem 4.2.3. [29, Theorem 2.4.] (*The inhomogeneous equation*) With respect to the previous notation, if σ contains two disjoint families of different non-zero elements

$$\{\mu_j\}_{j\in J} \cup \{\lambda_i\}_{i\in I} \subset \sigma, \tag{4.14}$$

where $\{\mu_j\}_{j\in J}$ and $\{\lambda_i\}_{i\in I}$ have the following properties:

1. For every $j \in J$ let $u'_j \in \mathcal{D}_B \cap \mathcal{D}_C \cap V'_1$ such that $Bu'_j = \mu_j u'_j$ and $C(u'_j) = 0$.
2. For every $i \in I$, let $u_i \in \mathcal{D}_B \cap \mathcal{D}_C \cap V'_1$ such that $Bu_i = \lambda_i u_i$ and $C(u_i) \neq 0$, $C(u_i) \in \mathcal{R}(A - \lambda_i I)$ and $C(u_i)$ is linearly independent with vectors from $\{(A - \lambda_k I)^{-1} P_{V'_2} C(u_k)\}_{k\in I}$. We also require that $\{C(u_i)\}_{i\in I}$ are linearly independent different vectors.

Then there exists a weak solution to the inhomogeneous equation (4.1), defined on

$$(\mathcal{N}(B) \cap \mathcal{D}_C) \dotplus \left(Lin(\{u'_j\}_{j\in J})\right) \dotplus (Lin(\{u_i\}_{i\in I}))\,.$$

Remark. Notice that $Lin(\{u'_j\}_{j\in J}) \cap Lin(\{u_i\}_{i\in I}) = \{0\}$, where u'_j and u_i are eigenvectors for B which correspond to different eigenvalues μ_j and λ_i of B. Therefore, the direct sum $Lin(\{u'_j\}_{j\in J}) \dotplus Lin(\{u_i\}_{i\in I})$ exists. We now proceed to prove the stated theorem.

Proof. Since B and A are closed operators, the corresponding null spaces are closed subspaces in V_1, V_2, respectively. The subspaces $\mathcal{N}(B)$ and $\mathcal{N}(A)$ have topological complements, so V_1 and V_2 can be split into direct sums. Let $V_1 = \mathcal{N}(B) \dotplus V'_1$ and $V_2 = \mathcal{N}(A) \dotplus V'_2$ as stated in the theorem. Put $V_1(B) := V'_1 \cap \mathcal{D}_B$ and $V_2(A) := V'_2 \cap \mathcal{D}_A$. Define one-to-one operators $B_1 \in L(V_1(B), V_1)$ and $A_1 \in L(V_2(A), V_2)$ like in the proof of Theorem 4.2.2. We now have $\sigma_p(A_1) = \sigma_p(A) \setminus \{0\}$ and $\sigma_p(B_1) = \sigma_p(B) \setminus \{0\}$.

Let $u \in \mathcal{N}(B) \cap \mathcal{D}_C$. Since $C(u) \in \mathcal{R}(A) = \mathcal{R}(A_1)$ there exists a unique $v \in V_2(A)$ such that $C(u) = A_1 v = Av$. Put $N(u) := v$. It follows that

$$AN(u) - NB(u) = AN(u) = A(v) = C(u), \tag{4.15}$$

for every $u \in \mathcal{N}(B) \cap \mathcal{D}_C$.

Now observe V'_1, V'_2 and B_1 and A_1. We define closed one-to-one operators $\Psi_1^{(0)} \in L(Lin(\{u'_j\}_{j\in J}), V_1)$ and $\Psi_2^{(0)} \in L(Lin(\{v'_j\}_{j\in J}), V_2)$ such that $\Psi_1^{(0)} u'_j := \frac{\mu_j}{2} u'_j$, $\Psi_2^{(0)} v'_j := \frac{\mu_j}{2} v'_j$, for every $j \in J$. Then

$$\left\{\frac{\mu_j}{2}\right\}_{j\in J} \subset \sigma_p(\Psi_1^{(0)}) \cap \sigma_p(\Psi_2^{(0)}) \neq \emptyset,$$

and $\mu_i = \mu_j \Rightarrow i = j$. Since $\mathcal{N}(\Psi_1^{(0)}) = 0_{V_1}$ and $\mathcal{N}(\Psi_2^{(0)}) = 0_{V_2}$, then $\mathcal{N}(\Psi_1^{(0)})$ and $\mathcal{N}(\Psi_2^{(0)})$ have topological complements in V_1' and V_2', respectively. Now Theorem 4.2.2 implies that there exists a non-trivial weak solution

$$Y^{(0)} \in L(Lin(\{u_j'\}_{j\in J}), Lin(\{v_j'\}_{j\in J})),$$

such that

$$\Psi_2^{(0)} Y^{(0)} - Y^{(0)} \Psi_1^{(0)} = 0 \tag{4.16}$$

holds. Further, for every $j \in J$ we have $Y^{(0)}(u_j') = v_j'$ (see proof of Theorem 4.2.2). Note that

$$0 \notin \left\{\frac{\mu_j}{2}\right\}_{j\in J} \subset \sigma_p(B_1 - \Psi_1^{(0)}) \cap \sigma_p(A_1 - \Psi_2^{(0)}) \neq \emptyset,$$

and $\{u_j'\}_{j\in J}$ and $\{v_j'\}_{j\in J}$ are the corresponding eigenvectors, respectively. Due to the assumption 1. of the theorem, $C(u_j') = 0$, so $(Y^{(0)} + C)(u_j') = v_j'$, for every $j \in J$. Since $B_1 - \Psi_1^{(0)}$ is one-to-one on $Lin(\{u_j'\}_{j\in J})$ and $A_1 - \Psi_2^{(0)}$ is one-to-one on $Lin(\{v_j'\}_{j\in J})$, we can apply Theorem 4.2.2 and conclude that $Y^{(0)} + C$ is a weak solution to the injective equation

$$(A_1 - \Psi_2^{(0)})X - X(B_1 - \Psi_1^{(0)}) = 0, \tag{4.17}$$

defined on $Lin(\{u_j'\}_{j\in J})$. But then for every $u' \in Lin(\{u_j'\}_{j\in J})$,

$$\begin{aligned}
0 =&(A_1 - \Psi_2^{(0)})(Y^{(0)} + C)(u') - (Y^{(0)} + C)(B_1 - \Psi_1^{(0)})(u') \\
=&A_1(Y^{(0)} + C)(u') - \Psi_2^{(0)} Y^{(0)}(u') - \Psi_2^{(0)} C(u') \\
&-(Y^{(0)} + C)B_1(u') + Y^{(0)}\Psi_1^{(0)}(u') + C\Psi_1^{(0)}(u') \\
=&A_1(Y^{(0)} + C)(u') - (Y^{(0)} + C)B_1(u') \\
&-(\Psi_2^{(0)} Y^{(0)} - Y^{(0)}\Psi_1^{(0)})(u') - (\Psi_2^{(0)} C - C\Psi_1^{(0)})(u') \\
=&A_1(Y^{(0)} + C)(u') - (Y^{(0)} + C)B_1(u') - C(u'),
\end{aligned} \tag{4.18}$$

where we used (4.16) and

$$\Psi_2^{(0)} C(u') - C\Psi_1^{(0)}(u') = 0 = C(u'), \quad u' \in Lin(\{u_j'\}_{j\in J}).$$

Put $X^{(0)} := C + Y^{(0)}$.

Condition 2. of the theorem implies the following. For every $i \in I$, define

$$\Psi_1(u_i) := \frac{1}{2} B_1(u_i).$$

Then $\sigma_p(\Psi_1) \supset \left\{\frac{\lambda_i}{2}\right\}_{i\in I}$ and $\{u_i\}_{i\in I}$ are the corresponding eigenvectors. Also note that

$$\left\{\frac{\lambda_i}{2}\right\}_{i\in I} \subset \sigma_p(B_1 - \Psi_1)$$

and u_i are the corresponding eigenvectors. Now define

$$\Psi_2(C(u_i)) := \left(1 + \frac{\lambda_i}{2}\right) C(u_i).$$

Further, since $C(u_i) \in \mathcal{R}(A - \lambda_i I)$, there exists unique $v_i \in V_2' \cap \mathcal{D}_A$ such that

$$v_i = (A_1 - \lambda_i I)^{-1}(C(u_i)), \tag{4.19}$$

that is,

$$(A_1 - \lambda_i I)v_i = C(u_i). \tag{4.20}$$

Since $\{C(u_i)\}_{i\in I}$ are linearly independent vectors, it follows that $\{v_i\}_{i\in I}$ are linearly independent vectors. Define

$$\Psi_2(v_i) := \frac{\lambda_i}{2} v_i + C(u_i).$$

Since $\{C(u_i)\}_{i\in I}$ are linearly independent vectors with respect to $\{v_i\}_{i\in I}$, we conclude that Ψ_2 is well defined on $Lin(\{C(u_i)\}_{i\in I}) \dot{+} Lin(\{v_i\}_{i\in I})$. Now

$$\Psi_2(v_i - C(u_i)) = \frac{\lambda_i}{2}(v_i - C(u_i)).$$

In other words, $\{\frac{\lambda_i}{2}\}_{i\in I} \subset \sigma_p(\Psi_2)$ and $v_i - C(u_i)$ are the corresponding eigenvectors. Also

$$(A_1 - \Psi_2)v_i = A_1(v_i) - \frac{\lambda_i}{2} v_i - C(u_i) = A_1(v_i) - \frac{\lambda_i}{2} v_i - (A_1 - \lambda_i I)v_i = \frac{\lambda_i}{2} v_i,$$

so

$$\left\{\frac{\lambda_i}{2}\right\}_{i\in I} \subset \sigma_p(A_1 - \Psi_2)$$

and v_i are the corresponding eigenvectors. Now

$$\left\{\frac{\lambda_i}{2}\right\}_{i\in I} \subset \sigma_p(A_1 - \Psi_2) \cap \sigma_p(B_1 - \Psi_1).$$

Since $\mathcal{N}(A_1 - \Psi_2) = 0_{V_2}$ and $\mathcal{N}(B_1 - \Psi_1) = 0_{V_1}$, it follows that $\mathcal{N}(A_1 - \Psi_2)$ and $\mathcal{N}(B_1 - \Psi_1)$ have topological complements in V_2' and V_1', respectively, so (applying Theorem 4.2.2) there exists $X^{(1)}$, which is a weak solution to the equation (4.13), and it is defined as

$$X^{(1)}(u_i) := v_i \tag{4.21}$$

(see proof of Theorem 4.2.2). Put

$$Y(u_i) := X^{(1)}(u_i) - C(u_i) = v_i - C(u_i).$$

We verify that (4.12) holds:

$$\begin{aligned}
&\Psi_2 Y(u_i) - Y\Psi_1(u_i) = \Psi_2(v_i - C(u_i)) - Y\left(\frac{\lambda_i}{2}u_i\right) = \\
&\frac{\lambda_i}{2}v_i + C(u_i) - \Psi_2(C(u_i)) - \frac{\lambda_i}{2}u_i + \frac{\lambda_i}{2}C(u_i) = \\
&(1+\frac{\lambda_i}{2})C(u_i) - \Psi_2(C(u_i)) = 0.
\end{aligned} \tag{4.22}$$

Finally, we verify that (4.10) holds:

$$\Psi_2 C(u_i) - C\Psi_1(u_i) - C(u_i) = \left(1+\frac{\lambda_i}{2}\right)C(u_i) - \frac{\lambda_i}{2}C(u_i) - C(u_i) = 0. \tag{4.23}$$

Put $X = N \dot{+} X^{(0)} \dot{+} X^{(1)}$. Combining the observations from (4.15) to (4.23), we see that X is a weak solution to (4.1), defined on

$$(\mathcal{N}(B) \cap \mathcal{D}_C) \dot{+} \left(Lin(\{u'_j\}_{j\in J})\right) \dot{+} \left(Lin(\{u_i\}_{i\in I})\right).$$

□

Remark. Once again, if $\sigma_p(A) \cap \sigma_p(B) = \{0\}$ then then every solution to the inhomogeneous Sylvester equation is obtained with operator N from the equation (4.15).

4.2.3 Extensions to Schauder bases

The previous results provide weak solutions to the equation (4.1), defined on finite linear combinations of the corresponding eigenvectors. One naturally wonders under which circumstances the can aforementioned solutions be extended i.e. do the solutions have to be defined on finite linear combinations of the eigenvectors.

When dealing with the (partial) differential operators, the solutions to the provided (P)DEs are always represented as the Fourier series of the given eigenfunctions. Hence we wonder whether the solutions to the Sylvester operator equation (4.1) can be defined on infinite sums generated by the corresponding eigenvectors (see [65]).

Definition 4.2.3. Let V be a Banach space over the field F. A Schauder basis is an ordered sequence $\{b_n\}_{n\in\mathbb{N}}$ of elements from V such that for every element $v \in V$ there exists a unique sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of scalars in F such that

$$v = \sum_{n\in\mathbb{N}} \alpha_n b_n,$$

where the convergence is understood in the norm topology

$$\lim_{n\to+\infty} \|v - \sum_{k=1}^{n} \alpha_n b_n\| = 0.$$

From unique representation of v via $\{b_n\}_{n\in\mathbb{N}}$ it follows that $\{b_n\}_{n\in\mathbb{N}}$ is a family of linearly independent vectors. There is no exact criterion which yields when a given Banach space has a Schauder basis. However, the necessary condition is obtained in the following two well-known theorems (see [37], [43], [59], [86], [99] and [102]).

Theorem 4.2.4. Let V be Banach space. Then its algebraic basis is either finite or has the cardinality of at least c (continuum).

Theorem 4.2.5. Let V be Banach space and suppose it has a Schauder basis. Then V must be separable.

Contrary, if the provided Banach space V is separable, that does not imply that it has a Schauder basis. A counterexample was provided by P. Enflo [51] in 1973.
The most important examples of Schauder bases are probably the power sequence basis $\{1, t, t^2, \ldots\}$ in c_0 and ℓ^p spaces, when $1 \leq p < \infty$ and the sequence of trigonometric polynomials $\{1, \sin\frac{t\pi}{d}, \cos\frac{t\pi}{d}, \sin\frac{2t\pi}{d}, \cos\frac{2t\pi}{d}\ldots\}$ in the $L^2[0, 2d]$ space, for some $d > 0$. In that case, the corresponding scalars α_n are the Fourier coefficients of the given function with respect to the provided basis. It is a well-known fact that ℓ^∞ space does not have a Schauder basis.

Recall arbitrary linear spaces V_1 and V_2 and one-to-one operators $B \in L(\mathcal{D}_B, V_1)$, $\mathcal{D}_B \subset V_1$ and $A \in L(\mathcal{D}_A, V_2)$, $\mathcal{D}_A \subset V_2$ defined on them.

Suppose there exists $W < \mathcal{D}_B$ a $B-$invariant subspace of V_1, which allows a Schauder basis $\mathcal{W} = \{w_n : n \in \mathbb{N}\}$ (consequently, W must be separable). It is not difficult to see that there exists a bijective operator $T \in L(B(W), W)$ such that $T(w_n) \in \mathcal{W}$, for every $n \in \mathbb{N}$, because B is assumed to be one-to-one. Now for every $w \in \mathcal{W}$ define

$$[w] = \{(TB)^n(w) : n \in \mathbb{Z}\}$$

and define a binary operation $\cdot_B$ on $[w]$ as

$$(\forall n, m \in \mathbb{Z})(TB)^n(w) \cdot_B (TB)^m(w) := (TB)^{n+m}(w).$$

Lemma 4.2.1 yields that $([w], \cdot_B)$ is isomorphic to exactly one element from the set $\{(\mathbb{Z}, +)\} \cup \{(\mathbb{Z}_k, +_k) : k \in \mathbb{N}\}$.

Analogously, assume there exists $Z < \mathcal{D}_A$ which is an $A-$invariant subspace of V_2, which allows a Schauder basis $\mathcal{Z} = \{z_n : n \in \mathbb{N}\}$ and define a bijective operator $S \in L(A(Z), Z)$ such that for every $n \in \mathbb{N}$ it follows that $S(z_n) \in \mathcal{Z}$. For every $z \in \mathcal{Z}$, define

$$[z] = \{(ZA)^n(z) : n \in \mathbb{Z}\}$$

and define $\cdot_A$ on every class $[z]$ as

$$(\forall n, m \in \mathbb{Z})(SA)^n(w) \cdot_A (SA)^m(w) := (SA)^{n+m}(w).$$

Lemma 4.2.1 yields that $([z], \cdot_A)$ is isomorphic to exactly one element from the set $\{(\mathbb{Z}, +)\} \cup \{(\mathbb{Z}_k, +_k) : k \in \mathbb{N}\}$.

The following Corollaries are immediate consequences of Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3, respectively.

Corollary 4.2.4. [29, Corollary 2.5.] *(The shifted injective homogeneous equation)* With respect to the previous notation, there exists $X \in L(W_0, Z)$ which is a weak solution to the equation

$$XTB = SAX, \tag{4.24}$$

defined on

$$W_0 := \left\{ \sum_{n\in\mathbb{N}} \alpha_n w_n : w_n \in \mathcal{W}, \alpha_n \in \mathbb{C}, n \in \mathbb{N} \text{ and } \sum_{n\in\mathbb{N}} \alpha_n X(w_n) \text{ converges in } Z \right\}.$$

Proof. 1) For $w \in \mathcal{W}$, define $X(w)$ as described below.
1.1) If $([w], \cdot_B) \cong (\mathbb{Z}, +)$, and if there is some $z \in \mathcal{Z}$ such that $([z], \cdot_A) \cong (\mathbb{Z}, +)$, then $X(w) := z$. Further, for every $m \in \mathbb{Z}$, put $X((TB)^m(w)) := (SA)^m(z)$. If there is no $z \in \mathcal{Z}$ such that $([z], \cdot_A) \cong (\mathbb{Z}, +)$, then $X([w]) := \{0_{V_2}\}$.

1.2) If $([w], \cdot_B) \cong (\mathbb{Z}_k, +_k)$, for some $k \in \mathbb{N}_0$, then there exists w' such that it is the generating element of $[w]$. If there exists $z \in \mathcal{Z}$ such that $([z], \cdot_A) \cong$

$(\mathbb{Z}_k, +_k)$, then there exists $z' \in [z]$ which is the generating element of $[z]$. Put $X(w') := z'$ and for $m = \overline{1, k-1}$, put $X((TB)^m(w')) := (SA)^m(z')$. If no such $z \in \mathcal{Z}$ exists, then $X([w]) := \{0_{V_2}\}$.

2) For any given $w \in W_0$, with unique Schauder representation in $\mathcal{W}$

$$w = \sum_{n\in\mathbb{N}} \alpha_n w_n, \quad w_n \in \mathcal{W}, \quad \alpha_n \in \mathbb{C}, \quad n \in \mathbb{N},$$

define $X(w) := \sum\limits_{n\in\mathbb{N}} \alpha_n X(w_n)$ (which converges by the choice of W_0). Defined this way, X is a weak solution to the equation (4.24), defined on W_0. □

Corollary 4.2.5. [29, Corollary 2.6.] *(The homogeneous equation)* Let V_1 and V_2 be given Banach spaces, $B \in L(V_1)$ and $A \in L(V_2)$ closed operators, such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ have topological complements in V_1, V_2, respectively, i.e. $V_1 = \mathcal{N}(B) \dot{+} V_1'$ and $V_2 = \mathcal{N}(A) \dot{+} V_2'$. If $\sigma_p(B) \cap \sigma_p(A) \neq \emptyset$ and the corresponding eigenvectors form Schauder bases for some $S_1 < \mathcal{D}_B \cap V_1'$ and $S_2 < \mathcal{D}_A \cap V_2'$, respectively, then the homogeneous equation

$$AX - XB = 0 \tag{4.25}$$

has a non-trivial weak solution, defined on some subset of S_1.

Corollary 4.2.6. [29, Corollary 2.7.] *(The inhomogeneous equation)* Let V_1 and V_2 be Banach spaces, $B \in L(V_1)$, $A \in L(V_2)$ closed operators such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ have topological complements in V_1, V_2, respectively. In that sense, put $V_1 = \mathcal{N}(B) \dot{+} V_1'$ and $V_2 = \mathcal{N}(A) \dot{+} V_2'$. Let $C \in L(V_1, V_2)$ such that $\mathcal{D}_C \cap \mathcal{D}_B \neq \{0\}$ and $C(\mathcal{D}_C \cap \mathcal{D}_B) \subset \mathcal{R}(A)$.
If

$$\{\mu_j\}_{j\in\mathbb{N}} \cup \{\lambda_i\}_{i\in\mathbb{N}} \subset (\sigma_p(B) \cap \sigma_p(A)) \setminus \{0\} \tag{4.26}$$

where $\{\mu_j\}_{j\in\mathbb{N}}$ and $\{\lambda_i\}_{i\in\mathbb{N}}$ are disjoint families of different elements with following properties:

1. For every $j \in \mathbb{N}$ let $u_j' \in \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$ such that $Bu_j' = \mu_j u_j'$ and $C(u_j') = 0$. Assume $\{u_j'\}_{j\in\mathbb{N}}$ to form a Schauder basis for some $S_J < \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$.

2. For every $i \in \mathbb{N}$ let $u_i \in \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$ such that $Bu_i = \lambda_i u_i$ and $\{u_i\}_{i\in\mathbb{N}}$ forms a Schauder basis for some $S_I < \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$. Assume that $\{C(u_i)\}_{i\in\mathbb{N}}$ are linearly independent different non-zero vectors, which form a Schauder basis for some $S_C < \mathcal{R}(A) \cap \mathcal{R}(A - \lambda_i I)$, and vectors $\{P_{V_2'}(A - \lambda_i I)^{-1} C(u_i)\}_{i\in\mathbb{N}}$ to form a Schauder basis for some $S_V < \mathcal{D}_A \cap V_2'$, such that $S_C \cap S_V = \{0\}$.

3. We require $S_J \cap S_I = \{0\}$.

Then there exists a weak solution to the inhomogeneous equation (4.1), defined on

$$(\mathcal{N}(B) \cap \mathcal{D}_C) \dot{+} (S_J) \dot{+} (S_I).$$

4.3 Applications to Sturm-Liouville operators

In this section we will illustrate our results on Sturm-Liouville operators. Entire theoretical background regarding Sturm-Liouville operators is taken from [65]. For more on differential and pseudo-differential operators, consult [81], [82] and rich references therein.

Definition 4.3.1. [65] Let $d > 0$ and let $p \in \mathcal{C}^1[0,d]$, $p(x) \neq 0$, $q \in \mathcal{C}[0,d]$ be real-valued functions. Operator $\mathcal{L} : \mathcal{C}^2[0,d] \to \mathcal{C}[0,d]$, given as

$$\mathcal{L}(\varphi) = \frac{\mathrm{d}}{\mathrm{d}\,x}\left(p(x)\frac{\mathrm{d}\,\varphi(x)}{\mathrm{d}\,x}\right) - q(x)\varphi(x)$$

is called a Sturm-Liouville operator.

For the given Sturm-Liouville operator $\mathcal{L}$, we formulate the boundary problem: find the non-trivial solution to the ordinary differential equation

$$\mathcal{L}(\varphi) + \lambda w(x)\varphi(x) = 0, \quad 0 < x < d, \tag{4.27}$$

where $\lambda \in \mathbb{C}$ and $w \in \mathcal{C}[0,d]$, which satisfies boundary conditions

$$\begin{cases} \alpha\varphi'(0) - \beta\varphi(0) = 0, \\ \gamma\varphi'(d) + \delta\varphi(d) = 0, \\ \alpha^2 + \beta^2 > 0, \quad \gamma^2 + \delta^2 > 0. \end{cases} \tag{4.28}$$

Definition 4.3.2. [65] Complex values λ for which problem (4.27)-(4.28) has a non-trivial solution are called eigenvalues and the corresponding solutions are called eigenfunctions (eigenvectors) of Sturm-Liouville operator $\mathcal{L}$.

Theorem 4.3.1. [65] For a provided weight function $w \in \mathcal{C}[0,d]$, the space

$$L_{2,w}[0,d] = \left\{\varphi : \int_0^d \varphi^2(x)w(x)\,\mathrm{d}\,x < \infty\right\}$$

is the $w-$weighted Hilbert space, with the scalar product

$$\langle f, g\rangle_w = \int_0^d f(x)\overline{g(x)}w(x)\,\mathrm{d}\,x.$$

The following theorem provides sufficient conditions for the existence of solutions to the problem (4.27)-(4.28).

Theorem 4.3.2. [65] *(Regular Sturm-Liouville boundary problem)* Assume that in (4.27) functions satisfy the following conditions

$$\begin{cases} p \in \mathcal{C}^1[0,d], & p(x) > 0, \ \ 0 \leq x \leq d; \\ q \in \mathcal{C}[0,d], & q(x) \geq 0, \ \ 0 \leq x \leq d; \\ w \in \mathcal{C}[0,d], & w(x) > 0, \ \ 0 < x < d, \end{cases}$$

and in (4.28) constants satisfy

$$\alpha, \beta, \gamma, \delta \geq 0, \quad \alpha + \beta > 0, \quad \gamma + \delta > 0.$$

Then:

1. Eigenvalues of Sturm-Liouville operator are non-negative (if $q(x) \neq 0$ or $\beta\delta > 0$, then they are all positive), non-repeating and form a strictly increasing unbounded sequence $0 \leq \lambda_1 < \lambda_2 < \ldots$.

2. The corresponding eigenvectors are $w-$orthogonal and form a complete system in the Hilbert space $L_{2,w}[0,d]$.

3. For every function $f \in \mathcal{C}^2[0,d]$ which satisfies boundary conditions (4.27)-(4.28) and

$$p(x)f'(x) - q(x)f(x) \leq C\sqrt{w(x)}, \quad x \in (0,d),$$

(which is always satisfied whenever $w(0) > 0$ and $w(d) > 0$), the series

$$\sqrt{w(x)}f(x) = \sum_{k=1}^{\infty} a_k \sqrt{w(x)}\varphi_k(x),$$

converges absolutely and uniformly on $[0,d]$, where $a_k = \frac{\langle f, \varphi_k \rangle_w}{\|\varphi_k\|_w^2}$ are the Fourier coefficients.

In what follows, we illustrate how the singular Sylvester equation applies to Sturm-Liouville eigenvalue problems.

Example 4.3.1. [29, Example 3.1.] Assume that $w(x) = 1$, for every $x \in [0,d]$, $d > 0$, and let $\mathcal{L}_1$, $\mathcal{L}_2$ be two different Sturm-Liouville operators whose point spectra intersect. In this example we will illustrate how to find X such that the homogeneous equation holds

$$\mathcal{L}_2 X = X\mathcal{L}_1. \tag{4.29}$$

Let $\mathcal{L}_1$ be a Sturm-Liouville operator such that its eigenvalues are $\lambda_k = \frac{6k\pi}{d}$, and the corresponding eigenfunctions are

$$u_k(x) = d\sin(\sqrt{\lambda_k}x),$$

for $x \in [0, d]$.
Let $\mathcal{L}_2$ be a Sturm-Liouville operator such that its eigenvalues are $\mu_k = \frac{9k\pi}{d}$, and the corresponding eigenfunctions are

$$v_k(x) = \frac{1}{2}\sin(\sqrt{\mu_k}x),$$

when $x \in [0, d]$. For every $(k \in \mathbb{N})$ $\lambda_{3k} = \mu_{2k}$, so we put

$$X(u_{3k}(x)) := v_{2k}(x). \tag{4.30}$$

Now let $f(x) \in \mathcal{C}^2(0, d) \cap \mathcal{C}^1[0, d]$ be represented as

$$f(x) = \sum_{k=1}^{+\infty} \alpha_{3k} u_{3k}(x), \tag{4.31}$$

where

$$\alpha_{3k} = \frac{\int_0^d df(x)\sin(\sqrt{\lambda_{3k}}x)\,\mathrm{d}\,x}{\|d\sin(\sqrt{\lambda_{3k}}x)\|^2} = \frac{\langle f, u_{3k}\rangle}{\|u_{3k}\|^2} \tag{4.32}$$

are the Fourier coefficients for the function f on $[0, d]$ with respect to functions u_{3k}. It is known that Fourier series converges uniformly and is uniformly bounded, when dealing with functions from the class $\mathcal{C}^2(0, d) \cap \mathcal{C}^1[0, d]$, hence we can resume to prove our identity. We also require that series

$$\sum_{k=0}^{+\infty} \alpha_{3k} v_{2k}(x) = \sum_{k=0}^{+\infty} \frac{\langle f, u_{3k}\rangle}{\|u_{3k}\|^2} v_{2k}(x) \tag{4.33}$$

converges uniformly and is unifromly bounded on $\mathcal{C}^2(0, d) \cap \mathcal{C}^1[0, d]$. Observe

$$\begin{aligned} X\mathcal{L}_1(f(x)) &= X\mathcal{L}_1\left(\sum_{k=1}^{+\infty} \alpha_{3k} u_{3k}(x)\right) = X\left(\sum_{k=1}^{+\infty} \alpha_{3k}\mathcal{L}_1(u_{3k}(x))\right) \\ &= X\left(\sum_{k=1}^{+\infty} \alpha_{3k}\lambda_{3k} u_{3k}(x)\right) = \sum_{k=1}^{+\infty} \alpha_{3k}\lambda_{3k} X\left(u_{3k}(x)\right) \\ &= \sum_{k=1}^{+\infty} \alpha_{3k}\mu_{2k} v_{2k}(x). \end{aligned} \tag{4.34}$$

On the other hand, we have

$$\begin{aligned}\mathcal{L}_2 X(f(x)) &= \mathcal{L}_2 X\left(\sum_{k=1}^{+\infty}\alpha_{3k}u_{3k}(x)\right) = \mathcal{L}_2\left(\sum_{k=1}^{+\infty}\alpha_{3k}X(u_{3k}(x))\right)\\ &= \mathcal{L}_2\left(\sum_{k=1}^{+\infty}\alpha_{3k}v_{2k}(x)\right) = \sum_{k=1}^{+\infty}\alpha_{3k}\mathcal{L}_2\left(v_{2k}(x)\right) \qquad (4.35)\\ &= \sum_{k=1}^{+\infty}\alpha_{3k}\mu_{2k}v_{2k}(x) = X\mathcal{L}_1\left(f(x)\right),\end{aligned}$$

so the solution (4.30) is a weak solution to the homogeneous equation (4.29), defined where ever the series (4.33) converges. ♣

In linear algebra, a vector v is said to be the generalized eigenvector for the given operator A with respect to the eigenvalue $\lambda(A)$ if $(A-\lambda(A)I)v$ is an eigenvector for A with respect to the eigenvalue $\lambda(A)$. In other words,

$$(A-\lambda(A)I)^2 v = 0.$$

Let w be positive non-constant weight function, continuous on $[0,d]$, for some $d>0$. We define $w-$generalized eigenfunction of first and second order for the given Sturm-Liouville operator.

Definition 4.3.3. [29] Function f is said to be $w-$generalized eigenfunction of the first order for Sturm-Liouville operator $\mathcal{L}$ with respect to the eigenvalue λ if

$$(\mathcal{L}+\lambda w)^2 f = 0.$$

Function g is said to be $w-$generalized eigenfunction of the second order for Sturm-Liouville operator $\mathcal{L}$ with respect to the eigenvalue λ if

$$(\mathcal{L}+\lambda I)(\mathcal{L}+\lambda w)g = 0.$$

In the following example we will illustrate how to transform $w-$weighted Sturm-Liouville problem (4.27)–(4.28) with the weight function w to the Sturm-Liouville problem (4.27)–(4.28) where $w(x)=1$, $x\in[0,d]$.

Example 4.3.2. [29, Example 3.2.] Let $\mathcal{L}$ be a Sturm-Liouville operator such that eigenvalues of the problem

$$\mathcal{L}(\varphi)+\lambda\varphi = 0 \qquad (4.36)$$

are $\{\lambda_k\}_{k\in\mathbb{N}}$ and the corresponding eigenfunctions are $\beta_k(x), k \in \mathbb{N}$. Let $w(x)$ be such that $w(x) > 0$, for every $x \in [0, d]$, and the problem

$$\mathcal{L}(\varphi(x)) + \lambda w(x)\varphi(x) = 0 \tag{4.37}$$

has the same eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$.

Define an operator

$$B(\varphi) := -\mathcal{L}(\varphi).$$

Then

$$(B - \lambda I)(\varphi) = 0 \Leftrightarrow \mathcal{L}(\varphi) + \lambda\varphi = 0,$$

so $\sigma_p(B) = \{\lambda_k\}_{k\in\mathbb{N}}$. Further, put

$$A(\varphi) := -\frac{1}{w(x)}\mathcal{L}(\varphi).$$

It follows that

$$(A - \lambda I)(\varphi) = 0 \Leftrightarrow \mathcal{L}(\varphi) + \lambda w(x)\varphi(x) = 0,$$

so $\sigma_p(A) = \{\lambda_k\}_{k\in\mathbb{N}}$. Finally, put

$$C(\beta_k(x)) := \frac{1}{w(x)}\beta_k(x).$$

It follows that

$$\left(\frac{1}{w(x)}\beta_k(x)\right)_{k\in\mathbb{N}}$$

is a family of linearly independent functions. By solving the inhomogeneous equation

$$AX - XB = C \tag{4.38}$$

over the space of corresponding eigenfunctions, $\big(\beta_k(x)\big)_{k\in\mathbb{N}}$, we will obtain a transformation that transforms eigenfunctions of (4.36) into $w-$generalized eigenfunctions of second order for (4.37).

In order to solve (4.38), we must find $g_k(x)$ such that (see proof of Theorem 4.2.3, expressions (4.19), (4.20) and (4.21))

$$(A - \lambda_k I)g_k(x) = C(\beta_k(x)) = \frac{1}{w(x)}\beta_k(x).$$

Multiplying by $w(x)$, we obtain the following

$$-\mathcal{L}(g_k(x)) - \lambda_k w(x)g_k(x) = \beta_k(x). \tag{4.39}$$

In other words,

$$(\mathcal{L} + \lambda_k I)(\beta_k(x)) = (\mathcal{L} + \lambda_k I)(\mathcal{L} + \lambda_k w(x))(g_k(x)) = 0 \tag{4.40}$$

so the weak solution X to the inhomogeneous Sylvester equation (4.38) maps eigenfunctions $\beta_k(x)$ to $w-$generalized eigenfunctions of second order $g_k(x)$:

$$X(\beta_k(x)) := g_k(x). \tag{4.41}$$

Since X is obviously a one-to-one map, defined on the space whose Schauder basis is $\big(\beta_k(x)\big)_{k\in\mathbb{N}}$, the operator X^{-1} maps the $w-$generalized eigenfunctions of second order $g_k(x)$ onto the eigenfunctions $\beta_k(x)$:

$$X^{-1}(g_k(x)) := \beta_k(x) \quad k \in \mathbb{N}. \tag{4.42}$$

♣

4.4 A special case: self-adjoint operators on Hilbert spaces

Weak solutions introduced in this chapter are defined on particular eigenspaces of B, which correspond to the shared eigenvalues with operator A. The previous section illustrates how those solutions apply in the associated eigenproblems from Sturm-Liouville theory. A natural question rises, and that is when can the weak solutions be extended to the largest domains possible? Furthermore, when can the inhomogeneous equation (4.1) be solved, if the spectral intersection of operators A and B occurs in parts of the spectra which are not the eigenvalues? Luckily, problems where closed operators appear (and the corresponding operator equations), usually require the spaces V_1 and V_2 to be separable Hilbert spaces, and the operators to be self-adjoint or symmetric operators on those spaces (consult [37], [43], [59], [77], [99] and [102]).

Example 4.4.1. Let $L^2(\mathbb{R})$ be the standard Hilbert space, equipped with the usual $\|\cdot\|_2$ norm. It is known that the Schwarz space $\mathcal{S}(\mathbb{R})$ equipped with the sup $-$norm is dense in $L^2(\mathbb{R})$ (see [37], [102]). The position operator P and the momentum operator Q are defined on $\mathcal{S}(\mathbb{R})$, and their domains are therefore dense in $L^2(\mathbb{R})$. They are unitarily equivalent, by virtue of the Fourier transform (see [99]), and are essentially self-adjoint (meaning that they have extensions in the graph topology, which are self-adjoint operators).

Furthermore, they both have purely absolutely continuous spectra, which consist of the entire real line,

$$\sigma(P) = \sigma_{ac}(P) = \sigma(Q) = \sigma_{ac}(Q) = \mathbb{R}.$$

Operators P and Q satisfy the basic equation of quantum mechanics (consult [70], [99] and [101]):

$$PQ - QP = \frac{h}{2\pi i} I, \tag{4.43}$$

where h is the Planck constant[1]. This example is essential in the sense that it does not have an analogue in the bounded operator setting, because the identity can never be represented as a commutator of two bounded operators (more generally, the unity in a unital Banach algebra can never be represented as a commutator of two elements from that algebra). ♣

The equation (4.43) can be viewed as a Sylvester equation, where $A = B = P$ and $Q = X$ (or vice verca) and $C = \frac{h}{2\pi i} I$. However, this implies that $\sigma(A) \cap \sigma(B) = \mathbb{R}$ and the spectral intersection occurs in the absolute continuous parts of the spectra, and not in the point spectra. Therefore, it is convenient to analyze the problem of singular Sylvester equations under these circumstances. For more on operator equations that stem from quantum mechanics, consult [70], [99] and [101].

From this point on, we assume that V_1 and V_2 are separable Hilbert spaces and A and B are self-adjoint unbounded operators whose spectra intersect. Recall that, if S is a self-adjoint operator, it is then a closed and densely defined operator, and its spectrum $\sigma(S)$ is purely approximate point spectrum, that is,

$$\sigma(S) = \sigma_{app}(S) = \sigma_p(S) \cup \sigma_c(S), \qquad \sigma_c(S) = \sigma_{app}(S) \setminus \sigma_p(S).$$

We formulate the Spectral mapping theorem for self-adjoint operators (consult books [37], [59] and [102]):

Theorem 4.4.1. *(Spectral mapping theorem for self-adjoint operators)* For a self-adjoint operator L, densely defined on a separable Hilbert space V, there exists a unique decomposition of identity, $(E_\lambda : \ \lambda \in \mathbb{R})$, consisting of orthogonal projectiors E_λ, such that

1. The representation

$$L = \int_{-\infty}^{+\infty} \lambda \, \mathrm{d}\, E_\lambda \tag{4.44}$$

[1] $h \approx 6.62607004 \times 10^{-34} m^2 kg/s$.

holds, where $\mathcal{D}_L$ consists of those $x \in V$ such that the integral

$$\int_{-\infty}^{+\infty} \lambda^2 \, \mathrm{d}\, |E_\lambda x|^2 \tag{4.45}$$

converges.

2. The function $\lambda \mapsto E_\lambda$ is strongly continuous from above. Furthermore, points of discontinuity of the function are precisely the eigenvalues for the operator L. In that case, if λ_0 is an eigenvalue of L, then $E_{\lambda_0} - E_{\lambda_0 - 0}$ is the orthogonal projector from V onto the eigenspace W_{λ_0} of L, which corresponds to λ_0.

3. The operator L commutes with every E_λ. Furthermore, an operator S commutes with L if and only if it commutes with every projector E_λ.

An elegant proof involving a geometric way of thinking was provided by Leinfelder in [64]. Separability of the space V, as well as density of the domain $\mathcal{D}_L$ play essential roles in the proof: important consequences follow immediately, which are applied in this section as well.

Proposition 4.4.1. With respect to the previous Theorem, the space V allows an orthogonal decomposition

$$V = \oplus_n V_n, \tag{4.46}$$

where V_n is an $L-$invariant subspace of V, such that $L_n := L(\mathcal{D}_L \cap V_n)$ is a bounded linear self-adjoint operator on V_n with $\mathcal{D}_{L_n} = \mathcal{D}_L \cap V_n$. In that case,

$$L = \oplus_n L_n. \tag{4.47}$$

Proposition 4.4.2. Let V be a separable Hilbert space and let

$$V = \oplus_n V_n \tag{4.48}$$

be an orthogonal sum of mutually orthogonal closed spaces V_n. If $(L_n)_n$ is a sequence of self-adjoint bounded linear operators, $L_n \in \mathcal{B}(V_n)$, then there exists a unique self-adjoint operator $L \in L(V)$, such that every V_n is $L-$invariant, and that L restricted to V_n coincides with L_n. The domain $\mathcal{D}_L$ consists of those vectors $x \in V$ such that the series

$$\sum_{n=1}^{+\infty} |L_n x_n|^2$$

converges, where $x_n = P_{V_n} x$. If $\sup\{\|L_n\| : n \in \mathbb{N}\}$ is finite, then L is a bounded operator.

We proceed with the problem at hand, and that is to solve the singular Sylvester equation (4.1), if A and B are self-adjoint operators, defined on separable Hilbert spaces, whose spectra intersect. Once again, we denote

$$\emptyset \neq \sigma(A) \cap \sigma(B) =: \sigma.$$

4.4.1 The case when $\sigma = \sigma_p(A) \cap \sigma_p(B)$

In this particular subsection, it is assumed that

$$\sigma = \sigma_p(A) \cap \sigma_p(B),$$

that is, the only shared points of the spectra are some eigenvalues. For more elegant notation, denote by $E_B^\lambda = \mathcal{N}(B - \lambda I)$ and $E_A^\lambda = \mathcal{N}(A - \lambda I)$ whenever $\lambda \in \sigma$. Different eigenvalues generate mutually orthogonal eigenvectors, so the spaces E_B^λ form an orthogonal sum. Put $E_B := \overline{\sum_\lambda E_B^\lambda}$. It is a closed subspace of V_1 and there exists $E_B^\perp$ such that $V_1 = E_B \oplus E_B^\perp$. Take $B = B_E \oplus B_1$ with respect to that decomposition and denote $C_0 = CP_{E_B^\perp}$.

Theorem 4.4.2. [24, Theorem 2.1.] *(The point spectrum case)* For given separable Hilbert spaces V_1 and V_2, let $A \in \mathcal{C}(V_2)$ and $B \in \mathcal{C}(V_1)$ be densely defined self-adjoint operators such that $\sigma(A) \cap \sigma(B) = \sigma_p(A) \cap \sigma_p(B) = \sigma$. Further, let $C \in L(V_1, V_2)$ be an arbitrary densely defined linear operator, such that $\mathcal{D}_B \subset \mathcal{D}_C$.

1. If the condition
$$C\left(\mathcal{N}(B - \lambda I)\right) \subset \mathcal{R}(A - \lambda I), \tag{4.49}$$
holds for every $\lambda \in \sigma$, then there exist infinitely many solutions X_E to the equation (4.1), defined on D_E
$$\left\{ u \in \mathcal{N}(B - \lambda I) : \ \lambda \in \sigma, \sum_{\lambda \in \sigma} P_{\mathcal{N}(A-\lambda I)^\perp}(A - \lambda I)^{-1} Cu \text{ converges} \right\}. \tag{4.50}$$

2. In addition, B_1 is a densely defined closed self-adjoint operator as well, $B_1 : \mathcal{D}_{B_1} \to E_B^\perp$. Assume that $\mathcal{D}_{B_1} \subset \mathcal{D}_{C_0}$, and that the following implication holds

$$0 \in \sigma(A) \cap \sigma(B_1) \Rightarrow 0 \in \sigma_p(A) \cap \sigma_p(B_1), \qquad \text{and} \tag{4.51}$$
$$C(\mathcal{N}(B_1)) \subset \mathcal{R}(A). \tag{4.52}$$

Then there exist infinitely many solutions X_1 to the eq. (4.1), defined on (with respect to inclusion) the largest subspace $\mathcal{D}_{X_1} \subset E_B^\perp$.

3. The solutions $X := X_E + X_1$, obtained in parts 1 and 2 of this theorem, defined on their largest domains (with respect to inclusion) $\mathcal{D}_E + \mathcal{D}_{X_1}$, are unique in the quotient class of operators from

$$L(V_1/(E_B + \mathcal{N}(B_1)), V_2/(E_A + \mathcal{N}(A))), \tag{4.53}$$

defined on the same domain.

Remark. This theorem is proved in a very similar manner Theorem 2.1.1 was proved. In addition, Statement 3. and expression (4.53) naturally generalize the characterization of matrix solutions obtained in the eigen-problem (2.5).

Proof. For every $\lambda \in \sigma$, let E_B^λ, E_B, $E_B^\perp$, B_E, B_1 and C_0 be provided as in the previous paragraph.

1. Define $E_A = \overline{\sum_\lambda E_A^\lambda}$ and split $\mathcal{D}_A$ into orthogonal sum

$$\mathcal{D}_A = (\mathcal{D}_A \cap E_A) \oplus \left(E_A^\perp \cap \mathcal{D}_A\right).$$

Decompose $A = A_E \oplus A_1$ with respect to that sum. Then A_1 is injective on $E_A^\perp \cap \mathcal{D}_A$ and $A_1 v = Av$, for every $v \in E_A^\perp \cap \mathcal{D}_A$. For every $\lambda \in \sigma$ let $N_\lambda \in L(E_B^\lambda, E_A^\lambda)$ be arbitrary. For every $u \in E_B^\lambda$, by assumption (4.49), there exists a unique $d(u) \in \left(E_A^\lambda\right)^\perp \cap \mathcal{D}_A$ such that

$$(A - \lambda I)d(u) = Cu. \tag{4.54}$$

Define

$$X_E^\lambda : u \mapsto N_\lambda u + d(u), \qquad u \in E_B^\lambda \cap D_E. \tag{4.55}$$

Then $X_E^\lambda : D_E \cap E_B^\lambda \to E_A^\lambda \oplus \left(P_{E_A^\lambda{}^\perp}(A_1 - \lambda I)^{-1} C E_B^\lambda\right)$ defines a linear map. What is left is to check whether $X_E := \sum_{\lambda \in \sigma} X_E^\lambda$ is a solution to the equation

$$AX_E - X_E B_E = CP_{E_B} \tag{4.56}$$

restricted to $E_B \cap D_E$. However, this is directly verifiable. For any $u \in E_B \cap D_E$ there exist unique scalars α_λ and unique $u_\lambda \in E_B^\lambda \cap D_E$, $\lambda \in \sigma$, such that $u = \sum \alpha_\lambda u_\lambda$. Then

$$(AX_E - X_E B)\,u = A\sum_{\lambda\in\sigma} \alpha_\lambda X_E^\lambda u_\lambda - \sum_{\lambda\in\sigma} \lambda\alpha_\lambda X_E^\lambda u_\lambda =$$
$$\sum_{\lambda\in\sigma} (\alpha_\lambda(A-\lambda I))\,(N_\lambda u_\lambda + d(u_\lambda)) = \sum_{\lambda\in\sigma} \alpha_\lambda C u_\lambda = Cu.$$

2. We now conduct analysis on $E_B^\perp$. The space E_B is $B-$invariant subspace of V_1, then for every $a \in E_B \cap \mathcal{D}_B$ and every $b \in E_B^\perp \cap \mathcal{D}_B = \mathcal{D}_{B_1}$ we have (recall that $\mathcal{D}_B$ is dense in the enitre V_1)

$$0 = \langle Ba, b\rangle = \langle a, Bb\rangle = \langle a, B_1 b\rangle,$$

so $B_1 : \mathcal{D}_{\mathcal{B}_1} \to E_B^\perp$ defines a closed, densely defined, self-adjoint operator in $E_B^\perp$. Since $\sigma \subseteq \sigma(B_E) \subseteq \{0\} \cup \sigma$, it follows that

$$\sigma(B_1) \subseteq \{0\} \cup \sigma(B) \setminus \sigma.$$

Case 1. Assume that $\sigma(B_1) \cap \sigma(A) = \emptyset$. Closed operators A and B_1 allow spectral decompositions with respect to the Spectral Mapping Theorem: let $(P_\lambda : \quad \lambda \in \mathbb{R})$ and $(Q_\mu : \quad \mu \in \mathbb{R})$ be spectral resolutions of the corresponding identities such that

$$A = \int_{-\infty}^{+\infty} \lambda \,\mathrm{d}\, P_\lambda, \qquad\qquad B_1 = \int_{-\infty}^{+\infty} \mu \,\mathrm{d}\, Q_\mu,$$

where the spaces $E_B^\perp$ and V_2 are decomposed as

$$E_B^\perp = \oplus_i V_{1i}, \qquad V_2 = \oplus_j V_{2j}, \tag{4.57}$$

and every V_{1i} (V_{2j}) corresponds to $(Q_{\mu_i} - Q_{\mu_i - 0})\, E_B^\perp$ (respectively, $\left(P_{\lambda_j} - P_{\lambda_j - 0}\right) V_2$). For fixed $i \in \mathbb{N}$, let $C_i = C\,(Q_{\mu_i} - Q_{\mu_i - 0})$ and let $J(i)$ be the set of indices j such that

$$J(i) = \{j : \quad \mathcal{R}(C_i) \cap V_{2j} \neq \emptyset\}^2.$$

Define

$$V_{2J(i)} = \overline{\oplus_{j\in J(i)} V_{2j}}.$$

[2] $J(i)$ must be non-empty, because (4.57) exausts the entire space V_2.

There exists unique X_i which is a bounded operator in $\mathcal{B}\left(V_{1i}, V_{2J(i)}\right)$, which is a solution to

$$A\left(P_{\lambda_{J(i)}} - P_{\lambda_{J(i)}-0}\right) X_i - X_i B\left(Q_{\mu_i} - Q_{\mu_i-0}\right) = C_i. \tag{4.58}$$

Note that C_i is defined wherever B_i is defined, as is bounded on that set. Then $X_1 = \oplus_i X_i$ is a unique solution to (4.1), defined on its natural domain $\mathcal{D}_{X_1}$, that is, wherever the sum $\oplus_i X_i$ converges.

Case 2. Assume that $\sigma(A)\cap\sigma(B_1) \neq \emptyset$. It follows that $\sigma(A)\cap\sigma(B_1) = \{0\}$. Define $\widehat{A} : \mathcal{D}_A/\mathcal{N}(A) \to V_2$ in a natural way, that is, for every $v \in \mathcal{D}_A$, decompose $v = v_1 + v_2$, where $v_1 \in \mathcal{N}(A)$ and $v_2 \in \mathcal{D}(A)\cap\mathcal{N}(A)^{\perp}$. Then $\widehat{A}\left(v_2 + \mathcal{N}(A)\right) := Av_2 = Av$. Similarly, define

$$\widehat{C_0} : \mathcal{D}_{C_0} \to \mathcal{D}(A)/\mathcal{N}(A), \qquad \widehat{C_0}(u) = P_{\mathcal{N}(A)^{\perp}} C_0 u, \quad u \in \mathcal{D}_{C_0},$$

and

$$\widehat{B_1} : \mathcal{D}(B_1)/\mathcal{N}(B_1) \to \mathcal{R}(B_1), \qquad \widehat{B_1}(u + \mathcal{N}(B_1)) := B_1 u, \;\; u \in \mathcal{D}(B_1).$$

Then $\sigma(\widehat{A}) \cap \sigma(\widehat{B_1}) = \emptyset$, and Case 1. of this proof applies, i.e. there exists a unique $\widehat{X}$ defined on its natural domain $\mathcal{D}_{\widehat{X}}$ such that

$$\widehat{A}\widehat{X} - \widehat{X}\widehat{B_1} = \widehat{C_0} \tag{4.59}$$

holds. Similarly to part 1. the condition (4.52) implies the following analysis. Let $N_0 \in L(\mathcal{N}(B_1), \mathcal{N}(A))$ be an arbitrary linear operator, and take

$$X_0 : \mathcal{N}(B_1) \to \mathcal{N}(A) + P_{\mathcal{N}(A)^{\perp}}\left(\widehat{A}\right)^{-1}\widehat{C}\left(\mathcal{N}(B_1)\right)$$

as in (4.55), where $\lambda = 0$, that is, define

$$X_0 u := N_0 u + P_{\mathcal{N}(A)^{\perp}}\left(\widehat{A}\right)^{-1}\widehat{C}u, \qquad u \in \mathcal{N}(B_1). \tag{4.60}$$

Finally, adding $X_1 := \widehat{X} + X_0$ gives (one of the) desired solution(s), defined on $\mathcal{D}_{X_1} = \mathcal{D}_{\widehat{X}} + \mathcal{D}_{X_0}$.

3. Adding the operators X_E and X_1, obtained in parts 1. and 2. gives the solutions of the form $X = X_E + X_1$. Zorn's lemma proves that there exist domains D_E and $\mathcal{D}_{X_1}$ such that $\mathcal{D}_X$ is the largest set possible, with respect to inclusion. Now assume there exits another Y that is

a solution to the said Sylvester equation, defined on the same domain as X. Decompose $Y = Y_E \oplus Y_1$ with respect to E_B and $E_B^{\perp}$. Since the equation $AX_1 - X_1B = C_1$ has a unique solution in the class $L(E_B^{\perp}/\mathcal{N}(B_1), V_2/\mathcal{N}(A))$, it follows that $Y_1 \in X_1 + L(\mathcal{N}(B_1), \mathcal{N}(A))$. Further, for every shared eigenvalue λ, the element $d_\lambda \in \left(E_A^\lambda\right)^\perp$ defined in (4.54) is uniquely determined. Thus $Y^\lambda : E_B^\lambda \to d_\lambda + E_A^\lambda$. This proves that $X - Y = 0 + L(E_B + \mathcal{N}(B_1), E_A + \mathcal{N}(A))$.

□

Corollary 4.4.1. [24, Corollary 2.1.] (*Number of solutions*) With respect to the previous notation, let all assumptions from Theorem 4.4.2 hold. Denote by Σ and Ω the sets of linear operators such that

$$\Sigma = \left\{N_\sigma : \quad N_\sigma = \oplus_{\lambda\in\sigma} N_\lambda, \quad N_\lambda \in L(E_B^\lambda, E_A^\lambda), \quad \lambda \in \sigma\right\}$$

and

$$\Omega = \{N_0 \in L(\mathcal{N}(B_1), \mathcal{N}(A))\}.$$

Let S be the set of all solutions to (4.1), which are defined on the largest domains possible. Then $|\Omega| \cdot |\Sigma| = |S|$.

Proof. Proof follows directly from Theorem 4.4.2, because choices for solutions depend solely on N_λ and N_0, whenever $\lambda \in \sigma$, as illustrated in (4.55) and (4.60). □

Remark. Due to Corollary 4.4.1, the solution $X_{(N_\sigma+N_0)} \in S$, with $N_\sigma \in \Sigma$ and $N_0 \in \Omega$, can be referred to as a *particular solution*. However, this implies that the particular solutions depend on the choice of the corresponding eigenvectors of A and B. Consequently, the solutions are unstable to small perturbations, because even the slightest changes in operators A and B can generate drastically different corresponding eigenvectors.

4.4.2 The case when $\sigma = \sigma_{app}(A) \cap \sigma_{app}(B)$

We now investigate the general case, where the spectral intersection occurs in the approximate point spectra of A and B. Let $L \in \{A, B\}$, and assume that $\lambda \in \sigma_{app}(L)$, that is, there exists a sequence $(x_n) \subset \mathcal{D}_L$ such that $\|x_n\| = 1$ while $\|(L - \lambda I)x_n\| \to 0$ as $n \to \infty$. The main idea is to construct a set which resembles an approximate eigenspace with respect to λ, in order to apply the same method from the previous case.

The problem of transferring the approximate point spectrum to the set of eigenvalues was firstly solved by Berberian in [8], which was further applied to bounded Fredholm operators by Wickstead, Buoni and Harte in [12] and [44]. To start, assume that L is a bounded normal operator on a Hilbert space V. Then for fixed μ and $\lambda \in \sigma_{app}(L)$, there exist two normed sequences (x_n) and (y_n), such that $\|(L-\lambda I)x_n\|$ and $\|(L-\mu I)y_n\|$ simultaneously tend to zero as n approaches infinity. Then for every n:

$$\begin{aligned}|(\mu-\lambda)\langle x_n, y_n\rangle| &= |\langle \lambda x_n - Lx_n, y_n\rangle + \langle x_n, L^* y_n - \bar{\mu} y_n\rangle| \\ &\leq \|\lambda x_n - Lx_n\| + \|Ly_n - \nu y_n\|,\end{aligned}$$

which tends to zero as $n \to +\infty$. This implies that approximate eigenvectors corresponding to different approximate eigenvalues tend to behave in an orthogonal manner, similarly to the exact eigenvectors corresponding to the actual different eigenvalues. This motivates the characterization of the approximate point spectrum of all bounded linear operators $L \in \mathcal{B}(V)$, which goes as the following (see [8]). Denote by $\ell_\infty(V)$ the space of all bounded sequences with values in V, equipped with the sup $-$norm. The set of all sequences which converge to zero is denoted by $c_0(V)$. It follows that c_0 is, with respect to the relative topology inherited from $\ell_\infty(V)$, a proper closed subspace, and defines a quotient space $\ell_\infty(V)/c_0(V)$ in a natural way. What is left is to enclose this space, in a manner that $\overline{\ell_\infty(V)/c_0(V)}$ forms a complete inner product space, with inner product defined via the generalized limits (called Banach limits) in $\ell_\infty(V)$ (see [8] for a more detailed construction). For a sequence $(x_n)_n \in \ell_\infty(V)$, a bounded linear operator $L \in \mathcal{B}(V)$ defines a bounded linear map on $\ell_\infty(V)$ as

$$L'((x_n)_n) := (Lx_n)_n \in \ell_\infty(V).$$

Furthermore, it follows that $L'(x_n) \in c_0(V)$, whenever $(x_n) \in c_0(V)$. Hence, $L_0' : \ell_\infty(V)/c_0(V) \to \ell_\infty(V)/c_0(V)$ defines a bounded linear operator, such that $L_0'((x)_n/c_0(V)) := (L'(x_n))/c_0(V)$, for every $(x_n) \in \ell_\infty(V)$. This implies that $\|L\| = \|L_0'\|$, and that L_0' extends continuously to the entire space $\overline{\ell_\infty(V)/c_0(V)}$, and that extension is denoted again by L_0'.

Theorem 4.4.3. [8, Theorem 1] For every $L \in \mathcal{B}(V)$, $\sigma_{app}(L) = \sigma_{app}(L_0') = \sigma_p(L_0')$.

Combining the previous discussion with the spectral mapping theorem for self-adjoint operators (Theorem 4.4.1), we modify Theorem 4.4.3 and apply it to our own problem.

Lemma 4.4.1. [24, Lemma 2.1.] Let V be a Hilbert space and let L be a densely defined bounded self-adjoint operator on V. Then there exists L_0' defined in the previous manner. For that L_0' we have $\sigma_{app}(L) = \sigma_p(L_0') = \sigma_{app}(L_0')$.

Proof. Again, observe the spaces $\ell(V)$ and $c_0(V)$. For every $(x_n)_n \in \ell(V)$ such that $x_n \in \mathcal{D}_L$, for every n, boundedness of L implies that $(Lx_n)_n \in \ell(V)$. This defines a bounded operator $L'(x_n)_n \mapsto (Lx_n)_n$, from $\mathcal{D}_{L'}$ to $\ell(V)$. The operator L is densely defined in V then so is L' in $\ell(V)$. Similarly, boundedness of L implies that $L' : \mathcal{D}_{L'} \cap c_0(V) \to c_0(V)$, and $\mathcal{D}_{L'}$ is dense in $c_0(V)$. For simpler notation, denote by $\mathcal{D}_L^\infty = \mathcal{D}_{L'}$ and $\mathcal{D}_L^0 = \mathcal{D}_{L'} \cap c_0(V)$. Now define an operator $L_0' : \mathcal{D}_{L_0'} \to \overline{\ell(V)/c_0(V)}$, where $\mathcal{D}_{L_0'} = \mathcal{D}_L^\infty/\mathcal{D}_L^0$ is densely defined in $\overline{\ell(V)/c_0(V)}$. Now trivially, if $\lambda \in \sigma_{app}(L) \setminus \sigma_p(L)$, then there exists a normed sequence $(x_n)_n \subset \mathcal{D}_L$, such that $((L-\lambda I)\,x_n)_n \in \mathcal{D}_{L'}^0$, but this means that

$$(L-\lambda I)_0' \left((L-\lambda I)\,x_n\right)_n = 0 \in \overline{\ell(V)/c_0(V)},$$

thus proving that

$$\lambda \in \sigma_p(L_0').$$

On the other hand, if $\lambda \in \sigma_{app}(L_0')$, then direct computation shows that $\lambda \in \sigma_{app}(L)$, thus

$$\sigma_{app}(L_0') \subset \sigma_{app}(L) \subset \sigma_p(L_0') \subset \sigma_{app}(L_0').$$

□

Theorem 4.4.4. [24, Theorem 2.3] *(The general case)* Let $A \in L(V_1)$ and $B \in L(V_2)$ be closed densely defined self-adjoint operators on separable Hilbert spaces V_1 and V_2, with spectral resolutions of identities

$$B = \int_{-\infty}^{+\infty} \mu \,\mathrm{d}\, F_\mu, \qquad V_1 = \oplus_n V_{1n}, \quad B_n : V_{1n} \to V_{1n} \text{ is a bounded operator} \tag{4.61}$$

and

$$A = \int_{-\infty}^{+\infty} \lambda \,\mathrm{d}\, E_\lambda, \qquad V_2 = \oplus_n V_{2n}, \quad A_n : V_{2n} \to V_{2n} \text{ is a bounded operator.} \tag{4.62}$$

Assume that $\sigma_{app}(B) \cap \sigma_{app}(A) =: \sigma \neq \emptyset$ and let $C \in L(V_1, V_2)$ be arbitrary densely defined linear operator, such that $\mathcal{D}_B \subset \mathcal{D}_C$. For every n, let operators $(B_n)_0'$ and $(A_n)_0'$ be defined as in the previous paragraph and let $(C_n)_0'$ be defined accordingly. If operators $(A_n)_0'$, $(B_n)_0'$ and $(C_n)_0'$ satisfy conditions (4.49)–(4.52) from Theorem 4.4.2, then there exist infinitely many solutions to (4.1), defined on the largest subsets of V_1 possible.

Proof. The first step is to apply spectral decomposition as in (4.61) and (4.62). Now if $\sigma = \sigma_p(A) \cap \sigma_p(B)$, then Theorem 4.4.2 applies. Otherwise, apply Lemma 4.4.1 to each B_n and A_n, respectively. Then the problem is transferred to the first case, that is, the spectral intersection occurs in the point spectra. If the conditions (4.49)–(4.52) are satisfied, then Theorem 4.4.2 applies and the proof is complete. $\square$

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Biography

Bogdan Đorđević was born in Leskovac, Serbia, on May 5th in 1993. He graduated from elementary school "Učitelj Tasa" in Niš, Serbia, class of 2008, as a bearer of "Vuk Karadžić" award (perfect GPA and awards in mathematics and physics competitions). He graduated from grammar school "Svetozar Marković" in Niš, class of 2012, module "advanced mathematics, physics and computer science", again as a bearer of "Vuk Karadžić" award. He attended Mathematics Summer School at research center "Petnica" during summer in 2011 and in 2012, where he presented his research titled "Representations of braids via rotational matrices" (in Serbian).

On fall 2012 he started BA studies in "Mathematics" at Faculty of Sciences and Mathematics, University of Niš, and graduated from the University in 2015, as an Honoring student, with perfect GPA (10/10). In summer 2015, Bogdan had already started his scientific research, at that moment, in the field of matrix analysis, and had presented his work on scientific congress GFTA held in Ohrid, FYR Macedonia, in July 2015 (these results were later published in 2016 in Annallele stiintifite Un. Al. I. Cuiza din Yassi, ser. Mathematica). On fall 2015, Bogdan started MA studies in "General mathematics" at Faculty of Sciences and Mathematics, University of Niš, and had graduated once again as an Honoring student in 2017, thesis title: "Solving the Sylvester matrix equation" (in Serbian). During his BA and MA studies, Bogdan was funded by national grants and stipends issued by Ministry of Education Science and Technical Development of Republic of Serba and by Ministry of Youth and Sports (Republic of Serbia). He received a plaque from city of Niš for being the best graduate student from Faculty of Sciences and Mathematics, University of Niš, in the school year 2016/17. In 2017 He won an award given by Mathematical Institute of the Serbian Academy of Science and Arts for having the best master thesis in the field of mathematics, mechanics.

On fall 2017, Bogdan started PhD School of Mathematics, module analysis, at the same faculty. He completed his PhD exams and finals in February 2020, with the perfect GPA (10/10). He has broadened his field of research to functional analysis, operator theory and their applications. He has published five scientific papers in this field and is active reviewer for prestigious scientific mathematical journals, such as Linear Multilinear Algebra (T&F) and Complex Analysis and Operator Theory (Springer). So far, he has given seven talks in scientific congresses and workshops, including "IWOTA", that took place in Lisbon on July, 2019.

From 2015 to 2019, Bogdan was involved in teaching activities at Faculty of Sciences and Mathematics, University of Niš, Serbia, and volunteered as a teaching assistant in courses of Real and Complex Mathematical Analysis. Since 2018, he has been teaching mathematics part time in grammar school "Svetozar Marković" in Niš, module "advanced mathematics, physics and computer science." He participates as a mentor and a lecturer at "Petnica" research center for talented high-schoolers.

From January 2018 to December 2019, Bogdan was a research trainee at the Faculty of Sciences and Mathematics, University of Niš, involved in the project "Functional analysis, stochastic analysis and applications", project no. 174004, funded by the Ministry of Education, Science and Technical development of Republic of Serbia. In December 2019 he transferred to Mathematical Institute of the Serbian Academy of Sciences and Arts, where he is currently working as a researcher, working on the same scientific project.

List of publications:

1. B. D. Djordjević, On a singular Sylvester equation with unbounded self-adjoint A and B, Complex Analysis and Operator Theory 14:43 (2020) https://doi.org/10.1007/s11785-020-01000-7
   IF2019=0.711, SCIe, M22

2. B. D. Djordjević, Operator algebra generated by an element from the module $\mathcal{B}(V_1, V_2)$, Complex Analysis and Operator Theory (ISSN 1661-8254), 13:5 (2019) 2381–2409 https://doi.org/10.1007/s11785-019-00899-x
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3. B. D. Djordjević, N. Č. Dinčić, Classification and approximation of solutions to Sylvester matrix equation, Filomat 33 (13) (2019) 4261-4280 https://doi.org/10.2298/FIL1913261D
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4. B. D. Djordjević, N. Č. Dinčić, Solving the operator equation $AX - XB = C$ with closed A and B, Integral Equations and Operator Theory (ISSN: 0378-620X (print version), ISSN: 1420-8989 (electronic version)), 90:51 (2018) https://doi.org/10.1007/s00020-018-2473-3 IF2016=0.787, SCI, SCIe, M22

5. B. D. Djordjević, The singular value of $A + B$ and $\alpha A + \beta B$, Scientific Annals of the Alexandru Ioan Cuza University of Iasi (NS). Mathematics An. Stiint. Univ. Al. I. Cuza Ia?i. Mat. (N.S.), f.2 3 (2016) 737-743

List of conferences and talks:

1. GFTA 2015, Ohrid, Macedonia, talk title: Singular values of some matrix pencils.

2. Algebra, Analysis with Applications 2018, Ohrid, Macedina, talk title: Some operator equations and algebras generated by them.

3. Second annual seminar on topology of configurational spaces 2018, Mathematical Institute Belgrade, talk title: Solving Sylvester operator equation and its applications.

4. IWOTA 2019, Lisbon, Portugal, talk title: On operator algebra generated by the general solution of Sylvester equation.

5. Kongres mladih matematičara u Novom Sadu 2019, Novi Sad, talk title: On some properties of singular Sylvester operator equations.

6. Susret matematičara Srbije i Crne Gore 2019, Budva, Montenegro, talk title: On singular matrix equations and connections with singular pdes.

7. Studentski seminar 2019, Matematički institute SANU, talk title: Ne-Banahove algebra operatora i singularne operatorske jednačine.

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Извод, **ИЗ**:	У овој дисертацији се изучава сингуларна Силвестерова операторска једначина, односно, операторска једначина облика *AX-XB=C*, под претпоставком да је она или нерешива или да има бесконачно много решења. Једначина се посматра у више разчличитих случаја, најпре у матричном случају, затим у случају када су у питању ограничени линеарни оператори на Банаховим просторима и коначно у случају када су у питању затворени линеарни оператори на Банаховим или Хилбертовим просторима. У сваком од поменутих сценарија се прво изводе довољни услови решивости полазне једначине, а онда се под тим претпоставкама прелази на њено решавање. Долази се до егзактних решења у затвореној форми, те се прелази на њихову класификацију и карактеризацију, односно, показује се да су изведеним посупцима обухваћена сва могућа решења сингуларне Силвестерове једначине. Посебна пажња је посвећена апроксимацијама решења. Добијени резултати су илустровани на неким савременим проблемима из теорије оператора, као што су спектрални проблеми ограничених и неограничених линеарних оператора, инверзни проблеми Штурм-Лиувилове теорије и операторске једначине које се јављају у квантној механици.
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Abstract, **AB**:		This thesis concerns singular Sylvester operator equations, that is, equations of the form $AX-XB=C$, under the premise that they are either unsolvable or have infinitely many solutions. The equation is studied in different cases, first in the matrix case, then in the case when A, B and C are bounded linear operators on Banach spaces, and finally in the case when A and B are closed linear operators defined on Banach or Hilbert spaces. In each of these cases, solvability conditions are derived and then, under those conditions, the initial equation is solved. Exact solutions are obtained in their closed forms, and their classification is conducted. It is shown that all solutions are obtained in the manner illustrated in this thesis. Special attention is dedicated to approximation schemes of the solutions. Obtained results are illustrated on some contemporary problems from operator theory, among which are spectral problems of bounded and unbounded linear operators, Sturm-Liouville inverse problems and some operator equations from quantum mechanics.
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