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DIFFERENT INVERTIBILITY MODIFICATIONS IN OPERATOR SPACES AND C^*-ALGEBRAS AND ITS APPLICATIONS

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MODIFIKACIJE INVERTIBILNOSTI NA PROSTORIMA OPERATORA I C^*-ALGEBRAMA I NJIHOVE PRIMENE

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Contents

Preface

In this thesis different modifications of invertibility in various settings and their applications are investigated. In particular, the reverse order law is considered for certain classes of generalized inverses in C^*-algebras with the goal of finding appropriate characterizing algebraic conditions, applications of generalized invertibility to solving certain types of equations in rings with unit and determining the general form of solutions. Another research concerns when the linear combinations of two operators belonging to the class of Fredholm operators. Some cases where the Fredholmness of linear combination is independent of the choice of the scalars will be described in detail.

In the first chapter we introduce fundamental concepts of the theory of generalized inverses. In Section 1.1 we present the need for the creation of the pseudoinverses. In Section 1.2 we describe the development of theory of generalized inverses. We also cite some influential books which cover this theory. Section 1.3 contains definitions and the most prominent results of different generalized inverses in different settings. We present the differences of one generalized inverse for various mathematical objects. We introduce the Moore-Penrose inverse, Drazin inverse, weighted Moore-Penrose inverse and the classes of generalized inverses which satisfy just some of Penrose equations in matrix, operator, C^*-algebra, Banach algebra and ring case. In Section 1.4 we describe some standard notation used in this dissertation.

The second chapter contains various reverse order laws. First, we mention some results related to the reverse order law for generalized inverses in different settings which had influence in our research. In Section 2.1 we present the original results published in [36] on the reverse order laws for $\{1,3\}$ and $\{1,4\}$- generalized inverses in C^*-algebras and particulary in the vector space of linear bounded operators on separable Hilbert spaces. Section 2.2 contains elementary algebraic proof of Hartwig's triple reverse order law for the regular elements in C^*-algebra from our paper [90]. Discussion on Hartwig's triple reverse order law continue in Section 2.3 which contains several significant improvements on Hartwig's triple reverse order law given in [33]. This new results are the product of our cooperation with the colleagues from The Johannes Kepler University Linz in Austria. In Section 2.4 we discuss the reverse order laws on $\{1,3\},\{1,4\},\{1,3,4\},\{1,2,3\}$ and $\{1,2,4\}$-inverses in a ring setting. We present therein new results from [34] which contain improvements of some known results in C^*-algebra case. The assumptions of the regularity of some elements are omitted.

In the third chapter, we present our original results, published in [35], about Fredholmness of a linear combination of two operators. Specially, we discuss some special cases when Fredholmness of a linear combination of two operators is independent of the scalars' choice, as well as some classes of operators for which is dependent.

The topic of the fourth chapter are the equations. First, we introduce basic results on this subject. In Section 4.1 we consider the algebraic conditions for the existence of a solution and the expression for the general solution of the system of three linear equations in a ring with a unit. Results from this section are published in [89]. In Section 4.2 we present some possible directions of further research.

We came to the most beautiful and the easiest part of this dissertation. After all I want to thank to my supervisor Professor Dragana Cvetković-Ilić for remarkable guidelines and selfless help in our research. This kind of dedication to job is exceptional. I am also thankful to all my professors who took me into world of mathematics. I want to thank to my family, my backbone, which took part in my every single act. At the end, thanks to my friends and all the people who helped to become what I am.

Chapter 1

Introduction

1.1 If I could turn back time...

For an element a from some algebraic structure with operation $\cdot$ and identity 1 we say that it is invertible if there exists an element b from that structure such that

$$a \cdot b = 1 = b \cdot a.$$

In everyday language we can say that invertibility of some process is ability of annulment that process with another one and vice verse.

Is this more invertible or non invertible process in real life probably depends of the eyes of prospector, but what about mathematical object? We will describe this on the set of $n \times n$ matrices over the field $\mathbb{F}, n \in \mathbb{N}$. We write $GL_n(\mathbb{F})$ for the set of $n \times n$ invertible matrices over the field $\mathbb{F}$ and $Sing_n(\mathbb{F})$ for the set of singular $n \times n$ matrices over $\mathbb{F}$.

Let $\mathbb{F}$ first be an infinite field of the cardinality c. If $n = 1$, then there is only one singular matrix (the zero matrix) and infinitely many invertible matrices. So there are more invertible matrices. If $n > 1$, then it is easy to construct a family of c singular matrices and to construct a family of c invertible matrices too. Since the cardinality of the set of all $n \times n$ matrices is also c, it follows that there are "the same number" of invertible and singular matrices in this case.

Now, let $\mathbb{F} = \mathbb{F}_q$ be a finite field with q elements. Over any field $\mathbb{F}$, the set of invertible $n \times n$ matrices is naturally bijective with the set of ordered bases for $\mathbb{F}_q^n$. For the first vector in a basis we can choose any non zero vector, so there are $q^n - 1$ choices. For the second one we can choose any vector not in the span of the first, so there are $q^n - q$ choices. Inductively, it follows that

$$GL_n(\mathbb{F}_q) = \prod_{k=0}^{n-1} (q^n - q^k).$$

This number is a monic polynomial of q of degree n^2. Hence $Sing_n(\mathbb{F}_q) = q^{n^2} - GL_n(\mathbb{F}_q)$ is a polynomial of q of degree strictly less than n^2. It follows that for fixed n, for q large enough, there will always be more invertible matrices.

But if we treat dimension as our notion of size, then the set of invertible matrices is always "bigger" than set of singular matrices. The set of all $n \times n$ matrices over a field $\mathbb{F}$ is isomorphic to the affine space $\mathbb{A}_{\mathbb{F}}^{n^2}$. The space of singular matrices is a hypersurface in this affine space, since it is precisely the zero set of the regular function $det : \mathbb{A}_{\mathbb{F}}^{n^2} \to \mathbb{F}$, and its complement, $GL_n(\mathbb{F})$, is therefore an open subvariety of the same affine space. It follows that $\dim(Sing_n(\mathbb{F})) = n^2 - 1$ and $\dim(GL_n(\mathbb{F})) = n^2$, where dimension here is in the sense of that of varieties over $\mathbb{F}$.

For intuition, this is totally analogous to the fact that a line and a plane (over the reals) have the same cardinality, but the plane has a greater dimensionality, so we think of it as bigger in that sense.

However, if we speak about invertibility in a matrix settings, matrix can have an inverse only if it is square, and even then it has an inverse only if its columns (or rows) are linearly independent. But what about singular or even rectangular matrices? Also, how to consider some basic problems (for example equations) whose participants are non invertible elements? In recent years needs have been felt in numerous areas of applied mathematics for some kind of partial inverse of non invertible elements. So, the concept of "pseudoinverse" or equivalently "generalized inverse" is created.

1.2 All starts in practice

In [118] Robinson gave a brief description of Gauss's participation in the theory of generalized inverses. Namely, motivated by his considerations of problems in geodesy, Gauss developed the method of least squares in 1794, but did not publish his results until several years later. In his works "Theoria motus" [60] and "Theoria combinations" [61], he improved the method of least square and even though he did not use the term generalized inverse, equivalent expressions and explicit formulaes for some specific class of matrices can be found. There is no evidence that he was willed to proceed in this direction (see also [107]).

The founder of the term pseudoinverse is Ivar Fredholm. In his famous paper [56] from 1903, Fredholm considered the problem of finding a continuous solution $f(x)$ of the functional equation, now named by him Fredholm integral equation of the second kind,

$$f(x) - \lambda \int_0^1 K(x,t)f(t)dt = g(x), \qquad 0 \leq x \leq 1, \tag{1.1}$$

in which $g(x)$ and the kernel $K(x,t)$ are given continuous functions, and λ is a given complex number. In the paper [114] from 1976, Rall described Fredholm pseudoinverse and its connection with the theory of generalized inverses developed in later years. Namely, the equation (1.1) can be treated as operator equation

$$(I - \lambda K)f = g,$$

where K is the linear integral operator with the continuous kernel $K(x,t)$ on the space of continuous functions on the segment $[0,1]$ denote by $C[0,1]$. Looking for the inverse of operator $I - \lambda K$ in the form $I + \lambda\Gamma$, where the Γ is linear integral operator with the continuous resolvent kernel $\Gamma(x, t; \lambda)$, and inspired by linear system of equations, Fredholm discovered that, except for certain isolated values of λ, equation (1.1) has a unique solution given by

$$f(x) = g(x) + \lambda \int_0^1 \Gamma(x, t; \lambda) g(t) dt, \qquad 0 \leq x \leq 1.$$

For the exceptional values of λ, now generally called eigenvalues of $K(x, t)$, Fredholm showed that equation (1.1) has no solutions or has an infinite family of solutions. If the equation (1.1) has infinitely many solutions, he constructed the pseudoinverse of the operator $I - \lambda K$ again in the form $I + \lambda H$, where the H is linear integral operator with the continuous pseudoresolvent kernel $H(x, t; \lambda)$.

The Fredholm's success led to a rapid development of the theory of integral equations and formed the basis of numerous concepts of functional analysis. However, the idea of the pseudoinverse of an integral operator was not pursued as intensively as some of the other Fredholm's concepts. Hilbert [68] in 1904, wrote about generalized Green's functions. In 1912, Hurwitz [70] used the finite dimensionality of the null-space of Fredholm operators to give a simple algebraic construction of class of all pseudoinverses. Let us mention also the works of Myller (1906), Westfall (1909), Bounitzky [16] in 1909, Elliott (1928), and Reid (1931). In [117] Reid gave a great purview of history of generalized inverses of differential and integral operators.

In the spirit of the phrase, from more complicated to simpler, generalized inverses first appeared in the settings operators and then in the settings of matrices. Moore was the first who gave a precise definition of generalized inverse for every finite matrix with complex entries (square or rectangular). In his paper [93] from 1920, he called that inverse *general reciprocal* and established its existence and uniqueness for any matrix A. He also gave an explicit form for general reciprocal of matrix A in terms of the subdeterminants of A and its conjugate transpose A^* (see [92] too). However, since Moore used unnecessarily complicated notation, his work was illegible for all but very dedicated readers. So it was not attract a lot of attention for 30 years after its first publication, during which time generalized inverses were given for matrices by Siegel [125] in 1937 and for operators by Tseng [139, 136, 137, 138] in 1933 and 1949, Murray and von Neumann [95] in 1936, Atkinson [3, 2] in 1951 and 1953 and others.

So, Moore's inverse needed to be revived. Bjerhammar [11, 12, 13] rediscovered it and also noted the relationship of generalized inverses to solutions of linear systems in the 1950's. In 1955, Penrose [103], unaware of Moore's work as Bjerhammar, extended Bjerhammar's results and gave different algebraic definition for Moore's inverse and showed its existence and uniqueness for any rectangular complex matrix A too. This discovery has been so important that this unique inverse is now commonly called the *Moore-Penrose inverse*.

Confirming the importance of simplicity of inscription, since 1955 the theory of generalized inverses has been started to develop rapidly. Generalized inverses which satisfy some of the Penrose's equations have been investigated, as well as some with different characteristics. Authors have been started to define and study generalized inverses in different settings (linear spaces of unbounded or bounded operators, C^*-algebras, Banach algebras, rings). Methods for their computation are also discovered because of their applicability. Therefore, we can notice generalized inverses in numerical analysis, control theory, cryptography, theory of differential equations and Markov chains, robotics and statistics.

Huge number of the original papers and some excellent books cover this topic. One of the fundamental books on that subject is "*Generalized inverses - Theory and Applications*" by A. Ben-Israel and T.N.E. Greville from 1974. and its second edition from 2003. [9]. It represent a comprehensive survey of generalized inverses illustrating the theory with applications in many areas mostly on matrix settings and additionally on operators between Hilbert spaces. "*Generalized inverses of linear transformations*" by S.L. Campbell and C.D. Meyer (1979) [17], another important book on this topic, describes utility of the concept of generalized inverses by presenting many diverse applications in which generalized inverses have an integral role. Let us mention Proceedings on of the Advanced Seminar on "*Generalized inverses and Applications*" edited by M.Z. Nashed, too. This book consists of 14 papers processing basic properties of generalized inverses, the Fredholm pseudoinverse (mentioned before), perturbations and approximations for generalized inverses, linear operator equations and applications to programming, games, networks and aggregation in econometrics. Book from 2003. by G. Wang, Y. Wei, S. Qiao [144] comprises a lot of published papers since mid-1970s on generalized inverses in the areas of perturbation theory, condition numbers, recursive algorithms. Generalized inverses of linear bounded operators on Banach and Hilbert spaces, and generalized inverses of elements in Banach and C^*-algebras are illustrated in [47] by V. Rakočević and D. Djordjević. Last but not least recent "*Algebraic Properties of Generalized Inverses*" by D.S. Cvetković Ilić and Y. Wei [40], describes the developments in the research directions on selected topics such as "reverse order law" problem, certain problems involving completions of operator matrices and Drazin inverse. The book also discusses the relevant open problems for each presented topic. One open problem stated in this book was a subject of one paper which will be presented in this dissertation.

1.3 The variety of generalized inverses in various settings

In this section it will be presented different generalized inverses for different mathematical objects. But always, we allude to the existence of identity and associativity of operation.

A basic intention of generalized inverse is to extend the concept of regular inverse. So, by generalized inverse of some element a we mean some element x such that

- exists for a wider class of elements than the class of invertible elements,
- has some of the properties of the usual inverse,

- reduces to the usual inverse for invertible element.

Thus, for some element can be defined different generalized inverses for different purposes i.e. with different characteristics of inverse of invertible element. Also, unlike the case of the invertible element, which has an unique inverse, there are more generalized inverses of some type.

The systematic study of generalized inverse firstly appears in matrix settings. As, we mentioned before, Moore constructed his "general reciprocal" for every finite matrix with complex entries and showed its uniqueness. Because of the complexity of the notation of the original Moore paper, we will give the interpreted Moore's definition by Ben-Israel and Charnes [8]:

Definition 1.3.1 *If $A \in \mathbb{C}^{m\times n}$, then the generalized inverse of A is the unique matrix $A^\dagger$ such that*

a) $AA^\dagger = P_{R(A)}$,

b) $A^\dagger A = P_{R(A^\dagger)}$.

Enlivened Moore's inverse in Penrose's work [103] has the following characterization:

Definition 1.3.2 *If $A \in \mathbb{C}^{m\times n}$, then the generalized inverse of A is the unique matrix $A^\dagger$ such that*

(1) $AA^\dagger A = A$,

(2) $A^\dagger AA^\dagger = A^\dagger$,

(3) $(AA^\dagger)^* = AA^\dagger$,

(4) $(A^\dagger A)^* = A^\dagger A$.

It is not hard to see that these two definitions are equivalent, or this proof can be seen in [17]. As we said, the unique matrix from Definitions 1.3.1 and 1.3.2 is called the *Moore-Penrose inverse* of matrix A and is usually denoted by $A^\dagger$. Equations $(1)-(4)$ are called Penrose equations. If A is regular matrix, then it can be easy checked that $A^\dagger = A^{-1}$.

Let now, $\mathcal{R}$ be arbitrary ring with a unit $1 \neq 0$ and an involution $a \mapsto a^*$ satisfying

$$(a^*)^* = a, \qquad (a+b)^* = a^* + b^*, \qquad (ab)^* = b^*a^*,$$

for all elements $a, b \in \mathcal{R}$. Since, Penrose's definition is algebraic it can be 'extended' on this setting. So, if for an element $a \in \mathcal{R}$ exists an element $x \in \mathcal{R}$ which satisfies the four Penrose equations we say that element a is Moore-Penrose invertible (MP-invertible). Element x is called the Moore-Penrose inverse of a and is denoted by $a^\dagger$. Similarly, from the matrix case we can carried uniqueness of Moore-Penrose inverse and conclusion that both $a^\dagger a$ and $aa^\dagger$ are projections, where by a projection we mean an element $p \in \mathcal{A}$ which is a Hermitian idempotent, i.e. such that $p^2 = p = p^*$.

Generalized inverses which satisfy some, but not all, of the four Penrose equations are very significant and they will be continuously present in this thesis. For $K \subseteq \{1,2,3,4\}$, we shall call $x \in \mathcal{R}$ a *K-inverse* of $a \in \mathcal{R}$ if it satisfies the Penrose equation (j) for each $j \in K$. We shall write aK for the collection of all K-inverses of $a \in \mathcal{R}$, and a^K for an unspecified element $x \in aK$. It is interesting to mention that first Fredholm's "pseudoinverse" is $\{1\}$-inverse of considered operator, but not necessary satisfies other Penrose equations. On the other hand, Hurwitz defined pseudoinverse which is $\{1,3,4\}$-inverse of discussed integral operator. This can be found in [114]. $\{1\}$-inverse is often called inner generalized inverse, $\{2\}$-inverse is outer and $\{1,2\}$-inverse is reflexive generalized inverse. In matrix settings $\{1,3\}$- and $\{1,4\}$-inverses are, because of their properties, also called least squares g-inverse and minimum norm g-inverses, respectively.

As we mentioned before, first application of generalized inverses was in a solving equations. In the matrix settings basic problem is to solve linear system $Ax = b$ where $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. This equation has a solution $x \in \mathbb{C}^n$ if and only if b is in the range of A. Otherwise, the residual vector $r = b - Ax$ is nonzero for all $x \in \mathbb{C}^n$, and it is desirable to find the best approximate solution in some sense. In other words, we may found the vector x which minimizes some norm of r. The most frequently used norm on $\mathbb{C}^m$ is the Euclidean norm which gives approximate solution x which minimizes the expression

$$\|b - Ax\|^2 = \sum_{i=1}^{m} \left|b_i - \sum_{j=1}^{n} a_{ij}x_j\right|^2.$$

Because of the definition of the Euclidean norm this solution is called the least-square solution of given system. The following theorem from [9] establishes the connection between the $\{1,3\}$-inverses and the least-squares solutions of $Ax = b$.

Theorem 1.3.1 [*9*] *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then $\|b - Ax\|$ is the smallest for $x = A^{(1,3)}b$, where $A^{(1,3)} \in A\{1,3\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, $\|b - Ax\|$ is the smallest when $x = Xb$, then $X \in A\{1,3\}$.*

Another problem is to find a solution of minimum norm of consistent equation $Ax = b$. The following theorem from [9] asserts that there is a unique minimum-norm solution of consistent equation $Ax = b$ and connects that solution with $\{1,4\}$-inverses of A.

Theorem 1.3.2 [*9*] *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. If $Ax = b$ is solvable, then the unique solution for which $\|x\|$ is the smallest, is given by*

$$x = A^{(1,4)}b,$$

where $A^{(1,4)} \in A\{1,4\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ is such that, whenever $Ax = b$ is solvable, $x = Xb$ is the solution of minimum norm, then $X \in A\{1,4\}$.

Again, the leading part goes to the the Moore-Penrose inverse $A^\dagger$. Combining the previous two theorems, the formulae for the unique minimum-norm least-squares solution of $Ax = b$ is deduced.

Corollary 1.3.1 *[104] Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then, among the least-squares solutions of $Ax = b$, $A^\dagger b$ is the one of minimum-norm. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, Xb is the minimum-norm least-squares solution of $Ax = b$, then $X = A^\dagger$.*

Directly, the minimum-norm least-squares solution, $x_0 = A^\dagger b$ (*approximate solution* [104]) of $Ax = b$, is vector x which satisfies the following two inequalities:

$$\|Ax_0 - b\| \leq \|Ax - b\|, \quad \text{for all} \quad x \in \mathbb{C}^n \tag{1.2}$$

and

$$\|x_0\| < \|x\|,$$

for any $x \neq x_0$ which gives equality in (1.2).

Unlike uniqueness of Moore-Penrose inverse, its existence is not guaranteed in different structures. An element a from a ring $\mathcal{R}$ (not necessarily with involution) is called *regular* (in the sense of von Neumann) if it has an inner inverse. In the excellent paper [65], Harte and Mbekhta gave the sufficient and necessary conditions for the the existence of the Moore-Penrose inverse of some element of C^*-algebra. Unifying the results from this paper we give the following

Theorem 1.3.3 *[65] Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$. The following conditions are equivalent:*

(i) a is regular,

(ii) $a\mathcal{A}$ is closed,

(iii) $\mathcal{A}a$ is closed,

(iv) $a^\dagger$ exists.

Clearly, a is MP-invertible if and only if a^* is MP-invertible. In this case

$$(a^*)^\dagger = (a^\dagger)^*.$$

If a is MP-invertible, then so are a^*a and aa^*, while

$$(a^*a)^\dagger = a^\dagger(a^*)^\dagger, \quad (aa^*)^\dagger = (a^*)^\dagger a^\dagger.$$

In C^*-algebra we have more

Theorem 1.3.4 *[65] If $a \in \mathcal{A}$ for a C^*-algebra $\mathcal{A}$, then*

$$a \text{ is regular} \;\Leftrightarrow\; aa^* \text{ is regular} \;\Leftrightarrow\; a^*a \text{ is regular}.$$

Example 1.3.1 Let $\mathcal{R}$ be an algebra of 2×2 matrices over $\mathbb{C}$ with involution $A \mapsto A^T$, for $A \in \mathcal{R}$. Element $A = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ is regular but it is not MP-invertible. Indeed, by a straightforward computation from Penrose equations, it can be derived:

$$A\{1\} = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} \mid x + iz = 1, x, y, z, t \in \mathbb{C} \right\},$$

$$A\{3\} = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} \mid y + it = 0, x, y, z, t \in \mathbb{C} \right\},$$

$$A\{4\} = \left\{ \begin{bmatrix} x & y \\ ix & t \end{bmatrix} \mid x, y, z, t \in \mathbb{C} \right\}.$$

We can see that $A\{1,4\} = \emptyset$, while $A\{1\} \neq \emptyset, A\{1,3\} \neq \emptyset$. Let us recall that if $a\{1,3\} \neq \emptyset$ and $a\{1,4\} \neq \emptyset$, then a is MP-invertible and $a^\dagger = a^{(1,4)}aa^{(1,3)}$.

To present equivalent conditions for MP-invertibility of some element of a ring with involution we must introduce another important generalized inverse. Drazin inverse in rings and semigroups, name after his founder, is introduced in [49] in 1958.

Definition 1.3.3 *Let $\mathcal{R}$ be a ring. The* Drazin inverse *of $a \in \mathcal{R}$ is the element $a^D \in \mathcal{R}$ which satisfies*

$$a^D a a^D = a^D, \quad a a^D = a^D a, \quad a^{k+1} a^D = a^k,$$

for some nonnegative integer k. The least such k is the index *of a, denoted by $ind(a)$. If $ind(a) \leq 1$, then the Drazin inverse a^D is called the* group inverse *and is denoted by a^g or $a^\sharp$.*

Drazin inverse is unique if it exists and is sometimes called spectral inverse because of its spectral properties which are similar to ordinary inverse. Namely, in the algebra of square matrices of dimension n the nonzero eigenvalues of the Drazin inverse are the reciprocals of the nonzero eigenvalues of the given matrix, and the corresponding generalized eigenvectors have the same grade. It is also known that it exists for every square matrix and it is its polynomial. The inverse was a subject of huge number of research papers and applications in various areas not only in matrix settings [17, 9, 113, 88] but in the setting of bounded linear operators and Banach algebras [87, 74, 81, 22].

Definition 1.3.4 *An element $a \in \mathcal{R}$ is* left $*$-cancellable *if $a^*ax = a^*ay$ implies $ax = ay$, it is* right $*$-cancellable *if $xaa^* = yaa^*$ implies $xa = ya$, and* $*$-cancellable *if it is both left and right $*$-cancellable.*

We observe that a is left $*$-cancellable if and only if a^* is right $*$-cancellable. In a C^*-algebra every element is $*$-cancellable.

The basic existence theorem for the Moore Penrose inverse in the setting of rings with involution was given in Theorem 8.25 [10] or Theorem 5.3 [76]:

Theorem 1.3.5 *Let $\mathcal{R}$ be a ring with involution and let $a \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *a is MP-invertible,*

(ii) *a is left $*$-cancellable and a^*a is group invertible,*

(iii) *a is right $*$-cancellable and aa^* is group invertible,*

(iv) *a is $*$-cancellable and both a^*a and aa^* are group invertible.*

The MP-inverse of a is given by $a^\dagger = (a^*a)^g a = a^*(aa^*)^g$.

Let us mention the following theorem, which characterize MP-invertibility of an element in ring with involution, too. First we give the necessary definition.

Definition 1.3.5 *An element a of a ring $\mathcal{R}$ with involution is* well-supported *if there exists a Hermitian idempotent p such that $ap = a$ and $a^*a + 1 - p$ is invertible. The idempotent p is called the* support *of a.*

Theorem 1.3.6 [*75*] *Let $\mathcal{R}$ be a ring with involution. An element $a \in \mathcal{R}$ is MP-invertible if and only if a is left $*$-cancellable and well-supported. The support p of a is given by $p = a^\dagger a$.*

The question is how to generalize definition of MP- inverse in the Banach algebra case? This did Rakočević in [110] using Vidav's definition of Hermitian element in complex Banach algebra with a unite.

Definition 1.3.6 [*140*] *Let $\mathcal{A}$ be a complex Banach algebra with a unite. An element $a \in \mathcal{A}$ is said to be* Hermitian *if $\|e^{ita}\| = 1$ for all $t \in \mathbb{R}$.*

In the case of the algebra of bounded linear operators in a Hilbert space, this definition is equivalent with ordinary definition of Hermitian operator, i.e. self-adjoint operator ($A = A^*$). For this reason, Rakočević treated elements $a \in \mathcal{A}$ for which there exists an element $x \in \mathcal{A}$ satisfying the following four conditions:

$$\begin{array}{ll} (1) & axa = a, \\ (2) & xax = x, \\ (3) & ax \text{ is Hermitian}, \\ (4) & xa \text{ is Hermitian}. \end{array}$$

For an element $a \in \mathcal{A}$ by L_a and R_a we denote the left and the right regular representation of $a \in \mathcal{A}$, i.e. functions from $\mathcal{A}$ to $\mathcal{A}$ defined by $L_a x = ax$ and $R_a x = xa$, for $x \in \mathcal{A}$. By Definition 1.3.6 can be seen that if a is Hermitian in $\mathcal{A}$, then L_a and R_a are Hermitian in $\mathcal{B}(\mathcal{A})$ ([14]). Like in other settings in [110] is proved the following

Lemma 1.3.1 [*110*] *For $a \in \mathcal{A}$ there is at most one x such that conditions* $(1) - (4)$ *are satisfied.*

Proof of previous lemma is not the same like in matrix case and is based on L_a, R_a and the fact that Hermitian idempotent operator on Banach space is determined with its range (see [97]). Now, analogously as in the ring with involution case, MP-inverse of an element $a \in \mathcal{A}$, if it exists, is an element $x \in \mathcal{A}$ which satisfies conditions $(1) - (4)$. From [110] we also cite the equivalent of Corollary 1.3.1 in Banach algebra. Namely, one more constatation from [97] shows that a Hermitian idempotent operator on an arbitrary Banach space retains several of the nice properties of a Hermitian idempotent on Hilbert space. Its range and null space are "orthogonal" in the rather strong sense that the norm of the sum of two vectors, one from each subspace, is unaffected by multiplying each vector by a different complex number of absolute value 1.

Lemma 1.3.2 *[110] Let $a, b \in \mathcal{A}$ and a be MP-invertible. Then*

$$\inf_{x \in \mathcal{A}} \|ax - b\| = \|ax_0 - b\|,$$

where $x_0 = a^\dagger b$.

Rakočević continued this research in [111, 112] where he gave a sufficient and necessary conditions for the continuity of MP-inverse in Banach algebras and particulary in C^*-algebras.

One more fruit of the investigation about generalized inverses is so-called weighted MP-inverse. In [20] Chipman introduced it for matrices, using positive definite weight matrices. After that Prasad and Bapat in [106] generalized its definition using arbitrary invertible, not necessarily positive definite, weights. (Recall that a complex $n \times n$ matrix M is *positive definite* if the scalar z^*Mz is strictly positive for every non-zero column vector z of n complex numbers, where z^* denotes the conjugate transpose of z. Evidently, every positive definite matrix is invertible.) Here we will give the definition in ring with involution case from [75].

Definition 1.3.7 *Let $\mathcal{R}$ be a ring with involution and e, f two invertible elements in $\mathcal{R}$. We say that an element $a \in R$ has a* weighted MP-inverse with weights e, f *if there exists $b \in \mathcal{R}$ such that*

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$

Correspondingly as for MP-inverse can be deduced, by algebraic proof, that an element $a \in \mathcal{R}$ can have at most one weighted MP-inverse with given weights e, f. We give the proof of uniqueness as illustration. If $b, c \in \mathcal{R}$ are two weighted MP-inverses with weights e, f of an element $a \in \mathcal{R}$, then

$$\begin{aligned} c &= cac = cabac = ce^{-1}eabac = ce^{-1}b^*a^*e^*ac = ce^{-1}b^*a^*eac = ce^{-1}b^*a^*c^*a^*e^* \\ &= ce^{-1}b^*a^*e^* = ce^{-1}eab = cab = cabab = caf^{-1}fbab = caf^{-1}a^*b^*f^*b \\ &= ca(f^{-1})^*a^*b^*f^*b = f^{-1}a^*c^*a^*b^*f^*b = f^{-1}a^*b^*f^*b = f^{-1}fbab = b. \end{aligned}$$

The unique weighted MP-inverse with weights e, f is usually denoted by $a^\dagger_{e,f}$, if it exists. In the following theorem we present results from [75] which consider a weighted

MP-inverse in a C^* -algebra $\mathcal{A}$ under the hypothesis that e, f are positive invertible elements in $\mathcal{A}$. For the results on weighted MP-inverse in a Banach algebra see [15]. An element a from C^*-algebra is *positive* if a is Hermitian and if its spectrum is a subset of $[0, +\infty)$. This definition can be found in wonderful Conway's book [21].

Theorem 1.3.7 [75] *Let $\mathcal{A}$ be a C^*-algebra with a unit and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then $a^\dagger_{e,f}$, exists.*

The proof of previous theorem is based on the fact that for a positive and invertible element e from complex C^*-algebra $\mathcal{A}$ with a unit and an involution $*$, $\mathcal{A}_e = (\mathcal{A}, *^e, \|\cdot\|_e)$ is also a C^*-algebra with a unit and the involution $x \mapsto x^{*e} = e^{-1}x^*e,\ x \in \mathcal{A}$ and the norm $x \mapsto \|x\|_e = \|e^{\frac{1}{2}}xe^{-\frac{1}{2}}\|,\ x \in \mathcal{A}$. The useful connection between the weighted MP-inverse and the ordinary MP-inverse is given in the coming theorem.

Theorem 1.3.8 [75] *Let $\mathcal{A}$ be a C^*-algebra with a unit and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then*

$$a^\dagger_{e,f} = f^{-\frac{1}{2}}(e^{\frac{1}{2}}af^{-\frac{1}{2}})^\dagger e^{\frac{1}{2}}.$$

We will finish this section with particular results on generalized inverses of bounded linear operators on Banach and Hilbert spaces. Let X, Y be Banach spaces and let $\mathcal{B}(X, Y)$ denote the set of all linear bounded operators from X to Y. Even though, X and Y can be different spaces, the $\{1\}$ and $\{2\}$-inverses of $A \in \mathcal{B}(X, Y)$ can be defined at the same manner as in ring settings (similarly to rectangular matrix case). If $B \in \mathcal{B}(Y, X)$ is an inner generalized inverse of A, then BAB is a reflexive generalized inverse of A. Because of that, A has reflexive generalized inverse if and only if A is regular. The following theorems can be found in [109, 47].

Theorem 1.3.9 *An operator $A \in \mathcal{B}(X, Y)$ is regular, if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of Y and X, respectively.*

By a well-known theorem about existence of topological complement of closed subspace of Banach space the previous theorem can be paraphrased.

Theorem 1.3.10 *An operator $A \in \mathcal{B}(X, Y)$ is regular, if and only if exist idempotents $P \in \mathcal{B}(Y)$ and $Q \in \mathcal{B}(X)$, such that $\mathcal{R}(P) = \mathcal{R}(A)$ and $\mathcal{R}(Q) = \mathcal{N}(T)$.*

It is clearly that $0 \in \mathcal{B}(Y, X)$ is outer inverse of every $A \in \mathcal{B}(X, Y)$. It was of interest to find operators which have nonzero outer inverse.

Theorem 1.3.11 [109] *An operator $A \in \mathcal{B}(X, Y)$ has nonzero outer inverse $B \in \mathcal{B}(Y, X)$ if and only if $A \neq 0$.*

If $A \in \mathcal{B}(X, Y)$ is regular and T and S are closed subspaces of X and Y respectively, such that $X = T \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus S$, then there is unique reflexive inverse $B \in \mathcal{B}(Y, X)$ of A, such that $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$. This inverse is denoted by $A^{(1,2)}_{T,S}$.

If we are looking for bounded linear operator between Hilbert spaces Theorem 1.3.9 can be simplified.

Theorem 1.3.12 *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. A is regular if and only if $\mathcal{R}(A)$ is closed.*

This simplification comes from the fact that every closed subspaces of a Hilbert space is complemented. Precisely, for an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with closed range we know $\mathcal{K} = \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$ and $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(A)^{\perp} = \mathcal{N}(A) \oplus \mathcal{R}(A^*)$. Even more, it is readily seen that uniquely determined $A^{(1,2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfies third and fourth Penrose equation, i.e. $A^{(1,2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$ is the Moore-Penrose inverse of A. We in fact deduced Theorem 1.3.3 in the context of $\mathcal{B}(\mathcal{H}, \mathcal{K})$.

Theorem 1.3.13 *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The following conditions are equivalent:*

(*i*) *A is regular,*

(*ii*) *$\mathcal{R}(A)$ is closed,*

(*iii*) *$\mathcal{R}(A^*)$ is closed,*

(*iv*) *$A^\dagger$ exists.*

In the same spirit, for the positive and invertible operators $M \in \mathcal{B}(\mathcal{K})$ and $N \in \mathcal{B}(\mathcal{H})$, introducing new inner products

$$\langle x, y\rangle_N = \langle Nx, y\rangle \quad \text{in } \mathcal{H}, \qquad \langle u, v\rangle_M = \langle Mu, v\rangle \quad \text{in } \mathcal{K},$$

can be derived Theorem 1.3.7 for $A^\dagger_{M,N}$, the weighted Moore-Penrose inverse of A with respect to the weights M and N.

1.4 Notation

In this section we give a description of symbols which will be used throughout this dissertation. By $\mathbb{C}$ we denote the set of complex numbers, while by $\mathbb{N}$ we mean the set of natural numbers and $\mathbb{F}$ is an arbitrary field. Complex Banach spaces will be often denoted by $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and complex Hilbert spaces with $\mathcal{H}, \mathcal{K}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2$ with inner product $\langle\cdot,\cdot\rangle$. $\mathcal{A}$ is a complex C^*-algebra with a unite and $\mathcal{R}$ is a ring with a unit $1 \neq 0$. If $m, n \in \mathbb{N}$, the vector space of all $m \times n$ complex matrices over complex field will be denoted by $\mathbb{C}^{m\times n}$ and $\mathbb{C}^n$ is the vector space of all n-tuples of complex numbers. For the set of all $m \times n$ matrices with entries in $\mathcal{R}$ and $\mathbb{F}$, we use symbol $\mathcal{R}^{m\times n}$ and $\mathbb{F}^{m\times n}$, respectively.

A linear map $A \in \mathbb{C}^{m\times n} : \mathbb{C}^n \to \mathbb{C}^m$ will be presented in the respect to the standard basis of $\mathbb{C}^n$ and $\mathbb{C}^m$. A^* is the conjugate transpose matrix and $r(A)$ is the rank of a matrix A. For Banach spaces $\mathcal{X}, \mathcal{Y}$ by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ we will denote the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. For simplicity, we also write $\mathcal{B}(\mathcal{X}, \mathcal{X})$ as $\mathcal{B}(\mathcal{X})$. I denotes the identity operator. For a given operator $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. If $\mathcal{H}$ is a complex Hilbert space, by $\mathcal{B}(\mathcal{H})^+$ we denote the cone of nonnegative definite operators of $\mathcal{B}(\mathcal{H})$, i.e. $\mathcal{B}(\mathcal{H})^+ := \{A \in \mathcal{B}(\mathcal{H}) : \langle Ax, x\rangle \geq 0, \forall x \in \mathcal{H}\}$. If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ then A^* and A' denote the Hilbert adjoint operator and the adjoint operator of A, respectively.

For a subspace $\mathcal{M}$ of Banach space $\mathcal{X}$, $dim\mathcal{M}$ will stand for the dimension of $\mathcal{M}$ if $\mathcal{M}$ is finite dimensional and otherwise will mean ∞. The set of all annihilators of $\mathcal{M}$ is $\mathcal{M}^\circ = \{f \in \mathcal{X}' = \mathcal{B}(\mathcal{X}, \mathbb{C}) \mid (\forall x \in \mathcal{M}) f(x) = 0\}$. For subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{X}$ with $\mathcal{M} \subseteq \mathcal{N}$, we set $codim_{\mathcal{N}}\,\mathcal{M} = dim\,\mathcal{N}/\mathcal{M}$ and $codim_{\mathcal{X}}\,\mathcal{M} = dim\,\mathcal{X}/\mathcal{M}$. For $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, we use the notations $n(A) = \dim \mathcal{N}(A), \beta(A) = \mathrm{codim}\mathcal{R}(A)$ and for $\mathcal{A} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we use notation $d(A) = \dim \mathcal{R}(A)^\perp$. If $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{M}$ is a subspace of $\mathcal{X}$ then the restriction of the operator A to the subspace $\mathcal{M}$ will be denoted by $A|_{\mathcal{M}}$. Given a closed subspace S of $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, the matrix operator decomposition of A induced by S is

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} S \\ S^\perp \end{bmatrix} \to \begin{bmatrix} S \\ S^\perp \end{bmatrix},$$

where $A_{11} = P_S A P_S|_S \in \mathcal{B}(S)$, $A_{12} = P_S A(I - P_S)|_{S^\perp} \in \mathcal{B}(S^\perp, S)$, $A_{21} = (I - P_S)AP_S|_S \in \mathcal{B}(S, S^\perp)$ and $A_{22} = (I - P_S)A(I - P_S)|_{S^\perp} \in \mathcal{B}(S^\perp)$.

A mapping $P \in \mathcal{B}(\mathcal{X})$ is called an idempotent if $P^2 = P$. A Hermitian idempotent from $\mathcal{B}(\mathcal{H})$ will be called orthogonal projection. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, we use the symbol $P_{\mathcal{M}}$ to denote the orthogonal projection onto $\mathcal{M}$ and $P_{\mathcal{M}}^{\mathtt{rst}}$ to denote an operator from $\mathcal{B}(\mathcal{H}, \mathcal{M})$ defined by $P_{\mathcal{M}}^{\mathtt{rst}} x = P_{\mathcal{M}} x$, for all $x \in \mathcal{H}$. If $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ by $P_{\mathcal{M},\mathcal{N}}$ we denote the projection onto $\mathcal{M}$ parallel to $\mathcal{N}$ and $P_{\mathcal{M},\mathcal{N}}^{\mathtt{rst}} \in \mathcal{B}(\mathcal{X}, \mathcal{M})$ is defined analogously as $P_{\mathcal{M}}^{\mathtt{rst}}$ above. The set of all Fredholm operators from the space $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is denoted by $F(\mathcal{H}, \mathcal{K})$. By $F_+(\mathcal{H}, \mathcal{K})$ $(F_-(\mathcal{H}, \mathcal{K}))$ we denote the set of all upper (lower) semi-Fredholm operators from $\mathcal{B}(\mathcal{H}, \mathcal{K})$. The index of a Fredholm operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is denoted by $ind(A)$.

$\mathcal{R}^-$ is the set of all regular elements of $\mathcal{R}$. Also we will use notations $r_a = 1 - aa^-$ and $l_a = 1 - a^- a$ for $a \in \mathcal{R}^-$, where a^- is an arbitrary inner inverse of a. For $A \in \mathcal{R}^{m\times n}$ with A^T we denote the transpose of matrix A. For given sets A, B, by $A \cdot B$ we denote the setwise product $A \cdot B = \{ab : a \in A,\ b \in B\}$.

Chapter 2

Different types of the reverse order law and improvements of some results

If a and b are invertible elements in a semigroup, then their product ab is an invertible element too, and its inverse is given by

$$(ab)^{-1} = b^{-1}a^{-1}. \tag{2.1}$$

This equality is called the reverse order law and its looking in the context of generalized inverses that led to investigation of so-called *"reverse order law"*, which resulted in a huge number of the papers. Namely, if we are looking for the generalized inverses of product of two matrices, the identity (2.1) cannot be trivially extended.

The investigation on the reverse order laws for the generalized inverses started with the reverse order law for the Moore-Penrose inverse which was discussed by Greville [63], in the 1960s, who gave a necessary and sufficient condition for the reverse order law

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}, \tag{2.2}$$

for matrices A and B. He proved that (2.2) holds if and only if $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$. This has been followed by further equivalents of (2.2). For example, Arghiriade [1] proved that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ holds if and only if $\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A)$. Further research on this subject has branched in several directions:

- The reverse order law for the products with more than two elements;
- The reverse order law for the different classes of generalized inverses;
- The reverse order law for the different settings.

The reverse order law problems are very applicable, both in theoretic research and in numerical computations in many areas, including the singular matrix problem, ill-posed problems, optimization problems, and statics problems (see for instance [9, 62, 116, 128]). In the following lines, we will present some selected theoretical results. The comprehensive survey of the results on this subject is presented in [40].

Koliha et al.[75] studied the reverse order law for the product of two Moore-Penrose invertible elements in the setting of rings with involution. It is known that the product of two Moore-Penrose invertible elements is not Moore-Penrose invertible element in general (see Remark 4.1.1). We restate the main result from [75]:

Theorem 2.0.1 [*75*] *Let $\mathcal{R}$ be a ring with involution, let $a, b \in \mathcal{R}$ be MP-invertible and let $(1 - a^\dagger a)b$ be left $*$-cancellable. Then the following conditions are equivalent:*

(i) *ab is MP-invertible and $(ab)^\dagger = b^\dagger a^\dagger$,*

(ii) $[a^\dagger a, bb^*] = 0$ *and* $[bb^\dagger, a^* a] = 0$.

In the 1980s, Hartwig [66] considered triple reverse order law for the Moore-Penrose inverse which was continued by other authors [134, 133, 45]. These results will be presented later. Tian [132] first studied the reverse order law for the Moore-Penrose inverse of product of n matrices. By using rank of matrices, he derived necessary and sufficient conditions for $A_n^\dagger A_{n-1}^\dagger \cdots A_1^\dagger$ to be $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$-, $\{1,2,3\}$-, $\{1,2,4\}$-inverse or Moore-Penrose inverse of $A_1 A_2 \cdots A_n$.

The next step was to extend the discussion of (2.2) to the more general case of reverse order law for K-inverses where $K \subseteq \{1,2,3,4\}$. When we consider the reverse order law for K-inverse, we consider one of the following inclusions:

$$\begin{aligned}
&BK \cdot AK \subseteq (AB)K,\\
&(AB)K \subseteq BK \cdot AK,\\
&(AB)K = BK \cdot AK.
\end{aligned}$$

Seems that the inclusion $(AB)K \subseteq BK \cdot AK$ is harder to be treated then the inclusion $BK \cdot AK \subseteq (AB)K$, because for the first one it is necessary to show that each element of $(AB)K$ can be represented as a product of some elements from BK and AK, respectively, while in the second one it is necessary to checked some identities. This is evident by the chronology of scientific results too.

Reverse order law for the inner inverses of matrices were investigated by many authors (see [124, 123, 115, 148]). We mention the result of Wei [147], who gave the necessary and sufficient conditions for inclusions $(AB)\{1\} = B\{1\}A\{1\}$ to hold, using product singular value decomposition of a matrix (P-SVD).

Theorem 2.0.2 [*147*] *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. The following conditions are equivalent:*

(i) $(AB)\{1\} = B\{1\}A\{1\}$,

(ii) *One of the following conditions holds:*

  (a) $\mathcal{R}(B) \subseteq \mathcal{N}(A)$ *and* $n \geq \min\{m + r(B), p + r(A)\}$,

(b) $\mathcal{N}(A) \subseteq \mathcal{R}(B)$ *and* ($m = r(A)$ *or* $p = r(B)$),

(iii) *One of the following conditions holds:*

(a) $r(AB) = 0$ *and* $n \geq \min\{m + r(B), p + r(A)\}$,

(b) $r(A) + r(B) - r(AB) = n$ *and* ($m = r(A)$ *or* $p = r(B)$).

We restate also the result of Pavlović and Cvetković-Ilić [99] who studied the reverse order $(AB)\{1\} \subseteq B\{1\}A\{1\}$ for $\{1\}$-inverses of operators on separable Hilbert spaces. Some results on completions of operator matrices are involved in the proof of this result.

Theorem 2.0.3 [99] *Let regular operators* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *and* $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ *be given by*

$$\begin{aligned} A &= \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \\ B &= \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}, \end{aligned}$$

and let AB *be regular. Then the following conditions are equivalent:*

(i) $(AB)\{1\} \subseteq B\{1\}A\{1\}$,

(ii) *One of the following conditions is satisfied:*

(a) $\dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(B)$, $\dim\mathcal{N}(A_1^*) + \dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(B^*)$ *and* $\dim\mathcal{N}(B^*) < \infty$,

(b) $\dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(B)$, $\dim\mathcal{N}(A^*) \leq \dim\mathcal{N}(A_2'') + \dim\mathcal{N}(A_2)$ *and* $\dim\mathcal{N}(B^*) = \infty$,

(c) $\dim\mathcal{N}(B) \leq \dim\mathcal{N}(A^*)$, $\dim\mathcal{N}(B_1^*) + \dim\mathcal{N}(B) \leq \dim\mathcal{N}(A)$ *and* $\dim\mathcal{N}(A) < \infty$,

(d) $\dim\mathcal{N}(B) \leq \dim\mathcal{N}(A^*)$, $\dim\mathcal{N}(B) \leq \dim\mathcal{N}(B_2'') + \dim\mathcal{N}(B_2)$ *and* $\dim\mathcal{N}(A) = \infty$,

where $A_2'' = P_{\mathcal{N}(A_1^*)} A_2|_{\mathcal{R}\left(A_2^*\right)}$, $B_1 = P_{\mathcal{R}(B^*)} B^*|_{\mathcal{R}(A^*)}$, $B_2 = P_{\mathcal{R}(B^*)} B^*|_{\mathcal{N}(A)}$ *and* $B_2'' = P_{\mathcal{N}(B_1^*)} B_2|_{\mathcal{R}(B_2^*)}$.

Shinozaki and Sibuya [123] proved that $(AB)\{1,2\} \subseteq B\{1,2\}A\{1,2\}$ always holds in the matrix settings. The generalization of their result in the case of regular bounded operators on Hilbert spaces which product is also regular can be found in [40]. The opposite inclusion $B\{1,2\}A\{1,2\} \subseteq (AB)\{1,2\}$ was studied by De Pierro and Wei [105] for matrices and by Cvetković-Ilić and Nikolov [37] in operator on Hilbert spaces case. We cite the generalization of this result by Nikolov Radenković for n regular bounded operators on Hilbert spaces.

Theorem 2.0.4 [*108*] *Let $A_i \in \mathcal{B}(\mathcal{H}_{i+1}, \mathcal{H}_i)$, be such that A_i, $i = 1, 2 \dots, n$ and all $A_1 A_2 \cdots A_j$, $A_{j-1} A_j$, $j = 2, 3, \dots, n$, are regular operators. The following conditions are equivalent:*

(*i*) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} = (A_1 A_2 \cdots A_n)\{1,2\}$,

(*ii*) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1 A_2 \cdots A_n)\{1,2\}$,

(*iii*) *There exists an integer i, $1 \le i \le n$, such that $A_i = 0$,*
or
$A_1 A_2 \cdots A_n \neq 0$ and $A_i \in \mathcal{B}_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = 2, 3, \dots, n$,
or
$A_1 A_2 \cdots A_n \neq 0$ and $A_i \in \mathcal{B}_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = 1, 2, \dots, n-1$,
or
$A_1 A_2 \cdots A_n \neq 0$ and there exists an integer k, $2 \le k \le n-1$, such that $A_i \in \mathcal{B}_l^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = 1, 2, \dots, k-1$, and $A_i \in \mathcal{B}_r^{-1}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ for $i = k+1, k+2, \dots, n$.

The reverse order laws for the $\{1,3\}$ and $\{1,4\}$-inverses were for the first time considered in the paper of M. Wei and Guo [146] in the matrix case, where some equivalent conditions for

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\} \tag{2.3}$$

and

$$(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\}, \tag{2.4}$$

are presented, using product singular value decomposition (P-SVD). Precisely, they have proved the following results:

Theorem 2.0.5 [*146*] *Suppose that $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$. Then $B\{1,3\} \cdot A\{1,3\} \subseteq (AB)\{1,3\}$ if and only if the following two conditions hold:*

$$Z_{12} = 0 \text{ and } Z_{14} = 0,$$

where submatrices Z_{12} and Z_{14} are described in the P-SVD of matrices A and B.

Theorem 2.0.6 [*146*] *Suppose that $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$. Then $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$ if and only if the following conditions hold:*

$$\begin{aligned}&dim(\mathcal{R}(Z_{14}) = dim(\mathcal{R}(Z_{12}, Z_{14}))\\ &0 \le \min\{p - r_2, m - r_1\} \le n - r_1 - r_2^2 - r(Z_{14}),\end{aligned}$$

where submatrices Z_{12}, Z_{14} and constants r_1, r_2, r_2^2 are described in the P-SVD of matrices A and B.

The further results on these reverse order laws can be found in the papers [46, 130, 32, 108, 85, 24]. The reverse order law for $\{1,3,4\}-$ generalized inverses was studied in [86, 39, 26] in different settings.

The inclusion $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$ in the matrix settings was considered by Xiong and Zheng [155] using some expressions for the maximal and the minimal ranks of the generalized Schur complement. This result was generalized to the C^*-algebra case by Cvetković-Ilić and Harte [32]. Some additional results, based on a block-operator matrix techniques, concerning this inclusion can be found in [84]. The opposite inclusion $(AB)\{1,2,3\} \subseteq B\{1,2,3\}A\{1,2,3\}$ in matrix case is considered in [38] where purely algebraic equivalent conditions with this reverse order law are derived.

Sun and Wei [126] extended the investigation of the reverse order laws to the case of weighted Moore-Penrose inverses of matrices:

Theorem 2.0.7 *[126] Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times l}$ and let $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$, $L \in \mathbb{C}^{l\times l}$ be positive definite Hermite matrices. Then*

$$(AB)^\dagger_{M,L} = B^\dagger_{N,L}A^\dagger_{M,N}$$

if and only if

$$\mathcal{R}(A^\sharp AB) \subseteq \mathcal{R}(B) \quad \textit{and} \quad R(BB^\sharp A^\sharp) \subseteq R(A^\sharp),$$

where $A^\sharp = N^{-1}A^*M$ and $B^\sharp = L^{-1}B^*N$.

The reverse order law for the Drazin inverse was first studied by Greville [63]. He showed that if $AB = BA$, then $(AB)^D = B^DA^D$ holds. The reverse order law for Drazin inverse of product od 2 and n matrices was considered by Tian [131] and Wang [143], respectively.

In Section 2.1 we will present our results that concern the necessary and sufficient conditions for the inclusion $(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\}$ in a C^*-algebra case to hold, and as a corollary in a case of bounded operators on separable Hilbert spaces. In Section 2.2 we will extend Hartwig's triple reverse order law to more general setting and continue with relaxing some conditions in Section 2.3. In Section 2.4 we discuss the reverse order laws on $\{1,3\},\{1,4\},\{1,3,4\},\{1,2,3\}$ and $\{1,2,4\}$ -inverses in ring setting. We will present a generalization of the results from [32] on the reverse order laws for $\{1,3\}$ and $\{1,4\}-$ generalized inverses, in the sense that the assumptions of the regularity of some elements will be removed. Also we will prove that some conditions of results from [39, 32], which concerns the reverse order law for $\{1,3,4\}$ -inverses and $\{1,2,3\}$ -inverses, respectively, can be relaxed.

2.1 The reverse order laws for $\{1,3\}$-generalized inverses

In this section we present our results from [36] that concern the reverse order laws for $\{1,3\}$ and $\{1,4\}$- generalized inverses in C^*-algebras. We give the corresponding results for the case of linear bounded operators on separable Hilbert spaces, too.

As we mentioned before, the reverse order laws for the $\{1,3\}$ and $\{1,4\}$-inverses in the matrix case were considered by M. Wei and Guo [146]. Evidently a disadvantage of the results presented in [146] Theorems 2.0.5 and 2.0.6 lies in the fact that the necessary and sufficient conditions for (2.3) and (2.4) to be satisfied depend on subblocks produced by P-SVD. In order to overcome this shortcoming, two methods are employed. One of the methods use some certain operator matrix representations (see [46]) and the other one is based on some maximal and minimal ranks of matrix expressions (see [130]). Using these two different methods, in both of the papers [46, 130] it is proved that

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\} \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B),$$

but in the first one in the case of regular operators and in the second one in the setting of matrices. These results are more elegant because they require no information about the P-SVD of appropriate matrices. Note that in the matrix case the condition $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ is equivalent to $r([B \quad A^*AB]) = r(B)$. In the case of three linear bounded operators, the reverse order law (2.3) was considered in [154]:

Theorem 2.1.1 *[154] Let $\mathcal{J},\mathcal{K},\mathcal{I}$ be Hilbert spaces and $T_1 \in \mathcal{B}(\mathcal{J},\mathcal{K}), T_2 \in \mathcal{B}(\mathcal{I},\mathcal{J}), T_3 \in \mathcal{B}(\mathcal{H},\mathcal{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. Then the following statements are equivalent:*

(i) $\mathcal{R}(T_1^*T_1T_2T_3) \subseteq \mathcal{R}(T_2)$ *and* $\mathcal{R}((T_1T_2)^*T_1T_2T_3) \subseteq \mathcal{R}(T_3)$,

(ii) $T_3\{1,3\}T_2\{1,3\}T_1\{1,3\} \subseteq (T_1T_2T_3)\{1,3\}$.

The above mentioned results were generalized in the paper of Cvetković-Ilić et al. [32] where purely algebraic necessary and sufficient conditions for (2.3) in C^*-algebras are offered, extending rank conditions for matrices and range conditions for Hilbert space operators:

Theorem 2.1.2 *[32] If $a, b \in \mathcal{A}$ are such that a, b, ab and $a(1 - bb^\dagger)$ are regular, then the following conditions are equivalent:*

(i) $b\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$,

(ii) $bb^\dagger a^*ab = a^*ab$,

(iii) $b^\dagger a^\dagger \in (ab)\{1,3\}$,

(iv) $b^\dagger a^\dagger \in (ab)\{1,2,3\}$.

This was followed by the paper of Nikolov Radenković [107, 108] where the inclusion (2.3) was considered in the case of n regular bounded linear operators on Hilbert spaces whose product is also regular:

Theorem 2.1.3 *[108] Let $A_i \in \mathcal{B}(\mathcal{H}_{i+1},\mathcal{H}_i)$ be regular operators such that $A_1A_2\cdots A_n$ is regular. The following statement are equivalent:*

(i) $A_n\{1,3\} \cdot A_{n-1}\{1,3\} \cdots A_1\{1,3\} \subseteq (A_1A_2\cdots A_n)\{1,3\}$,

(ii) $\mathcal{R}(A_k^* A_{k-1}^* \cdots A_1^* A_1 A_2 \cdots A_n) \subseteq \mathcal{R}(A_{k+1})$, *for* $k = \overline{1, n-1}$.

Notice that the result above can be generalized in the prime C^*-algebra settings.

On the other hand, concerning the inclusion (2.4) there are only several results: Beside the mentioned result of M. Wei and Guo [146] which involves certain information about the subblocks produced by P-SVD, we could find in the literature only two papers on this subject before our paper [36], and both concern the matrix settings. One of them is the paper of Liu and Yang [85] where is given the following result:

Theorem 2.1.4 *[85] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then $(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\}$ if and only if*

$$r([A^*AB \quad B]) + r(A) = r(AB) + t,$$

where $t = \min\{r([A^* \;\; B]), \max\{n + r(A) - m, n + r(B) - k\}\}$.

Some equivalent conditions with the one given in [85, 146] can be found in [24]:

Theorem 2.1.5 *[24] Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Then the following conditions are equivalent:*

(i) $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$,

(ii) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$ *and* $r(C) \geq \min\{n - r(A), k - r(B)\}$,

where $S = B^\dagger(I - A^\dagger A)$ *and* $C = I - A^\dagger A - S^\dagger S$.

Before proving the main results from [36], we present some auxiliary results. Let $\mathcal{A}$ denotes a complex C^*-algebra with a unite. A characterization of the set $a\{1,3\}$, for $a \in \mathcal{A}$ is given in [32] and presents a generalization to C^*-algebras of the result given in [9] in the complex matrix setting:

Lemma 2.1.1 *[32] Let $a \in \mathcal{A}$ be regular and $b \in \mathcal{A}$. Then $b \in a\{1,3\}$ if and only if $a^\dagger ab = a^\dagger$. Hence,*

$$a\{1,3\} = \{a^\dagger + (1 - a^\dagger a)x : \; x \in \mathcal{A}\}.$$

The following lemma will be used very often throughout the proof of the main result.

Lemma 2.1.2 *Let $a \in \mathcal{A}$ and let $p, q \in \mathcal{A}$ be projections such that ap and qa are regular. Then:*

(i) $(ap)^\dagger = p(ap)^\dagger$,

(ii) $(qa)^\dagger = (qa)^\dagger q$.

Proof. *(i)* Since $(ap)^\dagger = (ap)^*(app^*a^*)^\dagger = pa^*(apa^*)^\dagger$ it is clear that $(ap)^\dagger = p(ap)^\dagger$. *(ii)* This can be proved similarly. □

Lemma 2.1.3 *Let $a, b \in \mathcal{A}$ be such that a, b, ab and $s = b^\dagger(1 - a^\dagger a)$ are regular. Then the following conditions are equivalent:*

(i) $(1-ss^\dagger)\left((ab)^\dagger - b^\dagger a^\dagger\right)=0$,

(ii) $(1-ss^\dagger)\left((ab)^\dagger a - b^\dagger\right)=0$,

(iii) *For any* $z\in\mathcal{A}$,

$$(1-ss^\dagger)\left((ab)^\dagger - b^\dagger a^\dagger + b^\dagger b\left(1-(ab)^\dagger ab\right)z\right)=0.$$

Proof. $(i)\Rightarrow(ii)$: If (i) holds, then we have

$$(1-ss^\dagger)((ab)^\dagger a - b^\dagger)=(1-ss^\dagger)(b^\dagger a^\dagger a - b^\dagger)=-(1-ss^\dagger)s=0.$$

$(ii)\Rightarrow(i)$: If we multiply (ii) from the right by $a^\dagger$ and use the fact that $(ab)^\dagger aa^\dagger=(ab)^\dagger$ (see Lemma 2.1.2) we get that (i) holds.

$(i)\Rightarrow(iii)$: If (i) holds, then (ii) holds, so by Lemma 2.1.2 for any $z\in\mathcal{A}$

$$(1-ss^\dagger)\left((ab)^\dagger - b^\dagger a^\dagger + b^\dagger b\left(1-(ab)^\dagger ab\right)z\right)=(1-ss^\dagger)\left(b^\dagger-(ab)^\dagger a\right)bz=0.$$

$(iii)\Rightarrow(i)$: This follows taking $z=0$. □

Lemma 2.1.4 *Let* $a,b\in\mathcal{A}$ *be such that* a,b,ab *and* $s=b^\dagger(1-a^\dagger a)$ *are regular. Let* $z,u,v\in\mathcal{A}$. *There exists* $y\in\mathcal{A}$ *such that*

$$z=y\left(a^\dagger+s^\dagger\left((ab)^\dagger-b^\dagger a^\dagger+v\right)+(1-a^\dagger a)u\right),\tag{2.5}$$

if and only if there exists $x\in\mathcal{A}$ *such that*

$$z=x\left(a^\dagger+s^\dagger v+(1-a^\dagger a)u\right).\tag{2.6}$$

Proof. Note that $as^\dagger=as^*(ss^*)^\dagger=a(1-a^\dagger a)(b^\dagger)^*(ss^*)^\dagger=0$. Now, using Lemma 2.1.2, it is easy to verify that if (2.5) holds then for $x=y\left(1+s^\dagger\left((ab)^\dagger-b^\dagger a^\dagger\right)a\right)$ we have that (2.6) holds. Conversely, if (2.6) holds then for $y=x\left(1-s^\dagger\left((ab)^\dagger-b^\dagger a^\dagger\right)a\right)$ it follows that (2.5) is satisfied. □

Lemma 2.1.5 *Let* $a,b\in\mathcal{A}$ *be such that* a,b,ab *and* $s=b^\dagger(1-a^\dagger a)$ *are regular. Then the following conditions are equivalent:*

(i) *For any* $z\in\mathcal{A}$ *there exist* $y,u\in\mathcal{A}$ *such that*

$$(1-b^\dagger b)z=y\left(a^\dagger+s^\dagger\left(1-(ab)^\dagger ab\right)z+(1-a^\dagger a)(1-s^\dagger s)u\right),\tag{2.7}$$

(ii) $(1-b^\dagger b)\mathcal{A}\subseteq\bigcap_{v\in\mathcal{A}}\mathcal{A}\left(a^\dagger+s^\dagger(1-(ab)^\dagger ab)v+(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}\right)$,

(iii) $(1-b^\dagger b)\mathcal{A}\subseteq\mathcal{A}\left(a^\dagger+(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}\right)$,

(iv) $(1-b^\dagger b)\mathcal{A}(1-aa^\dagger)\subseteq\mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}$.

Proof. $(i) \Rightarrow (ii)$: If (i) holds, take arbitrary but fixed $z_1 \in (1-b^\dagger b)\mathcal{A}$. For arbitrary $v \in \mathcal{A}$, let $z = z_1 + b^\dagger bv$. Then for such z there exist $y, u \in \mathcal{A}$ such that

$$\begin{aligned}(1-b^\dagger b)z &= y\left(a^\dagger + s^\dagger\left(1-(ab)^\dagger ab\right)z + (1-a^\dagger a)(1-s^\dagger s)u\right)\\ &= y\left(a^\dagger + s^\dagger\left(1-(ab)^\dagger ab\right)b^\dagger bz + (1-a^\dagger a)(1-s^\dagger s)u\right),\end{aligned}$$

where for the last equality we used $s^\dagger(1-(ab)^\dagger ab)b^\dagger b = s^\dagger(1-(ab)^\dagger ab)$. Since $(1-b^\dagger b)z = z_1$ and $b^\dagger bz = b^\dagger bv$ we get

$$z_1 = y\left(a^\dagger + s^\dagger\left(1-(ab)^\dagger ab\right)v + (1-a^\dagger a)(1-s^\dagger s)u\right).$$

Now, since v is arbitrary it follows

$$z_1 \in \bigcap_{v\in\mathcal{A}} \mathcal{A}\left(a^\dagger + s^\dagger\left(1-(ab)^\dagger ab\right)v + (1-a^\dagger a)(1-s^\dagger s)\mathcal{A}\right).$$

Hence (ii) is satisfied.

$(ii) \Rightarrow (i)$: This is evident.

$(iii) \Rightarrow (ii)$: Suppose that (iii) holds. We will prove that

$$\mathcal{A}\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)\mathcal{A}\right) \subseteq \bigcap_{v\in\mathcal{A}} \mathcal{A}\left(a^\dagger + s^\dagger(1-(ab)^\dagger ab)v + (1-a^\dagger a)(1-s^\dagger s)\mathcal{A}\right),$$

so (ii) will follow directly. Take any $v \in \mathcal{A}$ and $x \in \mathcal{A}\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)\mathcal{A}\right)$. There exist $y \in \mathcal{A}$ and $u \in \mathcal{A}$ such that $x = y\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)u\right)$. Since, $s^\dagger s a^\dagger a = a^\dagger a s^\dagger s = 0$ we have that $s^\dagger s$ and $1-a^\dagger a$ commute, so using that $1 = a^\dagger a + s^\dagger s(1-a^\dagger a) + (1-s^\dagger s)(1-a^\dagger a)$ we get

$$\begin{aligned}x &= y\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)u\right)\\ &= y\left(a^\dagger a + s^\dagger s(1-a^\dagger a) + (1-s^\dagger s)(1-a^\dagger a)\right)\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)u\right)\\ &= y\left(a^\dagger a + (1-s^\dagger s)(1-a^\dagger a)\right)\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)u\right).\end{aligned}$$

Now, since $as^\dagger = 0$ and $1-s^\dagger s$ and $1-a^\dagger a$ commute, we get that

$$\left(a^\dagger a + (1-s^\dagger s)(1-a^\dagger a)\right)s^\dagger = 0,$$

so

$$\begin{aligned}x &= y\left(a^\dagger a + (1-s^\dagger s)(1-a^\dagger a)\right)\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)u\right)\\ &= y\left(a^\dagger a + (1-s^\dagger s)(1-a^\dagger a)\right)\left(a^\dagger + s^\dagger(1-(ab)^\dagger ab)v + (1-a^\dagger a)(1-s^\dagger s)u\right)\\ &\in \mathcal{A}\left(a^\dagger + s^\dagger(1-(ab)^\dagger ab)v + (1-a^\dagger a)(1-s^\dagger s)\mathcal{A}\right).\end{aligned}$$

Since v is an arbitrary, the inclusion follows.

$(ii) \Rightarrow (iii)$: This is evident.

$(iii) \Rightarrow (iv)$: If (iii) holds, then we get (iv) after multiplying (iii) by $(1-aa^\dagger)$ from the right.

$(iv) \Rightarrow (iii)$: To prove (iii), take arbitrary $z \in \mathcal{A}$. Then

$$\begin{aligned}(1-b^\dagger b)z &= (1-b^\dagger b)z(1-aa^\dagger) + (1-b^\dagger b)zaa^\dagger \\ &\in \mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A} + \mathcal{A}a^\dagger = \mathcal{A}(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)\mathcal{A}),\end{aligned}$$

where the last equality can be checked. □

In the following theorem we completely answer the question when

$$(ab)\{1,3\} \subseteq b\{1,3\}a\{1,3\}, \tag{2.8}$$

in case when $a, b \in \mathcal{A}$ are such that a, b, ab and $s = b^\dagger(1-a^\dagger a)$ are regular.

Theorem 2.1.6 *Let $a, b \in \mathcal{A}$ be such that a, b, ab and $s = b^\dagger(1-a^\dagger a)$ are regular. Then the following conditions are equivalent:*

(i) $(ab)\{1,3\} \subseteq b\{1,3\}a\{1,3\}$,

(ii) $(1-ss^\dagger)((ab)^\dagger - b^\dagger a^\dagger) = 0$ *and* $(1-b^\dagger b)\mathcal{A}(1-aa^\dagger) \subseteq \mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}$.

Proof. By Lemma 2.1.1, we have that (i) is equivalent with $\{(ab)^\dagger + (1-(ab)^\dagger ab)z :\ z \in \mathcal{A}\} \subseteq \{b^\dagger + (1-b^\dagger b)y :\ y \in \mathcal{A}\}\{a^\dagger + (1-a^\dagger a)x :\ x \in \mathcal{A}\}$, i.e. that for every $z \in \mathcal{A}$ there exist $x, y \in \mathcal{A}$ such that

$$\begin{aligned}&(ab)^\dagger + (1-(ab)^\dagger ab)z \\ &= b^\dagger a^\dagger + (1-b^\dagger b)ya^\dagger + b^\dagger(1-a^\dagger a)x + (1-b^\dagger b)y(1-a^\dagger a)x.\end{aligned} \tag{2.9}$$

Multiplying (2.9) by $b^\dagger b$ and then by $(1-b^\dagger b)$ from the left and using Lemma 2.1.2, we can show that (i) is equivalent with the fact that for any $z \in \mathcal{A}$ there exist $x, y \in \mathcal{A}$ such that

$$(ab)^\dagger - b^\dagger a^\dagger + b^\dagger b(1-(ab)^\dagger ab)z = sx \tag{2.10}$$

and

$$(1-b^\dagger b)z = (1-b^\dagger b)ya^\dagger + (1-b^\dagger b)y(1-a^\dagger a)x. \tag{2.11}$$

The last equality is equivalent with the fact that the equation (2.10) is solvable for any $z \in \mathcal{A}$ and that for any z the equation (2.11) is satisfied for some solution x of (2.10) and some $y \in \mathcal{A}$.

$(i) \Rightarrow (ii)$: Since the equation (2.10) is solvable for any $z \in \mathcal{A}$, by Lemma 4.0.1, this means that for any $z \in \mathcal{A}$

$$(1-ss^\dagger)\left((ab)^\dagger - b^\dagger a^\dagger + b^\dagger b\left(1-(ab)^\dagger ab\right)z\right) = 0. \tag{2.12}$$

Taking $z = 0$, we get that the first condition from (ii) is satisfied. Since the set of the solutions of equation (2.10) is given by

$$S_z = \{s^\dagger\left((ab)^\dagger - b^\dagger a^\dagger + (1-(ab)^\dagger ab)b^\dagger bz\right) + (1-s^\dagger s)u :\ u \in \mathcal{A}\},$$

taking $x \in S_z$ in equation (2.11), we get

$$(1-b^\dagger b)z = (1-b^\dagger b)y(a^\dagger + (1-a^\dagger a)s^\dagger \cdot \\ \cdot((ab)^\dagger - b^\dagger a^\dagger + (1-(ab)^\dagger ab)b^\dagger bz) + (1-a^\dagger a)(1-s^\dagger s)u).$$

Now using the fact that $s^\dagger = (1-a^\dagger a)s^\dagger$ we have that for any $z \in \mathcal{A}$ there exist $y, u \in \mathcal{A}$ such that

$$(1-b^\dagger b)z = y\left(a^\dagger + s^\dagger\left((ab)^\dagger - b^\dagger a^\dagger + \left(1-(ab)^\dagger ab\right)b^\dagger bz\right) \\ +(1-a^\dagger a)(1-s^\dagger s)u\right),$$

which is by Lemma 2.1.4, equivalent with the fact that for any $z \in \mathcal{A}$ there exist $y, u \in \mathcal{A}$ such that

$$(1-b^\dagger b)z = y\left(a^\dagger + s^\dagger\left(1-(ab)^\dagger ab\right)b^\dagger bz + (1-a^\dagger a)(1-s^\dagger s)u\right).$$

Now, taking $(1-b^\dagger b)z$ instead of z, we get that for any $z \in \mathcal{A}$ there exist $y, u \in \mathcal{A}$ such that

$$(1-b^\dagger b)z = y\left(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)u\right).$$

Therefore,

$$(1-b^\dagger b)\mathcal{A} \subseteq \mathcal{A}(a^\dagger + (1-a^\dagger a)(1-s^\dagger s)\mathcal{A})$$

which is equivalent with the second condition from (ii), by Lemma 2.1.5.

$(ii) \Rightarrow (i)$: If $(1-ss^\dagger)((ab)^\dagger - b^\dagger a^\dagger) = 0$ then by Lemma 2.1.3, we have that for any $z \in \mathcal{A}$,

$$(1-ss^\dagger)\left((ab)^\dagger - b^\dagger a^\dagger + b^\dagger b\left(1-(ab)^\dagger ab\right)z\right) = 0,$$

which means that for any $z \in \mathcal{A}$, the equation (2.10) is solvable and the set of the solutions, is described by

$$S_z = \{s^\dagger\left((ab)^\dagger - b^\dagger a^\dagger + \left(1-(ab)^\dagger ab\right)b^\dagger bz\right) + (1-s^\dagger s)u :\ u \in \mathcal{A}\}.$$

Now to prove (i) it is sufficient to show that for any $z \in \mathcal{A}$ there exist $y, u \in \mathcal{A}$ such that

$$(1-b^\dagger b)z = y\left(a^\dagger + s^\dagger\left((ab)^\dagger - b^\dagger a^\dagger + \left(1-(ab)^\dagger ab\right)b^\dagger bz\right) + (1-a^\dagger a)(1-s^\dagger s)u\right),$$

which is by Lemma 2.1.4, equivalent with the fact that for any $z \in \mathcal{A}$ there exist $y, u \in \mathcal{A}$ such that

$$(1-b^\dagger b)z = y\left(a^\dagger + s^\dagger\left(1-(ab)^\dagger ab\right)z + (1-a^\dagger a)(1-s^\dagger s)u\right).$$

Using Lemma 2.1.5 we can conclude that this is satisfied if and only if $(1-b^\dagger b)\mathcal{A}(1-aa^\dagger) \subseteq \mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}$. $\square$

Remark 2.1.1 If b is left invertible or a is right invertible then the second condition of (ii), $(1-b^\dagger b)\mathcal{A}(1-aa^\dagger) \subseteq \mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}$ is satisfied.

Remark 2.1.2 Notice that the inclusion $(1-b^\dagger b)\mathcal{A}(1-aa^\dagger) \subseteq \mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}$ holds if and only if the inclusion $(1-aa^\dagger)\mathcal{A}(1-b^\dagger b) \subseteq \mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}$ holds, since $(1-a^\dagger a)(1-s^\dagger s) = (1-s^\dagger s)(1-a^\dagger a)$.

Remark 2.1.3 In the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is an algebra of linear bounded operators on a Hilbert space, we have that the condition $(1-b^\dagger b)\mathcal{A}(1-aa^\dagger) \subseteq \mathcal{A}(1-a^\dagger a)(1-s^\dagger s)\mathcal{A}$ is equivalent to the one that for any $X \in \mathcal{B}(\mathcal{H})$, there exist $Y, Z \in \mathcal{B}(\mathcal{H})$ such that $P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)} = YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z$.

Literally repeating the proof of Theorem 2.1.6 one can obtain the following result (note that the operators A and B belong to different linear spaces which are not C^*-algebras).

Theorem 2.1.7 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that A, B, AB and $S = B^\dagger(I-A^\dagger A)$ are regular, where $\mathcal{H},\mathcal{K},\mathcal{L}$ are Hilbert spaces. Then the following conditions are equivalent:*

(i) $(AB)\{1,3\} \subseteq B\{1,3\}\cdot A\{1,3\}$,

(ii) $(I-SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$ *and for any* $X \in \mathcal{B}(\mathcal{K},\mathcal{L})$*, there exist* $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$ *and* $Z \in \mathcal{B}(\mathcal{K},\mathcal{H})$*such that* $P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)} = YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z$.

In the following theorem, in the case of the space of linear bounded operators on separable Hilbert spaces we present some conditions that are perhaps more easily verifiable for the reverse order law (2.4) to hold. It is surprising that the conditions are exactly the same as in the matrix case (see Theorem 2.1.5).

Theorem 2.1.8 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that A, B, AB and $S = B^\dagger(I-A^\dagger A)$ are regular, where $\mathcal{H},\mathcal{K},\mathcal{L}$ are separable Hilbert spaces. Then the following conditions are equivalent:*

(i) $(AB)\{1,3\} \subseteq B\{1,3\}\cdot A\{1,3\}$,

(ii) $(I-SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$ *and* $\dim(\mathcal{N}(A)\cap\mathcal{N}(B^*)) \geq \min\{\dim\mathcal{N}(B), \dim\mathcal{N}(A^*)\}$.

Proof. $(i) \Rightarrow (ii)$: It is clear by Theorem 2.1.7, that the first condition from (ii) is satisfied. Take $X \in \mathcal{B}(\mathcal{K},\mathcal{L})$ such that

$$\dim\mathcal{R}(P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)}) = \min\{\dim\mathcal{N}(B), \dim\mathcal{N}(A^*)\}.$$

By Theorem 2.1.7, there exist $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$ and $Z \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)} = YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z.$$

Now,

$$\dim \mathcal{R}(YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z) = \dim \mathcal{R}(P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)}).$$

Since $\dim \mathcal{R}(YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z) \leq \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$, we get that inequality from (ii) is satisfied.

$(ii) \Rightarrow (i)$: Suppose that (ii) holds and let us prove that (ii) of Theorem 2.1.7 is satisfied. We will consider two cases:
Case 1: If $\dim\mathcal{N}(A^*) > \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*)) \geq \dim\mathcal{N}(B)$: Take any $X \in \mathcal{B}(\mathcal{K}, \mathcal{L})$. There exists $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $\mathcal{R}(Y \mid_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}) = \mathcal{R}(P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)})$. Now, $\mathcal{R}(YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}) = \mathcal{R}(P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)})$ implies by Theorem 4.2.3 that there exists $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)} = YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z.$$

Case 2: If $\dim\mathcal{N}(A^*) \leq \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$: Take any $X \in \mathcal{B}(\mathcal{K}, \mathcal{L})$. There exists $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{R}(V \mid_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}) = \mathcal{N}(A^*)$. By Theorem 4.2.3 it follows that there exists $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$P_{\mathcal{N}(A^*)} = VP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z.$$

Now, let $Y = P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)}V$. Evidently,

$$YP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z = P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)}VP_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}Z = P_{\mathcal{N}(B)}XP_{\mathcal{N}(A^*)}.\square$$

The case $K = \{1, 4\}$ can be treated completely analogously and the corresponding result follows by taking adjoint elements.

Theorem 2.1.9 *Let $a, b \in \mathcal{A}$ be such that a, b, ab and $v = (1-bb^\dagger)a^\dagger$ are regular. Then the following conditions are equivalent:*

(i) $(ab)\{1, 4\} \subseteq b\{1, 4\}a\{1, 4\}$,

(ii) $((ab)^\dagger - b^\dagger a^\dagger)(1 - v^\dagger v) = 0$ *and* $(1 - aa^\dagger)\mathcal{A}(1 - b^\dagger b) \subseteq \mathcal{A}(1 - bb^\dagger)(1 - vv^\dagger)\mathcal{A}$.

It is interesting to mention that the second conditions appearing in (ii) in Theorem 2.1.6 and in Theorem 2.1.9 are the same, which will be proved in the following lemma:

Lemma 2.1.6 *Let $a, b \in \mathcal{A}$ be such that $a, b, ab, s = b^\dagger(1 - a^\dagger a)$ and $v = (1 - bb^\dagger)a^\dagger$ are regular. Then the following conditions are equivalent:*

(i) $(1 - b^\dagger b)\mathcal{A}(1 - aa^\dagger) \subseteq \mathcal{A}(1 - a^\dagger a)(1 - s^\dagger s)\mathcal{A}$,

(ii) $(1 - aa^\dagger)\mathcal{A}(1 - b^\dagger b) \subseteq \mathcal{A}(1 - bb^\dagger)(1 - vv^\dagger)\mathcal{A}$.

Proof. Set $p = (1 - a^\dagger a)(1 - s^\dagger s) = 1 - a^\dagger a - s^\dagger s = (1 - s^\dagger s)(1 - a^\dagger a)$ and $q = (1 - bb^\dagger)(1 - vv^\dagger) = 1 - bb^\dagger - vv^\dagger = (1 - vv^\dagger)(1 - bb^\dagger)$. We will show that $p = q$. Using Lemma 2.1.2 we have

$$\begin{aligned} qa^\dagger a &= (1 - bb^\dagger - vv^\dagger)a^\dagger a \\ &= (1 - bb^\dagger)a^\dagger a - (1 - bb^\dagger)a^\dagger((1 - bb^\dagger)a^\dagger)^\dagger(1 - bb^\dagger)a^\dagger a = 0, \end{aligned} \tag{2.13}$$

$$\begin{aligned} qs^\dagger s &= (1-bb^\dagger-vv^\dagger)s^\dagger s = (1-bb^\dagger-vv^\dagger)s^*(s^*)^\dagger \\ &= (1-bb^\dagger-vv^\dagger)(1-a^\dagger a)(b^*)^\dagger((1-a^\dagger a)(b^*)^\dagger)^\dagger \\ &= (1-bb^\dagger-vv^\dagger)(b^*)^\dagger((1-a^\dagger a)(b^*)^\dagger)^\dagger \\ &= -(b^\dagger vv^\dagger)^*((1-a^\dagger a)(b^*)^\dagger)^\dagger = 0, \end{aligned} \tag{2.14}$$

$$\begin{aligned} pbb^\dagger &= (1-a^\dagger a-s^\dagger s)bb^\dagger \\ &= (1-a^\dagger a)bb^\dagger - (1-a^\dagger a)(b^*)^\dagger((1-a^\dagger a)(b^*)^\dagger)^\dagger(1-a^\dagger a)bb^\dagger \\ &= 0, \end{aligned} \tag{2.15}$$

$$\begin{aligned} pvv^\dagger &= (1-a^\dagger a-s^\dagger s)vv^\dagger \\ &= (1-a^\dagger a-s^\dagger s)(1-bb^\dagger)a^\dagger((1-bb^\dagger)a^\dagger)^\dagger \\ &= (1-a^\dagger a-s^\dagger s)a^\dagger((1-bb^\dagger)a^\dagger)^\dagger \\ &= -s^\dagger sa^\dagger((1-bb^\dagger)a^\dagger)^\dagger = 0. \end{aligned} \tag{2.16}$$

From equations (2.13) $-$ (2.16) we get $qp = q(1-a^\dagger a)(1-s^\dagger s) = q(1-s^\dagger s) = q$ and $pq = p(1-bb^\dagger)(1-vv^\dagger) = p(1-vv^\dagger) = p$. Since p and q are projections, follows $qp = q = q^* = pq = p$.
Now,

$$\begin{aligned} &(1-b^\dagger b)\mathcal{A}(1-aa^\dagger) \subseteq \mathcal{A}p\mathcal{A} \Leftrightarrow (1-aa^\dagger)\mathcal{A}(1-b^\dagger b) \subseteq \mathcal{A}p\mathcal{A} \Leftrightarrow \\ &(1-aa^\dagger)\mathcal{A}(1-b^\dagger b) \subseteq \mathcal{A}q\mathcal{A}.\Box \end{aligned}$$

Hence, we have the following corollary:

Corollary 2.1.1 *Let $a, b \in \mathcal{A}$ be such that $a, b, ab, s = b^\dagger(1-a^\dagger a)$ and $v = (1-bb^\dagger)a^\dagger$ are regular. If $(ab)^\dagger = b^\dagger a^\dagger$, then the following conditions are equivalent:*

(*i*) $(ab)\{1,3\} \subseteq b\{1,3\}a\{1,3\}$,

(*ii*) $(ab)\{1,4\} \subseteq b\{1,4\}a\{1,4\}$.

2.2 Hartwig's triple reverse order law in C^*-algebras

As we mentioned before, Hartwig [66] considered the necessary and sufficient conditions for the reverse order law

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger, \tag{2.17}$$

where A, B and C are complex matrices for which ABC is defined, known now as a *Hartwig's triple reverse order law*.

Theorem 2.2.1 *[66] Let A, B, C be complex matrices such that ABC is defined and let $P = A^\dagger ABCC^\dagger$, $Q = CC^\dagger B^\dagger A^\dagger A$. The following conditions are equivalent:*

(i) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$,

(ii) $Q \in P\{1, 2\}$ *and both of* A^*APQ *and* $QPCC^*$ *are Hermitian,*

(iii) $Q \in P\{1, 2\}$ *and both of* A^*APQ *and* $QPCC^*$ *are EP,*

(iv) $Q \in P\{1\}$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$ *and* $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$,

(v) $PQ = PQPQ$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$ *and* $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$.

Hartwig's proof of Theorem 2.2.1 can be generalized for the operators on infinite dimensional Hilbert spaces except the proof of implication $(v) \Rightarrow (ii)$ which use matrix rank. Notice that one generalization on Hartwig's result is given in [45] for the case of closed-range bounded linear operators on infinite dimensional Hilbert spaces. The proof presented in [45] is based on operator matrices. In this section we present a very simple algebraic proof of Hartwig's result for the regular elements in C^*-algebra which can be found in our paper [90].

For regular elements a, b and c of a complex C^*-algebra with a unite $\mathcal{A}$ we use the following notations

$$p = a^\dagger abcc^\dagger \qquad \text{and} \qquad q = cc^\dagger b^\dagger a^\dagger a, \tag{2.18}$$

analogously as in Theorem 2.2.1. An element a from a C^*-algebra (a ring with involution) is *EP* if a is MP-invertible and $aa^\dagger = a^\dagger a$.

Theorem 2.2.2 *Let $\mathcal{A}$ be a complex C^*-algebra with a unite and let $a, b, c \in \mathcal{A}$ be such that a, b, c and abc are regular. Then the following conditions are equivalent:*

(i) $(abc)^\dagger = c^\dagger b^\dagger a^\dagger$,

(ii) $q \in p\{1, 2\}$ *and both of* a^*apq *and* $qpcc^*$ *are Hermitian,*

(iii) $q \in p\{1, 2\}$ *and both of* a^*apq *and* $qpcc^*$ *are EP,*

(iv) $q \in p\{1\}$, $a^*ap\mathcal{A} = q^*\mathcal{A}$ *and* $cc^*p^*\mathcal{A} = q\mathcal{A}$,

(v) $pq = pqpq$, $a^*ap\mathcal{A} = q^*\mathcal{A}$ *and* $cc^*p^*\mathcal{A} = q\mathcal{A}$.

Proof. $(i) \Leftrightarrow (ii)$: This can be showed exactly as in [66]. We give the proof because of the completeness. It is easy to notice that the condition $abcc^\dagger b^\dagger a^\dagger abc = abc$ is equivalent to the condition $pqp = p$, while $c^\dagger b^\dagger a^\dagger abcc^\dagger b^\dagger a^\dagger = c^\dagger b^\dagger a^\dagger$ holds precisely when $qpq = q$. Next, if $abcc^\dagger b^\dagger a^\dagger$ is Hermitian, so is $a^*abcc^\dagger b^\dagger a^\dagger a = a^*apq$. The converse follows, since $(a^*)^\dagger(a^*apq)a^\dagger = abcc^\dagger b^\dagger a^\dagger$. Lastly, if $c^\dagger b^\dagger a^\dagger abc$ is Hermitian, so is $c(c^\dagger b^\dagger a^\dagger abc)c^* = qpcc^*$. Again, the converse relation follows from the condition $c^\dagger(qpcc^*)(c^*)^\dagger = c^\dagger b^\dagger a^\dagger abc$.

$(ii) \Rightarrow (iii)$: This will follow if we show that a^*apq and $qpcc^*$ are regular. Indeed, we can check that $a^\dagger(a^\dagger)^* \in (a^*apq)\{1\}$ and $(c^\dagger)^*c^\dagger \in (qpcc^*)\{1\}$:

$$a^*apqa^\dagger(a^\dagger)^*a^*apq = a^*apqa^\dagger apq = a^*apqpq = a^*apq,$$

$$qpcc^*(c^\dagger)^*c^\dagger qpcc^* = qpcc^\dagger qpcc^* = qpqpcc^* = qpcc^*.$$

$(iii) \Rightarrow (iv)$: Since $a^*ap\mathcal{A} = q^*\mathcal{A}$ is equivalent with the facts that $a^*ap \in q^*\mathcal{A}$ and $q^* \in a^*ap\mathcal{A}$, we have

$$\begin{aligned} a^*ap \quad &= a^*apqp = a^*apq(a^*apq)^\dagger a^*apqp = (a^*apq)^\dagger a^*apqa^*ap \\ &= q^*p^*a^*a((a^*apq)^\dagger)^*a^*ap \in q^*\mathcal{A}, \end{aligned}$$

and

$$\begin{aligned} q^* \quad &= q^*p^*q^* = q^*p^*a^\dagger aq^* = q^*p^*a^*(a^\dagger)^*q^* = q^*p^*a^*aa^\dagger(a^\dagger)^*q^* \\ &= (a^*apq(a^*apq)^\dagger a^*apq)^*a^\dagger(a^\dagger)^*q^* = a^*apq(a^*apq)^\dagger q^* \in a^*ap\mathcal{A}. \end{aligned}$$

Similarly, $cc^*p^*\mathcal{A} = q\mathcal{A}$ is equivalent with the facts that $cc^*p^* \in q\mathcal{A}$ and $q \in cc^*p^*\mathcal{A}$, so we have

$$\begin{aligned} cc^*p^* \quad &= cc^*p^*q^*p^* = (qpcc^*(qpcc^*)^\dagger qpcc^*)^*p^* \\ &= qpcc^*(qpcc^*)^\dagger cc^*p^* \in q\mathcal{A}, \end{aligned}$$

and

$$\begin{aligned} q \quad &= qpq = qpcc^\dagger q = qpcc^*(c^\dagger)^*c^\dagger q = qpcc^*(qpcc^*)^\dagger qpcc^*(c^\dagger)^*c^\dagger q \\ &= cc^*p^*q^*((qpcc^*)^\dagger)^*q \in cc^*p^*\mathcal{A}. \end{aligned}$$

$(iv) \Rightarrow (v)$: It is evident.

$(v) \Rightarrow (ii)$: Firstly we will show that pc and $qa^\dagger$ are regular. Indeed, $pc = a^\dagger abc$ and $a^\dagger abc(abc)^\dagger aa^\dagger abc = a^\dagger abc$. Also, $cc^*p^*((pc)^\dagger)^*c^\dagger cc^*p^* = cc^*p^*$, so cc^*p^* is regular and then, since $qa^\dagger \in q\mathcal{A} = cc^*p^*\mathcal{A}$ and $cc^*p^*(cc^*p^*)^\dagger \in cc^*p^*\mathcal{A} = q\mathcal{A}$ we have $qa^\dagger = cc^*p^*x = cc^*p^*(cc^*p^*)^\dagger cc^*p^*x = qycc^*p^*x = qyqa^\dagger = qa^\dagger ayqa^\dagger$. Hence $qa^\dagger$ is regular. Now, analogously using $cc^*p^*\mathcal{A} = q\mathcal{A}$, we get

$$p = pcc^\dagger = pc(pc)^\dagger pcc^\dagger = pcc^*p^*((pc)^\dagger)^*c^\dagger = pqu,$$

and consequently $pqp = pqpqu = pqu = p$. This shows that $q \in p\{1\}$ and $qpqp = qp$. Also, using $a^*ap\mathcal{A} = q^*\mathcal{A}$, we get

$$q = qa^\dagger a = qa^\dagger(qa^\dagger)^\dagger qa^\dagger a = qa^\dagger(a^\dagger)^*q^*((qa^\dagger)^\dagger)^*a = qa^\dagger(a^\dagger)^*a^*apv = qa^\dagger apv = qpv,$$

which gives $qpq = qpqpv = qpv = q$. To complete the proof notice that, by $a^*ap\mathcal{A} = q^*\mathcal{A}$ and $cc^*p^*\mathcal{A} = q\mathcal{A}$, we set

$$q^*p^*a^*apq = q^*p^*q^*t = q^*t = a^*apq$$

and

$$qpcc^*p^*q^* = qpqz = qz = cc^*p^*q^*$$

which imply that a^*apq and $qpcc^*$ are Hermitian. $\square$

Remark 2.2.1 In the case when A, B and C are bounded linear operators on Hilbert space $\mathcal{H}$, by Theorem 4.2.3, we can conclude that conditions (iv) $((v))$ from Theorems 2.2.1 and 2.2.2 are equivalent.

Remark 2.2.2 Let $\mathcal{H}_i, i = \overline{1,4}$ be arbitrary Hilbert spaces, $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), B \in \mathcal{B}(H_2, \mathcal{H}_3)$ and $A \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ be closed range operators such that ABC has closed range. Hartwig's proof of Theorem 2.2.1 can be improved for the case of closed range operators with pure algebraic technique similarly as in proof of Theorem 2.2.2. Namely, the regularity of elements A^*APQ and $QPCC^*$ can be shown as in the proof of Theorem 2.2.2. Now, as we said, the proof given by Hartwig's stay valid except for the implication $(v) \Rightarrow (ii)$. The regularity of element PC can be verified as in the proof of Theorem 2.2.2, and now as in [66] we can get that $PQP = P$ and consequently $QPQP = QP$:

$$\mathcal{R}(CC^*P) = \mathcal{R}(Q) \Rightarrow \mathcal{R}(P) = \mathcal{R}(PC) = \mathcal{R}(PCC^*P^*) = \mathcal{R}(PQ) \overset{PQ=(PQ)^2}{\Rightarrow} PQP = P.$$

To see that the element $QA^\dagger$ is regular notice that $\mathcal{R}(PCC^*) = \mathcal{R}(PC)$ is closed and consequently $\mathcal{R}(QA^\dagger) = \mathcal{R}(Q) = \mathcal{R}(CC^*P^*)$ is closed. Now, using $\mathcal{R}(Q^*) = \mathcal{R}(A^*AP)$, we have

$$\begin{aligned}\mathcal{R}(Q) = \mathcal{R}(QA^\dagger) &= \mathcal{R}(QA^\dagger(QA^\dagger)^*) = \mathcal{R}(QA^\dagger(A^\dagger)^*Q^*) = \mathcal{R}(QA^\dagger(A^\dagger)^*A^*AP) \\ &= \mathcal{R}(QA^\dagger AP) = \mathcal{R}(QP)\end{aligned}$$

and now, since QP is idempotent with range $\mathcal{R}(Q)$ then $QPQ = Q$. The rest of the proof is as in [66]. Since $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q) = \mathcal{R}(QP)$ follows that $QPCC^*P^* = CC^*P^*$ and consequently $QPCC^*(QP)^* = CC^*(QP)^*$, which shows that $QPCC^*$ is Hermitian. Lastly,

$$\mathcal{R}(A^*AP) = \mathcal{R}(Q^*) = \mathcal{R}(Q^*P^*) \Rightarrow Q^*P^*A^*AP = A^*AP \Rightarrow (PQ)^*A^*APQ = A^*APQ,$$

and again, A^*APQ is Hermitian.

Remark 2.2.3 Let us mention some special cases when triple reverse order low for the Moore-Penrose inverse of the products of three regular elements a, b and c of C^*-algebra $\mathcal{A}$ holds. Recall that an element $x \in \mathcal{A}$ is unitary if $xx^* = 1 = x^*x$.
If a is unitary we get that

$$(abc)^\dagger = c^\dagger b^\dagger a^\dagger \Leftrightarrow (bc)^\dagger = c^\dagger b^\dagger.$$

Similarly, if c is unitary

$$(abc)^\dagger = c^\dagger b^\dagger a^\dagger \Leftrightarrow (ab)^\dagger = b^\dagger a^\dagger.$$

The case when b is unitary is not trivial like previous two, but can be deduced easily from known result. For elements $x, y \in \mathcal{A}$ set $[x, y] = xy - yx$. In an analogical manner as in Theorem 2.0.1 can be shown the following theorem. We give its proof for the completeness.

Theorem 2.2.3 *Let $\mathcal{A}$ be a complex C^*-algebra with a unite, let $a, b, c \in \mathcal{A}$ be regular elements and let b be unitary. Then the following conditions are equivalent:*

(i) *abc is regular and $(abc)^\dagger = c^\dagger b^\dagger a^\dagger$,*

(ii) *$[bcc^\dagger b^\dagger, a^*a] = 0$ and $[b^\dagger a^\dagger ab, cc^*] = 0$.*

Proof. $(i) \Rightarrow (ii)$: The left hand side of

$$c^\dagger b^\dagger a^\dagger abc = (c^*c)^\dagger(c^*b^\dagger a^\dagger abc)$$

is Hermitian, which implies

$$[(c^*c)^\dagger, c^*b^\dagger a^\dagger abc] = 0.$$

Further,

$$\begin{aligned} abcc^*c &= abcc^\dagger b^\dagger a^\dagger abcc^*c = abc(c^*c)^\dagger(c^*b^\dagger a^\dagger abc)c^*c \\ &= abc(c^*b^\dagger a^\dagger abc)c^\dagger c = abcc^*b^\dagger a^\dagger abc. \end{aligned}$$

Hence $abcc^*(1 - b^\dagger a^\dagger ab)c = 0$, and consequently

$$abcc^*(1 - b^\dagger a^\dagger ab)(abcc^*(1 - b^\dagger a^\dagger ab))^* = 0,$$

which gives $abcc^* = abcc^*b^\dagger a^\dagger ab$. Next we find that

$$b^\dagger a^\dagger abcc^* = b^\dagger a^\dagger abcc^*b^\dagger a^\dagger ab = b^\dagger a^\dagger abc(b^\dagger a^\dagger abc)^*$$

is Hermitian, implying

$$[b^\dagger a^\dagger ab, cc^*] = 0.$$

To prove the second result of (ii), notice that by taking adjoints in $(abc)^\dagger = c^\dagger b^\dagger a^\dagger$ we obtain $(c^*b^*a^*)^\dagger = (a^*)^\dagger(b^*)^\dagger(c^*)^\dagger$. From the first part of the implication $(i) \Rightarrow (ii)$, we get $[(b^*)^\dagger(c^*)^\dagger c^*b^*, a^*a] = 0$, which is equivalent to

$$[bcc^\dagger b^\dagger, a^*a] = 0.$$

$(ii) \Rightarrow (i)$: Suppose that (ii) holds. Because the Drazin inverse of an element double commutes with that element and a^*a and cc^* are Hermitian, we can conclude that $[bcc^\dagger b^\dagger, (a^*a)^\dagger] = 0$ and $[b^\dagger a^\dagger ab, (cc^*)^\dagger] = 0$. This implies $[bcc^\dagger b^\dagger, a^\dagger a] = [b^\dagger a^\dagger ab, cc^\dagger] = 0$. Then we have

$$\begin{aligned} &abcc^\dagger b^\dagger a^\dagger abc = abb^*a^\dagger abc = abc, \\ &c^\dagger b^\dagger a^\dagger abcc^\dagger b^\dagger a^\dagger = c^\dagger b^\dagger a^\dagger abb^*a^\dagger = c^\dagger b^\dagger a^\dagger, \\ &abcc^\dagger b^\dagger a^\dagger = abcc^\dagger b^\dagger(a^*a)^\dagger a^* = (a^\dagger)^*bcc^\dagger b^\dagger a^* = (abcc^\dagger b^\dagger a^\dagger)^*, \\ &c^\dagger b^\dagger a^\dagger abc = c^*(cc^*)^\dagger b^\dagger a^\dagger abc = c^*b^\dagger a^\dagger ab(c^\dagger)^* = (c^\dagger b^\dagger a^\dagger abc)^*, \end{aligned}$$

which gives that element abc is regular and $(abc)^\dagger = c^\dagger b^\dagger a^\dagger$. $\square$

2.3 Improvements on Hartwig's triple reverse order law in different settings

In this section, we present several significant improvements on Hartwig's triple reverse order law given in [33]. Ideas for the improvements comes from the fact that a lot of recently published results for generalized inverses and their applications were proved only under restrictive assumptions which limit their applications to certain very particular cases. One reason for that is that, in contrast to the setting of matrices, generalized inverses are not defined for each element of more general settings considered (algebras of operators, C^*-algebras, rings, ...). In order to benefit from the rich theory of generalized inverses and many already developed useful techniques, researchers usually impose existence of generalized inverses when proving statements. This leads to many results with redundant instances of assuming regularity of certain elements which makes them less applicable. So, in recent years a lot of effort has been made to widen the range of applicability of these results by considering more general cases of the problems without imposing any such additional assumptions.

This result is exactly one such important step in generalizing Hartwig's triple reverse order law which was initiated by the software `OperatorGB` [69]. The advantage of the framework developed in this software is that a single computation in an abstract setting proves analogous statements in various concrete settings (e.g. for matrices, linear bounded operators, C^*-algebras, ...) without having to inspect every step of the abstract computation. Just like in any ring, computations with noncommutative polynomials allow any two elements to be added or multiplied. Therefore, it is not clear a priori that a given proof of a statement in a ring is valid also for rectangular matrices or operators with different domain and codomain. Using the framework for algebraic proofs, the following steps have to be carried out once in a suitable ring of noncommutative polynomials:

1. All the assumptions on the operators involved have to be phrased in terms of identities involving those operators. Likewise, the claimed properties have to be expressed as identities of these operators.

2. These identities are converted into polynomials by uniformly replacing the individual operators by noncommutative indeterminates in the differences of the left and right hand sides.

3. Prove that the polynomials corresponding to the claim lie in the ideal generated by the set of polynomials corresponding to the assumptions.

More explicitly, in the last step, one has to find a concrete representation of the polynomials corresponding to the claim as a two-sided linear combination of polynomials corresponding to the assumptions, where coefficients are polynomials. Such *cofactor representations* serve as certificates for ideal membership that can be checked independently, but finding them is a hard problem. In practice, cofactor representations often can be found with the help of the computer. It the previous steps are carried out, then, to rigorously prove a statement for various concrete settings it suffices to check that the polynomials corresponding to the assumptions and claims are compatible with domains and codomains of operators.

The software package `OperatorGB` provides the command `Certify`, which not only tries to compute cofactor representations but also does the compatibility checks of assumptions and claims. Inspecting the explicit cofactor representations found by the software can also give hints how assumptions could be relaxed by dropping the assumptions that do not appear in the cofactor representations. More generally, the software makes it easy to experiment with different sets of assumptions for proving a desired claim. Improvements of Hartwig's triple reverse order law found by such experiments were the basis for the main results presented in this section. These results represent an important improvement of Hartwig's result in several senses:

- we consider the problem in rings with involution, thus generalizing all the known results on this subject.

- as for the original result of Hartwig (Theorem 2.2.1) we relax conditions (iv) i (v) by replacing the respective equalities of ranges assumed in both of them with appropriate inclusions of ranges. Furthermore, we show that only certain combinations of inclusions (four of them in total), along with the assumption that the element pq is idempotent, imply (2.17), while the other combinations do not guarantee the claimed conclusion (there are two such). As for the analogous results for algebras of operators and C^*-algebras (see [45] and [90]) we improve them in a similar way by replacing equalities with appropriate inclusions.

- as regards the results for algebras of operators and C^*-algebras in general (see [45] and [90]), we significantly reduce the set of starting assumptions upon which these are based and which considerably restrict the set of elements to which they apply. Namely, if one is interested in validity of (2.17), it is possible to omit the requirement that the product abc be MP-invertible, since it follows directly from some of the assumptions (iv) or (v). In the case of rings, MP-invertibilily of the product abc can be replaced with the weaker condition of right $*$- cancellability of abc.

- also, it is possible to generalize the result by showing that $b^\dagger$ can be replaced by an arbitrary element $\widetilde{b}$ as well as that $a^\dagger$ and $c^\dagger$ can be replaced with arbitrary $a^{(1,2,3)}$ and $c^{(1,2,4)}$, respectively. This way the assumption of MP-invertibility of the element b is dropped and MP-invertibility of the elements a and c is replaced with the existence of $a^{(1,2,3)}$ and $c^{(1,2,4)}$. This, although the last two are equivalent conditions in operator algebras and C^*-algebras, improves the results significantly in rings with involution since there the existence of a $\{1,2,3\}$-inverse of an element is equivalent with the existence of its $\{1,3\}$-inverse and the latter is a much weaker condition than MP-invertibility (see Example 1.3.1).

From now by $\mathcal{R}$ we denote a ring with a unit $1 \neq 0$ and an involution $a \mapsto a^*, a \in \mathcal{R}$.

Theorem 2.3.1 *Let $a, b, c \in \mathcal{R}$ be such that a, c are MP-invertible. Let $p = a^\dagger abcc^\dagger$ and $q = cc^\dagger \widetilde{b} a^\dagger a$, for $\widetilde{b} \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *abc is Moore-Penrose invertible and* $(abc)^\dagger = c^\dagger \widetilde{b} a^\dagger$,

(ivH) $q \in p\{1\}$, $a^*ap\mathcal{R} \supseteq q^*\mathcal{R}$ *and* $cc^*p^*\mathcal{R} \subseteq q\mathcal{R}$,

(vH) *abc is right ∗- cancellable,* $pq = (pq)^2$, $a^*ap\mathcal{R} \supseteq q^*\mathcal{R}$ *and* $cc^*p^*\mathcal{R} \subseteq q\mathcal{R}$,

(vi) $q \in p\{2\}$, $a^*ap\mathcal{R} \supseteq q^*\mathcal{R}$ *and* $cc^*p^*\mathcal{R} \subseteq q\mathcal{R}$.

Proof. Let $m = abc$ and $\widetilde{m} = c^\dagger \widetilde{b} a^\dagger$. Evidently pq is idempotent if and only if $m\widetilde{m}$ is idempotent. Also we have that the following equivalents hold:

$$a^*ap\mathcal{R} \supseteq q^*\mathcal{R} \Leftrightarrow m\mathcal{R} \supseteq (\widetilde{m})^*\mathcal{R} \Leftrightarrow \mathcal{R}m^* \supseteq \mathcal{R}\widetilde{m} \Leftrightarrow \widetilde{m} \in \mathcal{R}m^*;$$
$$cc^*p^*\mathcal{R} \subseteq q\mathcal{R} \Leftrightarrow m^*\mathcal{R} \subseteq \widetilde{m}\mathcal{R} \Leftrightarrow m^* \in \widetilde{m}\mathcal{R};$$

$(i) \Rightarrow (vH)$: If $m^\dagger = \widetilde{m}$ than clearly $m\widetilde{m}$ is idempotent. Also

$$\widetilde{m} = m^\dagger = m^\dagger m m^\dagger = m^\dagger (m^\dagger)^* m^* \in \mathcal{R}m^*,$$
$$m^* = (mm^\dagger m)^* = m^\dagger m m^* = \widetilde{m} m m^* \in \widetilde{m}\mathcal{R}.$$

$(vH) \Rightarrow (i)$: If (vH) holds, then there exist $u, v \in \mathcal{R}$ such that $\widetilde{m} = um^*$ and $m^* = \widetilde{m}v$. Now, multiplying $m\widetilde{m} = (m\widetilde{m})^2$ by v from the right side, we get $mm^* = m\widetilde{m}mm^*$ i.e. $(1 - m\widetilde{m})mm^* = 0$ which gives $(1 - m\widetilde{m})m = 0$ by right ∗- cancellability of m. So $\widetilde{m}$ is an inner inverse of m. Further, we have that

$$\widetilde{m} = um^* = u(m\widetilde{m}m)^* = \widetilde{m}(m\widetilde{m})^*,$$

which implies that $m\widetilde{m}$ is Hermitian and further

$$\widetilde{m} = \widetilde{m}(m\widetilde{m})^* = \widetilde{m}m\widetilde{m}.$$

Also

$$m = v^*(\widetilde{m})^* = v^*(\widetilde{m}m\widetilde{m})^* = m(\widetilde{m}m)^*,$$

which implies that $\widetilde{m}m$ is Hermitian.

$(ivH), (vi) \Rightarrow (vH)$: It is evident.

$(i) \Rightarrow (ivH)$: That $q \in p\{1\}$ follows directly from the fact that $\widetilde{m}$ is an inner inverse of m. The rest of the proof follows as in the part $(i) \Rightarrow (vH)$.

$(i) \Rightarrow (vi)$: That $q \in p\{2\}$ follows from the fact that $\widetilde{m}$ is an outer inverse of m. The rest of the proof follows as in the part $(i) \Rightarrow (vH)$. □

Notations $(ivH), (vH)$ come from Theorem 2.2.1

It is interesting to mention that if we take the reverse inclusion from (ii) of Theorem 2.3.1 and replace in the statement of the theorem the assumption of right ∗-cancellability of abc with the assumption of left ∗- cancellability of $c^\dagger \widetilde{b} a^\dagger$, we will get the analogous result:

Theorem 2.3.2 *Let $a, b, c, \widetilde{b} \in \mathcal{R}$ be such that a, c are MP-invertible. Let $p = a^\dagger abcc^\dagger$ and $q = cc^\dagger \widetilde{b} a^\dagger a$. Then the following conditions are equivalent:*

(i) *abc is Moore-Penrose invertible and $(abc)^\dagger = c^\dagger \widetilde{b} a^\dagger$,*

(ivH) *$q \in p\{1\}$, $a^*ap\mathcal{R} \subseteq q^*\mathcal{R}$ and $cc^*p^*\mathcal{R} \supseteq q\mathcal{R}$,*

(vH) *$c^\dagger \widetilde{b} a^\dagger$ is left $*$- cancellable, $pq = (pq)^2$, $a^*ap\mathcal{R} \subseteq q^*\mathcal{R}$ and $cc^*p^*\mathcal{R} \supseteq q\mathcal{R}$,*

(vi) *$q \in p\{2\}$, $a^*ap\mathcal{R} \subseteq q^*\mathcal{R}$ and $cc^*p^*\mathcal{R} \supseteq q\mathcal{R}$.*

Remark 2.3.1 It is worth noting that it can be seen easier than in the proof of Theorem 2.3.1 (Theorem 2.3.2) the condition (vH) from that theorem implies condition (vH) from Theorem 2.2.1, i.e. (2.17) in a matrix case. Note that

$$r(A^*AP) = r(P^*A^*A) = r(P^*) = r(P) \quad \text{and} \quad r(CC^*P^*) = r(PCC^*) = r(P).$$

So, if the condition (vH) from Theorem 2.3.1 (Theorem 2.3.2) is satisfied, then $r(Q) = r(P) = r(A^*AP) = r(CC^*P^*)$ which implies $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$ and $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$.

Remark 2.3.2 Similarly, for closed range bounded linear operators A, B, C defined on Hilbert spaces such that ABC can be defined, the implication $(vH) \Rightarrow (i)$ in Theorem 2.3.1 (Theorem 2.3.2) can be verified using operator matrices. Condition (vH) from Theorem 2.3.1 is equivalent with $M\widetilde{M} = (M\widetilde{M})^2, \mathcal{R}((\widetilde{M})^*) \subseteq \mathcal{R}(M)$ and $\mathcal{R}(M^*) \subseteq \mathcal{R}(\widetilde{M})$. By noting that $\mathcal{R}(M) \subseteq \overline{\mathcal{R}(M)} = \overline{\mathcal{R}(MM^*)} \subseteq \overline{\mathcal{R}(M\widetilde{M})} = \mathcal{R}(M\widetilde{M})$, we can conclude that $M\widetilde{M}M = M$. If we use the representations

$$\begin{aligned} M &= \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(M^*) \\ \mathcal{N}(M) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M^*) \end{bmatrix}, \\ \widetilde{M} &= \begin{bmatrix} X & Y \\ Z & T \end{bmatrix} : \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M^*) \\ \mathcal{N}(M) \end{bmatrix}, \end{aligned}$$

from $\mathcal{R}((\widetilde{M})^*) \subseteq \mathcal{R}(M)$ we get $\mathcal{N}(M^*) \subseteq \mathcal{N}(\widetilde{M})$ i.e. $Y = T = 0$, and from $M\widetilde{M}M = M$, that $X = M_1^{-1}$. From $\mathcal{R}(M^*) \subseteq \mathcal{R}(\widetilde{M})$, it follows that for each $x \in \mathcal{R}(M)$ there exists $y \in \mathcal{R}(M)$ such that $M_1^*x = M_1^{-1}y + Zy$ i.e. $M_1^*x = M_1^{-1}y$ and $0 = Zy$, which implies $ZM_1M_1^* = 0$ i.e. $Z = 0$. Finally, we get $\widetilde{M} = M^\dagger$. The implication $(vH) \Rightarrow (i)$ in Theorem 2.3.2 can be derived from the above arguments just by replacing M with $(\widetilde{M})^*$ and $\widetilde{M}$ with M^*.

The following example illustrates the fact that the remaining two combinations of inclusions in the original result of Hartwig (Theorem 2.2.1 (vH)) do not necessarily imply (2.17).

Example 2.3.1 Let

$$A = \begin{bmatrix} -3 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$A^\dagger = \frac{1}{17}\begin{bmatrix} -3 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad B^\dagger = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad C^\dagger = C.$$

If we define P and Q as in Theorem 2.2.1, we get that $PQ = 0$ is idempotent and $\mathcal{R}(A^*AP) \subseteq \mathcal{R}(Q^*)$ and $\mathcal{R}(CC^*P^*) \subseteq \mathcal{R}(Q)$ but $(ABC)^\dagger \neq C^\dagger B^\dagger A^\dagger$.
If matrices A, B, C are defined respectively as $C^\dagger, B^\dagger$ and $A^\dagger$ given above, we conclude that neither does the second pair of inclusions $\mathcal{R}(Q^*) \subseteq \mathcal{R}(A^*AP)$ and $\mathcal{R}(Q) \subseteq \mathcal{R}(CC^*P^*)$ and the assumption that the matrix PQ is idempotent imply (2.17).

On the other side above mentioned pairs of inclusions imply (2.17) with some assumptions on p and q:

Theorem 2.3.3 *Let $a, b, c \in \mathcal{R}$ be such that a, c are MP-invertible. Let $p = a^\dagger abcc^\dagger$ and $q = cc^\dagger \widetilde{b} a^\dagger a$, for $\widetilde{b} \in \mathcal{R}$. Then the following conditions are equivalent:*

(*i*) *abc is Moore-Penrose invertible and $(abc)^\dagger = c^\dagger \widetilde{b} a^\dagger$,*

(*ivH*) *$q \in p\{1\}$, $a^*ap\mathcal{R} \supseteq q^*\mathcal{R}$ and $cc^*p^*\mathcal{R} \supseteq q\mathcal{R}$,*

(*vi*) *$q \in p\{2\}$, $a^*ap\mathcal{R} \subseteq q^*\mathcal{R}$ and $cc^*p^*\mathcal{R} \subseteq q\mathcal{R}$.*

Following previous mentioned results we can deduced that MP-invertibility of the elements a and c can be replaced with the existence of $a^{(1,2,3)}$ and $c^{(1,2,4)}$.

Theorem 2.3.4 *Let $a, b, c, \widetilde{b} \in \mathcal{R}$ be such that exist $a^{(1,3)}$ and $c^{(1,4)}$ and abc is right $*$- cancellable. Let $a^{(1,2,3)}, c^{(1,2,4)}$ be given such that $c^{(1,2,4)}\widetilde{b}a^{(1,2,3)}$ is left $*$- cancellable and let $p = a^{(1,2,3)}abcc^{(1,2,4)}$ and $q = cc^{(1,2,4)}\widetilde{b}a^{(1,2,3)}a$. Then the following conditions are equivalent:*

(*i*) *abc is Moore-Penrose invertible and $(abc)^\dagger = c^{(1,2,4)}\widetilde{b}a^{(1,2,3)}$,*

(*ii*) *$q \in p\{1, 2\}$ and both of a^*apq and $qpcc^*$ are Hermitian,*

(*iii*) *$q \in p\{1, 2\}$ and both of $a^*apq\mathcal{R} = (a^*apq)^*\mathcal{R}$ and $qpcc^*\mathcal{R} = (qpcc^*)^*\mathcal{R}$,*

(*ivH*) *$pq = (pq)^2$, $a^*ap\mathcal{R} \supseteq q^*\mathcal{R}$ and $cc^*p^*\mathcal{R} \subseteq q\mathcal{R}$,*

(*vH*) *$pq = (pq)^2$, $a^*ap\mathcal{R} \subseteq q^*\mathcal{R}$ and $cc^*p^*\mathcal{R} \supseteq q\mathcal{R}$.*

Notice that if in Theorem 2.3.4 we replace $a^{(1,2,3)}, c^{(1,2,4)}$ with $a^{(1,3)}$ and $c^{(1,4)}$, respectively the assertion of the theorem will not hold anymore which will be shown in the next example:

Example 2.3.2 Let $B = C = \widetilde{B} = I$ and take any matrix A such that $A\{1,3,4\} \neq \{A^\dagger\}$ (such A can be any projection different than unit). If we take that $A^{(1,3)} = A^{(1,3,4)} \neq A^\dagger$ we will get that all conditions from $(ii) - (vH)$ are satisfied while (i) from Theorem 2.3.4 is not satisfied.

Finally by the above discussion we end this section with the improved version of the original Hartwig's result:

Theorem 2.3.5 *Let A, B, C be complex matrices such that ABC is defined and let $P = A^\dagger ABCC^\dagger$, $Q = CC^\dagger B^\dagger A^\dagger A$. The following conditions are equivalent:*

(i) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$,

(ii) $Q \in P\{1,2\}$ *and both of* A^*APQ *and* $QPCC^*$ *are Hermitian,*

(iii) $Q \in P\{1,2\}$ *and both of* A^*APQ *and* $QPCC^*$ *are EP,*

(iv') $Q \in P\{1\}$ $,\mathcal{R}(Q^*) \subseteq \mathcal{R}(A^*AP)$ *and* $\mathcal{R}(CC^*P^*) \subseteq \mathcal{R}(Q)$,

(iv'') $Q \in P\{1\}$, $\mathcal{R}(A^*AP) \subseteq \mathcal{R}(Q^*)$ *and* $\mathcal{R}(Q) \subseteq \mathcal{R}(CC^*P^*)$,

(iv''') $Q \in P\{1\}$, $\mathcal{R}(Q^*) \subseteq \mathcal{R}(A^*AP)$ *and* $\mathcal{R}(Q) \subseteq \mathcal{R}(CC^*P^*)$,

(iv'''') $Q \in P\{2\}$ $,\mathcal{R}(Q^*) \subseteq \mathcal{R}(A^*AP)$ *and* $\mathcal{R}(CC^*P^*) \subseteq \mathcal{R}(Q)$,

(iv''''') $Q \in P\{2\}$, $\mathcal{R}(A^*AP) \subseteq \mathcal{R}(Q^*)$ *and* $\mathcal{R}(Q) \subseteq \mathcal{R}(CC^*P^*)$,

(iv'''''') $Q \in P\{2\}$, $\mathcal{R}(A^*AP) \subseteq \mathcal{R}(Q^*)$ *and* $\mathcal{R}(CC^*P^*) \subseteq \mathcal{R}(Q)$,

(v') $PQ = (PQ)^2$, $\mathcal{R}(Q^*) \subseteq \mathcal{R}(A^*AP)$ *and* $\mathcal{R}(CC^*P^*) \subseteq \mathcal{R}(Q)$,

(v'') $PQ = (PQ)^2$, $\mathcal{R}(A^*AP) \subseteq \mathcal{R}(Q^*)$ *and* $\mathcal{R}(Q) \subseteq \mathcal{R}(CC^*P^*)$.

2.4 Improvements of results on reverse order laws for $\{1,3\}$ and $\{1,2,3\}$-generalized inverses

Following the spirit of the previous section in this section we present several improvements on some results that concern the reverse order laws from [34]. In C^*-algebra case we will remove the assumptions of the regularity of some elements. Precisely, we discuss the reverse order laws on $\{1,3\}, \{1,4\}, \{1,3,4\}, \{1,2,3\}$ and $\{1,2,4\}$-inverses in a ring setting.

As in Section 2.3 the main settings that we consider is a ring $\mathcal{R}$ with a unit $1 \neq 0$ and an involution $a \mapsto a^*, a \in \mathcal{R}$, while $\mathcal{A}$ denotes a C^*-algebra. If p and q are projections in $\mathcal{R}$, then we can represent an element $x \in \mathcal{R}$ as 2×2 matrix over $\mathcal{R}$, writing

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p,q},$$

where $x_1 = pxq$, $x_2 = px(1-q)$, $x_3 = (1-p)xq$, $x_4 = (1-p)x(1-q)$; note that $x = x_1 + x_2 + x_3 + x_4$.

2.4.1 Improvements on the reverse order law for $\{1,3\}$ and $\{1,3,4\}$-generalized inverses

In this subsection, we present improvement of the Theorem 2.1.2 [32] that concerns the reverse order law (2.3) in a C^*-algebra, first. Strictly speaking, we will remove assumptions of the regularity of ab and $a(1-bb^\dagger)$, by showing that under the assumption of regularity of a and b we have that condition $bb^\dagger a^*ab = a^*ab$ implies the regularity of their product ab. Before the main result, we prove the following auxiliary lemma.

Lemma 2.4.1 *Let $p, q \in \mathcal{R}$ be given projection and let $c = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_{p,q}$. Then c is MP-invertible if and only if a and b are MP-invertible. In this case $c^\dagger = \begin{bmatrix} a^\dagger & 0 \\ 0 & b^\dagger \end{bmatrix}_{q,p}$.*

Proof. ($\Leftarrow$:) If a and b are MP-invertible it can be checked, by straightforward computation, that $c^\dagger = \begin{bmatrix} a^\dagger & 0 \\ 0 & b^\dagger \end{bmatrix}_{q,p}$.

($\Rightarrow$:) Suppose that c is MP-invertible and $c^\dagger = \begin{bmatrix} x & y \\ z & t \end{bmatrix}_{q,p}$. From the Penrose's equations can be deduced that $x \in a\{1,3,4\}$ and $t \in b\{1,3,4\}$ which implies that xax and tbt are MP-inverses of a and b, respectively. Like in the opposite implication, the form of MP-inverse of c can be checked. □

Theorem 2.4.1 *Let $a, b \in \mathcal{R}$ be such that a, b are MP-invertible and $a(1-bb^\dagger)$ is right $*$-cancellable. The following conditions are equivalent:*

(i) $bb^\dagger a^*ab = a^*ab$,

(ii) $b\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$,

(iii) $b^\dagger a^\dagger \in (ab)\{1,3\}$,

(iv) $b^\dagger a^\dagger \in (ab)\{1,2,3\}$.

Proof. Let $a_1 = abb^\dagger$, $a_2 = a - a_1$ and $d = a_1a_1^* + a_2a_2^*$.

$(iii) \Rightarrow (i)$: Since

$$aa^*ab = aa^*abb^\dagger a^\dagger ab = aa^*(abb^\dagger a^\dagger)^*ab = aa^*(a^\dagger)^*bb^\dagger a^*ab = abb^\dagger a^*ab$$

we have that

$$a(1-bb^\dagger)a^*ab = 0.$$

By

$$a(1-bb^\dagger)\left(a(1-bb^\dagger)\right)^* ab = a(1-bb^\dagger)a^*ab = 0$$

and right $*$-cancellability of a_2 we get $bb^\dagger a^*ab = a^*ab$.

$(i) \Rightarrow (iv)$: Notice that the MP-invertibility of a implies the MP-invertibility of a^*a. Since $a^*a = \begin{bmatrix} a_1^*a_1 & 0 \\ 0 & a_2^*a_2 \end{bmatrix}_{bb^\dagger, bb^\dagger}$, by Lemma 2.4.1 and Theorem 1.3.5, we can conclude that a_1 and a_2 are MP-invertible. Let $s = a_1 a_1^\dagger$. Since $a_1^* a_2 = 0$ we have that $a_1 a_1^* \in s\mathcal{R}s$ and $a_2 a_2^* \in (1-s)\mathcal{R}(1-s)$. So d can be represented by

$$d = \begin{bmatrix} a_1 a_1^* & 0 \\ 0 & a_2 a_2^* \end{bmatrix}_{s,s}.$$

To prove that $abb^\dagger a^\dagger$ is Hermitian, using Lemma 2.4.1, we have that

$$\begin{aligned} abb^\dagger a^\dagger &= abb^\dagger a^*(aa^*)^\dagger = abb^\dagger (abb^\dagger)^* d^\dagger = a_1 a_1^* \left((a_1 a_1^*)^\dagger + (a_2 a_2^*)^\dagger \right) \\ &= a_1 a_1^* (a_1 a_1^*)^\dagger. \end{aligned}$$

Hence $abb^\dagger a^\dagger$ is Hermitian. Since

$$ab = (a^\dagger)^* a^* ab = (a^\dagger)^* bb^\dagger a^* ab = abb^\dagger a^\dagger ab$$

we get $b^\dagger a^\dagger \in (ab)\{1\}$. To prove that $b^\dagger a^\dagger \in (ab)\{2\}$, notice that

$$\begin{aligned} bb^\dagger a^\dagger abb^\dagger a^\dagger &= bb^\dagger a^*(aa^*)^\dagger (abb^\dagger) a^* (aa^*)^\dagger = a_1^* d^\dagger a_1 (a_1 + a_2)^* d^\dagger \\ &= a_1^* (a_1 a_1^*)^\dagger a_1 a_1^* (a_1 a_1^*)^\dagger = a_1^\dagger. \end{aligned}$$

On the other side $bb^\dagger a^\dagger = bb^\dagger a^*(aa^*)^\dagger = a_1^* d^\dagger = a_1^*(a_1 a_1^*)^\dagger = a_1^\dagger$. Hence

$$bb^\dagger a^\dagger abb^\dagger a^\dagger = bb^\dagger a^\dagger$$

i.e. $b^\dagger a^\dagger \in (ab)\{2\}$.

$(i) \Rightarrow (ii)$: Notice that any $\{1,3\}$-inverse of a has the form $a^{(1,3)} = a^\dagger + (1 - a^\dagger a)x$, for some x and that for any $\{1,3\}$-inverse of b, we have that $bb^{(1,3)} = bb^\dagger$. Let $a^{(1,3)}, b^{(1,3)}$ be arbitrary but fixed $\{1,3\}$-inverses of a and b, respectively. We will prove that $abb^\dagger(1 - a^\dagger a) = 0$. Indeed, using Lemma 2.4.1 (see part $(i) \Rightarrow (iv)$), we have

$$abb^\dagger a^\dagger a = a_1 a^*(aa^*)^\dagger a = a_1 a^* \left((a_1 a_1^*)^\dagger + (a_2 a_2^*)^\dagger \right) a = a_1 = abb^\dagger.$$

Now

$$abb^{(1,3)} a^{(1,3)} = abb^\dagger a^{(1,3)} = abb^\dagger a^\dagger.$$

Since we already proved that $(1) \Rightarrow (4)$, we have $b^{(1,3)} a^{(1,3)} \in (ab)\{1,3\}$.

$(ii) \Rightarrow (iii)$: It is evident.

$(iv) \Rightarrow (iii)$: It is evident. $\square$

The analogous result can be obtained in the case of $\{1,4\}$-generalized inverses.

Theorem 2.4.2 *If $a, b \in \mathcal{R}$ are such that a, b are MP-invertible and $(1 - a^\dagger a)b$ is left $*$-cancellable. The following conditions are equivalent:*

(i) $abb^* a^\dagger a = abb^*$,

(ii) $b\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$,

(iii) $b^\dagger a^\dagger \in (ab)\{1,4\}$,

(iv) $b^\dagger a^\dagger \in (ab)\{1,2,4\}$.

Notice that the conditions that $a(1-bb^\dagger)$ is right $*$-cancellable from Theorem 2.4.1 and that $(1-a^\dagger a)b$ is left $*$-cancellable from Theorem 2.4.2 are always satisfied in C^*-algebra. Since the property of the closedness of the range of operators is essential and we know that for operators $A, B \in \mathcal{B}(\mathcal{H})$ whose range is closed the range of their product need not to be closed, the following result which is a corollary of the Theorem 2.4.1 can be useful:

Corollary 2.4.1 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators with closed ranges. If $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ or $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$, then $\mathcal{R}(AB)$ is closed.*

By Theorems 2.4.1 and 2.4.2, we can prove the following results:

Theorem 2.4.3 *Let $a, b \in \mathcal{R}$ be MP-invertible, $a(1-bb^\dagger)$ be right $*$-cancellable and $(1-a^\dagger a)b$ be left $*$-cancellable. The following conditions are equivalent:*

(i) $b\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{1,3,4\}$,

(ii) $bb^\dagger a^*ab = a^*ab$ *and* $abb^*a^\dagger a = abb^*$,

(iii) $b^\dagger a^\dagger = (ab)^\dagger$.

Notice that in the C^*-algebra case, Theorem 2.4.2 presents a generalization of Theorem 2.1 from [26] since the conditions for the regularity of $a(1-bb^\dagger), (1-a^\dagger a)b$ and ab are removed.

It is interesting to mention that using the previous result for $\{1,3\}$-inverses, in the same manner as in Section 2.1, the approaching theorem can be proven.

Theorem 2.4.4 *Let $a, b \in \mathcal{R}$ be such that a, b are MP-invertible and $a(1-bb^\dagger)$ is right $*$-cancellable. The following conditions are equivalent:*

(i) $(ab)\{1,3\} = b\{1,3\} \cdot a\{1,3\}$,

(ii) $bb^\dagger a^*ab = a^*ab, (ab)^\dagger - b^\dagger a^\dagger \in b^\dagger(1-a^\dagger a)\mathcal{R}$ *and* $(1-b^\dagger b)\mathcal{R}(1-aa^\dagger) \subseteq \mathcal{R}(1-bb^\dagger)(1-(a(1-bb^\dagger))^\dagger a(1-bb^\dagger))\mathcal{R}$.

2.4.2 Improvements on the reverse order law for $\{1,2,3\}$- generalized inverses

In this subsection, by the similar techniques as in the case of $\{1,3\}$-inverses, we present a generalization of some recently published results on the reverse order law for $\{1,2,3\}$-inverses by moving some regularity conditions. As we mentioned, the inclusion,

$$B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}, \tag{2.19}$$

was considered by Xiong and Zheng [155] in the matrix settings:

Theorem 2.4.5 [*155*] *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. The following are equivalent:*

(i) $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$,

(ii) $r\left(\begin{bmatrix} B & A^*AB \end{bmatrix}\right) = r(B)$ *and* $r(AB) = \min\{r(A), r(B)\} = r(A) + r(B)$ $-r\left(\begin{bmatrix} A \\ B^* \end{bmatrix}\right)$.

This result was generalized to the C^*-algebra case by Cvetković-Ilić and Harte [32] as follows:

Theorem 2.4.6 [*32*] *Let $a, b \in \mathcal{A}$ be such that a, b, ab and $a - abb^\dagger$ are regular. The following are equivalent:*

(i) $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$,

(ii) $bb^\dagger a^*ab = a^*ab$ *and* $(bb^\dagger - (abb^\dagger)^\dagger abb^\dagger)\mathcal{A}(aa^\dagger - (ab)(ab)^\dagger) = \{0\}$.

Using the same method as in the case of $\{1,3\}$-inverses, we deduce that the regularity condition of ab and $a - abb^\dagger$ in C^*-algebra case (Theorem 2.4.6) can be removed. Our proof concerns ring case with some parts of the proof that are the same as in the proof of Theorem 2.4.6 but we will give it for the completeness. First, we need the following elementary lemma:

Lemma 2.4.2 *Let $a \in \mathcal{R}$ be such that $a\{1,3\} \neq \emptyset$. Then $b \in a\{1,2,3\}$ if and only if $a^*ab = a^*$ and $baa^{(1,3)} = b$, for some $a^{(1,3)} \in a\{1,3\}$.*

Proof. ($\Rightarrow$:) Let $b \in a\{1,2,3\}$. Then

$$a^*ab = a^*(ab)^* = a^*b^*a^* = (aba)^* = a^*$$

and

$$baa^{(1,3)} = babaa^{(1,3)} = bb^*a^*aa^{(1,3)} = bb^*(aa^{(1,3)}a)^* = bb^*a^* = bab = b.$$

($\Leftarrow$:) If we multiply $a^*ab = a^*$ by $(a^{(1,3)})^*$ from the left side, we have $aa^{(1,3)}ab = aa^{(1,3)}$ i.e. $ab = aa^{(1,3)}$ which implies that $b \in a\{1,3\}$. Also, $bab = baa^{(1,3)} = b$. Hence $b \in a\{1,2,3\}$.$\square$

The main result on the reverse order law (2.19) in the ring case follows.

Theorem 2.4.7 *Let $a, b \in \mathcal{R}$ be MP-invertible and let $a(1-bb^\dagger)$ be right $*$-cancellable. The following conditions are equivalent:*

(*i*) $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$,

(*ii*) $bb^\dagger a^*ab = a^*ab$ *and* $(bb^\dagger - (abb^\dagger)^\dagger abb^\dagger)\mathcal{R}(aa^\dagger - (ab)(ab)^\dagger) = \{0\}$.

Proof. Let $p = bb^\dagger$, $q = b^\dagger b$ and $r = aa^\dagger$. Then $b = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{p,q}$ and $a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{r,p}$. We have that $b\{1,2,3\} = \left\{ \begin{bmatrix} b^\dagger & 0 \\ u & 0 \end{bmatrix}_{q,p} : u \in (1-q)\mathcal{R}p \right\}$ and $a^\dagger = a^*(aa^*)^\dagger = \begin{bmatrix} a_1^* d^\dagger & 0 \\ a_2^* d^\dagger & 0 \end{bmatrix}_{p,r}$, where $d = a_1a_1^* + a_2a_2^*$. By Lemma 2.4.2, $a\{1,2,3\} = \left\{ \begin{bmatrix} z_1 & 0 \\ z_3 & 0 \end{bmatrix}_{p,r} : a_1^*a_1z_1 + a_1^*a_2z_3 = a_1^*, a_2^*a_1z_1 + a_2^*a_2z_3 = a_2^*, z_1 \in p\mathcal{R}r, z_3 \in (1-p)\mathcal{R}r \right\}$.
Hence $x \in b\{1,2,3\} \cdot a\{1,2,3\}$ if and only if $x = \begin{bmatrix} b^\dagger z_1 & 0 \\ uz_1 & 0 \end{bmatrix}_{q,r}$, for some $u \in (1-q)\mathcal{R}p$ and some $z_1 \in p\mathcal{R}r$ for which there exist $z_3 \in (1-p)\mathcal{R}r$ such that the following hold:

$$a_1^*a_1z_1 + a_1^*a_2z_3 = a_1^*, \quad a_2^*a_1z_1 + a_2^*a_2z_3 = a_2^*. \tag{2.20}$$

Hence, such x given by $x = \begin{bmatrix} b^\dagger z_1 & 0 \\ uz_1 & 0 \end{bmatrix}_{q,r}$, for some $u \in (1-q)\mathcal{R}p$ and some $z_1 \in p\mathcal{R}r$ such that (2.20) is satisfied for some $z_3 \in (1-p)\mathcal{R}r$, belongs to $(ab)\{1,2,3\}$ if and only if $z_1 \in a_1\{1,2,3\}$. Indeed, $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$ if and only if for any $z_1 \in p\mathcal{R}r$ such that (2.20) is satisfied for some $z_3 \in (1-p)\mathcal{R}r$ it follows that $z_1 \in a_1\{1,2,3\}$. This conclusion will be crucial in the rest of the proof and we will call it conclusion $(*)$.

$(i) \Rightarrow (ii)$: If (i) holds, then $b^\dagger a^\dagger \in (ab)\{1,2,3\}$. By Theorem 2.4.1, we have that $bb^\dagger a^*ab = a^*ab$. Now (2.20) has the form

$$a_1^*a_1z_1 = a_1^*, \quad a_2^*a_2z_3 = a_2^*. \tag{2.21}$$

Notice that, by Lemma 2.4.1, we have that a_1 and a_2 are MP-invertible, so the second equation from (2.21) is satisfied for $z_3 = a_2^\dagger$ while the first one is equivalent with $a_1z_1 = a_1a_1^\dagger$. By the conclusion $(*)$, we have that for each $z_1 \in p\mathcal{R}r$ such that $a_1z_1 = a_1a_1^\dagger$ it follows that $z_1a_1z_1 = z_1$. Using the fact that $z_1 \in p\mathcal{R}r$, we get that any solution of the equation $abb^\dagger zaa^\dagger = abb^\dagger(abb^\dagger)^\dagger$ satisfies that $a_1 \in bb^\dagger zaa^\dagger\{1\}$. Since the set of all $bb^\dagger zaa^\dagger$ for which z is a solution of the equation $abb^\dagger zaa^\dagger = abb^\dagger(abb^\dagger)^\dagger$ is described by $Z = \{(abb^\dagger)^\dagger + bb^\dagger yaa^\dagger - (abb^\dagger)^\dagger abb^\dagger yaa^\dagger : y \in \mathcal{R}\}$, we get

$$\Big(bb^\dagger - (abb^\dagger)^\dagger(abb^\dagger)\Big)y\Big(aa^\dagger - (ab)(ab)^\dagger\Big) = 0,$$

for any $y \in \mathcal{R}$.

$(ii) \Rightarrow (i)$: Suppose that (ii) holds. Since $bb^\dagger a^*ab = a^*ab$, is equivalent to $a_2^*a_1 = 0$ i.e. $a_1^*a_2 = 0$, we have that (2.20) is equivalent to (2.21). As in the previous direction since a_1 and a_2 are MP-invertible, we have that the second equation from (2.21) is satisfied for $z_3 = a_2^\dagger$ while the first one is equivalent with $a_1 z_1 = a_1 a_1^\dagger$. Now, to prove that $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$, by the conclusion $(*)$, it is sufficient to prove that $z_1 a_1 z_1 = z_1$ holds, for every $z_1 \in p\mathcal{R}r$ which satisfies the equation $a_1 z_1 = a_1 a_1^\dagger$. But this follows since $(bb^\dagger - (abb^\dagger)^\dagger abb^\dagger)\mathcal{R}(aa^\dagger - (ab)(ab)^\dagger) = \{0\}$. $\square$

The case $K = \{1,2,4\}$ is treated completely analogously, and the corresponding result follows by taking adjoint elements, or by reversal of products.

Theorem 2.4.8 *Let $a, b \in \mathcal{R}$ be such that a, b are MP-invertible and $(1 - a^\dagger a)b$ is left $*$-cancellable. The following are equivalent:*

(*i*) $b\{1,2,4\}a\{1,2,4\} \subseteq (ab)\{1,2,4\}$,

(*ii*) $a^\dagger abb^* a^* = bb^* a^*$ *and* $(a^\dagger a - a^\dagger ab(a^\dagger ab)^\dagger)\mathcal{R}(b^\dagger b - (ab)^\dagger (ab)) = \{0\}$.

Chapter 3

The Fredholm property of the sum of operators

In this chapter we will present our results from the paper [35] in which we gave necessary and sufficient conditions for the Fredholmness of a sum of two operators and considered some special cases when the Fredholmness of a linear combination of two operators is independent of the scalars' choice. Particulary, the Fredholmness of a linear combination of two idempotents is discussed and, as a corollary, the known result for the case of orthogonal projections is derived, and contrary some special classes of operators for which a linear combination of two operators depends of the choice of the scalars are mentioned. We start with some basic properties of Fredholm operators.

The class of Fredholm operators is the generalization of the class of invertible operators which is frequently present in applications.

Definition 3.0.1 [*94*] *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. We say that:*

(i) T is upper semi-Fredholm *if $\mathcal{R}(T)$ is closed and $n(T) = \dim \mathcal{N}(T) < \infty$,*

(ii) T is lower semi-Fredholm *if $\beta(T) = \text{codim}\mathcal{R}(T) < \infty$,*

(iii) T is Fredholm *if $n(T) < \infty$ and $\beta(T) < \infty$.*

The sets of all Fredholm operators, upper semi-Fredholm operators and lower semi-Fredholm operators from the space $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ are denoted by $F(\mathcal{X}, \mathcal{Y}), F_+(\mathcal{X}, \mathcal{Y})$ and $F_-(\mathcal{X}, \mathcal{Y})$, respectively. The index of a semi-Fredholm operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ (either upper or lower) is defined as $ind(T) = n(T) - \beta(T)$. It is known that, if $\beta(T) < \infty$, then $\mathcal{R}(T)$ is closed, so $F(\mathcal{X}, \mathcal{Y}) = F_+(\mathcal{X}, \mathcal{Y}) \cap F_-(\mathcal{X}, \mathcal{Y})$. The approaching theorem illustrate the connection between Fredholm operator and invertible operator.

Theorem 3.0.1 [*122*] *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T \in F(\mathcal{X}, \mathcal{Y})$. There is a closed subspace $\mathcal{X}_0$ of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{N}(T)$, and a subspace $\mathcal{Y}_0$ of $\mathcal{Y}$ of dimension $\beta(T)$ such that $\mathcal{Y} = \mathcal{R}(T) \oplus \mathcal{Y}_0$, holds. Moreover, there is an operator $T_0 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that:*

(i) $\mathcal{N}(T_0) = \mathcal{Y}_0$,

(ii) $\mathcal{R}(T_0) = \mathcal{X}_0$,

(iii) $T_0T = I$ *on* $\mathcal{X}_0$,

(iv) $TT_0 = I$ *on* $\mathcal{R}(T)$,

(iii) $T_0T = I - F_1$ *on* $\mathcal{X}$, *where* $F_1 \in \mathcal{B}(\mathcal{X})$ *is an operator of finite rank with* $\mathcal{R}(F_1) = \mathcal{N}(T)$,

(iii) $TT_0 = I - F_2$ *on* $\mathcal{Y}$, *where* $F_2 \in \mathcal{B}(\mathcal{Y})$ *is an operator of finite rank with* $\mathcal{R}(F_2) = \mathcal{Y}_0$.

It is well-known that an operator $I - K$, where K is a compact operator on Banach space $\mathcal{X}$ is Fredholm. Recall that, if $\mathcal{X}, \mathcal{Y}$ are normed vector spaces, then a linear operator K from $\mathcal{X}$ to $\mathcal{Y}$ is called *compact (completely continuous)* if it is defined on $\mathcal{X}$ and for every bounded sequence $\{x_n\} \subseteq \mathcal{X}$, the sequence $\{Kx_n\}$ has a subsequence which converges in $\mathcal{Y}$.

Theorem 3.0.2 [*122*] (Fredholm alternative) *Let* $\mathcal{X}$ *be a Banach space and let* K *be a compact operator on* $\mathcal{X}$. *Then,* $\mathcal{R}(I - K)$ *is closed in* X *and* $n(I - K) = \beta(I - K)$ *is finite. In particular, either* $\mathcal{R}(I - K) = \mathcal{X}$ *and* $\mathcal{N}(I - K) = \{0\}$, *or* $\mathcal{R}(I - K) \neq \mathcal{X}$ *and* $\mathcal{N}(I - K) \neq \{0\}$.

Some of the basic characteristics of Fredholm operators are contained in the following theorems.

Theorem 3.0.3 [*94*] *Let* $\mathcal{X}, \mathcal{Y}$ *and* $\mathcal{Z}$ *be Banach spaces,* $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ *and* $S \in \mathcal{B}(\mathcal{Y}, \mathcal{Z})$. *Then:*

(i) *if* T *and* S *are lower semi-Fredholm, then* ST *is lower semi-Fredholm, and* $ind(ST) = ind(S) + ind(T)$,

(ii) *if* T *and* S *are upper semi-Fredholm, then* ST *is upper semi-Fredholm, and* $ind(ST) = ind(S) + ind(T)$,

(iii) *if* T *and* S *are Fredholm, then* ST *is Fredholm, and* $ind(ST) = ind(S) + ind(T)$.

Theorem 3.0.4 [*94*] *Let* $\mathcal{X}, \mathcal{Y}$ *be Banach spaces and* $T \in F_+(\mathcal{X}, \mathcal{Y})$. *If* $\mathcal{M}$ *is closed subspace of* $\mathcal{X}$, *then* $T(M)$ *is closed subspace of* $\mathcal{Y}$.

Theorem 3.0.5 [*94*] *Let* $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ *be a (upper semi-, lower semi-) Fredholm operator and let* $K \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ *be a compact operator. Then* $T + K$ *is (upper semi-, lower semi-) Fredholm and* $ind(T + K) = ind(T)$.

The Fredholmness of a difference, sum and in general a linear combination of idempotents and orthogonal projections has been considered in several papers (see [51, 58, 59, 77, 78, 79, 153, 157]). Namely, in the literature, there are a several papers which contain the so-called stability theorems for linear combinations of idempotents. Baksalary and Baksalary [4] proved that, for idempotent matrices P and Q the invertibility of the linear combination $c_1P + c_2Q$ is independent of the constants $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$. In [52] this result is extended to the case of two idempotent operators on a Hilbert space. This result is followed by Koliha and Rakočević [79] and they proved, using arguments based on the stability of the nullity of linear combinations of two idempotent operators, that if $P, Q \in \mathcal{B}(\mathcal{X})$ are idempotent operators on a Banach space $\mathcal{X}$ then the Fredholmness of the linear combination $c_1P + c_2Q$ is independent of the constants $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$.

Theorem 3.0.6 [*79*] *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be idempotents. Then:*

(*i*) *If $c_1P + c_2Q$ is upper semi-Fredholm for some $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$, then it is upper semi-Fredholm for all $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$, and $n(c_1P + c_2Q)$ is constant for these constants.*

(*ii*) *If $c_1P + c_2Q$ is lower semi-Fredholm for some $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$, then it is lower semi-Fredholm for all $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$, and $\beta(c_1P + c_2Q)$ is constant for these constants.*

(*iii*) *If $c_1P + c_2Q$ is Fredholm for some $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$, then it is Fredholm for all $c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0$, and $n(c_1P + c_2Q), \beta(c_1P + c_2Q)$ and $ind(c_1P + c_2Q)$ are constant for these constants.*

Also, in the case when P and Q are orthogonal projections some necessary and sufficient conditions for $P - Q$ being Fredholm are known [77, 58].

Theorem 3.0.7 [*77*] *Let R and K be closed subspaces of a Hilbert space $\mathcal{H}$ and let P and Q be the orthogonal projections with the ranges R and K, respectively. The following are equivalent:*

(*i*) *$P - Q$ is Fredholm operator,*

(*ii*) *$I - PQ$ and $I - (I - P)(I - Q)$ are Fredholm operators,*

(*iii*) *$R + K$ is closed in $\mathcal{H}$ and $dim[(R \cap K) \oplus (R^\perp \cap K^\perp)] < \infty$,*

(*iv*) $\|P + Q - I\|_e = \inf_{K \in \mathcal{B}(\mathcal{H}), K \text{ is compact}} \|P + Q - I + K\| < 1$,

(*v*) *$P + Q$ and $I - PQ$ are Fredholm operators.*

Corollary 3.0.2 [*58*] *Let P and Q be idempotents on $\mathcal{H}$. If $P - Q$ is Fredholm than so is $P + Q$. Also, if $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) < \infty$, than $P - Q$ is Fredholm if and only if $P + Q$ is.*

In [35], using some results on completion problems of operator matrices, we derived the results about Fredholmness of a sum of two bounded linear operators. If Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ are represent with topological sums $\mathcal{X} = M \oplus N$ and $\mathcal{Y} = L \oplus R$, then an operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ can be seen as an 2×2 operator matrix

$$T = \begin{bmatrix} A & C \\ D & B \end{bmatrix} : \begin{bmatrix} M \\ N \end{bmatrix} \to \begin{bmatrix} L \\ R \end{bmatrix},$$

where A, B, C and D are bounded linear operators between appropriate spaces. By using this decomposition of operator, some problems of operator theory can be simplified. An operator $T \in \mathcal{B}(\mathcal{X})$ for which there exists a closed, complemented T-invariant subspace of $\mathcal{X}$, can be represented by an upper-triangular operator matrix.

One of the basic problem in the context of operator matrices is its completions. For example, if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, there are many published results which consider the necessary and sufficient conditions for the existence of an operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that the upper-triangular operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

belongs to certain class of operators, as well as the set of all such C is described. Here $M_C \in \mathcal{B}(\mathcal{H} \times \mathcal{K})$, where the inner product in $\mathcal{H} \times \mathcal{K}$ is as usual given by $\langle (h_1, k_1), (h_2, k_2) \rangle = \langle h_1, h_2 \rangle + \langle k_1, k_2 \rangle$. Additionally, in some papers the set

$$\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_*(M_C),$$

for different types of spectrum, is treated and the existence of an operator C' such that $\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_*(M_C) = \sigma_*(M_{C'})$.

Speaking of invertibility throughout the text, and now we will start with the basic result about completion to invertibility of M_C. The invertibility of M_C, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, first was considered in [50], in the case of separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. In [64], authors showed that result from [50] stay valid in the case of Banach spaces. We will present results given in [23], where are given necessary and sufficient conditions for the invertibility of M_C, using a method which allowed author to completely describe the set of all $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ for which M_C is invertible for given operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, in the case when $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces.

Theorem 3.0.8 [23] *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators. The operator matrix M_C is invertible for some $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ if and only if*

(i) A is left invertible,

(ii) B is right invertible,

(iii) $\mathcal{N}(B) \cong \mathcal{X}/R(A)$.

If conditions $(i)-(iii)$ are satisfied, the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is invertible is given by

$$S(A,B)=\{C\in\mathcal{B}(\mathcal{Y},\mathcal{X}):C=\begin{bmatrix} C_1 & 0\\ 0 & C_4\end{bmatrix}:\begin{bmatrix}\mathcal{P}\\ \mathcal{N}(B)\end{bmatrix}\to\begin{bmatrix}\mathcal{R}(A)\\ \mathcal{S}\end{bmatrix},$$
$$C_4 \text{ is invertible}, \mathcal{X}=\mathcal{R}(A)\oplus\mathcal{S} \text{ and } \mathcal{Y}=\mathcal{P}\oplus\mathcal{N}(B)\}.$$

In [27], the author established the necessary and sufficient conditions for the existence of an operator C such that M_C is injective and described the set of all such C.

Theorem 3.0.9 [*27*] *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. There exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that the operator matrix M_C is injective if and only if A is injective and one of the following conditions is satisfied:*

- (i) *$\mathcal{R}(A)$ is closed, $n(B) \leq \beta(A)$,*
- (ii) *$\mathcal{R}(A)$ is not closed.*

Furthermore, if (i) is satisfied, then the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is injective is given by

$$S_I(A,B)=\big\{C\in\mathcal{B}(\mathcal{K},\mathcal{H})\mid C_3 \textit{ is injectiv}\big\},$$

while if (ii) holds, it is given by

$$S_I(A,B)=\big\{C\in\mathcal{B}(\mathcal{K},\mathcal{H})\mid \mathcal{N}(C_1)\cap\mathcal{N}(C_3)=\{0\},\ C_1(\mathcal{N}(C_3))\cap\mathcal{R}(A)=\{0\}\big\},$$

where C is given by

$$C=\begin{bmatrix} C_1 & C_2\\ C_3 & C_4\end{bmatrix}:\begin{bmatrix}\mathcal{N}(B)\\ \mathcal{N}(B)^\perp\end{bmatrix}\to\begin{bmatrix}\overline{\mathcal{R}(A)}\\ \mathcal{R}(A)^\perp\end{bmatrix}.$$

There are several papers which consider when an upper-triangular operator matrix M_C is a Fredholm operator. One of them is [18] where the completion problem to upper (lower) semi-Fredholmness of an operator matrix M_C was studied while the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is upper (lower) semi-Fredholm was described in [31]. Furthermore, in [31] necessary and sufficient conditions for M_C to be a Fredholm operator were given. For given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is Fredholm will be denoted by $S_F(A,B)$.

Theorem 3.0.10 [*31*] *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators, where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces. Then M_C is Fredholm for some $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if one of the following conditions is satisfied:*

- *(i) A and B are Fredholm;*
- *(ii) A is upper semi-Fredholm, B is lower semi-Fredholm and $d(A)=n(B)=\infty$.*

Furthermore, if (i) is satisfied then $S_F(A,B)=\mathcal{B}(\mathcal{K},\mathcal{H})$, while if (ii) holds,

$$S_F(A,B)=\left\{C\in\mathcal{B}(\mathcal{K},\mathcal{H}):P^{rst}_{\mathcal{R}(A)^\perp}CP|_{\mathcal{N}(B)}\in F(\mathcal{N}(B),\mathcal{R}(A)^\perp)\right\}.$$

3.1 Fredholmness of the linear combination of operators

In this section we will first present the modification of Theorem 3.0.10 which concerns the Fredholmness of an upper triangular operator matrices on Hilbert spaces and then apply the obtained results to our consideration of the Fredholmness of the sum of two operators. Special emphasis will be put on some particular cases when the Fredholmness of the linear combination of two operators is independent of the choice of the scalars.

Notice that in Theorem 3.0.10 the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are isomorphically embedded in $\mathcal{H}\times\mathcal{K}$ as two mutually orthogonal closed subspaces whose direct sum is $\mathcal{H}\times\mathcal{K}$. Here we will consider the case when $M_C:\mathcal{H}\to\mathcal{K}$ and $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$, $\mathcal{K}=\mathcal{K}_1\oplus\mathcal{K}_2$ are topological decompositions which are in general nonorthogonal. Also we will not need the separability condition that is required in Theorem 3.0.10.

Theorem 3.1.1 *Let $\mathcal{H},\mathcal{K}$ be infinite dimensional Hilbert spaces and let $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$, $\mathcal{K}=\mathcal{K}_1\oplus\mathcal{K}_2$. An operator matrix $M_C=\begin{bmatrix} A & C \\ 0 & B\end{bmatrix}:\mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{K}_1\oplus\mathcal{K}_2$ is Fredholm if and only if*

(i) A is upper semi-Fredholm,

(ii) B is lower semi-Fredholm,

(iii) $P^{rst}_{\mathcal{S},\mathcal{R}(A)}C|_{\mathcal{N}(B)}$ is Fredholm,

where $\mathcal{K}_1=\mathcal{R}(A)\oplus\mathcal{S}$.

Proof. First, notice that if M_C is a Fredholm operator then $\mathcal{R}(A)=\mathcal{R}(M_C|_{\mathcal{H}_1})$ is a closed subspace of $\mathcal{K}_1$ by Theorem 3.0.4. Hence, in the proof of either implication we can suppose that $\mathcal{K}_1=\mathcal{R}(A)\oplus\mathcal{S}$, for some closed subspace $\mathcal{S}$. So, M_C has the following representation

$$\begin{bmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1\end{bmatrix}:\begin{bmatrix}\mathcal{H}_1\\ \mathcal{N}(B)\\ \mathcal{T}\end{bmatrix}\to\begin{bmatrix}\mathcal{R}(A)\\ \mathcal{S}\\ \mathcal{K}_2\end{bmatrix}. \tag{3.1}$$

Let M_C be a Fredholm operator. Since $\mathcal{N}(A)\subseteq\mathcal{N}(M_C)$, we have that A is upper semi-Fredholm. Also, since $\mathcal{R}(M_C)\subseteq\mathcal{R}\left(\begin{bmatrix} A & C\end{bmatrix}\right)\oplus\mathcal{R}(B)$, we have that B is lower semi-Fredholm. Now, evidently A_1 is right and B_1 is left invertible. Let

$$U=\begin{bmatrix} I & 0 & 0 \\ 0 & I & -C_4(B_1)_l^{-1} \\ 0 & 0 & I\end{bmatrix}:\begin{bmatrix}\mathcal{R}(A)\\ \mathcal{S}\\ \mathcal{K}_2\end{bmatrix}\to\begin{bmatrix}\mathcal{R}(A)\\ \mathcal{S}\\ \mathcal{K}_2\end{bmatrix}$$

and

$$V = \begin{bmatrix} I & -(A_1)_r^{-1}C_1 & -(A_1)_r^{-1}C_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{N}(B) \\ \mathcal{T} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{N}(B) \\ \mathcal{T} \end{bmatrix}.$$

We can check that U, V are invertible and

$$UM_CV = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{bmatrix}. \tag{3.2}$$

The operator matrix UM_CV is Fredholm if and only if A_1, C_3 and B_1 are Fredholm operators. Since $C_3 = P^{\mathtt{rst}}_{S,\mathcal{R}(A)}C|_{\mathcal{N}(B)}$, the condition (iii) follows. Hence, all the conditions $(i) - (iii)$ are satisfied.

If we suppose that $(i) - (iii)$ are satisfied, then M_C can be represented by (3.1) and using (3.2) we can see that M_C is a Fredholm operator. □

Remark 3.1.1 The advantage of Theorem 3.1.1 lies in the fact that the decompositions of the spaces which are used are not orthogonal unlike to what is the case in Theorem 3.0.10 in [31] and all other related recently published results. This will particularly be useful in the following results which concern Fredholmness of a linear combination of idempotents.

Remark 3.1.2 It is worth mentioning that Theorem 3.1.1 is also correct if $\mathcal{H}$ and $\mathcal{K}$ are Banach spaces with the additional assumptions that subspaces $\mathcal{R}(A)$ and $\mathcal{N}(B)$ are complemented.

Remark 3.1.3 From the proof of Theorem 3.1.1, we have that if M_C is Fredholm than $ind(M_C) = ind(UM_CV) = ind(A_1) + ind(C_3) + ind(B_1) = n(A_1) + ind(C_3) - \beta(B_1) = n(A) + ind(C_3) - \beta(B)$. Moreover, $n(M_C) = n(A) + n(C_3)$ and $\beta(M_C) = \beta(C_3) + \beta(B)$, where A_1, B_1 and C_3 are defined as in the proof of this theorem. Let us mention that this formulas are correct even when just $R(A)$ is closed. This is very elementary fact but we mention it since it will be used in the proof of the next theorem.

Now using the previous results on Fredholmness of upper-triangular operator matrices and appropriate matrix representations of the given operators $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we derive some necessary and sufficient conditions for the Fredholmness of their sum.

Theorem 3.1.2 *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be given operators. Let $\mathcal{H} = \mathcal{N}(B) \oplus \mathcal{P}$ and $\mathcal{K} = \overline{\mathcal{R}(A)} \oplus \mathcal{Q}$. Then the operator $A + B$ is Fredholm if and only if the following conditions hold:*

(i) $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$ *and* $A|_{\mathcal{N}(B)}$ *has closed range,*

(ii) $\dim(\mathcal{N}(A') \cap \mathcal{N}(B')) < \infty$ *and* $P_{\mathcal{Q},\overline{\mathcal{R}(A)}}B$ *has closed range,*

(iii) $\dim(\mathcal{P} \cap \mathcal{N}(A + B)) < \infty, \dim(\mathcal{R}(A|_{\mathcal{N}(B)}) \cap \mathcal{R}((A + B)\,|_{\mathcal{P}})) < \infty$ *and* $\dim \overline{\mathcal{R}(A)}/(\mathcal{R}(A|_{\mathcal{N}(B)}) + (\overline{\mathcal{R}(A)} \cap \mathcal{R}((A + B)\,|_{\mathcal{P}}))) < \infty.$

Furthermore, if $(i)-(iii)$ *hold, then*

$$\begin{aligned} n(A+B) &= \dim(\mathcal{N}(A)\cap\mathcal{N}(B)) + \dim(\mathcal{P}\cap\mathcal{N}(A+B)) \\ &\quad + \dim(\mathcal{R}(A|_{\mathcal{N}(B)})\cap\mathcal{R}((A+B)|_{\mathcal{P}})) \\ \beta(A+B) &= \dim(\mathcal{N}(A')\cap\mathcal{N}(B')) + \dim\overline{\mathcal{R}(A)}/(\mathcal{R}(A|_{\mathcal{N}(B)}) \\ &\quad + (\overline{\mathcal{R}(A)}\cap\mathcal{R}((A+B)\,|_{\mathcal{P}}))). \end{aligned}$$

Proof. With respect to the decompositions given below, $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ have the following representations:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{P} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{Q} \end{bmatrix}, \tag{3.3}$$

$$B = \begin{bmatrix} 0 & B_1 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{P} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{Q} \end{bmatrix}. \tag{3.4}$$

So $A+B$ is a Fredholm operator if and only if the operator matrix given by

$$\begin{bmatrix} A_1 & B_1 + A_2 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{P} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{Q} \end{bmatrix}$$

is Fredholm.

Let $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_1)} \oplus \mathcal{S}$ and $\mathcal{P} = \mathcal{N}(B_2) \oplus \mathcal{W}$. By Theorem 3.1.1 we have that $A+B$ is a Fredholm operator if and only if the following three conditions are satisfied:

(*) A_1 is upper semi-Fredholm,

(**) B_2 is lower semi-Fredholm,

(***) $P^{\mathtt{rst}}_{\mathcal{S},\overline{\mathcal{R}(A_1)}\oplus\mathcal{Q}}\,(A+B)\,|_{\mathcal{N}(B_2)}$ is Fredholm.

Evidently, $(*)$ holds if and only if $\dim(\mathcal{N}(A)\cap\mathcal{N}(B)) < \infty$ and $\mathcal{R}(A|_{\mathcal{N}(B)})$ is closed.

Also, $(**)$ holds if and only if B_2' is upper semi-Fredholm. Since $\mathcal{N}(B_2') = \mathcal{N}(B')\cap\mathcal{N}(A')$ it follows that $(**)$ holds if and only if the range of $P_{\mathcal{Q},\overline{\mathcal{R}(A)}}B$ is closed and $\dim(\mathcal{N}(B')\cap\mathcal{N}(A')) < \infty$.

Now, we will consider condition $(***)$ taking into account that $\mathcal{R}(A_1)$ is closed in both directions. The condition $(***)$ is satisfied if and only if

$$\dim\mathcal{N}(P_{\mathcal{S},\mathcal{R}(A_1)\oplus\mathcal{Q}}\,(A+B)\,|_{\mathcal{N}(B_2)}) < \infty$$

and

$$\dim\mathcal{S}/\mathcal{R}(P_{\mathcal{S},\mathcal{R}(A_1)\oplus\mathcal{Q}}\,(A+B)\,|_{\mathcal{N}(B_2)}) < \infty.$$

Since

$$\mathcal{N}(P_{\mathcal{S},\mathcal{R}(A_1)\oplus\mathcal{Q}}\,(A+B)\,|_{\mathcal{N}(B_2)}) = \{x\in\mathcal{P} \mid (A+B)\,x\in\mathcal{R}(A_1)\},$$

we have that $\dim \mathcal{N}(P_{\mathcal{S},\mathcal{R}(A_1)\oplus\mathcal{Q}}\,(A+B)\,|_{\mathcal{N}(B_2)}) = \dim(\mathcal{P}\cap\mathcal{N}\,(A+B)) + \dim(\mathcal{R}(A|_{\mathcal{N}(B)})\cap \mathcal{R}((A+B)\,|_{\mathcal{P}}))$. Also,

$$\dim \mathcal{S}/\mathcal{R}(P_{\mathcal{S},\mathcal{R}(A_1)\oplus\mathcal{Q}}\,(A+B)\,|_{\mathcal{N}(B_2)}) < \infty,$$

if and only if there exists a finite dimensional subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{M} \oplus \mathcal{R}(P_{\mathcal{S},\mathcal{R}(A_1))\oplus\mathcal{Q}}\,(A+B)\,|_{\mathcal{N}(B_2)})$. The last assertions is equivalent with the existence of a finite dimensional subspace $\mathcal{N}$ such that $\overline{\mathcal{R}(A)} = \mathcal{N} \oplus (\mathcal{R}(A|_{\mathcal{N}(B)}) + (\overline{\mathcal{R}(A)} \cap \mathcal{R}((A+B)\,|_{\mathcal{P}})))$, i.e. $\dim \overline{\mathcal{R}(A)}/(\mathcal{R}(A|_{\mathcal{N}(B)}) + (\overline{\mathcal{R}(A)} \cap \mathcal{R}((A+B)\,|_{\mathcal{P}})) < \infty$. The rest of the proof follows by Remark 3.1.3. □

Notice that the condition $\dim(\mathcal{N}(A') \cap \mathcal{N}(B')) < \infty$ from item (ii) of the previous theorem can be replaced by $\dim(\mathcal{R}(A)^{\perp} \cap \mathcal{R}(B)^{\perp}) < \infty$ which will be proved later in the Section 3.2.

In the special case when we take the orthogonal decompositions of spaces $\mathcal{H}, \mathcal{K}$ in proof of Theorem 3.1.2 we get the following result:

Theorem 3.1.3 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be given operators. Then $A+B$ is Fredholm if and only if the following conditions hold:*

(i) $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$, $\dim(\mathcal{R}(A)^{\perp} \cap \mathcal{R}(B)^{\perp}) < \infty$,

(ii) $AP_{\mathcal{N}(B)}$ *and* $P_{\mathcal{R}(A)^{\perp}}B$ *have closed ranges,*

(iii) $P_{\mathcal{S}}^{rst}\,(A+B)\,|_{\mathcal{T}}$ *is Fredholm,*

where $\mathcal{S} = \mathcal{N}\left(P_{\mathcal{N}(B)}A^*\right) \cap \overline{\mathcal{R}(A)}$ *and* $\mathcal{T} = \mathcal{N}\left(P_{\mathcal{R}(A)^{\perp}}B\right) \cap \mathcal{N}(B)^{\perp}$.
Furthermore, if $(i)-(iii)$ *hold then*

$$ind(A+B) = \dim(\mathcal{N}(A) \cap \mathcal{N}(B)) - \dim(\mathcal{R}(A)^{\perp} \cap \mathcal{R}(B)^{\perp}) + ind(P_{\mathcal{S}}^{rst}\,(A+B)\,|_{\mathcal{T}}).$$

Proof. Let $\mathcal{H} = \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp}$ and $\mathcal{K} = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^{\perp}$. With respect to these decompositions $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ have the following representations:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}, \tag{3.5}$$

$$B = \begin{bmatrix} 0 & B_1 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}. \tag{3.6}$$

Using the orthogonal decomposition $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_1)} \oplus (\mathcal{R}(A_1)^{\perp} \cap \overline{\mathcal{R}(A)})$ in the proof of Theorem 3.1.1, we get that the sum $A+B$ is a Fredholm operator if and only if the following three conditions are satisfied:

(i) A_1 is upper semi-Fredholm,

(ii) B_2 is lower semi-Fredholm,

(iii) $P^{\mathtt{rst}}_{\mathcal{R}(A_1)^\perp \cap \overline{\mathcal{R}(A)}}(A_2 + B_1)\,|_{\mathcal{N}(B_2)}$ is Fredholm.

Evidently, (i) holds if and only if $\dim(\mathcal{N}(A)\cap\mathcal{N}(B)) < \infty$ and $AP_{\mathcal{N}(B)}$ has closed range. Also, (ii) holds if and only if B_2^* is upper semi-Fredholm. Since

$$\mathcal{N}(B_2^*) = \ \mathcal{N}(B^*)\cap\mathcal{R}(A)^\perp = \mathcal{R}(B)^\perp\cap\mathcal{R}(A)^\perp,$$

it follows that (ii) holds if and only if the range of $P_{\mathcal{R}(A)^\perp}B$ is closed and $\dim(\mathcal{R}(A)^\perp\cap\mathcal{R}(B)^\perp) < \infty$. The third condition follows directly from the fact that $\mathcal{R}(A_1)^\perp\cap\overline{\mathcal{R}(A)} = \mathcal{N}(P_{\mathcal{N}(B)}A^*)\cap\overline{\mathcal{R}(A)}$ and $\mathcal{N}(B_2) = \mathcal{N}(P_{\mathcal{R}(A)^\perp}B)\cap\mathcal{N}(B)^\perp$. $\square$

Remark 3.1.4 If operators A_1 and B_2 from (3.3) and (3.4) are Fredholm, then $\mathcal{N}(B_2)$ and $\mathcal{S}$ defined in Theorem 3.1.2 are finite dimensional, so the condition $(***)$ from the proof of Theorem 3.1.2 is satisfied. Hence, in this case taking into consideration αA instead of A and βB instead of B, we get that for all constants $\alpha,\beta\in\mathbb{C}\setminus\{0\}$, the linear combination $\alpha A+\beta B$ is a Fredholm operator.

Also if one of the operators A_1 and B_2 is Fredholm and the other one is not, at least one of the conditions $(*)-(***)$ from the proof of Theorem 3.1.2 is not satisfied. Indeed, if conditions $(*)$ and $(**)$ are satisfied one of spaces $\mathcal{N}(B_2)$ and $\mathcal{S}$ is finite dimensional and other one is infinite dimensional, so the condition $(***)$ is not satisfied and $A+B$ is not a Fredholm operator. Moreover, in this case the linear combination $\alpha A+\beta B$ is not a Fredholm operator for any constants $\alpha,\beta\in\mathbb{C}\setminus\{0\}$. Thus we have the following result:

Theorem 3.1.4 *Let $A,B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ be given operators and let $\mathcal{H}=\mathcal{N}(B)\oplus\mathcal{P}$ and $\mathcal{K}=\overline{\mathcal{R}(A)}\oplus\mathcal{Q}$. Let $A_1 = P^{rst}_{\overline{\mathcal{R}(A)},\mathcal{Q}}A|_{\mathcal{N}(B)}$ and $B_2 = P^{rst}_{\mathcal{Q},\overline{\mathcal{R}(A)}}B|_{\mathcal{P}}$.*

- (i) *If the operators A_1 and B_2 are Fredholm, then for all constants $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ the linear combination $\alpha A+\beta B$ is a Fredholm operator.*
- (ii) *If one of operators A_1 and B_2 is Fredholm and the other one is not, then for all constants $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ the linear combination $\alpha A+\beta B$ is not a Fredholm operator.*

Remark 3.1.5 If we take orthogonal decompositions of spaces $\mathcal{H}$ and $\mathcal{K}$, as in Theorem 3.1.3, we have that $\mathcal{S}=\mathcal{N}(A_1^*)=\{x\in\mathcal{N}(A^*)^\perp \mid A^*x\in\overline{\mathcal{R}(B^*)}\}$ and $\mathcal{T}=\mathcal{N}(B_2)=\{x\in\mathcal{N}(B)^\perp \mid Bx\in\overline{\mathcal{R}(A)}\}$. It can be checked that $\dim\mathcal{N}(A_1^*)=\dim(\mathcal{R}(A^*)\cap\overline{\mathcal{R}(B^*)})$ and $\dim\mathcal{N}(B_2)=\dim(\overline{\mathcal{R}(A)}\cap\mathcal{R}(B))$. So, we are ready to present some particular cases when the linear combination $\alpha A+\beta B$ is independent of the choice of scalars $\alpha,\beta\in\mathbb{C}\setminus\{0\}$:

Theorem 3.1.5 *Let $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and $B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and let $m=\dim(\mathcal{R}(A^*)\cap\overline{\mathcal{R}(B^*)})$ and $n=\dim(\overline{\mathcal{R}(A)}\cap\mathcal{R}(B))$. If one of the following conditions hold*

- (i) $\max\{m,n\}<\infty$

(*ii*) $\min\{m,n\} < \max\{m,n\} = \infty$

then Fredholmness of the linear combination $\alpha A + \beta B$ *is independent of the choice of scalars* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.
Furthermore, we have that in the case (*i*)*:* $A+B$ *is Fredholm if and only if* $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$, $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$ *and* $AP_{\mathcal{N}(B)}$, $P_{\mathcal{R}(A)^\perp}B$ *have closed ranges, while in the case* (*ii*) *we have that* $A+B$ *is not Fredholm.*

Proof. The proof follows by Theorem 3.1.3 and Remark 3.1.5. $\square$

Remark 3.1.6 From previous theorem, we have that if m, n are both finite or if one is finite and the other infinite then Fredholmness of the linear combination $\alpha A + \beta B$ is independent of the choice of scalars $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. What will happen in the case when both m, n are infinite? By the next example we will give an answer on this question:

Example 3.1.1 Let $\mathcal{H}$ be infinite dimensional Hilbert space and let $A = I_{\mathcal{H}}$ and $B = -2I_{\mathcal{H}}$. Evidently $\alpha A + \beta B$ is a Fredholm operator except in the case when $\alpha = 2\beta$ which means that Fredholmness of the linear combination $\alpha A + \beta B$ depends on the choice of $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. On the other side if $A \in \mathcal{B}(H)$ is non Fredholm and $B = A$ we will get that $\alpha A + \beta B$, is a non Fredholm operator for any $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ which means that Fredholmness of the linear combination $\alpha A + \beta B$ is independent of the choice of scalars $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

From Theorem 3.1.3 we can deduce the following well known result a proof of which will be presently given since it is quite different from all others that can be found in the literature:

Corollary 3.1.1 *Let* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *be an operator of finite rank and* $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. *Then* $A + B$ *is Fredholm if and only if* B *is Fredholm.*

Proof. Since $\dim \mathcal{R}(A) < \infty$ from Theorem 3.1.5, we get that $A + B$ is Fredholm if and only if the following conditions hold:

(*i*) $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$, $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$,

(*ii*) $P_{\mathcal{R}(A)^\perp}B$ has closed range.

These conditions are satisfied if and only if B is Fredholm. Indeed, if B is Fredholm, the condition (*i*) is obviously satisfied. Also $P_{\mathcal{R}(A)^\perp}B$ is an upper semi-Fredholm operator as the product of two such operators, so its range is closed. Conversely, if conditions (*i*) and (*ii*) are satisfied, then from closedness of $\mathcal{R}(P_{\mathcal{R}(A)^\perp}B)$ we have that $\mathcal{R}(B^*P_{\mathcal{R}(A)^\perp})$ is closed. Now, from $\mathcal{R}(B^*) = \mathcal{R}(B^*P_{\mathcal{R}(A)^\perp}) + \mathcal{R}(B^*P_{\mathcal{R}(A)})$, we get that $\mathcal{R}(B)$ is closed. Using the fact that in any vector space $\mathcal{X}$ if $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{M}$ are subspaces of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\dim \mathcal{X}_2 < \infty$, and $dim(\mathcal{M} \cap \mathcal{X}_1) < \infty$ then $dim\mathcal{M} < \infty$, from $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$, $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{R}(A^*)$ and $\dim \mathcal{R}(A^*) < \infty$ it follows that $\dim \mathcal{N}(B) < \infty$. In the same manner from $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty, \mathcal{K} = \mathcal{R}(A)^\perp \oplus \mathcal{R}(A)$ and $\dim \mathcal{R}(A) < \infty$ we get that $\dim \mathcal{R}(B)^\perp < \infty$. So B is a Fredholm operator. $\square$

3.2 Some particular cases

In this section we present the necessary and sufficient conditions for $c_1P_1 + c_2P_2$ to be Fredholm in the case when P_1 and P_2 are idempotents and as a corollary we get the known result for the case of orthogonal projections. We will end this chapter with some examples which illustrate that, unlike the case of orthogonal projectors and idempotents, for some classes of operators Fredholmness of linear combinations of operators in general depends on the choice of scalars $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha + \beta \neq 0$. First, we give the following two results whose proofs will simultaneously be given below as one:

Theorem 3.2.1 *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be idempotents and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\alpha + \beta \neq 0$. Then $\alpha P + \beta Q$ is Fredholm if and only if the following conditions hold:*

(i) $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$ *and* $\dim(\mathcal{R}(P)^{\perp} \cap \mathcal{R}(Q)^{\perp}) < \infty$.

(ii) $P(I - Q)$ *and* $(I - P)Q$ *have closed ranges.*

(iii) $\dim(\mathcal{R}(P(I-Q)) \cap \mathcal{R}(Q)) < \infty$ *and* $\dim(\mathcal{R}(P)/(\mathcal{R}(P(I-Q)) + (\mathcal{R}(P) \cap \mathcal{R}(Q)))) < \infty$

Furthermore, if $(i) - (iii)$ hold, then

$$\begin{aligned} n(\alpha P + \beta Q) &= \dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) + \dim(\mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q)), \\ \beta(\alpha P + \beta Q) &= \dim(\mathcal{R}(P)^{\perp} \cap \mathcal{R}(Q)^{\perp})) + \dim(\mathcal{R}(P)/(\mathcal{R}(P(I - Q)) \\ &\quad + (\mathcal{R}(P) \cap \mathcal{R}(Q)))). \end{aligned}$$

Theorem 3.2.2 *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be idempotents. Then $P - Q$ is Fredholm if and only if the following conditions hold:*

(i) $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$ *and* $\dim(\mathcal{R}(P)^{\perp} \cap \mathcal{R}(Q)^{\perp}) < \infty$.

(ii) $(I - P)Q$ *has closed ranges.*

(iii) $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) < \infty$ *and* $\dim(\mathcal{R}(P)/\mathcal{R}(P(I - Q))) < \infty$.

Furthermore, if $(i) - (iii)$ hold, then

$$\begin{aligned} n(P - Q) &= \dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) + \dim(\mathcal{R}(P) \cap \mathcal{R}(Q)), \\ \beta(P - Q) &= \dim(\mathcal{R}(P)^{\perp} \cap \mathcal{R}(Q)^{\perp}) + \dim(\mathcal{R}(P)/\mathcal{R}(P(I - Q))). \end{aligned}$$

Proof. Let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. By Theorem 3.1.2, we have that $\alpha P + \beta Q$ is a Fredholm operator if and only if

(i) $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$ and $\dim(\mathcal{N}(P') \cap \mathcal{N}(Q')) < \infty$.

(ii) $P(I - Q)$ and $(I - P)Q$ have closed ranges.

(*iii*) $\dim(\mathcal{R}(Q)\cap\mathcal{N}(\alpha P+\beta Q))<\infty, \dim(\mathcal{R}(P|_{\mathcal{N}(Q)})\cap\mathcal{R}((\alpha P+\beta Q)\,|_{\mathcal{R}(Q)}))<\infty$ and $\dim(\mathcal{R}(P)/(\mathcal{R}(P|_{\mathcal{N}(Q)})+(\mathcal{R}(P)\cap\mathcal{R}((\alpha P+\beta Q)\,|_{\mathcal{R}(Q)}))))<\infty$.

Notice that

$$\mathcal{N}(P')\cap\mathcal{N}(Q')=\mathcal{R}(P)^\circ\cap\mathcal{R}(Q)^\circ=(\mathcal{R}(P)+\mathcal{R}(Q))^\circ$$

which implies that

$$\dim(\mathcal{N}(P')\cap\mathcal{N}(Q'))=\dim(\mathcal{R}(P)+\mathcal{R}(Q))^\circ=\dim(\mathcal{H}/\overline{\mathcal{R}(P)+\mathcal{R}(Q)})'.$$

Hence, we have that $\dim(\mathcal{N}(P')\cap\mathcal{N}(Q'))<\infty$ if and only if $\dim(\mathcal{R}(P)^\perp\cap\mathcal{R}(Q)^\perp)<\infty$. Now the theorems follow from the following equalities

$$\mathcal{R}(Q)\cap\mathcal{N}(\alpha P+\beta Q)=\begin{cases}\{0\} & \text{if } \alpha+\beta\neq 0\\ \mathcal{R}(P)\cap\mathcal{R}(Q) & \text{if } \alpha+\beta=0\end{cases},$$

$$\mathcal{R}(P|_{\mathcal{N}(Q)})\cap\mathcal{R}((\alpha P+\beta Q)\,|_{\mathcal{R}(Q)})=\begin{cases}\mathcal{R}(P(I-Q))\cap\mathcal{R}(Q) & \text{if } \alpha+\beta\neq 0\\ \{0\} & \text{if } \alpha+\beta=0\end{cases},$$

$$\mathcal{R}(P)\cap\mathcal{R}((\alpha P+\beta Q)\,|_{\mathcal{R}(Q)})=\begin{cases}\mathcal{R}(P)\cap\mathcal{R}(Q) & \text{if } \alpha+\beta\neq 0\\ \{0\} & \text{if } \alpha+\beta=0\end{cases}.$$

In the case $\alpha+\beta=0$, the condition

$$\dim(\mathcal{R}(P)/(\mathcal{R}(P|_{\mathcal{N}(Q)})+(\mathcal{R}(P)\cap\mathcal{R}((\alpha P+\beta Q)\,|_{\mathcal{R}(Q)}))))<\infty$$

is equivalent to the condition $\dim(\mathcal{R}(P)/\mathcal{R}(P(I-Q)))<\infty$ which implies that $\mathcal{R}(P(I-Q))$ is closed. $\square$

In the case when $P,Q\in\mathcal{B}(\mathcal{H})$ are orthogonal projections, the well-known result follows from our Theorems 3.2.1 and 3.2.2. First we give the following lemma which will be used in the proof of the following theorem.

Lemma 3.2.1 [77] *Let P and Q be orthogonal projections in $\mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(*i*) $\mathcal{R}(P-Q)$ *is closed,*

(*ii*) $\mathcal{R}(P+Q)$ *is closed,*

(*iii*) $\mathcal{R}(P)+\mathcal{R}(Q)$ *is closed,*

(*iv*) $\mathcal{N}(P)+\mathcal{N}(Q)$ *is closed,*

(*v*) $\mathcal{R}(P(I-Q))$ *is closed,*

(*vi*) $\mathcal{R}((I-P)Q)$ *is closed.*

If any of the conditions $(i)-(vi)$ *is satisfied, then* $\mathcal{R}(P+Q)=\mathcal{R}(P)+\mathcal{R}(Q)$.

Theorem 3.2.3 *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be orthogonal projections on a Hilbert space $\mathcal{H}$.*

(1) *If $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\alpha + \beta \neq 0$, then the following are equivalent:*

   (i) *$\alpha P + \beta Q$ is Fredholm,*

   (ii) *The range of $\mathcal{R}(P) + \mathcal{R}(Q)$ is closed and $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$.*

(2) *$P - Q$ is Fredholm if and only if $P + Q$ is Fredholm and $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) < \infty$.*

Furthermore, if $\alpha P + \beta Q$ is Fredholm then $ind(\alpha P + \beta Q) = 0$.

Proof. It is evident that condition (i) from Theorem 3.2.1 is equivalent with $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$ while using Lemma 3.2.1 we have that the condition (ii) from Theorem 3.2.1 is equivalent with the fact that $\mathcal{R}(P) + \mathcal{R}(Q)$ is closed. Now, the proof follows having in mind that

$$\mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q) = \{0\}$$

and from the fact that $\mathcal{R}(P) + \mathcal{R}(Q)$ is closed implies

$$\mathcal{R}(P)/(\mathcal{R}(P(I - Q)) + (\mathcal{R}(P) \cap \mathcal{R}(Q)) = \{0\}.\square$$

Let us mention that using the result from [79] which says that for any two idempotents P, Q and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha + \beta \neq 0$,

$$n(\alpha P + \beta Q) = dim(\mathcal{N}((I - P)Q) \cap \mathcal{N}(P))\,,$$

and our Theorem 3.2.1 we get the following result:

Theorem 3.2.4 *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be idempotents and $R(P(I - Q))$ be closed. Then*

$$\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) + \dim(\mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q)) = dim(\mathcal{N}((I - P)Q) \cap \mathcal{N}(P)).$$

Notice that if remove the condition that $R(P(I - Q))$ is closed, the equality from the previous theorem will be still valid.

The next example shows that the formula given in Corollary 4 from [58] stating that $ind(P - Q) = ind(P + Q) + \dim(\mathcal{R}(P) \cap \mathcal{R}(Q))$, in the case when $P, Q \in \mathcal{B}(\mathcal{H})$ are orthogonal projections is not true:

Example 3.2.1 Let $P, Q \in \mathcal{B}(l_2)$ be defined by

$$Px = (x_1, 0, x_3, 0, x_5, \ldots),$$

$$Qx = (x_1, x_2, 0, x_4, 0, x_6, \ldots),$$

for any $x = (x_n)_{n=1}^{\infty} \in l_2$. It is easy to see that P, Q are orthogonal projections and that $ind(P + Q) = ind(P - Q) = 0$ and $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) = 1$. Hence, $ind(P - Q) \neq ind(P + Q) + \dim(\mathcal{R}(P) \cap \mathcal{R}(Q))$.

Unlike the case of orthogonal projectors and idempotents, for classes of k-potent, nilpotent and partial isometry operators, Fredholmness of linear combinations of operators in general depends on the choice of scalars $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $\alpha + \beta \neq 0$. The following examples illustrate this facts. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is k-potent if $A^k = A$, where $k \in \mathbb{N}, k \geq 2$, while A is nilpotent operator if $A^n = 0$, for some $n \in \mathbb{N}$. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is partial isometry if $\|Ax\| = \|x\|$ for all $x \in \mathcal{N}(A)^{\perp}$.

Example 3.2.2 (*k-potent operators*) Let $A, B \in \mathcal{B}(l_2)$ be defined by

$$Ax = (\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, \ldots, \frac{1}{2}x_k - \frac{\sqrt{3}}{2}x_{k+1}, \frac{\sqrt{3}}{2}x_k + \frac{1}{2}x_{k+1}, \ldots),$$

$$Bx = (\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2, \ldots, \frac{1}{2}x_k + \frac{\sqrt{3}}{2}x_{k+1}, \frac{\sqrt{3}}{2}x_k - \frac{1}{2}x_{k+1}, \ldots),$$

for any $x = (x_n)_{n=1}^{\infty} \in l_2$. It is easy to see that $A^7 = A$ and $B^3 = B$, so A and B are k-potent operators. Also, for $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, if $\alpha \neq \pm\beta$ we have that $\alpha A + \beta B$ is invertible, so is Fredholm. On the other hand, $A + B$ and $A - B$ are not Fredholm since $n(A + B) = n(A - B) = \infty$.

Example 3.2.3 (*nilpotent operators*) Let $A, B \in \mathcal{B}(l_2)$ be defined by

$$Ax = (x_2 + x_3, x_3, 0, x_5 + x_6, x_6, 0, \ldots, x_{3k-1} + x_{3k}, x_{3k}, 0, \ldots),$$

$$Bx = (0, 2x_3, x_1, 0, 2x_6, x_4, \ldots, 0, 2x_{3k}, x_{3k-2}, \ldots),$$

for any $x = (x_n)_{n=1}^{\infty} \in l_2$. Than $A^3 = B^3 = 0$. It is easy to see that $A + B$ is invertible, so is Fredholm and that $2A - B$ is not Fredholm since $n(2A - B) = \beta(2A - B) = \infty$.

Example 3.2.4 (*partial isometry*) Let $A, B \in \mathcal{B}(l_2)$ be defined by

$$Ax = (\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, -\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2, \ldots, \frac{1}{2}x_k - \frac{\sqrt{3}}{2}x_{k+1}, -\frac{\sqrt{3}}{2}x_k - \frac{1}{2}x_{k+1}, \ldots),$$

$$Bx = (\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, \ldots, \frac{1}{2}x_k - \frac{\sqrt{3}}{2}x_{k+1}, \frac{\sqrt{3}}{2}x_k + \frac{1}{2}x_{k+1}, \ldots),$$

for any $x = (x_n)_{n=1}^{\infty} \in l_2$. It is easy to check that $A + B$ is not a Fredholm operator while $A + 2B$ is Fredholm.

Chapter 4

A system of three linear equations in a ring

Probably the most familiar application of matrices is in solving the systems of simultaneous linear equations. Let

$$Ax = b \tag{4.1}$$

be such a system, where b is a given vector and x is an unknown vector. If A is nonsingular, there is a unique solution for x given by $x = A^{-1}b$. In the general case, when A may be singular or rectangular, there may sometimes be no solutions or a multiplicity of solutions. What can be done in the case when A is rectangular or nonsingular? Namely, the principal application of $\{1\}$-generalized inverses is in solving the linear systems, where they are used in much the same way as ordinary inverses in the nonsingular case. The following theorem is a fundamental, well-known result given by Penrose in 1955. In this chapter, because of simplicity, unspecified inner inverse of an element A will be denoted by A^-.

Theorem 4.0.1 *[103] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$. Then the matrix equation*

$$AXB = C \tag{4.2}$$

is consistent if and only if for some (any) A^- and B^-,

$$AA^-CB^-B = C, \tag{4.3}$$

in which case the general solution is

$$X = A^-CB^- + Y - A^-AYBB^-,$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$.

It is important to emphasize that Penrose's proof of the previous theorem stay valid in different settings (C^* algebra, ring, vector space of bounded linear operators) with additional assumptions on regularity of A and B. Specializing Theorem 4.0.1 to the system (4.1) gives

Corollary 4.0.1 [*11*] *Let* $A \in \mathbb{C}^{m\times n}$ *and* $b \in \mathbb{C}^m$*. Then the system of linear equations* $Ax = b$ *is consistent if and only if for some* A^-*,*

$$AA^-b = b,$$

in which case the general solution is

$$x = A^-b + (I - A^-A)y,$$

for arbitrary $y \in \mathbb{C}^n$*.*

The least-squares, minimum-norm and least-squares minimum-norm solutions of the equation (4.1) are already described in Theorem 1.3.1, Theorem 1.3.2 and Corollary 1.3.1.

A necessary and sufficient condition for the system of equations

$$\begin{aligned} AX &= C, \\ XB &= D \end{aligned} \tag{4.4}$$

to have a common solution was given by Cecioni [19], and an expression for the general common solution by Rao and Mitra [116].

Theorem 4.0.2 [*19*, *116*] *A necessary and sufficient conditions for the solvability of the matrix system* (4.4) *over* $\mathbb{C}$ *are given by*

$$AA^-C = C, \qquad DB^-B = D, \qquad AD = CB.$$

In that case a general solution is given by

$$X = A^-C + DB^- - A^-ADB^- + (I - A^-A)V(I - BB^-),$$

where V *is arbitrary.*

The system of two linear matrix equations

$$\begin{aligned} A_1XB_1 &= C_1, \\ A_2XB_2 &= C_2 \end{aligned} \tag{4.5}$$

seems first to have been studied by Mitra [91] over the complex field. Using the systems of the form (4.4), they first solved the system (4.5) when A_1, A_2, B_1 and B_2 are non-negative definite matrices, and then considered the general case.

Theorem 4.0.3 [*91*] *A necessary and sufficient condition for the consistency of the system* (4.5) *is*

$$\begin{aligned} &A_1^*A_1(A_1^*A_1 + A_2^*A_2)^-A_2^*C_2B_2^*(B_1B_1^* + B_2B_2^*)^-B_1B_1^* \\ &= A_2^*A_2(A_1^*A_1 + A_2^*A_2)^-A_1^*C_1B_1^*(B_1B_1^* + B_2B_2^*)^-B_2B_2^*, \end{aligned}$$

in which case the general common solution is

$$\begin{aligned} X = {} & (A_1^*A_1 + A_2^*A_2)^-(A_1^*C_1B_1^* + Y + Z + A_2^*C_2B_2^*)(B_1B_1^* + B_2B_2^*)^- \\ & + U - (A_1^*A_1 + A_2^*A_2)^-(A_1^*A_1 + A_2^*A_2)U(B_1B_1^* + B_2B_2^*)(B_1B_1^* + B_2B_2^*)^-, \end{aligned}$$

where U is arbitrary, Y and Z are arbitrary solutions of the systems

$$\begin{aligned} A_2^*A_2(A_1^*A_1 + A_2^*A_2)^-Y &= A_1^*A_1(A_1^*A_1 + A_2^*A_2)^-A_2^*C_2B_2^*, \\ Y(B_1B_1^* + B_2B_2^*)^-B_1B_1^* &= A_1^*C_1B_1^*(B_1B_1^* + B_2B_2^*)^-B_2B_2^* \end{aligned}$$

and

$$\begin{aligned} A_1^*A_1(A_1^*A_1 + A_2^*A_2)^-Z &= A_2^*A_2(A_1^*A_1 + A_2^*A_2)^-A_1^*C_1B_1^*, \\ Z(B_1B_1^* + B_2B_2^*)^-B_2B_2^* &= A_2^*C_2B_2^*(B_1B_1^* + B_2B_2^*)^-B_1B_1^*, \end{aligned}$$

respectively.

There have been many generalizations of this problem in different settings. Let us mention some of them: Van der Woude [150], [149] investigated it over a field, Özgüler and Akar [96] considered it over a principle domain, Wang [142, 141] studied it over an arbitrary division ring and arbitrary regular ring with identity, Dajić [43] discussed it in a ring with a unit. Here, we give Dajić's result. In the rest of the chapter $\mathcal{R}$ denotes a ring with a unit $1 \neq 0$ and $\mathcal{R}^-$ is the set of all regular elements of $\mathcal{R}$. We also use notation $r_a = 1 - aa^-$ and $l_a = 1 - a^-a$ for $a \in \mathcal{R}^-$ where a^- is an arbitrary inner inverse of a.

Theorem 4.0.4 [*43*] *Let a_i, b_i, c_i be elements of a ring $\mathcal{R}$ with a unit such that a_i, b_i are regular, $a_ia_i^-c_ib_i^-b_i = c_i$ for $i = 1, 2$ and $s = a_2(1 - a_1^-a_1), t = (1 - b_1b_1^-)b_2$ are regular elements. The system of equations*

$$\begin{aligned} a_1xb_1 &= c_1, \\ a_2xb_2 &= c_2 \end{aligned} \tag{4.6}$$

is consistent if and only if

$$(1 - ss^-)(c_2 - gc_1f)(1 - t^-t) = 0, \tag{4.7}$$

where $g = (1 - ss^-)a_2a_1^-$ and $f = b_1^-b_2(1 - t^-t)$. In that case the general solution is given by

$$\begin{aligned} x = {} & [a_1^-c_1 - (1 - a_1^-a_1)s^-(a_2a_1^-c_1 - w)]b_1^-[1 - b_2t^-(1 - b_1b_1^-)] \\ & + [(1 - (1 - a_1^-a_1)s^-a_2)a_1^-v + (1 - a_1^-a_1)s^-c_2]t^-(1 - b_1b_1^-) \\ & + z - (a_1^-a_1 + (1 - a_1^-a_1)s^-s)z(b_1b_1^- + tt^-(1 - b_1b_1^-)), \end{aligned}$$

where v, w are given by

$$\begin{aligned} v &= c_1 f + g^-(1 - ss^-)c_2 t^- t + (a_1 a_1^- - g^- g)z_2 t^- t, \\ w &= g c_1 + ss^- c_2 (1 - t^- t) f^- + ss^- z_1 (b_1^- b_1 - f f^-), \end{aligned} \tag{4.8}$$

z_1, z_2 and z are arbitrary elements of $\mathcal{R}$ and $f^- = b_2^- b_1$, $g^- = a_1 a_2^-$.

When we talk about equations it is inevitably not to mention *Sylvester's equation*

$$AX - XB = C, \tag{4.9}$$

where $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}$ and $C \in \mathbb{C}^{n\times m}$, discussed among others by Silvester [129]. This equation can be seen as equation (4.2), i.e. $\begin{bmatrix} A & I_n \end{bmatrix} X \begin{bmatrix} I_m \\ B \end{bmatrix} = C$. One of the basic result on Equation (4.9) is the following theorem.

Theorem 4.0.5 *For given matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$, Sylvester's equation has a unique solution X for all $C \in \mathbb{C}^{n\times n}$ if and only if A and B have no common eigenvalues.*

First generalization of Equation (4.9) is a generalized Sylvester's matrix equation

$$AX - YB = C, \tag{4.10}$$

where $A \in \mathbb{C}^{m\times r}, B \in \mathbb{C}^{s\times n}$ and $C \in \mathbb{C}^{m\times n}$. Many problems in systems and control theory require the solution of Equation (4.10). Roth [120] in 1952 gave a necessary and sufficient condition for the consistency of the Sylvester's matrix equation and generalized Sylvester's matrix equation.

Theorem 4.0.6 [*120*] (Roth's removal rule) *The necessary and sufficient condition that Equation* (4.9), *where A, B and C are square matrices of order n with elements in a field $\mathbb{F}$, has a solution X with elements in $\mathbb{F}$ is that the matrices*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

are similar.

Theorem 4.0.7 [*120*] (Roth's equivalence theorem) *The necessary and sufficient condition that Equation* (4.10), *where $A \in \mathbb{F}^{m\times r}, B \in \mathbb{F}^{s\times n}$ and $C \in \mathbb{F}^{m\times n}$, has a solution $X \in \mathbb{F}^{r\times n}, Y \in \mathbb{F}^{m\times s}$ is that the matrices*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

are equivalent.

Recall that two rectangular matrices U and V are equivalent if $V = Q^{-1}UP$, for some invertible matrix $P \in \mathbb{F}^{n\times n}$ and some invertible matrix $Q \in \mathbb{F}^{m\times m}$, while two square matrices $U \in \mathbb{F}^{n\times n}$ and $V \in \mathbb{F}^{n\times n}$ are similar if $V = P^{-1}UP$, for some invertible matrix $P \in \mathbb{F}^{n\times n}$. Other characterization of the solvability of generalized Sylvester's matrix equation is given by Baksalary and Kala [6].

Theorem 4.0.8 [*6*] *The necessary and sufficient condition that the equation* (4.10), *where* $A \in \mathbb{F}^{m\times r}, B \in \mathbb{F}^{s\times n}$ *and* $C \in \mathbb{F}^{m\times n}$, *has a solution* $X \in \mathbb{F}^{r\times n}, Y \in \mathbb{F}^{m\times s}$ *is that*

$$(I - AA^-)C(I - B^-B) = 0.$$

If this is the case, the general solution of (4.10) *has the form*

$$\begin{aligned} X &= A^-C + A^-ZB + (I - A^-A)W, \\ Y &= -(I - AA^-)CB^- + Z - (I - AA^-)ZBB^-, \end{aligned}$$

with $W \in \mathbb{F}^{r\times n}$ *and* $Z \in \mathbb{F}^{m\times s}$ *are arbitrary matrices.*

Further generalization of Equation (4.10) is more generalized Sylvester equation which in the ring case can be written as

$$axb + cyd = e, \tag{4.11}$$

for $a, b, c, d, e \in \mathcal{R}$. Equation (4.11) was considered by many authors [5, 141, 43]. In the previous research the solvability of Sylvester equation (4.11) was considered as the solvability of a special case of (4.6) given by

$$\begin{aligned} gyd &= r_a e, \\ cyh &= el_b, \end{aligned} \tag{4.12}$$

where $g = r_a c, h = dl_b$ and Theorem 4.0.4 (or analogue result in different setting) was applied. As the result of that particular research the condition for the existence of a joint solution of the equations from (4.12) is superfluous. Therefore, Theorem 4.0.4 is not really necessary.

Indeed, if y_1 and y_2 are solutions of the first and second equations from (4.12) respectively then we can check that both, y_1 and y_2, are solutions of the equation $gyh = r_a el_b$. Hence by Lemma 4.0.1 there exists $u \in \mathcal{R}$ such that $y_1 = y_2 + u - g^-guhh^-$. It proves that $z = y_1 - l_g u = y_2 + g^-gur_h$ is a solution of system (4.12).

However, combining the previous results after some considerations it can be concluded that one regularity and one algebraic condition can be omitted from the appropriate result of [43], so we give the following result which is similar to the result of Bekselary and Kala [5] for matrices over a field.

Theorem 4.0.9 *Let* $e \in \mathcal{R}$ *and* $a, b, c, d \in \mathcal{R}^-$ *be such that* $g = r_a c$ *and* $h = dl_b$ *are regular. Equation* (4.11) *is consistent if and only if*

$$gg^- r_a ed^- d = r_a e \qquad and \qquad cc^- el_b h^- h = el_b,$$

where all inner inverses involved are arbitrary but fixed. In that case the general solution is given by

$$\begin{aligned} y &= g^- r_a ed^- + (l_g c^- + l_c g^- r_a) el_b h^- + w - c^- cl_g whh^- - g^- gwdd^-, \\ x &= a^-(e - cyd)b^- + z - a^- azbb^-, \end{aligned}$$

where $z, w \in \mathcal{R}$ are arbitrary.

4.1 Algebraic solvability conditions

Even though there are a certain number of papers concerning the system (4.6) in the literature, there are just few papers that considered the system of three linear matrix equations

$$\begin{aligned} A_1XB_1 &= C_1, \\ A_2XB_2 &= C_2, \\ A_3XB_3 &= C_3. \end{aligned} \tag{4.13}$$

Precisely, we are familiar with two papers [67], [135] and in both of them the methods used in the proofs are based on some properties of rank of matrices, so the results cannot be applied in the ring case. Here, we present the result from [135].

Theorem 4.1.1 [*135*] *The system of matrix equations* (4.13) *have a common solution if and only if the following eleven rank equalities hold:*

$$r\left(\begin{bmatrix} A_i & C_i \end{bmatrix}\right) = r(A_i), \quad r\left(\begin{bmatrix} B_i \\ C_i \end{bmatrix}\right) = r(B_i), \quad i = 1, 2, 3,$$

$$r\left(\begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 & B_2 \end{bmatrix}\right),$$

$$r\left(\begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_3 & A_3 \\ B_1 & B_3 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 & B_3 \end{bmatrix}\right),$$

$$r\left(\begin{bmatrix} C_2 & 0 & A_2 \\ 0 & -C_3 & A_3 \\ B_2 & B_3 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_2 \\ A_3 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_2 & B_3 \end{bmatrix}\right),$$

$$r\left(\begin{bmatrix} C_1 & 0 & 0 & A_1 & A_1 \\ 0 & -C_2 & 0 & A_2 & 0 \\ 0 & 0 & -C_3 & 0 & A_3 \\ B_1 & B_2 & B_3 & 0 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 & A_1 \\ A_2 & 0 \\ 0 & A_3 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}\right),$$

$$r\left(\begin{bmatrix} C_1 & 0 & 0 & A_1 \\ 0 & -C_2 & 0 & A_2 \\ 0 & 0 & -C_3 & A_3 \\ B_1 & B_2 & 0 & 0 \\ B_1 & 0 & B_3 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 & B_2 & 0 \\ B_1 & 0 & B_3 \end{bmatrix}\right).$$

In this section, we will present the purely algebraic conditions for the solvability of the system of the equations (4.13), in the ring case, as well as its general solution form presented in [89]. We generalize the method given in [43] for the system of two linear equations (4.6) and we start with some auxiliary results. The forms of inner inverses of the elements in $\mathcal{R}^{2\times 1}$ are given in the next lemma which is a consequence of Theorem 4 from [98] and can be found in [44].

Lemma 4.1.1 *[44] Let $u, v \in \mathcal{R}^-$. Then*

$$v(1-u^-u) \in \mathcal{R}^- \Leftrightarrow \begin{bmatrix} u \\ v \end{bmatrix} \text{ is regular} \Leftrightarrow \begin{bmatrix} v \\ u \end{bmatrix} \text{ is regular} \Leftrightarrow u(1-v^-v) \in \mathcal{R}^-.$$

If $\begin{bmatrix} u \\ v \end{bmatrix}$ is regular, two of its inner inverses are given by

$$\begin{bmatrix} u \\ v \end{bmatrix}^- = \begin{bmatrix} u^- - (1-u^-u)m^-vu^- & (1-u^-u)m^- \end{bmatrix}$$

and

$$\begin{bmatrix} u \\ v \end{bmatrix}^- = \begin{bmatrix} (1-v^-v)n^- & v^- - (1-v^-v)n^-uv^- \end{bmatrix},$$

where m^- and n^- are inner inverses of $m = v(1-u^-u)$ and $n = u(1-v^-v)$, respectively.

Remark 4.1.1 Notice that if u, v are regular, then $v(1-u^-u)$ is not always regular. For example, if $\mathcal{R}$ is a space of bounded linear operators on separable Hilbert space $\mathcal{H}$ there is an operator with closed range which does not preserve closeness of subspaces of $\mathcal{H}$. Precisely, let $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $\mathcal{M}$ and $\mathcal{M}^\perp$ are infinite dimensional. Let $U = P_\mathcal{M}$ and V be given by

$$V = \begin{bmatrix} 0 & 0 \\ V_1 & V_2 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix},$$

such that V_1 is isomorphism and V_2 is not regular. Notice that $\mathcal{R}(V) = \mathcal{M}^\perp$ so V is regular, while $\mathcal{R}(V(I-U^\dagger U)) = \mathcal{R}(V_2)$. So $V(I-U^\dagger U)$ is not regular.

Furthermore, by Lemma 4.1.1 we derive the forms of the inner inverse of the elements in $\mathcal{R}^{3\times 1}$, which will be used in the proof of the main result.

Lemma 4.1.2 *Let $u, v, w \in \mathcal{R}$.*

(i) *If* $u, vl_u \in \mathcal{R}^-$, *then* $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ *is regular* $\Leftrightarrow wl_ul_{vl_u} \in \mathcal{R}^-$, *in which case*

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^- = \begin{bmatrix} (1 - l_ul_{vl_u}(wl_ul_{vl_u})^-w)(u^- - l_u(vl_u)^-vu^-) \\ (1 - l_ul_{vl_u}(wl_ul_{vl_u})^-w)l_u(vl_u)^- \\ l_ul_{vl_u}(wl_ul_{vl_u})^- \end{bmatrix}^T. \tag{4.14}$$

(ii) *If* $v, ul_v \in \mathcal{R}^-$, *then* $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ *is regular* $\Leftrightarrow wl_vl_{ul_v} \in \mathcal{R}^-$, *in which case*

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^- = \begin{bmatrix} (1 - l_vl_{ul_v}(wl_vl_{ul_v})^-w)l_v(ul_v)^- \\ (1 - l_vl_{ul_v}(wl_vl_{ul_v})^-w)(v^- - l_v(ul_v)^-uv^-) \\ l_vl_{ul_v}(wl_vl_{ul_v})^- \end{bmatrix}^T.$$

(iii) *If* $w, ul_w \in \mathcal{R}^-$, *then* $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ *is regular* $\Leftrightarrow vl_wl_{ul_w} \in \mathcal{R}^-$, *in which case*

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^- = \begin{bmatrix} (1 - l_wl_{ul_w}(vl_wl_{ul_w})^-v)l_w(ul_w)^- \\ l_wl_{ul_w}(vl_wl_{ul_w})^- \\ (1 - l_wl_{ul_w}(vl_wl_{ul_w})^-v)(w^- - l_w(ul_w)^-uw^-) \end{bmatrix}^T.$$

(iv) *If* $u, wl_u \in \mathcal{R}^-$, *then* $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ *is regular* $\Leftrightarrow vl_ul_{wl_u} \in \mathcal{R}^-$, *in which case*

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^- = \begin{bmatrix} (1 - l_ul_{wl_u}(vl_ul_{wl_u})^-v)(u^- - l_u(wl_u)^-wu^-) \\ l_ul_{wl_u}(vl_ul_{wl_u})^- \\ (1 - l_ul_{wl_u}(vl_ul_{wl_u})^-v)l_u(wl_u)^- \end{bmatrix}^T.$$

(v) *If* $v, wl_v \in \mathcal{R}^-$, *then* $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ *is regular* $\Leftrightarrow ul_vl_{wl_v} \in \mathcal{R}^-$, *in which case*

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^- = \begin{bmatrix} l_vl_{wl_v}(ul_vl_{wl_v})^- \\ (1 - l_vl_{wl_v}(ul_vl_{wl_v})^-u)(v^- - l_v(wl_v)^-wv^-) \\ (1 - l_vl_{wl_v}(ul_vl_{wl_v})^-u)l_v(wl_v)^- \end{bmatrix}^T.$$

(vi) *If* $w, vl_w \in \mathcal{R}^-$, *then* $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ *is regular* $\Leftrightarrow ul_wl_{vl_w} \in \mathcal{R}^-$, *in which case*

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^- = \begin{bmatrix} l_wl_{vl_w}(ul_wl_{vl_w})^- \\ (1 - l_wl_{vl_w}(ul_wl_{vl_w})^-u)l_w(vl_w)^- \\ (1 - l_wl_{vl_w}(ul_wl_{vl_w})^-u)(w^- - l_w(vl_w)^-vw^-) \end{bmatrix}^T.$$

Proof. (i) The result follows from Lemma 4.1.1 when we replace u and v by $\begin{bmatrix} u \\ v \end{bmatrix}$ and w respectively and omit unnecessary regularities by using the first form of inner inverse of $\begin{bmatrix} u \\ v \end{bmatrix}$. (Lemma 4.1.1 is still correct for $u \in \mathcal{R}^{2\times 1}$.) The concrete proof follows.

If $wl_u l_{vl_u} \in \mathcal{R}^-$, we can check that (4.14) is correct. Conversely, if $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ is regular with an inner inverse $\begin{bmatrix} p & q & r \end{bmatrix}$, multiplying the equation

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \begin{bmatrix} p & q & r \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

by $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ wl_u(vl_u)^- vu^- - wu^- & -wl_u(vl_u)^- & 1 \end{bmatrix}$ from the left and by $l_u l_{vl_u}$ from the right, we get

$$\begin{bmatrix} 0 \\ 0 \\ wl_u l_{vl_u} r w l_u l_{vl_u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ wl_u l_{vl_u} \end{bmatrix}.$$

$(ii)-(vi)$ This follows from (i) because matrix $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ is regular if and only if matrix with any permutation of its rows is regular with an inner inverse whose columns are appropriate permutation of columns of inner inverse of matrix $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$. $\square$

Corollary 4.1.1 *Let $u, v, w, ul_v, vl_w, wl_u \in \mathcal{R}^-$. Then*

$$\begin{aligned} ul_v l_{wl_v} \in \mathcal{R}^- \Leftrightarrow ul_w l_{vl_w} \in \mathcal{R}^- \Leftrightarrow vl_u l_{wl_u} \in \mathcal{R}^- \Leftrightarrow vl_w l_{ul_w} \in \mathcal{R}^- \\ \Leftrightarrow wl_u l_{vl_u} \in \mathcal{R}^- \Leftrightarrow wl_v l_{ul_v} \in \mathcal{R}^-. \end{aligned}$$

At the same manner, or just taking adjoint elements in rings with involution, we can get

Lemma 4.1.3 *Let $u, v, w \in \mathcal{R}$.*

(i) *If $u, r_u v \in \mathcal{R}^-$, then $\begin{bmatrix} u & v & w \end{bmatrix}$ is regular $\Leftrightarrow r_{r_u v} r_u w \in \mathcal{R}^-$, in which case*

$$\begin{bmatrix} u & v & w \end{bmatrix}^- = \begin{bmatrix} (u^- - u^- v(r_u v)^- r_u)(1 - w(r_{r_u v} r_u w)^- r_{r_u v} r_u) \\ (r_u v)^- r_u (1 - w(r_{r_u v} r_u w)^- r_{r_u v} r_u) \\ (r_{r_u v} r_u w)^- r_{r_u v} r_u \end{bmatrix}. \tag{4.15}$$

(ii) If $v, r_v u \in \mathcal{R}^-$*, then* $\begin{bmatrix} u & v & w \end{bmatrix}$ *is regular* $\Leftrightarrow r_{r_v u} r_v w \in \mathcal{R}^-$*, in which case*

$$\begin{bmatrix} u & v & w \end{bmatrix}^- = \begin{bmatrix} (r_v u)^- r_v (1 - w(r_{r_v u} r_v w)^- r_{r_v u} r_v) \\ (v^- - v^- u (r_v u)^- r_v)(1 - w(r_{r_v u} r_v w)^- r_{r_v u} r_v) \\ (r_{r_v u} r_v w)^- r_{r_v u} r_v \end{bmatrix}.$$

(iii) If $w, r_w u \in \mathcal{R}^-$*, then* $\begin{bmatrix} u & v & w \end{bmatrix}$ *is regular* $\Leftrightarrow r_{r_w u} r_w v \in \mathcal{R}^-$*, in which case*

$$\begin{bmatrix} u & v & w \end{bmatrix}^- = \begin{bmatrix} (r_w u)^- r_w (1 - v(r_{r_w u} r_w v)^- r_{r_w u} r_w) \\ (r_{r_w u} r_w v)^- r_{r_w u} r_w \\ (w^- - w^- u (r_w u)^- r_w)(1 - v(r_{r_w u} r_w v)^- r_{r_w u} r_w) \end{bmatrix}.$$

(iv) If $u, r_u w \in \mathcal{R}^-$*, then* $\begin{bmatrix} u & v & w \end{bmatrix}$ *is regular* $\Leftrightarrow r_{r_u w} r_u v \in \mathcal{R}^-$*, in which case*

$$\begin{bmatrix} u & v & w \end{bmatrix}^- = \begin{bmatrix} (u^- - u^- w (r_u w)^- r_u)(1 - v(r_{r_u w} r_u v)^- r_{r_u w} r_u) \\ (r_{r_u w} r_u v)^- r_{r_u w} r_u \\ (r_u w)^- r_u (1 - v(r_{r_u w} r_u v)^- r_{r_u w} r_u) \end{bmatrix}.$$

(v) If $v, r_v w \in \mathcal{R}^-$*, then* $\begin{bmatrix} u & v & w \end{bmatrix}$ *is regular* $\Leftrightarrow r_{r_v w} r_v u \in \mathcal{R}^-$*, in which case*

$$\begin{bmatrix} u & v & w \end{bmatrix}^- = \begin{bmatrix} (r_{r_v w} r_v u)^- r_{r_v w} r_v \\ (v^- - v^- w (r_v w)^- r_v)(1 - u(r_{r_v w} r_v u)^- r_{r_v w} r_v) \\ (r_v w)^- r_v (1 - u(r_{r_v w} r_v u)^- r_{r_v w} r_v) \end{bmatrix}.$$

(vi) If $w, r_w v \in \mathcal{R}^-$*, then* $\begin{bmatrix} u & v & w \end{bmatrix}$ *is regular* $\Leftrightarrow r_{r_w v} r_w u \in \mathcal{R}^-$*, in which case*

$$\begin{bmatrix} u & v & w \end{bmatrix}^- = \begin{bmatrix} (r_{r_w v} r_w u)^- r_{r_w v} r_w \\ (r_w v)^- r_w (1 - u(r_{r_w v} r_w u)^- r_{r_w v} r_w) \\ (w^- - w^- v (r_w v)^- r_w)(1 - u(r_{r_w v} r_w u)^- r_{r_w v} r_w) \end{bmatrix}.$$

Corollary 4.1.2 *Let* $u, v, w, r_u v, r_v w, r_w u \in \mathcal{R}^-$*. Then*

$$\begin{aligned} r_{r_v w} r_v u \in \mathcal{R}^- &\Leftrightarrow r_{r_w v} r_w u \in \mathcal{R}^- \Leftrightarrow r_{r_u w} r_u v \in \mathcal{R}^- \Leftrightarrow r_{r_w u} r_w v \in \mathcal{R}^- \\ &\Leftrightarrow r_{r_u v} r_u w \in \mathcal{R}^- \Leftrightarrow r_{r_v u} r_v w \in \mathcal{R}^-. \end{aligned}$$

Let us compute $\begin{bmatrix} u \\ v \\ w \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}^-$ and $\begin{bmatrix} u & v & w \end{bmatrix}^- \begin{bmatrix} u & v & w \end{bmatrix}$, using formulaes (4.14) and (4.15) respectively, since it will be used later on. After a straightforward computation, we get the following

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}^- = \begin{bmatrix} uu^- & 0 & 0 \\ r_{vl_u} v u^- & vl_u (vl_u)^- & 0 \\ r_{wl_u l_{vl_u}} w (1 - l_u (vl_u)^- v) u^- & r_{wl_u l_{vl_u}} w l_u (vl_u)^- & wl_u l_{vl_u} (wl_u l_{vl_u})^- \end{bmatrix} \tag{4.16}$$

and

$$\left[\begin{array}{ccc} u & v & w \end{array}\right]^{-}\left[\begin{array}{ccc} u & v & w \end{array}\right]=\left[\begin{array}{ccc} u^{-}u & u^{-}vl_{r_uv} & u^{-}(1-v(r_uv)^{-}r_u)wl_{r_{r_uv}r_uw} \\ 0 & (r_uv)^{-}r_uv & (r_uv)^{-}r_uwl_{r_{r_uv}r_uw} \\ 0 & 0 & (r_{r_uv}r_uw)^{-}r_{r_uv}r_uw \end{array}\right]. \tag{4.17}$$

The following two lemmas will be useful in the proving of Theorem 4.1.2.

Lemma 4.1.4 [*43*] *Let $u, v \in \mathcal{R}^{-}$ be such that $s = vl_u, t = r_uv$ are regular. Then*

$$g = r_svu^{-} \qquad \textit{and} \qquad f = u^{-}vl_t$$

are regular with inner inverses $f^{-} = v^{-}u$ and $g^{-} = uv^{-}$, respectively. Element r_sv is also regular with inner inverse v^{-}.

Lemma 4.1.5 *Let $p, q \in \mathcal{R}$ be idempotents and $c \in \mathcal{R}$. The equation*

$$x - pxq = c \tag{4.18}$$

is solvable if and only if $pcq = 0$. In that case the set of solutions is given by $\{c + ptq \mid t \in \mathcal{R}\}$.

Proof. If Equation (4.18) is solvable, then there exists $x_0 \in \mathcal{R}$ such that

$$x_0 - px_0q = c. \tag{4.19}$$

Multiplying the equality (4.19) by p from the left and by q from the right, we get $0 = pcq$. Conversely, if $pcq = 0$, then $x = c$ is a solution of Equation (4.18).
Now, let Equation (4.18) be solvable. If x is a solution of this equation, then $x = c + pxq \in \{c + ptq \mid t \in \mathcal{R}\}$. Also, can be checked, since $pcq = 0$, that all the elements of the set $\{c + ptq \mid t \in \mathcal{R}\}$ are solutions of Equation (4.18).□

Theorem 4.1.2 *Let a_i, b_i, c_i be elements of a ring $\mathcal{R}$ with a unit such that a_i, b_i are regular and $a_ia_i^{-}c_ib_i^{-}b_i = c_i$ for $i = \overline{1,3}$. Additionally, let $s = a_2l_{a_1}, j = a_3l_{a_1}, m = jl_s, t = r_{b_1}b_2, k = r_{b_1}b_3, n = r_tk$ be regular elements. The system of equations*

$$\begin{aligned} &a_1xb_1 = c_1, \\ &a_2xb_2 = c_2, \\ &a_3xb_3 = c_3 \end{aligned} \tag{4.20}$$

is consistent if and only if the following condition holds

$$r_s(c_2 - a_2a_1^{-}c_1b_1^{-}b_2)l_t = 0 \tag{4.21}$$

and the given equation is solvable by $y_1, y_3 \in \mathcal{R}$

$$\begin{aligned} &r_ma_3l_{a_2}(1 - l_{a_1}s^{-}a_2)a_1^{-}y_1t^{-}kl_n + r_mjs^{-}y_3b_1^{-}(1 - b_2t^{-}r_{b_1})r_{b_2}b_3l_n \\ &= r_m[c_3 - js^{-}c_2t^{-}k - a_3a_2^{-}r_sc_2t^{-}k - js^{-}c_2l_tb_2^{-}b_3 \\ &\quad - (a_3 - js^{-}a_2)a_1^{-}c_1b_1^{-}(b_3 - b_2t^{-}k)]l_n. \end{aligned} \tag{4.22}$$

Proof. The solvability of the system of equations (4.20) is equivalent to the existence of $z_i \in \mathcal{R}, i = \overline{1,6}$ such that the equation

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} x \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} c_1 & z_1 & z_2 \\ z_3 & c_2 & z_4 \\ z_5 & z_6 & c_3 \end{bmatrix} \tag{4.23}$$

is solvable by x. The consistency of Equation (4.23) is, by Lemma 4.0.1, equivalent with

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^- \begin{bmatrix} c_1 & z_1 & z_2 \\ z_3 & c_2 & z_4 \\ z_5 & z_6 & c_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^- \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} c_1 & z_1 & z_2 \\ z_3 & c_2 & z_4 \\ z_5 & z_6 & c_3 \end{bmatrix}. \tag{4.24}$$

Using (4.16) and (4.17), by a straightforward computation, we conclude that the solvability of the system (4.20) is equivalent to the existence of $z_i \in \mathcal{R}, i = \overline{1,6}$ such that the approaching nine conditions from the matrix equation (4.24) are satisfied.

Positions $(1.1), (1.2), (2.1)$ and (2.2) in the matrix equation (4.24) are equivalent to the existence of the common solution of the first and second equations from (4.20) (which is exactly condition (4.21)) and the condition that z_1 and z_3 are given by

$$\begin{aligned} z_1 &= c_1 f + g^- r_s c_2 t^- t + a_1 a_1^- l_g y_1 t^- t, \\ z_3 &= g c_1 + s s^- c_2 l_t f^- + s s^- y_3 r_f b_1^- b_1, \end{aligned} \tag{4.25}$$

where $g = r_s a_2 a_1^-$, $f = b_1^- b_2 l_t$, $g^- = a_1 a_2^-$, $f^- = b_2^- b_1$, and $y_1, y_3 \in \mathcal{R}$ are arbitrary (this is a direct result of Theorem 4.0.4).

Positions (1.3) and (3.1) in (4.24) are exactly

$$\begin{aligned} &c_1 b_1^-(b_3 - b_2 t^- k) l_n + a_1 a_1^- z_1 t^- k l_n = z_2 - a_1 a_1^- z_2 n^- n, \\ &r_m(a_3 - j s^- a_2) a_1^- c_1 + r_m j s^- z_3 b_1^- b_1 = z_5 - m m^- z_5 b_1^- b_1, \end{aligned}$$

therefore by Lemma 4.1.5, the above equations are solvable by z_2, z_5 respectively, for all z_1, z_3 and the general forms of solutions are given by

$$\begin{aligned} z_2 &= c_1 b_1^-(b_3 - b_2 t^- k) l_n + a_1 a_1^- z_1 t^- k l_n + a_1 a_1^- y_2 n^- n, \\ z_5 &= r_m(a_3 - j s^- a_2) a_1^- c_1 + r_m j s^- z_3 b_1^- b_1 + m m^- y_5 b_1^- b_1, \end{aligned}$$

where $y_2, y_5 \in \mathcal{R}$ are arbitrary.

In addition, by replacing in the above two equations the appropriate z_1 and z_3 given by (4.25), and using that $tt^- k l_n = k l_n, r_m j s^- s = r_m j$, we get

$$\begin{aligned} z_2 &= (c_1 b_1^-(b_3 - b_2 t^- k) + g^- r_s c_2 t^- k) l_n + a_1 a_1^- l_g y_1 t^- k l_n + a_1 a_1^- y_2 n^- n, \\ z_5 &= r_m((a_3 - j s^- a_2) a_1^- c_1 + j s^- c_2 l_t f^-) + r_m j s^- y_3 r_f b_1^- b_1 + m m^- y_5 b_1^- b_1. \end{aligned} \tag{4.26}$$

Similarly, the positions (2.3) and (3.2) in (4.24) are

$$\begin{aligned} &(g c_1 + s s^- z_3) b_1^-(b_3 - b_2 t^- k) l_n + (g z_1 + s s^- c_2) t^- k l_n + g z_2 n^- n = z_4 - s s^- z_4 n^- n, \\ &r_m(a_3 - j s^- a_2) a_1^-(c_1 f + z_1 t^- t) + r_m j s^-(z_3 f + c_2 t^- t) + m m^- z_5 f = z_6 - m m^- z_6 t^- t. \end{aligned}$$

By Lemma 4.1.5, the above equations are solvable by z_4, z_6 respectively, for all z_1, z_2, z_3, z_5 and general forms of solutions are given by

$$\begin{aligned} z_4 = &\ (gc_1 + ss^- z_3)b_1^-(b_3 - b_2t^-k)l_n + (gz_1 + ss^-c_2)t^-kl_n + gz_2n^-n + ss^-y_4n^-n, \\ z_6 = &\ r_m(a_3 - js^-a_2)a_1^-(c_1f + z_1t^-t) + r_mjs^-(z_3f + c_2t^-t) + mm^-z_5f + mm^-y_6t^-t, \end{aligned}$$

where $y_4, y_6 \in \mathcal{R}$ are arbitrary.

Next, when we replace in the above two equations the appropriate z_1, z_2, z_3, z_5 given by (4.25) and (4.26), and using that

$$\begin{aligned} & ga_1a_1^- = g, \\ & b_1^-b_1f = f, \\ & r_m(a_3 - js^-a_2)a_1^-a_1 = r_m(a_3 - js^-a_2), \\ & b_1b_1^-(b_3 - b_2t^-k)l_n = (b_3 - b_2t^-k)l_n, \\ & r_m(a_3 - js^-a_2)a_1^-a_1a_2^-r_sc_2 = r_ma_3a_2^-r_sc_2, \\ & c_2l_tb_2^-b_1b_1^-(b_3 - b_2t^-k)l_n = c_2l_tb_2^-b_3l_n, \end{aligned} \tag{4.27}$$

we get

$$\begin{aligned} z_4 = &\ (gc_1 + ss^-y_3r_f)b_1^-(b_3 - b_2t^-k)l_n + ss^-c_2l_tb_2^-b_3l_n + (gg^-r_sc_2 + ss^-c_2)t^-kl_n \\ &+ gy_2n^-n + ss^-y_4n^-n, \\ z_6 = &\ r_m(a_3 - js^-a_2)a_1^-(c_1f + l_gy_1t^-t) + r_ma_3a_2^-r_sc_2t^-t + r_mjs^-(c_2l_tf^-f + c_2t^-t) \\ &+ mm^-y_5f + mm^-y_6t^-t, \end{aligned} \tag{4.28}$$

where $y_1, y_2, y_3, y_4, y_5, y_6 \in \mathcal{R}$ are arbitrary. The first two equalities from (4.27) are evident. Let us prove the third and fifth equalities (the fourth and sixth will follow in the similar way).

Since $0 = r_mm$, we have $r_ma_3l_{a_1} = r_mjs^-a_2l_{a_1}$, i.e. $r_m(a_3 - js^-a_2)l_{a_1} = 0$ which is equivalent with the third equality from (4.27). To prove the fifth equality from (4.27) notice that

$$\begin{aligned} r_m(a_3 - js^-a_2)a_1^-a_1a_2^-r_sc_2 &= r_m(a_3 - js^-a_2)a_2^-r_sc_2 = r_ma_3a_2^-r_sc_2 - r_mjs^-a_2a_2^-r_sc_2 \\ &= r_ma_3a_2^-r_sc_2 - r_mjs^-r_sc_2 = r_ma_3a_2^-r_sc_2 - r_mms^-c_2 = r_ma_3a_2^-r_sc_2. \end{aligned}$$

The last position (3.3) of (4.24) is given by

$$\begin{aligned} & r_m[(a_3 - js^-a_2)a_1^-z_1t^-k + js^-z_3b_1^-(b_3 - b_2t^-k)]l_n + \\ & r_m[(a_3 - js^-a_2)a_1^-z_2 + js^-z_4]n^-n + mm^-[z_5b_1^-(b_3 - b_2t^-k) + z_6t^-k]l_n = \\ & c_3 - mm^-c_3n^-n - r_mjs^-c_2t^-kl_n - r_m(a_3 - js^-a_2)a_1^-c_1b_1^-(b_3 - b_2t^-k)l_n, \end{aligned}$$

which is if we replace z_i, $i = \overline{1,6}$ equivalent to

$$\begin{aligned}
&r_m[(a_3 - js^-a_2)a_1^-l_g y_1 t^-k + js^-y_3 r_f b_1^-(b_3 - b_2t^-k)]l_n + \\
&r_m[(a_3 - js^-a_2)a_1^-y_2 + js^-y_4]n^-n + mm^-[y_5 b_1^-(b_3 - b_2t^-k) + y_6t^-k]l_n \\
&= c_3 - mm^-c_3n^-n - r_m[js^-c_2t^-k + a_3a_2^-r_sc_2t^-k + js^-c_2l_tb_2^-b_3 \\
&+(a_3 - js^-a_2)a_1^-c_1b_1^-(b_3 - b_2t^-k)]l_n.
\end{aligned} \tag{4.29}$$

Notice that $y_1, y_3, y_2, y_4, y_5, y_6 \in \mathcal{R}$ are the solutions of (4.29) if and only if y_1 and y_3 are the solutions of

$$\begin{aligned}
&r_m[(a_3 - js^-a_2)a_1^-l_g y_1 t^-k + js^-y_3 r_f b_1^-(b_3 - b_2t^-k)]l_n = \\
&r_m[c_3 - js^-c_2t^-k - a_3a_2^-r_sc_2t^-k - js^-c_2l_tb_2^-b_3 \\
&-(a_3 - js^-a_2)a_1^-c_1b_1^-(b_3 - b_2t^-k)]l_n,
\end{aligned} \tag{4.30}$$

y_2 and y_4 are the solutions of

$$r_m[(a_3 - js^-a_2)a_1^-y_2 + js^-y_4]n^-n = r_mc_3n^-n, \tag{4.31}$$

and y_5 and y_6 are the solutions of

$$mm^-[y_5b_1^-(b_3 - b_2t^-k) + y_6t^-k]l_n = mm^-c_3l_n. \tag{4.32}$$

Namely, let $y_1, y_3, y_2, y_4, y_5, y_6$ be the solutions of (4.29). Multiplying Equation (4.29) by r_m from the left and by l_n from the right we get that y_1 and y_3 are the solutions of Equation (4.30). At the same manner, multiplying Equation (4.29) by r_m from the left and by n^-n from the right, we get that y_2 and y_4 are the solutions of Equation (4.31), and multiplying Equation (4.29) by mm^- from the left and by l_n from the right we get that y_5 and y_6 are the solutions of Equation (4.32). Conversely, let y_1, y_3, and y_2, y_4, and y_5, y_6 be the solutions of Equations (4.30), (4.31) and (4.32), respectively. If we sum up these three equations we exactly get that $y_1, y_3, y_2, y_4, y_5, y_6$ are the solutions of Equation (4.29).

As a result, there are three general Sylvester equations, where it will be shown that (4.31) and (4.32) always have a solution. First, note that elements r_mjs^- and t^-kl_n are regular by Lemma 4.1.4 with inner inverses sj^- and k^-t respectively. By Lemma 4.0.1, the solvability of Equation (4.31) is equivalent to the solvability of the equation

$$(1 - r_mjj^-)r_m(a_3 - js^-a_2)a_1^-y_2n^-n = (1 - r_mjj^-)r_mc_3n^-n. \tag{4.33}$$

Since

$$(1 - r_mjj^-)r_m = r_m - r_mjj^-(1 - mm^-) \overset{m=jl_s}{=} r_m - r_mjj^- + r_mmm^- = r_mr_j$$

and

$$(1 - r_m jj^-) r_m (a_3 - js^- a_2) a_1^- = r_m r_j (a_3 - js^- a_2) a_1^- = r_m r_j a_3 a_1^-,$$

we get that (4.33) is equivalent to

$$r_m r_j a_3 a_1^- y_2 n^- n = r_m r_j c_3 n^- n. \tag{4.34}$$

Using the fact that $a_3 a_3^- c_3 = c_3$ and that $r_m r_j a_3 a_1^-$ is regular with inner inverse $a_1 a_3^-$, it can be seen that (4.34) is always solvable with the general solution

$$y_2 = a_1 a_3^- c_3 + u_2 - a_1 a_3^- r_m r_j a_3 a_1^- u_2 n^- n. \tag{4.35}$$

So, Equation (4.31) is solvable with the general solutions given by (4.35) and (4.36),

$$\begin{aligned} y_4 &= sj^- [r_m c_3 - r_m (a_3 - js^- a_2) a_1^- y_2] n^- n + u_4 - sj^- r_m js^- u_4 n^- n \\ &= sj^- r_m j [s^- a_2 a_3^- c_3 + (s^- a_2 (1 - a_3^- r_m r_j a_3) - j^- a_3) a_1^- u_2] n^- n \\ &\quad + u_4 - sj^- r_m js^- u_4 n^- n, \end{aligned} \tag{4.36}$$

where $u_2, u_4 \in \mathcal{R}$ are arbitrary.

At the same manner, the consistency of Equation (4.32) is equivalent to the consistency of the equation

$$mm^- y_5 b_1^- (b_3 - b_2 t^- k) l_n (1 - k^- k l_n) = mm^- c_3 l_n (1 - k^- k l_n),$$

i.e.

$$mm^- y_5 b_1^- b_3 l_k l_n = mm^- c_3 l_k l_n. \tag{4.37}$$

Having in mind that $c_3 b_3^- b_3 = c_3$ and $b_3^- b_1 \in b_1^- b_3 l_k l_n \{1\}$, Equation (4.37) is consistent with the general solution

$$y_5 = c_3 b_3^- b_1 + u_5 - mm^- u_5 b_1^- b_3 l_k l_n b_3^- b_1 \tag{4.38}$$

which together with

$$\begin{aligned} y_6 &= mm^- [c_3 l_n - y_5 b_1^- (b_3 - b_2 t^- k) l_n] k^- t + u_6 - mm^- u_6 t^- k l_n k^- t \\ &= mm^- [c_3 b_3^- b_2 t^- + u_5 b_1^- ((1 - b_3 l_k l_n b_3^-) b_2 t^- - b_3 k^-)] k l_n k^- t \\ &\quad + u_6 - mm^- u_6 t^- k l_n k^- t \end{aligned} \tag{4.39}$$

gives the general solution of (4.32) for arbitrary $u_5, u_6 \in \mathcal{R}$.
We get that (4.30) is actually (4.22), using that

$$r_m (a_3 - js^- a_2) a_1^- l_g = r_m a_3 l_{a_2} (1 - l_{a_1} s^- a_2) a_1^-,$$

$$r_f b_1^- (b_3 - b_2 t^- k) l_n = b_1^- (1 - b_2 t^- r_{b_1}) r_{b_2} b_3 l_n.$$

Indeed,

$$r_m(a_3 - js^-a_2)a_1^- l_g = r_m a_3 a_1^- - r_m js^- a_2 a_1^- - r_m(a_3 - js^- a_2)a_2^- r_s a_2 a_1^- =$$
$$r_m a_3 a_1^- - r_m js^- a_2 a_1^- - r_m a_3 a_2^- r_s a_2 a_1^- + r_m js^- r_s a_2 a_1^- =$$
$$r_m a_3 l_{a_2} a_1^- - r_m a_3 l_{a_1} s^- a_2 a_1^- + r_m a_3 a_2^- a_2 l_{a_1} s^- a_2 a_1^- = r_m a_3 l_{a_2}(1 - l_{a_1} s^- a_2)a_1^-,$$

and the second equality can be obtained in the same manner. □

Remark 4.1.2 Notice that by Remark 4.1.1 we cannot omit the assumed regularities.

In order to derive algebraic conditions for the solvability of the system of the equations (4.13), we discuss the solvability of Equation (4.22). Since

$$r_m a_3 l_{a_2}(1 - l_{a_1} s^- a_2)a_1^- a_1 = r_m a_3 l_{a_2}(1 - l_{a_1} s^- a_2),$$

we have that the regularity of the element $r_m a_3 l_{a_2}$ is sufficient (and necessary) for the regularity of the element $a = r_m a_3 l_{a_2}(1 - l_{a_1} s^- a_2)a_1^-$. Furthermore, the inner inverse of a is $a_1 l_{a_2}(r_m a_3 l_{a_2})^-$ and $r_a = r_{r_m a_3 l_{a_2}}$. Similarly, since

$$b_1 b_1^-(1 - b_2 t^- r_{b_1})r_{b_2} b_3 l_n = (1 - b_2 t^- r_{b_1})r_{b_2} b_3 l_n,$$

the inner inverse of element $d = b_1^-(1 - b_2 t^- r_{b_1})r_{b_2} b_3 l_n$ is $(r_{b_2} b_3 l_n)^- r_{b_2} b_1$. Let $G = r_a r_m js^- = r_{r_m a_3 l_{a_2}} r_m js^-$ and $H = b_1^-(1 - b_2 t^- r_{b_1})r_{b_2} b_3 l_k l_n$. We have that $G^- = s(r_{r_m a_3 l_{a_2}} r_m j)^-$ and, since

$$b_1 b_1^-(1 - b_2 t^- r_{b_1})r_{b_2} b_3 l_k l_n = (1 - b_2 t^- r_{b_1})r_{b_2} b_3 l_k l_n,$$

we get that $H^- = (r_{b_2} b_3 l_k l_n)^- r_{b_2} b_1$. Let $p = a_3 l_{a_2}, q = r_{b_2} b_3$ and $e = c_3 - js^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - js^- c_2 l_t b_2^- b_3 - (a_3 - js^- a_2)a_1^- c_1 b_1^-(b_3 - b_2 t^- k)$. Theorem 4.0.9 implies that Equation (4.22) is solvable if and only if

$$r_{r_m p} r_m j(r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- q l_n = r_{r_m p} r_m e l_n, \tag{4.40}$$

$$r_m j j^- e l_k l_n (q l_k l_n)^- q l_k l_n = r_m e l_k l_n. \tag{4.41}$$

Let us simplify the above conditions. Since $l_{ql_n} + (ql_n)^- ql_n = 1$, if we multiply (4.40) from the right, first by l_{ql_n} and then by $(ql_n)^- ql_n$, we get that (4.40) is equivalent with the following two conditions:

$$r_{r_m p} r_m e l_n l_{ql_n} = 0,$$
$$r_{r_{r_m p} r_m j} r_{r_m p} r_m e l_n (q l_n)^- q l_n = 0,$$

i.e.

$$\begin{aligned} &r_{r_m p} r_m e l_n l_{ql_n} = 0,\\ &r_{r_{r_m p} r_m j} r_{r_m p} r_m e l_n = 0. \end{aligned} \tag{4.42}$$

Similarly, using that $l_{ql_k l_n} + (ql_k l_n)^- q l_k l_n = 1$, if we multiply (4.41) from the right, first by $l_{ql_kl_n}$ and then by $(ql_kl_n)^-ql_kl_n$, we get that (4.41) is equivalent with the following two conditions

$$\begin{aligned} &r_m e l_k l_n l_{ql_kl_n} = 0, \\ &r_m r_j e l_k l_n (ql_kl_n)^- ql_kl_n = 0. \end{aligned} \tag{4.43}$$

By $r_j = r_j r_m$, (4.43) is further equivalent with

$$\begin{aligned} &r_m e l_k l_n l_{ql_kl_n} = 0, \\ &r_m r_j e l_k l_n = 0. \end{aligned} \tag{4.44}$$

When we replace the formulae for e in the conditions (4.42) and (4.44), we get that Equation (4.22) is consistent if and only if the following four conditions hold

$$\begin{aligned} &r_m r_j (c_3 - a_3 a_1^- c_1 b_1^- b_3) l_k l_n = 0, \\ &r_m (c_3 - js^- c_2 l_t b_2^- b_3 - (a_3 - js^- a_2) a_1^- c_1 b_1^- b_3) l_k l_n l_{ql_kl_n} = 0, \\ &r_{r_{r_mp} r_m j} r_{r_mp} r_m (c_3 - a_3 a_2^- r_s c_2 t^- k - a_3 a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n = 0, \\ &r_{r_mp} r_m e l_n l_{ql_n} = 0. \end{aligned} \tag{4.45}$$

Having in mind that

$$r_j (c_3 - a_3 a_1^- c_1 b_1^- b_3) l_k = 0 \tag{4.46}$$

is the exact condition for the existence of the common solution of the first and third equations from (4.20), we can conclude that the first condition from (4.45) can be replaced by (4.46) in the case when the rest three conditions from (4.45) and the condition (4.21) are satisfied and all equations from (4.20) are solvable separately. Also, by Theorem 4.0.9, the solutions y_3 and y_1 of Equation (4.22) are given by

$$\begin{aligned} y_3 = {} & s(r_{r_mp} r_m j)^- r_{r_mp} r_m e l_n (ql_n)^- r_{b_2} b_1 \\ & + s(l_{r_{r_mp} r_m j} j^- + (1 - j^- r_m j)(r_{r_mp} r_m j)^- r_{r_mp}) r_m e l_k l_n (ql_kl_n)^- r_{b_2} b_1 \\ & + u_3 - sj^- r_m j l_{r_{r_mp} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) ql_k l_n (ql_kl_n)^- r_{b_2} b_1 \\ & - s(r_{r_mp} r_m j)^- r_{r_mp} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) ql_n (ql_n)^- r_{b_2} b_1, \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} y_1 = {} & a_1 l_{a_2} (r_m p)^- r_m [e - j(r_{r_mp} r_m j)^- r_{r_mp} r_m e \\ & - j l_{r_{r_mp} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{ql_kl_n} q \\ & - (1 - j(r_{r_mp} r_m j)^- r_{r_mp} r_m) e l_k l_n (ql_kl_n)^- q] l_n k^- t + u_1 \\ & - a_1 l_{a_2} (r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n k^- t, \end{aligned} \tag{4.48}$$

where $u_1, u_3 \in \mathcal{R}$ are arbitrary.

Let us now compute the general solution of system (4.20). We have that

$$x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^- \begin{bmatrix} c_1 & z_1 & z_2 \\ z_3 & c_2 & z_4 \\ z_5 & z_6 & c_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^- + u - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^- \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} u \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^-,$$

where $u \in \mathcal{R}$ is arbitrary and $z_i \in \mathcal{R}, i = \overline{1,6}$ are such that Equation (4.23) is solvable. By (4.14) and (4.15) we get

$$\begin{aligned} x = {} & (1 - l_{a_1} l_s m^- a_3)(a_1^- - l_{a_1} s^- a_2 a_1^-)[(c_1 b_1^-(1 - b_2 t^- r_{b_1}) + z_1 t^- r_{b_1})(1 - b_3 n^- r_t r_{b_1}) \\ & + z_2 n^- r_t r_{b_1}] + (1 - l_{a_1} l_s m^- a_3) l_{a_1} s^- [(z_3 b_1^-(1 - b_2 t^- r_{b_1}) \\ & + c_2 t^- r_{b_1})(1 - b_3 n^- r_t r_{b_1}) + z_4 n^- r_t r_{b_1}] + l_{a_1} l_s m^- [(z_5 b_1^-(1 - b_2 t^- r_{b_1}) \\ & + z_6 t^- r_{b_1})(1 - b_3 n^- r_t r_{b_1}) + c_3 n^- r_t r_{b_1}] + u - (1 - l_{a_1} l_s l_m) u (1 - r_n r_t r_{b_1}), \end{aligned} \tag{4.49}$$

where $z_i, i = \overline{1,6}$ are given by (4.25), (4.26), (4.28); $y_i, i = \overline{1,6}$ are given by (4.35), (4.36), (4.38), (4.39), (4.47), (4.48) and $u, u_i \in \mathcal{R}, i = \overline{1,6}$ are arbitrary. Finally, we get the main result of this section:

Theorem 4.1.3 *Let a_i, b_i, c_i be elements of a ring $\mathcal{R}$ with a unit such that a_i, b_i are regular and $a_i a_i^- c_i b_i^- b_i = c_i$ for $i = \overline{1,3}$. Additionally, let $s = a_2 l_{a_1}, j = a_3 l_{a_1}, m = j l_s, t = r_{b_1} b_2, k = r_{b_1} b_3, n = r_t k, p = a_3 l_{a_2}, q = r_{b_2} b_3$ be such that $s, j, m, t, k, n, r_m p, q l_n, r_{r_m p} r_m j, q l_k l_n \in \mathcal{R}^-$. The following are equivalent:*

- (i) *The system of equations* (4.20) *is consistent.*
- (ii) *The conditions*

$$\begin{aligned} & r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t = 0, \\ & r_{r_m p} r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- q l_n = r_{r_m p} r_m e l_n, \\ & r_m j j^- e l_k l_n (q l_k l_n)^- q l_k l_n = r_m e l_k l_n \end{aligned}$$

are satisfied, where $e = c_3 - j s^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - j s^- c_2 l_t b_2^- b_3 - (a_3 - j s^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)$.

- (iii) *The conditions*

$$\begin{aligned} & r_s(c_2 - a_2 a_1^- c_1 b_1^- b_2) l_t = 0, \\ & r_j(c_3 - a_3 a_1^- c_1 b_1^- b_3) l_k = 0, \\ & r_m(c_3 - j s^- c_2 l_t b_2^- b_3 - (a_3 - j s^- a_2) a_1^- c_1 b_1^- b_3) l_k l_n l_{q l_k l_n} = 0, \\ & r_{r_{r_m p} r_m j} r_{r_m p} r_m (c_3 - a_3 a_2^- r_s c_2 t^- k - a_3 a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n = 0, \\ & r_{r_m p} r_m (c_3 - j s^- c_2 t^- k - a_3 a_2^- r_s c_2 t^- k - j s^- c_2 l_t b_2^- b_3 \\ & -(a_3 - j s^- a_2) a_1^- c_1 b_1^- (b_3 - b_2 t^- k)) l_n l_{q l_n} = 0 \end{aligned}$$

are satisfied.

In that case, the general solution of (4.20) *is given by* (4.49), *where*

$$
\begin{aligned}
z_1 &= c_1 f + g^- r_s c_2 t^- t + a_1 l_{a_2}(r_m p)^- r_m [e - j(r_{r_m p} r_m j)^- r_{r_m p} r_m e \\
&\quad - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q \\
&\quad - (1 - j(r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] l_n k^- t + a_1 a_1^- l_g u_1 t t^- \\
&\quad - a_1 l_{a_2}(r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n k^- t, \\
z_2 &= (c_1 b_1^- (b_3 - b_2 t^- k) + g^- r_s c_2 t^- k) l_n + a_1 a_3^- c_3 n^- n + a_1 (1 - a_3^- r_m r_j a_3) a_1^- u_2 n^- n \\
&\quad a_1 l_{a_2}(r_m p)^- r_m [e - j(r_{r_m p} r_m j)^- r_{r_m p} r_m e - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q \\
&\quad - (1 - j(r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] k^- k l_n + a_1 a_1^- l_g u_1 t^- k l_n \\
&\quad - a_1 l_{a_2}(r_m p)^- r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- k l_n, \\
z_3 &= g c_1 + s s^- c_2 l_t f^- + s(r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 \\
&\quad + s(l_{r_{r_m p} r_m j} j^- + (1 - j^- r_m j)(r_{r_m p} r_m j)^- r_{r_m p}) r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
&\quad + s s^- u_3 r_f b_1^- b_1 - s j^- r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
&\quad - s(r_{r_m p} r_m j)^- r_{r_m p} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n (q l_n)^- r_{b_2} b_1, \\
z_4 &= g c_1 b_1^- (b_3 - b_2 t^- k) l_n + s s^- c_2 l_t b_2^- b_3 l_n + (g g^- r_s c_2 + s s^- c_2) t^- k l_n + r_s a_2 a_3^- c_3 n^- n \\
&\quad + r_s a_2 (1 - a_3^- r_m r_j a_3) a_1^- u_2 n^- n + s j^- r_m j [s^- a_2 a_3^- c_3 + (s^- a_2 (1 - a_3^- r_m r_j a_3) \\
&\quad - j^- a_3) a_1^- u_2] n^- n + s(1 - j^- r_m j) s^- u_4 n^- n + s(r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- q l_n \\
&\quad + s(l_{r_{r_m p} r_m j} j^- + (1 - j^- r_m j)(r_{r_m p} r_m j)^- r_{r_m p}) r_m e l_k l_n (q l_k l_n)^- q l_n \\
&\quad + s l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n \\
&\quad - s j^- r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- q l_n, \\
z_5 &= r_m ((a_3 - j s^- a_2) a_1^- c_1 + j s^- c_2 l_t f^-) + m m^- c_3 b_3^- b_1 + m m^- u_5 b_1^- (1 - b_3 l_k l_n b_3^-) b_1 \\
&\quad + r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m e l_n (q l_n)^- r_{b_2} b_1 + r_m j l_{r_{r_m p} r_m j} j^- r_m e l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
&\quad + r_m j s^- u_3 r_f b_1^- b_1 - r_m j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_k l_n (q l_k l_n)^- r_{b_2} b_1 \\
&\quad - r_m j (r_{r_m p} r_m j)^- r_{r_m p} r_m j s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) q l_n (q l_n)^- r_{b_2} b_1, \\
z_6 &= r_m (a_3 - j s^- a_2) a_1^- c_1 f + r_m a_3 a_2^- r_s c_2 t^- t + r_m j s^- (c_2 l_t f^- f + c_2 t^- t) \\
&\quad + m m^- c_3 b_3^- b_2 l_t + m m^- u_5 b_1^- (1 - b_3 l_k l_n b_3^-) b_2 l_t + m m^- [c_3 b_3^- b_2 t^- \\
&\quad + u_5 b_1^- ((1 - b_3 l_k l_n b_3^-) b_2 t^- - b_3 k^-)] k l_n k^- t + m m^- u_6 t^- (1 - k l_n k^-) t \\
&\quad + r_m p (1 - l_{a_1} s^- a_2) a_1^- u_1 t^- (1 - k l_n k^-) t \\
&\quad + r_m p (r_m p)^- r_m [e - j l_{r_{r_m p} r_m j} s^- u_3 b_1^- (1 - b_2 t^- r_{b_1}) r_{q l_k l_n} q \\
&\quad - j(r_{r_m p} r_m j)^- r_{r_m p} r_m e - (1 - j(r_{r_m p} r_m j)^- r_{r_m p} r_m) e l_k l_n (q l_k l_n)^- q] l_n k^- t,
\end{aligned}
$$

$u, u_i \in \mathcal{R}, i = \overline{1,6}$ *are arbitrary,* $g = r_s a_2 a_1^-$, $f = b_1^- b_2 l_t$, $g^- = a_1 a_2^-$, $f^- = b_2^- b_1$, *and* e *is as in* (*ii*).

As an application of Theorem 4.1.3 we get necessary and sufficient conditions for the existence of common inner inverse of three regular elements.

Corollary 4.1.3 *Let* $a_i, i = \overline{1,3}$ *be regular elements of a unitary ring* $\mathcal{R}$, *with inner inverses* $a_i^-, i = \overline{1,3}$, *and let* $s = a_2 l_{a_1}, j = a_3 l_{a_1}, m = j l_s, t = r_{a_1} a_2, k = r_{a_1} a_3, n =$

$r_t k, p = a_3 l_{a_2}, q = r_{a_2} a_3$ be such that $s, j, m, t, k, n, r_m p, ql_n,\ r_{r_m p} r_m j, ql_k l_n \in \mathcal{R}^-$. Then elements $a_i, i = \overline{1,3}$ have a common inner inverse if and only if the following conditions are satisfied

$$\begin{aligned}
&r_s a_2(a_2^- - a_1^-)a_2 l_t = 0,\\
&r_j a_3(a_3^- - a_1^-)a_3 l_k = 0,\\
&r_m[a_3(a_3^- - a_1^-) - js^- a_2(a_2^- - a_1^-)]a_3 l_k l_n l_{ql_k l_n} = 0,\\
&r_{r_{r_m p} r_m j} r_{r_m p} r_m a_3[(a_3^- - a_1^-)a_3 - (a_2^- - a_1^-)a_2 t^- k]l_n = 0,\\
&r_{r_m p} r_m[a_3(a_3^- - a_1^-)a_3 - a_3(a_2^- - a_1^-)a_2 t^- k - js^- a_2(a_2^- - a_1^-)a_3\\
&+js^- a_2(a_2^- - a_1^-)a_2 t^- k]l_n l_{ql_n} = 0.
\end{aligned}$$

Proof. Result follows from $(i) \Leftrightarrow (iii)$ in Theorem 4.1.3 after some computations. □

4.2 Possible directions of further research

In the setting of operators the consideration of the solvability of the equation $AXB = C$ is much more difficult than in matrix settings. If A and B are closed range operators on Hilbert spaces, then Theorem 4.0.1 can be applied. But, Theorem 4.0.1 can not be applied in the general case. To present the solvability conditions for Equation (4.2) it is necessary to introduce the concept of generalized inverses for arbitrary operators. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ be complex Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. In this section, for an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ by its inner inverse we mean the arbitrary linear operator $A' : \mathcal{D}(A') \subseteq \mathcal{H}_2 \to \mathcal{H}_1$ with $\mathcal{R}(A) \subseteq \mathcal{D}(A')$ and

$$AA'A = A.$$

Thus, $A' \notin \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, in general. If, in addition, A' verifies

$$A'AA' = A',$$

then A' is called a reflexive inverse of A. Moreover, there exists a unique reflexive inverse of A which also verifies

$$(AA')^*|_{\mathcal{D}(A')} = AA'$$

and

$$(A'A)^* = A'A,$$

which is called the *Moore-Penrose generalized inverse* of A and it will be denoted by $A^\dagger$. Therefore, $A^\dagger$ is the unique reflexive inverse of A such that

$$A^\dagger A = P_{\overline{\mathcal{R}(A^*)}} \qquad \text{and} \qquad AA^\dagger = P_{\overline{\mathcal{R}(A)}}|_{\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp}.$$

The following lemma allows us to use the results on bounded operators for the Moore-Penrose generalized inverse.

Lemma 4.2.1 [*80*] *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$ such that $\mathcal{R}(A) \subseteq \mathcal{R}(C)$. Then $C^\dagger A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$.*

Let us remark that by a solution of the equation $AXB = C$, for $A \in \mathcal{B}(\mathcal{H}_4, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we mean an operator $X \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ which satisfies (4.2). Equation (4.2) was considered for the first time in 2010 in [80] in general case. Authors treated (4.2), first if A, B or C has closed range, and second in general case. It is interesting to mention that if just one of the three operators, A, B or C, has closed range, the solvability of the equation $AXB = C$ is equivalent with Penrose's condition (4.3), while if neither A, B nor C has closed range then the Penrose's assertion fails. For example, if $A = B = C$ and $\mathcal{R}(A)$ is not closed then $AA^-AA^-A = A$, while Equation (4.2) it is not solvable, because an inner inverse of A must be unbounded.

Theorem 4.2.1 [*80*] *Let $A \in \mathcal{B}(\mathcal{H}_4, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. If $\mathcal{R}(A), \mathcal{R}(B)$ or $\mathcal{R}(C)$ is closed then the following conditions are equivalent:*

(1) *There exists $X \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ such that $AXB = C$,*

(2) *$AA^-CB^-B = C$ for every inner inverses, A^-, B^-, of A and B, respectively,*

(3) *$\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$.*

In the general case we have the following result.

Theorem 4.2.2 [*80*] *Let $A \in \mathcal{B}(\mathcal{H}_4, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. The following conditions are equivalent:*

(1) *There exists $X \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ such that $AXB = C$,*

(2) *$\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}\left((A^\dagger C)^*\right) \subseteq \mathcal{R}(B^*)$,*

(3) *$\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and there exists $Y \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ such that $YB = A^\dagger C$.*

In the proofs of previous two results Lemma 4.2.1 and well-known Douglas lemma are used. Douglas lemma [48] relates factorization, range inclusion, and majorization of Hilbert space operators. It has been extensively used in operator theory, in particular in studies of division and quotients of operators (see [158, 71]), operator range inclusions [55], and operator inequalities [82]. In the context of Banach spaces results around the Douglas lemma are given in a recent paper [7]. It also has been generalized for unbounded operator case, Krein space operator and Hilbert modules case (see [57, 119, 54]).

Theorem 4.2.3 [*48*] *Let $A, B \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent:*

(1) *$\mathcal{R}(A) \subseteq \mathcal{R}(B)$.*

(2) *$AA^* \leq \lambda^2 BB^*$ for some constant $\lambda > 0$.*

(3) *$A = BC$, for some $C \in \mathcal{B}(\mathcal{H})$.*

If the equivalent condition $(1)-(3)$ *hold, then there is a unique operator* C *such that*

(a) $\|C\|^2 = \inf\{\lambda^2 : AA^* \leq \lambda^2 BB^*\}$

(b) $\mathcal{N}(A) = \mathcal{N}(C)$

(c) $\mathcal{R}(C) \subseteq \overline{\mathcal{R}(B^*)}$

This solution is called the Douglas reduced solution and can be expressed as B^+A.

There are many papers considering some special subclasses of the solutions of different types of the equations. Existence of a nonnegative definite solution of the matrix equation $AXB = C$ first was considered by Khatri and Mitra [73] in 1976 in terms of range conditions.

Theorem 4.2.4 [73] *Let* $A, B, C \in \mathbb{C}^{n\times n}$ *be such that the equation* $AXB = C$ *is solvable. There exists a nonnegative definite solution* $X \in \mathbb{C}^{n\times n}$ *of the equation* $AXB = C$ *if and only if*

$$r(B(A+B)^-C(A+B)^-A) = r(A(A+B)^-C^*) = r(B(A+B)^-C).$$

Evidently in the settings of operators this problem looks much more difficult. The special case when $B = A^*$ (see [41]) of this question was considered for the operator case, and one generalization was presented by Xu et al. [156] under some regularity conditions and special condition that $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(A^*)}$. Arias [80] succeeded to remove regularity conditions but not the condition $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(A^*)}$.

Theorem 4.2.5 [80] *Let* $A \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$*,* $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$*,* $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *be such that* $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(A^*)}$. *If the equation* $AXB = C$ *is solvable then the following conditions are equivalent:*

(1) *The exists* $X \in \mathcal{B}(\mathcal{H}_3)^+$ *such that* $AXB = C$.

(2) *The exists* $Y \in \mathcal{B}(\mathcal{H}_3)^+$ *such that* $YB = A^\dagger C$.

(3) $B^*A^\dagger C \geq 0$ *and* $\mathcal{R}\left((A^\dagger C)^*\right) \subseteq \mathcal{R}\left((B^*A^\dagger C)^{1/2}\right)$.

If one of these conditions holds and we consider the matrix operator decomposition induced by $\overline{\mathcal{R}(A^*)}$ *then the general form of the positive solutions is:*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & \left(\left(X_{11}^{1/2}\right)^\dagger X_{12}\right)^* \left(X_{11}^{1/2}\right)^\dagger X_{12} + F \end{bmatrix},$$

where $X_{11} = P_{\overline{\mathcal{R}(A^*)}} Y|_{\overline{\mathcal{R}(A^*)}}$ *with* $Y \in \mathcal{B}(\mathcal{H}_3)$ *a positive solution of* $YB = A^\dagger C$, $\mathcal{R}(X_{12}) \subseteq \mathcal{R}\left(X_{11}^{1/2}\right)$ *and* $F \in \mathcal{B}(\mathcal{N}(A))$ *is positive.*

The existence of a nonnegative definite solution of $AXB = C$ was considered in [53] in the case of Hilbert C^*-modules under the assumption that $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(A^*)}$.

However in the case of the equation $XB = C$ we have a complete answer obtained for the first time by Sebestyén [121]:

Theorem 4.2.6 *[121] Let $B, C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that the equation $XB = C$ has a solution. There exists a positive solution if and only if $C^*C \leq \lambda B^*C$, for some constant $\lambda \geq 0$.*

Theorem 4.2.6 was used in proving Theorem 4.2.5 too. On the other side, in [42] a more general result from [121] which concerns necessary and sufficient conditions for the existence of a nonnegative definite linear operator on Hilbert space whose restriction to a subset of this space is given, was used.

Theorem 4.2.7 *[121] Let $\mathcal{H}$ be a (complex) Hilbert space, $\mathcal{H}_0$ its subset, and b a function on $\mathcal{H}_0$ with values in $\mathcal{H}$. There exists $B \in \mathcal{B}(\mathcal{H})^+$ with restriction to $\mathcal{H}_0$ identical to b if and only if*

$$\left\| \sum_h c_h b(h) \right\|^2 \leq M \left\langle \sum_h c_h b(h), \sum_h c_h h \right\rangle, \tag{4.50}$$

holds with some constant $M \geq 0$, for any finite sequence $\{c_h\}_{h \in \mathcal{H}_0}$ of complex numbers indexed by elements of $\mathcal{H}_0$. In this case, $\|B\| \leq M$.

Using Douglas and Sebestyén's lemmas 4.2.3 and 4.2.7, Cvetković-Ilić and al. gave an equivalent conditions for the existence of a nonnegative definite solution of the operator Equation (4.2) as well as a general form of a nonnegative definite solution under the condition of regularity of $B^*(I - A^\dagger A)$, which is weaker than the condition $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(A^*)}$.

Theorem 4.2.8 *[42] Let $A \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$, $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be such that the equation $AXB = C$ is solvable and $\mathcal{R}(B^*(I - A^\dagger A))$ is closed. There exist a nonnegative definite solution of the equation $AXB = C$ if and only if*

(1) $(I - P)B^*A^\dagger C(I - P) \geq 0$,

(2) $\mathcal{R}((I - P)B^*A^\dagger C) \subseteq \mathcal{R}(K)$,

(3) $\mathcal{R}\left((I - P)(A^\dagger C)^*\right) \subseteq \mathcal{R}(K)$,

where $P = B^*(I - A^\dagger A)(B^*(I - A^\dagger A))^\dagger$ and $K = \left((I - P)B^*A^\dagger C(I - P)\right)^{1/2}$. If the equation $AXB = C$ has a nonnegative definite solution set $Q = (B^*(I - A^\dagger A))^\dagger B^*(I - A^\dagger A)$ and $Z_{12} = (I - P)B^*A^\dagger CP$. Then any nonnegative definite solution X of $AXB = C$ is given on $\mathcal{R}(B) \oplus \mathcal{R}(B)^\perp$ by

$$X = A^\dagger CB^\dagger + (B^*(I - A^+A))^\dagger(Z - B^*A^\dagger C)B^\dagger + W - (A^\dagger A + Q)WBB^\dagger,$$

where

$$Z = (I-P)B^*A^\dagger C + P(B^*A^\dagger C)^*(I-P) + (K^\dagger Z_{12})^* K^\dagger Z_{12} + PFP,$$

and $W \in \mathcal{B}(\mathcal{H}_3)$ *is such that* $\mathcal{R}\left(((I-Q)(I-A^\dagger A)WB)^*\right) \subseteq \mathcal{R}(K + (K^\dagger Z_{12})^* + (PFP)^{1/2})$, $F \in \mathcal{B}(\mathcal{H}_1)^+$ *and* $\mathcal{R}\left(P(A^\dagger C)^* - (K^\dagger Z_{12})^* K^\dagger (A^\dagger C)^*)\right) \subseteq \mathcal{R}\left((PFP)^{1/2}\right)$.

The research from [42] can be continued. The idea presented in [42] can be used with some modifications to generalize the result from [53] in the case of Hilbert C^*-modules without any range condition. Also, interesting application of Theorem 4.2.8 can be obtained in finding necessary and sufficient conditions for the existence of a nonnegative definite solution of the system of equations (4.5).

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Biography

Jovana Milošević was born on December 9th, 1991, in Niš, Serbia. She completed Učitelj Tasa Elementary School in Niš in 2006 as valedictorian, and Bora Stanković Grammar School in 2010. In the school year 2010/2011, she entered the Faculty of Sciences and Mathematics, University of Niš, at the Department of Mathematics, and got a bachelors degree in 2013 with a grade point average of 9.96/10 and a masters degree in 2015 with a grade point average of 10/10. In 2009/2010, she enrolled in PhD studies at the Department of Mathematics, the Faculty of Sciences and Mathematics, University of Niš, and passed all exams with a grade point average of 10/10.

Since 2015, Jovana has been working at the Faculty of Sciences and Mathematics in Niš in the Department of Mathematics, one year as a teaching associate, after that as a teaching assistant. She works on courses Intro to Algebraic Structures, Calculus II, Topics in High School Mathematics II, Mathematics II (Department of Physics) with undergraduate students, and Algebraic Structures, Generalized Inverses of matrices on graduate level. She participates as a researcher on a project Problems in nonlinear analysis, operator theory, topology and applications, No.174205 supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia. She also took a part in international projects Austria-Serbian Bilateral Research Project: Generalized inverses, symbolic computation and operator algebras (2016-2017) and China-Serbian Bilateral Research Project: The theory of tensor, operator matrices and applications (2018-2020). In 2016. she began to work in primary school with mathematically-talented children in Svetozar Marković Grammar School. She published four and has two papers under review in international journals with IF.

Publications

1. D.S. Cvetković-Ilić, J. Milošević, *Fredholmness of a linear combination of operators*, Journal of Math. Anal. Appl. 458(1) (2018), 555–565. (**M21**)

2. D.S. Cvetković-Ilić, J. Milošević, *Reverse order laws for* $\{1,3\}$*-generalized inverses*, Linear and Multilinear Algebra, 67(3) (2019), 613–624. (**M21**)

3. J.S. Milošević, *Hartwig's triple reverse order law in* C^**-algebras*, Filomat 32 (12) (2019), 4229–4232. (**M22**)

4. J.S. Milošević, *Algebraic conditions for the solvability of system of three linear*

equations in a ring, Linear and Multilinear Algebra, https://doi.org/10.1080/03081087.2020.1743634 (published online). (**M22**)

5. D.S. Cvetković-Ilić, C. Hofstadler, J.H. Poor, J. Milošević, C.G. Raab, G. Regensburger, *Algebraic proof methods for identities of matrices and operators: improvements of Hartwig's triple reverse order law*, (submitted).

6. D.S. Cvetković-Ilić, J. Milošević, *Different improvements on the reverse order laws*, (submitted).

ИЗЈАВА О АУТОРСТВУ

Изјављујем да је докторска дисертација, под насловом

DIFFERENT INVERTIBILITY MODIFICATIONS IN OPERATOR SPACES AND C*-ALGEBRAS AND ITS APPLICATIONS

која је одбрањена на Природно-математичком факултету Универзитета у Нишу:

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У Нишу, 18.05.2020.

Потпис аутора дисертације:

Јована Милошевић

Јована С. Милошевић

ИЗЈАВА О ИСТОВЕТНОСТИ ШТАМПАНОГ И ЕЛЕКТРОНСКОГ ОБЛИКА ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

Наслов дисертације:

DIFFERENT INVERTIBILITY MODIFICATIONS IN OPERATOR SPACES AND C*-ALGEBRAS AND ITS APPLICATIONS

Изјављујем да је електронски облик моје докторске дисертације, коју сам предала за уношење у **Дигитални репозиторијум Универзитета у Нишу,** истоветан штампаном облику.

У Нишу, 18.05.2020.

Потпис аутора дисертације:

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ИЗЈАВА О КОРИШЋЕЊУ

Овлашћујем Универзитетску библиотеку „Никола Тесла“ да у Дигитални репозиторијум Универзитета у Нишу унесе моју докторску дисертацију, под насловом:

DIFFERENT INVERTIBILITY MODIFICATIONS IN OPERATOR SPACES AND C*-ALGEBRAS AND ITS APPLICATIONS

Дисертацију са свим прилозима предала сам у електронском облику, погодном за трајно архивирање.

Моју докторску дисертацију, унету у Дигитални репозиторијум Универзитета у Нишу, могу користити сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons), за коју сам се одлучила.

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У Нишу, 19.05.2020.

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