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Sandwich semigroups in locally small categories

Sendvič polugrupe u lokalno malim kategorijama

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Abstract

Let S be a locally small category, and fix two (not necessarily distinct) objects i, j in S. Let S_{ij} and S_{ji} denote the set of all morphisms $i \to j$ and $j \to i$, respectively. Fix $a \in S_{ji}$ and define $(S_{ij}, \star_a)$, where $x \star_a y = xay$ for $x, y \in S_{ij}$. Then, $(S_{ij}, \star_a)$ is a semigroup, known as a *sandwich semigroup*, and denoted by S^a_{ij}. In this thesis, we conduct a thorough investigation of sandwich semigroups (in locally small categories) in general, and then apply these results to infer detailed descriptions of sandwich semigroups in a number of categories.

Firstly, we introduce the notion of a partial semigroup, and establish a framework for describing a category in "semigroup language". Then, we prove various results describing Green's relations and preorders, stability and regularity of S^a_{ij}. In particular, we emphasize the relationships between the properties of the sandwich semigroup and the properties of the category containing it. Also, we highlight the significance of the properties of the sandwich element a. In this process, we determine a natural condition on a called *sandwich regularity* which guarantees that the regular elements of S^a_{ij} form a subsemigroup tightly connected to certain non-sandwich semigroups. We explore these connections in detail and infer major structural results on $\mathrm{Reg}(S^a_{ij})$ and the generation mechanisms in it. Finally, we investigate ranks and idempotent ranks of the regular subsemigroup $\mathrm{Reg}(S^a_{ij})$ and idempotent-generated subsemigroup $\mathbb{E}(S^a_{ij})$ of S^a_{ij}. In general, we are able to infer expressions for lower bounds for these values. However, we show that in the case when $\mathrm{Reg}(S^a_{ij})$ is MI-dominated (a property which has to do with the "covering power" of certain local monoids), the mentioned lower bounds are sharp.

We apply the general theory to sandwich semigroups in various transformation categories (partial maps $\mathcal{PT}$, injective maps $\mathcal{I}$, totally defined maps $\mathcal{T}$, and matrices $\mathcal{M}(\mathbb{F})$ – corresponding to linear transformations of vector spaces over a field $\mathbb{F}$) and diagram categories (partition $\mathcal{P}$, planar partition $\mathscr{P}\mathcal{P}$, Brauer $\mathcal{B}$, partial Brauer $\mathcal{PB}$, Motzkin $\mathscr{M}$, and Temperley-Lieb $\mathcal{TL}$ categories), one at a time. In each case, we investigate the partial semigroup itself in terms of Green's relations and regularity and then focus on a sandwich semigroup in it. We apply the general results to thoroughly describe its structural and combinatorial properties. Furthermore, since in each category that we consider all elements are sandwich-regular, we may apply the theory concerning the regular subsemigroup in all of these cases. In particular, $\mathrm{Reg}(S^a_{ij})$ turns out to be tightly connected to a certain non-sandwich monoid for each category S we consider, and we are able to describe $\mathrm{Reg}(S^a_{ij})$ and $\mathbb{E}(S^a_{ij})$. However,

we conduct the combinatorial part of the investigation only for the sandwich semigroups in transformation categories ($\mathcal{PT}$, $\mathcal{I}$, $\mathcal{T}$, and $\mathcal{M}(\mathbb{F})$) and sandwich semigroups in the Brauer category $\mathcal{B}$ since only these have MI-dominated regular subsemigroups (and some other properties that make them more amenable to investigation). For these sandwich semigroups, we enumerate regular Green's classes and idempotents, and we calculate the ranks (and idempotent ranks, where appropriate) of $\mathrm{Reg}(S^a_{ij})$, $\mathbb{E}(S^a_{ij})$ and S^a_{ij}.

Izvod

Neka je S lokalno mala kategorija. Fiksirajmo proizvoljne (ne nužno različite) objekte i i j iz S. Neka S_{ij} i S_{ji} označavaju skupove svih morfizama $i \to j$ i $j \to i$, redom. Fiksirajmo morfizam $a \in S_{ji}$ i definišimo strukturu $(S_{ij}, \star_a)$, gde je $x \star_a y = xay$ za sve $x, y \in S_{ij}$. Tada je $(S_{ij}, \star_a)$ *sendvič polugrupa*, koju označavamo sa S_{ij}^a. U tezi ćemo sprovesti detaljno ispitivanje sendvič polugrupa (u lokalno maloj kategoriji) u opštem slučaju, a zatim ćemo primeniti dobijene rezultate u cilju opisivanja sendvič polugrupa u konkretnim kategorijama.

Najpre uvodimo pojam parcijalne polugrupe i postavljamo osnovu koja nam omogućava da opišemo kategoriju na "jeziku polugrupa". Zatim slede brojni rezultati koji opisuju Grinove relacije i poretke, kao i stabilnost i regularnost polugrupe $(S_{ij}, \star_a)$. Tu posebno ističemo veze između osobina sendvič polugrupe i parcijalne polugrupe koja je sadrži. Takođe, posebnu pažnju posvećujemo uticaju sendvič elementa a na osobine sendvič polugrupe $(S_{ij}, \star_a)$. Kao najbitniji primer se izdvaja osobina *sendvič-regularnosti*; naime, dokazujemo da, ako je a sendvič-regularan, onda regularni elementi iz S_{ij}^a formiraju podgrupu koja je usko povezana sa određenim "ne-sendvič" polugrupama. U tezi detaljno ispitujemo te veze i dobijamo važne rezultate o strukturi polugrupe $\mathrm{Reg}(S_{ij}, \star_a)$ i mehanizmima generisanja u njoj. Za kraj, ispitujemo rangove i idempotentne rangove regularne potpolugrupe $\mathrm{Reg}(S_{ij}, \star_a)$ i idempotentno-generisane potpolugrupe $\mathbb{E}(S_{ij}, \star_a)$. U opštem slučaju možemo dati donja ograničenja za ove vrednosti. Međutim, u slučaju kada je regularna polugrupa $\mathrm{Reg}(S_{ij}, \star_a)$ MI-dominirana (što znači da je određeni lokalni monoidi pokrivaju), ta donja ograničenja su dostignuta.

U ostatku teze, primenjujemo opštu teoriju na sendvič polugrupe u brojnim kategorijama transformacija (parcijalne funkcije $\mathcal{PT}$, injektivne parcijalne funkcije $\mathcal{I}$, potpuno definisane funkcije $\mathcal{T}$ i matrice $\mathcal{M}(\mathbb{F})$, koje predstavljaju linearne transformacije vektorskih prostora nad poljem $\mathbb{F}$) i kategorijama dijagrama (particije $\mathcal{P}$, planarne particije $\mathscr{P}\mathcal{P}$, Brauerove $\mathcal{B}$, parcijalne Brauerove $\mathcal{PB}$, Mockinove $\mathscr{M}$, i Temperli-Lib $\mathcal{TL}$ particije). U svakom od ovih slučajeva, prvo istražujemo parcijalnu polugrupu iz aspekta Grinovih relacija i regularnosti, a zatim se fokusiramo na (proizvoljnu) sendvič polugrupu u njoj. Pri tome, primenjujemo opšte rezultate da bismo detaljno opisali njenu strukturu i kombinatorne osobine. Osim toga, u svim slučajevima primenjujemo i teoriju vezanu za regularnu potpolugrupu, pošto su svi elementi u našim kategorijama sendvič-regularni. To znači da je u svakoj kategoriji S koju razmatramo, $\mathrm{Reg}(S_{ij}, \star_a)$ usko povezana sa određenim monoidom, i preko te veze možemo opisati polugrupe $\mathrm{Reg}(S_{ij}, \star_a)$ i $\mathbb{E}(S_{ij}, \star_a)$. Ipak, kombinatorni deo ispitivanja sprovodimo samo za sendvič polugrupe u kategorijama transformacija ($\mathcal{PT}$, $\mathcal{I}$, $\mathcal{T}$ i $\mathcal{M}(\mathbb{F})$) i sendvič polugrupe u Brauerovoj kategoriji $\mathcal{B}$, pošto samo one imaju MI-dominirane regularne potpolugrupe (i još neke osobine koje ih čine pogodnijim za ispitivanje). U ovim sendvič polugrupama računamo broj regularnih Grinovih klasa i idempotenata, i izračunavamo rangove (i idempotentne rangove, ako postoje) polugrupa $\mathrm{Reg}(S_{ij}, \star_a)$, $\mathbb{E}(S_{ij}, \star_a)$ i S_{ij}^a.

Preface

In the Serbian education system, a PhD candidate is required to publish at least one scientific article related to the topic of the thesis in order to be allowed to defend that thesis. In fact, it is a common practice to publish all the results in scientific journals first, and then to compile them in a thesis. For this reason, we present here results that have already been published in [33], [34] and [28]. The goal was to give a detailed and comprehensive account of the properties of sandwich semigroups in general (from [33]), and then to present the applications and further results obtained in [34] and [28]. To supplement this material, we also present the results from [30], using the theory and techniques developed in [33] (and simplifying the proofs significantly).

The field of sandwich semigroups, to which this thesis belongs, was born in the '50s, and it developed into an important area of research, due to the variety of fields in which sandwich operations arise naturally. Of course, this led to a number of related articles (see Section 1.1). However, until recently, the results were situation-specific, and there was no attempt to create a unifying theory which will apply to all sandwich semigroups, not depending on the category, or on the type of the underlying hom-set. So, aside from the scientific contribution of the articles [33], [34], and [28], the thesis contributes to the field in compiling the recent results of general type, along with numerous results in the domain of the combinatorial theory of semigroups. Moreover, we conduct a thorough investigation of the sandwich semigroups in four transformation categories (partial maps $\mathcal{PT}$, injective maps $\mathcal{I}$, "classical" maps $\mathcal{T}$, and matrices $\mathcal{M}(\mathbb{F})$ over a field $\mathbb{F}$) and six diagram categories (partition $\mathcal{P}$, planar partition $\mathscr{P}\mathcal{P}$, Brauer $\mathcal{B}$, partial Brauer $\mathcal{PB}$, Motzkin $\mathscr{M}$, and Temperley-Lieb $\mathcal{TL}$), which provides valuable insight and offers illustrative examples to demonstrate the differences and similarities among the sandwich semigroups of these types.

The thesis is organised as follows. In Chapter 1, we give the historical background for our topic and provide a short introduction, presenting the notions, notation and results needed for understanding the rest of the thesis. Then, in Chapter 2, we present the general theory developed in [30], [33] and [28]. We introduce partial semigroups and the notions needed to describe their structure (such as Green's relations, regular elements, etc.). Next, we focus on a fixed sandwich semigroup S_{ij}^a: we characterise its Green's relations and preorders, and then study structural issues such as regularity, stability, and generation. Then, we introduce the condition of sandwich-regularity on the sandwich element, and under that assumption investigate the structure of S_{ij}^a and $\mathrm{Reg}(S_{ij}^a)$ via the connection to certain non-sandwich semigroups. In particular, if a is sandwich-regular, there exists $b \in S_{ij}$ with $a = aba$ and $b = bab$, and we prove that the regular subsemigroup $\mathrm{Reg}(S_{ij}^a)$ is a pullback product of certain regular subsemigroups of S_{ii} and S_{jj}, and is also closely related to a certain regular submonoid of S_{ji}^b. This allows us to describe the structure of $\mathrm{Reg}(S_{ij}^a)$ and the idempotent-generated subsemigroup $\mathbb{E}(S_{ij}^a)$. In Section 2.4, we introduce the condition of MI-domination, and show its ties to the issues of generation; namely, we present lower bounds for $\mathrm{rank}(\mathrm{Reg}(S_{ij}^a))$ and $\mathrm{rank}(\mathbb{E}(S_{ij}^a))$, which turn out to be sharp if $\mathrm{Reg}(S_{ij}^a)$ is MI-dominated. We end the chapter by presenting some results concerning inverse categories and the rank of a sandwich semigroup.

In Chapter 3, which is based on [33], we apply the results presented in the previous chapter. Section 3.1 is dedicated to the partial semigroup $\mathcal{PT}$ and sandwich semigroups in it. First, we study Green's relations, regularity and stability in $\mathcal{PT}$ as well as invertibility and the combinatorial structure of a hom-set $\mathcal{PT}_{XY}$. Then, we focus on the sandwich semigroup $\mathcal{PT}_{XY}^a$. We describe its Green's relations and preorders, regularity and stability; we show that the regular subsemigroup $\mathrm{Reg}(\mathcal{PT}_{XY}^a)$ is a kind of "inflation" of $\mathcal{PT}_A$, where A is the image of the sandwich element a. This allows us to describe the combinatorial structure of $\mathrm{Reg}(\mathcal{PT}_{XY}^a)$, and the elements of the idempotent-generated subsemigroup $\mathbb{E}(\mathcal{PT}_{XY}^a)$. Further, we show that $\mathrm{Reg}(\mathcal{PT}_{XY}^a)$ is always MI-dominated, so we are able to calculate $\mathrm{rank}(\mathrm{Reg}(S_{ij}^a))$ and $\mathrm{rank}(\mathbb{E}(\mathcal{PT}_{XY}^a))$. Finally, we obtain formulae for $\mathrm{rank}(\mathcal{PT}_{XY}^a)$ depending on the properties of a. In Sections 3.2 and 3.3, the same program is also carried out for the sandwich semigroups in partial semigroups $\mathcal{T}$ and $\mathcal{I}$, respectively.

In Chapter 4, we present the results of [30], but we prove them as applications of the theory from Chapter 2. Following the program established in the previous chapter, we investigate the partial semigroup $\mathcal{M}(\mathbb{F})$ of all matrices over a field $\mathbb{F}$, the sandwich semigroup $\mathcal{M}_{mn}^A(\mathbb{F})$, its regular subsemigroup $\mathrm{Reg}(\mathcal{M}_{mn}^A(\mathbb{F}))$ (proving that it is an inflation of $\mathcal{M}_{\mathrm{rank}(A)}(\mathbb{F})$) and idempotent-generated subsemigroup $\mathbb{E}(\mathcal{M}_{mn}^A(\mathbb{F}))$. We are able to prove that $\mathrm{Reg}(\mathcal{M}_{mn}^A(\mathbb{F}))$ is always MI-dominated, and so we obtain the formulae for the ranks of all these semigroups.

Finally, Chapter 5 is dedicated to diagram categories. Again, we follow the same program of investigation for the partial semigroups $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{M}$, and $\mathcal{TL}$, as far as we are able to. In particular, we describe structural and combinatorial properties: Green's relations and preorders, regularity, stability, mid-identities and idempotent-generated subsemigroups. However, it turns out that the sandwich semigroups in $\mathcal{B}$ differ substantially from the sandwich semigroups in the other diagram categories we study. For instance, in $\mathcal{B}$ we always have MI-domination in the regular subsemigroup, while in the others we do not. Hence, in $\mathcal{B}_{mn}^\alpha$ we are able to conduct a more thorough investigation: we include results on isomorphism classification, the combinatorial structure of the regular subsemigroup, enumeration of idempotents, and the ranks of $\mathrm{Reg}(\mathcal{B}_{mn}^\alpha)$, $\mathbb{E}(\mathcal{B}_{mn}^\alpha)$ and $\mathcal{B}_{mn}^\alpha$.

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Chapter 1

Introduction

Here, we give some historical background on sandwich semigroups. For more information and additional references, the reader is advised to consult the introductions to the articles [28–30, 33, 34], which were the primary sources for the first section. In the second part of the chapter, we introduce the theory and notation needed for understanding the content of the thesis.

1.1 The story of sandwich semigroups

The idea of sandwich semigroups is based on the notion of a sandwich operation. This type of operation arises naturally in the theory of semigroups. Indeed, it is essential in the structural theory of finite semigroups, and its first appearance can be traced back to this particular field. Namely, any finite semigroup can be decomposed (see [58], section 3.1) in a certain way into principal factors, which are always semigroups of one of the two following types: completely 0-simple semigroups or zero-semigroups. As shown by Rees [106] in 1940, any completely 0-simple semigroup is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(\Lambda, I, P; G)$, in which multiplication of nonzero elements depends on a *sandwich matrix* P. More precisely (but without going into too much detail),

$$(i, g_1, \lambda)(j, g_2, \kappa) = (i, g_1 \cdot p_{\lambda j} \cdot g_2, \kappa),$$

where $p_{\lambda j}$ is the (λ, j)-element of the matrix P. The term *"sandwich operation"* clearly stems from the fact that g_1 and g_2 are not simply multiplied, but are combined into a "sandwich" with $p_{\lambda j}$. In fact, the same term is used in all cases where we introduce a new type of binary multiplication, based on some "simpler" (binary) one, in the following manner: insert an element between the factors and multiply all three of them via the "base" multiplication rule.

Naturally, *sandwich semigroups* are semigroups whose multiplication is a sandwich operation. The first time such a semigroup was considered (at least indirectly) was in a 1955 article [100], by Munn. Probably motivated by Rees' work, he investigated rings of $m \times n$ matrices, with matrix addition and sandwich multiplication

$$X \circ Y = XPY,$$

where the sandwich matrix P had a form prescribed by Rees' theorem. This article became extremely influential in the theory of semigroup representations (see for example [3, 21, 22, 51, 61, 75, 94, 104, 112, 113] and monographs [103, 107, 114]), so these rings became known as Munn rings.

In the same year, in [11], Brown considered such rings as well (he named these structures *generalised matrix algebras*), but since he was motivated by a connection with classical groups (see [10, 12, 128]), he did not restrict the form of the sandwich element. This paper also had a profound impact on the development of representation theory, which can be seen in [35, 47, 50, 73, 74, 80, 121, 129, 130].

Finally, five years after that, sandwich semigroups were mentioned on their own merits for the first time. Namely, Lyapin, in his monograph [82], introduced a few interesting semigroup constructions, and among them the following type of sandwich semigroups: for any two non-empty sets V, W let $\mathcal{T}_{VW}$ denote the set of all mappings from V to W; if we fix V, W and an arbitrary function $\theta \in \mathcal{T}_{WV}$, then $\mathcal{T}^{\theta}_{VW} = (\mathcal{T}_{VW}, \star_\theta)$, where

$$f \star_\theta g = f \circ \theta \circ g, \text{ for all } f, g \in \mathcal{T}_{VW},$$

is a *sandwich semigroup of functions.* Note that, if V and W are vector spaces over the same field and we restrict our attention to the set of linear transformations $V \to W$ (denoted $\mathcal{L}_{VW}$) and fix a linear transformation $\theta \in \mathcal{L}_{WV}$, we arrive at a *sandwich semigroup of linear transformations* (or equivalently *linear sandwich semigroup*) $\mathcal{L}^{\theta}_{VW} = (\mathcal{L}_{VW}, \star_\theta\restriction_{\mathcal{L}_{VW}})$. This semigroup is isomorphic to a sandwich semigroup of matrices (see chapter 4) and therefore also isomorphic to the underlying multiplicative semigroup of the corresponding generalised matrix algebra.

Magill was the first to actually investigate any type of sandwich semigroups; in [84], which appeared in 1967, he studied $\mathcal{T}^{\theta}_{VW}$. This paper was followed by two more articles on the same topic which he wrote with Subbiah, [86, 87], and an article [116] from Sullivan on sandwich semigroups of partial functions.

The 80's brought some fresh ideas in the field, when Hickey published [53] and [54], where he introduced and investigated a new type of sandwich semigroups − a variant of a semigroup: for a semigroup S and any element $a \in S$, the semigroup $S^a = (S, \star_a)$, where

$$b \star_a c = bac, \quad \text{for all } b, c \in S,$$

is the *variant* of semigroup S corresponding to the element a.

Thus, two main directions in the studies of sandwich semigroups were formed:

1. investigation of sandwich semigroups within a fixed category (e.g. sandwich semigroups of matrices, sandwich semigroups of functions) and

2. investigation of variants of semigroups.

Both of these topics have induced considerable interest, which can be seen in more recent articles, in particular. There have been papers on sandwich semigroups of functions: [15, 96, 118, 126], on linear sandwich semigroups: [20, 66, 71, 97], on sandwich semigroups of binary relations: [16–18, 120] and on variants of semigroups: [72, 123–125]. Moreover, in the monograph [45], a whole chapter was devoted to variants of various kinds of transformation semigroups. However, all these texts deal primarily with structural properties such as Green's relations, (von Neumann) regularity, ideals, classification up to isomorphism, and so on. In [29], Dolinka and East have undertaken a different task: they have investigated variants of a finite full transformation semigroup from the perspective of combinatorial semigroup theory. This project required further development of the general theory of variants. Inspired by this study, in 2016 they wrote another paper, [30], studying the same problems for sandwich semigroups of matrices (i.e. linear sandwich semigroups), for which they proved a number of general results concerning sandwich semigroups. In particular, they have introduced the notions of a partial semigroup and a sandwich semigroup in a locally small category, which incorporates all the different forms of sandwich semigroups previously mentioned. The results of this paper motivated the authors to study sandwich semigroups of (totally defined, partial and injective) transformations from the same point of view, in cooperation with other authors (among them is the author of this thesis). This, in turn, prompted further investigation of sandwich semigroups in general (that is, in locally small categories), and the whole project resulted in papers [33] and [34]. In the first one, we give an in-depth investigation of sandwich semigroups in locally small categories and their combinatorial properties. These results are applied to the sandwich semigroups of transformations in the second one, and additional theory and calculations concerning this special case are provided, as well. The same idea has driven the creation of [28], in which we study sandwich semigroups of diagrams.

In this thesis, we hope to compile the results of [33], [30], [34], and [28], by giving a comprehensive base from the first paper, subsequently applying it and further developing the results for the sandwich semigroups of transformations, matrices, and diagrams, respectively.

These studies (and this thesis) might prove extremely beneficial not only for semigroup theory in general but also in any field in which semigroup operations arise naturally:

- representation theory [50, 100],
- classical groups [11],
- category theory [99],
- automata theory [16, 17],
- topology [85, 87],
- computational algebra [37], and more.

1.2 Basics

Here, we give a short base for understanding the content.

Remark 1.2.1. It is important to point out that we work in the universe of the standard ZFC theory (see Chapters 1,5 and 6 of [62]).

A *groupoid* is an ordered pair $(G, \cdot)$ consisting of a non-empty set G, and a binary operation $\cdot$ on G (in other words, a $G \times G \to G$ function). If the exact operation is either known or implied or not essential for our discussion, we usually make an omission and denote that groupoid by G. To further shorten the notation, we may even leave out the sign of the operator in expressions; for instance, $a \cdot b$ will be denoted ab.

A *semigroup* is a groupoid G, which has the *associative* property:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad \text{for all } a, b, c \in G. \tag{1.1}$$

If an element e of a groupoid G satisfies

- $a \cdot e = a$ for each $a \in G$, it is a *right-identity* of the groupoid G;
- $e \cdot a = a$ for each $a \in G$ it is a *left-identity* of the groupoid G;
- $a \cdot e = e \cdot a = a$ for each $a \in G$, it is a *(two-sided) identity* of the groupoid G.

A *monoid* is a semigroup possessing a two-sided identity. It can be easily shown that such an identity is unique, if it exists.

Let S be a semigroup with a left-identity e_l. If $a \cdot b = e_l$ holds in S, we say that a is a *left-inverse* of b (i.e. b is *left-invertible*). In the case that e_r is a right-identity of S and $a \cdot b = e_r$, we say that b is a *right-inverse* of a (i.e. a is *right-invertible*). Moreover, if monoid M satisfies the following:

> for each $a \in M$ there exists $b \in M$, which is both a left- and a right-inverse for a,

then M is a *group*. If it exists, the element b can be shown to be unique for a fixed a; we call it the *inverse* of a and denote it a^{-1} (in this case, we say a is *invertible*). It is important to mention that there is a different notion of an inverse element for semigroups, and we will use that one exclusively from a certain point on; but, for now, by inverse we mean a group inverse.

If a group G has the *commutative* property:

$$a \cdot b = b \cdot a, \quad \text{for all } a, b \in G,$$

it is an *Abelian* group. An important example of a group is the *symmetric group* on a set X, S_X, whose elements are precisely all the permutations of the set X, and the operation is composition. More about symmetric groups will be said later.

For any listed type of structure, groupoid/semigroup/monoid/group, we define the term of the substructure of the corresponding type: *subgroupoid/ subsemigroup/submonoid/subgroup*, whose elements constitute a subset of some bigger structure S and form a groupoid/semigroup/monoid/group, under the operation of S. Such a substructure is *trivial* if it contains either all the elements of S or only the identity (of course, this is possible only in the case of monoids and groups).

A more complex structure, *field*, is a 3-tuple $(F, +, \cdot)$ consisting of a set F and two operations on it, such that $(F, +)$ is an Abelian group with identity e, $(F \setminus \{e\}, \cdot)$ is also an Abelian group and the second operation is *distributive* over the first:

$$(a + b) \cdot c = a \cdot c + b \cdot c, \quad \text{for all } a, b, c \in G \quad \text{and}$$
$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad \text{for all } a, b, c \in G.$$

Having covered some of the basic algebraic structures, now we turn to notions and notation concerning relations and functions. Given arbitrary sets X and Y, their *direct product* is the set

$$X \times Y = \{(x, y) \ : \ x \in X, y \in Y\}.$$

A *binary relation* on a non-empty set X is any subset of the direct product $X \times X$. As the elements of such a relation σ are ordered pairs, we often simplify the notation and write $x\sigma y$ instead of $(x, y) \in \sigma$. A binary relation σ on X is

reflexive if $x\sigma x$ for all $x \in X$;

symmetric if $x\sigma y$ implies $y\sigma x$ for all $x, y \in X$;

antisymmetric if $x\sigma y$ and $y\sigma x$ together imply $x = y$ for all $x, y \in X$;

transitive if $x\sigma y$ and $y\sigma z$ together imply $x\sigma z$ for all $x, y, z \in X$.

If a binary relation is reflexive, antisymmetric, and transitive, it is a *partial order*. If σ is a partial order on X and any two elements $x, y \in X$ are in a relation (i.e. we have $x\sigma y$ or $y\sigma x$), then σ is a *total order*, and X is a *chain* (in other words, a totally ordered set). If a binary relation is reflexive, symmetric, and transitive it is an *equivalence (relation)*. For any $x \in X$, an equivalence σ on X defines the *equivalence class* of x

$$[x]_\sigma = \{y \in X : \ x\sigma y\}.$$

The union of all equivalence classes of σ is the *partition* of the set X corresponding to the equivalence σ. An equivalence σ where, for all $x, y \in X$ the relation $(x, y) \in \sigma$ implies that $(cx, cy) \in \sigma$ for all $c \in X$, is called a *left-congruence*. A *right-congruence* is defined symmetrically. Finally, an equivalence is a *congruence* if it is both a left- and a right-congruence.

Since relations are essentially sets, we may check if two relations are comparable (i.e. one of them includes the other), and we may obtain their intersection, as well as their union. Furthermore, we may *compose* them: for binary relations σ, τ on X,

$$\sigma \circ \tau = \{(x, y) \in X \times X : \text{ there exists } z \in X \text{ such that } (x, z) \in \sigma \text{ and } (z, y) \in \tau\}.$$

Finally, note that the *full relation* $X \times X$ and the *diagonal relation* $\Delta_X = \{(x, x) : x \in X\}$ are the biggest and the smallest (in terms of inclusion) equivalences on a set X, respectively.

Next, we introduce some terms related to functions. For a function f, the map of an element x of its domain, $\mathrm{dom}\, f$, is denoted xf. If f maps G to H (which is denoted $f : G \to H$) with $X \subseteq G$ and $Y \subseteq H$, then $Xf = \{xf \ : \ x \in X\}$ is the *direct image* of X under f and $Yf^{-1} = \{x \in G \ : \ xf \in Y\}$ is the *inverse image* of Y under f. A *partial function* f, mapping G to H, is a function whose domain is a subset of the set G. Elements of $G \setminus \mathrm{dom}\, f$ are characterised as elements without a map.

The *kernel* of a function f, denoted $\ker f$, is (clearly) an equivalence relation on its domain, defined as follows:

$$(x, y) \in \ \ker f \Leftrightarrow xf = yf.$$

The set consisting of all the equivalence classes of this relation is the partition of the domain which corresponds to $\ker f$. The number of these classes equals the cardinality of the image of f ($\mathrm{im}\, f$), and we call it the *rank* of the function f and denote it $\mathrm{Rank}\, f$. We will use the following notation

$$f = \begin{pmatrix} F_i \\ f_i \end{pmatrix}_{i \in I},$$

where $\{F_i : i \in I\}$ is the partition corresponding to $\ker f$, and, for fixed $i \in I$, each member of F_i maps to f_i (in case where F_i is a singleton, we often omit the brackets). To shorten the notation, we use $f = \left(\begin{smallmatrix} F_i \\ f_i \end{smallmatrix}\right)$ if the index set is implied. If we want to describe a partial function, we either add a column, having the set $\overline{\mathrm{dom}\, f} = G \backslash \mathrm{dom}\, f$ on top and a dash ($-$) below it (describing the non-mapping part), or we emphasise both the defining sets ($f : G \to H$) and the domain ($\mathrm{dom}\, f \subseteq G$).

If $(G, \star)$ and $(H, \cdot)$ are groupoids and if f is a function mapping G to H which satisfies

$$(x \star y)f = xf \cdot yf \text{ for all } x, y \in G,$$

then f is a *homomorphism.* Furthermore, if G and H are monoids and the identity of G maps into the identity of H, f is a *monoid homomorphism.* Homomorphic images inherit most of the structural properties from the domain: for example, an identity of the domain maps to an identity of the image (note that this need not be the identity of the codomain) and invertible elements of the domain map to invertible elements of the image; an associative (commutative) structure has an associative (commutative) image, etc. This means that a homomorphic image of a structure of any mentioned type is a structure of the same type.

Next, we list some important types of homomorphisms:

monomorphism: for each element of H there exists at most one element of G mapping into it (this property is called *injectivity*);

epimorphism: for each element of H there exists at least one element of G mapping into it (this property is called *surjectivity*);

endomorphism: the domain and codomain are the same set; the set of all endomorphisms of a groupoid G is denoted $\operatorname{End} G$;

isomorphism: a homomorphism that is both surjective and injective; a groupoid G is *isomorphic* to a groupoid H if there exists an isomorphism mapping G to H (we denote this relation $G \cong H$);

Remark 1.2.2. For any two sets of the same cardinality, it is easily proved that the corresponding symmetric groups are isomorphic. Hence, for a fixed cardinality n we always consider the symmetric group S_n of all permutations of the set $\{1, 2, \ldots, n\}$.

automorphism: an endomorphism, which is an isomorphism as well; the set of all automorphisms of a groupoid G is denoted $\operatorname{Aut} G$.

Let $f : G \to H$ be a homomorphism of groupoids/semigroups/monoids /groups. Its kernel is a congruence, since $(x, y) \in \ker f$ implies both $(cx, cy) \in \ker f$ and $(xc, yc) \in \ker f$, for any element $c \in G$. This means that we can define a *quotient (factor)* groupoid/semigroup/monoid/group, $G/\ker f$, whose elements are the equivalence classes of $\ker f$, and the operation is defined by $[a]_{\ker f}[b]_{\ker f} = [ab]_{\ker f}$ (where $[x]_{\ker f}$ denotes the $\ker f$-class of the element x). If f is an epimorphism, it is easy to show that $G/\ker f \cong H$.

Finally, we will introduce some notions from category theory. For a comprehensive introduction to this field, see [83]. A *category* consists of nodes (objects) and their connections, called *morphisms*. These connections have a binary nature but are not symmetric, therefore can be shown in the form of arrows. The origin node and the ending node of an arrow x are called the *domain* ($x\,\boldsymbol{\delta}$) and the *range* ($x\,\boldsymbol{\rho}$) of x, respectively. The composition operation is defined in the usual way, meaning that two arrows can be composed if and only if the domain of the second one is the range of the first. The "concatenation" of two such arrows always results with an existing arrow, sharing its domain and range nodes with the first and the second arrow, respectively. This composition is associative, as well, in the sense of (1.1). Furthermore, each node is both the starting point and the ending point of at least one morphism.

Members of a special type of categories, *locally small categories*, obey some further rules. If described in the so-called Ehresmann-style "arrows only" fashion (see [41]), besides previously mentioned conditions, they also satisfy the following: for any two nodes i and j (not necessarily distinct), the class of all morphisms from i to j has to be a set (often called *hom-set*, or *morphism set*). When the starting and the ending node coincide, such a hom-set is an *endomorphism semigroup*. In this thesis, we deal with locally small categories, unless stated otherwise.

Remark 1.2.3. The term epimorphism has a slightly different meaning in category theory. Namely, it denotes a morphism $f : A \to B$ such that $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$, for all objects C and all morphisms $g_1, g_2 : B \to C$. Of course, if we work in the category of maps where the objects are sets, this corresponds exactly to the surjective functions. However, in general, it has a broader meaning. To avoid confusion, from now on, we use the term *surmorphism* to denote a surjective homomorphism.

1.3 Semigroups

In this section, we give the rudiments of semigroup theory but focus on those topics which will be used in this thesis. For most of the notions, notation and results we do not reference a particular source since they became a part of semigroup theory "folklore". For the same reason, we do not provide examples. However, we refer an interested reader to consult sources containing a detailed introduction to the subject, say [58] and [22].

Since monoids are more convenient for work than "plain" semigroups, sometimes we add an identity artificially: for a semigroup $(S, \cdot)$, we introduce

$$S^1 = \begin{cases} S, & \text{if } S \text{ has an identity;} \\ S \cup \{1\}, & \text{otherwise.} \end{cases}$$

and define $s \cdot 1 = 1 \cdot s = 1$ for each $s \in S$, and $1 \cdot 1 = 1$. Thus, S^1, together with the modified version of operation $\cdot$, forms a monoid.

If $(S, \cdot)$ is a semigroup, $X, Y \subseteq S$ its subsets, and $a \in S$ any element, we use the following notation:

$$XY = \{x \cdot y : x \in X,\ y \in Y\}, \quad aX = \{a \cdot x : x \in X\}, \quad Xa = \{x \cdot a : x \in X\}.$$

Furthermore, a subset $\emptyset \neq I \subseteq S$ is a

- *right ideal* of S if $IS \subseteq I$;
- *left ideal* of S if $SI \subseteq I$;
- *(two-sided) ideal* of S if $IS \cup SI \subseteq I$.

The most important ideals of a semigroup are its principal ideals: for a fixed element $a \in S$,

- aS^1 is the *principal right ideal* of S corresponding to the element a;
- S^1a is the *principal left ideal* of S corresponding to the element a;
- S^1aS^1 is the *principal (two-sided) ideal* of S corresponding to the element a.

Clearly, the names are fitting, since it is easily proved that these are the smallest right, left and two-sided ideal containing a, respectively. These ideals determine the structure of the semigroup S, through the famed *Green's relations* and *preorders*. Namely, the three preorders are defined on S as follows:

$$\begin{array}{lll} a \leq_{\mathscr{R}} b & \Leftrightarrow aS^1 \subseteq bS^1 & (\Leftrightarrow \text{ there exists } c \in S^1 \text{ such that } a = bc), \\ a \leq_{\mathscr{L}} b & \Leftrightarrow S^1a \subseteq S^1b & (\Leftrightarrow \text{ there exists } c \in S^1 \text{ such that } a = cb), \\ a \leq_{\mathscr{J}} b & \Leftrightarrow S^1aS^1 \subseteq S^1bS^1 & (\Leftrightarrow \text{ there exist } c, d \in S^1 \text{ such that } a = cbd). \end{array}$$

It is clear that reflexivity and transitivity hold, because our preorders are defined using inclusion. Now, the corresponding relations, $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{J}$ are introduced in a natural manner, while $\mathscr{H}$ and $\mathscr{D}$ are combinations of $\mathscr{R}$ and $\mathscr{L}$: for any two elements $a, b \in S$,

$$\begin{aligned} a \mathscr{R} b &\Leftrightarrow aS^1 = bS^1, \\ a \mathscr{L} b &\Leftrightarrow S^1a = S^1b, \\ a \mathscr{J} b &\Leftrightarrow S^1aS^1 = S^1bS^1, \\ a \mathscr{H} b &\Leftrightarrow a \mathscr{L} b \textit{ and } a \mathscr{R} b, \\ a \mathscr{D} b &\Leftrightarrow \text{ there exists } c \in S \text{ such that } a \mathscr{R} c \text{ and } c \mathscr{L} b. \end{aligned}$$

Remark 1.3.1. If $x = xyz$ ($x = zyx$), we will often conclude $x \mathscr{R} xy$ ($x \mathscr{L} yx$) with no further explanation, because $xy \leq_{\mathscr{R}} x$ ($yx \leq_{\mathscr{L}} y$) is clear.

The first three are clearly equivalence relations, by virtue of equality being one, and the fourth is, in fact, $\mathscr{R} \cap \mathscr{L}$, therefore an equivalence relation as well. Only the fifth one remains. Obviously, $\mathscr{D} = \mathscr{R} \circ \mathscr{L}$, by the definition of the composition of relations ($\circ$) and it is reflexive since $\mathscr{R}$ and $\mathscr{L}$ are. Moreover, we will show that the following lemma implies the symmetric and transitive properties.

Lemma 1.3.2. $\mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$

Proof. We will prove only $\subseteq$, as a dual argument will give the other inclusion. Suppose there exists an element $c \in S$ such that $a \mathscr{R} c$ and $c \mathscr{L} b$. These relations imply the existence of elements $d, e, f, g \in S^1$ such that $a = cd$, $c = ae$, $c = fb$ and $b = gc$. Note that, for the element $x = bd$ we have

$$x = bd = gcd = ga \qquad \text{and} \qquad a = cd = fbd = fx,$$

so $a \mathscr{L} x$. Furthermore,

$$x = bd \qquad \text{and} \qquad b = gc = gae = gcde = bde = xe$$

prove $x \mathscr{R} b$, hence $a \mathscr{L} x \mathscr{R} b$, i.e. $(a, b) \in \mathscr{L} \circ \mathscr{R}$. □

Having proved this, we may conclude (using the associative property of the composition of relations and the reflexivity of $\mathscr{R}$ and $\mathscr{L}$):

$$\begin{gathered} \mathscr{D}^{-1} = \{(a, b) : b \mathscr{D} a\} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L} = \mathscr{D}, \quad \text{and} \\ \mathscr{D} \circ \mathscr{D} = \mathscr{R} \circ \mathscr{L} \circ \mathscr{R} \circ \mathscr{L} = \mathscr{R} \circ \mathscr{R} \circ \mathscr{L} \circ \mathscr{L} = \mathscr{R} \circ \mathscr{L} = \mathscr{D} \end{gathered}$$

Hence, $\mathscr{D}$ is also an equivalence relation.

We can easily determine the inclusion relations among $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{J}$. Obviously, $\mathscr{H} \subseteq \mathscr{R} \subseteq \mathscr{J}$ and $\mathscr{H} \subseteq \mathscr{L} \subseteq \mathscr{J}$. Furthermore, since $\mathscr{R}$ and $\mathscr{L}$ are reflexive, we have $\mathscr{R} \subseteq \mathscr{D}$ and $\mathscr{L} \subseteq \mathscr{D}$. It is also easy to show that $\mathscr{D}$ is the smallest equivalence relation containing both $\mathscr{R}$ and $\mathscr{L}$ (because $\mathscr{D} = (\mathscr{R} \circ \mathscr{L})^\infty = \bigcup_{i=1}^{\infty} (\mathscr{R} \circ \mathscr{L})^i$), thus $\mathscr{D} \subseteq \mathscr{J}$. Figure 1.1 shows the described relations.

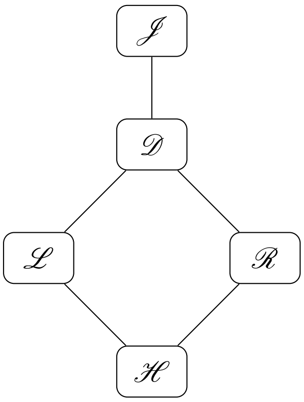


Figure 1.1: The Hasse diagram of Green's relations.

Remark 1.3.3. In fact, the relation $\mathscr{D}$ is often defined as $\mathscr{R} \vee \mathscr{L}$, the smallest equivalence containing $\mathscr{R} \cup \mathscr{L}$. Of course, the two definitions are equivalent, by virtue of Lemma 1.3.2.

Also, note that we may define a partial order $\leq_{\mathscr{J}}$ on the set of all $\mathscr{J}$-classes of a semigroup S, through the relation $\leq_{\mathscr{J}}$: for $\mathscr{J}$-classes J_1 and J_2,

$$\mathrm{J}_1 \leq_{\mathscr{J}} \mathrm{J}_2 \Leftrightarrow (\exists a \in \mathrm{J}_1)(\exists b \in \mathrm{J}_2)(a \leq_{\mathscr{J}} b).$$

Clearly, the relation is well defined, since for arbitrary elements $c \in \mathrm{J}_1$ and $d \in \mathrm{J}_2$, we have

$$S^1cS^1 = S^1aS^1 \subseteq S^1bS^1 = S^1dS^1.$$

Moreover, it is easy to see that, for any $a \in S$,

$$S^1aS^1 = \bigcup_{b \leq_{\mathscr{J}} a} S^1bS^1 = \{b \in S : b \leq_{\mathscr{J}} a\},$$

which implies that S^1aS^1 is the union of the $\mathscr{J}$-class containing a, and all the $\mathscr{J}$-classes $\leq_{\mathscr{J}}$-below it. In the special case, when S contains only one $\mathscr{J}$-class, we say it is a *simple* semigroup.

The $\mathscr{H}$-, $\mathscr{L}$-, $\mathscr{R}$-, $\mathscr{D}$- and $\mathscr{J}$- classes containing a chosen element $a \in S$ are usually denoted H_a, L_a, R_a, D_a and J_a, respectively.

The $\mathscr{H}$-, $\mathscr{R}$- and $\mathscr{L}$-classes in a fixed $\mathscr{D}$-class can be presented in a convenient way, using the so-called egg-box diagrams. Here, the rows represent the $\mathscr{R}$-classes, and the columns are the $\mathscr{L}$-classes, so the boxes represent the $\mathscr{H}$-classes. These boxes are always non-empty: for any row r and column c we may pick elements $a \in r$ and $b \in c$ and, since $a \mathscr{D} b$, there exists an element y such that $a \mathscr{R} y \mathscr{L} b$. This situation is depicted in the following figure.

a		y
x		b

Figure 1.2: A layout of an egg-box diagram.

The next lemma provides some more information regarding the $\mathscr{R}$-, $\mathscr{L}$-, and $\mathscr{H}$-classes in a fixed $\mathscr{D}$-class.

Lemma 1.3.4 (Green's Lemma)**.** *Let a, b be any elements of a semigroup S.*

(i) Suppose $a \mathscr{R} b$ and the elements $s, t \in S^1$ are such that $a = bs$ and $b = at$. Then the maps $\rho_t : \mathrm{L}_a \to \mathrm{L}_b : x \to xt$ and $\rho_s : \mathrm{L}_b \to \mathrm{L}_a : x \to xs$ are mutually inverse bijections. These maps restrict to mutually inverse bijections $\rho_t\restriction_{\mathrm{H}_a} : \mathrm{H}_a \to \mathrm{H}_b$ and $\rho_s\restriction_{\mathrm{H}_b} : \mathrm{H}_b \to \mathrm{H}_a$.

(ii) Suppose $a \mathscr{L} b$ and the elements $s, t \in S^1$ are such that $a = sb$ and $b = ta$. Then the maps $\lambda_t : \mathrm{R}_a \to \mathrm{R}_b : x \to tx$ and $\lambda_s : \mathrm{R}_b \to \mathrm{R}_a : x \to sx$ are mutually inverse bijections. These maps restrict to mutually inverse bijections $\lambda_t\restriction_{\mathrm{H}_a} : \mathrm{H}_a \to \mathrm{H}_b$ and $\lambda_t\restriction_{\mathrm{H}_b} : \mathrm{H}_b \to \mathrm{H}_a$.

(iii) If $a \mathscr{D} b$, then $|\,\mathrm{R}_a\,| = |\,\mathrm{R}_b\,|$, $|\,\mathrm{L}_a\,| = |\,\mathrm{L}_b\,|$ and $|\,\mathrm{H}_a\,| = |\,\mathrm{H}_b\,|$ (where $|T|$ denotes the cardinality of the set T).

Proof. We will prove only the first part, as (*ii*) follows by duality, and (*iii*) is a direct consequence of the previous two. First, note that the maps are well-defined, since for any element $c \in L_a$ (L_b) the definition of $\mathscr{L}$ implies $ct \mathscr{L} at = b$ ($cs \mathscr{L} bs = a$), so $ct \in \mathrm{L}_b$ ($cs \in \mathrm{L}_a$). Furthermore, for such a c there exists $y \in S^1$ such that $c = ya$, hence

$$c\rho_t\rho_s = cts = yats = ybs = ya = c,$$

i.e. $\rho_t\rho_s = \mathrm{id}_{\mathrm{L}_a}$. Clearly, $\rho_s\rho_t = \mathrm{id}_{\mathrm{L}_b}$ is proved similarly. Also note that, if we denote $d = c\rho_t = ct$, then $ds = c$ and $ct = d$, so $c \mathscr{R} d = c\rho_t$. This means that ρ_t (and ρ_s, similarly) preserves the $\mathscr{R}$-class, thus for any $x \in \mathrm{H}_a$ (H_b) we have $x\rho_t \in \mathrm{R}_a \cap \mathrm{L}_b = \mathrm{H}_b$ ($x\rho_s \in \mathrm{R}_b \cap \mathrm{L}_a = \mathrm{H}_a$). □

In order to investigate further the properties of $\mathscr{H}$-classes, we introduce a new type of elements − idempotents − and examine their properties. An element e is an *idempotent* of a semigroup $(S, \cdot)$ if $e \cdot e = e$. The set of all idempotents of S is denoted $\mathrm{E}(S)$. A subsemigroup of a semigroup is *full* if it contains all its idempotents. Note that an idempotent has to be a left-identity of its $\mathscr{R}$-class, since, for $a \in \mathrm{R}_e$ there exists $s \in S^1$ such that $a = es$ and we have $ea = ees = es = a$. Similarly, it has to be a right-identity of its $\mathscr{L}$-class, and therefore an identity of its $\mathscr{H}$-class. Moreover, the following holds:

Lemma 1.3.5. *If G is a subgroup of a semigroup S, then $G \subseteq \mathrm{H}_e$, where e is the identity of G. Indeed, H_e is the maximal subgroup of S with identity e.*

Proof. If G is a subgroup with identity e, and a any of its elements, then we have $ae = ea = a$ and $a^{-1}a = aa^{-1} = e$, so $e \mathscr{R} a \mathscr{L} e$, i.e. $a \in \mathrm{H}_e$. For the other statement, we have to prove that H_e is a group for any $e \in \mathrm{E}(S)$. Clearly, e acts as an identity. The restriction $\cdot\restriction_{\mathrm{H}_e}$ is an operation, since for any $a, b \in \mathrm{H}_e$ (from $b \mathscr{R} e$ and $a \mathscr{L} e$) we have $ab \mathscr{R} ae = a \mathscr{R} e$ and $ab \mathscr{L} eb = b \mathscr{L} e$, so $ab \mathscr{H} e$. Finally, we need to find an inverse for an arbitrary element $a \in \mathrm{H}_e$. Obviously, $a \mathscr{H} e$ implies the existence of elements $s, t \in S^1$ such that $at = e$ and $sa = e$. Let us examine elements $x = ete$ and $y = ese$:

$$ax = aete = ate = ee = e, \qquad ya = esea = esa = ee = e,$$
$$x = ete = eete = ex = (ya)x = y(ax) = ye = esee = ese = y.$$

From these, we conclude that x is an element with required properties, such that $x \in \mathrm{H}_e$. □

In the special case, when S is a monoid with identity e, the class H_e is called *the group of units* of S.

Since idempotents play a vital role in a semigroup, we use a special term for a semigroup in which each element can generate one. A semigroup S is *periodic* if each of its elements has a power that is an idempotent. In other words, for each $x \in S$ there exists $n \in \mathbb{N}$ such that $(x^n)^2 = x^n$ (recall that, according to our definition, the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$ does not include zero). For instance, all finite semigroups are periodic. To elaborate, for any $x \in S$, the subsemigroup $\{x^n : n \geq 1\}$ has to be finite, thus there exists the minimal exponent m and the minimal integer i such that $x^m = x^{m+i}$. It is easy to prove that $\{x^m, x^{m+1}, \dots, x^{m+i-1}\}$ is a subgroup of S, which obviously has an identity.

We introduce another class of semigroups containing the class of finite semigroups. Not surprisingly, its defining property is a significant argument when proving results in finite semigroup theory. A semigroup S is *stable* if for all $x, a \in S$,

$$x \mathscr{J} xa \Rightarrow x \mathscr{R} xa \quad \text{and} \quad x \mathscr{J} ax \Rightarrow x \mathscr{L} ax. \tag{1.2}$$

Stability will be a crucial property in our investigations of sandwich semigroups, and the structures containing them − partial semigroups. Let us prove that all finite semigroups are stable. Let S be a finite semigroup and let $x, a \in S$ be elements such that $x \mathscr{J} xa$. Then, $x = bxac$ for some $b, c \in S^1$, so $x = b^n x(ac)^n$ for all $n \geq 1$. Since S is finite, it has to be periodic, so there exists $m \in \mathbb{N}$ such that $(ac)^{2m} = (ac)^m$. Therefore, we have

$$x = b^m x(ac)^m = b^m x(ac)^{2m} = (b^m x(ac)^m)(ac)^m = x(ac)^m = xac(ac)^{m-1},$$

i.e. $x \mathscr{R} xa$. A similar argument shows the second implication. Furthermore, in a stable semigroup $\mathscr{J} = \mathscr{D}$ holds true (see Lemma 2.2.19).

An additional important matter for the description of a semigroup is its likeness to a group, i.e. the level of "invertibility" of its elements. Namely, an element a of a semigroup S is *(von Neumann) regular* if there exists an element $b \in S$ such that $aba = a$. In that case, b is a *pre-inverse* of a, and a is a *post-inverse* of b. The element b is a *(semigroup) inverse* of the element a, if it is both a pre-inverse and a post-inverse of a. It is easy to prove that $aba = a$ implies that a has a semigroup inverse bab, so every regular element has at least one inverse.

Remark 1.3.6. From now on, by an inverse element, we mean a semigroup (i.e. von Neumann) inverse, unless stated otherwise.

The sets of all pre-inverses, post-inverses and inverses of an element a are denoted $\mathrm{Pre}(a)$, $\mathrm{Post}(a)$ and $\mathrm{V}(a)$, and the set of all regular elements of S is denoted $\mathrm{Reg}(S)$. If $\mathrm{Reg}(S) = S$, then S is a *(von Neumann) regular semigroup*. Furthermore, if every element of S has a unique inverse, semigroup S is an *inverse* semigroup. If, on the other hand, there exists a mapping $S \to S : a \mapsto a^*$ such that

$$(a^*)^* = a, \quad (ab)^* = b^* a^*, \qquad \text{for all } a, b \in S,$$

then S is a $*$-*semigroup* (or a *semigroup with involution*). If in this semigroup also holds

$$aa^*a = a, \qquad \text{for all } a \in S,$$

it is a *regular* $*$-*semigroup*. In such a semigroup, elements of the form aa^* are called *projections* and may be characterised as the elements a for which $a^2 = a = a^*$.

Remark 1.3.7. Each inverse semigroup is a regular $*$-semigroup. Indeed, if the unique inverse of $x \in S$ is denoted by x^{-1}, and we define $x^* = x^{-1}$, then for any $x, y \in S$ we have $x^{**} = x$ (since $x^{**}x^*x^{**} = x^*$ and $x^*x^{**}x^* = x^*$), $xx^*x = x$, and $(xy)^* = y^*x^*$ because

$$(y^{-1}x^{-1})xy(y^{-1}x^{-1}) = (y^{-1}x^{-1}) \text{ and } xy(y^{-1}x^{-1})xy = xy.$$

However, not every regular $*$-semigroup is inverse: as in [102], consider any square rectangular band (see page 14 for the definition of a rectangular band; here, "square" means that $|I| = |J|$).

In a regular semigroup S we may introduce the natural partial order $\preceq$:

$$x \preceq y \ \text{ if and only if } \ x = ey = yf \ \text{ for some idempotents } e, f \in \mathrm{E}(S).$$

From this, for any idempotents $e, f \in \mathrm{E}(S)$ we may conclude $e \preceq f \Leftrightarrow e = fef \Leftrightarrow e = ef = fe$ (the proof is elementary, but requires a bit of semigroup acrobatics). This partial order may be defined on $\mathrm{E}(S)$ for any semigroup S, regardless of its regularity. Minimal idempotents with respect to this relation are called *primitive* idempotents.

Remark 1.3.8. Regular elements and regular (sub)semigroups have received much attention in the development of semigroup theory, and there are many important results concerning them. We will mention only three of those, which are needed for subsequent proofs:

- If x is a regular element in S, then every element of D_x is regular. (If $xyx = x$ and $x \mathscr{R} z$ with $x = zt$ and $z = xs$ for some $t, s \in S^1$, then $z = xs = xyxs = xyz = ztyz$ and z is regular. In case when $x \mathscr{L} z$, the proof is dual.)

- In a regular $\mathscr{D}$-class, each $\mathscr{L}$-class (and dually each $\mathscr{R}$-class) contains an idempotent (since $x = xyx$ implies $x \mathscr{L} yx$ with $yxyx = yx$ and $x \mathscr{R} xy$ with $xyxy = xy$).

- If x, y are elements of the same $\mathscr{D}$-class of S, then $\mathrm{L}_x \cap \mathrm{R}_y$ contains an idempotent if and only if $xy \in \mathrm{R}_x \cap \mathrm{L}_y$. (The forwards implication follows from the fact that, for such an idempotent e, we have $ey = y$, so Green's Lemma implies that the map $H_x \to \mathrm{R}_x \cap \mathrm{L}_y : w \mapsto wy$ is a bijection. For the reverse, we may suppose $xyz = x$ for some $z \in S^1$, so from Green's Lemma we infer that the maps $\mathrm{H}_y = \mathrm{L}_{xy} \cap \mathrm{R}_y \to \mathrm{L}_x \cap \mathrm{R}_y : w \to wz$ and $\mathrm{L}_x \cap \mathrm{R}_y \to \mathrm{L}_{xy} \cap \mathrm{R}_y = \mathrm{H}_y : w \to wy$ are mutually inverse bijections; thus, yz is an idempotent from $\mathrm{L}_x \cap \mathrm{R}_y$.)

As a closing for this section, we introduce several important types of semigroups. A *left-zero* semigroup consists solely of left zeroes; in other words, for any two elements a, b we have $ab = a$. A *left-group* is (isomorphic to) a direct product of a left-zero semigroup and a group. *The degree* of a left-group is the size of the associated left-zero semigroup. We define accordingly a right-zero semigroup, a right-group and its degree. The following lemma (Lemma 2.6 in [28]) and its dual describe a case in which a left-group or a right-group arises naturally. These results follow from the Rees Theorem (Theorem 3.2.3 in [58]), but we provide a direct proof for convenience.

Lemma 1.3.9. *If a regular $\mathscr{D}$-class of a semigroup is an $\mathscr{L}$-class, then it is a left-group.*

Proof. Suppose D is a regular $\mathscr{D}$-class, as well as an $\mathscr{L}$-class of a semigroup S. First, we prove that D is a subsemigroup. Let $x, y \in D$. Then, Remark 1.3.8 implies $y \mathscr{R} e$ for some $e \in \mathrm{E}(D)$, and $x \mathscr{L} e$ because D is an $\mathscr{L}$-class, so we have $e \in \mathrm{R}_y \cap \mathrm{L}_x$. Thus, Remark 1.3.8 implies $xy \in \mathrm{L}_y \cap \mathrm{R}_x \subseteq D$. Furthermore, since each $\mathscr{R}$-class contains an idempotent and each of them is a right-identity of D (again, by Remark 1.3.8), we conclude that D is a union of groups and $\mathrm{E}(D)$ is a left-zero semigroup. Let $e \in \mathrm{E}(D)$ be arbitrary, and H_e its associated group in D. It is easily seen now that $\mathrm{E}(D) \times \mathrm{H}_e \to D : (f, g) \mapsto fg$ is an isomorphism. □

Now, let I and J be non-empty sets, and let $(T, \cdot)$ be defined by $T = I \times J$ and $(a, b) \cdot (c, d) = (a, d)$. Then T is a *rectangular band*. If $|I| = i$ and $|J| = j$, we say that T is a $i \times j$ *rectangular band*. Associativity is easily checked; note also that

T is the direct product of the left-zero semigroup I and the right-zero semigroup J. Thus, each element is an idempotent and the $\mathscr{R}$-classes are the sets $\{x\} \times J$ for $x \in I$, while the $\mathscr{L}$-classes are the sets $I \times \{y\}$ for $y \in J$. Therefore, an $i \times j$ rectangular band has i $\mathscr{R}$-classes and j $\mathscr{L}$-classes. In the next lemma, we state a useful equivalent condition for a semigroup to be a rectangular band, which will be of use throughout the thesis.

Lemma 1.3.10. *A semigroup T is a rectangular band if and only if $aba = a$ for all $a, b \in T$.*

Proof. The direct implication is obvious. For the reverse, suppose that in the semigroup T we have $aba = a$ for all $a, b \in T$. We need to show that the semigroup T has the required form. Choose an arbitrary element $z \in T$ and denote the sets Tz and zT with I and J, respectively. If we choose arbitrary elements $x, y \in I$, there exist $a, b \in T$ such that $x = az$ and $y = bz$, so

$$xy = (az)(bz) = a(zbz) = az = x.$$

Similarly, for arbitrary elements $x, y \in J$ we have $xy = y$. Therefore, for $(a, b), (c, d) \in I \times J$ holds $(a, b) \cdot (c, d) = (a, d)$. Let us define a map $\varphi : T \to I \times J : x \mapsto (xz, zx)$. It suffices to show is that φ is an isomorphism. It is a homomorphism, since for all $x, y \in T$, we have

$$\begin{aligned} x\varphi \cdot y\varphi &= (xz, zx)(yz, zy) = (xzyz, zxzy) = (xz, zy) \\ &= (xyxzyz, zxzyxy) = (xyz, zxy) = (xy)\varphi. \end{aligned}$$

It is injective because $x\varphi = (xz, zx) = (yz, zy) = y\varphi$ implies $x = (xz)x = y(zx) = yzy = y$, and surjective since for $qz \in I$ and $zw \in J$ we have $(qz, zw) = (qwqzwz, zqzwqw) = (qwz, zqw) = (qw)\varphi$. ☐

In this thesis, we will encounter a somewhat more complex structure, a *rectangular group over a group G*, which is (isomorphic to) a direct product of a rectangular band and a group G. If the rectangular band in question has dimensions $i \times j$, we are dealing with a $i \times j$ *rectangular group over G*. In the next section, we give a result concerning its rank, which will be the base for some of our calculations later. Here, we prove a lemma (Proposition 1.6 from [4]) providing an equivalent condition for a semigroup to be a rectangular group.

Lemma 1.3.11. *A semigroup S is a rectangular group if and only if it is regular and $\mathrm{E}(S)$ is a rectangular band.*

Proof. The direct implication is easy to prove. Let us prove the reverse. Let S be such a semigroup, and suppose without loss of generality that $\mathrm{E}(S) = (I \times J, \cdot)$ for some nonempty sets I and J. Choose an arbitrary idempotent $e = (l, k)$ and let H_e denote the corresponding $\mathscr{H}$-class. We claim that S is isomorphic to the rectangular group $(I \times J) \times \mathrm{H}_e$, moreover, that $\phi : (I \times J) \times \mathrm{H}_e \to S : (i, j, g) \mapsto (i, j)g(i, j)$ is an isomorphism. Since for any two elements $(i_1, j_1, g), (i_2, j_2, h) \in (I \times J) \times \mathrm{H}_e$ we have

$$\begin{aligned}(i_1, j_1, g)\phi(i_2, j_2, h)\phi &= (i_1, j_1)g(i_1, j_1)(i_2, j_2)h(i_2, j_2) \\ &= (i_1, j_1)ege(i_1, j_1)(i_2, j_2)ehe(i_2, j_2) \\ &= (i_1, j_1)eghe(i_2, j_2) = (i_1, j_2)eghe(i_1, j_2) \\ &= (i_1, j_2)gh(i_1, j_2) = (i_1, j_2, gh)\phi \\ &= ((i_1, j_1, g)(i_2, j_2, h))\phi,\end{aligned}$$

so the map is homomorphic. To prove injectivity, suppose that

$$(i_1, j_1)g(i_1, j_1) = (i_2, j_2)h(i_2, j_2); \tag{1.3}$$

then we have

$$e(i_1, j_1)ege(i_1, j_1)e = e(i_2, j_2)ehe(i_2, j_2)e$$

so $g = h$. Thus, multiplying (1.3) by $g^{-1}e$ on the left gives $e(i_1, j_1) = e(i_2, j_2)$, while multiplying by eg^{-1} on the right gives $(i_1, j_1)e = (i_2, j_2)e$, so $j_1 = j_2$ and $i_1 = i_2$. The only property left to prove is surjectivity. As S is regular, all its $\mathscr{D}$-classes are regular, each $\mathscr{R}$-class contains an idempotent, and each $\mathscr{L}$-class contains an idempotent (by Remark 1.3.8). Since $E(S)$ is a rectangular band, Green's relations of $E(S)$ hold in S as well, so it consists of a single $\mathscr{D}$-class and there are no non-group $\mathscr{H}$-classes (because such a class would belong to a "new" $\mathscr{R}$- or $\mathscr{L}$-class). Thus, for an arbitrary $z \in S$ there exists and idempotent $(i, j) \in \mathrm{H}_z$, and we have $eze \in \mathrm{H}_e$ with $(i, j, eze)\phi = z$. □

The last lemma (Lemma 2.4 in [28]) we present in this section shares some similarities with Lemma 1.3.9. Firstly, it can be proved as a consequence of the Rees Theorem (Theorem 3.2.3 in [58]). Secondly, it characterises a regular $\mathscr{D}$-class of a stable (in particular, finite) semigroup under certain assumptions. In fact, since a left-group L is a $|L| \times 1$ rectangular group and a left-zero semigroup K is a $|K| \times 1$ rectangular band, Lemma 1.3.9 follows from Lemma 1.3.12 if the semigroup in question is stable.

Lemma 1.3.12. *Let D be a regular $\mathscr{D}$-class of a stable semigroup S. If $\mathrm{E}(D)$ is a subsemigroup of S, then $\mathrm{E}(D)$ is a rectangular band, and D is a rectangular group.*

Proof. By Lemma 1.3.11, it suffices to prove that $\mathrm{E}(D)$ is a rectangular band and that D is a subsemigroup. For the first one, we use Lemma 1.3.10. Suppose x, y are arbitrary elements of the semigroup $\mathrm{E}(D)$; then, $xy \in \mathrm{E}(D)$ and $xy \,\mathscr{D}\, x$, so stability implies $xy \,\mathscr{R}\, y$, i.e. $xyz = x$ for some $z \in S^1$. Hence, $xyx = xyxyz = (xy)^2z = xyz = x$.

Now we prove that D is a subsemigroup. Let $x, y \in D$ be arbitrary. Firstly, S is stable, so $\mathscr{D} = \mathscr{J}$ (by Lemma 2.2.19). Secondly, D is regular, so there exist $a \in \mathrm{V}(x)$ and $b \in \mathrm{V}(y)$, and we have $ax \,\mathscr{J}\, x$ (since $xax = x$ and $a \cdot x = ax$) and $yb \,\mathscr{J}\, y$. Therefore, $ax, yb \in \mathrm{E}(S) \cap D = \mathrm{E}(D)$ and $xy = xaxyby \leq_{\mathscr{J}} axyb \leq_{\mathscr{J}} xy$. Thus, $xy \,\mathscr{J}\, axyb$, which means that $xy \,\mathscr{D}\, axyb$. As $axyb \in \mathrm{E}(D)$ (since $\mathrm{E}(D)$ is a rectangular band), we may conclude $xy \in \mathrm{D}_{axyb} = D$. □

Remark 1.3.13. When comparing Lemmas 1.3.9 and 1.3.12, one might wonder if the assumption of stability is necessary in the second one. However, it is easily seen that the bicyclic monoid (see [58, page 32]) is a single regular $\mathscr{D}$-class whose idempotents form a subsemigroup, but it is not a rectangular group (for instance, each $\mathscr{H}$-class is a singleton, but not all elements are idempotents).

1.4 Elements of combinatorial semigroup theory

Here, we give a short overview of those concepts, notation and results specific to combinatorial semigroup theory, which we need for our investigation.

One of the key notions in this field is that of set generation. A subset Y of a semigroup S *generates* the set $T \subseteq S$ if

$$\{a_1 a_2 \cdots a_n : n \geq 1,\ a_1, \ldots, a_n \in Y\} = T.$$

To shorten that, we write $\langle Y \rangle = T$. If $\langle Y \rangle = S$, we say that Y is a *generating set* of the semigroup S. The *rank* of a semigroup S is the minimal size of a generating set for it:

$$\operatorname{rank}(S) = \min\{|Y| : Y \subseteq S,\ \langle Y \rangle = S\}.$$

Remark 1.4.1. It should be easy to distinguish between this and the previously defined notion of rank (Rank), since this one concerns sets, and that one maps.

Sometimes, we are interested in using only generating elements of a special type. For example, we often pose a question whether a semigroup can be generated solely by its idempotents; if so, we say it is *idempotent-generated*, and we may define its *idempotent rank*:

$$\operatorname{idrank}(S) = \min\{|Y| : Y \subseteq \mathrm{E}(S),\ \langle Y \rangle = S\}.$$

Even if a semigroup cannot be generated by its idempotents, we may be interested in all its elements that can be. These form the *idempotent-generated subsemigroup* of S, which is denoted $\mathbb{E}(S)$.

Other times, we face the task of generating a semigroup with some of the elements already provided. The *relative rank* of a semigroup S with respect to a subset $A \subseteq S$ measures the minimal number of additional elements needed:

$$\operatorname{rank}(S : A) = \min\{|B| : B \subseteq S,\ \langle A \cup B \rangle = S\}.$$

If S is idempotent-generated, we may also define the *relative idempotent rank* of S with respect to a subset $A \subseteq \mathrm{E}(S)$:

$$\operatorname{idrank}(S : A) = \min\{|B| : B \subseteq \mathrm{E}(S),\ \langle A \cup B \rangle = S\}.$$

We will also be interested in the "covering power" of an idempotent. Namely, for an idempotent e of a semigroup S, the set eSe is the *local monoid* of S with respect to the idempotent e. At certain points in this thesis, it will be important to us how much of a chosen semigroup can be covered by local monoids corresponding to idempotents of a special type.

As promised in the previous section, we give a result of Ruškuc [110], which concerns the rank of a rectangular group. In his article, he dealt with (more general) completely 0-simple semigroups, so we provide a short proof in the special case of rectangular groups, as in [33](Proposition 4.11).

Proposition 1.4.2. *Let T be an $r \times l$ rectangular group over G. Then*

(i) $\operatorname{rank}(T) = \max(r, l, \operatorname{rank}(G))$,

(ii) any generating set for T contains elements from every $\mathscr{R}$-class, and from every $\mathscr{L}$-class of T,

(iii) if $\operatorname{rank}(T) = r$*, then there is a minimum-size generating set for T that is a cross-section of the $\mathscr{R}$-classes of T,*

(iv) if $\operatorname{rank}(T) = l$*, then there is a minimum-size generating set for T that is a cross-section of the $\mathscr{L}$-classes of T.*

Here, a *cross-section* of an equivalence relation is a set containing exactly one member of each class.

Proof. Suppose that $T = I \times G \times J$, where $(i, g, j)(k, h, m) = (i, gh, m)$, $|I| = r$, $|J| = l$, and let π_1, π_2, π_3 be projections (not to be confused with projections in regular $*$-semigroups)

$$\pi_1 : T \to I : (i, x, j) \mapsto i, \qquad \pi_2 : T \to G : (i, x, j) \mapsto x,$$
$$\pi_3 : T \to J : (i, x, j) \mapsto j.$$

(ii) It is easy to conclude that the leftmost and the rightmost element of any product determine the first and the last coordinate (respectively) of the resulting element. Thus, the $\mathscr{R}$- and $\mathscr{L}$-class of an element are determined by the first and last coordinate, respectively. Therefore, to generate an arbitrary element $(i, g, j) \in X$ we will necessarily need an element from the same $\mathscr{R}$-class, and an element from the same $\mathscr{L}$-class.

(i) First, we prove $\operatorname{rank}(T) \leq \max(r, l, \operatorname{rank}(G))$. Let us fix a set $X \subseteq T$ such that:

(1) $|X| = \max(r, l, \operatorname{rank}(G))$,

(2) $|\operatorname{im}(\pi_2\restriction_X)| = \operatorname{rank}(G)$ and $\langle \operatorname{im}(\pi_2\restriction_X)\rangle = G$,

(3) restrictions $\pi_1\restriction_X$ and $\pi_3\restriction_X$ are surjective mappings; moreover, if $|X| = r$ then $\pi_1\restriction_X$ is a bijection, and if $|X| = l$ then $\pi_3\restriction_X$ is a bijection.

(Such a set clearly exists.) We will prove that the set X is a generating set for T. Let $(i, g, j) \in T$ be an arbitrary element. By (3), there exist $(i_x, g_x, j_x), (i_y, g_y, j_y) \in T$ such that $i_x = i$ and $j_y = j$; furthermore, by (2), there exist $(i_1, g_1, j_1), \dots, (i_n, g_n, j_n) \in X$ such that $g_x^{-1} g g_y^{-1} = g_1 \cdots g_n$. Hence,

$$(i_x, g_x, j_x)(i_1, g_1, j_1) \cdots (i_n, g_n, j_n)(i_y, g_y, j_y) = (i, g, j).$$

Let us prove now that $\operatorname{rank}(T) \geq \max(r, l, \operatorname{rank}(G))$. From (ii) follows $\operatorname{rank}(T) \geq \max(r, l)$. Let Ω be a generating set for T. Suppose the opposite, that $|\Omega| < \operatorname{rank}(G)$. Then $|\operatorname{im}(\pi_2\restriction_\Omega)| < \operatorname{rank}(G)$, so $\langle \operatorname{im}(\pi_2\restriction_\Omega)\rangle \neq G$ and therefore $\langle \Omega \rangle \neq T$, because we cannot generate all the elements of G. We have come to a contradiction, thus $|\Omega| \geq \operatorname{rank}(G)$.

(iii) and (iv) follow from (3), because we choose X so that $\pi_1\restriction_X$, or $\pi_3\restriction_X$ respectively, is a bijection. □

There is one more notion we need to introduce in this section. Namely, a *transformation* of a set X is a mapping $X \to X$. This term corresponds to the term of a permutation in group theory. The semigroup consisting of all the transformations of X under the composition of functions is the *full transformation semigroup* over X, $\mathcal{T}_X$. Any subsemigroup of $\mathcal{T}_X$ is a *transformation semigroup* over X. A *partial transformation* over X is a partial function $X \to X$. The set of all partial transformations over X is denoted $\mathcal{PT}_X$. Moreover, the set of all injective partial transformations over X forms a subsemigroup of $\mathcal{PT}_X$, which is called the *symmetric inverse semigroup* over X and is denoted $\mathcal{I}_X$.

In this thesis, our program of investigation is based on the one followed in [29] and [30]. That one was, in fact, inspired by a series of articles by one of the most influential mathematicians in combinatorial semigroup theory, John M. Howie. This series was commenced in his 1966 article [55], where he proved that the semigroup Sing_X, consisting of all singular (non-invertible) transformations over a finite set X, is idempotent-generated. In the following years, he continued this research (in single-author papers, and in cooperation with others): he calculated the rank and idempotent rank of Sing_X [48,57], classified its idempotent generating sets of minimal size [57], calculated the rank and idempotent rank of its ideals [60], investigated the gravity function of its elements (the length of the shortest product of idempotents of defect 1 giving the chosen element) [59] and expanded these results to other kinds of transformation semigroups [5,6,48,49,56]. These articles made an immense impact in the field and laid the ground for new studies and development of important directions in the research of transformation semigroups, endomorphism semigroups, diagram semigroups, and more.

Chapter 2

Sandwich semigroups

In this chapter, we aim to study sandwich semigroups in general. In order to do that, we define an additional type of structure - a partial semigroup (also known as semicategory, semigroupoid or precategory), which can be regarded as the "natural habitat" of sandwich semigroups. Studying it, we get the base for understanding the structure of sandwich semigroups contained within. After that, we delve into an investigation of a sandwich semigroup itself, examining Green's relations and their classes, benefits of stability, regularity, and invertibility, and the properties of its subsemigroup consisting of all regular elements (which exists under the assumption of sandwich-regularity), including its rank (for this, we study MI-domination). We also devote a section to the changes in general theory in the case when the researched category is inverse, and finally, we introduce some results on the rank of a sandwich semigroup. The major part of this chapter is based on [33], so we cite this paper as the source of the results and proofs unless stated otherwise.

First and foremost, we define a sandwich semigroup in general, so that the examples mentioned in the Section 1.1 fit into the definition.

Definition 2.0.1. Let $\mathcal{S}$ and I be a locally small category and its class of objects, respectively. Let $i, j \in I$ be two fixed objects (nodes) and let a be a fixed morphism $j \to i$. The semigroup $\mathcal{S}_{ij}^a = (\mathcal{S}_{ij}, \star_a)$, whose set of elements $\mathcal{S}_{ij}$ consists of all morphisms $i \to j$, with the operation $\star_a$ on it, defined with

$$x \star_a y = xay, \qquad \text{for all } x, y \in \mathcal{S}_{ij},$$

is the *sandwich semigroup* of $\mathcal{S}_{ij}$ with respect to a.

Remark 2.0.2. It is easy to see that, by choosing a specific category (transformations, diagrams, etc.), we choose the type of elements of the sandwich semigroup, and by fixing $i = j$, we choose our sandwich semigroup to be the variant $\mathcal{S}^a$ of the semigroup $(\mathcal{S}_{ii}, \circ)$, (in which the operation is simply the concatenation of arrows).

Note that, in this setting, I may be a proper class, and all the morphisms among its elements may form a proper class as well. However, it is important that $\mathcal{S}_{ij}$

be a set, in order for $\mathcal{S}_{ij}^a$ to be a semigroup. This is the justification for working specifically with locally small categories.

2.1 Partial semigroups

In order to understand the structure of sandwich semigroups, we introduce and study partial semigroups, in the same manner as in [30]. However, we add some new content from [28], in order to broaden the scope and depth of our investigation in the following sections. Thus, these two papers are the references for the results of this section, the most important ones being Green's Lemma for partial semigroups (Lemma 2.1.8) and its special version for the set $S_{ij} = \{z \in S : z\,\boldsymbol{\delta} = i, z\,\boldsymbol{\rho} = j\}$ (Lemma 2.1.9).

Definition 2.1.1. A *partial semigroup* is a 5-tuple $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ consisting of a class S, a partial binary map $(x, y) \mapsto x \cdot y$ (defined on some subset of $S \times S$), a class of "coordinates" I, and functions $\boldsymbol{\delta}, \boldsymbol{\rho} : S \to I$, which determine the left and right coordinates of elements of S, respectively; these five have to satisfy the following four conditions: for all $x, y, z \in S$,

(i) $x \cdot y$ is defined if and only if $x\,\boldsymbol{\rho} = y\,\boldsymbol{\delta}$
(two elements can be multiplied if and only if their "meeting coordinates" coincide);

(ii) if $x \cdot y$ is defined, then $(x \cdot y)\,\boldsymbol{\delta} = x\,\boldsymbol{\delta}$ and $(x \cdot y)\,\boldsymbol{\rho} = y\,\boldsymbol{\rho}$
(if two elements can be multiplied, the product keeps the "non-meeting coordinates" of the factors);

(iii) if $x \cdot y$ and $y \cdot z$ are defined, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
(we have associativity, provided that the products included are defined);

(iv) for any $i, j \in I$, the class $S_{ij} = \{x \in S : x\,\boldsymbol{\delta} = i,\ x\,\boldsymbol{\rho} = j\}$ is a set
(when we choose and fix two coordinates as left and right, the elements that have those coordinates form a set).

Moreover, a partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ is *monoidal* if it also satisfies the following:

(v) there exists a function $I \to S : i \mapsto e_i$ such that, for all $x \in S$, $x \cdot e_{x\,\boldsymbol{\rho}} = x = e_{x\,\boldsymbol{\delta}} \cdot x$
(for each coordinate there exists an identity element).

Remark 2.1.2. If S is a proper class, $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$ are not functions, strictly speaking, because their domain is not a set. However, we use the same familiar term, since the main quality, mapping each element from the domain to exactly one element of the codomain, stays the same.

Note that, if we interpret I as a class of nodes and S as the class of morphisms among them, the conditions (i-v) ensure that we are in fact dealing with a locally small category. Conversely, a locally small category can, in an obvious way, be interpreted as a monoidal partial semigroup. Therefore, we can use these terms interchangeably.

This synonymity leads us to the alternative definition of a sandwich semigroup (which is a generalisation of the previous one, since we do not demand partial semigroups to be monoidal):

Definition 2.1.3. Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup, $i, j \in I$ two fixed coordinates and let $a \in S_{ji}$ be an arbitrarily chosen, fixed element. Semigroup $S^a_{ij} = (S_{ij}, \star_a)$, where

$$x \star_a y = xay, \qquad \text{for all } x, y \in S_{ij},$$

is called the *sandwich semigroup* of S_{ij} with respect to a (which is called the *sandwich element*).

Since matching coordinates enable multiplication in all cases, and the definition of partial semigroups guarantees associativity, the term is well-defined. **From now on, by a sandwich semigroup, we mean a structure of the type described in Definition 2.1.3.**

To improve the readability of the thesis, we use the expressions hom-set and endomorphism semigroup, even in the case that S is not monoidal (i.e. not a locally small category).

Now, our plan is to investigate partial semigroups first, in order to get the "big picture", and then to "zoom in" on sandwich semigroups.

As usual, we shorten the notation $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ to S if the rest of the information is either unimportant for the discussion or clear from the context. From now on, the object of our interest is an arbitrary partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$, until stated otherwise.

Intuitively, a partial semigroup feels like a loose semigroup: some elements cannot be multiplied because they are not "connected", but whenever multiplication is possible, we have associativity. That feeling is further strengthened when one realises that any semigroup is a partial semigroup, with $|I| = 1$. Moreover, in a partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$, for every $i \in I$, the set S_{ii} (we usually denote it S_i) is a semigroup with respect to $\cdot\restriction_{S_i \times S_i}$, since all the elements have coinciding coordinates. Because of these similarities, we will use definitions and techniques similar to the ones we used for semigroups. For instance, an element x of a partial semigroup S is *regular* if there exists $y \in S$ such that $x \cdot y \cdot x = x$, in which case $y \in \text{Pre}(x)$ (y is a *pre-inverse* of x) and $x \in \text{Post}(y)$ (x is a *post-inverse* of y). The class of all regular elements in S is denoted $\text{Reg}(S)$ and S itself is *regular* if $\text{Reg}(S) = S$. If the partial semigroup in question is monoidal, it is natural to say that the corresponding category is regular. Note, however, that the term regular category has a different meaning in category theory. Here we always mean (von Neumann) regular.

Furthermore, if an element $x \in S$ is regular with $x = xyx$ for some $y \in S$, then the element $z = yxy$ is an *inverse* of x, i.e. it satisfies $x = xzx$ and $z = zxz$ (all the products obviously exist). The set of all inverses of x is denoted $\mathrm{V}(x)$. If each of its elements has a unique inverse, the partial semigroup itself is *inverse*.

Additionally, a *partial $*$-semigroup* is a 6-tuple $(S, I, \boldsymbol{\delta}, \boldsymbol{\rho}, \cdot, ^*)$ such that the structure $(S, I, \boldsymbol{\delta}, \boldsymbol{\rho}, \cdot)$ is a partial semigroup and $^* : S \to S : x \mapsto x^*$ is a mapping such that for all $x, y \in S$,

(a) $(x^*)\,\boldsymbol{\delta} = x\,\boldsymbol{\rho}$, $(x^*)\,\boldsymbol{\rho} = x\,\boldsymbol{\delta}$, and $(x^*)^* = x$;

(b) if $x \cdot y$ is defined, then $(x \cdot y)^* = y^* x^*$.

Finally, a *regular partial $*$-semigroup* is a partial $*$-semigroup such that $xx^*x = x$ for all $x \in S$. As in the case of regular $*$-semigroups, the elements of the form xx^* are called *projections* (and may be characterised as the elements x for which $x^2 = x = x^*$) and each inverse partial semigroup is a regular partial $*$-semigroup.

Continuing in this fashion, we define a map mimicking the natural embedding of a semigroup S into the corresponding monoid S^1. Namely, for our partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ we create a monoidal partial semigroup $S^{(1)}$: for each coordinate $i \in I$, we add an element e_i to S_{ii} acting as an identity (in cases in which it can be multiplied), if such an element does not already exist. The embedding $S \to S^{(1)} : x \mapsto x$ is the *natural embedding* of S into the corresponding monoidal partial semigroup.

Next, we introduce some examples of partial semigroups. The first one is trivial, and the other two describe types of partial semigroups that will be considered in the following chapters of the thesis.

Example 2.1.4. Let $\{S_i : i \in I\}$ be any family of pairwise disjoint semigroups (i.e. for all $i, j \in I$ holds $S_i \cap S_j = \emptyset$), and put $S = \bigcup_{i \in I} S_i$. For any $x \in S$ there exists exactly one $i \in I$ such that $x \in S_i$, so we define $x\,\boldsymbol{\delta} = x\,\boldsymbol{\rho} = i$. Thus, for any two elements $x, y \in S$, the multiplication $x \cdot y$ is defined only in the case that x and y belong to the same set S_i, and its result is the same as in semigroup S_i. It is clear that $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ is a partial semigroup, and it is monoidal (regular) if and only if S_i is a monoid (regular semigroup) for each $i \in I$. Moreover, there exists a mapping $^* : S \to S$ such that $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a (regular) partial $*$-semigroup if and only if S_i is a (regular) $*$-semigroup for each $i \in I$.

Example 2.1.5. Let $\mathbb{F}$ be a field and let $\mathcal{M}$ denote the set of all finite-dimensional, non-empty matrices over $\mathbb{F}$. We use the usual matrix multiplication $\cdot$, and introduce functions $\boldsymbol{\delta}, \boldsymbol{\rho} : \mathcal{M} \to \mathbb{N}$ describing the number of columns and the number of rows of a matrix, respectively. Then $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho})$ is a monoidal and regular partial semigroup, which we will discuss in detail in Chapter 4. Further, if we define $^* : \mathcal{M} \to \mathcal{M} : A_{i,j} \mapsto A_{j,i}$, to be the operation of *transposition* (turning rows into columns and vice versa), then $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a partial $*$-semigroup, but not a regular partial $*$-semigroup (for instance, for the square matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ we have $AA^*A \neq A$, since $1 + 1 \neq 1$ in any field). In fact, even though $\mathcal{M}$ is regular, there does not exist an operation $^* : \mathcal{M} \to \mathcal{M}$ such that $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a regular partial $*$-semigroup. For a detailed discussion, see Lemma 4.1.6.

Example 2.1.6. Let **Set** denote the class of all sets, and for all $A, B \in \mathbf{Set}$ let

$$\mathbf{T}_{AB} = \{f : f \text{ is a function } A \to B\}.$$

Now, define the class $\mathcal{T} = \{(A, f, B) : A, B \in \mathbf{Set},\ f \in \mathbf{T}_{AB}\}$, and maps $\boldsymbol{\delta} : \mathcal{T} \to \mathbf{Set} : (A, f, B) \to A$ and $\boldsymbol{\rho} : \mathcal{T} \to \mathbf{Set} : (A, f, B) \to B$. We are now in position to introduce the partial semigroup $(\mathcal{T}, \circ, \mathbf{Set}, \boldsymbol{\delta}, \boldsymbol{\rho})$. As discussed in Chapter 3, this partial semigroup is both regular and monoidal. However, no unary operation * on $\mathcal{T}$ satisfies the requirements for $(\mathcal{T}, \circ, \mathbf{Set}, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ to be a partial $*$-semigroup (see Proposition 3.0.3).

From the definition of a sandwich semigroup, we see a hint of duality, because the pair $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$ clearly refer to the left side and right side (coordinates) of an element. As it turns out, any partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ indeed has a *dual partial semigroup* $(S, \bullet, I, \boldsymbol{\rho}, \boldsymbol{\delta})$, in which the operation is defined in the following way:

$$x \bullet y = \begin{cases} y \cdot x, & y\,\boldsymbol{\rho} = x\,\boldsymbol{\delta}; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Furthermore, if $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a partial (regular) $*$-semigroup, it is easy to see that $(S, \bullet, I, \boldsymbol{\rho}, \boldsymbol{\delta}, ^*)$ is a partial (regular) $*$-semigroup since the map $* : S \to S$ determines an isomorphism from $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ to $(S, \bullet, I, \boldsymbol{\rho}, \boldsymbol{\delta}, ^*)$, and vice versa. This duality helps us to keep the proofs shorter and neater.

Before continuing to examine partial semigroups, in order to describe their internal structure, we will prepare proper notation and definitions, similar to the ones used for semigroups. Namely, for all $x, y \in S$, let

$$\begin{aligned} x \leq_{\mathscr{R}} y \quad &\Leftrightarrow \quad \text{there exists } s \in S^{(1)} \text{ such that } x = ys, \\ x \leq_{\mathscr{L}} y \quad &\Leftrightarrow \quad \text{there exists } s \in S^{(1)} \text{ such that } x = sy, \\ x \leq_{\mathscr{H}} y \quad &\Leftrightarrow \quad x \leq_{\mathscr{L}} y \text{ and } x \leq_{\mathscr{R}} y, \\ x \leq_{\mathscr{J}} y \quad &\Leftrightarrow \quad \text{there exist } s, t \in S^{(1)} \text{ such that } x = syt. \end{aligned}$$

Further, for all $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{J}\}$ we define the relation $\mathscr{K} = \leq_{\mathscr{K}} \cap \geq_{\mathscr{K}}$. Note that $x \,\mathscr{R}\, y$ ($x \,\mathscr{L}\, y$) implies $x\,\boldsymbol{\delta} = y\,\boldsymbol{\delta}$ ($x\,\boldsymbol{\rho} = y\,\boldsymbol{\rho}$), hence $x \,\mathscr{H}\, y$ implies both $x\,\boldsymbol{\delta} = y\,\boldsymbol{\delta}$ and $x\,\boldsymbol{\rho} = y\,\boldsymbol{\rho}$. Also, it is easy to prove that $\mathscr{R}$ is a left-congruence (in other words, $a \,\mathscr{R}\, b$ implies $sa \,\mathscr{R}\, sb$ for any $s \in S$ with $s\,\boldsymbol{\rho} = a\,\boldsymbol{\delta}$), and $\mathscr{L}$ is a right-congruence (i.e. $a \,\mathscr{L}\, b$ implies $as \,\mathscr{L}\, bs$ for any $s \in S$ with $s\,\boldsymbol{\delta} = a\,\boldsymbol{\rho}$). In the case of regular partial $*$-semigroups we may provide elegant characterisations for $\mathscr{R}$ and $\mathscr{L}$ (and thus for $\mathscr{H}$, as well) from [28]:

Lemma 2.1.7. *If S is a regular partial $*$-semigroup with $x, y \in S$, then*

(i) $x \leq_{\mathscr{R}} y \Leftrightarrow xx^* = yy^*xx^*$,

(ii) $x \,\mathscr{R}\, y \Leftrightarrow xx^* = yy^*$,

(iii) $x \leq_{\mathscr{L}} y \Leftrightarrow x^*x = x^*xy^*y$,

(iv) $x \,\mathscr{L}\, y \Leftrightarrow x^*x = y^*y$,

Proof. We prove only (i) and (ii), as the other two follow by duality.

(i) Let $x \leq_{\mathscr{R}} y$, i.e. $x = ys$ for some $s \in S$. Then,

$$xx^* = (ys)x^* = yy^*ysx^* = yy^*xx^*.$$

To prove the converse, suppose $xx^* = yy^*xx^*$. Then we have $x = (xx^*)x = yy^*xx^*x \leq_{\mathscr{R}} y$.

(ii) Since $\mathscr{R} = \leq_{\mathscr{R}} \cap \geq_{\mathscr{R}}$, by (i), we need to prove that

$$xx^* = yy^* \Leftrightarrow xx^* = yy^*xx^* \wedge yy^* = xx^*yy^*.$$

The direct implication is clear, and for the reverse note that $xx^* = (x^*)^*x^* = (xx^*)^* = (yy^*xx^*)^* = xx^*yy^* = yy^*$. □

Furthermore, we introduce the fifth relation $\mathscr{D} = \mathscr{R} \circ \mathscr{L}$. As is the case with semigroups, it can be proved that $\mathscr{D}$ is the smallest equivalence relation containing both $\mathscr{R}$ and $\mathscr{L}$, and that $\mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$. (The proof is analogous to the proof of Lemma 1.3, since all of the necessary products exist, due to elements being in the same $\mathscr{R}$- or $\mathscr{L}$-class.) Enhancing the notation, for each $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{J}, \mathscr{D}\}$ and each $x \in S$, we define

$$[x]_{\mathscr{K}} = \{y \in S : \ x \mathscr{K} y\}.$$

Since partial semigroups contain sandwich semigroups, and the latter are the real objects of our interest, we have to find a way to avoid confusion between the properties of a partial semigroup and the properties of sandwich semigroups it contains. Firstly, we denote Green's relations in a sandwich semigroup S_{ij}^a with $\mathscr{L}^a, \mathscr{R}^a, \mathscr{H}^a, \mathscr{J}^a$ and $\mathscr{D}^a$. Secondly, for each $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{J}, \mathscr{D}\}$, each $\mathrm{K} \in \{\mathrm{L}, \mathrm{R}, \mathrm{H}, \mathrm{J}, \mathrm{D}\}$ and each $x \in S_{ij}$ we define

$$\mathrm{K}_x^a = \{y \in S_{ij} : \ x \mathscr{K}^a y\} \qquad \text{and} \qquad \mathrm{K}_x = [x]_{\mathscr{K}} \cap S_{ij} = \{y \in S_{ij} : \ x \mathscr{K} y\}.$$

Thus, $[x]_{\mathscr{K}}$ is the *$\mathscr{K}$-class* of element x in S, K_x is the *$\mathscr{K}$-class* of element x in the hom-set S_{ij}, and K_x^a is the *$\mathscr{K}^a$-class* of element x in the sandwich semigroup S_{ij}^a. Lastly, we need the restriction of the $\leq_{\mathscr{J}}$ relation to the set $S_{ij} \times S_{ij}$. To simplify matters, we denote it with $\leq_{\mathscr{J}}$ as well, but emphasise that it is defined on S_{ij}.

Having done the necessary preparation, we may examine Green's relations of partial semigroups, as in [30]. The next lemma has the same formulation and proof (keeping in mind the information about coordinates that we gain from elements being in the same $\mathscr{R}$- or $\mathscr{L}$-class) as its semigroup counterpart, Lemma 1.3.4.

Lemma 2.1.8 (Green's lemma for partial semigroups)**.** *Let x, y be any elements of a partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$.*

(i) Suppose $x \mathscr{R} y$ and the elements $s, t \in S^{(1)}$ are such that $x = ys$ and $y = xt$. Then the maps $[x]_{\mathscr{L}} \to [y]_{\mathscr{L}} : w \to wt$ and $[y]_{\mathscr{L}} \to [x]_{\mathscr{L}} : w \to ws$ are mutually inverse bijections. These maps restrict to mutually inverse bijections $[x]_{\mathscr{H}} \to [y]_{\mathscr{H}}$ and $[y]_{\mathscr{H}} \to [x]_{\mathscr{H}}$.

(ii) *Suppose* $x \mathscr{L} y$ *and the elements* $s, t \in S^{(1)}$ *are such that* $x = sy$ *and* $y = tx$. *Then the maps* $[x]_{\mathscr{R}} \to [y]_{\mathscr{R}} : w \to tw$ *and* $[y]_{\mathscr{R}} \to [x]_{\mathscr{R}} : w \to sw$ *are mutually inverse bijections. These maps restrict to mutually inverse bijections* $[x]_{\mathscr{H}} \to [y]_{\mathscr{H}}$ *and* $[y]_{\mathscr{H}} \to [x]_{\mathscr{H}}$.

(iii) *If* I *is a set and* $x \mathscr{D} y$, *then* $|[x]_{\mathscr{R}}| = |[y]_{\mathscr{R}}|$, $|[x]_{\mathscr{L}}| = |[y]_{\mathscr{L}}|$ *and* $|[x]_{\mathscr{H}}| = |[y]_{\mathscr{H}}|$.

We can give even more information, if we focus on a single set S_{ij}:

Lemma 2.1.9. *Let* $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ *be a partial semigroup,* $i, j \in I$ *and let* x, y *be any elements of the set* $S_{ij} = \{z \in S : \ z\,\boldsymbol{\delta} = i, \ z\,\boldsymbol{\rho} = j\}$.

(i) *Suppose* $x \mathscr{R} y$ *and the elements* $s, t \in S^{(1)}$ *are such that* $x = ys$ *and* $y = xt$. *Then the maps* $\mathrm{L}_x \to \mathrm{L}_y : w \to wt$ *and* $\mathrm{L}_y \to \mathrm{L}_x : w \to ws$ *are mutually inverse bijections. These maps restrict to mutually inverse bijections* $\mathrm{H}_x \to \mathrm{H}_y$ *and* $\mathrm{H}_y \to \mathrm{H}_x$.

(ii) *Suppose* $x \mathscr{L} y$ *and the elements* $s, t \in S^{(1)}$ *are such that* $x = sy$ *and* $y = tx$. *Then the maps* $\mathrm{R}_x \to \mathrm{R}_y : w \to tw$ *and* $\mathrm{R}_y \to \mathrm{R}_x : w \to sw$ *are mutually inverse bijections. These maps restrict to mutually inverse bijections* $\mathrm{H}_x \to \mathrm{H}_y$ *and* $\mathrm{H}_y \to \mathrm{H}_x$.

(iii) *If* $x \mathscr{D} y$, *then* $|\,\mathrm{R}_x\,| = |\,\mathrm{R}_y\,|$, $|\,\mathrm{L}_x\,| = |\,\mathrm{L}_y\,|$ *and* $|\,\mathrm{H}_x\,| = |\,\mathrm{H}_y\,|$.

Proof. We prove (i), and (ii) will follow by duality. Let $x \mathscr{R} y$ and suppose the elements $s, t \in S^{(1)}$ are such that $x = ys$ and $y = xt$. Lemma 2.1.8(i) guarantees that the maps $[x]_{\mathscr{L}} \to [y]_{\mathscr{L}} : w \to wt$ and $[y]_{\mathscr{L}} \to [x]_{\mathscr{L}} : w \to ws$ are mutually inverse bijections. If we prove that these functions map L_x to L_y and L_y to L_x, respectively, Lemma 2.1.8(i) will imply (i) (the second part follows from the fact that $[x]_{\mathscr{H}} = \mathrm{H}_x$ and $[y]_{\mathscr{H}} = \mathrm{H}_y$). In fact, we need to prove only one of these statements, because the proof for the other one is analogous. Suppose $w \in \mathrm{L}_x$; thus, $w \in S_{ij}$ and there exists $q \in S^{(1)}$ such that $qx = w$. The element w maps to $wt = qxt = qy$, so $(wt)\,\boldsymbol{\rho} = (qy)\,\boldsymbol{\rho} = y\,\boldsymbol{\rho} = j$ and $(wt)\,\boldsymbol{\delta} = w\,\boldsymbol{\delta} = i$, hence $wt \in S_{ij}$. Since Lemma 2.1.8(i) implies $wt \in [y]_{\mathscr{L}}$, we have $wt \in \mathrm{L}_y$.

(iii) Suppose $x \mathscr{D} y$. Then there exists $z \in S$ such that $x \mathscr{R} z \mathscr{L} y$. Therefore $z\,\boldsymbol{\delta} = x\,\boldsymbol{\delta} = i$ and $z\,\boldsymbol{\rho} = y\,\boldsymbol{\rho} = j$, so $z \in \mathrm{R}_x \cap \mathrm{L}_y$. Now (i) and (ii) together imply the statement. □

We have gained the insight we needed into partial semigroups, so we move on to our main topic, sandwich semigroups.

2.2 Sandwich semigroups

In this section, we focus on a sandwich semigroup S^a_{ij} in a fixed partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$. As in [30], we prove the theorem on Green's relations of a sandwich semigroup, and then we focus on the results from [33] and [28]. Namely, we investigate maximal $\mathscr{J}$-classes in a sandwich semigroup, its stability and some properties of its regular elements. Furthermore, we examine S^a_{ij} in the case that a is (right-)invertible, and study the characteristics of partial subsemigroups (i.e. the appropriate substructure) in partial semigroups.

Naturally, properties of a sandwich semigroup have a lot to do with the partial semigroup containing it, but maybe even more with the chosen sandwich element. Namely, this element determines the so-called *P-sets*, defined with

$$\mathrm{P}_1^a = \{x \in S_{ij} : xa \mathscr{R} x\}, \qquad \mathrm{P}_2^a = \{x \in S_{ij} : ax \mathscr{L} x\},$$
$$\mathrm{P}_3^a = \{x \in S_{ij} : axa \mathscr{J} x\}, \qquad \mathrm{P}^a = \mathrm{P}_1^a \cap \mathrm{P}_2^a,$$

which (as we are about to show in the first subsection) shape the Green's relations of a sandwich semigroup.

Before we continue, we give alternative definitions for the above defined P-sets, which will be of help later on. For the first one, note that $x \in \mathrm{P}_1^a$ means that $x = xas$ for some $s \in S^{(1)}$, which implies $x = xas = (xas)as = xa(sas)$; since $sas \in S$ (not just $S^{(1)}$), we have $sas \in S_{ij}$ and $x \in xaS_{ij}$. As $xaS_{ij} \subseteq xaS^{(1)}$, we have proved the first of the following equalities (the rest are shown similarly):

$$\mathrm{P}_1^a = \{x \in S_{ij} : x \in xaS_{ij}\}, \qquad \mathrm{P}_2^a = \{x \in S_{ij} : x \in S_{ij}ax\},$$
$$\mathrm{P}_3^a = \{x \in S_{ij} : x \in S_{ij}axaS_{ij}\}.$$

In the case that S is a regular partial $*$-semigroup, we may provide even simpler characterisations for P_1^a and P_2^a, from [28]:

Lemma 2.2.1. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ be a regular partial $*$-semigroup with $i, j \in I$ and $a \in S_{ji}$. Then*

(i) $\mathrm{P}_1^a = \{x \in S_{ij} : x^*x \in \mathrm{Post}(aa^*)\} = \{x \in S_{ij} : aa^* \in \mathrm{Pre}(x^*x)\}$,

(ii) $\mathrm{P}_2^a = \{x \in S_{ij} : xx^* \in \mathrm{Post}(a^*a)\} = \{x \in S_{ij} : a^*a \in \mathrm{Pre}(xx^*)\}$,

Proof. We will show (i), and (ii) will follow by duality. The second equality is clear, since $u \in \mathrm{Pre}(v) \Leftrightarrow v \in \mathrm{Post}(u)$ for any $u, v \in S$. To prove the first one, we consider the two inclusions. Suppose $x \in \mathrm{P}_1^a$, i.e. $x \mathscr{R} xa$; Lemma 2.1.7(ii) then implies $xx^* = (xa)(xa)^* = xaa^*x^*$, so $x^*x = x^*(xx^*)x = x^*xaa^*x^*x$, which proves that $x^*x \in \mathrm{Post}(aa^*)$. For the reverse containment note that from $x^*x = x^*xaa^*x^*x$ follows

$$x = xx^*x = xx^*xaa^*x^*x = xaa^*x^*x \leq_{\mathscr{R}} xa. \qquad \square$$

2.2.1 Green's relations of sandwich semigroups

In the next proposition, we prove a few important properties of P-sets. The first part was proven in [30] and the other two in [29]. The whole proposition serves as a prelude to the crucial theorem following it.

Proposition 2.2.2. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup, with $i, j \in I$ and $a \in S_{ji}$. If $y \in S_{ij}$ is an arbitrary element, then*

(i) $\mathrm{Reg}(S_{ij}^a) \subseteq \mathrm{P}^a \subseteq \mathrm{P}_3^a$,

(ii) $y \in \mathrm{P}_1^a$ *if and only if* $\mathrm{L}_y \subseteq \mathrm{P}_1^a$,

(iii) $y \in \mathrm{P}_2^a$ *if and only if* $\mathrm{R}_y \subseteq \mathrm{P}_2^a$.

Proof. (i) If $x \in \mathrm{Reg}(S_{ij}^a)$, there exists $z \in S_{ij}$ such that $x = x \star_a z \star_a x = xazax$, so

$$x = xa \cdot zax \quad \text{and} \quad xa = x \cdot a \quad \text{imply} \quad x \mathscr{R} xa,$$
$$x = xaz \cdot ax \quad \text{and} \quad ax = a \cdot x \quad \text{imply} \quad x \mathscr{L} ax,$$

hence $x \in \mathrm{P}^a$.

If we assume $x \in \mathrm{P}^a$, then $x \mathscr{R} xa$ and $x \mathscr{L} ax$, thus there exist $u, v \in S_{ij}$ such that $x = xau$ and $x = vax$. Therefore $x = v \cdot axa \cdot u$; this, together with $axa = a \cdot x \cdot a$ gives $a \mathscr{J} axa$, i.e. $x \in \mathrm{P}_3^a$.

(ii) We will prove only the direct implication, since the other one is trivial. Suppose $z \in \mathrm{L}_y$ is arbitrary. Then there exists $s \in S^{(1)}$ so that $z = sy$, and from $y \in \mathrm{P}_1^a$ we have $y \mathscr{R} ya$. Since $\mathscr{R}$ is a left-congruence, $z = sy \mathscr{R} sya = za$ and therefore $z \in \mathrm{P}_1^a$.

(iii) Dual to (ii). □

Finally, we are able to prove the theorem from [30] describing Green's relations in a sandwich semigroup.

Theorem 2.2.3. *Let* $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ *be a partial semigroup with* $i, j \in I$ *and* $a \in \mathcal{S}_{ji}$. *If* $x \in S_{ij}$, *then*

(i) $\mathrm{R}_x^a = \begin{cases} \mathrm{R}_x \cap \mathrm{P}_1^a, & \text{if } x \in \mathrm{P}_1^a \\ \{x\}, & \text{if } x \in S_{ij} \setminus \mathrm{P}_1^a, \end{cases}$

(ii) $\mathrm{L}_x^a = \begin{cases} \mathrm{L}_x \cap \mathrm{P}_2^a, & \text{if } x \in \mathrm{P}_2^a \\ \{x\}, & \text{if } x \in S_{ij} \setminus \mathrm{P}_2^a, \end{cases}$

(iii) $\mathrm{H}_x^a = \begin{cases} \mathrm{H}_x, & \text{if } x \in \mathrm{P}^a \\ \{x\}, & \text{if } x \in S_{ij} \setminus \mathrm{P}^a, \end{cases}$

(iv) $\mathrm{D}_x^a = \begin{cases} \mathrm{D}_x \cap \mathrm{P}^a, & \text{if } x \in \mathrm{P}^a \\ \mathrm{L}_x^a, & \text{if } x \in \mathrm{P}_2^a \setminus \mathrm{P}_1^a \\ \mathrm{R}_x^a, & \text{if } x \in \mathrm{P}_1^a \setminus \mathrm{P}_2^a \\ \{x\}, & \text{if } x \in S_{ij} \setminus (\mathrm{P}_1^a \cup \mathrm{P}_2^a), \end{cases}$

(v) $\mathrm{J}_x^a = \begin{cases} \mathrm{J}_x \cap \mathrm{P}_3^a, & \text{if } x \in \mathrm{P}_3^a \\ \mathrm{D}_x^a, & \text{if } x \in S_{ij} \setminus \mathrm{P}_3^a. \end{cases}$

If $x \in S_{ij} \setminus \mathrm{P}^a$, *then* $\mathrm{H}_x^a = \{x\}$ *is a non-group* $\mathscr{H}^a$*-class in* S_{ij}^a.

Remark 2.2.4. Since the classes R_x^a and L_x^a are described in the same theorem, the expression for D_x^a in Theorem 2.2.3 may be simplified in the following way:

$$\mathrm{D}_x^a = \begin{cases} \mathrm{D}_x \cap \mathrm{P}^a, & \text{if } x \in \mathrm{P}^a \\ \mathrm{L}_x^a, & \text{if } x \in S_{ij} \setminus \mathrm{P}_1^a \\ \mathrm{R}_x^a, & \text{if } x \in S_{ij} \setminus \mathrm{P}_2^a \end{cases}$$

However, to avoid confusion, we use the former because its determining classes do not intersect.

Proof. (*i*) Let $y \in \mathrm{R}^a_x \setminus \{x\}$. This implies the existence of $z, q \in S_{ij}$ such that $x = y \star_a z = yaz$ and $y = x \star_a q = xaq$. Note that $x = xaqaz$, so $x \mathscr{R}\, xa$, i.e. $x \in \mathrm{P}^a_1$. Therefore, in every case in which there exists an element $y \in \mathrm{R}^a_x \setminus \{x\}$, we have $x \in \mathrm{P}^a_1$. We conclude that R^a_x is a singleton $\{x\}$ if $x \in S_{ij} \setminus \mathrm{P}^a_1$. Furthermore, from $x = y \star_a z = yaz$ and $y = x \star_a q = xaq$ we may also deduce $x \mathscr{R}\, y$ and $y = yazaq$, so $y \mathscr{R}\, ya$ and therefore $y \in \mathrm{R}_x \cap \mathrm{P}^a_1$. So, $\mathrm{R}^a_x \subseteq \mathrm{R}_x \cap \mathrm{P}^a_1$ in the case that $x \in \mathrm{P}^a_1$. We need to prove the reverse inclusion. Suppose $x \in \mathrm{P}^a_1$ and let $y \in \mathrm{R}_x \cap \mathrm{P}^a_1$. Then, there exist $z, q \in S^{(1)}$ and $t, s \in S_{ij}$ such that $y = xz$, $x = yq$, $y = yat$ and $x = xas$. Thus $y = xz = xasz = x \star_a sz$ and $x = yq = yatq = y \star_a tq$, where $(sz)\,\boldsymbol{\delta} = s\,\boldsymbol{\delta} = a\,\boldsymbol{\rho} = i$, $(sz)\,\boldsymbol{\rho} = z\,\boldsymbol{\rho} = y\,\boldsymbol{\rho} = j$ and $(tq)\,\boldsymbol{\delta} = t\,\boldsymbol{\delta} = a\,\boldsymbol{\rho} = i$, $(tq)\,\boldsymbol{\rho} = q\,\boldsymbol{\rho} = x\,\boldsymbol{\rho} = j$, so $sz, tq \in S_{ij}$ and $y \in \mathrm{R}^a_x$ immediately follows.

(*ii*) is dual to (*i*).

(*iii*) Since $\mathrm{P}^a = \mathrm{P}^a_1 \cap \mathrm{P}^a_2$, from (*i*) and (*ii*) one may immediately deduce

$$\mathrm{H}^a_x = \mathrm{R}^a_x \cap \mathrm{L}^a_x = \begin{cases} \mathrm{H}_x \cap \mathrm{P}^a, & \text{if } x \in \mathrm{P}^a \\ \{x\}, & \text{if } x \in S_{ij} \setminus \mathrm{P}^a. \end{cases}$$

Hence, we just need to prove that $\mathrm{H}_x \subseteq \mathrm{P}^a$ if $x \in \mathrm{P}^a$. But by Proposition 2.2.2(*ii*) and (*iii*), from $x \in \mathrm{P}^a_1 \cap \mathrm{P}^a_2$ we have $\mathrm{L}_x \subseteq \mathrm{P}^a_1$ and $\mathrm{R}_x \subseteq \mathrm{P}^a_2$. Thus, $\mathrm{H}_x = \mathrm{L}_x \cap \mathrm{R}_x \subseteq \mathrm{P}^a_1 \cap \mathrm{P}^a_2 = \mathrm{P}^a$.

(*iv*) It is easy to see that

$$\mathrm{D}^a_x = \bigcup_{y \in \mathrm{R}^a_x} \mathrm{L}^a_y = \bigcup_{y \in \mathrm{L}^a_x} \mathrm{R}^a_y\,. \tag{2.1}$$

In the case $x \notin \mathrm{P}^a_1$, (*i*) implies $\mathrm{R}^a_x = \{x\}$ so $\mathrm{D}^a_x = \mathrm{L}^a_x$. Similarly, if $x \notin \mathrm{P}^a_2$, then $\mathrm{L}^a_x = \{x\}$ and $\mathrm{D}^a_x = \mathrm{R}^a_x$. If both of these conditions hold (i.e. $x \notin \mathrm{P}^a_1 \cup \mathrm{P}^a_2$), (*i*) and (*ii*) together imply $\mathrm{D}^a_x = \{x\}$. Now, suppose $x \in \mathrm{P}^a_1 \cap \mathrm{P}^a_2$. Since $\mathrm{R}_x \subseteq \mathrm{P}^a_2$ (by Proposition 2.2.2(iii)), from (2.1), (*i*) and (*ii*), we deduce

$$\mathrm{D}^a_x = \bigcup_{y \in \mathrm{R}_x \cap \mathrm{P}^a_1} \left(\mathrm{L}_y \cap \mathrm{P}^a_2 \right) = \mathrm{P}^a_2 \cap \bigcup_{y \in \mathrm{R}_x \cap \mathrm{P}^a_1} \mathrm{L}_y\,. \tag{2.2}$$

From Proposition 2.2.2(ii) we know that

$$\mathrm{L}_y \cap \mathrm{P}^a_1 = \begin{cases} \mathrm{L}_y, & y \in \mathrm{P}^a_1; \\ \emptyset, & y \in S_{ij} \setminus \mathrm{P}^a_1. \end{cases} \tag{2.3}$$

In the case $y \in \mathrm{P}^a_1$ the equality is obvious, and for the other, assume that $z \in \mathrm{L}_y \cap \mathrm{P}^a_1$, and then $\mathrm{L}_y = \mathrm{L}_z \subseteq \mathrm{P}^a_1$ contradicts $y \notin \mathrm{P}^a_1$. Thus, we deduce $\bigcup_{y \in \mathrm{R}_x \cap \mathrm{P}^a_1} \mathrm{L}_y = \bigcup_{y \in \mathrm{R}_x \cap \mathrm{P}^a_1} (\mathrm{L}_y \cap \mathrm{P}^a_1)$ because $y \in \mathrm{P}^a_1$. Also from (2.3), we infer $\bigcup_{y \in \mathrm{R}_x \cap \mathrm{P}^a_1} (\mathrm{L}_y \cap \mathrm{P}^a_1) = \bigcup_{y \in \mathrm{R}_x} (\mathrm{L}_y \cap \mathrm{P}^a_1)$, as $\mathrm{L}_y \cap \mathrm{P}^a_1$ equals $\emptyset$ if $y \in \mathrm{R}_x \setminus \mathrm{P}^a_1$. So, continuing the line (2.2), we have

$$\mathrm{D}^a_x = \mathrm{P}^a_2 \cap \bigcup_{y\in \mathrm{R}_x} (\mathrm{L}_y \cap \mathrm{P}^a_1) = \mathrm{P}^a_2 \cap \mathrm{P}^a_1 \cap \bigcup_{y\in \mathrm{R}_x} \mathrm{L}_y = \mathrm{P}^a \cap \bigcup_{y\in \mathrm{R}_x} \mathrm{L}_y = \mathrm{P}^a \cap \mathrm{D}_x \,.$$

(v) Similarly as in the first three cases, suppose $y \in \mathrm{J}^a_x \setminus \{x\}$. The definition of the relation $\mathscr{J}^a$ implies that $x \leq_{\mathscr{J}^a} y$ and $y \leq_{\mathscr{J}^a} x$, so exactly one of (a-c) holds and exactly one of (d-f) holds:

(a) $x = yaz$, for some $z \in S_{ij}$,

(b) $x = nay$, for some $n \in S_{ij}$,

(c) $x = nayaz$, for some $z, n \in S_{ij}$,

(d) $y = xav$, for some $v \in S_{ij}$,

(e) $y = wax$, for some $w \in S_{ij}$,

(f) $y = waxav$, for some $v, w \in S_{ij}$.

Now, since $x \neq y$, we can make some useful conclusions in the following combinations of cases:

a,d: $x \,\mathscr{R}^a\, y$, so (i) gives $x, y \in \mathrm{P}^a_1$,

b,e: $x \,\mathscr{L}^a\, y$, so (ii) gives $x, y \in \mathrm{P}^a_2$.

In any other combination of cases, one may prove $x \,\mathscr{J}\, axa$ and $y \,\mathscr{J}\, aya$, so we have $x, y \in \mathrm{P}^a_3$. Thus, J^a_x not being a singleton implies $x \in \mathrm{P}^a_1 \cup \mathrm{P}^a_2 \cup \mathrm{P}^a_3$. In other words, if $x \in S_{ij} \setminus (\mathrm{P}^a_1 \cup \mathrm{P}^a_2 \cup \mathrm{P}^a_3)$ then $\mathrm{J}^a_x = \{x\} = \mathrm{D}^a_x$ (the last equality follows from (iv)).

We need to examine three more cases:

$\mathbf{x \in P^a_1 \setminus P^a_3}$ Note that $\mathrm{D}^a_x \subseteq \mathrm{J}^a_x$, since $\mathscr{D}^a \subseteq \mathscr{J}^a$. We prove the reverse inclusion. Suppose $y \in \mathrm{J}^a_x$ and $y \neq x$ (because $y = x$ clearly implies $y \in \mathrm{D}^a_x$). As above, we know that one of (a-c) and one of (d-f) holds, and that the combination (a),(d) implies $y \in \mathrm{R}^a_x \subseteq \mathrm{D}^a_x$. We have also proved that the combination (b),(e) implies $x, y \in \mathrm{P}^a_2$ and any other combination gives $x, y \in \mathrm{P}^a_3$. However, from $x \in \mathrm{P}^a_1 \setminus \mathrm{P}^a_3$ we deduce $x \notin \mathrm{P}^a_2$, since Proposition 2.2.2(i) guarantees $\mathrm{P}^a = \mathrm{P}^a_1 \cap \mathrm{P}^a_2 \subseteq \mathrm{P}^a_3$. Therefore, we have drawn contradicting conclusions in any combination, except for (a),(d), so that is the only case possible.

$\mathbf{x \in P^a_2 \setminus P^a_3}$ Dual to the previous one.

$\mathbf{x \in P^a_3}$ We need to show that $\mathrm{J}^a_x = \mathrm{J}_x \cap \mathrm{P}^a_3$. Suppose $y \in \mathrm{J}_x \cap \mathrm{P}^a_3$. This means that there exist $z, q, s, t \in S^{(1)}$ and $w, r, u, v \in S$ such that

$$y = zxq, \qquad x = syt, \qquad y = wayar, \qquad x = uaxav.$$

Hence, $y = zxq = zuaxavq = zu \star_a x \star_a vq$ and $x = syt = swayart = sw \star_a y \star_a rt$, which implies $zu, vq, sw, rt \in S_{ij}$, and therefore $y \in \mathrm{J}^a_x$.

Let us show the reverse inclusion. Suppose $y \in \mathrm{J}^a_x$. Obviously, $y \in \mathrm{J}_x$. If $y = x$, we evidently have $y \in \mathrm{P}^a_3$, so we focus on the case $y \neq x$. From the above discussion, we know that one of (a-c) and one of (d-f) holds, that the combinations (a),(d) and (b),(e) imply $x, y \in \mathrm{P}_1^a$ and $x, y \in \mathrm{P}_2^a$, respectively, and that any other combination implies $x, y \in \mathrm{P}_3^a$. Therefore, we need to discuss only the cases when (a),(d) or (b),(e) hold. In fact, we may focus solely on the case (a),(d), because the other one is symmetrical. So, let $z, v \in S_{ij}$ be elements such that $x = yaz$ and $y = xav$. Since $x \in \mathrm{P}_3^a$, there exist $t, s \in S$ such that $x = taxas$. We may conclude that $y = xav = taxasav = tayazasav$, so $y \in \mathrm{P}_3^a$.

For the final statement about $\mathscr{H}^a$-classes, we prove the contrapositive. Suppose that H_x^a is a group with identity e. Then $x = x \star_a e = e \star_a x = xae = eax$. Thus $x \mathscr{R} xa$ and $x \mathscr{L} ax$, so $x \in \mathrm{P}^a$. □

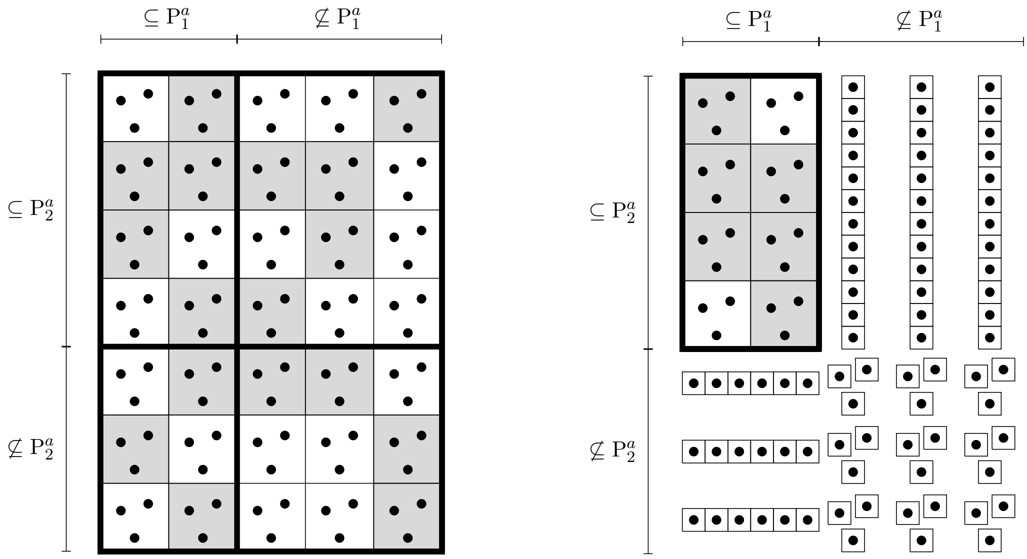


Figure 2.1: A schematic diagram from [29], giving a visual presentation of the way a $\mathscr{D}$-class of S_{ij} breaks up into $\mathscr{D}^a$-classes in S_{ij}^a. The reader should note that the elements belonging to P_1^a and P_2^a preserve their $\mathscr{R}$- and $\mathscr{L}$-classes, respectively. The group $\mathscr{H}$-classes are shaded, to illustrate that this property is not necessarily preserved.

Remark 2.2.5. The meaning of Theorem 2.2.3 is easier to discern using visual aids, so we provide the reader with Figure 2.1 showing the splitting of a $\mathscr{D}$-class of a hom-set S_{ij} into multiple $\mathscr{D}^a$-classes in S_{ij}^a. Furthermore, figures 3.4–3.8, 4.4–4.7 and 5.10–5.12 display the egg-box diagrams of various sandwich semigroups.

Having achieved this goal, let us linger on the same topic a little bit more, exploring the special case arising if the sandwich element has a left- and right-identity. This requirement is not terribly restrictive. As a matter of fact, all the partial semigroups that we study in this thesis are monoidal, so all their elements have a left- and right-identity.

The rest of the results in this subsection were proved in [28].

Lemma 2.2.6. *Suppose $a \in S_{ji}$ has a left- and right-identity in S. If $x, y \in S_{ij}$, then*

(i) $x \leq_{\mathscr{R}^a} y \Leftrightarrow x = y$ *or* $x \leq_{\mathscr{R}} ya$*,*

(ii) $x \leq_{\mathscr{L}^a} y \Leftrightarrow x = y$ *or* $x \leq_{\mathscr{L}} ay$*,*

(iii) $x \leq_{\mathscr{J}^a} y \Leftrightarrow x = y$ *or* $x \leq_{\mathscr{R}} ya$ *or* $x \leq_{\mathscr{L}} ay$ *or* $x \leq_{\mathscr{J}} aya$*.*

Proof. Consider the following equalities:

(a) $x = y$,

(b) $x = say$ for some $s \in S_{ij}$,

(c) $x = yam$ for some $m \in S_{ij}$,

(d) $x = sayam$ for some $s, m \in S_{ij}$.

By the definition of Green's preorders in S_{ij}^a, we have

$$x \leq_{\mathscr{L}^a} y \Leftrightarrow (a) \vee (b), \quad x \leq_{\mathscr{R}^a} y \Leftrightarrow (a) \vee (c), \quad \text{and} \quad x \leq_{\mathscr{J}^a} y \Leftrightarrow (a) \vee (b) \vee (c) \vee (d).$$

Since $(b) \Rightarrow x \leq_{\mathscr{L}} ay$, $(c) \Rightarrow x \leq_{\mathscr{R}} ya$ and $(d) \Rightarrow x \leq_{\mathscr{J}} aya$, the direct implications in $(i)-(iii)$ hold. For the converse ones, suppose $e, g \in S$ are left- and right-identities of a (i.e. $ea = ag = a$); then,

$$x \leq_{\mathscr{L}} ay \Leftrightarrow x = say = s(ea)y = (se)ay, \text{ for some } s \in S^{(1)},$$
$$x \leq_{\mathscr{R}} ya \Leftrightarrow x = yam = y(ag)m = ya(gm), \text{ for some } m \in S^{(1)},$$
$$x \leq_{\mathscr{J}} aya \Leftrightarrow x = sayam = s(ea)y(ag)m = (se)aya(gm), \text{ for some } s \in S^{(1)},$$

where se or/and gm belong to S_{ij}, in each case. Therefore, $x \leq_{\mathscr{L}} ay \Rightarrow (b)$, $x \leq_{\mathscr{R}} ya \Rightarrow (c)$ and $x \leq_{\mathscr{J}} aya \Rightarrow (d)$. □

In the next proposition, we show simplifications that occur in Lemma 2.2.6(iii) when one of the elements concerned belongs to one of the P-sets.

Proposition 2.2.7. *Suppose $a \in S_{ji}$ has a left- and right-identity in S and let $x, y \in S_{ij}$.*

(i) *If* $x \in \mathrm{P}_1^a$*, then* $x \leq_{\mathscr{J}^a} y \Leftrightarrow x \leq_{\mathscr{J}} aya$ *or* $x \leq_{\mathscr{R}} ya$*.*

(ii) *If* $x \in \mathrm{P}_2^a$*, then* $x \leq_{\mathscr{J}^a} y \Leftrightarrow x \leq_{\mathscr{J}} aya$ *or* $x \leq_{\mathscr{L}} ay$*.*

(iii) *If* $x \in \mathrm{P}_3^a$*, then* $x \leq_{\mathscr{J}^a} y \Leftrightarrow x \leq_{\mathscr{J}} aya$*.*

(iv) *If* $y \in \mathrm{P}_1^a$*, then* $x \leq_{\mathscr{J}^a} y \Leftrightarrow x \leq_{\mathscr{J}} ay$ *or* $x \leq_{\mathscr{R}} y$*.*

(v) *If* $y \in \mathrm{P}_2^a$*, then* $x \leq_{\mathscr{J}^a} y \Leftrightarrow x \leq_{\mathscr{J}} ya$ *or* $x \leq_{\mathscr{L}} y$*.*

(vi) If $y \in \mathrm{P}_3^a$, then $x \leq_{\mathscr{J}^a} y \Leftrightarrow x \leq_{\mathscr{J}} y$.

Proof. In the proof of Lemma 2.2.6 we have concluded that, under the assumption of a having a left- and right-identity, we have $x \leq_{\mathscr{J}^a} y \Leftrightarrow (a) \vee (b) \vee (c) \vee (d)$ and

$$(b) \Leftrightarrow x \leq_{\mathscr{L}} ay, \quad (c) \Leftrightarrow x \leq_{\mathscr{R}} ya \quad \text{and} \quad (d) \Leftrightarrow x \leq_{\mathscr{J}} aya.$$

Therefore, for the first three statements, we need to prove only the direct implications.

(i) By definition, $x \in \mathrm{P}_1^a$ means that $x = xav$ for some $v \in S_{ij}$, so (a) implies $x = xav = yav$ and we have (c). Furthermore, (b) gives $x = xav = sayav$, so we have (d) in this case. Therefore, we have $x \leq_{\mathscr{J}^a} y \Leftrightarrow (c) \vee (d)$. Part (ii) follows by duality.

(iii) From $x \in \mathrm{P}_3^a$ we have $x = uaxav$ for some $u, v \in S_{ij}$, so by substituting $uaxav$ for x in each case, we conclude that (a), (b) and (c) all imply (d).

Now, suppose that $e, g \in S$ are a left- and right- identity of a, i.e. that $a = ea = ag$.

(iv) Let $y \in \mathrm{P}_1^a$; then, $y = yav$ for some $v \in S_{ij}$. Thus, in case when we have (b), $x = say = sayav$, so (d) holds, as well. Now, evidently (a) and (c) both imply $x \leq_{\mathscr{R}} y$ and (d) implies $x \leq_{\mathscr{J}} ay$. For the converse, suppose that $x \leq_{\mathscr{J}} ay$ or $x \leq_{\mathscr{R}} y$. In the first case, $x = sayt$ for some $s, t \in S^{(1)}$, so

$$x = sayt = sayavt = (se)aya(gt),$$

with $se, gt \in S_{ij}$. Hence, (d) is true. In the second case, for some $t \in S^{(1)}$ holds $x = yt$, therefore $x = yt = yavt$ with $vt \in S_{ij}$, so (c) is true.

(v) is dual to (iv).

(vi) The direct implication is clear, as $x \leq_{\mathscr{J}} y$ follows from each (a)$-$(d). For the converse, note that from $y \in \mathrm{P}_3^a$ we have $y = uayav$ for some $u, v \in S_{ij}$ and $x \leq_{\mathscr{J}} y$ gives $x = syt$ for some $s, t \in S^{(1)}$; so $x = syt = suayavt$ with $su, vt \in S_{ij}$, implying (d). □

Remark 2.2.8. Note that, since $\mathrm{P}^a \subseteq \mathrm{P}_3^a$ (by Proposition 2.2.2), statements (iii) and (vi) of Proposition 2.2.7 hold for P^a, too.

Analysing the previous proposition, one may wonder whether we can make similar statements about the relations $\mathscr{R}^a$ and $\mathscr{L}^a$. It turns out that some combinations (e.g. $x \in \mathrm{P}_2^a$ and $x \leq_{\mathscr{R}^a} y$) do not provide any simplifications to Lemma 2.2.6. In fact, only four of them do.

Proposition 2.2.9. *Suppose $a \in S_{ji}$ has a left- and right-identity in S and let $x, y \in S_{ij}$.*

(i) If $x \in \mathrm{P}_1^a$, then $x \leq_{\mathscr{R}^a} y \Leftrightarrow x \leq_{\mathscr{R}} ya$.

(ii) If $y \in \mathrm{P}_1^a$, then $x \leq_{\mathscr{R}^a} y \Leftrightarrow x \leq_{\mathscr{R}} y$.

(iii) If $x \in \mathrm{P}_2^a$, then $x \leq_{\mathscr{L}^a} y \Leftrightarrow x \leq_{\mathscr{L}} ay$.

(iv) If $y \in \mathrm{P}_2^a$, *then* $x \leq_{\mathscr{L}^a} y \Leftrightarrow x \leq_{\mathscr{L}} y$.

Proof. We prove only (i) and (ii), because the other two are their dual statements. From the proof of Lemma 2.2.6, we have $x \leq_{\mathscr{R}^a} y \Leftrightarrow (a) \vee (c)$. We may immediately conclude (by the same lemma) that the reverse implication in (i) is true. The other one is also clear, since $x \in \mathrm{P}_1^a$ implies $x = xav$ for some $v \in S_{ij}$, and then (a) and (c) both give $x \leq_{\mathscr{R}} ya$. Now, we prove (ii); the direct implication being obvious, we focus on the reverse. Since $y \in \mathrm{P}_1^a$, for some $v \in S_{ij}$ we have $y = yav$. If $x \leq_{\mathscr{R}} y$, i.e. $x = yu$ for some $u \in S^{(1)}$, then $x = yavu$ for $vu \in S_{ij}$. Thus, $x \leq_{\mathscr{R}^a} y$. □

Note that, since $\mathrm{P}^a = \mathrm{P}_1^a \cap \mathrm{P}_2^a$, all the statements of Proposition 2.2.9 hold for P^a, as well. Furthermore, in the previous three results, the assumption of a having identities is not necessary for the direct implications.

2.2.2 Maximal $\mathscr{J}^a$-classes

The previous subsection was dedicated to the description of Green's relations, their classes, and the relations $\leq_{\mathscr{L}^a}$, $\leq_{\mathscr{R}^a}$ and $\leq_{\mathscr{J}^a}$. In this section, we focus on the last one. More precisely, we deal with *maximal* $\mathscr{J}^a$-classes of a sandwich semigroup with respect to this partial order. First, we divide these classes into two disjoint sets − trivial and nontrivial maximal $\mathscr{J}^a$-classes. Then, we investigate which semigroups contain nontrivial maximal $\mathscr{J}^a$-classes. We close the subsection with a series of examples offering more insight into the notions introduced. The results and examples presented in this subsection are from [28].

Lemma 2.2.10. *If* $x \in S_{ij}$ *is such that* $x \not\leq_{\mathscr{J}} a$ *in* S, *then* $\{x\}$ *is a maximal* $\mathscr{J}^a$*-class in* S_{ij}^a*; additionally,* $\{x\}$ *is a nonregular* $\mathscr{D}^a$*-class.*

Proof. Suppose $x \in S_{ij}$ with $x \not\leq_{\mathscr{J}} a$ in S. For the first statement, it suffices to prove that, for any $y \in S_{ij}$, the relation $x \leq_{\mathscr{J}^a} y$ implies $x = y$. Since $x \leq_{\mathscr{J}^a} y$ holds if and only if one of (a)−(d) from the proof of Lemma 2.2.6 hold, and the statements (b)−(d) all imply that $x \leq_{\mathscr{J}} a$ in S, the implication holds. Thus, $\{x\}$ is indeed a maximal $\mathscr{J}^a$-class in S_{ij}^a. Clearly, $\{x\} \subseteq \mathrm{D}_x^a \subseteq \mathrm{J}_x^a = \{x\}$, so $\{x\}$ is a $\mathscr{D}^a$-class, as well. Furthermore, x is not an idempotent, as $x = x \star_a x = xax$ would give $x \leq_{\mathscr{J}} a$ in S. □

Maximal $\mathscr{J}^a$-classes of this type in S_{ij}^a will be called *trivial*. Any other $\mathscr{J}^a$-class of S_{ij}^a will be called *nontrivial*. These notions turn out to be vital in Chapter 5, where we deal with sandwich semigroups of partitions.

Naturally, our first question is: how many of these can a sandwich semigroup have? As we are about to prove, if a is regular, the number of nontrivial maximal $\mathscr{J}^a$-classes in S_{ij}^a is either zero or one, while the number of trivial maximal $\mathscr{J}^a$-classes is not bounded.

Example 2.2.11. Fix an arbitrary nonempty set X and consider the sandwich semigroup of partial mappings $\mathcal{PT}_X^a$ with $a = \emptyset \in \mathcal{PT}_X$. By Proposition 3.1.2, elements of $\mathcal{PT}_X \setminus \{a\}$ cannot be $\mathscr{J}$-below a in $\mathcal{PT}$, so each of them forms a trivial maximal $\mathscr{J}^a$-class, by Lemma 2.2.10. Evidently, the greater the size of X is, the greater is the number of trivial maximal $\mathscr{J}^a$-classes of $\mathcal{PT}_X^a$.

Lemma 2.2.12. *Suppose $a \in S_{ji}$ is regular.*

(i) There is at most one nontrivial maximal $\mathscr{J}^a$-class in S_{ij}^a.

(ii) If a nontrivial maximal $\mathscr{J}^a$-class exists, then it contains Pre(a).

(iii) If a nontrivial maximal $\mathscr{J}^a$-class exists, and if it is a $\mathscr{D}^a$-class, then it is regular.

Proof. We prove only (ii) and (iii), since (i) follows directly from (ii).

(ii) Let J be a nontrivial maximal $\mathscr{J}^a$-class in S_{ij}^a. By definition, there exists $x \in J$ such that $x \leq_{\mathscr{J}} a$ in S. In other words, there exist $u, v \in S^{(1)}$ such that $x = uav$. Suppose $b \in \text{Pre}(a)$, i.e. $aba = a$. Then $b \in S_{ij}$ (as $a \in S_{ji}$), so

$$x = uav = uabav = u(aba)b(aba)v = (uab) \star_a b \star_a (bav),$$

where clearly $uab, bav \in S_{ij}$. Thus, $x \leq_{\mathscr{J}^a} b$. Since $J = \mathrm{J}_x^a$ is a maximal $\mathscr{J}^a$-class, we have $\mathrm{J}_b^a = J$.

(iii) Suppose J from above is a $\mathscr{D}^a$-class, as well. Let $d \in \mathrm{V}(a) \subseteq \text{Pre}(a)$ ($\mathrm{V}(a)$ is nonempty, since a is regular); then, $d = dad = d \star_a d$ and (ii) gives $d \in J$, so J contains an idempotent. □

However, (i) does not hold in general. For instance, take the semigroup $S = (\{a, b, 0\}, \cdot)$, where

$\cdot$	a	b	0
a	b	0	0
b	0	0	0
0	0	0	0

Then, S is a partial semigroup with a singleton set of nodes; moreover, we have $0 \leq_{\mathscr{J}} b \leq_{\mathscr{J}} a$. Also, it is easily seen that the element a is not regular, and that the variant $S^a = (S, \star_a)$ satisfies $x \star_a y = 0$ for all $x, y \in S$. Thus, the sets $\{a\}$, $\{b\}$, and $\{0\}$ are the $\mathscr{J}^a$-classes of the variant S^a, the first two evidently being maximal. These classes obviously cannot be trivial (in the sense of Lemma 2.2.10), so the sandwich semigroup $(S, \star_a)$ has two nontrivial maximal $\mathscr{J}^a$-classes.

Nonetheless, all the partial semigroups examined in Chapters 3–5 are regular, so we focus on the case when a is regular. Having proved that there can be at most one nontrivial maximal $\mathscr{J}^a$-class in S_{ij}^a, we want to identify the sandwich semigroups which contain such a class.

Proposition 2.2.13. *Suppose $a \in S_{ji}$ is regular. Then the following are equivalent:*

(i) S_{ij}^a has a nontrivial maximal $\mathscr{J}^a$-class,

(ii) for all $x \in S_{ij}$, $a \leq_{\mathscr{J}} axa \Rightarrow x \leq_{\mathscr{J}} a$,

(iii) for all $x \in S_{ij}$*,* $a \mathscr{J} axa \Rightarrow x \mathscr{J} a$*.*

Proof. Let $a \in S_{ij}$ be regular and fix some $b \in \mathrm{V}(a)$. Then, $a = aba$, so ab and ba are left- and right- identities for a, respectively. We also have $b \in \mathrm{P}_3^a$ ($b = b(aba)b$) and $b \mathscr{J} a$. Furthermore, from Lemma 2.2.12(*ii*) we may conclude: if there exists a nontrivial maximal $\mathscr{J}^a$-class in S_{ij}^a, it is unique and it is the class $J = \mathrm{J}_b^a$.

(*i*) $\Rightarrow$ (*ii*) We prove the contrapositive statement. Suppose that $x \in S_{ij}$ satisfies $a \leq_{\mathscr{J}} axa$, but $x \not\leq_{\mathscr{J}} a$. By Lemma 2.2.10, $\{x\}$ is a (trivial) maximal $\mathscr{J}^a$-class in S_{ij}^a, and x is nonregular. On the other hand, $a \leq_{\mathscr{J}} axa$ and $b \mathscr{J} a$ together imply $b \leq_{\mathscr{J}} axa$. Since $b \in \mathrm{P}_3^a$, Proposition 2.2.7(*iii*) gives $b \leq_{\mathscr{J}^a} x$. Now,

$$J = \mathrm{J}_b^a \neq \mathrm{J}_x^a = \{x\},$$

because b is regular and x is not. Thus, $J < \mathrm{J}_x^a$, so J is not a maximal $\mathscr{J}^a$-class.

(*ii*) $\Rightarrow$ (*iii*) Suppose that (*ii*) holds, and let $x \in S_{ij}$ be such that $a \mathscr{J} axa$; in particular, we have $a \leq_{\mathscr{J}} axa$, so (*ii*) gives $x \leq_{\mathscr{J}} a$. Since $a \leq_{\mathscr{J}} axa \leq_{\mathscr{J}} x$, we have $a \mathscr{J} x$.

(*iii*) $\Rightarrow$ (*i*) Suppose that the statement (*iii*) is true. It suffices to show that J is a maximal $\mathscr{J}^a$-class. Let $x \in S_{ij}$ be such that $b \leq_{\mathscr{J}^a} x$, i.e.

$$\mathrm{J}_b^a = J \leq_{\mathscr{J}^a} \mathrm{J}_x^a. \tag{2.4}$$

From Proposition 2.2.7(*iii*) follows $b \leq_{\mathscr{J}} axa$. Together with $b \mathscr{J} a$, this implies $a \leq_{\mathscr{J}} axa$ (so $a \mathscr{J} axa$); thus, by (*iii*) we have $x \mathscr{J} a$. In particular, $x \leq_{\mathscr{J}} a$, so $a \mathscr{J} b$ gives $x \leq_{\mathscr{J}} b$. Since $b \in \mathrm{P}_3^a$, by Proposition 2.2.7(*vi*) we have $x \leq_{\mathscr{J}^a} b$, so (2.4) implies $\mathrm{J}_x^a = J = \mathrm{J}_b^a$. □

Additionally, we state a sufficient condition for a sandwich semigroup not to have nontrivial maximal $\mathscr{J}^a$-classes; it arises directly from the equivalent condition (*iii*) of the previous proposition.

Corollary 2.2.14. *If* $a \in S_{ji}$ *is regular and has a pre-inverse that is not* $\mathscr{J}$*-related to* a *(in* S*), then* S_{ij}^a *has only trivial maximal* $\mathscr{J}^a$*-classes.*

Proof. Suppose $b \in \mathrm{Pre}(a)$ is such that a and b are not $\mathscr{J}$-related in S. Since $a = aba$, we have $a \mathscr{J} aba$, so the statement (*iii*) from Proposition 2.2.13 does not hold, which means that S_{ij}^a has no nontrivial maximal $\mathscr{J}^a$-classes. □

We can do even better, under additional assumptions. Recall the definition of a stable semigroup (1.2). Similarly, we say a partial semigroup S is *stable*, if the implications (1.2) hold for all $x, a \in S$. We have:

Lemma 2.2.15. *If* S *is stable, and* $a \in S_{ji}$ *and* $x \in S_{ij}$*, then*

(i) $a \mathscr{J} axa \Leftrightarrow a \mathscr{H} axa$*,*

(ii) if $a \mathscr{J} x$*, then* $x = xax \Leftrightarrow a = axa$*.*

Proof. (i) Since $\mathscr{H} \subseteq \mathscr{J}$, we prove only the forwards implication. Suppose $a \mathrel{\mathscr{J}} axa$; then, by stability we have $a \mathrel{\mathscr{R}} axa$ (since $a \mathrel{\mathscr{J}} a(xa)$) and $a \mathrel{\mathscr{L}} axa$ (since $a \mathrel{\mathscr{J}} (ax)a$), so $a \mathrel{\mathscr{H}} axa$.

(ii) Suppose $a \mathrel{\mathscr{J}} x$. Clearly, it suffices to prove only the direct implication. If $x = xax$, we have

$$ax \leq_{\mathscr{J}} a \mathrel{\mathscr{J}} x = xax \leq_{\mathscr{J}} ax,$$

so $ax \mathrel{\mathscr{J}} a$. By stability, we have $ax \mathrel{\mathscr{R}} a$, so $axs = a$ for some $s \in S$. Then, $a = axs = a(xax)s = ax(axs) = axa$. □

Remark 2.2.16. The first part of the previous lemma (in the case where S is a semigroup) is Exercise A.2.2.1 in [108]. Furthermore, stability is necessary in both statements: for instance, let S be a monoid with identity 1 and a nonidentity idempotent e with $e \mathrel{\mathscr{J}} 1$ (the bicyclic monoid is such a monoid); then $e = e1e$ and $(1, 1e1) = (1, e) \in \mathscr{J} \setminus \mathscr{H}$.

Now, we may prove

Proposition 2.2.17. *If S is stable and $\mathscr{H}$-trivial (i.e. $\mathscr{H} = \{(x, x) : x \in S\}$), and if $a \in S_{ji}$ is regular, then the following are equivalent:*

(i) *S^a_{ij} has a nontrivial maximal $\mathscr{J}^a$-class,*

(ii) *every pre-inverse of a is $\mathscr{J}$-related to a in S,*

(iii) $\mathrm{Pre}(a) = \mathrm{V}(a)$.

Proof. $(i) \Leftrightarrow (ii)$ Since S is stable and $\mathscr{H}$-trivial, by Lemma 2.2.15(i) we have $a \mathrel{\mathscr{J}} axa \Leftrightarrow a = axa$, so statement (iii) from Proposition 2.2.13 (under these assumptions) amounts to: for all $x \in S_{ij}$, $a = axa$ implies $x \mathrel{\mathscr{J}} a$. This is clearly the same as (ii).

$(ii) \Rightarrow (iii)$ Suppose that (ii) is true. It suffices to prove $\mathrm{Pre}(a) \subseteq \mathrm{V}(a)$. If $b \in \mathrm{Pre}(a)$, then $a = aba$ and $b \mathrel{\mathscr{J}} a$ in S, so Lemma 2.2.15(ii) gives $b = bab$.

$(iii) \Rightarrow (ii)$ The very definition of the set $\mathrm{V}(a)$ implies that its elements are $\mathscr{J}$-related to a, so the proof is complete. □

Remark 2.2.18. Proposition 2.2.17 does not hold if S is not $\mathscr{H}$-trivial (for instance, see Figure 5.12 and the comment below it in Subsection 5.3.6).

2.2.3 Stability and regularity

As hinted in its title, this subsection can be divided up into two parts: in the first one, we examine the effects of stability in a sandwich semigroup, and in the second, we give results concerning regular elements of a sandwich semigroup. The first part contains a number of results, the most important being Proposition 2.2.23, which states the impact of different kinds of stability on the relations among P-sets. In contrast, the second part consists of only three results, Proposition 2.2.29 being crucial because it gives some essential properties of P-sets and the full characterisation of the regular elements in a sandwich semigroup.

First of all, we introduce the terms of stability in a more meticulous way than previously.

An element a of a partial semigroup S is

- $\mathscr{R}$*-stable* if $xa \mathrel{\mathscr{J}} x \Rightarrow xa \mathrel{\mathscr{R}} x$ for all $x \in S$,
- $\mathscr{L}$*-stable* if $ax \mathrel{\mathscr{J}} x \Rightarrow ax \mathrel{\mathscr{L}} x$ for all $x \in S$,
- *stable*, if it is both $\mathscr{R}$-stable and $\mathscr{L}$-stable.

Furthermore, S itself is *stable* ($\mathscr{R}$*-stable*, $\mathscr{L}$*-stable*) if each of its elements is stable ($\mathscr{R}$-stable, $\mathscr{L}$-stable, respectively). These definitions are inspired by the definition of stability for semigroups from [108] and [39] (see (1.2)). In the same book, the authors give a useful result concerning stable semigroups, which can be trivially adapted to partial semigroups:

Lemma 2.2.19. *Let S be a stable (partial) semigroup. Then the following are equivalent for all $x, y \in S$:*

(i) $x \mathrel{\mathscr{J}} y$*;*

(ii) there exists $z \in S$ such that $x \mathrel{\mathscr{L}} z \mathrel{\mathscr{R}} y$;

(iii) there exists $w \in S$ such that $x \mathrel{\mathscr{R}} w \mathrel{\mathscr{L}} y$;

(iv) $x \mathrel{\mathscr{D}} y$.

Proof. Clearly, $(ii) \Rightarrow (iv)$, $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$. Let us prove $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$. Suppose $x \mathrel{\mathscr{J}} y$. This implies the existence of elements $q, s, u, v \in S^{(1)}$ such that $qxs = y$ and $uyv = x$. It follows that $uqxsv = x$, so $qx \mathrel{\mathscr{J}} x$, $xs \mathrel{\mathscr{J}} x$ and by stability we have $qx \mathrel{\mathscr{L}} x$, $xs \mathrel{\mathscr{R}} x$. Since $\mathscr{L}$ is a right-congruence, and $\mathscr{R}$ is a left-congruence, it follows that $x \mathrel{\mathscr{R}} xs \mathrel{\mathscr{L}} qxs$ and $x \mathrel{\mathscr{L}} qx \mathrel{\mathscr{R}} qxs = y$. □

One of the benefits of stability in a partial semigroup is the fact (proved in [30]) that it is inherited by the sandwich semigroups contained in the said partial semigroup.

Lemma 2.2.20. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a stable partial semigroup. Then S^a_{ij} is stable for all $i, j \in I$ and $a \in S_{ji}$.*

This statement is a direct corollary of Lemma 2.2.27(v), so we omit the proof.

Naturally, stability also has its effects on the partial semigroup itself:

Lemma 2.2.21. *Let S be a stable (partial) semigroup and let $u, v \in S$.*

(i) If $u \leq_{\mathscr{L}} v \leq_{\mathscr{J}} u$, then $u \mathrel{\mathscr{L}} v$. *(ii) If $u \leq_{\mathscr{R}} v \leq_{\mathscr{J}} u$, then $u \mathrel{\mathscr{R}} v$.*

Proof. We prove only (i), as (ii) is dual. Since $u \leq_{\mathscr{L}} v \leq_{\mathscr{J}} u$, we have $u \mathrel{\mathscr{J}} v$. Furthermore, $u \leq_{\mathscr{L}} v$ means that $u = xv$ for some $x \in S^{(1)}$. Thus, $v \mathrel{\mathscr{J}} u = xv$ from which follows $v \mathrel{\mathscr{L}} xv = u$ by stability. □

Remark 2.2.22. In addition, in the case of partial semigroups, we may conclude that, if the elements u and v belong to the same hom-set S_{ij}, then it suffices to assume that each element of S_{ij} is stable (rather than assuming that the whole partial semigroup is stable).

Our main question is: how does stability affect the structural properties of sandwich semigroups? The following series of results (from [33]) answers that question in detail.

Proposition 2.2.23. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup, $i, j \in I$ and $a \in S_{ji}$. Then*

(i) a is $\mathscr{R}$-stable $\Rightarrow \mathrm{P}_3^a \subseteq \mathrm{P}_1^a$,

(ii) a is $\mathscr{L}$-stable $\Rightarrow \mathrm{P}_3^a \subseteq \mathrm{P}_2^a$,

(iii) a is stable $\Rightarrow \mathrm{P}_3^a = \mathrm{P}^a$.

Proof. We prove only the first part, as the second follows by duality, and the third follows from the previous two, since in 2.2.2(i) we proved $\mathrm{P}^a \subseteq \mathrm{P}_3^a$.

Suppose a is $\mathscr{R}$-stable, i.e. for all $x \in S$ holds $xa \mathrel{\mathscr{J}} x \Rightarrow xa \mathrel{\mathscr{R}} x$. Recall, $x \in \mathrm{P}_3^a$ means that $uaxav = x$ for some $u, v \in S^{(1)}$, which implies $xa \mathrel{\mathscr{J}} x$. Now, from $\mathscr{R}$-stability we have $xa \mathrel{\mathscr{R}} x$, so $x \in \mathrm{P}_1^a$. □

Remark 2.2.24. Note that $\mathscr{R}$-stability ($\mathscr{L}$-stability) in the implication (i) ((ii), respectively), in the previous proposition may be replaced with *local $\mathscr{R}$-stability* on S_{ij} (*local $\mathscr{L}$-stability* on S_{ij}):

$$xa \mathrel{\mathscr{J}} x \Rightarrow xa \mathrel{\mathscr{R}} x \quad \text{for all} \quad x \in S_{ij}$$

($ax \mathrel{\mathscr{J}} x \Rightarrow ax \mathrel{\mathscr{L}} x$ for all $x \in S_{ij}$). The implication will still hold because, in the proof, $x \in \mathrm{P}_3^a$ implies $x\boldsymbol{\delta} = a\boldsymbol{\rho} = i$ and $x\boldsymbol{\rho} = a\boldsymbol{\delta} = j$. An analogous modification may be carried out in the part (iii), where, instead of stability, we may require only *local stability* on S_{ij}, which means both local $\mathscr{R}$-stability and local $\mathscr{L}$-stability S_{ij}.

Proposition 2.2.25. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup, fix $i, j \in I$ and $a \in S_{ji}$. If a is stable and $\mathscr{J} = \mathscr{D}$, then $\mathscr{J}^a = \mathscr{D}^a$.*

Proof. Suppose that a is stable and that $\mathscr{J} = \mathscr{D}$ in S. In the case that $x \in S_{ij} \setminus \mathrm{P}_3^a$, Theorem 2.2.3($v$) implies $\mathrm{J}_x^a = \mathrm{D}_x^a$. On the other hand, if $x \in \mathrm{P}_3^a$, we have

$$\mathrm{J}_x^a = \mathrm{J}_x \cap \mathrm{P}_3^a = \mathrm{D}_x \cap \mathrm{P}_3^a = \mathrm{D}_x \cap \mathrm{P}^a,$$

the second equality following from $\mathscr{J} = \mathscr{D}$, and the third from Proposition 2.2.23(iii). Since $x \in \mathrm{P}_3^a = \mathrm{P}^a$, the set $\mathrm{D}_x \cap \mathrm{P}^a$ is exactly D_x^a (by Theorem 2.2.3(v)). □

Since Lemma 2.2.19 guarantees $\mathscr{J} = \mathscr{D}$ in a stable partial semigroup, the previous proposition gives:

Corollary 2.2.26. *If S_{ij}^a is a sandwich semigroup in a stable partial semigroup S, then $\mathscr{J}^a = \mathscr{D}^a$.*

Having shown the benefits of stability, we take the next logical step by investigating in which circumstances it occurs. In order to do that, we have to make a few introductory notes.

Observe that, for a partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ and for fixed coordinates $i, j \in I$ and a sandwich element $a \in S_{ji}$, we have: $S_{ij}a$ is a subsemigroup of S_i, aS_{ij} is a subsemigroup of S_j, and $aS_{ij}a$ is a subset of S_{ji}. We use these sets in the following lemma (which is a combination of two results from [33]), but their true relevance to the sandwich semigroup S_{ij}^a will not be apparent until the subsection 2.3.1.

Lemma 2.2.27. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup, $i, j \in I$ and $a \in S_{ji}$.*

(i) If aS_{ij} is periodic, then a is $\mathscr{R}$-stable.

(ii) If $S_{ij}a$ is periodic, then a is $\mathscr{L}$-stable.

(iii) If each element of $aS_{ij}a$ is $\mathscr{R}$-stable in S, then S_{ij}^a is $\mathscr{R}$-stable.

(iv) If each element of $aS_{ij}a$ is $\mathscr{L}$-stable in S, then S_{ij}^a is $\mathscr{L}$-stable.

(v) If each element of $aS_{ij}a$ is stable in S, then S_{ij}^a is stable.

Proof. We prove (i), and part (ii) follows by a dual argument. The goal is to show that $xa \mathscr{J} x \Rightarrow xa \mathscr{R} x$, for each $x \in S$. So, suppose $x \in S$ and $xa \mathscr{J} x$. Since $x \cdot a = xa$, we have $xa \leq_{\mathscr{R}} x$. To prove $x \leq_{\mathscr{R}} xa$, note that $xa \mathscr{J} x$ implies that one of the following holds:

(a) $x = xa$,

(b) $x = xav$, for some $v \in S$,

(c) $x = uxa$, for some $u \in S$,

(d) $x = uxav$, for some $u, v \in S$.

Clearly, (a) and (b) both imply $x \leq_{\mathscr{R}} xa$. Case (c) reduces to (d), since $x = uxa = uu(xa)a$. So, only the case (d) remains to be considered. We deduce $x = uxav = uuxavav = u^n x(av)^n$, for each $n \geq 1$. Since $v \in S_{ij}$ (because $v\boldsymbol{\delta} = a\boldsymbol{\rho}$ and $v\boldsymbol{\rho} = x\boldsymbol{\rho}$) and aS_{ij} is periodic, there exists $m \geq 1$ such that $(av)^m$ is an idempotent and therefore

$$x = u^m x(av)^m = (u^m x(av)^m)(av)^m = x(av)^m,$$

so $x \leq_{\mathscr{R}} xa$.

Since (iv) follows from (iii) by duality and (v) clearly follows from (iii) and (iv), the only statement that we prove, in addition to (i), is (iii). Suppose that each element of $aS_{ij}a$ is $\mathscr{R}$-stable and let us prove that, for each $x \in S_{ij}^a$ the following holds: $x \star_a y \mathscr{J}^a x \Rightarrow x \star_a y \mathscr{R}^a x$. The proof follows the same outline as the proof of (i). Obviously, $xay \leq_{\mathscr{R}^a} x$. From $x \star_a y \mathscr{J}^a x$, we know that exactly one of the following equalities holds:

(a) $x = xay$, (c) $x = uaxay$, for some $u \in S_{ij}$,

(b) $x = xayav$, for some $v \in S_{ij}$, (d) $x = uaxayav$, for some $u, v \in S_{ij}$.

If any of (a) or (b) is true, then $x \star_a y \mathscr{R}^a x$; (c) again reduces to (d), by the virtue of $x = uaxay = uaua(xay)ay$. If (d) is the case, then $x \mathscr{J} xaya$, so $x \mathscr{R} xaya$ (because $aya \in aS_{ij}a$ is stable in S); thus, there exists $z \in S$ such that $x = xayaz$. We may deduce $z \boldsymbol{\delta} = a \boldsymbol{\rho} = i$ and $z \boldsymbol{\rho} = x \boldsymbol{\rho} = j$, so $z \in S_{ij}$ and $x \mathscr{R}^a xay$. □

Next, we aim to study regularity in sandwich semigroups. More specifically, the first result (Proposition 2.10 in [30]) states the connection between regularity of a partial semigroup and regularity of sandwich semigroups contained in it.

Lemma 2.2.28. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup with $i, j \in I$, $a \in S_{ji}$, and $aS_{ij}a \subseteq \mathrm{Reg}(S)$. Then $\mathrm{Reg}(S^a_{ij})$ is a subsemigroup of S^a_{ij}.*

Proof. We need to prove that any $\star_a$-product of elements of the set $\mathrm{Reg}(S^a_{ij})$ belongs to $\mathrm{Reg}(S^a_{ij})$. Suppose $x, y \in \mathrm{Reg}(S^a_{ij})$, and $xazax = x$, $yaway = y$, for some $z, w \in S_{ij}$. Since the elements of $aS_{ij}a$ are all regular, $azaxayawa$ is a regular element, so there exists $q \in S$ so that $(azaxayawa)q(azaxayawa) = azaxayawa$. Thus

$$\begin{aligned}(xay)a(waqaz)a(xay) &= ((xazax)ay)awaqaza(xa(yaway)) \\ &= x(azaxayawa)q(azaxayawa)y \\ &= x(azaxayawa)y = (xazax)a(yaway) = xay,\end{aligned}$$

and $x \star_a y = xay \in \mathrm{Reg}(S^a_{ij})$. □

The following proposition (proved in [33]) will be used in a number of occasions; however, its significance primarily lies in paving the way for investigating $\mathrm{Reg}(S^a_{ij})$ in the Section 2.3. For the proof, note that an empty set is considered a subsemigroup, and a left and right ideal of any semigroup.

Proposition 2.2.29. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup, $i, j \in I$ and $a \in S_{ji}$. Then*

(i) P^a_1 *is a left ideal of* S^a_{ij},

(ii) P^a_2 *is a right ideal of* S^a_{ij},

(iii) P^a *is a subsemigroup of* S^a_{ij},

(iv) $\mathrm{Reg}(S^a_{ij}) = \mathrm{P}^a \cap \mathrm{Reg}(S)$,

(v) $\mathrm{Reg}(S^a_{ij}) = \mathrm{P}^a \Leftrightarrow \mathrm{P}^a \subseteq \mathrm{Reg}(S)$.

Proof. (*i*) We need to prove $S_{ij}a\,\mathrm{P}^a_1 \subseteq \mathrm{P}^a_1$. Suppose $x \in \mathrm{P}^a_1$ and $y \in S_{ij}$. Then $xa \mathscr{R} x$, so $yaxa \mathscr{R} yax$ (because $\mathscr{R}$ is a left-congruence) and $yax \in \mathrm{P}^a_1$.

(*ii*) is dual to (*i*). Part (*iii*) follows from (*i*) and (*ii*), since any left/right/ two-sided ideal of a semigroup is clearly a subsemigroup, and an intersection of two subsemigroups is always a subsemigroup. Note that P^a is non-empty precisely when both P_1^a and P_2^a are non-empty (if $x \in \mathrm{P}_1^a$ and $y \in \mathrm{P}_2^a$, then $xy \in \mathrm{P}^a$, by (i) and (ii)).

(iv) First, note that we have $\subseteq$, since from Proposition 2.2.2(i) it follows that $\mathrm{Reg}(S_{ij}^a) \subseteq \mathrm{P}^a$, and $\mathrm{Reg}(S_{ij}^a) \subseteq \mathrm{Reg}(S)$ is obviously true. To prove the reverse inclusion, suppose $x \in \mathrm{P}^a \cap \mathrm{Reg}(S)$ and $x = xay = zax = xwx$, for some $y, z, w \in S$. Then, $x = xwx = (xay)w(zax) = x \star_a ywz \star_a x$, where $(ywz)\boldsymbol{\delta} = y\boldsymbol{\delta} = a\boldsymbol{\rho} = i$ and $(ywz)\boldsymbol{\rho} = z\boldsymbol{\rho} = a\boldsymbol{\delta} = j$. This proves $x \in \mathrm{Reg}(S_{ij}^a)$. Finally, (v) is a direct consequence of (iv). □

Finally, we introduce a result of [28], which adds a new "layer" to Lemma 2.2.28 in the special case of regular partial $*$-semigroups.

Lemma 2.2.30. *If $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a regular partial $*$-semigroup, and if $a \in S_i$ is a projection, then $\mathrm{Reg}(S_i^a)$ is a regular $*$-semigroup with involution inherited from S.*

Proof. Clearly, Lemma 2.2.28 implies that $\mathrm{Reg}(S_i^a)$ is a subsemigroup of S_i^a, while Proposition 2.2.29(v) gives $\mathrm{Reg}(S_i^a) = \mathrm{P}^a$. We need to prove

$$x^{**} = x, \qquad (x \star_a y)^* = y^* \star_a x^*, \qquad x = x \star_a x^* \star_a x \qquad \text{for all } x, y \in \mathrm{P}^a\,.$$

The first equality is obvious; for the second, note that $(x \star_a y)^* = (xay)^* = y^*a^*x^* = y^*ax^* = y^* \star_a x^*$, since a is a projection. Let us prove the third one. If $x \in \mathrm{P}^a \subseteq \mathrm{P}_1^a$, from Lemma 2.2.1$(i)$ we have $x^*x = x^*xaa^*x^*x = x^*xax^*x$, the last equality following from the fact that a is a projection. Therefore,

$$xx^* = x(x^*x)x^* = x(x^*xax^*x)x^* = (xx^*x)a(x^*xx^*) = xax^*,$$

and dually, $x \in \mathrm{P}_2^a$ so $xx^* = xx^*axx^*$ and $x^*x = x^*ax$. We may conclude $x \star_a x^* \star_a x = (xax^*)ax = x(x^*ax) = xx^*x = x$. Also, note that $\mathrm{Reg}(S_i^a)$ is closed for * since for all $x \in S_i$ we have $x^* = x^* \star_a x \star_a x^*$. □

Remark 2.2.31. In a special case when S is an inverse partial semigroup, $\mathrm{Reg}(S_i^a)$ is an inverse semigroup, since it is closed for inverting. Moreover, $\mathrm{Reg}(S_{ij}^a)$ is an inverse semigroup, regardless of a being a projection or not (see Proposition 2.5.2)! The same, however, does not necessarily hold when S is a regular $*$-semigroup. For instance, consider the egg-box diagram of $\mathrm{Reg}(\mathcal{B}_{64}^{\sigma_2})$ in Figure 5.13 and the corresponding description. The reader may verify that any $\mathscr{D}$-class contains unequal numbers of $\mathscr{R}$- and $\mathscr{L}$-classes, so $\mathrm{Reg}(\mathcal{B}_{64}^{\sigma_2})$ is not even a $*$-semigroup (as the rule $(xy)^* = y^*x^*$ implies an equal number of $\mathscr{R}$- and $\mathscr{L}$-classes in each $\mathscr{D}$-class).

Remark 2.2.32. In the case where S is a semigroup (i.e. $|I| = 1$) Lemma 2.2.30 applies to its variant corresponding to a projection: If S is a regular $*$-semigroup, and if $a \in S$ is a projection, then $\mathrm{Reg}(S^a)$ is a regular $*$-semigroup with involution inherited from S.

2.2.4 Right-invertibility

In this subsection, we investigate the properties of S^a_{ij} in the case when a is right-invertible in S_{ij}. This means that there exists an element $b \in S_{ij}$ such that $x = xab$ for any $x \in S_{ij}$ (i.e. ab is a right-identity for the set S_{ij}). Note that, in this case, a is not necessarily right-invertible in S. Let $\mathrm{RI}(a)$ denote the set of all right-inverses of a in S_{ij}. Clearly, all the notions and results in this subsection here have a "left" counterpart. However, we do not state these since they are easy to infer.

This subsection is relevant for sandwich semigroups in all the categories we investigate (see Lemma 3.0.2 and Propositions 4.1.7 and 5.1.7). All the results and examples presented in this section were proved in [28], except for Lemma 2.2.38, which was proved in [33].

Lemma 2.2.33.

(i) If $a \in S_{ji}$ is right-invertible, then $\mathrm{V}(a) = \mathrm{Pre}(a) \subseteq \mathrm{RI}(a) \subseteq \mathrm{Post}(a)$.

(ii) If $a \in S_{ji}$ is right-invertible and regular, then $\mathrm{V}(a) = \mathrm{Pre}(a) = \mathrm{RI}(a) \subseteq \mathrm{Post}(a)$.

Proof. (i) Since $\mathrm{V}(a) = \mathrm{Pre}(a) \cap \mathrm{Post}(a)$, it suffices to prove $\mathrm{Pre}(a) \subseteq \mathrm{RI}(a) \subseteq \mathrm{Post}(a)$. Suppose $x \in \mathrm{Pre}(a)$ (i.e. $axa = a$) and fix some $b \in \mathrm{RI}(a)$. Now, we have $y = yab$ for any $y \in S_{ij}$. In particular, $x = xab$, and $ax = a(xab) = (axa)b = ab$, so $x \in \mathrm{RI}(a)$. Thus, $x = x(ax)$ and $x \in \mathrm{Post}(a)$.

(ii) Having shown (i), we need to prove only $\mathrm{RI}(a) \subseteq \mathrm{Pre}(a)$. If $x \in \mathrm{RI}(a)$, then $yax = y$ for any $y \in S_{ij}$. Since $a \in S_{ji}$ is regular, we have $a = aza$ for some $x \in S_{ij}$, so

$$a = aza = a(zax)a = (aza)xa = axa,$$

which means that $x \in \mathrm{Pre}(a)$. □

Remark 2.2.34. Right-invertibility of an element $a \in S_{ji}$ in the hom-set S_{ij} does not imply its regularity. For example, take two distinct sets $X, Y \neq \emptyset$, let $I = \{X, Y\}$ and define

$$S = S_{X,X} \cup S_{X,Y} \cup S_{Y,Y} \cup S_{Y,X},$$

where $S_{Y,X} = \mathbf{PT}_{YX}$, $S_{Y,Y} = \mathbf{PT}_{Y,Y}$, $S_{X,X} = \{\emptyset_{X,X}\}$, and $S_{X,Y} = \{\emptyset_{X,Y}\}$ ($\mathbf{PT}_{A,B}$ denotes the set of all partial maps $A \to B$, and $\emptyset_{AB}$ denotes the empty map $A \to B$.) Choose a map $a \in S_{Y,X} \setminus \{\emptyset_{Y,X}\}$; then $\emptyset_{X,Y} a \emptyset_{Y,Y} = \emptyset_{X,Y}$, so a is right-invertible in $S_{X,Y} = \{\emptyset_{X,Y}\}$. However, $\emptyset_{X,Y}$ is the sole element in $S_{X,Y}$ and does not belong to $\mathrm{Pre}(a)$, so a is not regular.

However, if $a \in S_{ji}$ is right-invertible and our object of interest is the sandwich semigroup S^a_{ij} itself, we may assume without loss of generality that a is regular. Namely, if we pick any $b \in \mathrm{RI}(a)$ and let $c = aba \in S_{ji}$, then $xcy = xabay = xay$ for any $x, y \in S_{ij}$, so $S^a_{ij} \equiv S^c_{ij}$; furthermore, c is right-invertible ($cb = abab = ab$, so $b \in \mathrm{RI}(c)$) and regular ($cbc = abababa = aba = c$).

The following Proposition may be regarded as a supplement to Subsection 2.2.2, since it explores maximal $\mathscr{J}^a$-classes in a special case when a is right-invertible.

Proposition 2.2.35. *Suppose $a \in S_{ji}$ is right-invertible.*

(i) The sandwich semigroup S^a_{ij} has a maximum $\mathscr{J}^a$-class, and this contains $\mathrm{RI}(a)$.

(ii) If S^a_{ij} is stable, then the maximum $\mathscr{J}^a$-class of S^a_{ij} is in fact an $\mathscr{L}^a$-class, and is a left-group with set of idempotents $\mathrm{RI}(a)$.

Proof. (i) Let $b \in \mathrm{RI}(a)$. Since for any $x \in S_{ij}$ holds $x = xab$, we have $x \leq_{\mathscr{L}^a} b$ so $x \leq_{\mathscr{J}^a} b$. This proves the statement.

(ii) As above, let $b \in \mathrm{RI}(a)$. From (i) we know that J^a_b is the maximum $\mathscr{J}^a$-class. First, we prove $\mathrm{L}^a_b = \mathrm{J}^a_b$. The direct containment being clear, we show the reverse one. Let $x \mathrel{\mathscr{J}^a} b$. From the proof of (i), we have $x \leq_{\mathscr{L}^a} b$, so $x \leq_{\mathscr{L}^a} b \leq_{\mathscr{J}^a} x$. Since S^a_{ij} is stable, we may apply Lemma 2.2.21(i) in this semigroup. Therefore $x \mathrel{\mathscr{L}^a} b$.

We have proved that J^a_b is an $\mathscr{L}^a$-class, so it has to be a $\mathscr{D}^a$-class, as well. Since $b = bab \in \mathrm{J}^a_b$, it is regular, so it follows by Lemma 1.3.9 that it is a left-group.

The only thing left to prove is the equality $\mathrm{RI}(a) = \mathrm{E}_a(\mathrm{J}^a_b)$ (for any $U \subseteq S_{ij}$, $\mathrm{E}_a(U)$ denotes the set of all idempotents in U with respect to the sandwich multiplication $\star_a$). Since $x \in \mathrm{RI}(a)$ implies $x = xax$ and $x \in \mathrm{J}^a_b$, we proved the direct inclusion. For the reverse one, let $x \in \mathrm{E}_a(\mathrm{J}^a_b)$. Then we have $x = xax$ and $x \mathrel{\mathscr{L}^a} b$, so $b = uax$ for some $u \in S_{ij}$. Thus, for any $z \in S_{ij}$

$$zax = zabax = zauaxax = zauax = zab = z,$$

so $x \in \mathrm{RI}(a)$. □

Remark 2.2.36. We provide an example showing that stability is necessary in (ii). If X is an infinite set and if $a \in \mathcal{PT}_X$ is a full, injective and non-surjective mapping, then a is right-invertible in $\mathcal{PT}_X$ because it is full and injective (by Lemma 3.0.2). Let $b \in \mathrm{RI}(a)$; now, Proposition 2.2.35(i) implies that J^a_b is the maximum $\mathscr{J}^a$-class in $\mathcal{PT}^a_X$. However, a is not stable since it is not surjective (by Proposition 3.1.7(iii)), so there exists a map $f \in \mathrm{P}^a_3 \setminus \mathrm{P}^a$ (by Lemma 3.1.12(iii)) with $\operatorname{Rank} f = \operatorname{Rank} b$. Thus, Theorem 3.1.10 implies

$$\mathrm{J}^a_b = \mathrm{J}_b \cap \mathrm{P}^a_3 = \mathrm{D}_b \cap \mathrm{P}^a_3 \neq \mathrm{D}_b \cap \mathrm{P}^a = \mathrm{D}^a_b\,.$$

Therefore, J^a_b is not even a $\mathscr{D}^a$-class, let alone an $\mathscr{L}^a$-class.

It turns out that similar statements (as in Proposition 2.2.35) can be made for the $\mathscr{J}$-classes of S_{ij}, in the case when there exists a right-invertible element $a \in S_{ji}$.

Proposition 2.2.37. *Suppose $a \in S_{ji}$ is right-invertible.*

(i) The hom-set S_{ji} has a maximum $\mathscr{J}$-class, and this contains $\mathrm{RI}(a)$.

(ii) If each element of S_{ij} is stable in S, then the maximum $\mathscr{J}$-class of S_{ij} is in fact an $\mathscr{L}$-class.

Proof. By analysing the proof of Proposition 2.2.35, one can verify that (i) and (ii) can be proved in precisely the same way (in fact, for (ii) we need only the first paragraph of the proof for 2.2.35(ii)). Note that, instead of Lemma 2.2.21, we apply its altered version from Remark 2.2.22. $\square$

Finally, we provide an appropriate closing for this stage of investigation, by showing the key consequences of right-invertibility of a for the structure of the sandwich semigroup S_{ij}^a.

Lemma 2.2.38. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup, $i, j \in I$ and $a \in S_{ji}$. If a is right-invertible in S_{ij}, then $\mathrm{P}_1^a = S_{ij}$, $\mathrm{P}^a = \mathrm{P}_2^a$ and $\mathscr{R}^a = \mathscr{R}$ on S_{ij}^a.*

Proof. Let a be right-invertible and $b \in \mathrm{RI}(a)$. Then $xab = x$ for all $x \in S_{ij}$, so $x \mathrel{\mathscr{R}} xa$ and therefore $x \in \mathrm{P}_1^a$ for all $x \in S_{ij}$. Now, $\mathrm{P}^a = \mathrm{P}_2^a$ follows from the definition of P^a, and $\mathscr{R}^a = \mathscr{R}$ on S_{ij}^a from Theorem 2.2.3(i). $\square$

Remark 2.2.39. Figures 3.7 and 3.8 show egg-box diagrams for sandwich semigroups with a left-invertible and a right-invertible sandwich element, respectively.

2.2.5 Partial subsemigroups

In this subsection, we introduce the term *partial subsemigroup*. Predictably, it denotes a substructure of a partial semigroup that is also a partial semigroup. In other words, if $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ is a partial semigroup and $T \subseteq S$ is a class such that $(T, \cdot\restriction_{T\times T}, I, \boldsymbol{\delta}\restriction_T, \boldsymbol{\rho}\restriction_T)$ is a partial semigroup, then T is a partial subsemigroup of S. To avoid confusion, we denote Green's relations of T by $\mathscr{K}^T$, and Green's relations of S by $\mathscr{K}^S$, for $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$. In the case of sandwich semigroups (with sandwich element a) inside T and S, Green's relations are denoted by $\mathscr{K}^a(T)$ and $\mathscr{K}^a(S)$, respectively, and the corresponding classes containing a chosen element $x \in T$ are $\mathrm{K}_x^a(T)$ and $\mathrm{K}_x^a(S)$, respectively. Similarly, the notation for P-sets in S and T is modified so that it contains information about the partial semigroup considered: for $a \in S_{ji}$ we write

$$\mathrm{P}_1^a(S) = \{x \in S_{ij} : xa \mathrel{\mathscr{R}^S} x\} \qquad \mathrm{P}_1^a(T) = \{x \in T_{ij} : xa \mathrel{\mathscr{R}^T} x\}.$$

We rename the rest of the P-sets analogously.

As is the case with their semigroup counterparts (subsemigroups), partial subsemigroups inherit some properties from partial semigroups containing them. In the following series of propositions (from [33]), we provide some insight into these connections. We deal with Green's relations of partial subsemigroups and their sandwich semigroups, as well as stability. Interestingly enough, these connections do not exist in general but can be proved if we add some regularity assumptions.

Proposition 2.2.40. *Let T be a partial subsemigroup of S, and let $x, y \in T$ and $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}\}$.*

(i) If $y \in \mathrm{Reg}(T)$, then $x \leq_{\mathscr{K}^S} y \Leftrightarrow x \leq_{\mathscr{K}^T} y$.

(*ii*) *If* $x, y \in \mathrm{Reg}(T)$*, then* $x \mathscr{K}^S y \Leftrightarrow x \mathscr{K}^T y$.

Proof. Note that, for each $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}\}$, the implication ($\Leftarrow$) trivially holds in the equivalence of (i), as well as in the equivalence of (ii).

To prove (i), suppose first that $x \leq_{\mathscr{R}^S} y$ and $y \in \mathrm{Reg}(T)$. Then, $x = yz$ and $y = yqy$ for some $z \in S$ and $q \in T$, which (together) imply $x = yz = yqyz = yqx$, where $q, x \in T$, so $qx \in T$ and $x \leq_{\mathscr{R}^T} y$. A dual argument proves the statement for $\mathscr{K} = \mathscr{L}$, and the one for $\mathscr{K} = \mathscr{H}$ follows from the previous two. Part (ii) follows directly from (i). □

Remark 2.2.41. Obviously, the statements also apply if T is a subsemigroup of a semigroup S, as any semigroup is a partial semigroup, as well.

Proposition 2.2.42. *Let a be an element of a regular (partial) subsemigroup T of a (partial) semigroup S. Then the following hold:*

(*i*) *if a is $\mathscr{R}$-stable in S, then it is $\mathscr{R}$-stable in T;*

(*ii*) *if a is $\mathscr{L}$-stable in S, then it is $\mathscr{L}$-stable in T;*

(*iii*) *if a is stable in S, then it is stable in T.*

Proof. Again, we prove only (i), as (ii) is dual, and (iii) follows directly from (i) and (ii). Suppose a is $\mathscr{R}$-stable in S, i.e. $xa \mathscr{J}^S x \Rightarrow xa \mathscr{R}^S x$ for all $x \in S$. We need to prove $\mathscr{R}$-stability in T. If we assume $xa \mathscr{J}^T x$ for $x \in T$ with $x\boldsymbol{\rho} = a\boldsymbol{\delta}$, it follows that $xa \mathscr{J}^S x$ and, by stability of S, $xa \mathscr{R}^S x$. Since $x, a \in T = \mathrm{Reg}(T)$, we have $xa \in T = \mathrm{Reg}(T)$, thus Proposition 2.2.40(ii) implies $xa \mathscr{R}^T x$. □

Now, we describe P-sets and Green's relations of a sandwich semigroup in a partial subsemigroup T of a partial semigroup S. Note that the sets $T_{ij}a$ and aT_{ij} (counterparts of $S_{ij}a$ and aS_{ij}) make an appearance both in Proposition 2.2.43 and in Proposition 2.2.45, once more proving the significance of the investigation conducted in Section 2.3 below.

Proposition 2.2.43. *Let a be an element of T_{ji} in a partial semigroup $(T, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ with $i, j \in I$, and let T be a partial subsemigroup of S. Then*

(*i*) $\mathrm{P}_1^a(T) \subseteq \mathrm{P}_1^a(S) \cap T$*, with equality if* $T_{ij} \cup T_{ij}a \subseteq \mathrm{Reg}(T)$*,*

(*ii*) $\mathrm{P}_2^a(T) \subseteq \mathrm{P}_2^a(S) \cap T$*, with equality if* $T_{ij} \cup aT_{ij} \subseteq \mathrm{Reg}(T)$*,*

(*iii*) $\mathrm{P}^a(T) \subseteq \mathrm{P}^a(S) \cap T$*, with equality if* $T_{ij} \cup T_{ij}a \cup aT_{ij} \subseteq \mathrm{Reg}(T)$*,*

(*iv*) $\mathrm{P}_3^a(T) \subseteq \mathrm{P}_3^a(S) \cap T$*, with equality if a is stable in S and* $T_{ij} \cup T_{ij}a \cup aT_{ij} \subseteq \mathrm{Reg}(T)$*.*

Proof. (i) Clearly, $\mathrm{P}_1^a(T) \subseteq \mathrm{P}_1^a(S) \cap T$ because $\mathrm{P}_1^a(T) \subseteq T \subseteq S$ and $\mathscr{R}^T \subseteq \mathscr{R}^S$. Now, suppose that $T_{ij} \cup T_{ij}a \subseteq \mathrm{Reg}(T)$ and $x \in \mathrm{P}_1^a(S) \cap T$. This implies $x \in T \cap S_{ij} = T_{ij}$ and $xa \mathscr{R}^S x$, so Proposition 2.2.40(ii) guarantees $xa \mathscr{R}^T x$, because $xa \in T_{ij}a$ and $x \in T_{ij}$ are regular in T. Hence, $x \in \mathrm{P}_1^a(T)$.

(ii) is proved by a dual argument, and (iii) is a direct consequence of (i) and (ii).

(iv) The inclusion is proved analogously as the corresponding part of (i). For the second part of the statement, suppose a is stable and $T_{ij} \cup T_{ij}a \cup aT_{ij} \subseteq \mathrm{Reg}(T)$. Stability of a in S, by Proposition 2.2.23(iii), implies $\mathrm{P}_3^a(S) = \mathrm{P}^a(S)$. If we prove $\mathrm{P}_3^a(T) = \mathrm{P}^a(T)$, the result will follow from the statement (iii). In order to prove $\mathrm{P}_3^a(T) = \mathrm{P}^a(T)$, recall Remark 2.2.24, following the Proposition 2.2.23. It suffices to prove that, from stability in S follows local stability in T. The proof is analogous to the proof for Proposition 2.2.42, the only difference being that, instead of regularity of T, we use the fact that $x \in T_{ij} \subseteq \mathrm{Reg}(T)$, $xa \in T_{ij}a \subseteq \mathrm{Reg}(T)$ and (for the local $\mathscr{L}$-stability) $ax \in aT_{ij} \subseteq \mathrm{Reg}(T)$. □

The next result offers a different set of conditions which imply equalities mentioned in the previous proposition.

Lemma 2.2.44. *Let a be an element of T_{ji} in a partial semigroup $(T, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ with $i, j \in I$ and let T be a partial subsemigroup of S.*

(i) If $\mathscr{R}^T = \mathscr{R}^S \cap (T \times T)$, then $\mathrm{P}_1^a(T) = \mathrm{P}_1^a(S) \cap T$.

(ii) If $\mathscr{L}^T = \mathscr{L}^S \cap (T \times T)$, then $\mathrm{P}_2^a(T) = \mathrm{P}_2^a(S) \cap T$.

(iii) If $\mathscr{R}^T = \mathscr{R}^S \cap (T \times T)$, $\mathscr{L}^T = \mathscr{L}^S \cap (T \times T)$, then $\mathrm{P}^a(T) = \mathrm{P}^a(S) \cap T$.

(iv) If $\mathscr{J}^T = \mathscr{J}^S \cap (T \times T)$, then $\mathrm{P}_3^a(T) = \mathrm{P}_3^a(S) \cap T$.

Proof. Part (i) is easily proved, since Proposition 2.2.43(i) gives $\mathrm{P}_1^a(T) \subseteq \mathrm{P}_1^a(S) \cap T$, and for any $x \in T_{ij}$ we have $xa, x \in T$ so the implication $xa \,\mathscr{R}^S\, x \Rightarrow xa \,\mathscr{R}^T\, x$ is true. Part (ii) is dual, (iii) is a direct consequence of (i) and (ii), and part (iv) is proved analogously as (i), since $axa, x \in T$. □

In addition to the previously introduced notation, we include the following: for $a \in T_{ji} \subseteq S_{ji}$ and $x \in T_{ij}$ we write

$$\mathrm{K}_x(T) = \{y \in T_{ij} : x \,\mathscr{K}^T\, y\} \text{ and } \mathrm{K}_x(S) = \{y \in S_{ij} : x \,\mathscr{K}^S\, y\},$$

for all $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}, \mathrm{J}\}$. Using this, we may describe the correlation between Green's relations of a partial semigroup and Green's relations of its partial subsemigroup.

Proposition 2.2.45. *Let a be an element of T_{ji} in a partial semigroup $(T, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ with $i, j \in I$ and let T be a partial subsemigroup of S. Then*

(i) $\mathscr{R}^a(T) \subseteq \mathscr{R}^a(S) \cap (T \times T)$, with equality if $T_{ij} \cup T_{ij}a \subseteq \mathrm{Reg}(T)$,

(ii) $\mathscr{L}^a(T) \subseteq \mathscr{L}^a(S) \cap (T \times T)$, with equality if $T_{ij} \cup aT_{ij} \subseteq \mathrm{Reg}(T)$,

(iii) $\mathscr{H}^a(T) \subseteq \mathscr{H}^a(S) \cap (T \times T)$, with equality if $T_{ij} \cup T_{ij}a \cup aT_{ij} \subseteq \mathrm{Reg}(T)$,

Proof. Let us prove (i). Clearly, the first part is true, since $\mathscr{R}^a(T) \subseteq \mathscr{R}^a(S)$. Suppose $T_{ij} \cup T_{ij}a \subseteq \mathrm{Reg}(T)$. Then, Proposition 2.2.43(i) gives $\mathrm{P}_1^a(T) = \mathrm{P}_1^a(S) \cap T$. Let us describe the $\mathscr{R}$-classes $\mathrm{R}_x^a(S)$ and $\mathrm{R}_x^a(T)$ of an arbitrary element $x \in T$:

- if $x \in (S_{ij} \setminus \mathrm{P}_1^a(S)) \cap T = T_{ij} \setminus \mathrm{P}_1^a(T)$, then Theorem 2.2.3(*i*) gives $\mathrm{R}_x^a(S) = \{x\} = \mathrm{R}_x^a(T)$;
- if $x \in \mathrm{P}_1^a(S) \cap T = \mathrm{P}_1^a(T)$, then Proposition 2.2.40(*ii*) and the fact that $\mathrm{R}_x(T) \subseteq T_{ij} \subseteq \mathrm{Reg}(T)$ together imply $x \mathscr{R}^S y \Leftrightarrow x \mathscr{R}^T y$ for $y \in T_{ij}$. In other words, $\mathrm{R}_x(T) = \mathrm{R}_x(S) \cap T$. Hence, Theorem 2.2.3(*i*) gives

$$\begin{aligned} \mathrm{R}_x^a(T) &= \mathrm{R}_x(T) \cap \mathrm{P}_1^a(T) = (\mathrm{R}_x(S) \cap T) \cap (\mathrm{P}_1^a(S) \cap T) \\ &= (\mathrm{R}_x(S) \cap \mathrm{P}_1^a(S)) \cap T = \mathrm{R}_x^a(S) \cap T. \end{aligned}$$

Part (*ii*) is dual, and (*iii*) is a direct consequence of (*i*) and (*ii*). □

Example 2.2.46. In Chapter 3, we examine the partial semigroup $\mathcal{PT}$ and its partial subsemigroups $\mathcal{T}$ and $\mathcal{I}$, so the reader may see Sections 3.2 and 3.3 for the direct applications of these results.

2.3 Sandwich regularity and the structure of $\mathrm{Reg}(S_{ij}^a)$

Let us fix a partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ with $i, j \in I$ and an element $a \in S_{ji}$. In this entire section we study the sandwich semigroup S_{ij}^a, and the set consisting of its regular elements, $\mathrm{Reg}(S_{ij}^a)$. As in the "plot" of [33], we start off by examining the connections among the semigroups S_{ij}^a, $(S_{ij}a, \cdot)$, $(aS_{ij}, \cdot)$ and $(aS_{ij}a, \star_b)$ (under the assumption of regularity of a, with $b \in V(a)$). In a natural step forward, we restrict our attention to the four sets consisting of their regular elements. We introduce a condition ensuring that these sets define subsemigroups of the original semigroups. Having studied these subsemigroups and their links, we gain enough insight to investigate $\mathrm{Reg}(S_{ij}^a)$ (which coincides with P^a under the said condition) in terms of Green's relations and the structure of its $\mathscr{D}$-classes. Indeed, we give a fascinating result (see Theorem 2.3.12 and Remark 2.3.13) explicitly describing this structure through the semigroup $\mathrm{Reg}(aS_{ij}a, \star_b)$. We close off the section by using this result to study some problems of generation.

2.3.1 Commutative diagrams

We have already mentioned the sets $S_{ij}a$, aS_{ij} and $aS_{ij}a$ in the context of conditions providing stability in a sandwich semigroup. At the time, the only additional information we needed were the facts that $S_{ij}a$ is a subsemigroup of S_i, aS_{ij} is a subsemigroup of S_j and $aS_{ij}a$ is a subset of S_{ji}. Here, we will discover a lot more.

First, suppose a is regular. Then the set of its inverses $\mathrm{V}(a)$ is non-empty, and we may choose and fix an element $b \in \mathrm{V}(a)$. Since $aba = a$ and $b = bab$, we have $b \in S_{ij}$ and

$$S_{ij}a = S_{ij}aba = (S_{ij}a)ba \subseteq S_iba = (S_ib)a \subseteq S_{ij}a,$$

so $S_{ij}a = S_iba$ is the principal left ideal of S_i corresponding to the element ba. By a dual argument, aS_{ij} is the principal right ideal of S_j corresponding to ab.

Additionally, $aS_{ij}a$ turns out to be a subsemigroup of $(S_{ji}, \star_b)$, because for all $x, y \in S_{ij}$:

$$axa \star_b aya = axabaya = axaya \in aS_{ij}a.$$

Moreover, $(aS_{ij}a, \star_b)$ is a monoid with identity $aba = a$, since $axa \star_b a = axa = a \star_b axa$ for all $x \in S_{ij}$. In fact, the operation $\star_b\restriction_{aS_{ij}a}$ turns out to be independent of the choice of $b \in \mathrm{V}(a)$! In other words, if we choose an element $c \in \mathrm{V}(a)$, operations $\star_b$ and $\star_c$ coincide on $aS_{ij}a$, since $aba = a = aca$. To emphasise this independence, we will use a new sign $\circledast$ for the operation $\star_b\restriction_{aS_{ij}a}$.

All of the above mentioned semigroups are connected and these links may be visually presented as in the following diagram:

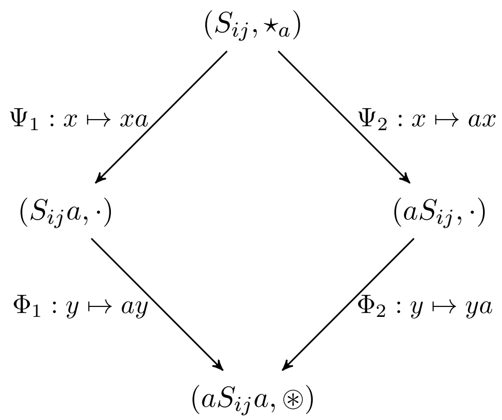


Figure 2.2: A diagram depicting the connections between S^a_{ij} and $(aS_{ij}a, \circledast)$.

This diagram obviously commutes. We may also conclude that, for all $q, w \in S_{ij}$, all $ta, sa \in S_{ij}a$ and all $az, ap \in aS_{ij}$, holds:

$$\begin{aligned} q\Psi_1 \cdot w\Psi_1 &= qawa = (qaw)\Psi_1 = (q \star_a w)\Psi_1, \\ q\Psi_2 \cdot w\Psi_2 &= aqaw = (qaw)\Psi_2 = (q \star_a w)\Psi_2, \\ (ta)\Phi_1 \circledast (sa)\Phi_1 &= atabasa = atasa = (tasa)\Phi_1 = (ta \cdot sa)\Phi_1, \\ (az)\Phi_2 \circledast (ap)\Phi_2 &= azabapa = azapa = (azap)\Phi_2 = (az \cdot ap)\Phi_2. \end{aligned}$$

Therefore, all the maps in the diagram are homomorphisms. Moreover, they are surmorphisms, because of the forms of their codomains. In order for them to be isomorphisms, we need injectivity.

It is easily seen that Ψ_1 is injective if and only if the following holds

$$xa = ya \Rightarrow x = y, \qquad \text{for all } x, y \in S_{ij}. \tag{2.5}$$

By symmetry, Ψ_2 is injective if and only if

$$ax = ay \Rightarrow x = y, \qquad \text{for all } x, y \in S_{ij}. \tag{2.6}$$

Assuming that the union of these two conditions holds, we can even prove that $\Phi = \Psi_1\Phi_1 = \Psi_2\Phi_2$ is an isomorphism. Indeed, it is a composition of surmorphisms, therefore a surmorphism itself, and we have injectivity because for all $x, y \in S_{ij}$

$$\begin{aligned} axa = aya \Rightarrow b(axa) = b(aya) \Rightarrow bax = bay \\ \Rightarrow a(bax) = a(bay) \Rightarrow ax = ay \Rightarrow x = y, \end{aligned}$$

the second and the last implication following from (2.5) and (2.6), respectively, because $bax, bay, x, y \in S_{ij}$. Thus, if (2.5) and (2.6) both hold, then all the semigroups in the diagram (2.2) are isomorphic.

Remark 2.3.1. Clearly, the duo (2.5) and (2.6) not only implies injectivity of Φ, but is also implied by it. The justification is simple: if $\Phi = \Psi_1 \circ \Phi_1 = \Psi_2 \circ \Phi_2$ is an isomorphism, then Ψ_1 (and similarly Ψ_2) has to be injective, because $x\Psi_1 = y\Psi_1$ implies $x\Phi = y\Phi$.

However, even if none of the conditions (2.5) and (2.6) hold, we can prove that the monoids $(aS_{ij}a, \star_b)$ and $(bS_{ji}b, \star_a)$ are isomorphic. Namely, the maps $aS_{ij}a \to bS_{ji}b : x \mapsto bxb$ and $bS_{ij}b \to aS_{ji}a : x \mapsto axa$ are mutually inverse isomorphisms:

$$a(b(awa)b)a = awa, \quad b(a(bqb)a)b = bqb, \quad \text{for all } w \in S_{ij}, \text{ and all } q \in S_{ji}.$$

Expanding the diagram downwards, we are able to show that

$$(aS_{ij}a, \circledast) \to (baS_{ij}a, \cdot) : x \mapsto bx \quad \text{and} \quad (aS_{ij}a, \circledast) \to (aS_{ij}ab, \cdot) : x \mapsto xb$$

are isomorphisms (since $baxa = baya \Rightarrow abaxa = abaya$ and $axab = ayab \Rightarrow axaba = ayaba$ for any $x, y \in S_{ij}$). Moreover, since $S_{ij}a = S_iba$ (as was shown at the beginning of this section), it follows that $(aS_{ij}a, \circledast)$ is isomorphic to $(baS_iba, \cdot)$, the local monoid of the semigroup S_i with respect to the idempotent $ba \in S_i$ (clearly, $(bab)a = ba$). By symmetry, $(aS_{ij}a, \circledast)$ is also isomorphic to the local monoid $(abS_jab, \cdot)$ of S_j with respect to the idempotent $ab \in S_j$.

The purpose of this lengthy discussion and the connection to P^a will be revealed when we reexamine the diagram on Figure 2.2, restricting our attention to the set $\mathrm{Reg}(S^a_{ij}) \subseteq S_{ij}$. Of course, this set might not be a subsemigroup of S^a_{ij}, so we introduce a condition (see [33]) ensuring that it is (and much more): we choose a *sandwich-regular* sandwich element $a \in S_{ji}$, which means that $\{a\} \cup aS_{ij}a \subseteq \mathrm{Reg}(S)$. As in [33], we prove

Proposition 2.3.2. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S and let $b \in \mathrm{V}(a)$. Then*

(i) $\mathrm{Reg}(S^a_{ij}) = \mathrm{P}^a$ *is a regular subsemigroup of* S^a_{ij}*,*

(ii) $\mathrm{Reg}(S_{ij}a, \cdot) = \mathrm{P}^a\, a = \mathrm{P}^a_2\, a$ *is a regular subsemigroup of* $(S_{ij}a, \cdot)$*,*

(iii) $\mathrm{Reg}(aS_{ij},\cdot) = a\,\mathrm{P}^a = a\,\mathrm{P}_1^a$ *is a regular subsemigroup of* $(aS_{ij},\cdot)$,

(iv) $aS_{ij}a = \mathrm{Reg}(aS_{ij}a,\circledast) = a\,\mathrm{P}^a\,a = a\,\mathrm{P}_1^a\,a = a\,\mathrm{P}_2^a\,a$ *is a regular subsemigroup of* S_{ji}^b.

Proof. (i) From Proposition 2.2.29(iii), we know that P^a is a subsemigroup of S_{ij}^a, and Proposition 2.2.2(i) gives $\mathrm{Reg}(S_{ij}^a) \subseteq \mathrm{P}^a$. Thus, it suffices to prove the reverse inclusion. Suppose $x \in \mathrm{P}^a$ and $x = xay = zax$ for $z, y \in S_{ij}$. Since a is sandwich-regular, $axa \in aS_{ij}a$ is regular in S, so there exists $t \in S_{ij}$ such that $axa = axataxa$. We may deduce

$$x = xay = zaxay = z(axataxa)y = (zax)ata(xay) = xatax = x \star_a t \star_a x,$$

so $x \in \mathrm{Reg}(S_{ij}^a)$.

(ii) From the definition of P^a, we have $\mathrm{P}^a\,a \subseteq \mathrm{P}_2^a\,a$. Conversely, if $x \in \mathrm{P}_2^a$ and $yax = x$ for some $y \in S_{ij}$, then $xa = x(aba) = (xab)a$, so $xab\,\mathscr{R}\,xa$ $(= xaba)$ and $xab = (yax)ab = y(axab)$, so $xab\,\mathscr{L}\,axab$. Thus $xab \in \mathrm{P}^a$ and $xa = xaba \in \mathrm{P}^a\,a$. We have proved $\mathrm{P}^a\,a = \mathrm{P}_2^a\,a$.

From (i) we deduce that $\mathrm{P}^a\,a = \mathrm{Reg}(S_{ij}^a)a$ is a subsemigroup of S_i (because $\mathrm{P}^a\,a\,\mathrm{P}^a\,a \subseteq \mathrm{P}^a\,a$). Also, $\mathrm{Reg}(S_{ij}^a)a \subseteq \mathrm{Reg}(S_{ij}a,\cdot)$, because any $x \in \mathrm{Reg}(S_{ij}^a)$ with $x = xayax$ (where $y \in S_{ij}$) satisfies $xa = xayaxa$, so $xa \in \mathrm{Reg}(S_{ij}a,\cdot)$. If we show $\mathrm{Reg}(S_{ij}a,\cdot) \subseteq \mathrm{P}_2^a\,a$, the statement will be proved in whole. Suppose $ya \in \mathrm{Reg}(S_{ij}a,\cdot)$, with $qa \in S_{ij}a$ such that $yaqaya = ya$. But then $yaqay = (ya)qay = (yaqaya)qay = yaq(ayaqay) \leq_{\mathscr{L}} ayaqay$, so $yaqay\,\mathscr{L}\,ayaqay$ and $ya = (yaqay)a \in \mathrm{P}_2^a\,a$. A dual argument proves (iii).

(iv) From (ii) and (iii) we have $a\,\mathrm{P}^a\,a = a\,\mathrm{P}_2^a\,a$ and $a\,\mathrm{P}^a\,a = a\,\mathrm{P}_1^a\,a$, respectively. Also, (i) implies that $a\,\mathrm{P}^a\,a = a\,\mathrm{Reg}(S_{ij}^a)a$ is a subsemigroup of S_{ji}^b, since $a\,\mathrm{P}^a\,a \star_b a\,\mathrm{P}^a\,a = a\,\mathrm{P}^a\,a\,\mathrm{P}^a\,a \subseteq a\,\mathrm{P}^a\,a$. Furthermore, we have $a\,\mathrm{P}^a\,a \subseteq \mathrm{Reg}(aS_{ij}a,\circledast) \subseteq aS_{ij}a$, since for any $z \in \mathrm{P}^a$, (i) gives $z \in \mathrm{Reg}(S_{ij}^a)$, so $z = zayaz$ for some $y \in S_{ij}$, which implies

$$aza = azayaza = az(aba)y(aba)za = aza \circledast aya \circledast aza,$$

hence $aza \in \mathrm{Reg}(aS_{ij}a,\circledast) \subseteq aS_{ij}a$. We still need to prove $aS_{ij}a \subseteq a\,\mathrm{P}^a\,a$. Let $x \in S_{ij}$ be arbitrary. Then $axa = (aba)x(aba) = a(baxab)a$, with $baxab = baxabab \leq_{\mathscr{R}} baxaba$ and $baxab = babaxab \leq_{\mathscr{L}} abaxab$, so $baxab \in \mathrm{P}^a$ and $axa = abaxaba \in a\,\mathrm{P}^a\,a$. □

Note that all the elements of a regular partial semigroup are sandwich-regular. Conveniently, this will be the case in all the partial semigroups we consider. Thus, from now on, we suppose that the chosen sandwich element a is sandwich-regular unless stated otherwise.

The discussion and results presented in this subsection so far add up to the following commutative diagram (of semigroup surmorphisms):

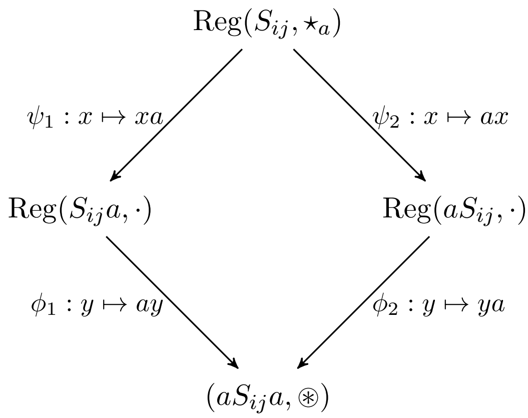


Figure 2.3: A diagram depicting the connections between $\mathrm{Reg}(S^a_{ij})$ and $aS_{ij}a$.

To simplify notation, we write

$$\mathrm{P}^a = \mathrm{Reg}(S_{ij},\star_a), \qquad T_1 = \mathrm{Reg}(S_{ij}a,\cdot) = \mathrm{P}^a\, a,$$
$$W = (aS_{ij}a,\circledast) = a\,\mathrm{P}^a\, a, \qquad T_2 = \mathrm{Reg}(aS_{ij},\cdot) = a\,\mathrm{P}^a\,.$$

As our investigation below will reveal, the semigroups P^a and W have very much in common. This will be shown using their connections via T_1 and T_2. Thus, we will refer the reader to Diagram 2.3 quite often.

2.3.2 Green's relations on P^a and W

Our next objective is to describe the connection between P^a and W. In order to do that, we need to examine Green's relations in both of these semigroups, as in [33].

Let us denote Green's relations of $\mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$ with $\mathscr{K}^{\mathrm{P}^a}$, for all $\mathscr{K} \in \{\mathscr{R},\mathscr{L},\mathscr{H},\mathscr{D},\mathscr{J}\}$, and the corresponding class containing $x \in \mathrm{P}^a$ with $\mathrm{K}^{\mathrm{P}^a}_x$. The following lemma is, in cases where $\mathscr{K} = \mathscr{L},\mathscr{R},\mathscr{H}$, a special case of a more general Proposition 2.4.2. in [58].

Lemma 2.3.3. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. If $\mathscr{K}^{\mathrm{P}^a}$ is any of Green's relations on $\mathrm{Reg}(S^a_{ij}) = \mathrm{P}^a$ other than $\mathscr{J}^{\mathrm{P}^a}$, then $\mathscr{K}^{\mathrm{P}^a} = \mathscr{K}^a \cap (\mathrm{P}^a \times \mathrm{P}^a)$. Moreover, for all $x \in \mathrm{P}^a$ holds $\mathrm{K}^{\mathrm{P}^a}_x = \mathrm{K}^a_x$.*

Proof. The equality $\mathscr{K}^{\mathrm{P}^a} = \mathscr{K}^a \cap (\mathrm{P}^a \times \mathrm{P}^a)$ for $\mathscr{K} \in \{\mathscr{R},\mathscr{L},\mathscr{H}\}$ follows from Remark 2.2.41. Let $\mathscr{K} = \mathscr{D}$. Clearly, $\mathscr{D}^{\mathrm{P}^a} \subseteq \mathscr{D}^a \cap (\mathrm{P}^a \times \mathrm{P}^a)$. Suppose that $(x,y) \in \mathscr{D}^a \cap (\mathrm{P}^a \times \mathrm{P}^a)$ and $x \,\mathscr{L}^a\, z \,\mathscr{R}^a\, y$ for some $z \in S_{ij}$. Since x is a regular element in S^a_{ij}, Remark 1.3.8 implies $\mathrm{L}^a_x \subseteq \mathrm{Reg}(S^a_{ij}) = \mathrm{P}^a$. Thus $z \in \mathrm{P}^a$ and $x \,\mathscr{L}^{\mathrm{P}^a}\, z \,\mathscr{R}^{\mathrm{P}^a}\, y$, so $x \,\mathscr{D}^{\mathrm{P}^a}\, y$.

For the second part, note that, since $x \in \mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$, Remark 1.3.8 implies $\mathrm{K}^a_x \subseteq \mathrm{Reg}(S^a_{ij}) = \mathrm{P}^a$, for all $\mathrm{K} \in \{\mathrm{L},\mathrm{R},\mathrm{H},\mathrm{D}\}$. This means that we have $\mathrm{K}^{\mathrm{P}^a}_x = \mathrm{K}^a_x$ in all these cases. □

The previous lemma prompts us to simplify the notation, since the Green's relations and their classes in P^a coincide with the ones in S_{ij}^a. We will therefore use $\mathscr{K}^a$ instead of $\mathscr{K}^{\mathrm{P}^a}$ for all $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}\}$, and for any $x \in \mathrm{P}^a$ the corresponding class will be denoted K_x^a. Of course, we are keeping the notation $\mathscr{J}^{\mathrm{P}^a}$ and $\mathrm{J}_x^{\mathrm{P}^a}$. However, there is a special case in which we do not need to consider $\mathscr{J}^{\mathrm{P}^a}$ separately.

Lemma 2.3.4. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. If $\mathscr{J} = \mathscr{D}$, then in S_{ij}^a we have $\mathscr{J}_x^{\mathrm{P}^a} = \mathscr{D}_x^{\mathrm{P}^a}$.*

Proof. From Theorem 2.2.3, Proposition 2.2.2(i) and Lemma 2.3.3, for each $x \in \mathrm{P}^a$ we have

$$\mathrm{J}_x^{\mathrm{P}^a} \subseteq \mathrm{J}_x^a \cap \mathrm{P}^a = \mathrm{J}_x \cap \mathrm{P}_3^a \cap \mathrm{P}^a = \mathrm{J}_x \cap \mathrm{P}^a = \mathrm{D}_x \cap \mathrm{P}^a = \mathrm{D}_x^a = \mathrm{D}_x^{\mathrm{P}^a} \subseteq \mathrm{J}_x^{\mathrm{P}^a}.$$

Thus, $\mathrm{J}_x^{\mathrm{P}^a} = \mathrm{D}_x^{\mathrm{P}^a}$. □

Next, we turn to Green's relations of W. To avoid confusion, we will denote them by $\mathscr{K}^{\circledast}$ for $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$, and the corresponding class containing $x \in W$ will be denoted by $\mathrm{K}_x^{\circledast}$. The next Lemma will give us a clearer picture on the way Green's relations of W relate to Green's relations of S_{ji}^b.

Lemma 2.3.5. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S and let $b \in \mathrm{V}(a)$. If $x \in W$, then $\mathrm{H}_x^{\circledast} = \mathrm{H}_x^b$.*

Proof. Clearly, $\mathrm{H}_x^{\circledast} \subseteq \mathrm{H}_x^b$. Suppose $y \in \mathrm{H}_x^b$. If $y = x$, then $y \in \mathrm{H}_x^{\circledast}$. In case $y \neq x$, there exist $s, t \in S_{ji}$ such that $y = sbx$ and $y = xbt$. Since $x \in W$ and W is a monoid with identity a, we have

$$\begin{aligned} y &= sbx = s \star_b x = s \star_b (x \star_b a) = (s \star_b x) \star_b a = y \star_b a \quad \text{and} \\ y &= xbt = x \star_b t = (a \star_b x) \star_b t = a \star_b (x \star_b t) = a \star_b y. \end{aligned}$$

Therefore, $y = a \star_b y = a \star_b y \star_b a = a(byb)a \in aS_{ij}a = W$, so $x, y \in W = \mathrm{Reg}(W)$. Thus, Remark 2.2.41($ii$) implies $x \mathscr{H}^{\circledast} y$. □

2.3.3 Pullback products and an embedding

Finally, we are in a position to show a new aspect of the connection between P^a and W, as in [33]. In order to do that, we need a short introduction to define the necessary terms.

First, we introduce the notion of a subdirect product of semigroups. The following definition is a specialised version of the general definition (which concerns subdirect product of algebras) used in Universal algebra (see Definition 8.1. in [14]).

Definition 2.3.6. Let $\{A_i : i \in I\}$ be a family of semigroups with direct product $\prod_{i \in I} A_i$. For each $j \in I$, let $\pi_j : \prod_{i \in I} A_i \to A_j$ be the j-th projection ($x\pi_j$ gives the j-th component of x). A subsemigroup A of $\prod_{i \in I} A_i$ is a *subdirect product* of semigroups $A_i : i \in I$ if, for each $j \in I$, the restriction $\pi_i{\restriction}_A$ is surjective.

The next term we define is closely related to that of subdirect products. In fact, its definition is a "recipe" to create a special kind of subdirect product of semigroups.

Definition 2.3.7. The *pullback product* of semigroups $A_i : i \in I$ with respect to a semigroup T and surmorphisms $f_i : A_i \to T$ (one surmorphism for each $i \in I$) is the semigroup

$$\{a \in \prod_{i \in I} A_i : a\pi_j f_j = a\pi_k f_k \text{ for all } j, k \in I\}.$$

Clearly, any pullback product A of semigroups $A_i : i \in I$ is a subsemigroup of $\prod_{i \in I} A_i$ (because projections $\pi_i : i \in I$ and maps $f_i : i \in I$ are all homomorphisms) and a subdirect product of semigroups $A_i : i \in I$. (The second assertion is true because, for a fixed $k \in I$ and a fixed $x \in A_k$, the element xf_k of T corresponds to at least one element of A, as the set $(xf_k)f_i^{-1}$ is non-empty for each $i \in I$ and contains the possible coordinates for each $i \in I$.)

Having introduced the necessary concepts, we continue to study P^a and W. Let us consider the map

$$\Psi = (\Psi_1, \Psi_2) : (S_{ij}, \star_a) \to (S_{ij}a, \cdot) \times (aS_{ij}, \cdot) : x \mapsto (xa, ax).$$

We may prove now that the set $\mathrm{im}(\Psi)$ is a subdirect product of $(S_{ij}a, \cdot)$ and $(aS_{ij}, \cdot)$. Firstly, as $\mathrm{im}(\Psi)$ is an image of a homomorphism, it is a subsemigroup of $(S_{ij}a, \cdot) \times (aS_{ij}, \cdot)$. Moreover, for any element of $S_{ij}a$ (aS_{ij}), there evidently exists an element from S_{ij} mapping to it via Ψ_1 (Ψ_2).

Obviously, Ψ might not be injective, nor surjective. If, however, the implication

$$xa = ya \quad \text{and} \quad ax = ay \quad \Rightarrow \quad x = y$$

holds for all $x, y \in S_{ij}$, then Ψ is an embedding.

Following the same "tactics" as in the case of the maps from diagram 2.2, we define functions

$$\phi : \psi_1\phi_1 = \psi_2\phi_2 = \mathrm{P}^a \to W : x \mapsto axa$$

and

$$\psi = (\psi_1, \psi_2) : \mathrm{P}^a \to T_1 \times T_2 : x \mapsto (xa, ax)$$

for the maps ϕ_1, ϕ_2, ψ_1 and ψ_2 from diagram 2.3. Being "specialised versions" of maps Ψ and Φ, the maps ψ and ϕ inherit some of their properties. In particular, ϕ is a surmorphism (just as Φ is), because it is a composition of surmorphisms, and $\mathrm{im}(\psi)$ is a subdirect product of T_1 and T_2. We may give an even stronger result:

Theorem 2.3.8. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. Then*

(i) *ψ is injective,*

(ii) $\mathrm{im}(\psi) = \{(s,t) \in T_1 \times T_2 : as = ta\} = \{(s,t) \in T_1 \times T_2 : s\phi_1 = t\phi_2\}$.

In particular, P^a is a pullback product of T_1 and T_2 with respect to W and surmorphisms ψ_1 and ψ_2.

Proof. To prove (i), suppose that $x, y \in \mathrm{P}^a$ are elements such that $x\psi = y\psi$. This implies $(xa, ax) = (ya, ay)$, so $xa = ya$ and $ax = ay$. Since $x, y \in \mathrm{P}^a \subseteq \mathrm{P}_1^a$, there exist $s, t \in S_{ij}$ such that $x = xas$ and $y = yat$. Thus, from $xa = ya$ we have $x = yas$ and $y = xat$, i.e. $x \mathscr{R}^a y$. Dually, $x \in \mathrm{P}_2^a$ and $ax = ay$ together imply $x \mathscr{L}^a y$. Lemma 1.3.4(i), applied to the semigroup P^a, guarantees that the maps $\mathrm{L}_x^a \to \mathrm{L}_y^a : w \mapsto wat$ and $\mathrm{L}_y^a \to \mathrm{L}_x^a : w \mapsto was$ are mutually inverse bijections, hence we have $w = watas$ for all $w \in \mathrm{L}_x^a$. Since $x \mathscr{L}^a y$, we have $x = xatas$ and $y = yatas$, so $x = xatas = yatas = y$, because $xa = ya$.

We show only the first equality in (ii) (the second one being obvious) by showing both inclusions. First, suppose that $(x, y) \in \mathrm{im}(\psi)$. This implies that there exists $q \in \mathrm{P}^a$ such that $(x, y) = q\psi = (qa, aq)$, so $ax = aqa = ya$. Thus, $(x, y) \in \{(s, t) \in T_1 \times T_2 : as = ta\}$. Conversely, if we suppose $(x, y) \in \{(s, t) \in T_1 \times T_2 : as = ta\}$, there exist $z, q \in \mathrm{P}^a$ with $x = za$ and $y = aq$, and we have $aza = ax = ya = aqa$. Since $z \in \mathrm{P}^a = \mathrm{Reg}(S_{ij}^a)$, there exists $p \in S_{ij}$ such that $z = zapaz$. Similarly, $ax = aza \in W = \mathrm{Reg}(W)$, so there exists $v \in \mathrm{P}^a$ such that $aza \circledast ava \circledast aza = aza$. In other words, $azavaza = aza$, so $axvax = ax$. Therefore

$$x = za = zapaza = zapax = zapaxvax = zapazavax = zavax = xvax.$$

By a symmetric argument, there exists $r \in S_{ij}$ such that $q = qaraq$. Similarly as above, $azavaza = aza$ implies $aqavaqa = aqa$ (because $aza = aqa$), so $yavya = ya$ and

$$y = aq = aqaraq = yaraq = yavyaraq = yavaqaraq = yavaq = yavy.$$

Finally, from $ax = ya$ follows

$$(x, y) = (xvax, yavy) = (xvya, axvy) = ((xvy)\Psi_1, (xvy)\Psi_2),$$

and since from Proposition 2.3.2(i) we have

$$xvy \in \mathrm{Reg}(S_{ij}a, \cdot)\, \mathrm{P}^a\, \mathrm{Reg}(aS_{ij}, \cdot) = \mathrm{P}^a\, a\, \mathrm{P}^a\, a\, \mathrm{P}^a \subseteq \mathrm{P}^a,$$

we may conclude that $(x, y) = ((xvy)\psi_1, (xvy)\psi_2)$. □

2.3.4 The internal structure of the $\mathscr{D}^a$-classes of P^a

Continuing the task of describing the connection between P^a and W, we focus on the way their internal structures (meaning Green's relations and their classes) are related via the map $\phi : \mathrm{P}^a \to W : x \mapsto axa$. Therefore, we will often use this particular function, and it will be useful to shorten the notation. For all $x \in \mathrm{P}^a$, we write $\overline{x} = x\phi = axa \in W$, and if $X \subseteq \mathrm{P}^a$, then we write $\overline{X} = \{\overline{x} : x \in X\}$.

Also, for all $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$ and all $x, y \in \mathrm{P}^a$, we define $x \widehat{\mathscr{K}^a} y$ if $\overline{x}\, \mathscr{K}^{\circledast}\, \overline{y}$ in W. Clearly, $\widehat{\mathscr{K}^a}$ is an equivalence relation for each $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$, since it is the ϕ-preimage of an equivalence relation. For an element $x \in \mathrm{P}^a$, its $\widehat{\mathscr{K}^a}$-class in P^a is denoted by $\widehat{\mathrm{K}_x^a}$. Obviously, $\widehat{\mathrm{K}_x^a}$ is the inverse image of the class

$\mathrm{K}^{\circledast}_{\overline{x}}$ with respect to the homomorphism ϕ, so it has to preserve the classes of the original Green's relation. Let us elaborate on this: two elements that are in the same $\mathscr{K}^a$-class of P^a map into the same $\mathscr{K}^{\circledast}$-class in W, since homomorphisms preserve Green's relations; therefore, an inverse image of the class $\mathrm{K}^{\circledast}_{\overline{x}}$ is a union of $\mathscr{K}^a$-classes in P^a.

In general, the idempotents of a sandwich semigroup are not idempotents in the partial semigroup containing it, so we introduce special notation: for $X \subseteq \mathrm{P}^a$ and $Y \subseteq W$, we write

$$\mathrm{E}_a(X) = \{x \in X \ : \ x = xax\}, \quad \text{and} \quad \mathrm{E}_b(Y) = \{y \in Y \ : \ y = yby\}.$$

(Note that, as $x = xax$ implies $xa \mathrel{\mathscr{R}} x \mathrel{\mathscr{L}} ax$, P^a contains all the idempotents of S^a_{ij}, even if a is not sandwich-regular.) Furthermore, for $x \in \mathrm{P}^a$, the set of all its inverses (with respect to $\star_a$) from P^a is denoted

$$\mathrm{V}_a(x) = \{y \in \mathrm{P}^a \ : \ x = x \star_a y \star_a x \text{ and } y = y \star_a x \star_a y\}.$$

Equipped with these new terms and notation, we may prove the following lemma from [33]:

Lemma 2.3.9. *If $a \in S_{ji}$ is a sandwich-regular element of a partial semigroup S, then in $\mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$ we have*

(i) $\mathscr{R}^a \subseteq \widehat{\mathscr{R}^a} \subseteq \mathscr{D}^a$,

(ii) $\mathscr{L}^a \subseteq \widehat{\mathscr{L}^a} \subseteq \mathscr{D}^a$,

(iii) $\mathscr{H}^a \subseteq \widehat{\mathscr{H}^a} \subseteq \mathscr{D}^a$,

(iv) $\widehat{\mathscr{D}^a} = \mathscr{D}^a \subseteq \widehat{\mathscr{J}^a} = \mathscr{J}^{\mathrm{P}^a}$.

Proof. First, we prove (i). The discussion above shows $\mathscr{R}^a \subseteq \widehat{\mathscr{R}^a}$. To prove the inclusion $\widehat{\mathscr{R}^a} \subseteq \mathscr{D}^a$, suppose $(x, y) \in \widehat{\mathscr{R}^a}$. Since $x, y \in \mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$, by Remark 1.3.8 and Lemma 2.3.3, in the semigroup P^a there exist idempotents e, f such that $x \mathrel{\mathscr{R}^a} e$ and $y \mathrel{\mathscr{R}^a} f$. We will find an element $w \in \mathrm{P}^a$ such that $e \mathrel{\mathscr{R}^a} w \mathrel{\mathscr{L}^a} f$, and it will follow that $x \mathrel{\mathscr{D}^a} e \mathrel{\mathscr{D}^a} w \mathrel{\mathscr{D}^a} f \mathrel{\mathscr{D}^a} y$, since $\mathscr{L}^a \subseteq \mathscr{D}^a$ and $\mathscr{R}^a \subseteq \mathscr{D}^a$. Let us choose $w = eaf$. Clearly, $w \leq_{\mathscr{L}^a} f$ and $w \leq_{\mathscr{R}^a} e$. To prove the inverse relations, we turn our attention to the situation in W. From $x \mathrel{\mathscr{R}^a} e$ and $y \mathrel{\mathscr{R}^a} f$ follows $x \mathrel{\widehat{\mathscr{R}^a}} e$ and $y \mathrel{\widehat{\mathscr{R}^a}} f$, respectively, so $e \mathrel{\widehat{\mathscr{R}^a}} x \mathrel{\widehat{\mathscr{R}^a}} y \mathrel{\widehat{\mathscr{R}^a}} f$, and therefore $\overline{e} \mathrel{\mathscr{R}^{\circledast}} \overline{f}$ in W. Since $e, f \in \mathrm{E}_a(\mathrm{P}^a)$, we have $\overline{e}, \overline{f} \in \mathrm{E}_b(W)$ (because $aea \circledast aea = aeaea = aea$, and a similar calculation holds for f). Being idempotents, both of them are left-identities for their $\mathscr{R}^{\circledast}$-class, so $\overline{e} \circledast \overline{f} = \overline{f}$ and $\overline{f} \circledast \overline{e} = \overline{e}$, i.e. $aeafa = afa$ and $afaea = aea$. Now, we may deduce

$$\begin{aligned} e &= eae = eaeae = eafaeae \leq_{\mathscr{R}^a} eaf, \\ f &= faf = fafaf = faeafaf = faeaf \leq_{\mathscr{L}^a} eaf, \end{aligned}$$

and finally $e \mathrel{\mathscr{R}^a} eaf \mathrel{\mathscr{L}^a} f$.

(ii) follows by a dual argument, and (iii) follows from (i) and (ii), because for all $x \in \mathrm{P}^a$ we have

$$\widehat{\mathrm{H}^a_x} = (\mathrm{H}^{\circledast}_{\overline{x}})\phi^{-1} = (\mathrm{R}^{\circledast}_{\overline{x}} \cap \mathrm{L}^{\circledast}_{\overline{x}})\phi^{-1} = (\mathrm{R}^{\circledast}_{\overline{x}})\phi^{-1} \cap (\mathrm{L}^{\circledast}_{\overline{x}})\phi^{-1} = \widehat{\mathrm{R}^a_x} \cap \widehat{\mathrm{L}^a_x},$$

which implies $\widehat{\mathscr{H}^a} = \widehat{\mathscr{R}^a} \cap \widehat{\mathscr{L}^a}$.

We need to show (iv). The inclusions $\mathscr{D}^a \subseteq \widehat{\mathscr{D}^a}$ and $\mathscr{J}^{\mathrm{P}^a} \subseteq \widehat{\mathscr{J}^a}$ are easy to prove. Furthermore, $\widehat{\mathscr{D}^a} \subseteq \widehat{\mathscr{J}^a}$, since for all $x \in \mathrm{P}^a$ holds

$$\widehat{\mathrm{D}^a_x} = (\mathrm{D}^{\circledast}_{\overline{x}})\phi^{-1} \subseteq (\mathrm{J}^{\circledast}_{\overline{x}})\phi^{-1} = \widehat{\mathrm{J}^a_x}.$$

Hence, it suffices to show $\widehat{\mathscr{D}^a} \subseteq \mathscr{D}^a$ and $\widehat{\mathscr{J}^a} \subseteq \mathscr{J}^{\mathrm{P}^a}$. The first inclusion follows from (i) and (ii), used in the following reasoning:

$$\begin{aligned} x \, \widehat{\mathscr{D}^a} \, y &\Leftrightarrow \overline{x} \, \mathscr{D}^{\circledast} \, \overline{y} \Leftrightarrow (\exists z \in \mathrm{P}^a) \ (\overline{x} \ \mathscr{R}^{\circledast} \ \overline{z} \ \mathscr{L}^{\circledast} \ \overline{y}) \\ &\Leftrightarrow (\exists z \in \mathrm{P}^a) \ (x \ \widehat{\mathscr{R}^a} \ z \ \widehat{\mathscr{L}^a} \ y) \Leftrightarrow x \ \widehat{\mathscr{R}^a} \circ \widehat{\mathscr{L}^a} \ y \cdot \\ &\Rightarrow x \, \mathscr{D}^a \circ \mathscr{D}^a \, y \Rightarrow x \, \mathscr{D}^a \, y. \end{aligned}$$

For the second, suppose $x \ \widehat{\mathscr{J}^a} \ y$, i.e. $\overline{x} \ \mathscr{J}^{\circledast} \, \overline{y}$ (so by definition, $\overline{x} \leq_{\mathscr{J}^{\circledast}} \overline{y}$ and $\overline{y} \leq_{\mathscr{J}^{\circledast}} \overline{x}$). This means that there exist $u, v, p, q \in \mathrm{P}^a$ such that $\overline{u} \circledast \overline{x} \circledast \overline{p} = \overline{y}$ and $\overline{v} \circledast \overline{y} \circledast \overline{q} = \overline{x}$, so $auaxapa = aya$ and $avayaqa = axa$. If we choose $z \in \mathrm{V}_a(x)$ and $r \in \mathrm{V}_a(y)$ (which exist, since $x, y \in \mathrm{Reg}(S^a_{ij})$), then

$$\begin{aligned} x &= xazax = xazaxazax = xazavayaqazax = xazav \star_a y \star_a qazax \leq_{\mathscr{J}^{\mathrm{P}^a}} y \\ y &= yaray = yarayaray = yarauaxaparay = yarau \star_a x \star_a paray \leq_{\mathscr{J}^{\mathrm{P}^a}} x, \end{aligned}$$

so $x \ \mathscr{J}^{\mathrm{P}^a} \ y$. □

Note that, in the proof for $\widehat{\mathscr{J}^a} \subseteq \mathscr{J}^{\mathrm{P}^a}$, we have proved the implication $\overline{x} \leq_{\mathscr{J}^{\circledast}} \overline{y} \Rightarrow x \leq_{\mathscr{J}^{\mathrm{P}^a}} y$. The reverse implication obviously holds, so we have

Corollary 2.3.10. *If $a \in S_{ji}$ is a sandwich-regular element of a partial semigroup S and if $x, y \in \mathrm{P}^a$, then* $\mathrm{J}^{\mathrm{P}^a}_x \leq_{\mathscr{J}^{\mathrm{P}^a}} \mathrm{J}^{\mathrm{P}^a}_y \Leftrightarrow \mathrm{J}^{\circledast}_{\overline{x}} \leq_{\mathscr{J}^{\circledast}} \mathrm{J}^{\circledast}_{\overline{y}}$.

In order to prove the main theorem of this section, we need to make a crucial step by proving the next lemma (from [33]), which turns out to be vital in the next section, as well.

Lemma 2.3.11. *If $a \in S_{ji}$ is a sandwich-regular element of a partial semigroup S, we have*

$$\mathrm{E}_a(\mathrm{P}^a) = \mathrm{E}_a(S^a_{ij}) = (\mathrm{E}_b(W))\phi^{-1}.$$

Proof. The first equality is clear. For the second, note that we have $\mathrm{E}_a(\mathrm{P}^a) \subseteq (\mathrm{E}_b(W))\phi^{-1}$ because $aba = a$. For the reverse inclusion, we suppose $x \in (\mathrm{E}_b(W))\phi^{-1}$ (which implies $axa = axa \circledast axa = axaxa$) and choose $y \in \mathrm{V}_a(x)$, so that

$$x = xayax = xay(axa)yax = (xayax)a(xayax) = xax = x \star_a x. \qquad \square$$

Finally, we may prove a result from [33], which is the most important theorem in the whole section.

Theorem 2.3.12. *Suppose $a \in S_{ji}$ is a sandwich-regular element of a partial semigroup S. Let $x \in \mathrm{P}^a$ and put $r = |\widehat{\mathrm{H}^a_x}/\mathscr{R}^a|$ and $l = |\widehat{\mathrm{H}^a_x}/\mathscr{L}^a|$. Then*

(i) the restriction of the map $\phi : \mathrm{P}^a \to W$ to the set H^a_x, $\phi\restriction_{\mathrm{H}^a_x} : \mathrm{H}^a_x \to \mathrm{H}^{\circledast}_{\overline{x}}$, is a bijection,

(ii) H^a_x is a group if and only if $\mathrm{H}^{\circledast}_{\overline{x}}$ is a group, in which case these groups are isomorphic,

(iii) if H^a_x is a group, then $\widehat{\mathrm{H}^a_x}$ is an $r \times l$ rectangular group over $\mathrm{H}^{\circledast}_{\overline{x}}$,

(iv) if H^a_x is a group, then $\mathrm{E}_a(\widehat{\mathrm{H}^a_x})$ is an $r \times l$ rectangular band.

Proof. First, note that the definition of $\widehat{\mathrm{H}^a_x}$ and Lemma 2.3.9(*iii*) together imply that ϕ maps the class H^a_x to the class $\mathrm{H}^{\circledast}_{\overline{x}}$. In addition, recall that $\phi\restriction_{\widehat{\mathrm{H}^a_x}} : \widehat{\mathrm{H}^a_x} \to \mathrm{H}^{\circledast}_{\overline{x}}$ is a surmorphism.

Our first step is to prove the equivalence in (*ii*). The direct implication clearly holds, since ϕ is a homomorphism and it maps the idempotent (identity) of H^a_x to an idempotent element in $\mathrm{H}^{\circledast}_{\overline{x}}$. To prove the reverse implication, suppose that $\mathrm{H}^{\circledast}_{\overline{x}}$ is a group, and let $\overline{e}$ be its idempotent and $\overline{y}$ be the group inverse of $\overline{x}$. Lemma 2.3.11 guarantees that $(\overline{e})\phi^{-1} \subseteq \mathrm{E}_a(\mathrm{P}^a)$. If we fix the element $w = xayaeayax$, we have

$$\begin{aligned}\overline{w} =& a(xayaeayax)a = ax(aba)y(aba)e(aba)y(aba)xa = \\ =& \overline{x} \circledast \overline{y} \circledast \overline{e} \circledast \overline{y} \circledast \overline{x} = \overline{e} \circledast \overline{e} \circledast \overline{e} = \overline{e},\end{aligned}$$

so $w \in \mathrm{P}^a$ is an idempotent. We will show that $w \mathscr{H}^a x$, and then Lemma 1.3.5 will imply that H^a_x is a group. It suffices to show $w \mathscr{R}^a x$, as $w \mathscr{L}^a x$ follows by a symmetric argument. Since $w = xayaeayax$, clearly $w \leq_{\mathscr{R}^a} x$; if $v \in \mathrm{V}_a(x)$, then

$$\begin{aligned}x &= xavax = xavaxavax = xav \cdot \overline{x} \cdot vax = xav \cdot \overline{e} \circledast \overline{x} \cdot vax \\ &= xav \cdot \overline{w} \circledast \overline{x} \cdot vax = xav \cdot (axayaeayaxa)b(axa) \cdot vax \\ &= (xavax)(ayaeayaxaba)(xavax) \\ &= (xayaeayax)ax = wax,\end{aligned}$$

so $x \leq_{\mathscr{R}^a} w$. Thus, $x \mathscr{R}^a w$.

Now, we need to prove that, in the case that H^a_x and $\mathrm{H}^{\circledast}_{\overline{x}}$ are groups, they are isomorphic. Clearly, the map $\phi\restriction_{\mathrm{H}^a_x} : \mathrm{H}^a_x \to \mathrm{H}^{\circledast}_{\overline{x}}$ is a group homomorphism, and w is a unique idempotent in H^a_x (by Lemma 1.3.5), hence Lemma 2.3.11 implies

$$\{u \in \mathrm{H}^a_x : u\phi = \overline{e}\} = \{w\},$$

so $\{w\}$ is the group kernel of $\phi\restriction_{H^a_x}$. Since w is the identity of the group H^a_x and $\phi\restriction_{H^a_x}$ is a homomorphism, for any $c, d \in \mathrm{H}^a_x$ (with group inverses c^{-1} and d^{-1}, respectively), the equality $c\phi = d\phi$ implies

$$\overline{w} = (c\phi)^{-1} \circledast d\phi = c^{-1}\phi \circledast d\phi = (c^{-1}d)\phi$$

(where $(c\phi)^{-1}$ is the group inverse of $c\phi$ in $\mathrm{H}^{\circledast}_{\overline{x}}$), so $c^{-1}d = w$ and therefore $c = d$. We have thus proved that the map $\phi\restriction_{\mathrm{H}^a_x} : \mathrm{H}^a_x \to \mathrm{H}^{\circledast}_{\overline{x}}$ is injective. It remains to show that it is surjective. Let $y \in \mathrm{H}^{\circledast}_{\overline{x}}$. Then there exists $z \in \mathrm{P}^a$ with $\overline{z} = y$. Since $\overline{w}$ is the identity of $\mathrm{H}^{\circledast}_{\overline{x}}$, the element $u = wazaw$ satisfies

$$\overline{u} = aua = awazawa = \overline{w} \circledast \overline{z} \circledast \overline{w} = \overline{z} = y.$$

Furthermore, if $v \in \mathrm{P}^a$ is an element such that $\overline{v}$ is the group inverse of $\overline{z}$ in $\mathrm{H}^{\circledast}_{\overline{x}}$, then

$$\begin{aligned} w = wawaw = w \cdot \overline{w} \cdot w = w \cdot \overline{z} \circledast \overline{v} \cdot w = w \cdot \overline{z} \circledast \overline{w} \circledast \overline{v} \cdot w = \\ = wazawavaw = uavaw \leq_{\mathscr{R}^a} u \end{aligned}$$

so $w \mathscr{R}^a u$. A dual argument shows that $w \mathscr{L}^a u$, so $u \in \mathrm{H}^a_w = \mathrm{H}^a_x$ and $\overline{u} = y$. Therefore, $\phi\restriction_{\mathrm{H}^a_x}$ is an isomorphism.

Next, we prove (i). Of course, in the proof for (ii), we have proved (i) in the case when H^a_x is a group. However, even if that is not the case, Remark 1.3.8 guarantees that there exists $e \in \mathrm{E}_a(\mathrm{P}^a)$ such that $x \mathscr{R}^a e$, for which $\phi\restriction_{\mathrm{H}^a_e} : \mathrm{H}^a_e \to \mathrm{H}^{\circledast}_{\overline{e}}$ is an isomorphism (by (ii)). The relation $x \mathscr{R}^a e$ implies the existence of elements $u, v \in \mathrm{P}^a$ such that $x \star_a u = e$ and $e \star_a v = x$, and Green's Lemma (1.3.4)(i) for semigroups H^a_x and $\mathrm{H}^{\circledast}_{\overline{x}}$ implies that the maps

$$\begin{aligned} &\theta_1 : \mathrm{H}^a_x \to \mathrm{H}^a_e : w \mapsto w \star_a u, &\qquad \theta_3 : \mathrm{H}^a_{\overline{x}} \to \mathrm{H}^a_{\overline{e}} : \overline{w} \mapsto \overline{w} \circledast \overline{u}, \\ &\theta_2 : \mathrm{H}^a_e \to \mathrm{H}^a_x : w \mapsto w \star_a v, &\qquad \theta_4 : \mathrm{H}^a_{\overline{e}} \to \mathrm{H}^a_{\overline{x}} : \overline{w} \mapsto \overline{w} \circledast \overline{v} \end{aligned}$$

are bijections, with $\theta_2 = \theta_1^{-1}$ and $\theta_4 = \theta_3^{-1}$. Now we may conclude that for all $q \in \mathrm{H}^a_x$ we have

$$q \overset{\theta_1}{\longmapsto} qau \overset{\phi}{\longmapsto} aqaua \overset{\theta_4}{\longmapsto} aqauabava = aqauava = a(q\theta_1\theta_2)a = aqa = q\phi,$$

so $\phi\restriction_{\mathrm{H}^a_x} = \theta_1 \circ \phi\restriction_{\mathrm{H}^a_e} \circ \theta_4$ is a composition of three bijections, therefore a bijection itself.

Finally, we prove (iii) and (iv). Suppose that H^a_x is a group. Without loss of generality, we may assume that x is an idempotent. Recall from the end of the Section 1.3, that a rectangular group is a semigroup isomorphic to a direct product of a group and a rectangular band. We need to prove that $\widehat{\mathrm{H}^a_x}$ is a rectangular group and $\mathrm{E}_a(\widehat{\mathrm{H}^a_x})$ is a rectangular band. By Lemma 1.3.11, it suffices to prove that:

(a) $\widehat{\mathrm{H}^a_x}$ is a semigroup;

(b) $\mathrm{E}_a(\widehat{\mathrm{H}^a_x})$ is a rectangular band;

(c) $\widehat{\mathrm{H}^a_x}$ is regular.

To prove (a), suppose that $s,t \in \widehat{\mathrm{H}^a_x}$ and let us inspect $s \star_a t$. Since $\overline{s}, \overline{t} \in \mathrm{H}^{\circledast}_{\overline{x}}$,

$$\overline{s \star_a t} = asata = asabata = \overline{s} \circledast \overline{t},$$

and $\mathrm{H}^{\circledast}_{\overline{x}}$ is a group, the element $\overline{s \star_a t}$ belongs to $\mathrm{H}^{\circledast}_{\overline{x}}$, so $s \star_a t \in \widehat{\mathrm{H}^a_x}$. For (b), recall that by Lemma 1.3.10, it is enough to prove $y \star_a z \star_a y = y$ for all $y, z \in \mathrm{E}_a(\widehat{\mathrm{H}^a_x})$. Suppose $y, z \in \mathrm{E}_a(\widehat{\mathrm{H}^a_x})$. Since homomorphism maps idempotents to idempotents, all the elements of $\mathrm{E}_a(\widehat{\mathrm{H}^a_x})$ map to the identity $\overline{x}$, so $\overline{y} \circledast \overline{z} \circledast \overline{y} = \overline{x} = \overline{y}$ and

$$y = yayay = y \cdot \overline{y} \cdot y = y \cdot \overline{y} \circledast \overline{z} \circledast \overline{y} \cdot y = y \cdot ayazaya \cdot y = yazay = y \star_a z \star_a y.$$

Finally, (c) is clear since $\widehat{\mathrm{H}^a_x}$ is an union of groups. □

Let us pause for a moment and analyse our findings. From the results presented in this subsection (in particular, from the previous theorem and its proof), we may conclude the following:

Remark 2.3.13. The structure of $\mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$ in terms of Green's relations, is a kind of "inflation" of the corresponding structure of $W = \mathrm{Reg}(aS_{ij}a, \circledast) = (aS_{ij}a, \circledast)$. In particular,

- a $\widehat{\mathscr{J}^a}$-class $\widehat{\mathrm{J}^a_x}$ in P^a is precisely the $\mathscr{J}^{\mathrm{P}^a}$-class $\mathrm{J}^{\mathrm{P}^a}_x$, and it corresponds to the $\mathscr{J}^{\circledast}$-class $\mathrm{J}^{\circledast}_{\overline{x}}$ in W; moreover, the partially ordered sets $(\mathrm{P}^a / \mathscr{J}^{\mathrm{P}^a}, \leq_{\mathscr{J}^{\mathrm{P}^a}})$ and $(W/\mathscr{J}^{\circledast}, \leq_{\mathscr{J}^{\circledast}})$ are order-isomorphic;

- a $\widehat{\mathscr{D}^a}$-class $\widehat{\mathrm{D}^a_x}$ in P^a is precisely the $\mathscr{D}^a$-class D^a_x, and it corresponds to the $\mathscr{D}^{\circledast}$-class $\mathrm{D}^{\circledast}_{\overline{x}}$ in W; this correspondence is one-one and onto, meaning that each $\mathscr{D}^{\circledast}$-class corresponds to exactly one $\widehat{\mathscr{D}^a}$-class;

- each $\widehat{\mathscr{K}^a}$-class (where $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}\}$) in P^a is a union of $\mathscr{K}^a$-classes;

- the structure of a single $\mathscr{D}^{\circledast}$-class $\mathrm{D}^{\circledast}_{\overline{x}}$ in terms of relations $\mathscr{R}^{\circledast}$, $\mathscr{L}^{\circledast}$ and $\mathscr{H}^{\circledast}$, is the same as the structure of $\widehat{\mathscr{D}^a_x}$ in terms of relations $\widehat{\mathscr{R}^a}$, $\widehat{\mathscr{L}^a}$ and $\widehat{\mathscr{H}^a}$, respectively, in the sense that each $\mathscr{K}^{\circledast}$-class $\mathrm{K}^{\circledast}_{\overline{x}}$ corresponds to a single $\widehat{\mathscr{K}^a}$-class, $\widehat{\mathrm{K}^a_x}$;

- an $\widehat{\mathscr{H}^a}$-class $\widehat{\mathrm{H}^a_x}$ is a union of $\mathscr{H}^a$-classes, and these are either all non-groups (if $\mathrm{H}^{\circledast}_{\overline{x}} = \mathrm{H}^b_{\overline{x}}$ is a non-group $\mathscr{H}^{\circledast}$-class of W) or else all groups (if $\mathrm{H}^{\circledast}_{\overline{x}} = \mathrm{H}^b_{\overline{x}}$ is a group); in the latter case, $\widehat{\mathrm{H}^a_x}$ is a rectangular group.

The last three observations are illustrated in the Figure 2.4, in the form of egg-box diagrams of a single $\mathscr{D}^a$-class of P^a and its corresponding $\mathscr{D}^{\circledast}$-class of W. The group $\mathscr{H}^a$- and $\mathscr{H}^{\circledast}$-classes are shaded gray, and solid lines in the left egg-box denote the boundaries between $\widehat{\mathscr{R}^a}$-classes and between $\widehat{\mathscr{L}^a}$-classes.

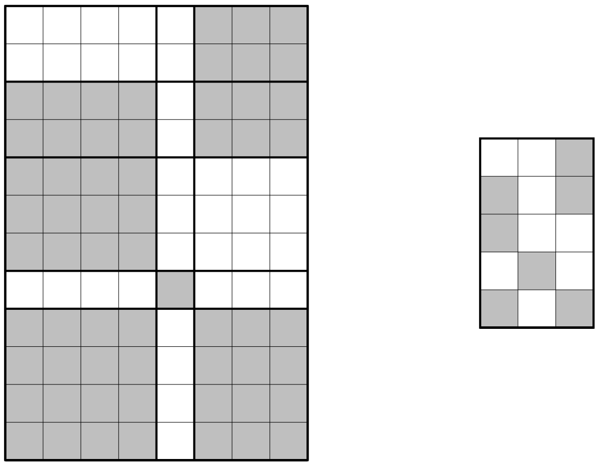


Figure 2.4: P^a is an "inflation" of W.

2.3.5 Generation and idempotent-generation

Having described the connection between P^a and W in detail, we want to use the acquired knowledge to study the problems of generation in P^a, assuming that we have the necessary information on generation in W. We give two results (both from [33]). The first one is technical and rather tiresome. But once we prove it, the second one, Theorem 2.3.15, follows quite smoothly.

For this, we need specific notation. Suppose $X \subseteq S_{ij}$ in a sandwich semigroup S_{ij}^a. Then $\langle X \rangle_a$ denotes the $\star_a$-subsemigroup of S_{ij}^a generated by X.

Lemma 2.3.14. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. If $X \subseteq \mathrm{P}^a$, then $(\langle \overline{X} \rangle_b)\phi^{-1} \subseteq \langle X \cup \mathrm{E}_a(\mathrm{P}^a) \rangle_a$.*

Proof. Suppose $X \subseteq \mathrm{P}^a$ and let $x \in (\langle \overline{X} \rangle_b)\phi^{-1}$. This means that there exist $x_1, \dots, x_n \in X$ such that $x\phi = \overline{x} = \overline{x_1} \circledast \cdots \circledast \overline{x_n}$. We want to prove that x can be generated in P^a by elements from $X \cup \mathrm{E}_a(\mathrm{P}^a)$. It suffices to show that

$$x = p \star_a x_1 \star_a \cdots \star_a x_n \star_a v \tag{2.7}$$

for some $p, v \in \mathrm{E}_a(\mathrm{P}^a)$.

First, we pick the elements p and v, and then we prove that they satisfy (2.7). Put $y = x_1 \star_a \cdots \star_a x_n$. Note that $\overline{y} = \overline{x_1} \circledast \cdots \circledast \overline{x_n} = \overline{x}$, so $y \in \widehat{\mathrm{H}_x^a}$. Thus, from Lemma 2.3.9(iii) follows $y \in \mathrm{D}_x^a$. Since P^a is a regular semigroup, D_x^a is a regular $\mathscr{D}^a$-class, so Remark 1.3.8 guarantees the existence of idempotents $u, v \in \mathrm{E}_a(\mathrm{P}^a)$ such that $u \mathscr{R}^a y$ and $v \mathscr{L}^a x$. We have picked the idempotent v, and p will be chosen using the properties of u. Since $x \mathscr{D}^a y \mathscr{R}^a u$, we have $x \mathscr{D}^a u$, which means that the set $\mathrm{R}_x^a \cap \mathrm{L}_u^a$ is non-empty; in fact, it is an $\mathscr{H}^a$-class in D_x^a, say H_p^a. We may conclude $p \widehat{\mathscr{L}^a} u$ (from $p \mathscr{L}^a u$ and Lemma 2.3.9(*ii*)) and $p \widehat{\mathscr{R}^a} u$ (because $p \mathscr{R}^a x \widehat{\mathscr{H}^a} y \mathscr{R}^a u$ and $\widehat{\mathscr{H}^a} \subseteq \widehat{\mathscr{R}^a}$). Hence, $p \in \widehat{\mathrm{H}_u^a}$, which contains the group H_u^a (u is an idempotent, so the corresponding $\mathscr{H}^a$-class is a group), so Theorem 2.3.12(*ii*) & (*iii*) imply that

$\widehat{\mathrm{H}^a_u}$ is a rectangular group, therefore H^a_p is a group, too. Because of this, we may assume without loss of generality that we have chosen p to be an idempotent. For these idempotents v and p we will prove (2.7).

Before we do that, we need to locate the element $p\star_a x_1 \star_a \cdots \star_a x_n \star_a v = p\star_a y \star_a v$, in terms of the $\mathscr{H}^a$-class containing it. Since $y\,\mathscr{D}^a\,v$, by a dual argument to the one above, we may show that $\mathrm{L}^a_y \cap \mathrm{R}^a_v = \mathrm{H}^a_q$ for some idempotent $q \in \mathrm{E}_a(\mathrm{P}^a)$. Recall that any idempotent in a semigroup is a right-identity of its $\mathscr{L}$-class, and a left-identity of its $\mathscr{R}$-class; hence, from $p\,\mathscr{L}^a\,u$ and $q\,\mathscr{R}^a\,v$ we know that idempotents u and q satisfy $p = p\star_a u$ and $v = q\star_a v$, respectively. Thus, Green's Lemma 1.3.4(i), applied to P^a, implies that:

- the maps
$$\theta_1 : \mathrm{R}^a_u \to \mathrm{R}^a_p : z \mapsto p\star_a z \quad \text{and} \quad \theta_2 : \mathrm{L}^a_q \to \mathrm{L}^a_v : z \mapsto z\star_a v$$
are bijections and
- $(z, z\theta_1) \in \mathscr{L}^a$ for all $z \in \mathrm{R}^a_u$ and $(w, w\theta_2) \in \mathscr{R}^a$ for all $w \in \mathrm{L}^a_q$.

Therefore, from $y \in \mathrm{R}^a_u$ it follows that $p\star_a y = y\theta_1 \in \mathrm{L}^a_y \cap \mathrm{R}^a_p = \mathrm{L}^a_q \cap \mathrm{R}^a_x$, and by a similar reasoning, from $p\star_a y \in \mathrm{L}^a_q$ follows

$$p\star_a y\star_a v = (p\star_a y)\theta_2 \in \mathrm{L}^a_v \cap \mathrm{R}^a_{p\star_a y} = \mathrm{L}^a_x \cap \mathrm{R}^a_x = \mathrm{H}^a_x$$

Finally, we are ready to prove (2.7). Since homomorphisms map idempotents to idempotents, and elements $p, u \in \mathrm{E}_a(\mathrm{P}^a)$ are in the same $\widehat{\mathscr{H}^a}$-class, we have $\overline{p} = \overline{u}$ (because $\overline{u}$ is the unique idempotent of $\mathrm{H}^{\circledast}_{\overline{u}} = (\widehat{\mathrm{H}^a_u})\phi$). Similarly, $\overline{q} = \overline{v}$. Again, by the left-identity and right-identity properties of idempotents, from $y\,\mathscr{R}^a\,u$ and $y\,\mathscr{L}^a\,q$, we have $y = u\star_a y = y\star_a q$. This implies $\overline{y} = \overline{u}\circledast\overline{y} = \overline{y}\circledast\overline{q}$, which in in turn implies
$$\overline{p\star_a y\star_a v} = \overline{p}\circledast\overline{y}\circledast\overline{v} = \overline{u}\circledast\overline{y}\circledast\overline{q} = \overline{y} = \overline{x}.$$
Hence $(p\star_a y\star_a v)\phi = x\phi$. Since $p\star_a y\star_a v \in \mathrm{H}^a_x$ and ϕ is injective on H^a_x (by Theorem 2.3.12), it follows that $x = p\star_a y\star_a v$. □

The previous lemma will help us discern which elements of the semigroup P^a are idempotent-generated. Since P^a is a subsemigroup of S^a_{ij}, these elements are exactly the idempotent-generated elements of S^a_{ij}, too. We introduce the following notation:

$$\mathbb{E}_a(S^a_{ij}) = \langle \mathrm{E}_a(S^a_{ij})\rangle_a \quad (=) \quad \mathbb{E}_a(\mathrm{P}^a) = \langle \mathrm{E}_a(\mathrm{P}^a)\rangle_a,$$

while the idempotent-generated subsemigroup of W is $\mathbb{E}_b(W) = \langle \mathrm{E}_b(W)\rangle_b$.

Theorem 2.3.15. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. We have $\mathbb{E}_a(S^a_{ij}) = \mathbb{E}_a(\mathrm{P}^a) = (\mathbb{E}_b(W))\phi^{-1}$.*

Proof. The idempotent-generated elements of the semigroup P^a map to idempotent-generated elements of W, since homomorphisms map idempotents to idempotents.

Therefore, $\mathbb{E}_a(\mathrm{P}^a) \subseteq (\mathbb{E}_b(W))\phi^{-1}$. For the reverse inclusion, recall from Lemma 2.3.11 that $(\mathrm{E}_a(\mathrm{P}^a))\phi = \mathrm{E}_b(W)$. If we put $X = \mathrm{E}_a(\mathrm{P}^a)$ in Lemma 2.3.14, the inclusion follows directly. □

2.4 MI-domination and ranks

Our next goal is to investigate the rank (and idempotent rank, where applicable) of P^a and $\mathbb{E}_a(S^a_{ij})$ (again, under the assumption that the sandwich element a is sandwich-regular). The results of the previous section will be of vital importance here.

For this investigation, we need a theoretical introduction on regularity preserving elements, mid-identities, RP-domination, and MI-domination, which will be provided in the first subsection. In a natural step forward, Subsection 2.4.2 studies mid-identities and regularity preserving elements in sandwich semigroups. Finally, in the last subsection, we present three results which create a base for Theorem 2.4.16 and Theorem 2.4.17, in which we calculate the rank of P^a and both the rank and idempotent rank of $\mathbb{E}_a(S^a_{ij})$, respectively. The chapter is based on the results from [33], with some added material, which is appropriately emphasised.

2.4.1 MI-domination

Here, we introduce the notions of regularity-preserving elements [53] and mid-identities [131], along with the concepts of RP-domination and MI-domination. We show that MI-domination is a natural extension of RP-domination, and we prove that, in case it holds, it enables us to calculate the said ranks. As we are about to see in the following chapters, we have MI-domination in the regular subsemigroups of sandwich semigroups in $\mathcal{PT}$, $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{M}$ (Chapters 3 and 4), while in Chapter 5 we encounter some natural examples where it does not hold, as well as one key category in which it holds.

Definition 2.4.1. Let T be a regular semigroup and let $u \in T$. Then, u is

- *regularity-preserving* if the variant semigroup $T^u = (T, \star_u)$ is regular;
- *mid-identity* if $xuy = xy$ for all $x, y \in T$.

We write $\mathrm{RP}(T)$ and $\mathrm{MI}(T)$ for the set of all regularity-preserving elements in T, and the set of all mid-identities of T, respectively.

Clearly, for an element $u \in \mathrm{MI}(T)$, the operation $\star_u$ on T simplifies in the following manner: $x \star_u y = xuy = xy$. Therefore, $(T, \star_u)$ is, in fact, the original regular semigroup T, so $u \in \mathrm{RP}(T)$. We have just proved $\mathrm{MI}(T) \subseteq \mathrm{RP}(T)$. We may also show that $\mathrm{MI}(T)$ is a rectangular band for all regular semigroups T. Recall the equivalent condition from Lemma 1.3.10. Suppose that T is a regular semigroup, fix arbitrary elements $x, y \in \mathrm{MI}(T)$ and an element $v \in \mathrm{V}(x)$. Then $xyx = x^2 =$

$x(xvx) = (xxv)x = xvx = x$, the fourth equality following from the fact that $x \in \mathrm{MI}(T)$. Therefore, $\mathrm{MI}(T)$ is a rectangular band, and in particular, $\mathrm{MI}(T) \subseteq \mathrm{E}(T)$.

Next, we want to introduce the two notions of "domination". The term RP-domination was first used in [8], and it inspired the introduction of the term of MI-domination, in [33]. Recall the natural partial order $\preceq$ on a regular semigroup, from Section 1.3.

Definition 2.4.2. The regular semigroup T is

- *RP-dominated* if every element of T is $\preceq$-below an element of $\mathrm{RP}(T)$;
- *MI-dominated* if every *idempotent* of T is $\preceq$-below an element of $\mathrm{MI}(T)$.

Note the difference between the two definitions. In the first one, we require every element to be below an element of $\mathrm{RP}(T)$, while in the second, only idempotents need to be below an element of $\mathrm{MI}(T)$. The cause of this is the nature of mid-identities; namely, any element $\preceq$-below a mid-identity has to be an idempotent (if $v \preceq u$ with $v = eu = uf$ for some $e, f \in \mathrm{E}(T)$ and $u \in \mathrm{MI}(T)$, then $v^2 = eueu = eu = v$).

Next, we introduce a criterion for a semigroup to be MI-dominated, which will turn out to be significant later on. Let $\mathrm{Max}_{\preceq}(T)$ denote the set of all $\preceq$-maximal idempotents of a semigroup T. It is easy to observe that $\mathrm{MI}(T) \subseteq \mathrm{Max}_{\preceq}(T)$ (if we suppose $u \preceq v$ for some $u \in \mathrm{MI}(T)$ and $v \in \mathrm{E}(T)$, then $u = vuv = vv = v$). This means that a semigroup T is MI-dominated if and only if $\mathrm{MI}(T) = \mathrm{Max}_{\preceq}(T)$.

We will prove that the two notions of domination are in direct relation, meaning that one implies the other. In order to do that, we give the following result, which is a combination of Lemma 2.5(1), Theorem 1.2 from [8], and Corollary 4.8 from [53]. We provide the proof for convenience.

Lemma 2.4.3. *Let T be a regular semigroup.*

(i) If $x \in T$ and $e, f \in E(T)$ are such that $e \preceq x$ and $x \mathscr{H} f$, then $e \preceq f$.

(ii) If T has a mid-identity, then $\mathrm{RP}(T)$ is a rectangular group consisting of those elements of T that are $\mathscr{H}$-related to a mid-identity.

Proof. (i) Since f is an idempotent, H_f is a group $\mathscr{H}$-class, so $x \mathscr{H} f$ implies $xf = fx = x$. Further, $e \preceq x$ means that $e = xk = lx$ for some $k, l \in \mathrm{E}(T)$. Thus, $ef = lxf = lx = e$ and $fe = fxk = xk = e$, so $e \preceq f$.

(ii) Recall from previous discussion that $\mathrm{MI}(T)$ is a rectangular band and $\mathrm{MI}(T) \subseteq \mathrm{RP}(T)$. First, we will show that

$$\mathrm{RP}(T) = \bigcup_{e \in \mathrm{MI}(T)} \mathrm{H}_e \,. \tag{2.8}$$

To prove $(\supseteq)$, suppose $x \in \mathrm{MI}(T)$ and $y \mathscr{H} x$. Since x is an idempotent, its $\mathscr{H}$-class is a group, so there exists an element $z \in \mathrm{H}_x$ such that $yz = zy = x$. We want to prove that $y \in \mathrm{RP}(T)$, i.e. that $T^y = (T, \star_y)$ is a regular semigroup. Let $q \in T$ be an arbitrary element. Since T is regular and $x \in \mathrm{MI}(T)$, there exists an element

$w \in T$ for which $q = qwq = qxwxq$. Thus, $q = q(yz)w(zy)q = q \star_y zwz \star_y q$ and q is regular in T^y.

Next, we show $(\subseteq)$. Suppose $x \in \mathrm{MI}(T)$ and let $y \in \mathrm{RP}(T)$ be arbitrary. Since T^y is regular, there exists an element $z \in T$ such that $x = x \star_y z \star_y x = xyzyx$. We claim that yzy is a mid-identity, and a member of the $\mathscr{H}$-class H_y. The first part is easily proved, since for all $s, t \in T$

$$s \star_{yzy} t = syzyt = s(xyzyx)t = sxt = st.$$

For the second part, note that from regularity of T follows $y = yqy$, for some $q \in T$. Thus, we have

$$\begin{aligned} y &= yqy = yq(yzy)y = (yqy)zy^2 = (yzy)y \qquad \text{and} \\ y &= yqy = y(yzy)qy = y^2z(yqy) = y(yzy) \end{aligned}$$

because $yzy \in \mathrm{MI}(T)$. Clearly, $y \,\mathscr{H}\, yzy$.

Having proved (2.8), the next task is to show that $\mathrm{RP}(T)$ is a subsemigroup of T. Let $u, v \in \mathrm{RP}(T)$ be arbitrary, and let $y, x \in \mathrm{MI}(T)$ be such that $u \,\mathscr{H}\, x$ and $v \,\mathscr{H}\, y$. We claim that $uv \in \mathrm{RP}(T)$. Since H_x and H_y are groups, there exist group inverses $u^{-1} \in \mathrm{H}_x$ and $v^{-1} \in \mathrm{H}_y$. Now, we have

$$\begin{aligned} (xy)(uv) &= (xyu)v = (xu)v = uv, \\ (uv)(v^{-1}u^{-1}y) &= u(vv^{-1})u^{-1}y = uyu^{-1}y = uu^{-1}y = xy, \\ (uv)(xy) &= u(vxy) = u(vy) = uv, \\ (xv^{-1}u^{-1})(uv) &= xv^{-1}(u^{-1}u)v = xv^{-1}xv = xv^{-1}v = xy, \end{aligned}$$

so $uv \,\mathscr{H}\, xy$. Hence, from $xy \in \mathrm{MI}(T)$ (this holds because MI is a rectangular band) and (2.8) follows $uv \in \mathrm{RP}(T)$.

Finally, we prove that $\mathrm{RP}(T)$ is a rectangular group. Since it is a semigroup, and $\mathrm{MI}(T)$ is a rectangular band, by Lemma 1.3.11, it suffices to show that $\mathrm{RP}(T)$ is regular. This is easily proved because from (2.8) it follows that $\mathrm{RP}(T)$ is a union of groups. □

Now, we are in position to prove the relation among MI- and RP-domination in a regular semigroup, as in [33]. In more detail, we show that, if MI-domination is possible in a semigroup (i.e. it contains at least one mid-identity), then RP-domination implies MI-domination.

Proposition 2.4.4. *Let T be a regular semigroup with a mid-identity. If T is RP-dominated, then T is MI-dominated, as well.*

Proof. Suppose T is a regular, RP-dominated semigroup with a mid-identity. Let $e \in \mathrm{E}(T)$ be arbitrary. Since T is RP-dominated, there exists $x \in \mathrm{RP}(T)$ such that $e \preceq x$. By Lemma 2.4.3(ii), $\mathrm{RP}(T)$ consists of elements that are $\mathscr{H}$-related to a mididentity, so there exists $u \in \mathrm{MI}(T) \subseteq \mathrm{E}(T)$ such that $x \,\mathscr{H}\, u$. Lemma 2.4.3(i) now implies $e \preceq u$. Therefore, every idempotent is $\preceq$-below a mid-identity. □

However, the converse does not hold. For instance, the semigroup $S = (\{a, b, c\}, \cdot)$, where

$\cdot$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	b

(i.e. $S = C^1$, where $C = \{b, c\}$ is the group of order 2) is regular with a mid-identity a, and is a counterexample. Namely, since $\mathrm{MI}(S) = \{a\}$ and $\mathrm{E}(S) = \{a, b\}$ with $b \preceq a$ (because $b = aba$), S is MI-dominated; on the other hand, $\mathrm{RP}(S) = \{a\}$ and $c \not\preceq a$, so S is not RP-dominated.

As Proposition 2.4.4 does not offer an equivalent condition for MI-domination, we provide one in the next result (from [33]).

Proposition 2.4.5. *Let T be a regular semigroup with a mid-identity. If we write $R = \mathrm{RP}(T)$ and $M = \mathrm{MI}(T)$, then*

(i) for all $e \in M$, the map $T \to eTe : x \mapsto exe$ is a surmorphism;

(ii) for $e, f \in M$, the map $eTe \to fTf : x \mapsto fxf$ is an isomorphism with inverse $fTf \to eTe : x \mapsto exe$;

(iii) the set $\bigcup_{e\in M} eTe = MTM = RTR$ is a subsemigroup of T;

(iv) T is MI-dominated if and only if

$$T = \bigcup_{e\in M} eTe.$$

Proof. (i) Let $e \in M$. Clearly, the proposed map is surjective. Since $(exe)(eye) = exeye = exye$ for all $x, y \in T$, the map is a homomorphism, as well.

(ii) Let $e, f \in M$. The proposed maps are homomorphisms, as restrictions of homomorphisms of the form presented in (i). Furthermore, for all $x \in T$, $e(f(exe)f)e = efexefe = exe$ and $f(e(fxf)e)f = fefxfef = fxf$, so we have mutually inverse isomorphisms.

(iii) Note that $RTRRTR \subseteq RTR$, so RTR is a subsemigroup of T. Moreover, clearly $\bigcup_{e\in M} eTe \subseteq MTM$ and $MTM \subseteq RTR$ (because $M \subseteq R$), so it suffices to prove $RTR \subseteq \bigcup_{e\in M} eTe$. Suppose $x, z \in R$ and $y \in T$ and consider the element xyz. Lemma 2.4.3(ii) guarantees that $x \mathscr{H} u$ and $z \mathscr{H} v$ for some $u, v \in M$, thus $x = ux$ and $z = zv$. Since M is a subsemigroup of $\mathrm{E}(T)$, we have $uv \in M$, so

$$xyz = (ux)y(zv) = (uvx)y(zuv) = uv(xyz)uv \in \bigcup_{e\in M} eTe,$$

the second equality following from the characteristics of mid-identities u and v.

(iv) For the direct implication, suppose T is MI-dominated. Clearly, $T \supseteq \bigcup_{e\in M} eTe$, so we need to prove the reverse inclusion. Let $x \in T$. Since T is regular, by Remark 1.3.8 there exist elements $e, f \in \mathrm{E}(T)$ such that $x \mathscr{R} e$ and $x \mathscr{L} f$, so $ex = x = xf$ and therefore $x = ex = exf$. MI-domination in T implies that the idempotents e and f are $\preceq$-below some mid-identities u and v, respectively. We may conclude that $e = ue$ and $f = fv$, so

$$x = exf = (ue)x(fv) = u(exf)v \in MTM = \bigcup_{e \in M} eTe,$$

using (iii) in the last step.

For the reverse implication, suppose $T = \bigcup_{e \in M} eTe$ and let $u \in \mathrm{E}(T)$. Since $u \in T$, we have $u = eve$ for some $e \in M$ and $v \in T$. Hence $eue = e(eve)e = eve = u$, so $u \preceq e$. □

Therefore, a regular semigroup with at least one mid-identity is MI-dominated if and only if it is covered (in a topological sense) by local monoids corresponding to mid-identities.

In Proposition 2.4.4, we have seen that RP-domination implies MI-domination. Our next task is to state sufficient conditions for the converse. In order to describe these conditions, we introduce the term of factorisable semigroups [19, 122].

Definition 2.4.6. Semigroup S is *factorisable* if $S = GE$ for some subgroup G of S and some set of idempotents $E \subseteq \mathrm{E}(S)$.

First, we study the "domination problems" in a monoid, as in [33].

Lemma 2.4.7. *A monoid T is*

(i) MI-dominated,

(ii) RP-dominated if and only if it is factorisable.

Proof. (i) Suppose T is a monoid with identity e. We claim that $\mathrm{MI}(T) = \{e\}$. Clearly, $e \in \mathrm{MI}(T)$, and for any $u \in \mathrm{MI}(T)$, we have $u = eue = ee = e$. Furthermore, the monoid T is MI-dominated, since $v = eve$ for all $v \in \mathrm{E}(T)$.

(ii) By Lemma 2.4.3(ii), from $\mathrm{MI}(T) = \{e\}$ it follows that $\mathrm{RP}(T)$ is the group $\mathscr{H}$-class H_e. Thus, T is RP-dominated if and only if for all $x \in T$ there exists $y \in \mathrm{H}_e$ such that $x \preceq y$.

We will modify our equivalent condition further, until we achieve the wanted form. By definition,

$$x \preceq y \;\Leftrightarrow\; (\exists f, h \in \mathrm{E}(T))(x = fy \text{ and } x = yh).$$

We prove that, in our case, only one of the equalities suffices. Suppose that, for a fixed element $x \in T$ we have $x = yf$ for some $f \in \mathrm{E}(T)$ and $y \in \mathrm{H}_e$. Since $y \in \mathrm{H}_e$, we have $e = sy$ for some $s \in \mathrm{H}_e$. Moreover, e being the identity implies $x = xe = xsy$ and

$$(xs)(xs) = (yfs)(yfs) = yf(sy)fs = yfefs = yffs = yfs = xs.$$

Thus, $xs \in \mathrm{E}(T)$ and $x = (xs)y$, so $x \preceq y$.

Therefore, T is RP-dominated if and only if $T = \mathrm{H}_e\,\mathrm{E}(T)$. By a symmetric argument it follows that T is RP-dominated if and only if $T = \mathrm{E}(T)\,\mathrm{H}_e$. □

Now, we are ready to give the necessary and sufficient condition for RP-domination, in the case where we have MI-domination. This is a result from [33].

Proposition 2.4.8. *Let T be an MI-dominated regular semigroup. Then T is RP-dominated if and only if the local monoid eTe is RP-dominated (equivalently, factorisable) for each mid-identity $e \in \mathrm{MI}(T)$.*

Proof. First, we prove the direct implication. Suppose T is a regular semigroup, both MI- and RP-dominated. Choose an arbitrary $e \in \mathrm{MI}(T)$ and consider the local monoid eTe with identity $eee = e$. Let us prove its regularity. Fix an element $exe \in eTe$. By the regularity of T, there exists $y \in T$ such that $xyx = x$. Thus

$$(exe)(eye)(exe) = exeyexe = exyxe = exe.$$

Therefore, eTe is regular, and we may study the "domination problems" in it. Let us prove it is RP-dominated. Choose and fix an arbitrary $exe \in eTe$. Since T is RP-dominated, $x \preceq y$ for some $y \in \mathrm{RP}(T)$. In other words, there exist $s, t \in \mathrm{E}(T)$ such that $x = ys = ty$, which implies

$$\begin{aligned} exe &= eyse = (eye)(ese), \qquad & (ese)(ese) &= esse = ese, \\ exe &= etye = (ete)(eye), \qquad & (ete)(ete) &= ette = ete. \end{aligned}$$

Hence, $exe \preceq eye$ in eTe. Since $y \in \mathrm{RP}(T)$, the variant $(T, \star_y)$ of T is regular, so the variant $(eTe, \star_{eye})$ of eTe also has to be regular. Let us elaborate. Clearly, $e \in \mathrm{MI}(T)$ implies $\star_y = \star_{eye}$ on T. Therefore, every element $ewe \in eTe$ has an inverse eqe in $(eTe, \star_{eye})$, where q is an inverse of w in $(T, \star_y)$. Putting all these facts together, we may conclude that $exe \preceq eye$ with $eye \in \mathrm{RP}(eTe)$, so eTe is RP-dominated.

To prove the reverse implication, suppose that T is an MI-dominated regular semigroup, such that eTe is an RP-dominated local monoid for all $e \in \mathrm{MI}(T)$. We need to show that T itself is RP-dominated. Fix an arbitrary element $x \in T$. The semigroup T is MI-dominated, so it is covered by local monoids corresponding to mid-identities, by Proposition 2.4.5(*iv*). Thus, $x \in eTe$ for some $e \in \mathrm{MI}(T)$. Since eTe is RP-dominated, there exists $y \in \mathrm{RP}(eTe)$ such that $x \preceq y$. Having in mind that eTe is a monoid, from the proof of Lemma 2.4.7 we conclude that $\mathrm{RP}(eTe) = \mathrm{H}_e$ in eTe, so $y \mathscr{H} e$ in eTe, and hence in T as well. From Lemma 2.4.3(*ii*) it follows that $y \in \mathrm{RP}(T)$ (because $e \in \mathrm{MI}(T)$). Therefore, $x \preceq y \in \mathrm{RP}(T)$, and T is RP-dominated. □

2.4.2 Mid-identities and regularity-preserving elements in P^a

In this subsection, we use the results of the previous (preparatory) one, to study further the regular subsemigroup P^a. The information we gain enables us to infer crucial theorems in the next section.

Recall that we have fixed a partial semigroup S, two coordinates $i, j \in I$, a sandwich-regular element $a \in S_{ji}$, and one of its semigroup inverses $b \in \mathrm{V}(a)$; we want to investigate the regular subsemigroup $\mathrm{Reg}(S_{ij}^a) = \mathrm{P}^a$ of the sandwich semigroup S_{ij}^a. In this process, we will often make use of the monoid $W = (aS_{ij}a, \circledast)$ (with identity $aba = a$) and the mapping $\phi : \mathrm{P}^a \to aS_{ij}a : x \mapsto axa = \overline{x}$.

The following result (from [33]) is, in fact, the key result of this subsection and it describes the mid-identities and regular-preserving elements of P^a.

Proposition 2.4.9. *If $a \in S_{ji}$ is sandwich-regular, then*

(i) $\mathrm{MI}(\mathrm{P}^a) = \mathrm{E}_a(\widehat{\mathrm{H}_b^a}) = \mathrm{V}(a) \subseteq \mathrm{Max}_{\preceq}(\mathrm{P}^a)$,

(ii) $\mathrm{RP}(\mathrm{P}^a) = \widehat{\mathrm{H}_b^a}$.

Proof. (i) When we were introducing the term of MI-domination above, we noted that we have $\mathrm{MI}(T) \subseteq \mathrm{Max}_{\preceq}(T)$ in all regular semigroups. Thus, we need to prove only the equalities. We will show that $\mathrm{MI}(\mathrm{P}^a) \subseteq \mathrm{V}(a) \subseteq \mathrm{E}_a(\widehat{\mathrm{H}_b^a}) \subseteq \mathrm{MI}(\mathrm{P}^a)$.

Let $u \in \mathrm{MI}(\mathrm{P}^a)$. Since all mid-identities are idempotents, we have $u = u \star_a u = uau$. Furthermore, the defining property of mid-identities guarantees

$$a = ababa = a(b \star_a b)a = a(b \star_a u \star_a b)a = (aba)u(aba) = aua,$$

so $u \in \mathrm{V}(a)$.

If $u \in \mathrm{V}(a)$, then $uau = u$ and $aua = a$. Hence, $u \in \mathrm{E}_a(\mathrm{P}^a)$, and for all $x \in \mathrm{P}^a$

$$\overline{u} \circledast \overline{x} = auabaxa = auaxa = axa = \overline{x}, \qquad \text{and}$$
$$\overline{x} \circledast \overline{u} = axabaua = axaua = axa = \overline{x}.$$

These equalities together imply that $\overline{u}$ is the identity of W, so $\overline{u} = \overline{b}$ and $u \in \widehat{\mathrm{H}_b^a}$. Therefore, $u \in \mathrm{E}_a(\widehat{\mathrm{H}_b^a})$.

Suppose $u \in \mathrm{E}_a(\widehat{\mathrm{H}_b^a})$. Thus, $\overline{u}$ is the unique idempotent of $\mathrm{H}_{\overline{b}}^{\circledast}$, so $aua = \overline{b} = a$. From this, we have: for all $x, y \in \mathrm{P}^a$,

$$x \star_a u \star_a y = x(aua)y = xay = x \star_a y.$$

Therefore, $u \in \mathrm{MI}(\mathrm{P}^a)$.

(ii) Since P^a is regular and $b \in \mathrm{MI}(\mathrm{P}^a)$, Lemma 2.4.3$(ii)$ implies

$$\mathrm{RP}(\mathrm{P}^a) = \bigcup_{e \in \mathrm{MI}(\mathrm{P}^a)} \mathrm{H}_e^a.$$

From (i) we have $\mathrm{MI}(\mathrm{P}^a) = \mathrm{E}_a(\widehat{\mathrm{H}_b^a})$, so $\mathrm{RP}(\mathrm{P}^a) = \widehat{\mathrm{H}_b^a}$, as each $\mathscr{H}^a$-class in $\widehat{\mathrm{H}_b^a}$ contains an idempotent. □

The next Proposition is a result from [28]. In it, we suppose $a \in S_{ji}$ is regular, fix an inverse $b \in \mathrm{V}(a)$ and examine the class J_b^a of S_{ij}^a. Even if we do not include the assumption of sandwich-regularity of a, by Theorem 2.2.3(v) we know that $\mathrm{V}(a) \subseteq \mathrm{J}_b^a$, since $\mathrm{V}(a) \subseteq \mathrm{P}^a \subseteq \mathrm{P}_3^a$ and $\mathrm{V}(a) \subseteq \mathrm{J}_a$. Moreover, we have the following:

Proposition 2.4.10. *Suppose S is stable, and $a \in S_{ji}$ is regular. Fix some $b \in \mathrm{V}(a)$. Then*

(i) $\mathrm{J}^a_b = \mathrm{D}^a_b$*,*

(ii) $\mathrm{E}_a(\mathrm{J}^a_b) = \mathrm{V}(a)$ *is a rectangular band under* $\star_a$*,*

(iii) J^a_b *is a rectangular group under* $\star_a$*,*

(iv) if S is regular, then $\mathrm{J}^a_b = \mathrm{RP}(\mathrm{P}^a)$ *and* $\mathrm{E}_a(\mathrm{J}^a_b) = \mathrm{MI}(\mathrm{P}^a)$*.*

Proof. (i) follows directly from Corollary 2.2.26.

(ii) and (iii). Note that, except in (iv), we do not have sandwich-regularity, so we have to be careful which of the previous results we use in our argument. First, we prove $\mathrm{E}_a(\mathrm{J}^a_b) = \mathrm{V}(a)$. If $x \in \mathrm{E}_a(\mathrm{J}^a_b)$, then $x = xax$ and $x \mathscr{J}^a b$. Thus, $x \mathscr{J} b$, so $a \mathscr{J} b$ gives $x \mathscr{J} a$. From Lemma 2.2.15(ii) now follows $axa = a$, so $x \in \mathrm{V}(a)$. For the reverse containment, let $x \in \mathrm{V}(a)$ so that $xax = x$ and $axa = a$. Then $x \in \mathrm{E}_a(S^a_{ij})$, and we have $x = xax = xabax$ and $b = bab = baxab$, which means that $x \mathscr{J}^a b$. Hence, $x \in \mathrm{E}_a(\mathrm{J}^a_b)$.

Since $\mathrm{J}^a_b = \mathrm{D}^a_b$ contains $b \in \mathrm{V}(a)$, it is a regular $\mathscr{D}$-class. Furthermore, for $x, y \in \mathrm{V}(a)$ we have $xay \in \mathrm{V}(a)$ (as $(xay)a(xay) = xaxay = xay$ and $a(xay)a = aya = a$), so $\mathrm{V}(a) = \mathrm{E}_a(\mathrm{J}^a_b)$ is a subsemigroup of J^a_b. Therefore, the statement follows from Lemma 1.3.12.

(iv) If S is regular, then the set $\{a\} \cup aS_{ij}a$ is also regular in S, i.e. a is sandwich-regular. Thus, Proposition 2.4.9(i) gives $\mathrm{MI}(\mathrm{P}^a) = \mathrm{V}(a)$ $(= \mathrm{E}_a(\mathrm{J}^a_b)$, by $(ii))$. Clearly, it suffices to prove $\mathrm{RP}(\mathrm{P}^a) = \mathrm{J}^a_b$. Since $\mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$ and $b \in \mathrm{V}(a) = \mathrm{MI}(\mathrm{P}^a)$, Lemma 2.4.3$(ii)$ gives

$$\mathrm{RP}(\mathrm{P}^a) = \bigcup_{x \in \mathrm{MI}(\mathrm{P}^a)} \mathrm{H}^a_x = \bigcup_{x \in \mathrm{E}(\mathrm{D}^a_b)} \mathrm{H}^a_x = \mathrm{D}^a_b = \mathrm{J}^a_b,$$

the penultimate equality following from (iii). □

The final result of this subsection, Proposition 2.4.11 (from [33]), enables us to describe what it means for P^a to be MI-dominated.

Recall the term of a local monoid of a semigroup from Section 1.4. For an idempotent $e \in \mathrm{MI}(\mathrm{P}^a) = \mathrm{V}(a)$, we write W_e for the local monoid of P^a with respect to e. The first part of Proposition 2.4.5 guarantees that such a local monoid is a homomorphic image of P^a. The second part of the same proposition implies that the local monoids of P^a are isomorphic to each other. The following result proves that they are in fact all isomorphic to the monoid W.

Proposition 2.4.11. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. For any $e \in \mathrm{V}(a)$, the restriction of ϕ to the local monoid W_e is an isomorphism $\phi\restriction_{W_e} : W_e \to W$.*

Proof. Since $e \in \mathrm{V}(a)$, we have $eae = a$ and $aea = a$, so the map $W \to W_e : x \mapsto exe$ and the restriction $\phi\restriction_{W_e} : W_e \to W$ are mutually inverse maps. □

Therefore, we may conclude that, if $\mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$ is MI-dominated, then it is a union of local monoids corresponding to its mid-identities (by Proposition 2.4.5(*iv*)), all of which are isomorphic copies of $W = (aS_{ij}a, \circledast)$ (by the previous proposition).

2.4.3 Rank and idempotent rank

This subsection contains the most significant results of the current section, because here we obtain the formulae for the rank of P^a and the rank and idempotent rank of $\mathbb{E}_a(S^a_{ij})$. The first two results are preparatory. They are followed by a major one, Proposition 2.4.14, which gives a lower bound for the rank of a subsemigroup of P^a (that satisfies certain conditions). In addition, we show that this bound is met if P^a is MI-dominated. Then, we apply this result to prove Theorem 2.4.16 and Theorem 2.4.17. All results of this subsection were originally proved in [33].

If M is a monoid, we write G_M for the group of units of M.

Lemma 2.4.12. *Let M be an idempotent-generated monoid. Then*

(i) $G_M = \{\mathrm{id}_M\}$,

(ii) $M \setminus G_M$ *is an ideal of* M,

(iii) $\mathrm{rank}(M) = 1 + \mathrm{rank}(M : G_M)$,

(iv) $\mathrm{idrank}(M) = 1 + \mathrm{idrank}(M : G_M)$.

Proof. (*i*) Since $\mathrm{id}_M \in G_M$ by the definition of G_M, we need to prove it is the only element in G_M. Suppose $g \in G_M$. M is idempotent-generated, so there exists a minimal integer k, such that some k idempotents can generate g. Fix any $e_1, \dots, e_k \in \mathrm{E}(M)$ with $g = e_1 \cdots e_k$. From $e_1 \in \mathrm{E}(M)$, we have

$$e_1 g = e_1 e_1 \cdots e_k = e_1 \cdots e_k = g.$$

Further, since g belongs to a group, it has a group inverse g^{-1}, so

$$e_1 = e_1 \,\mathrm{id}_M = e_1 g g^{-1} = g g^{-1} = \mathrm{id}_M\,.$$

Thus, $k = 1$ implies $g = e_1 = \mathrm{id}_M$, and $k \geq 2$ implies $g = e_1 \cdots e_k = \mathrm{id}_M e_2 \cdots e_k = e_2 \cdots e_k$, contradicting the minimality of k.

(*ii*) It suffices to prove that $(M \setminus G_M)G_M = G_M(M \setminus G_M) = M \setminus G_M$ and $G_M \cap (M \setminus G_M)(M \setminus G_M) = \emptyset$. The former clearly holds, since from (*i*) we have $G_M = \{\mathrm{id}_M\}$. For the latter, suppose the opposite, that $\mathrm{id}_M = xy$ for some $x, y \in M \setminus G_M$. Since M is idempotent-generated, there exists a minimal integer k and idempotents $e_1, \dots, e_k \in \mathrm{E}(M)$ such that $x = e_1 \cdots e_k$. But then $x = e_1 x$, so $e_1 = e_1 \,\mathrm{id}_M = e_1 x y = xy = \mathrm{id}_M$. As in the previous part, we may conclude that $k = 1$ and $x = e_1 = \mathrm{id}_M$, which contradicts the assumption $x \in M \setminus G_M$.

(*iii*) and (*iv*) By (*ii*), $M \setminus G_M$ is an ideal, so any product involving its elements cannot generate elements of G_M. Therefore, any generating set for M contains id_M, as it is the only element of G_M (by (*i*)). ∎

It is important to note that, in the following lemma, we suppose every idempotent of P^a is $\preceq$-below a maximal one (which rules out sandwich semigroups where P^a has infinite increasing chains of idempotents). In particular, this condition holds if P^a is MI-dominated.

Lemma 2.4.13. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. Suppose every idempotent of P^a is $\preceq$-below a maximal one. If $X \subseteq \mathrm{P}^a$ is a set such that $\mathrm{E}_b(W) \subseteq \langle \overline{X} \rangle_b$ and $\mathrm{Max}_{\preceq}(\mathrm{P}^a) \subseteq \langle X \rangle_a$, then $(\langle \overline{X} \rangle_b)\phi^{-1} = \langle X \rangle_a$.*

Proof. Since $X \subseteq \overline{X}\phi^{-1}$, we have $\langle X \rangle_a \subseteq (\langle \overline{X} \rangle_b)\phi^{-1}$. For the reverse inclusion, note that Lemma 2.3.14 gives $(\langle \overline{X} \rangle_b)\phi^{-1} \subseteq \langle X \cup \mathrm{E}_a(\mathrm{P}^a) \rangle_a$, so it suffices to prove $\mathrm{E}_a(\mathrm{P}^a) \subseteq \langle X \rangle_a$. Let $e \in \mathrm{E}_a(\mathrm{P}^a)$; we show that e can be generated by elements of X. Since every idempotent is $\preceq$-below a maximal one, $e \preceq f$, for some $f \in \mathrm{Max}_{\preceq}(\mathrm{P}^a)$. In other words, $e = f \star_a e \star_a f = faeaf$. Since $\overline{e} \in \mathrm{E}_b(W)$ and $\mathrm{E}_b(W) \subseteq \langle \overline{X} \rangle_b$, we have $\overline{e} = \overline{x_1} \circledast \overline{x_2} \circledast \cdots \circledast \overline{x_k}$ for some $x_1, x_2, \ldots, x_k \in X$. Hence, $aea = ax_1ax_2a \cdots ax_ka$ and

$$e = faeaf = fax_1ax_2a \cdots ax_kaf = f \star_a x_1 \star_a x_2 \star_a \cdots \star_a x_k \star_a f.$$

Now, we may conclude $e \in \langle X \rangle_a$, because $f \in \mathrm{Max}_{\preceq}(\mathrm{P}^a) \subseteq \langle X \rangle_a$. □

Next, we give the crucial result we announced earlier. Recall that the subsemigroup of a semigroup is S is full if it contains all its idempotents. Also, note that in a rectangular group $S = I \times G \times J$, we have $\mathrm{E}(S) = \{(i, e, j) : i \in I, j \in J\}$, where e is the identity of G. Thus, it is easily seen that any full subsemigroup of S is of the form $I \times K \times J$, for some submonoid K of the group G.

Proposition 2.4.14. *Let $a \in S_{ji}$ be a sandwich-regular element of a partial semigroup S. In P^a, put $r = |\widehat{\mathrm{H}_b^a}/\mathscr{R}^a|$ and $l = |\widehat{\mathrm{H}_b^a}/\mathscr{L}^a|$. Let M be a full submonoid of W, such that $M \setminus G_M$ is an ideal of M and $G_M = M \cap G_W$. Then, $N = M\phi^{-1}$ is a full subsemigroup of P^a, and we have*

$$\mathrm{rank}(N) \geq \mathrm{rank}(M : G_M) + \max(r, l, \mathrm{rank}(G_M)) \tag{2.9}$$

with equality if P^a is MI-dominated.

Proof. Since N is the reverse image of a submonoid of W under the homomorphism ϕ, it is clearly a subsemigroup of P^a. Furthermore, it is a full subsemigroup, because Lemma 2.3.11 implies $\mathrm{E}_a(\mathrm{P}^a) = (\mathrm{E}_b(W))\phi^{-1} \subseteq (M)\phi^{-1}$. Thus, $N \cap \widehat{\mathrm{H}_b^a}$ is a full subsemigroup of $\widehat{\mathrm{H}_b^a}$, i.e. a full subsemigroup of an $r \times l$ rectangular group over $\mathrm{H}_b^a \cong \mathrm{H}_a^{\circledast} = G_W$ (by Theorem 2.3.12(*iii*)). Therefore, $N \cap \widehat{\mathrm{H}_b^a}$ is isomorphic to a direct product of the $r \times l$ rectangular band and a submonoid $K = N \cap \mathrm{H}_b^a$ of the group $\mathrm{H}_b^a \cong G_W$. We may conclude

$$K\phi = (N \cap \mathrm{H}_b^a)\phi = (N)\phi \cap (\mathrm{H}_b^a)\phi = M \cap \mathrm{H}_a^{\circledast} = M \cap G_W = G_M,$$

the second equality following from the statement

$$x\phi = y\phi \quad \Rightarrow \quad (x \in N \Leftrightarrow y \in N), \qquad\qquad \text{for all } x, y \in \mathrm{P}^a,$$

which is true because $N = M\phi^{-1}$. By Theorem 2.3.12(ii), the map $\phi\restriction_{\mathrm{H}^a_b}$ is injective, which implies $\phi\restriction_K$ is injective as well, so $K \cong G_M$. Therefore, $N \cap \widehat{\mathrm{H}^a_b}$ is a $|\widehat{\mathrm{H}^a_b}/\mathscr{R}^a| \times |\widehat{\mathrm{H}^a_b}/\mathscr{L}^a|$ $(= r \times l)$ rectangular group over G_M. From Proposition 1.4.2, we have

$$\mathrm{rank}(N \cap \widehat{\mathrm{H}^a_b}) = \max(r, l, \mathrm{rank}(G_M)).$$

To prove the bound (2.9), suppose that X is a generating set for N, with $|X| = \mathrm{rank}(N)$, and put $Y = X \cap \widehat{\mathrm{H}^a_b}$ and $Z = X \setminus \widehat{\mathrm{H}^a_b}$. Let $u \in N \cap \widehat{\mathrm{H}^a_b}$. We show that u has to be generated solely by elements from Y. Consider an expression $u = x_1 \star_a \cdots \star_a x_k$, where $x_1, \dots, x_k \in X$. We have $\overline{u} = \overline{x_1} \circledast \cdots \circledast \overline{x_k}$, and

$$\overline{u} \in (N \cap \widehat{\mathrm{H}^a_b})\phi = (N \cap \mathrm{H}^a_b)\phi = G_M,$$

so $\overline{x_1}, \dots, \overline{x_k} \in G_M$, since $M \setminus G_M$ is an ideal of M. Thus, $x_1, \dots x_k \in (G_M)\phi^{-1} = \widehat{\mathrm{H}^a_b}$, and we may conclude $x_1, \dots x_k \in Y$. Hence, we have proved $N \cap \widehat{\mathrm{H}^a_b} = \langle Y \rangle_a$, so we infer that

$$|Y| \geq \mathrm{rank}(N \cap \widehat{\mathrm{H}^a_b}) = \max(r, l, \mathrm{rank}(G_M)). \tag{2.10}$$

Furthermore, since ϕ is a homomorphism, we have

$$M = N\phi = (\langle X \rangle_a)\phi = \langle \overline{X} \rangle_b = \langle \overline{Y} \cup \overline{Z} \rangle_b = \langle \langle \overline{Y} \rangle_b \cup \overline{Z} \rangle_b = \langle G_M \cup \overline{Z} \rangle_b,$$

the last equality following from the previous discussion. Hence,

$$|Z| \geq |\overline{Z}| \geq \mathrm{rank}(M : G_M). \tag{2.11}$$

From equations (2.10) and (2.11) directly follows (2.9), since $|X| = |Y| + |Z|$.

Let us show the last statement is true. Suppose that P^a is MI-dominated. Since we have proved the lower bound (2.9), it suffices to find a generating set of the stated size. Let $Y \subseteq \mathrm{P}^a$ be a generating set for $N \cap \widehat{\mathrm{H}^a_b}$ with $|Y| = \mathrm{rank}(N \cap \widehat{\mathrm{H}^a_b}) = \max(r, l, \mathrm{rank}(G_M))$; in addition, let $Z \subseteq \mathrm{P}^a$ be such that $M = \langle G_M \cup \overline{Z} \rangle_b$ and $|Z| = \mathrm{rank}(M : G_M)$. The set $X = Y \cup Z$ is clearly of desired size, so it suffices to show $N = \langle X \rangle_a$. Firstly, note that

$$\begin{aligned} M = \langle G_M \cup \overline{Z} \rangle_b &= \langle (N \cap \mathrm{H}^a_b)\phi \cup \overline{Z} \rangle_b = \langle (N \cap \widehat{\mathrm{H}^a_b})\phi \cup \overline{Z} \rangle_b \\ &= \langle (\langle Y \rangle_a)\phi \cup \overline{Z} \rangle_b = \langle \langle \overline{Y} \rangle_b \cup \overline{Z} \rangle_b = \langle \overline{Y} \cup \overline{Z} \rangle_b = \langle \overline{X} \rangle_b, \end{aligned}$$

which implies $\mathrm{E}_b(W) \subseteq M = \langle \overline{X} \rangle_b$, because M is full. Secondly, having in mind that P^a is MI-dominated (which means that $\mathrm{MI}(\mathrm{P}^a) = \mathrm{Max}_\preceq(\mathrm{P}^a)$), from Proposition 2.4.9(i) and the fact that $N \cap \widehat{\mathrm{H}^a_b}$ is a full subsemigroup of $\widehat{\mathrm{H}^a_b}$, we have

$$\mathrm{Max}_\preceq(\mathrm{P}^a) = \mathrm{MI}(\mathrm{P}^a) = \mathrm{V}(a) = \mathrm{E}_a(\widehat{\mathrm{H}^a_b}) \subseteq N \cap \widehat{\mathrm{H}^a_b} = \langle Y \rangle_a \subseteq \langle X \rangle_a.$$

Again, since P^a is MI-dominated, every idempotent from $\mathrm{E}_a(\mathrm{P}^a)$ is $\preceq$-below an element of $\mathrm{MI}(\mathrm{P}^a) = \mathrm{Max}_{\preceq}(\mathrm{P}^a)$, so Lemma 2.4.13 implies

$$N = M\phi^{-1} = (\langle \overline{X} \rangle_b)\phi^{-1} = \langle X \rangle_a. \qquad \square$$

Remark 2.4.15. In the statement of Proposition 2.4.14 (and in Theorem 2.4.16, and Theorem 2.4.17, as well), the condition "P^a is MI-dominated" is equivalent to "N is MI-dominated".

Proof. First, we prove the following claim:

> If U is a full subsemigroup of a regular semigroup T with a mid-identity, then $\mathrm{MI}(U) = \mathrm{MI}(T)$.

Clearly, if $u \in \mathrm{MI}(T)$, then $u \in \mathrm{E}(T) \subseteq U$, so $u \in \mathrm{MI}(U)$. Thus, we proved $\mathrm{MI}(T) \subseteq \mathrm{MI}(U)$. To prove the reverse inclusion, suppose that $u \in \mathrm{MI}(U)$. Let $e \in \mathrm{MI}(T)$ be arbitrary. Then $e \in \mathrm{E}(T) \subseteq U$, so $e = ee = eue$. Thus, for all $x, y \in T$ we have $xy = xey = xeuey = xuy$, so $u \in \mathrm{MI}(T)$.

Since N is a full subsemigroup of P^a (as shown at the beginning of the previous proof), we have $\mathrm{E}_a(N) = \mathrm{E}_a(\mathrm{P}^a)$ and the claim gives $\mathrm{MI}(N) = \mathrm{MI}(\mathrm{P}^a)$. These facts imply the stated equivalence. $\square$

Finally, we use Proposition 2.4.14 to obtain formulae for the ranks of P^a and its idempotent-generated subsemigroup $\mathbb{E}_a(\mathrm{P}^a) = \mathbb{E}_a(S_{ij}^a)$.

Theorem 2.4.16. *Suppose $a \in S_{ji}$ is a sandwich-regular element of a partial semigroup S. Let $r = |\widehat{\mathrm{H}_b^a} / \mathscr{R}^a|$ and $l = |\widehat{\mathrm{H}_b^a} / \mathscr{L}^a|$, and suppose $W \setminus G_W$ is an ideal of W. Then*

$$\mathrm{rank}(\mathrm{P}^a) \geq \mathrm{rank}(W : G_W) + \max(r, l, \mathrm{rank}(G_W)),$$

with equality if P^a is MI-dominated.

Proof. Put $M = W$; it is clearly a full submonoid of W, we have $G_W = W \cap G_W$ and $W \setminus G_W$ is an ideal of W, by assumption. Since $W\phi^{-1} = \mathrm{P}^a$, the result follows directly from Proposition 2.4.14. $\square$

Theorem 2.4.17. *Suppose $a \in S_{ji}$ is a sandwich-regular element of a partial semigroup S. Let $r = |\widehat{\mathrm{H}_b^a} / \mathscr{R}^a|$ and $l = |\widehat{\mathrm{H}_b^a} / \mathscr{L}^a|$. Then*

$$\mathrm{rank}(\mathbb{E}_a(\mathrm{P}^a)) \geq \mathrm{rank}(\mathbb{E}_b(W)) + \max(r, l) - 1$$

and

$$\mathrm{idrank}(\mathbb{E}_a(\mathrm{P}^a)) \geq \mathrm{idrank}(\mathbb{E}_b(W)) + \max(r, l) - 1,$$

with equality in both if P^a is MI-dominated.

Proof. Put $M = \mathbb{E}_b(W)$. Obviously, M is a full subsemigroup of W. Since it is an idempotent-generated monoid with identity a, Lemma 2.4.12(i) gives $G_M = \{\mathrm{id}_M\} = \{a\}$ (and thus we have $G_M = M \cap G_W$). The second part of the same lemma guarantees that $M \setminus G_M$ is an ideal of M. Therefore, the conditions of

Proposition 2.4.14 are satisfied, and we have a lower bound for the rank of $N = M\phi^{-1} = (\mathbb{E}_b(W))\phi^{-1} = \mathbb{E}_a(\mathrm{P}^a)$ (the last equality follows from Theorem 2.3.15):

$$\mathrm{rank}(\mathbb{E}_a(\mathrm{P}^a)) \geq \mathrm{rank}(\mathbb{E}_b(W) : \{a\}) + \max(r, l, \mathrm{rank}(\{a\})),$$

with equality in the MI-dominated case. From Lemma 2.4.12(iii), we have $\mathrm{rank}(\mathbb{E}_b(W) : \{a\}) = \mathrm{rank}(\mathbb{E}_b(W)) - 1$, and clearly $\mathrm{rank}(\{a\}) = 1$, so we may transform the previous inequality to

$$\mathrm{rank}(\mathbb{E}_a(\mathrm{P}^a)) \geq \mathrm{rank}(\mathbb{E}_b(W)) - 1 + \max(r, l).$$

Next, we show the statement concerning the idempotent rank. Firstly, note that $\mathbb{E}_a(\mathrm{P}^a)$ is an idempotent-generated semigroup, therefore it has an idempotent rank. Thus, we analyse an arbitrary generating set consisting of idempotents, as in the proof of Proposition 2.4.14. Put $M = \mathbb{E}_b(W)$, and let $N = \mathbb{E}_a(\mathrm{P}^a) = \langle X \rangle_a$, where $X \subseteq \mathrm{E}_a(\mathrm{P}^a)$. Since $G_M = \{a\}$ and N is a full subsemigroup of P^a, as in the proof of Proposition 2.4.14, we have $N \cap \widehat{\mathrm{H}_b^a} = G_M\phi^{-1} = a\phi^{-1} = \mathrm{V}(a) = \mathrm{E}_a(\widehat{\mathrm{H}_b^a})$ (the last two following by Proposition 2.4.9(i)). Put $Y = X \cap \mathrm{V}(a)$ and $Z = X \setminus \mathrm{V}(a)$. The same argument as in the proof of Proposition 2.4.14 gives:

(i) $\mathrm{V}(a) = \langle Y \rangle_a$, thus $|Y| \geq \mathrm{rank}(\mathrm{V}(a)) = \max(r, l)$ (the last equality follows from Proposition 1.4.2);

(ii) $M = \langle G_M \cup \overline{Z} \rangle_b = \langle \{a\} \cup \overline{Z} \rangle_b$, and since X contains only idempotents, we have

$$|Z| \geq |\overline{Z}| \geq \mathrm{idrank}(M : G_M) = \mathrm{idrank}(M) - 1,$$

the last equality following from Lemma 2.4.12(iv);

(iii) if P^a is MI-dominated, and we pick $Y_1, Z_1 \subseteq \mathrm{E}_a(\mathrm{P}^a)$ such that $\mathrm{V}(a) = \langle Y_1 \rangle_a$ and $\langle \{a\} \cup \overline{Z_1} \rangle = M$, with $|Y_1| = \max(r, l)$ and $|Z_1| = \mathrm{idrank}(M : G_M)$, then $N = \langle Y_1 \cup Z_1 \rangle_a$.

Thus, from (i) and (ii) we have $\mathrm{idrank}(N) \geq \mathrm{idrank}(\mathbb{E}_b(W)) - 1 + \max(r, l)$, and ($iii$) proves that MI-domination in P^a implies equality. □

Remark 2.4.18. If P^a is MI-dominated, then from Theorem 2.4.17 we have

$$\mathrm{rank}(\mathbb{E}_b(W)) = \mathrm{idrank}(\mathbb{E}_b(W)) \Rightarrow \mathrm{rank}(\mathbb{E}_a(\mathrm{P}^a)) = \mathrm{idrank}(\mathbb{E}_a(\mathrm{P}^a)).$$

Under the same assumption, the reverse implication holds if $r, l < \aleph_0$.

2.5 Inverse monoids

In this section, we study simplifications that occur if we are operating within an inverse category or, more generally, if the sandwich element of our sandwich semigroup is uniquely sandwich-regular. This will be the case with the category of partial injections (see Section 3.3).

First, we introduce the key terms of this section. Recall the term of an inverse semigroup from Section 1.3. Naturally, an *inverse monoid* is an inverse semigroup with an identity. A corresponding term in category theory is the term of an inverse category, as defined in [67]. We shall use a slightly different, yet equivalent definition (see Section 2.3.2 of [23]):

Definition 2.5.1. A category X is an *inverse category* if for every morphism $f : A \to B$ there exists a unique morphism $g : B \to A$ such that $fgf = f$ and $gfg = g$.

Indeed, a one-object inverse category is precisely an inverse monoid.

Recall that the results of Sections 2.3 and 2.4 are obtained under the assumption of sandwich-regularity of the sandwich element $a \in S_{ji}$. Here, we introduce some properties stronger than sandwich-regularity. An element $a \in S_{ji}$ is *uniquely regular* if $\mathrm{V}(x) = \{y \in S : x = xyx, y = yxy\}$ is a singleton. Furthermore, $a \in S_{ji}$ is *uniquely sandwich-regular* if every element of $\{a\} \cup aS_{ij}a$ is uniquely regular in the partial semigroup S. Obviously, unique sandwich-regularity of a implies its unique regularity, and also implies its sandwich-regularity. Note that in an inverse category, every element is uniquely sandwich-regular.

Now, suppose that $a \in S_{ji}$ is a uniquely sandwich-regular element and consider the results of Sections 2.3 and 2.4 (as in Section 3 of [33]).

Proposition 2.5.2. *Suppose $a \in S_{ji}$ is uniquely sandwich-regular and that $\mathrm{V}(a) = \{b\}$. Then all maps in the diagram 2.3 are isomorphisms (thus the map $\phi : \mathrm{P}^a \to W$ is an isomorphism), and all semigroups are inverse monoids.*

Proof. Our first step is to show that $\mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$ is a monoid with identity b. Let $x \in \mathrm{P}^a$ and fix any $y \in \mathrm{V}_a(x)$. Since $x = xayax$ and $y = yaxay$, we have

$$xab = xayaxab = xabayaxab \text{ and } aya = ayaxabaya,$$
$$bax = baxayax = baxayabax \text{ and } aya = ayabaxaya,$$

so $x, xab, bax \in \mathrm{V}(aya)$. From the unique sandwich-regularity of a it follows that every element of $\{a\} \cup aS_{ji}a$ is uniquely regular; in particular, $aya \in aS_{ij}a$ is uniquely regular, so $x = xab = bax$. Thus, $x = x \star_a b = b \star_a x$.

In Subsection 2.3.1 we have shown that the maps ϕ_1, ϕ_2, ψ_1 and ψ_2 are surmorphisms, so we need to prove all of them are injective. It suffices to show that $\phi = \psi_1\phi_1 = \psi_2\phi_2$ is injective (because ψ_1 and ψ_2 are surmorphisms). Suppose that $x, y \in \mathrm{P}^a$ are such that $x\phi = y\phi$, i.e. $axa = aya$. Having in mind that b is the identity of P^a, we may conclude

$$x = b \star_a x \star_a b = baxab = bayab = b \star_a y \star_a b = y.$$

Finally, we show that P^a is an inverse semigroup, and consequently an inverse monoid. Let $u \in \mathrm{P}^a$ and let $x, y \in \mathrm{V}_a(u)$. From $uaxau = u = uayau$ we have $auaxaua = aua = auayaua$; together with $x = xauax$ and $y = yauay$ these imply $x, y \in \mathrm{V}(aua)$. Hence $x = y$, as $aua \in aS_{ij}a$ is uniquely regular. Therefore, P^a is an inverse monoid, as required. Since the rest of the semigroups are its isomorphic images, they are inverse monoids, as well. □

Remark 2.5.3. Proposition 2.5.2 has a series of corollaries, in the form of significant simplifications of the results of Sections 2.3 and 2.4. We give a short summary of the most important ones:

- the map $\psi = (\psi_1, \psi_2)$ from Theorem 2.3.8 is trivially injective;
- since ϕ as an isomorphism, the $\widehat{\mathscr{K}^a}$-relations of Section 2.3 are identical to the $\mathscr{K}^a$-relations, so the rectangular groups in Theorem 2.3.12 are just groups;
- for the same reason, Theorem 2.3.15 is completely trivial;
- Proposition 2.4.9 says that $\mathrm{MI}(\mathrm{P}^a) = \{b\}$ and $\mathrm{RP}(\mathrm{P}^a) = \mathrm{H}_b^a$ consist only of the identity and the units, respectively, which is true in any monoid;
- clearly, P^a is MI-dominated, so Theorem 2.4.16 reduces to "$\mathrm{rank}(\mathrm{P}^a) = \mathrm{rank}(W : G_W) + \mathrm{rank}(G_W)$ if $W \setminus G_W$ is an ideal of W";
- for the same reason, Theorem 2.4.17 reduces to "$\mathrm{rank}(\mathbb{E}_a(\mathrm{P}^a)) = \mathrm{rank}(\mathbb{E}_b(W))$ and $\mathrm{idrank}(\mathbb{E}_a(\mathrm{P}^a)) = \mathrm{idrank}(\mathbb{E}_b(W))$".

2.6 The rank of a sandwich semigroup

We devote the final section to the results concerning the rank of S_{ij}^a. Unfortunately, they are quite limited, even under assumptions such as sandwich-regularity of the sandwich element. This comes as no surprise since the structure of the sandwich semigroup S_{ij}^a may, in general, be much more complex than the structure of its regular subsemigroup P^a, for instance. So, instead of exact values, we give some rough lower bounds.

Let us fix a partial semigroup $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ and a sandwich element $a \in S_{ji}$ for some $i, j \in I$. Note that we make no further assumptions.

In Section 1.3, for a semigroup S, we describe the partial order $\leq_{\mathscr{J}}$ on the set of its $\mathscr{J}$-classes, $S/\mathscr{J}$. Here, we claim the following: if X is a generating set for a semigroup S and J any maximal $\mathscr{J}$-class of S, then $X \cap J$ contains a generating set for J. Namely, if $y \in J$ and $y = x_1 \cdots x_k$ for some $k \in \mathbb{N}$ and some $x_1, \ldots, x_k \in X$, then $y \leq_{\mathscr{J}} x_i$ for all $1 \leq i \leq k$. Since J is a maximal $\mathscr{J}$-class, we have $y \mathscr{J} x_i$ for all $1 \leq i \leq k$, so J is generated exclusively by elements of $X \cap J$ (more generally, we may conclude that $S \setminus J$ is an ideal of the semigroup S), and the claim is proved. In the case of sandwich semigroups, this means that any generating set of S_{ij}^a must include a generating set for each maximal $\mathscr{J}^a$-class J, which consists of the elements of J.

On page 25, we defined Green's classes of a hom-set S_{ij} by $\mathrm{K}_x = \{y \in S_{ij} : x \mathscr{K} y\}$ for each $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{J}, \mathscr{D}\}$. Also, we introduced the restriction of the relation $\leq_{\mathscr{J}}$ on the set $S_{ij}/\mathscr{J}$. In the following two results (from [33]), we study how much of a maximal $\mathscr{J}$-class is necessary for generating the sandwich semigroup S_{ij}^a.

Lemma 2.6.1. *Suppose $\langle X\rangle_a = T \subseteq S_{ij}^a$ and that $J \subseteq T$ is a maximal $\mathscr{J}$-class in S_{ij}. Then*

(i) $X \cap J \neq \emptyset$,

(ii) if every element of aS_{ij} is $\mathscr{R}$-stable, then X has non-trivial intersection with each $\mathscr{R}$-class of J,

(iii) if every element of $S_{ij}a$ is $\mathscr{L}$-stable, then X has non-trivial intersection with each $\mathscr{L}$-class of J.

Proof. First, we prove (i). Suppose $x \in J$, and $x = x_1 \star_a \cdots \star_a x_k$ where $x_1, \ldots, x_k \in X$. This implies $x \leq_{\mathscr{J}} x_i$, for all i. Thus $x \mathscr{J} x_i$ for all i, as $\mathrm{J}_x = J$ is a maximal $\mathscr{J}$-class in S_{ij}. In particular, each x_i belongs to $X \cap J$.

It suffices to prove (ii), as (iii) is dual. Suppose that every element of aS_{ij} is $\mathscr{R}$-stable. Choose an arbitrary $x \in T \subseteq S_{ij}$ and suppose $x = x_1 \star_a \cdots \star_a x_k$. We claim that $x_1 \mathscr{R} x$ in S_{ij}. If $k = 1$, then $x = x_1$, so the relation obviously holds. If $k \geq 2$, put $z = x_2 \star_a \cdots \star_a x_k$, so that $x = x_1 \star_a z = x_1 a z$. From the proof of (i), we have $x_1 \mathscr{J} x = x_1 a z$, so $\mathscr{R}$-stability of $az \in aS_{ij}$ implies $x_1 \mathscr{R} x_1 a z = x$ in S_{ij}. □

The next result is a direct corollary of the previous lemma.

Corollary 2.6.2. *Let $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ be a partial semigroup with $i, j \in I$ and $a \in S_{ji}$. Suppose every element of aS_{ij} is $\mathscr{R}$-stable and every element of $S_{ij}a$ is $\mathscr{L}$-stable. Write $\{J_k : k \in K\}$ for the set of maximal $\mathscr{J}$-classes of S_{ij}. Then*

$$\operatorname{rank}(S_{ij}^a) \geq \sum_{k \in K} \max(|J_k/\mathscr{R}|, |J_k/\mathscr{L}|).$$

In some cases, but not all, the above lower bound is the exact value of $\operatorname{rank}(S_{ij}^a)$ (see Theorems 3.1.51 and 3.1.57).

Before moving on, we provide a modified version of Proposition 3.26 from [28], which will be of help for our calculations in the following chapters. Recall the results of Subsection 2.2.4, in particular Propositions 2.2.37 and 2.2.35. We have proved the following: if each element of $S_{ij} \cup aS_{ij}a$ is stable in S and $a \in S_{ji}$ is right-invertible with $b \in \mathrm{RI}(a)$, then $\mathrm{J}_b = \mathrm{L}_b$ is the maximum $\mathscr{J}$-class of the hom-set S_{ij}, while $\mathrm{J}_b^a = \mathrm{L}_b^a$ is the maximum $\mathscr{J}^a$-class of S_{ij}^a (because the fact that the elements of $aS_{ij}a$ are stable implies stability in S_{ij}^a, by Lemma 2.2.27(v)). Furthermore, J_b^a is a left-group over $\mathrm{H}_b^a = \mathrm{H}_b$ (the last equality following from Theorem 2.2.3(iii), since b is regular). If we choose X to be a cross-section of the $\mathscr{H}$-classes in J_b, then there clearly exist $X_1 \subseteq X$ such that $\mathrm{J}_b^a = \bigcup_{x \in X_1} \mathrm{H}_x = \bigcup_{x \in X_1} \mathrm{H}_x^a$. As for the elements of $X_2 = X \setminus X_1$, they always belong to singleton $\mathscr{H}^a$-classes, while their $\mathscr{H}$-classes are non-singletons, in general. In this setting, we may prove the following:

Proposition 2.6.3. *Suppose S is a partial semigroup and $a \in S_{ji}$ is right-invertible. Further, suppose that each element of $S_{ij} \cup aS_{ij}a$ is stable, and that each element of aS_{ij} is $\mathscr{R}$-stable. Keep the above notation and write $T = \langle \mathrm{J}_b \rangle_a$.*

(i) *We have $T = \langle \mathrm{J}_b^a \cup X_2 \rangle_a$.*

(ii) $\mathrm{rank}(T) = |X_2| + \max(|X_1|, \mathrm{rank}(\mathrm{H}_b^a))$.

(iii) *If* $\mathrm{rank}(\mathrm{H}_b^a) \leq |\, \mathrm{J}_b^a \,/\, \mathscr{H}^a \,|$, *then* $\mathrm{rank}(T) = |\, \mathrm{J}_b \,/\, \mathscr{H} \,|$.

Proof. (i) We have $\mathrm{J}_b^a \cup X_2 \subseteq \mathrm{J}_b$, so we may immediately conclude that $\langle \mathrm{J}_b^a \cup X_2 \rangle_a \subseteq T$. Thus, it suffices to show $\mathrm{J}_b \subseteq \langle \mathrm{J}_b^a \cup X_2 \rangle_a$. Let $y \in \mathrm{J}_b$. Then, there exists $x \in X$ such that $y \in \mathrm{H}_x$. Thus, we have $y \mathscr{R} x$ and so $y = xv$ for some $v \in S^{(1)}$. Since b is a right-inverse of a in S_{ij}, we have

$$y = xv = (xab)v = x \star_a bv.$$

As $x \in X \subseteq \mathrm{J}_b^a \cup X_2$, it suffices to prove $bv \in \mathrm{J}_b^a$. From $b \mathscr{J} y = xabv \leq_{\mathscr{J}} bv \leq_{\mathscr{J}} b$ we may conclude that all these elements are $\mathscr{J}$-related, so $bv \mathscr{J} b$. Moreover, since $\mathrm{J}_b = \mathrm{L}_b$ (see the discussion above), we have $bv \mathscr{L} b$. Now, note that $bv = (bab)v = b \cdot a(bv)$, so $bv \mathscr{L} a(bv)$ and therefore $bv \in \mathrm{P}_2^a$. Hence, $bv \in \mathrm{L}_b \cap \mathrm{P}_2^a$, and since $\mathrm{L}_b \cap \mathrm{P}_2^a = \mathrm{L}_b^a$ (by Theorem 2.2.3(ii)) and $\mathrm{L}_b^a = \mathrm{J}_b^a$ (by the discussion preceding this proposition), it follows that $bv \in \mathrm{J}_b^a$.

(ii) Since J_b^a is a left-group over H_b^a (by the discussion above), it is in fact a $|\, \mathrm{J}_b^a \,/\, \mathscr{R}^a \,| \times 1$ rectangular group over H_b^a. Then, Proposition 1.4.2(i) gives

$$\begin{aligned} \mathrm{rank}(\mathrm{J}_b^a) &= \max(|\, \mathrm{J}_b^a \,/\, \mathscr{R}^a \,|, 1, \mathrm{rank}(\mathrm{H}_b^a)) = \max(|\, \mathrm{J}_b^a \,/\, \mathscr{R}^a \,|, \mathrm{rank}(\mathrm{H}_b^a)) \\ &= \max(|\, \mathrm{J}_b^a \,/\, \mathscr{H}^a \,|, \mathrm{rank}(\mathrm{H}_b^a)) = \max(|X_1|, \mathrm{rank}(\mathrm{H}_b^a)), \end{aligned} \tag{2.12}$$

the penultimate equality following from $\mathrm{L}_b^a = \mathrm{J}_b^a$. Let Ω be a generating set for J_b^a of size $\mathrm{rank}(\mathrm{J}_b^a)$. Applying (i), we have

$$\langle \Omega \cup X_2 \rangle_a = \langle \langle \Omega \rangle_a \cup X_2 \rangle_a = \langle \mathrm{J}_b^a \cup X_2 \rangle_a = T,$$

therefore

$$\mathrm{rank}(T) \leq |\Omega \cup X_2| = |\Omega| \cup |X_2| = \max(|X_1|, \mathrm{rank}(\mathrm{H}_b^a)) + |X_2|.$$

It remains to show the reverse inequality. Firstly, recall that $\mathrm{J}_b^a \subseteq \mathrm{J}_b$ is a maximal $\mathscr{J}^a$-class of S_{ij}^a. Thus, by the discussion at the beginning of this section, $S_{ij} \setminus \mathrm{J}_b^a$ is an ideal of S_{ij}^a and so $T \setminus \mathrm{J}_b^a$ is an ideal of T (because T is a subsemigroup of S_{ij} containing J_b^a). Thus, any generating set of T contains a generating set for J_b^a, consisting of elements of J_b^a, so $\mathrm{rank}(T) \geq \mathrm{rank}(\mathrm{J}_b^a)$. Furthermore, since each element of aS_{ij} is $\mathscr{R}$-stable and $J_b \subseteq T$, Lemma 2.6.1(ii) implies that any generating set for T has non-trivial intersection with each $\mathscr{R}$-class of J. Thus, a generating set for T contains a cross-section of non-regular $\mathscr{R}$-classes of $\mathrm{J}_b = \mathrm{L}_b$ (which is clearly of size

$|(\mathrm{J}_b \setminus \mathrm{J}_b^a)/\mathscr{H}| = |X_2|$) and we have

$$\operatorname{rank}(T) \geq \operatorname{rank}(\mathrm{J}_b^a) + |X_2| = \max(|X_1|, \operatorname{rank}(\mathrm{H}_b^a)) + |X_2|.$$

(iii) If $\operatorname{rank}(\mathrm{H}_b^a) \leq |\,\mathrm{J}_b^a / \mathscr{H}^a\,|$, from (2.12) we have $\operatorname{rank}(\mathrm{J}_b^a) = |\,\mathrm{J}_b^a / \mathscr{H}^a\,| = |X_1|$, and from the proof of part *(ii)* we conclude that

$$\operatorname{rank}(T) = \operatorname{rank}(\mathrm{J}_b^a) + |X_2| = |X_1| + |X_2| = |X| = |\,\mathrm{J}_b / \mathscr{H}\,|. \qquad \square$$

In the dual situation, if each element of $S_{ij} \cup aS_{ij}a$ is stable in S and $a \in S_{ji}$ is left-invertible with a left-inverse b, then $\mathrm{J}_b = \mathrm{R}_b$ is the maximum $\mathscr{J}$-class of the hom-set S_{ij}, while $\mathrm{J}_b^a = \mathrm{R}_b^a$ is the maximum $\mathscr{J}^a$-class of S_{ij}^a; furthermore, J_b^a is a right-group over $\mathrm{H}_b^a = \mathrm{H}_b$. Again, we choose X to be a cross-section of the $\mathscr{H}$-classes in J_b, we fix $X_1 \subseteq X$ so that $\mathrm{J}_b^a = \bigcup_{x \in X_1} \mathrm{H}_x = \bigcup_{x \in X_1} \mathrm{H}_x^a$ and we write $X_2 = X \setminus X_1$. Now, we state the obvious dual of Proposition 2.6.3, which follows by a dual argument.

Proposition 2.6.4. *Suppose S is a partial semigroup and $a \in S_{ji}$ is left-invertible. Further, suppose that each element of $S_{ij} \cup aS_{ij}a$ is stable, and that each element of $S_{ij}a$ is $\mathscr{L}$-stable. Keep the above notation and write $T = \langle \mathrm{J}_b \rangle_a$.*

(i) *We have $T = \langle \mathrm{J}_b^a \cup X_2 \rangle_a$.*

(ii) $\operatorname{rank}(T) = |X_2| + \max(|X_1|, \operatorname{rank}(\mathrm{H}_b^a))$.

(iii) *If* $\operatorname{rank}(\mathrm{H}_b^a) \leq |\,\mathrm{J}_b^a / \mathscr{H}^a\,|$, *then* $\operatorname{rank}(T) = |\,\mathrm{J}_b / \mathscr{H}\,|$.

Chapter 3

Sandwich semigroups of transformations

In this chapter, we apply the results of Chapter 2 to obtain results on sandwich semigroups in three particular categories of functions: partial functions, "plain" functions (with a full domain) and injective partial functions. We extend those results, where possible, by investigating further. The results of this chapter were published in [34], so we cite this article unless stated otherwise.

First, we examine the partial semigroup $\mathcal{PT}$, and then we investigate comprehensively the sandwich semigroups contained in it. In more detail, after we examine regularity, stability, and Green's relations in $\mathcal{PT}$, we focus on a sandwich semigroup $\mathcal{PT}^a_{XY}$:

- we describe its P-sets, Green's relations, the structure of its $\mathscr{J}^a$-classes and the order $\leq_{\mathscr{J}^a}$;
- we examine its regular subsemigroup and its connections to the semigroups presented in the Diagrams 2.2 and 2.3; we also give neat alternative descriptions for these semigroups;
- we study the structure of $\operatorname{Reg}(\mathcal{PT}^a_{XY})$ via its connection to $W = \mathcal{PT}_A$ (where $A = \operatorname{im} a$), by describing its Green's relations and the inflation of W (see Remark 2.3.13); furthermore, we prove that $\operatorname{Reg}(\mathcal{PT}^a_{XY})$ is MI-dominated, so Theorem 2.4.16 enables us to calculate the rank of $\operatorname{Reg}(\mathcal{PT}^a_{XY})$;
- we characterise its idempotents, enumerate them and calculate the rank of the idempotent-generated subsemigroup $\mathbb{E}_a(\mathcal{PT}^a_{XY})$;
- finally, we calculate its rank.

Subsequently, we perform the same analysis for $\mathcal{T}$ and a sandwich semigroup of the form $\mathcal{T}^a_{XY}$, as well as for $\mathcal{I}$ and a sandwich semigroup of the form $\mathcal{I}^a_{XY}$.

We need to introduce some notions and notation specific for the topic of this chapter. Let **Set** denote the class of all sets. For $A, B \in \mathbf{Set}$, we define

$$\begin{aligned} \mathbf{T}_{AB} &= \{f : f \text{ is a function } A \to B\}, \\ \mathbf{PT}_{AB} &= \{f : f \text{ is a function } C \to B, \text{ for some } C \subseteq A\}, \\ \mathbf{I}_{AB} &= \{f : f \text{ is an injective function } C \to B, \text{ for some } C \subseteq A\}. \end{aligned}$$

We say that $f \in \mathbf{PT}_{AB}$ is *full* if $\operatorname{dom} f = A$ (i.e. if $f \in \mathbf{T}_{AB}$). Similarly as in the previous chapter, for any $A \in \mathbf{Set}$, we write $\mathbf{PT}_A = \mathbf{PT}_{AA}$, $\mathbf{T}_A = \mathbf{T}_{AA}$ and $\mathbf{I}_A = \mathbf{I}_{AA}$. Note that these are the partial transformation semigroup over A, the full transformation semigroup over A and the symmetric inverse semigroup over A, respectively. Occasionally, we will refer to $\mathbf{PT}_{\alpha\beta}$ or $\mathbf{PT}_{\alpha}$, where α and β are cardinals. In these cases, we simply regard cardinals as sets (the same goes for $\mathbf{T}_{\alpha\beta}$ and $\mathbf{I}_{\alpha\beta}$).

Clearly, $\mathbf{I}_{AB} \subseteq \mathbf{PT}_{AB}$ for any $A, B \in \mathbf{Set}$. Furthermore, the empty map $\emptyset$ belongs to both sets, so $\mathbf{I}_{AB} \cap \mathbf{I}_{CD} \neq \emptyset$ and $\mathbf{PT}_{AB} \cap \mathbf{PT}_{CD} \neq \emptyset$ for any $A, B, C, D \in \mathbf{Set}$. However, the same does not hold for $\mathbf{T}_{AB} \cap \mathbf{T}_{CD}$. Since the domains need to be full, $\mathbf{T}_{AB} \cap \mathbf{T}_{CD} \neq \emptyset$ if and only if $A = C = \emptyset$ or $A = C \neq \emptyset$ with $B \cap D \neq \emptyset$.

Let $\mathbf{Set}^+ = \mathbf{Set} \setminus \{\emptyset\}$ and define

$$\begin{aligned} \mathcal{PT} &= \{(A, f, B) : A, B \in \mathbf{Set},\ f \in \mathbf{PT}_{AB}\}, \\ \mathcal{T} &= \{(A, f, B) : A, B \in \mathbf{Set}^+,\ f \in \mathbf{T}_{AB}\}, \\ \mathcal{I} &= \{(A, f, B) : A, B \in \mathbf{Set},\ f \in \mathbf{I}_{AB}\}. \end{aligned}$$

We may define a partial binary operation on $\mathcal{PT}$:

$$(A, f, B) \cdot (C, g, D) = \begin{cases} (A, f \circ g, D), & \text{if } B = C; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Note that $\mathcal{T}$ and $\mathcal{I}$ are subclasses of $\mathcal{PT}$, both closed under the defined multiplication.

The choice of $\mathbf{Set}^+$ instead of $\mathbf{Set}$ for $\mathcal{T}$ arises from the fact that $\mathbf{T}_{A\emptyset} = \emptyset$ if and only if $A \neq \emptyset$. Therefore, the only full-domain maps we disregard are the functions of the form $\emptyset \to A$, for $A \in \mathbf{Set}$. This is only a matter of convenience; the results would essentially remain the same with the inclusion of these maps (since a map of such type can, by composition, produce only a map of the same type).

Next, we define

$$\boldsymbol{\delta} : \mathcal{PT} \to \mathbf{Set} : (A, f, B) \mapsto A \quad \text{and} \quad \boldsymbol{\rho} : \mathcal{PT} \to \mathbf{Set} : (A, f, B) \mapsto B.$$

Note that, for any two elements (A, f, B) and (C, g, D) from $\mathcal{PT}$, the product $(A, f, B)(C, g, D)$ exists if and only if $(A, f, B)\,\boldsymbol{\rho} = (C, g, D)\,\boldsymbol{\delta}$, and in that case we have

$$((A, f, B) \cdot (C, g, D))\,\boldsymbol{\delta} = A \quad \text{and} \quad ((A, f, B) \cdot (C, g, D))\,\boldsymbol{\rho} = D.$$

Also, whenever a product is defined, we have associativity, because the composition of maps is associative. Finally, for any $A, B \in \mathbf{Set}$, the class $\mathcal{PT}_{AB} = \{(A, f, B) : f \in \mathbf{PT}_{AB}\}$ is a set. Therefore, $(\mathcal{PT}, \cdot, \mathbf{Set}, \boldsymbol{\delta}, \boldsymbol{\rho})$ is a partial semigroup.

Furthermore, we may conclude (by a similar argument as in the previous paragraph) that $(\mathcal{T}, \cdot\restriction_{\mathcal{T}}, \mathbf{Set}^+, \boldsymbol{\delta}\restriction_{\mathcal{T}}, \boldsymbol{\rho}\restriction_{\mathcal{T}})$ and $(\mathcal{I}, \cdot\restriction_{\mathcal{I}}, \mathbf{Set}, \boldsymbol{\delta}\restriction_{\mathcal{I}}, \boldsymbol{\rho}\restriction_{\mathcal{I}})$ are both partial semigroups. In fact, they are partial subsemigroups of $(\mathcal{PT}, \cdot, \mathbf{Set}, \boldsymbol{\delta}, \boldsymbol{\rho})$.

We abbreviate the notation for $(\mathcal{PT}, \cdot, \mathbf{Set}, \boldsymbol{\delta}, \boldsymbol{\rho})$, $(\mathcal{T}, \cdot\restriction_{\mathcal{T}}, \mathbf{Set}^+, \boldsymbol{\delta}\restriction_{\mathcal{T}}, \boldsymbol{\rho}\restriction_{\mathcal{T}})$ and $(\mathcal{I}, \cdot\restriction_{\mathcal{I}}, \mathbf{Set}, \boldsymbol{\delta}\restriction_{\mathcal{I}}, \boldsymbol{\rho}\restriction_{\mathcal{I}})$ to $\mathcal{PT}$, $\mathcal{T}$ and $\mathcal{I}$, respectively. Since for any $X \in \mathbf{Set}^+$ all three of them contain the identity map $\mathrm{id}_X : X \to X : x \mapsto x$, and $(\emptyset, \mathrm{id}_\emptyset, \emptyset) \in \mathcal{I} \subseteq \mathcal{PT}$, these partial semigroups are monoidal, i.e. locally small categories. Moreover, we have

Proposition 3.0.1. *The partial semigroups $\mathcal{PT}$, $\mathcal{T}$ and $\mathcal{I}$ are all von-Neumann regular.*

Proof. First, we show the von-Neumann regularity of $\mathcal{PT}$. Let (A, f, B) be an arbitrary element of $\mathcal{PT}$. If f is the empty map ($f = \emptyset$), we have $fgf = f$ for $(B, g, A) \in \mathcal{PT}$, with g being an empty map. Hence (A, f, B) is regular. Suppose $f = \binom{F_i}{f_i}_{i \in I}$. For each $i \in I$ fix an element $a_i \in F_i$ and define $g = \binom{\{f_i\}}{a_i}_{i \in I}$ with $g : B \to A$. Obviously

$$(A, f, B) \cdot (B, g, A) \cdot (A, f, B) = (A, f, B), \tag{3.1}$$

i.e. (A, f, B) is regular.

Note that $g \in \mathbf{I}_{BA}$, so we have proved the regularity of $\mathcal{I}$, as well. As for $\mathcal{T}$, the equality (3.1) holds (and $g \in \mathbf{T}_{BA}$), if we define g so that $f_i \mapsto a_i$ and the elements of the set $B \setminus \mathrm{im}(f)$ map to any element of A. $\square$

Therefore, $\mathcal{T}$ and $\mathcal{I}$ are regular partial subsemigroups of $\mathcal{PT}$. Thus, once we investigate $\mathcal{PT}$, we may use the results of Subsection 2.2.5 to obtain information on $\mathcal{T}$ and $\mathcal{I}$, as well. Moreover, the regularity of these three partial semigroups implies the sandwich-regularity of their elements. Thus, we may apply the theory of Sections 2.3 and 2.4 to attain results on sandwich semigroups in $\mathcal{PT}$, $\mathcal{T}$ and $\mathcal{I}$.

Before focusing on $\mathcal{PT}$, we investigate two additional topics in order to make our study easier down the line. Firstly, having in mind the results of Subsections 2.2.1 and 2.2.4, we pose the following questions for $\mathcal{PT}$, $\mathcal{T}$ and $\mathcal{I}$:

- In which cases an element has a left-identity? In which cases an element has a right-identity?
- In which cases an element $a \in S_{ji}$ is left- or right-invertible in S_{ij}?

Clearly, since all three of them are monoidal, the answer to the first two questions in all three cases is: Always. Furthermore,

Lemma 3.0.2. *If $\mathcal{Z}$ is any of partial semigroups $\mathcal{PT}$,$\mathcal{T}$ and $\mathcal{I}$, then*

(i) $a \in \mathcal{Z}_{XY}$ is right-invertible in $\mathcal{Z}_{YX}$ if and only if it is full and injective;

(ii) $a \in \mathcal{Z}_{XY}$ *is left-invertible in* $\mathcal{Z}_{YX}$ *if and only if it is surjective.*

Proof. (i) Suppose $a \in \mathcal{Z}_{XY}$ is full and injective and let $b \in \mathrm{V}(a)$. Then, we may write $a = \binom{\{u_i\}}{v_i}_{i\in I}$, where $\{u_i : i \in I\} = X$. Since $aba = a$, we clearly have $(v_i)b = u_i$, so $ab = \mathrm{id}_{\mathrm{dom}\, a} = \mathrm{id}_X$. Thus, for any $x \in \mathcal{Z}$ with $x\,\boldsymbol{\rho} = X$ we have $xab = x\,\mathrm{id}_X = x$, which means that a is right-invertible in $\mathcal{Z}$.

Conversely, if $a \in \mathcal{Z}_{XY}$ is right-invertible in $\mathcal{Z}$, there exists some $b \in \mathcal{Z}_{YX}$ such that $xab = x$ for all $x \in \mathcal{Z}$ with $x\,\boldsymbol{\delta} = X$. In particular, for $x = \mathrm{id}_X$ we have $\mathrm{id}_X\, ab = ab = \mathrm{id}_X$. Thus, $X = \mathrm{dom}(ab) \subseteq \mathrm{dom}\, a$ and $\ker(a) \subseteq \ker(ab) = \{(x,x) : x \in X\}$. From these two we may conclude that a is both full and injective.

(ii) is shown in a similar manner, since $ba = \mathrm{id}_{\mathrm{im}(a)}$ for $b \in \mathrm{V}(a)$ and $\mathrm{im}(ca) \subseteq \mathrm{im}(c)$ for any $c \in \mathcal{Z}$. □

Secondly, we are interested in whether any of the partial semigroups $\mathcal{PT}$, $\mathcal{T}$ and $\mathcal{I}$ can be expanded to a partial $*$-semigroup. The first statement of the following result was proved in [34] as Lemma 4.1.

Proposition 3.0.3. *The partial semigroup* $\mathcal{I}$ *can be expanded to a partial* $*$*-semigroup, which is an inverse partial semigroup. However, neither* $\mathcal{PT}$ *nor* $\mathcal{T}$ *can be expanded to a partial* $*$*-semigroup.*

Proof. First, let us prove that each element $(A, f, B) \in \mathcal{I}$ has a unique inverse. If $f = \emptyset$, then $(A, \emptyset, B)$ trivially has a single inverse $(B, \emptyset, A)$. So, suppose $f = \binom{h_i}{f_i}_{i\in I}$, and $g \in \mathrm{V}(f)$ in $\mathcal{I}$. Since $(h_i)fgf = f_i$ for all $i \in I$, we have $(f_i)g = h_i$ for all $i \in I$. Suppose there exists $a \in \mathrm{dom}\, g \setminus \{f_i : i \in I\}$. Then $(a)g \notin \{h_i : i \in I\}$ (because g is injective), so $(a)gfg$ is not defined, which contradicts g being an inverse of f. Thus, $g = \binom{f_i}{h_i}_{i\in I}$ and (B, g, A) is the unique inverse for (A, f, B) in $\mathcal{I}$. For clarity, we denote g by f^{-1}.

Now, we define ${}^* : \mathcal{I} \to \mathcal{I} : (A, f, B) \mapsto (B, f^{-1}, A)$. Clearly, the 6-tuple $(\mathcal{I}, \cdot\restriction_{\mathcal{I}}, \mathbf{Set}, \boldsymbol{\delta}\restriction_{\mathcal{I}}, \boldsymbol{\rho}\restriction_{\mathcal{I}}, {}^*)$ is an inverse partial semigroup, so a partial $*$-semigroup, as well.

Finally, we prove the last statement. We consider only the category $\mathcal{PT}$, since the proof for $\mathcal{T}$ is similar. Suppose that there exists an operation ${}^* : \mathcal{PT} \to \mathcal{PT}$, which expands $\mathcal{PT}$ to a partial $*$-semigroup. By the definition of a partial $*$-semigroup, for any $f = (A, f', B) \in \mathcal{PT}$ we have

$$(f^*)\,\boldsymbol{\delta} = B, \qquad (f^*)\,\boldsymbol{\rho} = A, \qquad \text{and} \qquad f^{**} = f.$$

These three together imply that * defines a bijection $\mathbf{PT}_{AB} \to \mathbf{PT}_{BA}$ for all $A, B \in \mathbf{Set}$. But if $A = \{1\}$ and $B = \{1, 2\}$, then $|\,\mathbf{PT}_{AB}\,| = 3$ while $|\,\mathbf{PT}_{BA}\,| = 4$, so no such bijection exists. Therefore, such an operation * cannot be defined. □

3.1 The category $\mathcal{PT}$

Having introduced the necessary notation and general results, we are ready to discuss partial functions, the partial semigroup $\mathcal{PT}$ and the sandwich semigroups it contains. For the sake of brevity, in this section, the term map (function) denotes a partial function.

In dealing with functions, the kernel relation is of vital importance to us. For this reason, we need to expand our vocabulary on the "relational front". Let X and Y be sets such that $X \subseteq Y$, and let σ be an equivalence relation on Y. Then:

- $\sigma \cap (X \times X)$ is an equivalence relation on X called the *the restriction of* σ *to* X and denoted $\sigma\restriction_X$;
- if each σ-class contains at least one element of X, we say that X *saturates* σ;
- if each σ-class contains at most one element of X, we say that σ *separates* X.

If X both saturates σ and is separated by σ, then X is clearly a cross-section of σ.

In the following lemma, we use these terms to describe the properties of the composition of two partial functions.

Lemma 3.1.1. *Let* $A, B, C \in \mathbf{Set}$, $f \in \mathbf{PT}_{AB}$ *and* $g \in \mathbf{PT}_{BC}$. *Then*

(i) $\operatorname{dom}(fg) \subseteq \operatorname{dom} f$, *with equality if and only if* $\operatorname{im} f \subseteq \operatorname{dom} g$,

(ii) $\operatorname{im}(fg) \subseteq \operatorname{im} g$, *with equality if and only if* $\operatorname{im} f$ *saturates* $\ker g$,

(iii) $\ker(fg) \supseteq (\ker f)\restriction_{\operatorname{dom}(fg)}$, *with equality if and only if* $\ker g$ *separates* $\operatorname{im} f$,

(iv) $\operatorname{rank}[(fg) \leq \min(\operatorname{Rank} f, \operatorname{Rank} g)$.

Proof. Parts (i) and (ii) follow directly from the definition of composition. Furthermore, (ii) implies $|\operatorname{im}(fg)| \leq |\operatorname{im} g|$, i.e. $\operatorname{Rank}(fg) \leq \operatorname{Rank} g$. In addition,

$$\operatorname{Rank}(fg) = |\operatorname{im}(fg)| = |(\operatorname{im} f)g| \leq |\operatorname{im} f| = \operatorname{Rank} f,$$

so (iv) holds, as well. Only (iii) remains to be proved. Clearly, for any two elements $x, y \in \operatorname{dom}(fg)$ the equality $xf = yf$ implies $(x)fg = (y)fg$. Moreover, the reverse containment holds if and only if $\ker g$ does not "connect" any pair of elements from $\operatorname{im} f$ (i.e. if and only if it separates elements of $\operatorname{im} f$). □

Having proved this basic lemma and introduced the notation needed, we get to investigating the partial semigroup $\mathcal{PT}$.

Proposition 3.1.2. *Let* $(A, f, B), (C, g, D) \in \mathcal{PT}$. *Then*

(i) $(A, f, B) \leq_{\mathscr{R}} (C, g, D) \Leftrightarrow$
$$A = C,\ \operatorname{dom} f \subseteq \operatorname{dom} g \text{ and } \ker f \supseteq (\ker g)\restriction_{\operatorname{dom} f},$$

(ii) $(A, f, B) \leq_{\mathscr{L}} (C, g, D) \Leftrightarrow B = D$ *and* $\operatorname{im} f \subseteq \operatorname{im} g$,

(iii) $(A, f, B) \leq_{\mathscr{J}} (C, g, D) \Leftrightarrow \operatorname{Rank} f \leq \operatorname{Rank} g$,

(iv) $(A, f, B)\,\mathscr{R}\,(C, g, D) \Leftrightarrow A = C,\ \operatorname{dom} f = \operatorname{dom} g$ *and* $\ker f = \ker g$,

(v) $(A, f, B) \mathscr{L} (C, g, D) \Leftrightarrow B = D$ *and* $\operatorname{im} f = \operatorname{im} g$,

(vi) $(A, f, B) \mathscr{J} (C, g, D) \Leftrightarrow (A, f, B) \mathscr{D} (C, g, D) \Leftrightarrow \operatorname{Rank} f = \operatorname{Rank} g$.

Proof. (i) Suppose $(A, f, B) \leq_{\mathscr{R}} (C, g, D)$. In other words,

$$(A, f, B) = (C, g, D)(E, h, J) \tag{3.2}$$

for some $(E, h, J) \in \mathcal{PT}$. Thus $C = A$, $J = B$, $D = E$, and $f = gh$, which implies $\operatorname{dom} f \subseteq \operatorname{dom} g$ (by Lemma 3.1.1(i)) and $\ker f \supseteq (\ker g)\restriction_{\operatorname{dom} f}$ (by Lemma 3.1.1(iii)).

Conversely, we suppose $A = C$, $\operatorname{dom} f \subseteq \operatorname{dom} g$, and $\ker f \supseteq (\ker g)\restriction_{\operatorname{dom} f}$, and pick $(D, h, B) \in \mathcal{PT}$ where $h \in \mathbf{PT}_{DB}$ is defined as follows:

$$\operatorname{dom} h = (\operatorname{dom} f)g = \{xg : x \in \operatorname{dom} f\} \quad \text{and} \quad (xg)h = xf \text{ for } x \in \operatorname{dom} f.$$

Clearly, $\operatorname{dom} f \subseteq \operatorname{dom} g$ guarantees that the domain is well defined, and $\ker f \supseteq (\ker g)\restriction_{\operatorname{dom} f}$ guarantees the same for h. Let $E = D$ and $J = B$. Now, (3.2) is easy to show.

(ii) Obviously, $(A, f, B) \leq_{\mathscr{L}} (C, g, D)$ means that

$$(A, f, B) = (E, h, J)(C, g, D) \tag{3.3}$$

for some $(E, h, J) \in \mathcal{PT}$, which implies $D = B$ and $f = hg$, so $\operatorname{im} f \subseteq \operatorname{im} g$.

To prove the reverse implication, suppose $B = D$ and $\operatorname{im} f \subseteq \operatorname{im} g$, and write $f = \binom{F_i}{f_i}_{i \in I}$. Next, for each $i \in I$ choose an element $g_i \in (f_i)g^{-1}$ (such an element exists, since $\operatorname{im} f \subseteq \operatorname{im} g$) and let $h \in \mathbf{PT}_{AC}$ be defined with $h = \binom{F_i}{g_i}_{i \in I}$. Again, for $E = A$ and $J = C$, one may easily check (3.3).

(iii) If we suppose $(A, f, B) \leq_{\mathscr{J}} (C, g, D)$, we have

$$(A, f, B) = (E, h, J)(C, g, D)(K, q, L) \tag{3.4}$$

for some $(E, h, J), (K, q, L) \in \mathcal{PT}$, which implies $f = hgq$, so $\operatorname{Rank} f \leq \operatorname{Rank} g$ by Lemma 3.1.1(iv).

Conversely, suppose $\operatorname{Rank} f \leq \operatorname{Rank} g$ and write $f = \binom{F_i}{f_i}_{i \in I}$ and $g = \binom{G_t}{g_t}_{t \in T}$. Since $|I| \leq |T|$, we may assume $I \subseteq T$ without loss of generality. Now, for each $i \in I \subseteq T$ choose and fix an $e_i \in G_i$. Let us define $h \in \mathbf{PT}_{AC}$ and $q \in \mathbf{PT}_{DB}$ with $h = \binom{F_i}{e_i}_{i \in I}$ and $q = \binom{g_i}{f_i}_{i \in I}$. If we let $E = A$, $J = C$, $K = D$, and $L = B$, the equality (3.4) is easily shown.

(iv) follows directly from (i), noting that $\ker f = (\ker f)\restriction_{\operatorname{dom} f}$.

(v) follows directly from (ii).

(vi) Obviously, (iii) implies $(A, f, B) \mathscr{J} (C, g, D) \Leftrightarrow \operatorname{Rank} f = \operatorname{Rank} g$. Since $\mathscr{D} \subseteq \mathscr{J}$ (see Chapter 2), it suffices to show that

$$\operatorname{Rank} f = \operatorname{Rank} g \quad \Rightarrow \quad (A, f, B) \mathscr{D} (C, g, D).$$

Suppose $\operatorname{Rank} f = \operatorname{Rank} g$ and let $f = \binom{F_i}{f_i}_{i \in I}$ and $g = \binom{G_i}{g_i}_{i \in I}$. If we define

$h \in \mathbf{PT}_{AD}$ with $h = \binom{F_i}{g_i}_{i \in I}$, then (iv) and (v) imply

$$(A, f, B)\,\mathscr{R}(A, h, D)\,\mathscr{L}(C, g, D),$$

so $(A, f, B)\,\mathscr{D}(C, g, D)$. □

For any cardinal μ let D_μ denote the $\mathscr{J} = \mathscr{D}$-class of $\mathcal{PT}$ containing all partial maps of rank μ.

Now, we turn our attention to the sets of form

$$\mathcal{PT}_{AB} = \{(A, f, B) : f \in \mathbf{PT}_{AB}\}, \quad \text{for} \quad A, B \in \mathbf{Set}\,.$$

These are the underlying sets for our sandwich semigroups, so the interest in their properties is justified. First, we investigate the structure of $\mathcal{PT}_{AB}$ arising from Green's relations of $\mathcal{PT}$, restricted to $\mathcal{PT}_{AB}$. As in Chapter 2, these intersections will be called Green's relations of $\mathcal{PT}_{AB}$ (each $\mathscr{K}$-class of $\mathcal{PT}_{AB}$ corresponding to the $\mathscr{K}$-class of $\mathcal{PT}$ containing it), and we denote the restriction of the relation $\leq_{\mathscr{J}}$ on $\mathcal{PT}_{AB}$ also by $\leq_{\mathscr{J}}$. From the previous result we may draw the following conclusion:

Corollary 3.1.3. *Let $A, B \in \mathbf{Set}$. The $\mathscr{J} = \mathscr{D}$-classes of $\mathcal{PT}_{AB}$ are the sets*

$$\mathrm{D}^{AB}_\mu = \mathrm{D}_\mu \cap \mathcal{PT}_{AB} = \{(A, f, B) : f \in \mathbf{PT}_{AB},\ \mathrm{Rank}\, f = \mu\},$$

for each cardinal $0 \leq \mu \leq \min(|A|, |B|)$. These $\mathscr{J}$-classes form a chain in $\mathcal{PT}_{AB}$: $\mathrm{D}^{AB}_\mu \leq_{\mathscr{J}} \mathrm{D}^{AB}_\nu \Leftrightarrow \mu \leq \nu$.

Our next task is to describe D^{AB}_μ in terms of the number and the sizes of the Green's classes it contains. In order to do this, we introduce some additional notation. For background on basic cardinal arithmetic, the reader is referred to Chapter 5 in [62].

For $n, k \in \mathbb{N}_0$, the *Stirling number of the second kind* $\mathcal{S}(n, k)$ is the number of ways of partitioning a set of n elements into k non-empty sets (blocks). It can be calculated via the recurrence relation

$$\begin{aligned} &\mathcal{S}(n, k) = \mathcal{S}(n-1, k-1) + k\,\mathcal{S}(n-1, k), \quad \text{for } k > 0, n > 0,\\ &\mathcal{S}(0, 0) = 1, \quad \mathcal{S}(n, 0) = \mathcal{S}(0, n) = 0, \quad \text{for } n > 0, \end{aligned}$$

as well as via the formula

$$\mathcal{S}(n, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

Let κ, μ be cardinals with $\mu \leq \kappa$. In the following, we identify a κ- or μ-element set with the corresponding cardinal. Then

$\kappa!$ denotes the size of the symmetric group over a set of size κ. When κ is finite, this is the ordinary factorial; otherwise, it equals 2^κ, by [26].

$\binom{\kappa}{\mu}$ denotes the number of μ-element subsets of a κ-element set. If κ is finite, this is obviously the ordinary binomial coefficient. Otherwise, it equals κ^μ. Let us explain the latter. First, suppose $\mu < \kappa$. Choosing the elements one by one, we arrive at $\kappa^\mu = \kappa$ possibilities, since removing μ elements from a κ-element set leaves another κ elements in it. Thus, κ is an upper bound because we have counted each combination $\mu!$ times. It is also a lower bound, since there are κ ways to choose a singleton subset of κ, and the number of μ-element subsets cannot be smaller than that. Now, suppose $\mu = \kappa$. Note that there are 2^κ subsets of κ. The discussion of the previous case implies that there are in total κ subsets of cardinalities smaller than κ (the empty set, and κ sets of size ν, for each $\nu < \kappa$). Thus, there must be $2^\kappa (= \kappa^\mu)$ of those of size κ.

$\mathcal{S}(\kappa, \mu)$ denotes the number of ways to partition a κ-element set into μ blocks. Let κ be finite; by definition, $\mathcal{S}(\kappa, \mu)$ is the Stirling number of the second kind. Otherwise, $\mathcal{S}(\kappa, 1) = 1$ and $\mathcal{S}(\kappa, \mu) = 2^\kappa$ for $\mu \geq 2$. Let us elaborate on the latter. Recall that κ has 2^κ subsets, all of which (except $\emptyset$) can be members of our partition. To get an upper bound for $\mathcal{S}(\kappa, \mu)$, we choose μ times from the set of all those subsets (not setting any conditions), arriving at $(2^\kappa)^\mu = 2^\kappa$ possibilities. We claim that this is a lower bound, as well. Note that we may count the number of μ-block partitions by choosing the set A_1 containing the element 1 first (by partitioning κ into 2 subsets), and then partitioning $\kappa \setminus A_1$ into $\mu - 1$ sets. Since κ can be partitioned into two blocks in 2^κ ways (each of 2^κ subsets determines a partition, in which case each partition is counted twice), it can be partitioned into μ subsets in at least as many ways.

In the case when $\kappa < \mu$, we define $\binom{\kappa}{\mu} = \mathcal{S}(\kappa, \mu) = 0$.

These terms help us describe the structure of D_μ^{AB}:

Corollary 3.1.4. *Let $A, B \in$ **Set**, write $\alpha = |A|$ and $\beta = |B|$ and fix some cardinal $0 \leq \mu \leq \min(\alpha, \beta)$. Then*

(i) $|\,\mathrm{D}_\mu^{AB} / \mathscr{R}\,| = \mathcal{S}(\alpha + 1, \mu + 1)$,

(ii) $|\,\mathrm{D}_\mu^{AB} / \mathscr{L}\,| = \binom{\beta}{\mu}$,

(iii) $|\,\mathrm{D}_\mu^{AB} / \mathscr{H}\,| = \binom{\beta}{\mu}\,\mathcal{S}(\alpha + 1, \mu + 1)$,

(iv) each $\mathscr{H}$-class in D_μ^{AB} has size $\mu!$,

(v) $|\,\mathrm{D}_\mu^{AB}\,| = \mu!\binom{\beta}{\mu}\,\mathcal{S}(\alpha + 1, \mu + 1)$.

Proof. (*i*) Proposition 3.1.2(*iv*) implies that the $\mathscr{R}$-class of an element of $\mathcal{PT}_{AB}$ is determined by its kernel and domain. Thus, the number of $\mathscr{R}$-classes in D_μ^{AB} is the number of valid (i.e. possible) domain-kernel pairs in A, where the kernel has exactly μ classes. Hence, we need the number of partitions of the set A into $\mu + 1$ blocks, one (special) block being the non-mapping part of A (which might be empty), and the rest defining the kernel. We calculate this by adding a special element (∞, for example) to the set A, which will determine the non-mapping block by belonging to it. Therefore, the number of such choices is $\mathcal{S}(\alpha+1, \mu+1)$.

(ii) By 3.1.2(v), the $\mathscr{L}$-class of an element of $\mathcal{PT}_{AB}$ is determined by its image. The possible number of images of rank μ in B is $\binom{|B|}{\mu}$.

(iii) follows directly from (i) and (ii).

(iv) In a fixed $\mathscr{H}$-class H of D^{μ}_{AB}, both the kernel and the image of its elements are fixed. Hence, the number of maps in H equals the number of ways to connect the μ classes of the kernel with the μ elements of the image, which is $\mu!$.

(v) is a direct consequence of (iii) and (iv). □

Remark 3.1.5. We may calculate the size of $\mathcal{PT}_{AB}$ in two ways. Firstly, each of the α elements of A can either map into an element of B, or be outside of the domain. Secondly, we may sum the sizes of D^{AB}_{μ} for each possible rank $0 \leq \mu \leq \min(\alpha, \beta)$. Therefore, we have

$$|\mathcal{PT}_{AB}| = (\beta+1)^{\alpha} = \sum_{\mu=0}^{\min(\alpha,\beta)} \mu! \binom{\beta}{\mu} \mathcal{S}(\alpha+1, \mu+1).$$

Before we dive into the examination of sandwich semigroups in $\mathcal{PT}$, we need to explore stability of its elements. Using parts (i) and (ii) of Lemma 2.2.27, we may prove an element is stable, provided that we had already proved that certain semigroups are periodic. Thus, we define a suitable type of semigroups and prove them to be periodic. For $X \in \mathbf{Set}$, the set of all finite-rank elements of $\mathbf{PT}_X$ is denoted

$$\mathbf{PT}^{\mathrm{fr}}_X = \{f \in \mathbf{PT}_X : \mathrm{Rank}\, f < \aleph_0\}.$$

Lemma 3.1.1(iv) implies that $\mathbf{PT}^{\mathrm{fr}}_X$ is a subsemigroup of $\mathcal{PT}_X$. Moreover,

Lemma 3.1.6. $\mathbf{PT}^{\mathrm{fr}}_X$ *is a periodic semigroup for every* $X \in \mathbf{Set}$.

Proof. Let $X \in \mathbf{Set}$ and $f \in \mathbf{PT}^{\mathrm{fr}}_X$. By the definition of periodic semigroups, we need to show that f has a power which is an idempotent. Consider the sequence $\mathrm{Rank}\, f, \mathrm{Rank}\, f^2, \mathrm{Rank}\, f^3, \ldots$ It is non-increasing, by Lemma 3.1.1(iv). Moreover, since $\mathrm{Rank}\, f$ is finite, it is a non-increasing sequence of integers. Therefore, it must eventually become constant. Let k be an integer such that $\mathrm{Rank}\, f^k = \mathrm{Rank}\, f^{k+m}$ for all $m \geq 0$, and let $f = \binom{F_i}{f_i}_{i \in I}$. Since $\mathrm{im}\, f^t \subseteq \mathrm{im}\, f^{t+1}$ for all $t \geq 1$, we may conclude that, for any $m \geq 0$ we have $f^{k+m} = \binom{F_i}{f_{i\pi}}_{i \in I}$ for some permutation π of the set $\mathrm{im}\, f$. As there exist a finite number of these permutations, the set $\{f^{k+m} : m \geq 0\}$ is also finite. In fact, when paired with composition, it is the underlying set of a finite semigroup. Such a semigroup has an idempotent, as we proved in Section 1.3. □

Now, we have the base for the following result, in which we state equivalent conditions for an element of $\mathcal{PT}$ to be $\mathscr{R}$-stable, $\mathscr{L}$-stable or stable.

Proposition 3.1.7. *If* $(A, f, B) \in \mathcal{PT}$, *then* *(i)* (A,f,B) *is* $\mathscr{R}$*-stable* $\Leftrightarrow$ [$\operatorname{Rank} f < \aleph_0$ *or* f *is full and injective*],

(ii) (A,f,B) *is* $\mathscr{L}$*-stable* $\Leftrightarrow$ [$\operatorname{Rank} f < \aleph_0$ *or* f *is surjective*],

(iii) (A,f,B) *is stable* $\Leftrightarrow$ [$\operatorname{Rank} f < \aleph_0$ *or* f *is full and bijective*].

Proof. First, suppose $\operatorname{Rank} f < \aleph_0$. By Lemma 2.2.27$(i)$ and (ii), stability of f follows if we prove that the elements of the sets $(A,f,B)\mathcal{PT}_{BA}$ and $\mathcal{PT}_{BA}(A,f,B)$ are all periodic. Since f is a map of finite rank, Lemma 3.1.1(iv) implies

$$(A,f,B)\mathcal{PT}_{BA} \subseteq \{(A,g,A) : g \in \mathbf{PT}_A^{\mathrm{fr}})\},$$
$$\mathcal{PT}_{BA}(A,f,B) \subseteq \{(B,g,B) : g \in \mathbf{PT}_B^{\mathrm{fr}})\}.$$

Seeing that $\mathbf{PT}_A^{\mathrm{fr}}$ and $\mathbf{PT}_B^{\mathrm{fr}}$ are both periodic (by Lemma 3.1.6), we may conclude that all elements of $(A,f,B)\mathcal{PT}_{BA}$ and $\mathcal{PT}_{BA}(A,f,B)$ are periodic. Thus, we showed that (A,f,B) is stable, as required.

Next, suppose that f is full and injective. To prove that (A,f,B) is $\mathscr{R}$-stable, we need to show

$$(C,g,D)(A,f,B)\,\mathscr{J}\,(C,g,D) \Rightarrow (C,g,D)(A,f,B)\,\mathscr{R}(C,g,D) \tag{3.5}$$

for all $(C,g,D) \in \mathcal{PT}$. The implication trivially holds if the product is undefined. Thus, suppose $D = A$ and note that, since f is full and injective, parts (i) and (iii) of Lemma 3.1.1 imply $\operatorname{dom}(gf) = \operatorname{dom} g$ and $\ker(gf) = (\ker g)\restriction_{\operatorname{dom}(gf)} = (\ker g)\restriction_{\operatorname{dom} g} = \ker g$, respectively. Therefore, by Proposition 3.1.2(iv), we have

$$(C,g,D)(A,f,B)\,\mathscr{R}(C,g,D)$$

whenever the product $(C,g,D)(A,f,B)$ is defined. Hence, the implication (3.5) is true in all cases.

Similarly, if we suppose that f is surjective, Lemma 3.1.1(ii) implies $\operatorname{im}(fg) = \operatorname{im} g$. Hence,

$$(A,f,B)(C,g,D)\,\mathscr{L}(C,g,D),$$

whenever the product $(A,f,B)(C,g,D)$ is defined, implying $\mathscr{L}$-stability of (A,f,B).

Clearly, if we assume that f is full and bijective, it is both $\mathscr{R}$- and $\mathscr{L}$-stable, by the previous two arguments. Thus, we have established the reverse implications in all three statements of the proposition. We need to show the direct implications, as well. In both (i) and (ii), we prove the contrapositive. Once we prove these two, part (iii) immediately follows.

To show (i), suppose that f is a map of infinite rank, either non-full or non-injective. Write $f = \binom{F_i}{f_i}_{i\in I}$, and choose an element $g_i \in F_i$ for each $i \in I$. If f is non-full, fix some $a \in A \setminus \operatorname{dom} f$ and define $g = \left(\begin{smallmatrix} g_i & a \\ g_i & a \end{smallmatrix}\right)_{i\in I} \in \mathbf{PT}_A$. It is easy to see that $\operatorname{dom}(gf) \neq \operatorname{dom} g$, but $\operatorname{Rank}(gf) = \operatorname{Rank} g - 1 = \operatorname{Rank} g$. Then Proposition 3.1.2$(vi)$ and (iv) imply that $(A,g,A)\cdot(A,f,B)$ and (A,g,A) are $\mathscr{J}$-related, but not $\mathscr{R}$-related. Hence, (A,f,B) is not $\mathscr{R}$-stable. In the case where f is non-injective, choose some F_i with $|F_i| \geq 2$, and fix $a \in F_i$ such that $a \neq g_i$. For the map $g =$

$\binom{g_j\ a}{g_j\ a}_{j\in I} \in \mathbf{PT}_A$, we have $\ker(gf) \neq \ker g$ and $\operatorname{Rank}(gf) = \operatorname{Rank} g - 1 = \operatorname{Rank} g$. Thus, $(A, g, A) \cdot (A, f, B)$ and (A, g, A) are $\mathscr{J}$-related, but not $\mathscr{R}$-related, just as in the previous case.

Finally, we show (ii). Let f be a non-surjective map with infinite rank. Keep the notation $f = \binom{F_i}{f_i}_{i\in I}$. Pick an element $a \in B \setminus \operatorname{im} f$, and let $g = \binom{f_i\ a}{f_i\ a}_{i\in I} \in \mathbf{PT}_B$. Then $\operatorname{im}(fg) \neq \operatorname{im} g$, but $\operatorname{Rank}(fg) = \operatorname{Rank} g$, which means that $(A, f, B) \cdot (B, g, B)$ and (B, g, B) are $\mathscr{J}$-related, but not $\mathscr{L}$-related (by Proposition 3.1.2(vi) and (v)). Thus, (A, f, B) is not $\mathscr{L}$-stable. $\square$

3.1.1 Green's relations, regularity and stability in $\mathcal{PT}^a_{XY}$

Having acquired the necessary knowledge on $\mathcal{PT}$, we are ready to investigate sandwich semigroups of partial functions. This subsection is dedicated to the description of Green's relations, regularity and stability in such a sandwich semigroup. These are the three most important factors, which determine the structure of the sandwich semigroup. For the rest of the section, we fix some $X, Y \in \mathbf{Set}$ and a partial map $a \in \mathbf{PT}_{YX}$, with the purpose of investigating $\mathcal{PT}^a_{XY}$. We will frequently refer to characteristics of a, so we write

$$a = \binom{A_i}{a_i}_{i\in I}, \quad B = \operatorname{dom} a, \quad \sigma = \ker a, \quad A = \operatorname{im} a, \quad \alpha = \operatorname{Rank} a.$$
$$\beta = |X \setminus \operatorname{im} a|, \qquad \lambda_i = |A_i| \ \text{ for } i \in I, \qquad \Lambda_J = \prod_{j\in J} \lambda_j \ \text{ for } J \subseteq I.$$

Also, we will often need an inverse element of a, so we fix $b_i \in A_i$ for each $i \in I$, and define $b = \binom{a_i}{b_i}_{i\in I} \in \mathbf{PT}_{XY}$ (so that $a = aba$ and $b = bab$).

Furthermore, in order to simplify the notation, we identify the partial function $f \in \mathbf{PT}_{CD}$ with the corresponding element (C, f, D) of $\mathcal{PT}$. This makes one of the $\mathbf{PT}_{CD}$ and $\mathcal{PT}_{CD}$ redundant, so we use $\mathcal{PT}_{CD}$ in both cases.

From Theorem 2.2.3 we see that, in order to describe Green's relations in $\mathcal{PT}^a_{XY}$, we need to describe its P-sets first.

Proposition 3.1.8. *We have*

(i) $\mathrm{P}^a_1 = \{f \in \mathcal{PT}_{XY} \ : \ \operatorname{dom}(fa) = \operatorname{dom} f,\ \ker(fa) = \ker f\}$
$\quad = \{f \in \mathcal{PT}_{XY} \ : \ \operatorname{im} f \subseteq \operatorname{dom} a,\ \ker a \textit{ separates } \operatorname{im} f\}$,

(ii) $\mathrm{P}^a_2 = \{f \in \mathcal{PT}_{XY} \ : \ \operatorname{im}(af) = \operatorname{im} f\}$
$\quad = \{f \in \mathcal{PT}_{XY} \ : \ \operatorname{im} a \textit{ saturates } \ker f\}$,

(iii) $\mathrm{P}^a = \{f \in \mathcal{PT}_{XY} \ : \ \operatorname{dom}(fa) = \operatorname{dom} f,$
$\quad \ker(fa) = \ker f,\ \operatorname{im}(af) = \operatorname{im} f\}$
$\quad = \{f \in \mathcal{PT}_{XY} \ : \ \operatorname{im} f \subseteq \operatorname{dom} a,$
$\quad \ker a \textit{ separates } \operatorname{im} f,\ \operatorname{im} a \textit{ saturates } \ker f\}$,

(iv) $\mathrm{P}^a_3 = \{f \in \mathcal{PT}_{XY} \ : \ \operatorname{Rank}(afa) = \operatorname{Rank} f\}$.

Proof. Note that the first line in all four statements follows from the definition of P-sets and Proposition 3.1.2, and the second line (if exists) follows from Lemma 3.1.1. □

Remark 3.1.9. In cases where a has some special properties, these conditions simplify significantly. In particular, if a is full, $\operatorname{im} f \subseteq \operatorname{dom} a$ is trivially true; if a is injective, $\ker a$ always separates $\operatorname{im} f$; finally, if a is surjective, $\operatorname{im} a$ clearly saturates $\ker f$.

Note that, in the case when a is a full bijection, we clearly have $\mathrm{P}_1^a = \mathrm{P}_2^a = \mathrm{P}^a = \mathcal{PT}_{XY}$, and moreover $\mathcal{PT}_{XY}^a \cong \mathcal{PT}_X \cong \mathcal{PT}_Y$ (as the maps $\mathcal{PT}_{XY}^a \to \mathcal{PT}_X : x \mapsto xa$ and $\mathcal{PT}_{XY}^a \to \mathcal{PT}_Y : x \mapsto ax$ are clearly isomorphisms).

Having described Green's relations of $\mathcal{PT}$ (Proposition 3.1.2) and P-sets in $\mathcal{PT}_{XY}^a$ (the previous proposition), we may use Theorem 2.2.3 to describe Green's relations in $\mathcal{PT}_{XY}^a$. This result originally appeared in [96] (as Theorems 2.6, 2.7 and 2.8), although in a different form.

Theorem 3.1.10. *If $f \in \mathcal{PT}_{XY}$, then in $\mathcal{PT}_{XY}^a$ we have*

(i) $$\mathrm{R}_f^a = \begin{cases} \mathrm{R}_f \cap \mathrm{P}_1^a, & f \in \mathrm{P}_1^a; \\ \{f\}, & f \notin \mathrm{P}_1^a. \end{cases}$$

(ii) $$\mathrm{L}_f^a = \begin{cases} \mathrm{L}_f \cap \mathrm{P}_2^a, & f \in \mathrm{P}_2^a; \\ \{f\}, & f \notin \mathrm{P}_2^a. \end{cases}$$

(iii) $$\mathrm{H}_f^a = \begin{cases} \mathrm{H}_f, & f \in \mathrm{P}^a; \\ \{f\}, & f \notin \mathrm{P}^a. \end{cases}$$

(iv) $$\mathrm{D}_f^a = \begin{cases} \mathrm{D}_f \cap \mathrm{P}^a, & f \in \mathrm{P}^a; \\ \mathrm{L}_f^a, & f \in \mathrm{P}_2^a \setminus \mathrm{P}_1^a; \\ \mathrm{R}_f^a, & f \in \mathrm{P}_1^a \setminus \mathrm{P}_2^a; \\ \{f\}, & f \notin (\mathrm{P}_1^a \cup \mathrm{P}_2^a). \end{cases}$$

(v) $$\mathrm{J}_f^a = \begin{cases} \mathrm{J}_f \cap \mathrm{P}_3^a \; (= \mathrm{D}_f \cap \mathrm{P}_3^a), & f \in \mathrm{P}_3^a; \\ \mathrm{D}_f^a, & f \notin \mathrm{P}_3^a. \end{cases}$$

Further, if $f \notin \mathrm{P}^a$, then $\mathrm{H}_f^a = \{f\}$ is a non-group $\mathscr{H}^a$-class in $\mathcal{PT}_{XY}^a$.

Remark 3.1.11. The reader may inspect Figures 3.4–3.8 for some examples of sandwich semigroups of the form $\mathcal{PT}_{XY}^a$, presented in the form of egg-box diagrams. Since all of the sandwich semigroups in our examples are finite, in each of them we have $\mathscr{J}^a = \mathscr{D}^a$, so the diagrams give a clear picture of the $\leq_{\mathscr{J}}$-structure and the $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes (for details, see the introduction to Subsection 3.1.6).

Our next topic will be the structure of $\mathscr{J}^a$-classes in $\mathcal{PT}_{XY}^a$. In particular, we will investigate in which cases they coincide with $\mathscr{D}^a$-classes, the partial order $\leq_{\mathscr{J}^a}$, and the maximal $\mathscr{J}^a$-classes with respect to this order. In order to conduct this investigation, we need to examine P_3^a and its connections to the other P-sets.

Lemma 3.1.12. *Suppose μ is a cardinal with $\aleph_0 \le \mu \le \alpha = \operatorname{Rank} a$.*

(i) If a is not $\mathscr{R}$-stable, then there exists $f \in \mathrm{P}_3^a \setminus \mathrm{P}_1^a$ with $\operatorname{Rank} f = \mu$.

(ii) If a is not $\mathscr{L}$-stable, then there exists $f \in \mathrm{P}_3^a \setminus \mathrm{P}_2^a$ with $\operatorname{Rank} f = \mu$.

(iii) If a is not stable, then there exists $f \in \mathrm{P}_3^a \setminus \mathrm{P}^a$ with $\operatorname{Rank} f = \mu$.

Proof. We prove (i) and (ii) by constructing such maps. Then (iii) follows directly, since $\mathrm{P}_3^a \setminus \mathrm{P}^a \supseteq \mathrm{P}_3^a \setminus \mathrm{P}_q^a$, for $q = 1, 2$. Note also that, as $\alpha = |I| = \operatorname{Rank} a$ is infinite, there exists a set $J \subsetneq I$ such that $|J| = \mu$. Hence, there also exists an index $k \in I \setminus J$.

(i) Suppose a is not $\mathscr{R}$-stable. Since $a \notin \mathcal{PT}_{YX}^{\mathrm{fr}}$, Proposition 3.1.7$(i)$ implies that a is either non-full, or non-injective. In the first case, choose some $y \in Y \setminus \operatorname{dom} a$ and let $f = \binom{a_k\ a_j}{y\ \ b_j}_{j \in J} \in \mathcal{PT}_{XY}$. Clearly, $\operatorname{Rank}(afa) = \operatorname{Rank} f - 1 = \operatorname{Rank} f$. However, $a_k \in \operatorname{dom} f \setminus \operatorname{dom}(fa)$, so Proposition 3.1.8 implies $f \in \mathrm{P}_3^a \setminus \mathrm{P}_1^a$. Similarly, in the case that f is non-injective, there exists $i \in I$ with $|A_i| \ge 2$, and we may assume $i \in J$ without loss of generality. Thus, we may pick $y \in A_i \setminus \{b_i\}$ and define $f = \binom{a_k\ a_j}{y\ \ b_j}_{j \in J} \in \mathcal{PT}_{XY}$. Again, $\operatorname{Rank}(afa) = \operatorname{Rank} f - 1 = \operatorname{Rank} f$, but this time $\ker f \ne \ker(fa)$. These two together imply $f \in \mathrm{P}_3^a \setminus \mathrm{P}_1^a$.

(ii) Suppose a is not $\mathscr{L}$-stable. As in (i), from Proposition 3.1.7(ii) it follows that a is non-surjective. Hence, there exists some $y \in X \setminus \operatorname{im} a$, and we may define $f = \binom{y\ \ a_j}{b_k\ b_j}_{j \in J} \in \mathbf{PT}_{XY}$. We have $\operatorname{Rank}(afa) = \operatorname{Rank} f - 1 = \operatorname{Rank} f$ and $b_k \in \operatorname{im} f \setminus \operatorname{im}(af)$, so Proposition 3.1.8 gives $f \in \mathrm{P}_3^a \setminus \mathrm{P}_2^a$. □

Using the previous lemma, we are able to prove:

Proposition 3.1.13. *In $\mathcal{PT}_{XY}^a$ we have $\mathscr{J}^a = \mathscr{D}^a \Leftrightarrow a$ is stable.*

Proof. The reverse implication follows immediately from Proposition 2.2.25 and Proposition 3.1.2(vi). We show the direct one by proving the contrapositive. Suppose a is not stable. We will show $\mathscr{J}^a \ne \mathscr{D}^a$. More precisely, we are going to prove that $\mathrm{J}_b^a \ne \mathrm{D}_b^a$. Since b is an inverse of a, we have $b \in \mathrm{P}^a \subseteq \mathrm{P}_3^a$, so Theorem 3.1.10$(iv)$ and (v) imply

$$\mathrm{D}_b^a = \mathrm{D}_b \cap \mathrm{P}^a = \mathrm{J}_b \cap \mathrm{P}^a \quad \text{and} \quad \mathrm{J}_b^a = \mathrm{J}_b \cap \mathrm{P}_3^a .$$

Now, Lemma 3.1.12(iii) guarantees the existence of a map $f \in \mathrm{P}_3^a \setminus \mathrm{P}^a$ with $\operatorname{Rank} f = \operatorname{Rank} b = \operatorname{Rank} a$. Therefore $f \in \mathrm{J}_b$, by Proposition 3.1.2(vi), so $f \in \mathrm{J}_b^a \setminus \mathrm{D}_b^a$. □

Exploiting Lemma 3.1.12 further, we use it as the base for proving equivalent conditions for the sandwich element of $\mathcal{PT}_{XY}^a$ to be $\mathscr{R}$- or $\mathscr{L}$-stable. The following proposition is a strengthened version of Proposition 2.2.23, tailored to $\mathcal{PT}_{XY}^a$. The first statement in an old result (it is implied by Theorem 5.3 in [86]).

Proposition 3.1.14. *We have $\operatorname{Reg}(\mathcal{PT}_{XY}^a) = \mathrm{P}^a$. Moreover,*

(i) a is $\mathscr{R}$-stable $\Leftrightarrow \mathrm{P}_3^a \subseteq \mathrm{P}_1^a$,

(ii) a is $\mathscr{L}$-stable $\Leftrightarrow \mathrm{P}_3^a \subseteq \mathrm{P}_2^a$,

(iii) a *is stable* $\Leftrightarrow \mathrm{P}_3^a = \mathrm{P}^a$.

Proof. Since $\mathcal{PT}$ is a regular partial semigroup, Proposition 2.2.29(iv) implies the first statement. Furthermore, the direct implications in (i) – (iii) follow from Proposition 2.2.23. We prove the converse for (i) and (ii), hence (iii) follows as a direct consequence (because $\mathrm{P}^a = \mathrm{P}_1^a \cap \mathrm{P}_2^a$, and $\mathrm{P}^a \subseteq \mathrm{P}_3^a$ by 2.2.2). In fact, it suffices to prove (i), as the proof for (ii) is dual.

(i) We show this by proving the contrapositive. Suppose a is not $\mathscr{R}$-stable. By Proposition 3.1.7(i), $\operatorname{Rank} a \geq \aleph_0$, so Lemma 3.1.12($i$) implies the existence of $f \in \mathrm{P}_3^a \setminus \mathrm{P}_1^a$. □

Now, we focus on the relation $\leq_{\mathscr{J}^a}$, as promised. To simplify notation, we use the symbol $\leq$ instead. There is no chance of confusion, as it is the only relation we defined on the $\mathscr{J}$-classes of a semigroup.

Recall that any element of $\mathcal{PT}$ has a left- and a right-identity in $\mathcal{PT}$ (page 83). Thus, directly from Lemma 2.2.6(iii) and Proposition 3.1.2, we conclude the following:

Proposition 3.1.15. *Let* $f, g \in \mathcal{PT}_{XY}$. *Then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a$ *in* $\mathcal{PT}_{XY}^a$ *if and only if one of the following holds:*

(i) $f = g$,

(ii) $\operatorname{Rank} f \leq \operatorname{Rank}(aga)$,

(iii) $\operatorname{im} f \subseteq \operatorname{im}(ag)$,

(iv) $\operatorname{dom} f \subseteq \operatorname{dom}(ga)$ *and* $\ker f \supseteq (\ker(ga))\restriction_{\operatorname{dom} f}$.

Additionally, from Propositions 2.2.7 and 3.1.2 we immediately obtain

Proposition 3.1.16. *Let* $f, g \in \mathcal{PT}_{XY}$.

(i) If $f \in \mathrm{P}_1^a$, *then*

$$\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\operatorname{Rank} f \leq \operatorname{Rank}(aga) \text{ or } [[\operatorname{dom} f \subseteq \operatorname{dom}(ga) \text{ and } \ker f \supseteq (\ker(ga))\restriction_{\operatorname{dom} f}]].$$

(ii) If $f \in \mathrm{P}_2^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\operatorname{Rank} f \leq \operatorname{Rank}(aga)$ *or* $\operatorname{im} f \subseteq \operatorname{im}(ag)]$.

(iii) If $f \in \mathrm{P}_3^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow \operatorname{Rank} f \leq \operatorname{Rank}(aga)$.

(iv) If $g \in \mathrm{P}_1^a$, *then*

$$\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\operatorname{Rank} f \leq \operatorname{Rank}(ag) \text{ or } [\operatorname{dom} f \subseteq \operatorname{dom} g \text{ and } \ker f \supseteq (\ker g)\restriction_{\operatorname{dom} f}]].$$

(v) If $g \in \mathrm{P}_2^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\operatorname{Rank} f \leq \operatorname{Rank}(ga)$ *or* $\operatorname{im} f \subseteq \operatorname{im} g]$.

(vi) If $g \in \mathrm{P}_3^a$*, then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \;\Leftrightarrow\; \operatorname{Rank} f \leq \operatorname{Rank} g$.

Remark 3.1.17. From Proposition 2.2.2(i) we have $\mathrm{P}^a \subseteq \mathrm{P}_3^a$, so the parts ($iii$) and ($vi$) apply to elements of P^a, as well.

Recall from Section 1.3 that a $\mathscr{D}$-class (of a semigroup) is either regular in its entirety, or none of its elements are regular. Using the previous results, we are able to identify and describe all the regular $\mathscr{D}^a$-classes in $\mathcal{PT}_{XY}^a$.

Proposition 3.1.18. *The regular* $\mathscr{D}^a$*-classes of* $\mathcal{PT}_{XY}^a$ *are precisely the sets*

$$\mathrm{D}_\mu^a = \{f \in \mathrm{P}^a : \operatorname{Rank} f = \mu\}, \quad \textit{for each cardinal } 0 \leq \mu \leq \alpha = \operatorname{Rank} a.$$

Further, if $f \in \mathrm{P}^a$*, then* $\mathrm{D}_f^a = \mathrm{J}_f^a$ *if and only if* $\operatorname{Rank} f < \aleph_0$ *or* a *is stable.*

Proof. First, we prove that all the regular $\mathscr{D}^a$-classes are of the given form. Let $f \in \mathrm{P}^a$. Since $\mathrm{P}^a \subseteq \mathrm{P}_3^a$, Theorem 3.1.10 and Proposition 3.1.2(vi) give $\mathrm{D}_f^a = \mathrm{D}_f \cap \mathrm{P}^a = \mathrm{J}_f \cap \mathrm{P}^a = \mathrm{D}_\mu^a$, for $\mu = \operatorname{Rank} f$. Note that $\operatorname{Rank} f = \operatorname{Rank}(afa) \leq \operatorname{Rank} a$ (because $f \in \mathrm{P}^a$).

Next, for any cardinal $0 \leq \mu \leq \alpha = \operatorname{Rank} a$, we prove that D_μ^a is non-empty (by presenting a regular element of rank μ). Namely, for any set $J \subseteq I$ with $|J| = \mu$, the map $f_J = \binom{a_j}{b_j}_{j \in J}$ belongs to P^a and has $\operatorname{Rank} f_J = |J| = \mu$, so $f_J \in \mathrm{D}_\mu^a$.

Finally, we prove the last statement. For the direct implication, we show the contrapositive. Suppose that we have $f \in \mathrm{P}^a$ with $\operatorname{Rank} f \geq \aleph_0$ and suppose that a is not stable. Then, by Lemma 3.1.12(iii), there exists $g \in \mathrm{P}_3^a \setminus \mathrm{P}^a$ with $\operatorname{Rank} g = \operatorname{Rank} f$. Hence, $g \in \mathrm{J}_f^a$ (by Theorem 3.1.10(v) and Proposition 2.2.2(i)). However, $g \notin \mathrm{P}^a$, so $g \notin \mathrm{J}_f \cap \mathrm{P}^a = \mathrm{D}_f^a$, which gives $\mathrm{D}_f^a \neq \mathrm{J}_f^a$.

To prove the reverse implication, suppose that $f \in \mathrm{P}^a$ and either a is stable or $\operatorname{Rank} f < \aleph_0$. In the first case, Propositions 3.1.2(vi) and 2.2.25 guarantee $\mathrm{J}_f^a = \mathrm{D}_f^a$. In the second case, it suffices to show $\mathrm{J}_f^a \subseteq \mathrm{D}_f^a$. Since $f \in \mathrm{P}^a \subseteq \mathrm{P}_3^a$, parts ($iv$) and ($v$) of Theorem 3.1.10 give $\mathrm{J}_f^a = \mathrm{D}_f \cap \mathrm{P}_3^a$ and $\mathrm{D}_f^a = \mathrm{D}_f \cap \mathrm{P}^a$. Let $g \in \mathrm{D}_f \cap \mathrm{P}_3^a$. By Proposition 3.1.8($iv$) we have $\operatorname{Rank}(aga) = \operatorname{Rank} g$, which, together with

$$\operatorname{Rank}(aga) \leq \left\{ \begin{array}{c} \operatorname{Rank}(ag) \\ \operatorname{Rank}(ga) \end{array} \right\} \leq \operatorname{Rank} g,$$

gives $\operatorname{Rank}(ag) = \operatorname{Rank}(ga) = \operatorname{Rank} g = \operatorname{Rank} f$. Having in mind that $\operatorname{im}(ag) \subseteq \operatorname{im} g$, and $|\operatorname{im} g| = \operatorname{Rank} g = \operatorname{Rank}(ag) = |\operatorname{im}(ag)|$ is finite, we may conclude that $\operatorname{im}(ag) = \operatorname{im} g$. Thus, by Proposition 3.1.8(ii), $g \in \mathrm{P}_2^a$. Moreover, the equality $\operatorname{rank}(ga) = \operatorname{rank} g$ implies $\ker(ga) = \ker g$ and $\operatorname{dom}(ga) = \operatorname{dom} g$ (because $\operatorname{rank} g < \aleph_0$), so Proposition 3.1.8($i$) gives $g \in \mathrm{P}_1^a$. Therefore, $g \in \mathrm{J}_f \cap \mathrm{P}_1^a \cap \mathrm{P}_2^a = \mathrm{D}_f \cap \mathrm{P}^a = \mathrm{D}_f^a$, by Proposition 3.1.2($vi$). □

Expanding further on the results on the relation $\leq_{\mathscr{J}}$ gathered above, we can identify and characterise all the maximal $\mathscr{J}^a$-classes in $\mathcal{PT}_{XY}^a$. Due to this characterisation, we may easily deduce whether these classes are trivial (in terms of Lemma 2.2.10). We find that the form, type and number of these classes depend heavily on the rank of the sandwich element a. More precisely, they depend on the answer to the following question: Is $\alpha = \min(|X|, |Y|)$?

For convenience, we write $\xi = \min(|X|, |Y|)$.

Proposition 3.1.19.

(i) If $\alpha < \xi$, then the maximal $\mathscr{J}^a$-classes of $\mathcal{PT}^a_{XY}$ are precisely the singleton sets $\{f\}$, for $f \in \mathcal{PT}_{XY}$ with $\operatorname{Rank} f > \alpha$. Hence, all the maximal $\mathscr{J}^a$-classes of $\mathcal{PT}^a_{XY}$ are trivial in this case.

(ii) If $\alpha = \xi$, then we have a single maximum $\mathscr{J}^a$-class in $\mathcal{PT}^a_{XY}$, which is

$$\mathrm{J}^a_b = \{f \in \mathrm{P}^a_3 : \operatorname{Rank} f = \alpha\}.$$

This maximal $\mathscr{J}^a$-class is clearly nontrivial.

Proof. (i) Suppose that $\alpha < \xi$. Firstly, note that the singleton sets of the specified form are indeed maximal $\mathscr{J}^a$-classes. Namely, for any $f \in \mathcal{PT}_{XY}$ with $\operatorname{Rank} f > \alpha$, by Proposition 3.1.2$(iii)$ we have $f \not\leq_{\mathscr{J}} a$, so Lemma 2.2.10 implies that $\{f\}$ is a maximal $\mathscr{J}^a$-class. Thus, it suffices to prove that any maximal $\mathscr{J}^a$-class has the given form. Suppose there exists $g \in \mathrm{P}_{XY}$ with $\operatorname{Rank} g \leq \alpha$ such that J^a_g is a maximal $\mathscr{J}^a$-class. Write $g = \binom{G_j}{g_j}_{j\in J}$ with $|J| = \operatorname{Rank} g \leq \alpha$. Pick $h_1 = \binom{G_j}{b_j}_{j\in J} \in \mathcal{PT}_{XY}$ and $h_2 = \binom{a_j}{g_j}_{j\in J} \in \mathcal{PT}_{XY}$. Since $|J| \leq \alpha < \min(X, Y)$, there exists a map h_2' with $\operatorname{Rank} h_2' > \alpha$, extending h_2. We have $g = h_1 \star_a h_2'$, so $g \leq_{\mathscr{J}^a} h_2'$, i.e. $\mathrm{J}^a_g \leq \mathrm{J}^a_{h_2'}$. However, $\operatorname{Rank} h_2' > \alpha \geq \operatorname{Rank} g$ implies $\mathrm{J}^a_{h_2'} \neq \mathrm{J}^a_g$, so J^a_g is not a maximal $\mathscr{J}^a$-class, which contradicts our assumption.

(ii) Suppose $\alpha = \xi$. From Proposition 2.2.2(i) we have $b \in \mathrm{P}^a \subseteq \mathrm{P}^a_3$, so Theorem 3.1.10$(v)$ implies

$$\mathrm{J}^a_b = \mathrm{J}_b \cap \mathrm{P}^a_3 = \{f \in \mathrm{P}^a_3 : \operatorname{Rank} f = \alpha\}.$$

Furthermore, for any $g \in \mathcal{PT}_{XY}$ we have $\operatorname{Rank} g \leq \xi = \alpha = \operatorname{Rank} b$. Therefore, Proposition 3.1.16(vi) implies $\mathrm{J}^a_f \leq \mathrm{J}^a_g$. □

Remark 3.1.20. For a visual presentation, we refer the reader to the egg-box diagrams in Figures 3.4–3.8 and Figure 3.10. The Figures 3.4–3.6 represent the case when $\alpha < \xi$, and the Figures 3.7, 3.8 and 3.10 showcase some representatives of the case $\alpha = \xi$.

3.1.2 A structure theorem for $\operatorname{Reg}(\mathcal{PT}^a_{XY})$ and connections to (non-sandwich) semigroups of partial transformations

In this subsection, we examine the connections of $\mathcal{PT}^a_{XY}$ and $\operatorname{Reg}(\mathcal{PT}^a_{XY})$ to $(a\mathcal{PT}_{XY}a, \circledast)$. The idea is to apply the theory from Subsections 2.3.1 and 2.3.3 to the sandwich semigroup $\mathcal{PT}^a_{XY}$. First, we closely examine the semigroups from Diagrams 2.2 and 2.3 and give characterisations for them in the case of $S^a_{ij} = \mathcal{PT}^a_{XY}$. Furthermore, we describe the simplifications that can be made to the general theory in cases when a is full, injective or surjective. We close the subsection by describing $\mathrm{Reg}(\mathcal{PT}^a_{XY})$ as a pull-back product, using the results of Subsection 2.3.3.

We keep the previously introduced notation. In addition, recall that, in Subsection 2.3.1, we have dealt with the regular monoid $(aS_{ij}a, \circledast)$, where $aS_{ij}a \subseteq S_{ji}$ and $\circledast$ denotes the restriction of the map $\star_b$ to the set $aS_{ij}a$, which does not depend on the choice of the inverse b.

Note that all elements of $\mathcal{PT}$ are sandwich-regular since the whole partial semigroup is regular. Therefore, Diagrams 2.2 and 2.3, adjusted to the case of $\mathcal{PT}^a_{XY}$, are the following:

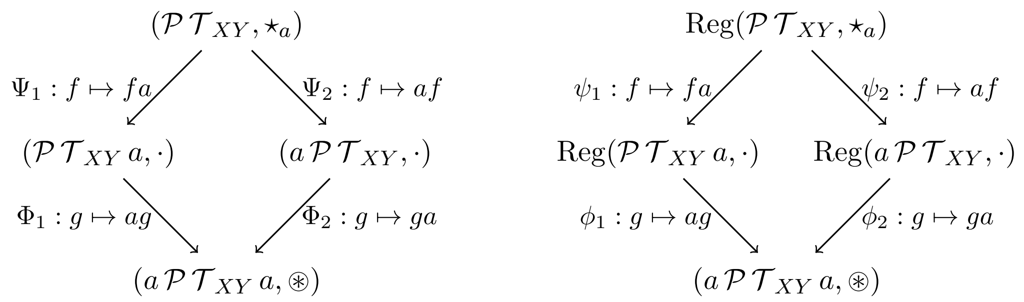


Figure 3.1: Diagrams illustrating the connections between $\mathcal{PT}^a_{XY}$ and $(a\,\mathcal{PT}_{XY}\, a, \circledast)$ (left) and between $\mathrm{Reg}(\mathcal{PT}^a_{XY})$ and $(a\,\mathcal{PT}_{XY}\, a, \circledast)$ (right).

We will examine the left diagram first, keeping in mind the results of Subsection 2.3.1. On the top, we have the semigroup $\mathcal{PT}^a_{XY}$, and in the bottom, the regular monoid $(a\,\mathcal{PT}_{XY}\, a, \circledast)$, which is a subsemigroup of $\mathcal{PT}^b_{YX}$. Recall from the discussion on page 50 that $(aS_{ij}a, \circledast) \to (baS_{ij}a, \cdot) : x \mapsto bx$ is an isomorphism, $S_{ij}a = S_i ba$, and $aS_{ij} = abS_j$. Hence, the map

$$\eta : (a\,\mathcal{PT}_{XY}\, a, \circledast) \to (ba\,\mathcal{PT}_{XY}\, a, \cdot) : x \mapsto bx$$

is an isomorphism, and $(ba\,\mathcal{PT}_{XY}\, a, \cdot) = (ba\,\mathcal{PT}_X\, ba, \cdot)$ is the local monoid of $\mathcal{PT}_X$ with respect to the idempotent $ba = \binom{a_i}{a_i}_{i\in I} \in \mathcal{PT}_X$. Moreover, since $\mathrm{dom}(ba) = \mathrm{im}(ba) = A$, we have $ba\,\mathcal{PT}_X\, ba \equiv \mathcal{PT}_A$ (because $f = bafba$ for any $f \in \{(X, g, X) : \mathrm{dom}(g), \mathrm{im}(g) \subseteq A\}$). Thus, $(a\,\mathcal{PT}_{XY}\, a, \circledast)$ is isomorphic to $\mathcal{PT}_A$, the semigroup of all partial transformations $A \to A$.

Now, we examine the sets in the middle. From $\mathrm{im}\, a = A$ we have $\mathrm{im}(fa) \subseteq A$ for any $f \in \mathcal{PT}_{XY}$. In fact, if we introduce the following notation

$$\mathcal{PT}(X, A) = \{f \in \mathcal{PT}_X \ : \ \mathrm{im}\, f \subseteq A\}$$

(for the set of all partial transformations on X with image restricted by A), then

$$\mathcal{PT}_{XY}\, a = \mathcal{PT}_X\, ba = \mathcal{PT}(X, A).$$

The first equality is a conclusion drawn earlier ($S_{ij}a = S_i ba$), and the second holds because $fba = f$ for all $f \in \mathcal{PT}(X, A)$. It is easily seen that $\mathcal{PT}(X, A)$ is a subsemigroup of $\mathcal{PT}_X$; indeed, it is a principal left ideal. By investigating the sandwich semigroup $\mathcal{PT}^a_{XY}$, we will obtain some information on $\mathcal{PT}(X,A)$, as well. This type of semigroups has been investigated before, in [44]. There, the authors describe Green's relations, classify regular elements and calculate the rank of the semigroup in the case that X is finite.

Only the semigroup $(a\mathcal{PT}_{XY},\cdot)$ is left to be examined. Note that $\ker(af) \supseteq \ker(a) = \sigma$ for all $f \in \mathcal{PT}_{XY}$. Similarly as in the previous case, we define

$$\mathcal{PT}(Y,\sigma) = \{f \in \mathcal{PT}_Y \ : \ \text{every } \ker f\text{-class is a union of } \sigma\text{-classes}\}$$

(the set of all partial transformations on Y with kernel restricted by σ) and infer

$$a\mathcal{PT}_{XY} = ab\mathcal{PT}_Y = \mathcal{PT}(Y,\sigma)$$

(because $abf = f$ for all $f \in \mathcal{PT}(Y,\sigma)$). Also, $\mathcal{PT}(Y,\sigma)$ is a subsemigroup of $\mathcal{PT}_Y$, more precisely, its principal right ideal. As we are about to see, in special cases, our results on $\mathcal{PT}^a_{XY}$ offer some information on $\mathcal{PT}(Y,\sigma)$, as well. To the author's knowledge, [34] is the first article to investigate such semigroups.

Finally, recall from the discussion in Subsection 2.3.1 that all the maps on Figure 3.1 are surmorphisms. The previous analysis yields a new commutative diagram, which is an "improved" version of the left diagram on Figure 3.1, with an addition of the isomorphism η:

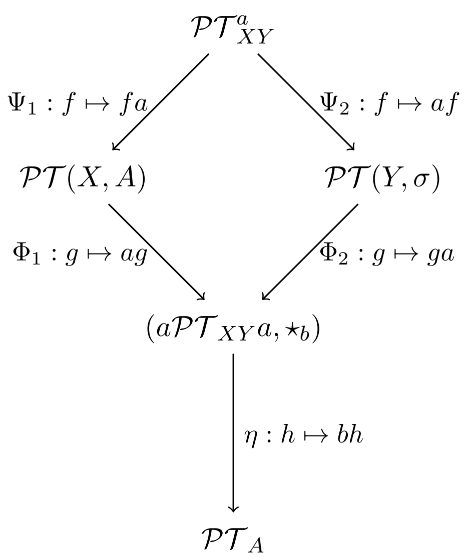


Figure 3.2: Diagram illustrating the connections between $\mathcal{PT}^a_{XY}$ and $(a\mathcal{PT}_{XY}\,a, \circledast)$.

Next, we examine the right-hand side diagram in Figure 3.1. For that reason, we restrict our attention to the regular elements of $\mathcal{PT}^a_{XY}$. The following lemma gives characterisations of the regular elements in $\mathcal{PT}(X, A)$ and $\mathcal{PT}(Y, \sigma)$, hence describing the semigroups in the middle of the diagram.

Let θ be an equivalence relation. Then $\mathrm{u}(\theta)$ and π_θ denote the underlying set of θ (i.e. the set on which θ is defined) and its partition corresponding to θ, respectively.

Lemma 3.1.21. *We have*

(i) $\mathrm{Reg}(\mathcal{PT}(X, A)) = \{f \in \mathcal{PT}(X, A) \ : \ \ker f$ *is saturated by* $A\}$,

(ii) $\mathrm{Reg}(\mathcal{PT}(Y, \sigma)) = \{f \in \mathcal{PT}(Y, \sigma) \ : \ \mathrm{im}\, f \subseteq \mathrm{u}(\sigma),$
$\mathrm{im}\, f$ *is separated by* $\sigma\}$.

Remark 3.1.22. Part (i) of the this lemma was proved in [44] (Theorem 1.2), but instead of "$\ker f$ is saturated by A", the authors used an equivalent condition: "$Xf = Af$", where $Zf = \{zf : z \in Z \cap \mathrm{dom}\, f\}$ for any $Z \subseteq X$.

Proof. Note that we are dealing here with non-sandwich semigroups.

(i) We show the equality by proving that both inclusions hold. Suppose first that $f \in \mathcal{PT}(X, A)$ is such that $\ker f$ is saturated by A and write $f = \binom{F_j}{f_j}_{j\in J}$. Our assumption guarantees the existence of an element $c_j \in F_j \cap A$ for each $j \in J$. Then the map $g = \binom{f_j}{c_j}_{j\in J} \in \mathcal{PT}(X, A)$ satisfies $\mathrm{im}\, g \subseteq A$ and $fgf = f$.

For the reverse inclusion, we prove the contrapositive. Suppose $f = \binom{F_j}{f_j}_{j\in J} \in \mathcal{PT}(X, A)$ is such that A does not saturate $\ker f$. Then there exists $l \in J$ with $F_l \cap A = \emptyset$, which implies $f_l \notin \mathrm{im}(fgf)$ for any $g \in \mathcal{PT}(X, A)$. Thus, we have $f \neq fgf$ for any $g \in \mathcal{PT}(X, A)$, i.e. f is not regular.

(ii) The proof is similar to the previous one. Suppose that $f \in \mathcal{PT}(Y, \sigma)$ is such that $\mathrm{im}\, f \subseteq \mathrm{u}(\sigma)$ and σ separates $\mathrm{im}\, f$. We will prove f is regular. Write $f = \binom{F_j}{f_j}_{j\in J}$ and $\pi_\sigma = \{A_i : i \in I\}$. The two assumptions together imply that, for each $j \in J$, there exists exactly one $l_j \in I$ such that $f_j \in A_{l_j}$. Fix some $w_j \in F_j$ for each $j \in J$, and define $g = \binom{A_{l_j}}{w_j}_{j\in J} \in \mathcal{PT}_Y$. Clearly, $g \in \mathcal{PT}(Y, \sigma)$ and $fgf = f$.

Let $f = \binom{F_j}{f_j}_{j\in J} \in \mathcal{PT}(Y, \sigma)$. Again, we prove the contrapositive: if either $\mathrm{im}\, f \not\subseteq \mathrm{u}(\sigma)$ or $\mathrm{im}\, f$ is not separated by σ, then f is not regular. In the case that $\mathrm{im}\, f \not\subseteq \mathrm{u}(\sigma)$, there exists $f_l \in \mathrm{im}\, f \backslash \mathrm{u}(\sigma)$, so $(f_l)g$ is undefined for each $g \in \mathcal{PT}(Y, \sigma)$. Thus, $F_l \subseteq \mathrm{dom}\, f \setminus \mathrm{dom}(fgf)$ for any $g \in \mathcal{PT}(Y, \sigma)$. In the second case, there exist distinct $l, k \in J$ such that $(f_l, f_k) \in \sigma$. The definition of $\mathcal{PT}(Y, \sigma)$ implies that, for any $g \in \mathcal{PT}(Y, \sigma)$, f_l and f_k either both belong to $Y \setminus \mathrm{dom}\, g$, or both belong to $\mathrm{dom}\, g$, in which case $(x, y) \in \ker g$. Therefore, either $F_l \cup F_k \in \mathrm{dom}\, f \setminus \mathrm{dom}(fgf)$ or the elements of F_l and F_k belong to the same class in $\ker(fgf)$, but not in $\ker f$. Hence, in both cases we have $f \neq fgf$, for any $g \in \mathcal{PT}(Y, \sigma)$. □

Thus, the second diagram on Figure 3.1 becomes

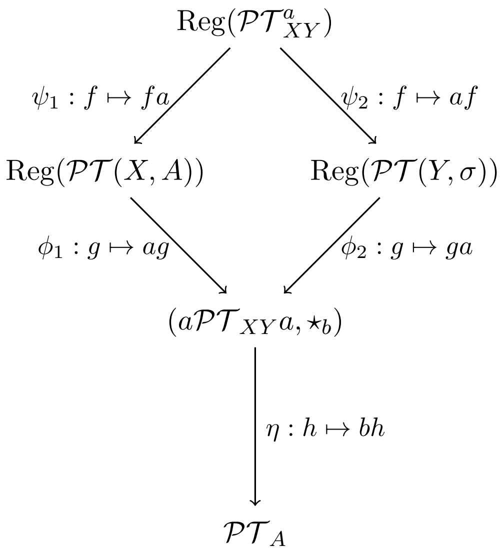


Figure 3.3: Diagram describing the connections between $P^a = \mathrm{Reg}(\mathcal{PT}^a_{XY})$ and $(a\,\mathcal{PT}_{XY}\,a, \circledast)$.

In the discussion below, we investigate the maps and semigroups in Diagrams 3.2 and 3.3 in cases when the sandwich element a has some special properties. Recall that $B = \operatorname{dom} a$.

- If a is full and injective, we have $\mathrm{Reg}(\mathcal{PT}(Y, \sigma)) = \mathcal{PT}(Y, \sigma) = \mathcal{PT}_Y$, since $\sigma = \{(y, y) : y \in Y\}$. Moreover, from the discussion in Subsection 2.3.1 it follows that Ψ_1 and ψ_1 are isomorphisms, because $ab = \mathrm{id}_Y$ implies that a is right-invertible (so the implication (2.5) is true). Therefore, $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(X, A)$, in this case. Figure 3.7 shows an egg-box diagram of a sandwich semigroup of such type (namely, it shows $\mathcal{PT}(X, A)$, where $X = \{1, 2, 3, 4\}$ and $A = \{1, 2, 3\}$).

- If a is surjective, then $\mathrm{Reg}(\mathcal{PT}(X, A)) = \mathcal{PT}(X, A) = \mathcal{PT}_X$, because $A = \operatorname{im} a = X$. Furthermore, a is left-invertible (since $ba = \mathrm{id}_X$), so Ψ_2 and ψ_2 are isomorphisms, which implies $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma)$. The structure of a sandwich semigroup of such type is depicted on the first egg-box diagram in Figure 3.8 (it shows $\mathcal{PT}(Y, \sigma)$, where $Y = \{1, 2, 3, 4, 5\}$ and $\pi_\sigma = \{\{1\}, \{2\}, \{3, 4\}\}$).

  – If a is both surjective and injective (but not necessarily full), then $\sigma = \{(y, y) : y \in \operatorname{dom} a\}$, so in addition to the benefits of surjectivity, we have

  $$\mathcal{PT}(Y, \sigma) = \{f \in \mathcal{PT}_Y \ : \ \operatorname{dom} f \subseteq B\} \cong \mathcal{PT}^a_{XY} \quad \text{and}$$
  $$\mathrm{Reg}(\mathcal{PT}(Y, \sigma)) = \{f \in \mathcal{PT}(Y, \sigma) \ : \ \operatorname{im} f \subseteq B\} \equiv \mathcal{PT}_B\,.$$

  Let us elaborate the second line. Obviously, the subsemigroup $R = \{f \in \mathcal{PT}_Y : \operatorname{dom} f, \operatorname{im} f \subseteq B\} \equiv \mathcal{PT}_B$ is regular, and any function

$g \in \mathcal{PT}(Y, \sigma) \setminus R$ has elements mapping outside of B, so these elements cannot be in the domain of ghg, for any $h \in \mathcal{PT}(Y, \sigma)$. Thus, $R = \text{Reg}(\mathcal{PT}(Y, \sigma))$. To the author's knowledge, [34] is the first article to investigate such semigroups.

– If a is full, injective, and surjective, all of the above holds, so

$$\mathcal{PT}^a_{XY} \cong \mathcal{PT}(X, A) = \mathcal{PT}_X = \mathcal{PT}_A, \text{ and}$$
$$\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma) = \mathcal{PT}_Y$$

In addition, all the maps in the Diagrams 3.2 and 3.3 are isomorphisms, rendering further investigation in this case unnecessary (since the problems we consider for sandwich semigroups have been solved for semigroups $\mathcal{PT}_X$). Therefore, in our study, we omit the case when a is a full bijection.

The rightmost egg-box diagram in Figure 3.8 shows the structure of a sandwich semigroup with a surjective, injective and non-full sandwich element, while the diagrams on Figure 3.10 illustrate the case when the sandwich element is full, injective and surjective.

We close the subsection by describing a different aspect of the connections among semigroups on Diagram 3.3, inspired and implied by Theorem 2.3.8.

Theorem 3.1.23. *The map*

$$\psi : \text{Reg}(\mathcal{PT}^a_{XY}) \to \text{Reg}(\mathcal{PT}(X, A)) \times \text{Reg}(\mathcal{PT}(Y, \sigma)) : f \mapsto (fa, af).$$

is injective, and

$$\text{im}(\psi) = \{(g, h) \in \text{Reg}(\mathcal{PT}(X, A)) \times \text{Reg}(\mathcal{PT}(Y, \sigma)) : ag = ha\}.$$

In particular, the semigroup $\text{Reg}(\mathcal{PT}^a_{XY})$ *is a pullback product of* $\text{Reg}(\mathcal{PT}(X, A))$ *and* $\text{Reg}(\mathcal{PT}(Y, \sigma))$ *with respect to* $\mathcal{PT}_A$*.*

3.1.3 The regular subsemigroup $\text{P}^a = \text{Reg}(\mathcal{PT}^a_{XY})$

Our next object of interest is the semigroup $\text{Reg}(\mathcal{PT}^a_{XY})$ itself. As we remarked earlier, all the elements of $\mathcal{PT}$ are sandwich regular, so Proposition 2.3.2(i) implies $\text{P}^a = \text{Reg}(\mathcal{PT}^a_{XY})$, and Proposition 3.1.8($iii$) provides a characterisation of its elements.

Inspired by Subsection 2.3.4, we want to investigate further and describe Green's relations and their classes, as well as their connection to Green's relations in $W \cong \mathcal{PT}_A$ (in other words, the inflation described in Theorem 2.3.12 and Remark 2.3.13). Furthermore, we provide additional results, specific to $\mathcal{PT}$, which include information about the sizes of Green's classes and their number, so we may calculate the size of P^a and determine equivalent conditions for it to be finite, countable or uncountable. Moreover, we prove that $\text{Reg}(\mathcal{PT}^a_{XY})$ is always MI-dominated. Hence, we may use the theory of MI-domination in order to calculate its rank. Throughout the subsection, we make remarks on the simplifications occurring in the cases where a is full, injective or surjective.

We keep the notation introduced earlier. Recall that, in Lemma 2.3.3, we have proved that for all $x \in \mathrm{P}^a$ and any $\mathrm{K} = \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}\}$ we have $\mathrm{K}_x^{\mathrm{P}^a} = \mathrm{K}_x^a$. Moreover, since $\mathscr{J} = \mathscr{D}$ in $\mathcal{PT}$ (by Proposition 3.1.2(vi)), from Lemma 2.3.4 it follows that $\mathscr{J}^{\mathrm{P}^a} = \mathscr{D}^{\mathrm{P}^a}$.

Therefore, for each $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}\}$, we continue to write $\mathscr{K}^a$ for the corresponding Green's relation of P^a, and K_f^a for its class containing an element $f \in \mathrm{P}^a$. Now, having in mind Lemmas 2.3.3 and 2.3.4, we have enough information to describe Green's relations in P^a. The parts (i)-(iv) of the following proposition were first proved in Theorem 5.7 in [86].

Proposition 3.1.24. *Let $f \in \mathrm{P}^a = \mathrm{Reg}(\mathcal{PT}_{XY}^a)$. Then*

(i) $\mathrm{R}_f^a = \mathrm{R}_f \cap \mathrm{P}^a = \{g \in \mathrm{P}^a \ : \ \mathrm{dom}\, g = \mathrm{dom}\, f,\ \ker g = \ker f\}$,

(ii) $\mathrm{L}_f^a = \mathrm{L}_f \cap \mathrm{P}^a = \{g \in \mathrm{P}^a \ : \ \mathrm{im}\, g = \mathrm{im}\, f\}$,

(iii) $\mathrm{H}_f^a = \mathrm{H}_f \cap \mathrm{P}^a$
$= \{g \in \mathrm{P}^a \ : \ \mathrm{dom}\, g = \mathrm{dom}\, f,\ \ker g = \ker f,\ \mathrm{im}\, g = \mathrm{im}\, f\}$,

(iv) $\mathrm{D}_f^a = \mathrm{D}_f \cap \mathrm{P}^a = \{g \in \mathrm{P}^a \ : \ \mathrm{Rank}\, g = \mathrm{Rank}\, f\}$.

The $\mathscr{J}^{\mathrm{P}^a} = \mathscr{D}^a$-classes of P^a are the sets

$$\mathrm{D}_\mu^a = \{g \in \mathrm{P}^a : \mathrm{Rank}\, g = \mu\} \qquad \textit{for each cardinal } 0 \le \mu \le \alpha = \mathrm{rank}\, a,$$

and these form a chain under the ordering $\le_{\mathscr{J}}$ on the $\mathscr{J}^{\mathrm{P}^a}$-classes:

$$\mathrm{D}_\mu^a \le \mathrm{D}_\nu^a \Leftrightarrow \mu \le \nu.$$

Proof. From Remark 1.3.8 we may conclude that for any $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}\}$ and any $f \in \mathrm{P}^a$ we have $\mathrm{K}_f^a \subseteq \mathrm{P}^a$ (equivalently, $\mathrm{K}_f^a = \mathrm{K}_f^a \cap \mathrm{P}^a$). Thus, since $\mathrm{P}^a \subseteq \mathrm{P}_q^a$ for all $q \in \{1, 2, 3\}$, from Theorem 3.1.10 we have the first equality in (i)-(iv). The second follows directly from Proposition 3.1.2. Finally, Propositions 3.1.18 and 3.1.16(vi) imply the last statement. □

Therefore, we have a minimum and a maximum $\mathscr{J}^{\mathrm{P}^a} = \mathscr{D}^a$-class in P^a:

$$\mathrm{D}_0^a = \{\emptyset\} \ \text{ and } \ \mathrm{D}_\alpha^a = \{f \in \mathrm{P}^a \ : \ \mathrm{Rank}\, f = \alpha\}.$$

In the case that $\alpha = \xi = \max(X, Y)$, the latter is also the maximum $\mathscr{J}^a$-class of $\mathcal{PT}_{XY}^a$, by Proposition 3.1.19. On the Figure 3.9, we show the structure of the regular subsemigroups of several sandwich semigroups. The reader may check the egg-box diagrams of the original sandwich semigroups on Figures 3.4–3.8 to locate the maximal $\mathscr{J}^a$-classes.

Recall the map $\phi : \mathrm{P}^a \to W : f \mapsto afa$ from Subsection 2.3.4. Instead of W, we want to deal with the (isomorphic) semigroup $\mathcal{PT}_A$, so we replace ϕ with

$$\varphi = \phi\eta : P^a \to \mathcal{PT}_A : f \mapsto bafa.$$

For simplicity, we slightly abuse the notation used for ϕ, and write $\overline{f} = f\varphi = bafa$ for all $f \in \mathrm{P}^a$. Using the map φ, we define new relations on P^a: for all $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}\}$, and all $f, g \in \mathrm{P}^a$,

$$f\ \widehat{\mathscr{K}^a}\ g \Leftrightarrow \overline{f}\ \mathscr{K}\ \overline{g} \text{ in } \mathcal{PT}_A.$$

Clearly, these correspond to the $\widehat{\mathscr{K}^a}$-classes defined in Section 2.3.4. We write $\widehat{\mathrm{K}^a_f}$ for the $\widehat{\mathscr{K}^a}$-class of an element $f \in \mathrm{P}^a$.

Next, we will find a suitable representation for the image $\overline{f}$ of an element $f = \binom{F_j}{f_j}_{j\in J} \in \mathrm{P}^a$. Recall that $a = \binom{A_i}{a_i}_{i\in I}$. Firstly, since $f \in \mathrm{P}^a$, we have $\operatorname{rank} f \leq \operatorname{rank} a$, and we may suppose $J \subseteq I$. Secondly, from Proposition 3.1.8(*iii*) we know that $\operatorname{im} f \subseteq \operatorname{dom} a$, $\ker a$ separates $\operatorname{im} f$ and $\operatorname{im} a = A$ saturates $\ker f$. Because of the first two properties, we may assume without loss of generality that $f_j \in A_j$ for each $j \in J$. The third property guarantees $F_j \cap A \neq \emptyset$ for all $j \in J$, so we write $F_j \cap A = \{a_i : i \in I_j\}$ where $\emptyset \neq I_j \subseteq I$. Clearly, the sets I_j are pairwise disjoint sets, but their union is not necessarily the whole set I. Therefore, $\operatorname{dom} \overline{f} \subseteq \operatorname{dom} b = A$ and

$$a_i\overline{f} = (a_i)bafa = (a_i)fa = \begin{cases} (f_j)a, & i \in I_j; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} a_j, & i \in I_j; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

More succinctly, $\overline{f} = \binom{F_j\cap A}{a_j}_{j\in J}$. We will use this notation from now on, unless stated otherwise. The discussion above implies $\operatorname{dom} \overline{f} = \operatorname{dom} f \cap A$, $\operatorname{im} \overline{f} = (\operatorname{im} f)a = \{a_j : j \in J\}$, and $\overline{f} = (fa)\restriction_A$.

The previous analysis serves as preparation for Theorem 3.1.26, which details the "inflation connection" between P^a and $\mathcal{PT}_A$ from Theorem 2.3.12. In the proof Theorem 3.1.26 (as well as in the following text) we will need some properties of the semigroup $\mathcal{PT}_A$, which we present in the following Lemma. These results have become a part of semigroup theory "folklore", so we omit the proofs. For a detailed account, see [45].

Lemma 3.1.25. *Let $f \in \mathcal{PT}_A$ with* $\operatorname{Rank} f = \mu$. *In $\mathcal{PT}_A$, we have*

(i) $\mathrm{R}_f = \{g \in \mathcal{PT}_A : \operatorname{dom} g = \operatorname{dom} f,\ \ker g = \ker f\}$;

(ii) $\mathrm{L}_f = \{g \in \mathcal{PT}_A : \operatorname{im} g = \operatorname{im} f\}$;

(iii) $\mathrm{H}_f = \{g \in \mathcal{PT}_A : \operatorname{dom} g = \operatorname{dom} f,\ \ker g = \ker f,\ \operatorname{im} g = \operatorname{im} f\}$;

(iv) $|\,\mathrm{H}_f\,| = \mu!$; *furthermore, if* H_f *contains an idempotent, then* $\mathrm{H}_f \cong \mathrm{S}_\mu$;

(v) $\mathrm{D}_f = \mathrm{J}_f = \{g \in \mathcal{PT}_A : \operatorname{Rank} g = \operatorname{Rank} f = \mu\} = \mathrm{D}_\mu$.

(vi) *If* $\alpha = |A|$ *is finite, then* $\mathrm{D}_\alpha = \mathrm{H}_{\mathrm{id}_A} \cong \mathrm{S}_A$ *and* $\mathcal{PT}_A \setminus \mathrm{D}_\alpha$ *is an ideal of the semigroup* $\mathcal{PT}_A$.

Finally, we may prove

Theorem 3.1.26. *Let* $f = \binom{F_j}{f_j}_{j\in J} \in \mathrm{P}^a$ *with* $\operatorname{Rank} f = \mu$. *Then*

(i) $\widehat{\mathrm{R}^a_f}$ *is the union of* $(\mu+1)^\beta$ $\mathscr{R}^a$*-classes of* P^a*;*

(ii) $\widehat{\mathrm{L}^a_f}$ *is the union of* Λ_J $\mathscr{L}^a$*-classes of* P^a*;*

(iii) $\widehat{\mathrm{H}^a_f}$ *is the union of* $(\mu+1)^\beta\Lambda_J$ $\mathscr{H}^a$*-classes of* P^a*, each of which has size* $\mu!$*;*

(iv) *if* $\mathrm{H}_{\overline{f}}$ *is a non-group* $\mathscr{H}$*-class of* $\mathcal{PT}_A$*, then each* $\mathscr{H}^a$*-class of* P^a *contained in* $\widehat{\mathrm{H}^a_f}$ *is a non-group;*

(v) *if* $\mathrm{H}_{\overline{f}}$ *is a group* $\mathscr{H}$*-class of* $\mathcal{PT}_A$*, then each* $\mathscr{H}^a$*-class of* P^a *contained in* $\widehat{\mathrm{H}^a_f}$ *is a group isomorphic to* S_μ*; further,* $\widehat{\mathrm{H}^a_f}$ *is a* $(\mu+1)^\beta \times \Lambda_J$ *rectangular group over* S_μ*, and its idempotents* $\mathrm{E}_a(\widehat{\mathrm{H}^a_f})$ *form a* $(\mu+1)^\beta \times \Lambda_J$ *rectangular band;*

(vi) $\widehat{\mathrm{D}^a_f} = \mathrm{D}^a_f = \mathrm{D}^a_\mu = \{g \in \mathrm{P}^a : \operatorname{Rank} g = \mu\}$ *is the union of:*

- *(a)* $(\mu+1)^\beta\, \mathcal{S}(\alpha+1,\mu+1)$ $\mathscr{R}^a$*-classes of* P^a,
- *(b)* $\sum\limits_{\substack{K\subseteq I\\ |K|=\mu}} \Lambda_K$ $\mathscr{L}^a$*-classes of* P^a,
- *(c)* $(\mu+1)^\beta\, \mathcal{S}(\alpha+1,\mu+1) \sum\limits_{\substack{K\subseteq I\\ |K|=\mu}} \Lambda_K$ $\mathscr{H}^a$*-classes of* P^a.

Proof. (*i*) Recall that $f \ \widehat{\mathscr{R}^a}\ g$ means that $\overline{f}\,\mathscr{R}\,\overline{g}$ in $\mathcal{PT}_A$, i.e. $\operatorname{dom}\overline{f} = \operatorname{dom}\overline{g}$ and $\ker\overline{f} = \ker\overline{g}$. Therefore, by fixing the domain $\operatorname{dom}\overline{f}$ and kernel $\ker\overline{f}$ in $\mathcal{PT}_A$, we completely determine the $\mathscr{R}$-class $R = \mathrm{R}_{\overline{f}}$ in $\mathcal{PT}_A$. We need to know how many $\mathscr{R}^a$-classes map into R via φ. In other words, we need the number of domain-kernel combinations (D, K) such that elements having both domain D and kernel K map into R. For an element $g \in \mathrm{P}^a$ mapping into R, we have $\operatorname{Rank} g = \operatorname{Rank} f = \mu$, so we write $g = \binom{G_j}{g_j}_{j\in J}$. We may conclude that

- $\operatorname{dom}\overline{g} = \operatorname{dom}\overline{f}$, so $D \cap A = \operatorname{dom} f \cap A$; note that the elements outside A are not restricted in terms of belonging to the domain D;
- $\ker\overline{g} = \ker\overline{f}$, therefore $\{G_j \cap A : j \in J\} = \{F_j \cap A : j \in J\}$; here too, the elements outside A are not restricted in terms of belonging to a specific class of the kernel.

Therefore, the properties of elements inside A are completely determined, while all $\beta = |X \setminus \operatorname{im} a|$ of them outside can be either in one of the μ classes of K, or outside the domain. Thus, we have $(\mu+1)^\beta$ pairs of form (D, K). Note that this is also the number of $\mathscr{R}^a$-classes in any $\widehat{\mathscr{R}^a}$-class of $\widehat{\mathrm{D}^a_f}$.

(ii) By definition, $f \; \widehat{\mathscr{L}^a} \; g$ if and only if $\overline{g} \mathscr{L} \overline{f}$, i.e. $\operatorname{im} \overline{g} = \operatorname{im} \overline{f}$. Hence, by the discussion preceding this theorem, $\operatorname{im} \overline{g} = \{a_j : j \in J\}$. Since $\overline{g} = (ga)\restriction_A$, the previous conclusion implies that, for each $j \in J$, the set A_j contains at least one element of $\operatorname{im} g$. Furthermore, since $g \in \mathrm{P}^a$, by Proposition 3.1.8(iii) we have $\operatorname{im} g \subseteq \operatorname{dom} a$ and $\ker a$ separates $\operatorname{im} g$. Therefore, $\operatorname{im} g$ is a cross-section of the partition $\{A_j : j \in J\}$. As $|A_j| = \lambda_j$ for each $j \in J$, the number of such cross-sections is $\Lambda_J = \prod_{j \in J} \lambda_j$.

(iii) Recall from Subsection 2.3.4 that $\widehat{\mathrm{H}^a_f} = \widehat{\mathrm{R}^a_f} \cap \widehat{\mathrm{L}^a_f}$. This equality, together with (i) and (ii), implies the statement about the number of $\mathscr{H}^a$-classes in $\widehat{\mathrm{H}^a_f}$. By Theorem 2.3.12(i), all these classes have size $|\,\mathrm{H}_{\overline{f}}\,|$ (in $\mathcal{PT}_A$), which is $\mu!$, since $\operatorname{Rank} \overline{f} = |J| = \mu$.

(iv) and (v). The elements of the $\mathscr{H}$-class $\mathrm{H}_{\overline{f}}$ are completely determined by their domain, kernel and image, $\operatorname{dom} \overline{f}$, $\ker \overline{f}$ and $\operatorname{im} \overline{f}$, respectively. Further, in a fixed $\mathscr{D}^a$-class D we know that: if we want to define a map $g \in D$ in a specific way,

- the choice of $\operatorname{im} g$ affects neither $\operatorname{dom} g$ nor $\ker g$, and
- neither the choice of $\operatorname{dom} g$, nor the choice of $\ker g$ affect $\operatorname{im} g$.

Therefore, from the proofs of (i) and (ii) it follows that $r = |\widehat{\mathrm{H}^a_f} / \mathscr{R}^a\,| = (\mu+1)^\beta$ and $l = |\widehat{\mathrm{H}^a_f} / \mathscr{L}^a\,| = \Lambda_J$. Furthermore, if $\mathrm{H}_{\overline{f}}$ is a group, it is isomorphic to S_μ. Thus, Theorem 2.3.12 implies (iv) and (v).

(vi) Recall from Lemma 2.3.9 that $\widehat{\mathscr{D}^a} = \mathscr{D}^a$ in P^a. Thus, the description of $\widehat{\mathrm{D}^a_f}$ follows from Proposition 3.1.24. We will prove (a) by considering the number of $\widehat{\mathscr{R}^a}$-classes in $\widehat{\mathrm{D}^a_f}$, and multiplying this value by the number of $\mathscr{R}^a$-classes in a $\widehat{\mathscr{R}^a}$-class (which is calculated in (i)). Clearly, $|\widehat{\mathrm{D}^a_f} / \widehat{\mathscr{R}^a}\,|$ equals $|\,\mathrm{D}_{\overline{f}} / \mathscr{R}\,|$ in $\mathcal{PT}_A$, which is $\mathcal{S}(\alpha+1, \mu+1)$, by Lemma 3.1.4($i$). Hence, ($a$) follows. To prove ($b$), note that the proof of (ii) implies that an $\widehat{\mathscr{L}^a}$-class in $\widehat{\mathrm{D}^a_f}$ is characterised by its corresponding set of indexes $J \subseteq I$ of cardinality μ (which determines $\operatorname{im} \overline{f}$). Therefore, ($ii$) implies ($b$). Finally, ($c$) follows directly from (a) and (b). □

Remark 3.1.27. The egg-box diagrams on Figures 3.9 and 3.10 illustrate the previous theorem.

Remark 3.1.28. The previous theorem may be simplified substantially in the cases where a has some special property:

- If a is injective, each class of $\ker a$ is a singleton, so $\lambda_i = 1$ for all $i \in I$. Thus, $\Lambda_J = 1$ for all $J \subseteq I$, which means that $\widehat{\mathscr{L}^a} = \mathscr{L}^a$, and $\widehat{\mathrm{H}^a_f}$ from Theorem 3.1.26(v) is a $(\mu+1)^\beta \times 1$ rectangular group over S_μ. In particular, all of this holds when $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(X, A)$. Figure 3.6 shows an egg-box diagram of such a sandwich semigroup.

- If a is surjective (i.e. when $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma)$), we have $\beta = |X \setminus \operatorname{im} a| = 0$. Hence, $(\mu + 1)^\beta = 1$, which means that $\widehat{\mathscr{R}^a} = \mathscr{R}^a$ and $\widehat{\mathrm{H}^a_f}$ from Theorem 3.1.26(v) is a $1 \times \Lambda_J$ rectangular group over S_μ. An example of such a sandwich semigroup may be seen in Figure 3.8 (the leftmost egg-box diagram).

- If a is a bijection (not necessarily full), then $\widehat{\mathscr{H}^a} = \mathscr{H}^a$ and in Theorem 3.1.26(v) we have $\widehat{\mathrm{H}^a_f} \cong \mathrm{S}_\mu$. An example of such a sandwich semigroup may be seen in Figure 3.8 (the rightmost egg-box diagram).

From parts (vi)(c) and (iii) of Theorem 3.1.26 and Proposition 3.1.24 directly follows

Corollary 3.1.29. *For any $0 \leq \mu \leq \alpha$ we have*

$$|\,\mathrm{D}^a_\mu\,| = \mu!(\mu+1)^\beta\,\mathcal{S}(\alpha+1, \mu+1) \sum_{\substack{K \subseteq I \\ |K| = \mu}} \Lambda_K.$$

Consequently,

$$|\,\mathrm{P}^a\,| = \sum_{\mu=0}^{\alpha} |\,\mathrm{D}^a_\mu\,| = \sum_{\mu=0}^{\alpha} \mu!(\mu+1)^\beta\,\mathcal{S}(\alpha+1, \mu+1) \sum_{\substack{K \subseteq I \\ |K| = \mu}} \Lambda_K.$$

It turns out that the formula for $|\,\mathrm{P}^a\,|$ can be simplified in the case that $\operatorname{rank} a = \alpha \geq 1$ and either $|X| \geq \aleph_0$, or $\lambda_i \geq \aleph_0$ for some $i \in I$. In the next Proposition we elaborate these simplifications, and we prove equivalent conditions for $|\,\mathrm{P}^a\,|$ to be finite, countable or uncountable.

Proposition 3.1.30.

(i) If $\alpha \geq 1$ and $|X| \geq \aleph_0$ then

$$|\,\mathrm{P}^a\,| = 2^{|X|}\Lambda_I = \max(2^{|X|}, \Lambda_I).$$

(ii) If $\alpha \geq 1$ and $|X| < \aleph_0$ and $\lambda_i \geq \aleph_0$ for some $i \in I$, then

$$|\,\mathrm{P}^a\,| = \Lambda_I = \max_{i \in I} \lambda_i.$$

(iii) $|\,\mathrm{P}^a\,| < \aleph_0 \Leftrightarrow \alpha = 0$ or $[|X| < \aleph_0$ and $\lambda_i < \aleph_0$ for all $i \in I]$.

(iv) $|\,\mathrm{P}^a\,| = \aleph_0 \Leftrightarrow \alpha \geq 1$ and $|X| < \aleph_0$ and $\max_{i\in I} \lambda_i = \aleph_0$.

(v) $|\,\mathrm{P}^a\,| > \aleph_0 \Leftrightarrow \alpha \geq 1$ and $[|X| \geq \aleph_0$ or $\lambda_i > \aleph_0$ for some $i \in I]$.

Proof. (i) Recall that $\alpha = |I| = \operatorname{rank} a \leq \xi = \min\{|X|, |Y|\}$ and $\beta = |X \setminus \operatorname{im} a| \leq |X|$. Suppose $\alpha \geq 1$ and $|X| \geq \aleph_0$. The second assumption clearly implies $2^{|X|}\Lambda_I = \max(2^{|X|}, \Lambda_I)$. We show the equality $|\,\mathrm{P}^a\,| = 2^{|X|}\Lambda_I$ by proving both $|\,\mathrm{P}^a\,| \leq 2^{|X|}\Lambda_I$ and $|\,\mathrm{P}^a\,| \geq 2^{|X|}\Lambda_I$. For the first one, let $0 \leq \mu \leq \alpha$, and note that

- $\mu! \leq \alpha! \leq 2^{|X|}$ (the last inequality evidently holds for $\alpha < \aleph_0$, and otherwise follows from $\alpha! = 2^\alpha$),
- $(\mu+1)^\beta \leq (|X|+1)^{|X|} = |X|^{|X|} = 2^{|X|}$,
- $\mathcal{S}(\alpha+1, \mu+1) \leq (2^{\alpha+1})^{\mu+1} = 2^{(\alpha+1)(\mu+1)} \leq 2^{|X|}$ (which is clear if $\alpha < \aleph_0$, and otherwise follows from $(\alpha+1)(\mu+1) = \alpha$),
- $\sum\limits_{\substack{K \subseteq I \\ |K| = \mu}} \Lambda_K \leq \sum\limits_{K \subseteq I} \Lambda_K \leq \sum\limits_{K \subseteq I} \Lambda_I = 2^{|I|}\Lambda_I \leq 2^{|X|}\Lambda_I$.

Thus, Corollary 3.1.29 implies $|\,\mathrm{D}^a_\mu\,| \leq 2^{|X|}\Lambda_I$ for each $0 \leq \mu \leq \alpha$, so

$$|\,\mathrm{P}^a\,| = \sum_{\mu=0}^{\alpha} |\,\mathrm{D}^a_\mu\,| \leq (\alpha+1)2^{|X|}\Lambda_I \leq (|X|+1)2^{|X|}\Lambda_I = 2^{|X|}\Lambda_I.$$

Let us prove now the second inequality. We have

$$|\,\mathrm{P}^a\,| \geq |\,\mathrm{D}^a_\alpha\,| = \alpha!(\alpha+1)^\beta\,\mathcal{S}(\alpha+1, \alpha+1) \sum_{\substack{K \subseteq I \\ |K| = \alpha}} \Lambda_K \geq \alpha! 2^\beta \Lambda_I.$$

Note that, if $\alpha < \aleph_0$, then $\beta = |X \setminus \operatorname{im} a| = |X|$, so $\alpha! 2^\beta \Lambda_I = 2^{|X|}\Lambda_I$. Otherwise,

$$\alpha! 2^\beta \Lambda_I = 2^\alpha 2^\beta \Lambda_I = 2^{\alpha+\beta}\Lambda_I = 2^{|X|}\Lambda_I.$$

(ii) Suppose $\alpha \geq 1$, $|X| < \aleph_0$, and $\lambda_i \geq \aleph_0$ for some $i \in I$. The second assumption implies $|I| = \alpha, \beta < \aleph_0$. Thus, for any $0 \leq \mu \leq \alpha$, the value of the expression $\mu!(\mu+1)^\beta\,\mathcal{S}(\alpha+1, \mu+1)$ is finite, and

$$\sum_{\substack{K \subseteq I \\ |K| = \mu}} \Lambda_K \leq \sum_{K \subseteq I} \Lambda_I = 2^\alpha \Lambda_I = \Lambda_I = \prod_{i \in I} \lambda_i = \max_{i \in I} \lambda_i.$$

Therefore, $|\,\mathrm{D}^a_\mu\,| \leq \Lambda_I$, and $|\,\mathrm{P}^a\,| \leq \sum_{\mu=0}^{\alpha} \Lambda_I = (\alpha+1)\Lambda_I = \Lambda_I$, by Corollary 3.1.29. The reverse inequality is easily seen, since

$$|\,\mathrm{P}^a\,| \geq |\,\mathrm{D}^a_\alpha\,| \geq \sum_{\substack{K \subseteq I \\ |K| = \alpha}} \Lambda_K \geq \Lambda_I.$$

(iii) First, suppose that $|\,\mathrm{P}^a\,| < \aleph_0$. If $\alpha = 0$, the implication holds. Otherwise, (i) implies $|X| < \aleph_0$ (because $|\,\mathrm{P}^a\,|$ would be infinite otherwise), and then (ii) implies $\lambda_i < \aleph_0$ for all $i \in I$ (for the same reason). Let us prove the reverse implication using

Corollary 3.1.29. Note that, if $\alpha = 0$, we have $|\,\mathrm{P}^a\,| = |\,\mathrm{D}_0^a\,| = 1$, whereas if $\alpha \geq 1$, $|X| < \aleph_0$, and $\lambda_i < \aleph_0$ for all $i \in I$, then D_μ^a is finite for any $0 \leq \mu \leq \alpha (\leq |X| < \aleph_0)$, so $|\,\mathrm{P}^a\,| < \aleph_0$.

(*iv*) Suppose $|\,\mathrm{P}^a\,| = \aleph_0$. Then (*iii*) implies $\alpha \geq 1$, so (*i*) implies $|X| < \aleph_0$ (otherwise we have $|P^a| \geq 2^{|X|} > \aleph_0$). Finally, from (*ii*) we have $\lambda_i \geq \aleph_0$ for some $i \in I$ (as the opposite implies the finiteness of $|\,\mathrm{P}^a\,|$), hence (*ii*) gives $\aleph_0 = |\,\mathrm{P}^a\,| = \max_{i \in I} \lambda_i$. The reverse implication follows directly from (*ii*).

(*v*) For the direct implication, suppose $|\,\mathrm{P}^a\,| > \aleph_0$. Then, (*iii*) implies $\alpha \geq 1$. If $|X| \geq \aleph_0$, the implication holds; otherwise, from (*iii*) we have $\lambda_i \geq \aleph_0$ for some $i \in I$ (because $|\,\mathrm{P}^a\,|$ is infinite), and from (*iv*) we have $\max_{i \in I} \lambda_i > \aleph_0$ (since $|\,\mathrm{P}^a\,|$ is uncountable). To prove the the reverse implication, note that

- by (*i*), $\alpha \geq 1$ and $|X| \geq \aleph_0$ imply $|\,\mathrm{P}^a\,| = 2^{|X|}\Lambda_I \geq 2^{|X|} > \aleph_0$, and
- in the case that $|X| < \aleph_0$, by (*ii*), the assumptions $\alpha \geq 1$ and $\lambda_i > \aleph_0$ for some $i \in I$ give $|\,\mathrm{P}^a\,| = \max_{i \in I} \lambda_i > \aleph_0$. □

Remark 3.1.31. Let us examine the value of the parameter $\Lambda_I = \prod_{i \in I} \lambda_i$ in different cases. It depends on the sizes of the sets in $\ker a$. We have

- $\Lambda_I = 1$ if and only if a is injective;
- if $\lambda_i \geq \aleph_0$ for some $i \in I$, and $|I| = \alpha < \aleph_0$, then $\Lambda_I = \max_{i \in I} \lambda_i$, as in part (*ii*);
- if $|I| = \alpha \geq \aleph_0$, we may suppose without loss of generality that the sequence $\langle \lambda_i : i \in I \rangle$ is nondecreasing and then Lemma 5.9 in [62] gives

$$\Lambda_I = (\sup_{i \in I} \lambda_i)^\alpha.$$

After investigating the cardinality of P^a, a logical follow-up is the calculation of the rank of P^a. In order to apply Theorem 2.4.16, we need to show that P^a is MI-dominated. Since $\mathcal{PT}$ is regular, and P^a is a regular semigroup with (at least one) mid-identity b, we may apply the theory from Subsection 2.4.1. For completeness, we also identify the cases when P^a is RP-dominated. For that, we need the following result from [122] (Theorems 3.1 and 3.2).

Lemma 3.1.32. *Each of the monoids $\mathcal{PT}_A$, $\mathcal{T}_A$ and $\mathcal{I}_A$ is factorisable if and only if A is finite.*

Proof. We prove the claim only for $\mathcal{PT}_A$, as the proofs for $\mathcal{T}_A$ and $\mathcal{I}_A$ are similar. Suppose A is finite. We claim that $\mathcal{PT}_A = \mathcal{S}_A \cdot \mathrm{E}(\mathcal{PT}_A)$. Let $f = \binom{F_i}{f_i}_{i \in I} \in \mathcal{PT}_A$ and for each $i \in I$ choose $a_i \in F_i$. Next, choose a permutation $g \in \mathcal{S}_A$ such that $(a_i)g = f_i$ for all $i \in I$. Then, let $h \in \mathcal{PT}_A$ with $\mathrm{dom}\, h = \mathrm{dom}\, f$ be defined by $(x)h = f_i$, if $(x)g^{-1} \in F_i$ for some $i \in I$. It is easily seen that h is an idempotent, and that $f = gh \in \mathcal{S}_A \cdot \mathrm{E}(\mathcal{PT}_A)$.

Conversely, suppose that A is not finite and suppose $\mathcal{PT}_A = G \cdot \mathrm{E}(\mathcal{PT}_A)$ for some subgroup G of $\mathcal{PT}_A$. Then, since $\mathcal{S}_A \subseteq G \cdot \mathrm{E}(\mathcal{PT}_A)$, we have

$$\mathrm{H}_{\mathrm{id}_A} = \mathcal{S}_A \subseteq G \subseteq \mathrm{H}_{\mathrm{id}_A}$$

(as id_A is the only surjective idempotent in $\mathcal{PT}_A$, and $\mathrm{H}_{\mathrm{id}_A}$ is the maximal subgroup containing it), so $G = \mathcal{S}_A$. Now, choose some $a \in A$. We have $|A| = |A \setminus \{a\}|$, so there exists a bijection $f : A \to A \setminus \{a\}$, which can be regarded as a map from $\mathcal{PT}_A$. Suppose that $f = gh$ for some $g \in \mathcal{S}_A$ and some idempotent $h \in \mathrm{E}(\mathcal{PT}_A)$. Then, we have $A \setminus \{a\} \subseteq \mathrm{im}(h)$, so $h\restriction_{A\setminus\{a\}} = \mathrm{id}_{A\setminus\{a\}}$. Hence $g(x) = f(x)$ for all $x \in A$. Since $a \in A \setminus \mathrm{im}(f) = A \setminus \mathrm{im}(g)$, g cannot be a permutation of A. Therefore, $\mathcal{PT}_A$ is not factorisable. □

Proposition 3.1.33.

(i) The semigroup $\mathrm{P}^a = \mathrm{Reg}(\mathcal{PT}^a_{XY})$ *is MI-dominated.*

(ii) The semigroup $\mathrm{P}^a = \mathrm{Reg}(\mathcal{PT}^a_{XY})$ *is RP-dominated if and only if* $\mathrm{Rank}\, a < \aleph_0$.

Proof. (i) By Proposition 2.4.9(ii), we have $\mathrm{RP}(\mathrm{P}^a) = \widehat{\mathrm{H}^a_b}$, so parts (iii) and (iv) of Proposition 2.4.5 imply that P^a is MI-dominated if and only if

$$\mathrm{P}^a = \widehat{\mathrm{H}^a_b} \star_a \mathrm{P}^a \star_a \widehat{\mathrm{H}^a_b}.$$

Since $\widehat{\mathrm{H}^a_b} \subseteq \mathrm{P}^a$, we clearly have $\widehat{\mathrm{H}^a_b} \star_a \mathrm{P}^a \star_a \widehat{\mathrm{H}^a_b} \subseteq \mathrm{P}^a$. For the reverse inclusion, suppose $f = \binom{F_j}{f_j}_{j\in J} \in \mathrm{P}^a$ with $J \subseteq I$. Since $f \in \mathrm{P}^a$, Proposition 3.1.8(iii) implies that $\mathrm{im}\, f \subseteq \mathrm{dom}\, a$, $\ker a$ separates $\mathrm{im}\, f$ and $\mathrm{im}\, a$ saturates $\ker f$. Therefore, we may assume without loss of generality that $f_j \in A_j$ for all $j \in J$. We also have $F_j \cap A = \{a_k : k \in I_j\} \neq \emptyset$, where the sets $I_j \subseteq I$ are pairwise disjoint. For each $j \in J$, there exists some partition $\{F_{j,k} : k \in I_j\}$ of the set F_j such that $a_k \in F_{j,k}$ for each $k \in I_j$. Thus, if we put $L = I \setminus \bigcup_{j\in J} I_j$, and if we let

$$g = \begin{pmatrix} F_{j,k} & a_l \\ b_k & b_l \end{pmatrix}_{j\in J,\ k\in I_j,\ l\in L} \quad \text{and} \quad h = \begin{pmatrix} a_j & a_m \\ f_j & b_m \end{pmatrix}_{j\in J,\ m\in I\setminus J},$$

then we have $f = gafah$. In addition, the discussion preceding Lemma 3.1.25 implies $\overline{g} = \overline{h} = \mathrm{id}_A$, so $\overline{g} = \overline{h} \in \mathrm{H}_{\overline{b}}$ in $\mathcal{PT}_A$ and thus $g, h \in \widehat{\mathrm{H}^a_b}$.

(ii) Proposition 2.4.8 and part (i) imply that P^a is RP-dominated if and only if the local monoid $e \star_a \mathrm{P}^a \star_a e$ is factorisable for each $e \in \mathrm{MI}(\mathrm{P}^a)$. By Proposition 2.4.9$(i)$, we have $\mathrm{MI}(\mathrm{P}^a) = \mathrm{V}(a)$, so Proposition 2.4.11 implies $e \star_a \mathrm{P}^a \star_a e \cong W$ for each $e \in \mathrm{MI}(\mathrm{P}^a)$. Since $W \cong \mathcal{PT}_A$, the semigroup P^a is RP-dominated if and only if $\mathcal{PT}_A$ is factorisable, which occurs if and only if A is finite, by Lemma 3.1.32. □

Finally, we may prove one of the key results of this section:

Theorem 3.1.34.

(i) If $|\,\mathrm{P}^a\,| \geq \aleph_0$, *then* $\mathrm{rank}(\mathrm{P}^a) = |\,\mathrm{P}^a\,|$.

(ii) If $|\,\mathrm{P}^a\,| < \aleph_0$*, then*

$$\mathrm{rank}(\mathrm{P}^a) = \begin{cases} 1, & \text{if } \alpha = 0; \\ 1 + \max(2^\beta, \Lambda_I), & \text{if } \alpha = 1; \\ 2 + \max(3^\beta, \Lambda_I), & \text{if } \alpha = 2; \\ 2 + \max((\alpha+1)^\beta, \Lambda_I, 2), & \text{if } \alpha \geq 3. \end{cases}$$

Proof. Firstly, note that μ elements can generate at most μ^n n-element products for any $n \in \mathbb{N}$. Thus, if $|\,\mathrm{P}^a\,| > \aleph_0$, a generating set cannot have less than $|\,\mathrm{P}^a\,|$ elements, because $\aleph_0$-many cardinals smaller than $|\,\mathrm{P}^a\,|$ cannot add up to $|\,\mathrm{P}^a\,|$. Secondly, if $\alpha = 0$, then a is the empty function, so $\mathrm{P}^a = \{b\}$ and $\mathrm{rank}(\mathrm{P}^a) = |\,\mathrm{P}^a\,| = 1$.

Now, we examine the rest of the cases. Suppose $\alpha \geq 1$ and $|\,\mathrm{P}^a\,| \leq \aleph_0$. By Proposition 3.1.30(*iii*) and (*iv*), we have either $|X| < \aleph_0$ and $\lambda_i < \aleph_0$ for all $i \in I$, or $|X| < \aleph_0$ and $\max_{i\in I} \lambda_i = \aleph_0$. Therefore, in both cases $|A| = \alpha \leq |X| < \aleph_0$, so $W \cong \mathcal{PT}_A \cong \mathcal{PT}_\alpha$ is finite, and hence $W \setminus G_W \cong \mathcal{PT}_\alpha \setminus \mathrm{S}_\alpha$ is an ideal of W, by Lemma 3.1.25(*vi*). Now, Theorem 2.4.16 gives

$$\mathrm{rank}(\mathrm{P}^a) = \mathrm{rank}(W : G_W) + \max(|\widehat{\mathrm{H}^a_b}/\mathscr{R}\,|, |\widehat{\mathrm{H}^a_b}/\mathscr{L}\,|, \mathrm{rank}(G_W)),$$

because P^a is MI-dominated (as proved in Proposition 3.1.33(*i*)). By Theorem 3.1.26(*i*) and (*ii*), we have $|\widehat{\mathrm{H}^a_b}/\mathscr{R}\,| = (\alpha+1)^\beta$ and $|\widehat{\mathrm{H}^a_b}/\mathscr{L}\,| = \Lambda_I$. In addition, $W \cong \mathcal{PT}_A$, so $G_W \cong \mathrm{S}_\alpha$. Thus,

$$\mathrm{rank}(\mathrm{P}^a) = \mathrm{rank}(\mathcal{PT}_\alpha : \mathrm{S}_\alpha) + \max((\alpha+1)^\beta, \Lambda_I, \mathrm{rank}(\mathrm{S}_\alpha)). \tag{3.6}$$

In the case that $|\,\mathrm{P}^a\,| = \aleph_0$, we have $\max_{i\in I} \lambda_i = \aleph_0$, so $\Lambda_I = \aleph_0$, and hence $|\,\mathrm{P}^a\,| = \aleph_0$.

Finally, part (*ii*) follows directly from(3.6), having in mind that

$$\mathrm{rank}(\mathrm{S}_\alpha) = \begin{cases} 1, & \text{if } \alpha \leq 2; \\ 2, & \text{if } 3 \leq \alpha < \aleph_0; \end{cases} \quad \text{and} \quad \mathrm{rank}(\mathcal{PT}_\alpha : \mathrm{S}_\alpha) = \begin{cases} 1, & \text{if } \alpha = 1; \\ 2, & \text{if } 2 \leq \alpha < \aleph_0. \end{cases}$$

The first equality is a well-known result, and the second an easily proved one: to generate $\mathcal{PT}_\alpha$, we need at least one element from $\mathcal{T}_\alpha \setminus \mathrm{S}_\alpha$ and at least one element from $\mathcal{PT}_\alpha \setminus \mathcal{T}_\alpha$, because $(\mathcal{PT}_\alpha \setminus \mathcal{T}_\alpha) \cup \mathrm{S}_\alpha$ and $\mathcal{T}_\alpha$ are subsemigroups of $\mathcal{PT}_\alpha$; it turns out that any pair from the set $(\mathrm{D}_{\alpha-1}(\mathcal{T}_\alpha),\ \mathrm{D}_{\alpha-1}(\mathcal{PT}_\alpha) \setminus \mathcal{T}_\alpha)$ will do. □

Remark 3.1.35. If $|\,\mathrm{P}^\alpha\,| < \aleph_0$ and if a is injective or surjective, we may simplify the above formula. Note that, by 3.1.30(*iii*), the first assumption implies $\alpha \leq |X| < \aleph_0$.

- If a is injective, then $\Lambda_I = 1$ and $\beta = |X \setminus \mathrm{im}\, a| = |X| - \alpha$. Thus, in this case we have

$$\mathrm{rank}(\mathrm{P}^a) = \begin{cases} 1, & \text{if } \alpha = 0; \\ 1 + 2^{|X|-1}, & \text{if } \alpha = 1; \\ 2 + 3^{|X|-2}, & \text{if } \alpha = 2; \\ 2 + \max((\alpha+1)^{|X|-\alpha}, 2), & \text{if } \alpha \geq 3. \end{cases}$$

If a is additionally full and non-surjective (recall that we omit the case where a is a full bijection), then $\mathcal{PT}^a_{XY} = \mathcal{PT}(X, A)$, and we have $\alpha < |X|$, so $2 \leq (\alpha+1)^{|X|-\alpha}$ (for $\alpha \geq 0$). Therefore, for any set $A \subsetneq X$,

$$\operatorname{rank}(\operatorname{Reg}(\mathcal{PT}(X, A))) = \begin{cases} 1, & \text{if } A = \emptyset; \\ 1 + 2^{|X|-1}, & \text{if } |A| = 1; \\ 2 + (|A|+1)^{|X|-|A|}, & \text{if } |A| \geq 2. \end{cases}$$

- If a is surjective, then $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma)$, and we have $A = X$, so $\beta = 0$. Recall that π_σ and $\mathrm{u}(\sigma)$ denote the partition and the underlying set corresponding to the equivalence relation σ, respectively. Then, for any equivalence relation σ with $\mathrm{u}(\sigma) \subseteq Y$, we have

$$\operatorname{rank}(\operatorname{Reg}(\mathcal{PT}(Y, \sigma))) = \begin{cases} 1, & \text{if } \pi_\sigma = \emptyset; \\ 1 + \Lambda_I, & \text{if } |\pi_\sigma| = 1; \\ 2 + \Lambda_I, & \text{if } |\pi_\sigma| = 2; \\ 2 + \max(\Lambda_I, 2), & \text{if } |\pi_\sigma| \geq 3. \end{cases}$$

- If a is a non-full bijection, we have the benefits of both injectivity and surjectivity, so $\operatorname{rank}(\mathrm{P}^a) = \operatorname{rank}(\mathcal{PT}_A)$.

3.1.4 Idempotents and idempotent-generation

Following the path paved in Chapter 2, now we investigate the idempotents and the idempotent-generated subsemigroup of $\mathcal{PT}^a_{XY}$. In particular, we characterise the idempotents and calculate their number; further, we describe the idempotent-generated subsemigroup in terms of its connection with $\mathcal{PT}_A$ via φ and infer the formula for its rank. In addition, we provide a neat description for it in the case where $\alpha < \aleph_0$. As usual, each result is also given in a simplified form corresponding to the cases when a is full, injective, or surjective.

Since all idempotents are obviously regular elements, we are in fact investigating the idempotents and the idempotent-generated subsemigroup of P^a. To ensure easier understanding, we introduce the corresponding notation: let

$$\mathrm{E}_a(\mathcal{PT}^a_{XY}) = \{f \in \mathcal{PT}_{XY} : f = f \star_a f\} \qquad (= \mathrm{E}_a(\mathrm{P}^a)), \text{ and}$$
$$\mathcal{E}^a_{XY} = \mathbb{E}_a(\mathcal{PT}^a_{XY}) = \langle E_a(\mathcal{PT}^a_{XY})\rangle_a \quad (= \mathbb{E}_a(\mathrm{P}^a)).$$

denote the set of idempotents of $\mathcal{PT}^a_{XY}$ and the idempotent-generated subsemigroup of $\mathcal{PT}^a_{XY}$, respectively.

We start with the properties of $\mathrm{E}_a(\mathcal{PT}^a_{XY})$.

Proposition 3.1.36.

(i) $\mathrm{E}_a(\mathcal{PT}^a_{XY}) = \{f \in \mathcal{PT}_{XY} : (af)\restriction_{\operatorname{im} f} = \operatorname{id}_{\operatorname{im} f}\}$.

(ii) If $|\,\mathrm{P}^a\,| \geq \aleph_0$*, then* $|\,\mathrm{E}_a(\mathcal{PT}^a_{XY})| = |\,\mathrm{P}^a\,|$.

(iii) If $|\,\mathrm{P}^a\,| < \aleph_0$, *then*

$$|\,\mathrm{E}_a(\mathcal{PT}^a_{XY})| = \sum_{\mu=0}^{\alpha} (\mu+1)^{|X|-\mu} \sum_{\substack{J\subseteq I\\ |J|=\mu}} \Lambda_J.$$

Proof. (i) Since the defining property of an idempotent $f \in \mathrm{P}^a$ is the equality $faf = f$, which is equivalent to $(af)\restriction_{\mathrm{im}\, f} = \mathrm{id}_{\mathrm{im}\, f}$, the characterisation follows.

(ii) Let $|\,\mathrm{P}^a\,| \geq \aleph_0$. Obviously, it suffices to prove $|\,\mathrm{E}_a(\mathcal{PT}^a_{XY})| \geq |\,\mathrm{P}^a\,|$. By Proposition 3.1.30$(iv)$ and (v), we have $\alpha \geq 1$ and exactly one of the following statements is true:

- $|X| < \aleph_0$ and $\max_{i\in I} \lambda_i \geq \aleph_0$,
- $|X| \geq \aleph_0$.

In the first case, we will use $\mathrm{E}_a(\mathcal{PT}^a_{XY}) \supseteq \mathrm{E}_a(\widehat{\mathrm{H}^a_b})$. Recall that the latter is an $(\alpha+1)^\beta \times \Lambda_I$ rectangular band, by Theorem 3.1.26(v). Hence, Proposition 3.1.30(ii) gives

$$|\,\mathrm{E}_a(\mathcal{PT}^a_{XY})| \geq |\,\mathrm{E}_a(\widehat{\mathrm{H}^a_b})| \geq \Lambda_I = \max_{i\in I} \lambda_i = |\,\mathrm{P}^a\,|.$$

In the second case, Proposition 3.1.30(i) yields $|\,\mathrm{P}^a\,| = \max(2^{|X|}, \Lambda_I)$. If $|\,\mathrm{P}^a\,| = \Lambda_I$, the proof is same as in the previous case, save the use of Proposition 3.1.30(ii). Otherwise, we need to prove $|\,\mathrm{E}_a(\mathcal{PT}^a_{XY})| \geq 2^{|X|}$. Fix an $i \in I$. The set $X \setminus a_i$ has $2^{|X|}$ subsets, since $|X| > \aleph_0$. For each $C \subseteq X \setminus a_i$, we define $\binom{C\cup\{a_i\}}{b_i} \in \mathcal{PT}_{XY}$, which is evidently a $\star_a$-idempotent, unique among the others defined in this manner, due to its domain. Thus, we have enumerated $2^{|X|}$ different idempotents.

(iii) Suppose $|\,\mathrm{P}^a\,| < \aleph_0$. By Proposition 3.1.30$(iii)$, we have either $\alpha = 0$, or $|X| < \aleph_0$ and $\lambda_i < \aleph_0$ for all $i \in I$. If $\alpha = 0$, then $\mathrm{P}^a = \mathrm{E}(\mathrm{P}^a) = \{b\}$, so the statement is true in that case. Suppose $\alpha \geq 1$ and let $f = \binom{F_j}{f_j}_{j\in J}$ be any idempotent. Since $f \in \mathrm{P}^a$, by Proposition 3.1.8(iii), the following hold: $\mathrm{im}\, f \subseteq \mathrm{dom}\, a$, $\ker a$ separates $\mathrm{im}\, f$ and $\mathrm{im}\, a$ saturates $\ker f$. Thus, we may assume without loss of generality that $f_j \in A_j$ for all $j \in J$. Then, from the condition $(af)\restriction_{\mathrm{im}\, f} = \mathrm{id}_{\mathrm{im}\, f}$ in (i), we have $a_j \in F_j$ for all $j \in J$. So, in order to fix an idempotent f, we need to fix the set $J \subseteq I$, the element $f_j \in A_j$ for each $j \in J$, and the sets $\mathrm{dom}\, f \setminus \mathrm{im}\, a$ and $\ker f$. Note that $0 \leq |J| \leq \alpha$, and if $|J| = \mu$, the choice of the pair (J, im_f) can be made in

$$\sum_{\substack{J\subseteq I\\ |J|=\mu}} \Lambda_J$$

ways. This leaves $|X| - \mu$ elements in $X \setminus \{a_j : j \in J\}$ to be placed either in one of the μ classes of $\ker f$, or outside $\mathrm{dom}\, f$. Therefore, there are

$$\sum_{\substack{J\subseteq I\\ |J|=\mu}} \Lambda_J \cdot (\mu+1)^{|X|-\mu}$$

idempotents of rank μ. As the possible values for the rank span from 0 to α, this concludes the proof. ☐

Remark 3.1.37. Suppose $|\operatorname{P}^a| < \aleph_0$. As before, we discuss the simplifications that can be made in special cases.

- If a is injective, $\Lambda_J = 1$ for all $J \subseteq I$, so

$$\sum_{\substack{J\subseteq I\\|J|=\mu}} \Lambda_J = \sum_{\substack{J\subseteq I\\|J|=\mu}} 1 = \binom{|I|}{\mu} = \binom{\alpha}{\mu}.$$

  Therefore $|\operatorname{E}_a(\mathcal{PT}^a_{XY})| = \sum_{\mu=0}^{\alpha}(\mu+1)^{|X|-\mu}\binom{\alpha}{\mu}$. In the case that a is both injective and full, we have $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(X, A)$, so

$$|\operatorname{E}(\mathcal{PT}(X, A))| = \sum_{\mu=0}^{|A|}(\mu+1)^{|X|-\mu}\binom{|A|}{\mu}.$$

- If a is surjective, then $X = A$ so $|X| = \alpha = |\pi_\sigma|$. Since $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma)$ in this case, we have

$$|\operatorname{E}(\mathcal{PT}(Y, \sigma))| = \sum_{\mu=0}^{|\pi_\sigma|}(\mu+1)^{|\pi_\sigma|-\mu} \sum_{\substack{J\subseteq I\\|J|=\mu}} \Lambda_J.$$

- If a is a (non-full) bijection, we have both $\Lambda_J = 1$ for all $J \subseteq I$ and $X = A$, so in this case

$$|\operatorname{E}_a(\mathcal{PT}^a_{XY})| = \sum_{\mu=0}^{\alpha}(\mu+1)^{\alpha-\mu}\binom{\alpha}{\mu}.$$

  This equals the number of idempotents in $\mathcal{PT}_\alpha$ (obtained in Corollary 2.7.5 in [45]), since $\operatorname{P}^a \cong \mathcal{PT}_\alpha$.

In the following, we need some additional information on the semigroup $W \cong \mathcal{PT}_A$. In order to present the needed results, we enhance our notation. Let $\operatorname{E}(\mathcal{PT}_A)$ and $\mathbb{E}(\mathcal{PT}_A)$ denote the set of idempotents and the idempotent-generated subsemigroup of $\mathcal{PT}_A$, respectively. Further, for a map $f \in \mathcal{PT}_A$, we introduce

$$\begin{aligned} \operatorname{sh} f &= |\{x \in \operatorname{dom} f : xf \neq x\}|, & \operatorname{def} f &= |A \setminus \operatorname{im} f|,\\ \operatorname{coll} f &= \sum_{x\in \operatorname{im} f}(|xf^{-1}| - 1), & \operatorname{codef} f &= |A \setminus \operatorname{dom} f|, \end{aligned}$$

named the *shift*, *collapse*, *defect* and *codefect* of f, respectively.

Having introduced these, we may state the required results. Since the proofs are lengthy and do not contribute to our understanding of sandwich semigroups, we skip them and only give references for them. For part (i), see [43] and [46]. Part

(ii) follows from Theorem III in [55], using the same approach for examination of $\mathcal{PT}_A$, as was used for $\mathcal{T}_A$.

Proposition 3.1.38.

(i) If $|A| < \aleph_0$, *then* $\mathbb{E}(\mathcal{PT}_A) = \{\mathrm{id}_A\} \cup (\mathcal{PT}_A \setminus \mathrm{S}_A)$, *and*

$$\mathrm{rank}(\mathbb{E}(\mathcal{PT}_A)) = \mathrm{idrank}(\mathbb{E}(\mathcal{PT}_A)) = \binom{\alpha+1}{2} + 1.$$

(ii) If $|A| \geq \aleph_0$, *then*

$$\begin{aligned}\mathbb{E}(\mathcal{PT}_A) = \{\mathrm{id}_A\} &\cup \{f \in \mathcal{PT}_A \setminus \mathrm{S}_A : \mathrm{sh}\, f + \mathrm{codef}\, f < \aleph_0\} \\ &\cup \{f \in \mathcal{PT}_A : \mathrm{sh}\, f + \mathrm{codef}\, f = \mathrm{coll}\, f + \mathrm{codef}\, f \\ &\qquad = \mathrm{def}\, f \geq \aleph_0\}\end{aligned}$$

and $\mathrm{rank}(\mathbb{E}(\mathcal{PT}_A)) = \mathrm{idrank}(\mathbb{E}(\mathcal{PT}_A)) = |\,\mathcal{PT}_A\,| = 2^{|A|}$.

Using these properties of $\mathcal{PT}_A$, we may calculate the rank and the idempotent rank of $\mathbb{E}_a(\mathcal{PT}^a_{XY})$, applying Theorem 2.4.17 and the fact that P^a is MI-dominated (Proposition 3.1.33(i)).

Theorem 3.1.39.

(i) $\mathcal{E}^a_{XY} = \mathbb{E}_a(\mathcal{PT}^a_{XY}) = (\mathbb{E}(\mathcal{PT}_A))\varphi^{-1}$,

(ii) $\mathrm{rank}(\mathcal{E}^a_{XY}) = \mathrm{idrank}(\mathcal{E}^a_{XY}) = \begin{cases} |\mathcal{E}^a_{XY}| = |\,\mathrm{P}^a\,|, & |\,\mathrm{P}^a\,| \geq \aleph_0; \\ \binom{\alpha+1}{2} + \max((\alpha+1)^\beta, \Lambda_I), & |\,\mathrm{P}^a\,| < \aleph_0. \end{cases}$

Proof. (i) follows directly from Lemma 2.3.11, since

$$(\mathbb{E}_b(W))\phi^{-1} = (\mathbb{E}(\mathcal{PT}_A))\varphi^{-1}.$$

(ii) In $\mathcal{PT}^a_{XY}$, the set $\mathrm{E}_a(\mathcal{PT}^a_{XY}) = \mathrm{E}_a(\mathrm{P}^a)$ is a subset of the subsemigroup P^a, so we have $\mathbb{E}_a(\mathcal{PT}^a_{XY}) = \mathbb{E}_a(\mathrm{P}^a)$. Thus,

$$|\,\mathrm{E}_a(\mathcal{PT}^a_{XY})| \leq |\mathbb{E}_a(\mathcal{PT}^a_{XY})| \leq |\,\mathrm{P}^a\,|.$$

In the case that $|\,\mathrm{P}^a\,| \geq \aleph_0$, Proposition 3.1.36$(ii)$ gives $|\,\mathrm{E}_a(\mathcal{PT}^a_{XY})| = |\,\mathrm{P}^a\,|$, hence $|\mathbb{E}_a(\mathcal{PT}^a_{XY})| = |\,\mathrm{P}^a\,|$.

Let us complete the proof in the case where $|\,\mathrm{P}^a\,| > \aleph_0$. Since $|\mathcal{E}^a_{XY}| = |\,\mathrm{P}^a\,|$ is an uncountable set, it cannot be generated by a set od smaller size.

Next, suppose $|\,\mathrm{P}^a\,| \leq \aleph_0$. Parts (iii) and (iv) of Proposition 3.1.30 give $\alpha = 0$ or $|X| < \aleph_0$. In either case, $\alpha < \aleph_0$. Having in mind that $W \cong \mathcal{PT}_A$ and the fact that P^a is MI-dominated, Theorem 2.4.17, Proposition 3.1.38(i), and parts (i) and

(ii) of Theorem 3.1.26 together give

$$\begin{aligned}\operatorname{rank}(\mathcal{E}^a_{XY}) = \operatorname{idrank}(\mathcal{E}^a_{XY}) &= \operatorname{rank}(\mathcal{PT}_A) + \max((\alpha+1)^\beta, \Lambda_I) - 1 \\ &= \binom{\alpha+1}{2} + \max((\alpha+1)^\beta, \Lambda_I).\end{aligned} \tag{3.7}$$

If $|\,\mathrm{P}^a\,| < \aleph_0$, the proof is complete. Further, in the case where $|\,\mathrm{P}^a\,| = \aleph_0$, Proposition 3.1.30$(iv)$ gives $\max_{i\in I} \lambda_i = \aleph_0$, so (3.7) implies

$$\operatorname{rank}(\mathcal{E}^a_{XY}) = \operatorname{idrank}(\mathcal{E}^a_{XY}) = \Lambda_I = \max_{i\in I} \lambda_i = \aleph_0 = |\,\mathrm{P}^a\,|. \qquad \square$$

Remark 3.1.40. Using the facts stated in the previous remarks, we give the simplified version of the formula from part (ii) in special cases:

- if a is full and injective, we have $\Lambda_I = 1$ (and $\beta = |X \setminus \operatorname{im} a| = |X| - |A|$ in the case where $|X| < \aleph_0$), so the following holds for $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(X, A)$:

$$\begin{aligned}\operatorname{rank}(\mathbb{E}(\mathcal{PT}(X,A))) &= \operatorname{idrank}(\mathbb{E}(\mathcal{PT}(X,A))) \\ &= \begin{cases} |\mathbb{E}(\mathcal{PT}(X,A))| = |\,\mathrm{P}^a\,|, & |\,\mathrm{P}^a\,| \geq \aleph_0; \\ \binom{|A|+1}{2} + (|A|+1)^{|X|-|A|}, & |\,\mathrm{P}^a\,| < \aleph_0. \end{cases}\end{aligned}$$

- if a is surjective, then $\alpha = |A| = |X| = |\pi_\sigma|$ and $\beta = |X \setminus \operatorname{im} a| = 0$; thus, for $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma)$ we have

$$\begin{aligned}\operatorname{rank}(\mathbb{E}(\mathcal{PT}(Y,\sigma))) &= \operatorname{idrank}(\mathbb{E}(\mathcal{PT}(Y,\sigma))) \\ &= \begin{cases} |\mathbb{E}(\mathcal{PT}(Y,\sigma))| = |\,\mathrm{P}^a\,|, & |\,\mathrm{P}^a\,| \geq \aleph_0; \\ \binom{|\pi_\sigma|+1}{2} + \Lambda_I, & |\,\mathrm{P}^a\,| < \aleph_0. \end{cases}\end{aligned}$$

In Theorem 3.1.39(i), we gave a description of $\mathcal{E}^a_{XY}$ via the map φ. If $\alpha < \aleph_0$, we can offer an elegant alternative description. In order to prove this result, we need the following lemma.

Recall that D^a_α is the regular class of $\mathcal{PT}^a_{XY}$ containing all the regular elements of rank α (see Proposition 3.1.18).

Lemma 3.1.41. *If $\alpha < \aleph_0$, then $\mathrm{J}^a_b = \mathrm{D}^a_\alpha = \widehat{\mathrm{H}^a_b}$. In the case that $\alpha = \xi = \max(|X|, |Y|)$ as well, J^a_b is the maximum $\mathscr{J}^a$-class of $\mathcal{PT}^a_{XY}$.*

Proof. Since $\alpha < \aleph_0$, the semigroup $W \cong \mathcal{PT}_A \cong \mathcal{PT}_\alpha$ is finite and hence stable (see Section 1.3). Thus, Proposition 2.4.10(i) gives $\mathrm{J}^a_b = \mathrm{D}^a_b = \mathrm{D}^a_\alpha$. Moreover, Propositions 2.4.9(ii) and 2.4.10(iv) together imply $\mathrm{J}^a_b = \widehat{\mathrm{H}^a_b}$, as $\mathcal{PT}$ is regular. The last statement follows from Proposition 3.1.19(ii). $\square$

Theorem 3.1.42. *If $\alpha = \operatorname{rank} a < \aleph_0$, then*

$$\mathcal{E}^a_{XY} = \mathbb{E}_a(\mathcal{PT}^a_{XY}) = \mathrm{E}_a(\mathrm{D}^a_\alpha) \cup (\mathrm{P}^a \setminus \mathrm{D}^a_\alpha).$$

Proof. Suppose $\alpha = \operatorname{rank} a < \aleph_0$. By Theorem 3.1.39($i$), Proposition 3.1.38($i$), and Theorem 3.1.26($v$), we have

$$\begin{aligned}\mathcal{E}^a_{XY} = (\mathbb{E}(\mathcal{PT}_A))\varphi^{-1} &= (\mathrm{id}_A)\varphi^{-1} \cup (\mathcal{PT}_A \setminus \mathrm{S}_A)\varphi^{-1} \\ &= (\mathrm{id}_A)\varphi^{-1} \cup (\mathcal{PT}_A)\varphi^{-1} \setminus (\mathrm{S}_A)\varphi^{-1} \\ &= \mathrm{E}_a(\widehat{\mathrm{H}^a_b}) \cup (\mathrm{P}^a \setminus \widehat{\mathrm{H}^a_b})\end{aligned}$$

(the last two equalities following from the fact that φ is a homomorphism). Thus, the result follows directly from Lemma 3.1.41. □

Remark 3.1.43. As always, we analyse the result in the special cases. Suppose $\alpha = \operatorname{rank} a < \aleph_0$.

- If a is injective and full, then $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(X, A)$. In $\mathcal{PT}(X, A)$, we have

$$\begin{aligned}\mathrm{E}(\mathrm{D}_\alpha) &= \{f \in \mathcal{PT}(X, A) : ff = f,\ \operatorname{rank} f = \alpha\} \\ &= \{f \in \mathcal{PT}(X, A) : f\restriction_A = \mathrm{id}_A\}.\end{aligned}$$

  Therefore, Lemma 3.1.21 implies

$$\begin{aligned}\mathbb{E}(\mathcal{PT}(X, A)) = \{&f \in \mathcal{PT}(X, A) : f\restriction_A = \mathrm{id}_A\} \\ &\cup \{f \in \mathcal{PT}(X, A) : \ker f \ \text{ is saturated by } A,\ \operatorname{rank} f < |A|\}.\end{aligned}$$

- If a is surjective, then $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma)$. In $\mathcal{PT}(Y, \sigma)$, we have

$$\begin{aligned}\mathrm{E}(\mathrm{D}_\alpha) &= \{f \in \mathcal{PT}(Y, \sigma) : ff = f,\ \operatorname{rank} f = \alpha\} \\ &= \{f \in \mathcal{PT}(Y, \sigma) : \ker f = \sigma,\ (S)f \in S \text{ for each } S \in \pi_\sigma\}.\end{aligned}$$

  Therefore, Lemma 3.1.21 implies

$$\begin{aligned}\mathbb{E}(\mathcal{PT}(Y, \sigma)) = \{&f \in \mathcal{PT}(Y, \sigma) : \ker f = \sigma,\ (S)f \in S \text{ for each } S \in \pi_\sigma\} \\ &\cup \{f \in \mathcal{PT}(Y, \sigma) : \operatorname{im} f \subseteq \mathrm{u}(\sigma),\ \operatorname{im} f \text{ is separated by } \sigma, \\ &\qquad \operatorname{rank} f < |\pi_\sigma|\}.\end{aligned}$$

3.1.5 The rank of a sandwich semigroup $\mathcal{PT}^a_{XY}$

In this section, we complete the investigation of $\mathcal{PT}^a_{XY}$ by calculating its rank. It turns out that, in the finite case, we have entirely different formulae depending on the answers to the following questions: Is a full? Is a injective? Is a surjective? The results presented in this section were obtained in [34].

We start with the simpler cases. First, note that, when defining an element of $\mathcal{PT}_{XY}$, each of the $|X|$ elements in X can either be mapped to one of the $|Y|$ elements in Y or be placed outside the domain. Hence, we have $|\mathcal{PT}^a_{XY}| = (|Y| + 1)^{|X|}$. Using this, we may deduce the rank in the following cases.

- **Suppose $\mathbf{X} = \emptyset$ or $\mathbf{Y} = \emptyset$.** Then, $\mathcal{PT}^a_{XY} = \{\emptyset\}$, and therefore $\operatorname{rank}(\mathcal{PT}^a_{XY}) = |\mathcal{PT}^a_{XY}| = 1$.

- **Suppose $\mathbf{X}, \mathbf{Y} \neq \emptyset$ and $\alpha = \mathbf{0}$.** Since $a = \emptyset$, we have $fag = \emptyset$ for all $f, g \in \mathcal{PT}^a_{XY}$. Therefore,

$$\operatorname{rank}(\mathcal{PT}^a_{XY}) = |\mathcal{PT}^a_{XY} \setminus \{\emptyset\}| = |\mathcal{PT}^a_{XY}| - 1 = (|Y| + 1)^{|X|} - 1$$

- **Suppose $\mathbf{X}, \mathbf{Y} \neq \emptyset$ and suppose $|\mathbf{X}| \geq \aleph_0$ or $|\mathbf{Y}| > \aleph_0$.** Obviously, this holds if and only if $\mathcal{PT}^a_{XY} > \aleph_0$; in such case $\operatorname{rank}(\mathcal{PT}^a_{XY}) = |\mathcal{PT}^a_{XY}|$.

- **Suppose $\mathbf{X}, \mathbf{Y} \neq \emptyset$, $|\mathbf{X}| < \aleph_0$, $|\mathbf{Y}| \leq \aleph_0$, $\alpha \geq \mathbf{1}$, and suppose a is a full bijection.** Since a being a full bijection implies $\mathcal{PT}^a_{XY} \cong \mathcal{PT}_A = \mathcal{PT}_X$ and we assumed $|X| < \aleph_0$, by Theorem 3.1.5 in [45], we have

$$\operatorname{rank}(\mathcal{PT}^a_{XY}) = \operatorname{rank}(\mathcal{PT}_{|X|}) = \begin{cases} |X| + 1, & |X| \leq 2; \\ 4, & |X| > 2. \end{cases} \tag{3.8}$$

Hence, for the remainder of this subsection, we assume that $\mathbf{X}, \mathbf{Y} \neq \emptyset$, $|\mathbf{X}| < \aleph_0$, $|\mathbf{Y}| \leq \aleph_0$, $\alpha \geq \mathbf{1}$ and that a is either non-full or non-injective or non-surjective.

To simplify navigating through results concerning different cases, we give the following table:

a full?	a injective?	a surjective	Reference	Egg-box diagram
N	N	N	Theorem 3.1.48	Figure 3.4
Y	N	N		Figure 3.5
N	Y	N		Figure 3.6
Y	Y	N	Theorem 3.1.51	Figure 3.7
N	N	Y	Theorem 3.1.57	Figure 3.8
Y	N	Y		Figure 3.8
N	Y	Y		Figure 3.8
Y	Y	Y	see (3.8)	Figure 3.10

Note that the assumption $|X| < \aleph_0$ implies $\alpha \leq \xi = \min(|X|, |Y|) < \aleph_0$, so a is stable, by Proposition 3.1.7(*iii*). Thus, Propositions 2.2.25 and 3.1.2(*vi*) imply that $\mathscr{J}^a = \mathscr{D}^a$ in $\mathcal{PT}^a_{XY}$. This information will be vital for the discussion of generation of the maximal $\mathscr{J}$-classes. As for the rest of the elements, we will be able to generate them by multiplying elements having higher ranks. For this reason, in this subsection we deal with the $\mathscr{J} = \mathscr{D}$-classes of $\mathcal{PT}_{XY}$. Recall the notation

$$\mathrm{D}_\mu = \mathrm{D}_\mu(\mathcal{PT}_{XY}) = \{f \in \mathcal{PT}_{XY} : \operatorname{Rank} f = \mu\} \text{ for each } \mu \in \{0, 1, \ldots, \xi\},$$

which gives $\mathcal{PT}_{XY} = \mathrm{D}_0 \cup \mathrm{D}_1 \cup \cdots \cup \mathrm{D}_\xi$.

We may also adapt the previously introduced notation to the assumptions made above. Since $\alpha < \aleph_0$, we may assume

$$a = \left(\begin{smallmatrix} A_1 & \cdots & A_\alpha \\ a_1 & \cdots & a_\alpha \end{smallmatrix}\right) \quad \text{and} \quad b = \left(\begin{smallmatrix} a_1 & \cdots & a_\alpha \\ b_1 & \cdots & b_\alpha \end{smallmatrix}\right).$$

Having done that, we present the lemma describing why the concept of "downwards generating" works.

Lemma 3.1.44.

(i) If $\mu \leq \alpha - 2$, then $\mathrm{D}_\mu \subseteq \mathrm{D}_{\mu+1} \star_a \mathrm{D}_{\mu+1}$.

(ii) If a is not surjective, then $\mathrm{D}_{\alpha-1} \subseteq \mathrm{D}_\alpha \star_a \mathrm{D}_\alpha$.

Proof. Since both parts may be proved using the same approach, we handle them together. Suppose $\mu \leq \alpha - 1$ and let $f = \binom{F_1\ \cdots\ F_\mu}{f_1\ \cdots\ f_\mu} \in \mathrm{D}_\mu$. From $\mu < \alpha \leq |X|$, we conclude that the assumptions of both parts ((*i*): $\mu \leq \alpha + 2$ and (*ii*): a is non-surjective) guarantee the existence of an element $z \in X \setminus \{a_1, \ldots, a_{\mu+1}\}$. Furthermore, one of the following must hold:

(a) f is non-full, in which case we fix some $x \in X \setminus \operatorname{dom} f$, or

(b) f is non-injective, in which case there exists at least one non-singleton class in $\ker f$; hence, we may assume without loss of generality that $|F_\mu| \geq 2$ and fix some partition $\{F'_\mu, F''_{\mu+1}\}$ of F_μ.

Furthermore, since $\mu < \alpha \leq |Y|$, we may fix an element $y \in Y \setminus \operatorname{im} f$. Now, we define maps

$$g = \begin{cases} \begin{pmatrix} F_1 & \cdots & F_\mu & x \\ b_1 & \cdots & b_\mu & b_{\mu+1} \end{pmatrix} & \text{in case (a);} \\ \begin{pmatrix} F_1 & \cdots & F_{\mu-1} & F'_\mu & F''_\mu \\ b_1 & \cdots & b_{\mu-1} & b_\mu & b_{\mu+1} \end{pmatrix} & \text{in case (b);} \end{cases}$$

and

$$h = \begin{cases} \begin{pmatrix} a_1 & \cdots & a_\mu & z \\ f_1 & \cdots & f_\mu & y \end{pmatrix} & \text{in case (a);} \\ \begin{pmatrix} a_1 & \cdots & a_\mu & a_{\mu+1} & z \\ f_1 & \cdots & f_\mu & f_\mu & y \end{pmatrix} & \text{in case (b);} \end{cases}$$

Clearly, both in (a) and in (b) we have $f = gah = g \star_a h \in \mathrm{D}_{\mu+1} \star_a \mathrm{D}_{\mu+1}$. □

Using the above concept, we can generate the whole semigroup, using only the elements of ranks α and $\alpha - 1$. Moreover, if a is non-surjective, the elements of D_α suffice.

Corollary 3.1.45.

(i) In $\mathcal{PT}^a_{XY}$ holds $\mathrm{D}_0 \cup \mathrm{D}_1 \cup \cdots \cup \mathrm{D}_\alpha = \langle \mathrm{D}_\alpha \cup \mathrm{D}_{\alpha-1} \rangle_a$.

(ii) If a is non-surjective, we have $\mathrm{D}_0 \cup \mathrm{D}_1 \cup \cdots \cup \mathrm{D}_\alpha = \langle \mathrm{D}_\alpha \rangle_a$.

Proof. Note that Lemma 3.1.1(*iv*) implies the reverse containment both in (*i*) and in (*ii*). Lemma 3.1.44(*i*) implies $\mathrm{D}_0 \cup \mathrm{D}_1 \cup \ldots \cup \mathrm{D}_{\alpha-2} \subseteq \langle \mathrm{D}_{\alpha-1} \rangle_a$. Thus, part (*i*) follows immediately, and the second part follows from Lemma 3.1.44(*ii*). □

This is where our path forks. Depending on the properties of the sandwich element a, we use different strategies for generating $\mathcal{PT}^a_{XY}$. Results 3.1.46 - 3.1.48 deal with the case where $\alpha < \xi = \min(|X|, |Y|)$, and results 3.1.50 - 3.1.57 concern the remaining case, where $\alpha = \xi$.

Suppose $\alpha < \xi$. Firstly, we give a lemma describing a type of elements in D_α that can be generated using the elements of $\mathrm{D}_{\alpha+1}$.

Lemma 3.1.46. *Suppose $\alpha < \xi$ and let $f \in \mathrm{D}_\alpha$. If a and f are both non-injective or both non-full, then $f \in \mathrm{D}_{\alpha+1} \star_a \mathrm{D}_{\alpha+1}$.*

Proof. Suppose $f = \begin{pmatrix} F_1 & \dots & F_\alpha \\ f_1 & \dots & f_\alpha \end{pmatrix}$. Since $\operatorname{Rank} f = \alpha = \operatorname{Rank} a < \xi$, there exist some $x \in X \setminus \operatorname{im} a$ and $y \in Y \setminus \operatorname{im} f$. If a and f are both non-injective, there exist both a non-singleton $\ker a$-class and a non-singleton $\ker f$-class. We may suppose without loss of generality that A_α and F_α are such classes (the same index does not jeopardise generality, since it is just a matter of convenient enumeration). Therefore, there exist an element $z \in A_\alpha \setminus \{b_\alpha\}$, and some partition $\{F'_\alpha, F''_\alpha\}$ of F_α. If we define $g = \begin{pmatrix} F_1 & \dots & F_{\alpha-1} & F'_\alpha & F''_\alpha \\ b_1 & \dots & b_{\alpha-1} & b_\alpha & z \end{pmatrix}$ and $h = \begin{pmatrix} a_1 & \dots & a_\alpha & x \\ f_1 & \dots & f_\alpha & y \end{pmatrix}$, it is easily seen that $f = gah \in \mathrm{D}_{\alpha+1} \star_a \mathrm{D}_{\alpha+1}$

In the alternative case, when a and f are both non-full, we may choose some $u \in X \setminus \operatorname{dom} f$ and $v \in Y \setminus \operatorname{dom} a$. Similarly as in the first case, for the map $g = \begin{pmatrix} F_1 & \dots & F_{\alpha-1} & F_\alpha & u \\ b_1 & \dots & b_{\alpha-1} & b_\alpha & v \end{pmatrix}$ and the map h defined above, we have $f = gah \in \mathrm{D}_{\alpha+1} \star_a \mathrm{D}_{\alpha+1}$. □

The following lemma gives us an inkling of the way in which the whole set D_α will eventually be generated.

Lemma 3.1.47. *Suppose $\alpha < \xi$ and $f = g \star_a h$, where $g, h \in \mathcal{PT}_{XY}$ and $f \in \mathrm{D}_\alpha$.*

(i) If a is injective and f full, then $f \mathscr{R} g$.

(ii) If a is full and f injective, then $f \mathscr{R} g$.

Proof. First, we draw some conclusions from the assumptions of the lemma. By Lemma 3.1.1(i) and (iii), from $f = g \star_a h$, we have

(a) $\operatorname{dom} f \subseteq \operatorname{dom} g$, and (b) $\ker f \supseteq (\ker g)\restriction_{\operatorname{dom} f}$.

Moreover, since $f = g \star_a h \in \mathrm{D}_\alpha$, we have

$$\alpha = \operatorname{Rank} a = \operatorname{Rank} f = \operatorname{Rank}(gah) \leq \operatorname{Rank}(ah) \leq \operatorname{Rank} a,$$

so $\operatorname{Rank}(ah) = \operatorname{Rank} a = \alpha < \aleph_0$. Thus, $\operatorname{dom}(ah) = \operatorname{dom} a$ and $\ker(ah) = \ker a$. Then, from Lemma 3.1.1(i) and (iii), follows

(c) $\operatorname{im} a \subseteq \operatorname{dom} h$, and (d) $\ker h$ separates $\operatorname{im} a$.

(i) Suppose a is injective and f is a full map. The second assumption and (a) together imply $\operatorname{dom} f = \operatorname{dom} g = X$. In order to prove $f \mathscr{R} g$, it suffices to prove $\ker f = \ker g$ (see Proposition 3.1.2(iv)). We have

$$\begin{aligned}(x, y) \in \ker f \Leftrightarrow xf = yf \Leftrightarrow (x)gah = (y)gah \\ \Leftrightarrow (x)ga = (y)ga \Leftrightarrow xg = yg \Leftrightarrow (x, y) \in \ker g,\end{aligned}$$

the third and fourth equivalence following from (d) and the assumption of injectivity of a, respectively.

(ii) Suppose a is full and f is injective. We have already proved $\operatorname{dom} f \subseteq \operatorname{dom} g$ (see (a)), and now we prove the reverse containment, having in mind that a is full and that we have (c):

$$\begin{aligned}x \in \operatorname{dom} g \Rightarrow xg \in Y = \operatorname{dom} a \Rightarrow (x)ga \in \operatorname{im} a \subseteq \operatorname{dom} h \\ \Rightarrow x \in \operatorname{dom}(gah) = \operatorname{dom} f.\end{aligned}$$

Thus, we have $\operatorname{dom} f = \operatorname{dom} g$, so (b) implies $\ker f \supseteq \ker g$. Moreover, we may conclude $\ker f = \ker g$, since f is injective. □

Having proved these technical results, we are ready to prove the theorem stating the rank of $\mathcal{PT}^a_{XY}$ in the case where $\operatorname{rank}(a) = \alpha < \xi = \min(|X|, |Y|)$ (i.e. when a is non-surjective, and either non-injective, or non-full, or both).

Theorem 3.1.48. *Suppose $|X| < \aleph_0$, $|Y| \leq \aleph_0$, and that $1 \leq \alpha < \xi$ (hence, a is non-surjective). We have*

$$\operatorname{rank}(\mathcal{PT}^a_{XY}) = \sum_{\mu=\alpha+1}^{\xi} \mu! \binom{|Y|}{\mu} \mathcal{S}(|X|+1, \mu+1) + \begin{cases} 0, & \text{if } a \text{ is non-injective and non-full;} \\ \mathcal{S}(|X|, \alpha), & \text{if } a \text{ is injective and non-full;} \\ \binom{|X|}{\alpha}, & \text{if } a \text{ is full and non-injective.} \end{cases}$$

Proof. By the discussion in Section 2.6, any generating set of $\mathcal{PT}^a_{XY}$ must include elements from every maximal $\mathscr{J}^a$-class. Under the assumptions of the theorem, Proposition 3.1.19(i) guarantees that the maximal $\mathscr{J}^a$-classes are exactly all the singletons $\{f\}$, such that $\operatorname{Rank} f > \alpha$ (hence, the possible value ranges from $\alpha + 1$ to $\min(|X|, |Y|) = \xi$). Therefore, any generating set contains all such elements, so

$$\begin{aligned}\operatorname{rank}(\mathcal{PT}^a_{XY}) &\geq |\{f \in \mathcal{PT}_{XY} : \operatorname{Rank} f > \alpha\}| \\ &= \sum_{\mu=\alpha+1}^{\xi} \mu! \binom{|Y|}{\mu} \mathcal{S}(|X|+1, \mu+1)\end{aligned}$$

(we summed the number of such elements in each $\mathscr{D} = \mathscr{J}$-class of $\mathcal{PT}_{XY}$, which was calculated in Corollary 3.1.4(v)). Now, let M denote the set $\{f \in \mathcal{PT}_{XY} : \operatorname{Rank} f > \alpha\}$. We may conclude that

$$\begin{aligned}\operatorname{rank}(\mathcal{PT}^a_{XY}) &= |M| + \operatorname{rank}(\mathcal{PT}^a_{XY} : M)\\ &= \sum_{\mu=\alpha+1}^{\xi} \mu! \binom{|Y|}{\mu} \mathcal{S}(|X|+1, \mu+1) + \operatorname{rank}(\mathcal{PT}^a_{XY} : M).\end{aligned}$$

The value of $\operatorname{rank}(\mathcal{PT}^a_{XY} : M)$ is calculated for each case separately. Note that it suffices to generate the class D_α, since Corollary 3.1.45(ii) implies that this set generates all the $\mathscr{D} = \mathscr{J}$-classes below it.

Case 1: a is non-injective and non-full. Fix an arbitrary element $f \in \mathrm{D}_\alpha$. Since $\alpha < |X|$, f is either non-injective or non-full. Thus, f and a are either both non-injective or both non-full, so Lemma 3.1.46 gives $f \in \langle \mathrm{D}_{\alpha+1}\rangle_a$ and the previous discussion implies $\operatorname{rank}(\mathcal{PT}^a_{XY} : M) = 0$.

Case 2: a is injective and non-full. Suppose $\Omega \subseteq \mathcal{PT}_{XY}$ is a set such that $\langle M \cup \Omega\rangle_a = \mathcal{PT}^a_{XY}$ and $|\Omega| = \operatorname{rank}(\mathcal{PT}^a_{XY} : M)$. Since $\mathrm{D}_\alpha \subseteq \langle M \cup \Omega\rangle_a$, we claim that, for each **full** transformation $f \in \mathrm{D}_\alpha$, there exists an element $g \in \Omega$ such that $g \mathscr{R} f$. Consider an expression $f = g_1 \star_a \cdots \star_a g_k$, with $g_1, \ldots, g_k \in M \cup \Omega$. If $k = 1$, we clearly have $f \mathscr{R} g_1$. If $k > 1$, the same is implied by Lemma 3.1.47(i). Since in both cases we have $\operatorname{Rank} g_1 = \operatorname{Rank} f = \alpha$ (which follows from $\mathscr{R} \subseteq \mathscr{J}$ and Proposition 3.1.2(vi)), we may infer that $g_1 \notin M$. Therefore, $\operatorname{rank}(\mathcal{PT}^a_{XY} : M)$ is grater than or equal to the number of $\mathscr{R}$-classes in D_α containing full transformations. By Proposition 3.1.2(iv), these classes are determined only by their kernel, i.e. their partition of X into α subsets. Thus, the number of such classes is $\mathcal{S}(|X|, \alpha)$ and

$$\operatorname{rank}(\mathcal{PT}^a_{XY} : M) \geq \mathcal{S}(|X|, \alpha).$$

Now, we show the reverse inequality by providing a generating set of the stated size. Let $\mathscr{E}$ be the set of all equivalence relations with α classes over the set X. For each $\varepsilon \in \mathscr{E}$, fix an $f_\varepsilon \in \mathrm{D}_\alpha$ with $\ker f_\varepsilon = \varepsilon$ and $\operatorname{im} f_\varepsilon = \{b_1, \ldots, b_\alpha\}$. We define $\Omega = \{f_\varepsilon : \varepsilon \in \mathscr{E}\}$. By the discussion preceding the cases, it suffices to show

$$\mathrm{D}_\alpha \subseteq \langle M \cup \Omega\rangle_a.$$

Recall that a is not full. Fix an arbitrary element $g \in \mathrm{D}_\alpha$. If g is non-full, too, then Lemma 3.1.46 gives $g \in \langle M\rangle_a$. If g is full, let $g = \left(\begin{smallmatrix} G_1 & \cdots & G_\alpha \\ g_1 & \cdots & g_\alpha\end{smallmatrix}\right)$. Note that $\ker g \in \mathscr{E}$, so $f_{\ker g} = \left(\begin{smallmatrix} G_1 & \cdots & G_\alpha \\ b_{1\pi} & \cdots & b_{\alpha\pi}\end{smallmatrix}\right)$ for some permutation π of the set $\{1, \ldots, \alpha\}$. Since $\operatorname{Rank} g = \operatorname{Rank} a = \alpha < \min(|X|, |Y|)$, there exist some $x \in X \setminus \operatorname{im} a$ and $y \in Y \setminus \operatorname{im} g$. Then, for the map $h = \left(\begin{smallmatrix} a_{1\pi} & \cdots & a_{\alpha\pi} & x \\ g_1 & \cdots & g_\alpha & y\end{smallmatrix}\right)$ evidently holds $h \in M$ and $f_{\ker g} \star_a h = g$. Thus, $g \in \langle M \cup \Omega\rangle_a$.

Case 3: a is non-injective and full. Again, we let $\Omega \subseteq \mathcal{PT}_{XY}$ be a set such that $M \cup \Omega$ generates $\mathcal{PT}^a_{XY}$ and $|\Omega| = \operatorname{rank}(\mathcal{PT}^a_{XY} : M)$. This time, we claim that for each **injective** partial transformation $f \in \mathrm{D}_\alpha$, there exists an element $g \in \Omega$ such that $g \mathscr{R} f$. The proof is virtually identical to the one for the previous case, the only difference being the use of part (ii) of Lemma 3.1.47, instead of part (i). By Proposition 3.1.2(iv), an $\mathscr{R}$-class containing injective maps is determined solely by the domain of its elements. Thus, D_α contains $\binom{|X|}{\alpha}$ such classes and

$$\operatorname{rank}(\mathcal{PT}^a_{XY} : M) \geq \binom{|X|}{\alpha}.$$

The reverse inequality is shown in a similar manner as in the previous case. We define the set of possible domains, $\mathscr{Q} = \{D \subseteq X : |D| = \alpha\}$, and for each $D \in \mathscr{Q}$, we choose an injective map $f_D \in \mathrm{D}_\alpha$ with $\operatorname{dom} f_D = D$ and $\operatorname{im} f_D = \{b_1, \ldots b_\alpha\}$. Now, we prove that the union of the sets $\Omega = \{f_D : D \in \mathscr{Q}\}$ and M generates D_α. Let $g \in \mathrm{D}_\alpha$. If g is non-injective, Lemma 3.1.46 gives $g \in \langle M \rangle_a$. Otherwise, we may write $g = \left(\begin{smallmatrix} g_1 & \cdots & g_\alpha \\ q_1 & \cdots & q_\alpha \end{smallmatrix}\right)$, so we have $f_{\operatorname{dom} g} = \left(\begin{smallmatrix} g_1 & \cdots & g_\alpha \\ b_{1\pi} & \cdots & b_{\alpha\pi} \end{smallmatrix}\right)$ for some permutation π. Again, there exist some $x \in X \setminus \operatorname{im} a$ and $y \in Y \setminus \operatorname{im} g$, so we may define $h = \left(\begin{smallmatrix} a_{1\pi} & \cdots & a_{\alpha\pi} & x \\ q_1 & \cdots & q_\alpha & y \end{smallmatrix}\right) \in M$ and we clearly have $g = f_{\operatorname{dom} g} \star_a h$. □

Remark 3.1.49. In particular, in the case where $1 \leq \alpha < |X| < |Y| = \aleph_0$, we have $\binom{|Y|}{\mu} = \aleph_0$ for all $\alpha + 1 \leq \mu \leq \xi$ (and all the other factors are finite), so

$$\operatorname{rank}(\mathcal{PT}^a_{XY}) = \aleph_0 = |\,\mathcal{PT}^a_{XY}\,|.$$

Next, we examine the case in which $\alpha = \xi$. We keep the assumptions $|X| < \aleph_0$ and $|Y| \leq \aleph_0$, so $\alpha < \aleph_0$. Since we omit the case where a is a full bijection (i.e. where $\alpha = |X| = |Y|$), we have two possible cases:

- either $\alpha = |Y| < |X| < \aleph_0$, so a is full, injective and non-surjective, and $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y, \sigma)$,
- or $\alpha = |X| < |Y| \leq \aleph_0$, so a is surjective and either non-full or non-injective, and $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(X, A)$.

The results concerning the rank of $\mathcal{PT}^a_{XY}$ in the second case were originally proved in Theorem 2.4 of [44].

We start by providing some additional information on the semigroup $\mathcal{PT}^a_{XY}$ in the cases which we investigate.

Lemma 3.1.50.

(i) If $\alpha = |Y| < \aleph_0$, then $\mathrm{P}^a_1 = \mathcal{PT}^a_{XY}$, $\mathrm{P}^a_2 = \mathrm{P}^a$, and $\mathscr{R}^a = \mathscr{R}$ on $\mathcal{PT}^a_{XY}$.

(ii) If $\alpha = |X| < \aleph_0$, then $\mathrm{P}^a_2 = \mathcal{PT}^a_{XY}$, $\mathrm{P}^a_1 = \mathrm{P}^a$, and $\mathscr{L}^a = \mathscr{L}$ on $\mathcal{PT}^a_{XY}$.

Proof. As we remarked before, a is full and injective in the case that $\alpha = |Y| < \aleph_0$. Hence, it is right-invertible, by Lemma 3.0.2(i). Similarly, if $\alpha = |X| < \aleph_0$, a is surjective and then 3.0.2(i) implies left-invertibility. Now, the result follows directly from Lemma 2.2.38. □

Recall Lemma 3.1.1(iv), which implies that any product containing an element with a non-maximum rank results in a map of a non-maximum rank. So, any generating set of $\mathcal{PT}_{XY}^{a}$ has to contain a generating set for D_α consisting purely of elements from D_α. Furthermore, Corollary 3.1.45(i) implies that, if we generate D_α and $\mathrm{D}_{\alpha-1}$, we may generate the whole $\mathcal{PT}_{XY}$! In particular, if a is non-surjective, part (ii) of the same corollary states that the elements of D_α suffice. This information will be used in the process of calculating the rank of $\mathcal{PT}_{XY}^{a}$ in the case where $\alpha = |Y| < |X|$.

Theorem 3.1.51. *Suppose* $1 \leq \alpha = \operatorname{Rank} a = |Y| < |X| < \aleph_0$. *Then*

$$\operatorname{rank}(\mathcal{PT}_{XY}^{a}) = \mathcal{S}(|X|+1, \alpha+1).$$

Proof. Since $\alpha = |Y| < |X| < \aleph_0$, a is full, injective and non-surjective, and we have $\operatorname{Rank} f \leq |Y| = \alpha$ for each $f \in \mathcal{PT}_{XY}$. Thus, D_α generates $\mathcal{PT}_{XY}^{a}$, by Corollary 3.1.45(ii). Furthermore, from Lemma 3.0.2(i) it follows that a is right-invertible, and from Proposition 3.1.7(iii) we have that

$$\mathcal{PT}_{XY} \cup a\mathcal{PT}_{XY}\,a \cup a\mathcal{PT}_{XY}$$

is stable because each of its elements has a finite rank (by Lemma 3.1.1(iv)). Therefore, we may apply Proposition 2.6.3, where $T = \langle \mathrm{J}_b \rangle_a = \langle \mathrm{D}_b \rangle_a = \langle \mathrm{D}_\alpha \rangle_a = \mathcal{PT}_{XY}$. We want to prove that $\operatorname{rank}(\mathrm{H}_b^a) \leq |\,\mathrm{J}_b^a \,/\, \mathscr{H}^a\,|$, in order to apply part (iii). Propositions 2.4.10(i), (iv) and 2.4.9(ii) give $\mathrm{J}_b^a = \mathrm{D}_b^a = \widehat{\mathrm{H}_b^a}$, while Theorem 3.1.26$(v)$ implies that $\mathrm{J}_b^a = \mathrm{D}_\alpha^a = \widehat{\mathrm{H}_b^a}$ is an $(\alpha+1)^\beta \times \Lambda_I = (\alpha+1)^\beta \times 1$ (a is injective, so $\Lambda_I = 1$) rectangular group over $\mathrm{H}_b^a \cong \mathrm{S}_\alpha$. Hence, from $\beta = |X \setminus \operatorname{im} a| \geq 1$ and from the fact that $\operatorname{rank}(\mathrm{S}_\alpha) \leq 2$, we have

$$|\,\mathrm{J}_b^a \,/\, \mathscr{H}^a\,| = |\,\mathrm{J}_b^a \,/\, \mathscr{R}^a\,| = (\alpha+1)^\beta \geq 2 \geq \operatorname{rank}(\mathrm{S}_\alpha) = \operatorname{rank}(\mathrm{H}_b^a).$$

Therefore, Propositions 3.1.2(vi) and 2.6.3(iii) give

$$\operatorname{rank}(\mathcal{PT}_{XY}^{a}) = |\,\mathrm{J}_b \,/\, \mathscr{H}\,| = |\,\mathrm{D}_\alpha \,/\, \mathscr{H}\,| = |\,\mathrm{D}_\alpha \,/\, \mathscr{R}\,| = \mathcal{S}(|X|+1, \alpha+1),$$

the last two equalities following from Lemma 2.2.37(ii) and Corollary 3.1.4(i), respectively. ☐

Remark 3.1.52. As we mentioned before, in the case that $\alpha < \aleph_0$, we have $\alpha = |Y|$ if and only if a is full and injective, if and only if $\mathcal{PT}_{XY}^{a} \cong \mathcal{PT}(X, A)$. Hence,

$$\operatorname{rank}(\mathcal{PT}(X,A)) = \mathcal{S}(|X|+1, |A|+1) \qquad \text{if } 1 \leq |A| < |X| < \aleph_0.$$

Finally, we turn our attention to the only case left, when $1 \leq \alpha = |X| < |Y| \leq \aleph_0$. This condition implies that a is surjective, and either non-injective or non-full or both. It turns out that these conditions heavily influence the generation of D_α and $\mathrm{D}_{\alpha-1}$, so we need to investigate each case separately. Firstly, in each case we describe a type of elements of $\mathrm{D}_{\alpha-1}$, which can be generated by the elements of D_α.

Lemma 3.1.53. *Suppose* $1 \le \alpha = |X| < |Y| \le \aleph_0$.

(i) If a is non-injective, and $f \in \mathrm{D}_{\alpha-1}$ is full, then $f \in \mathrm{D}_\alpha \star_a \mathrm{D}_\alpha$.

(ii) If a is non-full, and $f \in \mathrm{D}_{\alpha-1}$ is injective, then $f \in \mathrm{D}_\alpha \star_a \mathrm{D}_\alpha$.

(iii) If a is non-injective and non-full, then $\mathcal{PT}^a_{XY} = \langle \mathrm{D}_\alpha \rangle_a$.

Proof. (i) Suppose a is non-injective and let $f = \left(\begin{smallmatrix} F_1 & \cdots & F_{\alpha-1} \\ f_1 & \cdots & f_{\alpha-1} \end{smallmatrix}\right) \in \mathrm{D}_{\alpha-1}$ be a full map. Since f is full and $|X| = \alpha < \aleph_0$, we may assume without loss of generality that

$$|F_1| = 2 \qquad \text{and} \qquad |F_2| = \cdots = |F_{\alpha-1}| = 1.$$

Thus, suppose $F_1 = \{u, v\}$. Further, a being non-injective implies $|A_i| = 2$ for some $i \in \{1, \ldots, \alpha\}$, so we may assume (without loss of generality, as well) that $|A_1| \ge 2$. If we fix some $x \in A_1 \setminus \{b_1\}$, and some $y \in Y \setminus \operatorname{im} f$ (which exists because $|Y| > |X| = \alpha$) and define maps $g = \left(\begin{smallmatrix} u & v & F_2 & \cdots & F_{\alpha-1} \\ x & b_1 & b_2 & \cdots & f_{\alpha-1} \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} a_1 & a_2 & \cdots & a_{\alpha-1} & a_\alpha \\ f_1 & f_2 & \cdots & f_{\alpha-1} & y \end{smallmatrix}\right)$, then we have $g \star_a h = f$, so $f \in \mathrm{D}_\alpha \star_a \mathrm{D}_\alpha$.

(ii) Suppose a is non-full and let $f = \left(\begin{smallmatrix} f_1 & \cdots & f_{\alpha-1} \\ g_1 & \cdots & g_{\alpha-1} \end{smallmatrix}\right) \in \mathrm{D}_{\alpha-1}$ be an injective map. Since $|\operatorname{dom} f| = \alpha - 1 < \alpha = |X|$, there exists some $x \in X \setminus \operatorname{dom} f$. Also, $\operatorname{Rank} f < \alpha = |\operatorname{dom} a| < |Y|$ guarantees the existence of some $y \in Y \setminus \operatorname{dom} a$ and some $z \in Y \setminus \operatorname{im} f$. Thus, for $g = \left(\begin{smallmatrix} f_1 & \cdots & f_{\alpha-1} & x \\ b_1 & \cdots & b_{\alpha-1} & y \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} a_1 & a_2 & \cdots & a_{\alpha-1} & a_\alpha \\ g_1 & g_2 & \cdots & g_{\alpha-1} & z \end{smallmatrix}\right)$ we have $g \star_a h = f$, which proves $f \in \mathrm{D}_\alpha \star_a \mathrm{D}_\alpha$.

(iii) Suppose a is both non-injective and non-full. Choose an arbitrary $f \in \mathrm{D}_{\alpha-1}$. Since $\operatorname{Rank} f = |X| - 1$, f is either full, or injective. In both cases we have $f \in \mathrm{D}_\alpha \star_a \mathrm{D}_\alpha$, by (i) and (ii), respectively. Hence $\mathrm{D}_{\alpha-1} \subseteq \langle \mathrm{D}_\alpha \rangle_a$. Now, Corollary 3.1.45$(ii)$ implies the statement. □

Secondly, we add a lemma proving that, if a is injective or full, then D_α can generate only partial maps of the same type.

Lemma 3.1.54. *Suppose* $1 \le \alpha = |X| < |Y| \le \aleph_0$.

(i) If a is injective, then every element of $\langle \mathrm{D}_\alpha \rangle_a$ is injective.

(ii) If a is full, then every element of $\langle \mathrm{D}_\alpha \rangle_a$ is full.

Proof. Note that, in both cases, each element $f \in \mathrm{D}_\alpha$ is a full injection, since $|\operatorname{dom} f| = |X| < \aleph_0$. Therefore, if a is injective, each element in $\langle \mathrm{D}_\alpha \rangle_a$ is an injection, as a product of injective partial maps. Similarly, if a is full, each element of $\langle \mathrm{D}_\alpha \rangle_a$ is full. □

Finally, in the following two lemmas we prove that the rest of the elements can be generated by enhancing the generating set by a single element from $\mathrm{D}_{\alpha-1}$.

Lemma 3.1.55. *Suppose* $1 \le \alpha = |X| < |Y| \le \aleph_0$.

(i) If $f \in \mathrm{D}_{\alpha-1}$ *is injective, then* $f \in \mathrm{D}_\alpha \star_a g \star_a \mathrm{D}_\alpha$*, where*

$$g = \left(\begin{smallmatrix} a_1 & \cdots & a_{\alpha-1} \\ b_1 & \cdots & b_{\alpha-1} \end{smallmatrix}\right).$$

(ii) If $f \in \mathrm{D}_{\alpha-1}$ *is full, then* $f \in \mathrm{D}_\alpha \star_a g \star_a \mathrm{D}_\alpha$*, where*

$$g = \left(\begin{smallmatrix} a_1 & \cdots & a_{\alpha-2} & \{a_{\alpha-1}, a_\alpha\} \\ b_1 & \cdots & b_{\alpha-2} & b_{\alpha-1} \end{smallmatrix}\right).$$

Proof. (i) Let $f = \left(\begin{smallmatrix} f_1 & \cdots & f_{\alpha-1} \\ g_1 & \cdots & g_{\alpha-1} \end{smallmatrix}\right) \in \mathrm{D}_{\alpha-1}$ be an injective map. Since $|\operatorname{dom} f| < |X|$ and $\operatorname{Rank} f < |Y|$, there exist some $x \in X \setminus \operatorname{dom} f$ and $y \in Y \setminus \operatorname{im} f$. Thus, it is easy to see that, for

$$h_1 = \left(\begin{smallmatrix} f_1 & \cdots & f_{\alpha-1} & x \\ b_1 & \cdots & b_{\alpha-1} & b_\alpha \end{smallmatrix}\right) \quad \text{and} \quad h_2 = \left(\begin{smallmatrix} a_1 & \cdots & a_{\alpha-1} & a_\alpha \\ g_1 & \cdots & g_{\alpha-1} & y \end{smallmatrix}\right),$$

we have $f = h_1 a g a h_2 \in \mathrm{D}_\alpha \star_a g \star_a \mathrm{D}_\alpha$.

(ii) Let $f \in \mathrm{D}_{\alpha-1}$ be a full map. As $|\operatorname{dom} f| = |X| = |\ker f| + 1$, the equivalence relation $\ker f$ has exactly one two-element class and $\alpha - 1$ singleton classes. Therefore, without loss of generality we may write $f = \left(\begin{smallmatrix} f_1 & \cdots & f_{\alpha-2} & \{f_{\alpha-1}, x\} \\ g_1 & \cdots & g_{\alpha-2} & g_{\alpha-1} \end{smallmatrix}\right)$. Moreover, since $|Y| > \operatorname{Rank} f$, there exists some $y \in Y \setminus \operatorname{im} f$, so we may define maps h_1 and h_2 in the same manner as above. Here, too, we have $f = h_1 a g a h_2 \in \mathrm{D}_\alpha \star_a g \star_a \mathrm{D}_\alpha$. ☐

Lemma 3.1.56. *Suppose* $1 \leq \alpha = |X| < |Y| \leq \aleph_0$*.*

(i) If a *is full, then* $\mathcal{PT}^a_{XY} = \langle \mathrm{D}_\alpha \cup \{g\} \rangle_a$*, where*

$$g = \left(\begin{smallmatrix} a_1 & \cdots & a_{\alpha-1} \\ b_1 & \cdots & b_{\alpha-1} \end{smallmatrix}\right).$$

(ii) If a *is injective, then* $\mathcal{PT}^a_{XY} = \langle \mathrm{D}_\alpha \cup \{g\} \rangle_a$*, where*

$$g = \left(\begin{smallmatrix} a_1 & \cdots & a_{\alpha-2} & \{a_{\alpha-1}, a_\alpha\} \\ b_1 & \cdots & b_{\alpha-2} & b_{\alpha-1} \end{smallmatrix}\right).$$

Proof. Clearly, in both cases it suffices to prove $\mathrm{D}_{\alpha-1} \subseteq \langle \mathrm{D}_\alpha \cup \{g\} \rangle_a$, since $\mathrm{D}_{\alpha-1} \cup \mathrm{D}_\alpha$ generates $\mathcal{PT}^a_{XY}$, by Corollary 3.1.45(i).

(i) Let a be full. Since $\alpha < |Y|$, it is non-injective, as well. Fix an arbitrary $f \in \mathrm{D}_{\alpha-1}$. If f is full, $f \in \langle \mathrm{D}_\alpha \rangle_a$ by Lemma 3.1.53(i). If f is non-full, it has to be injective because $\operatorname{Rank} f = \alpha - 1 = |X| - 1$. Hence, Lemma 3.1.55$(i)$ gives $f \in \langle \mathrm{D}_\alpha \cup \{g\} \rangle_a$.

(ii) is proved similarly, because a is injective, thus non-full, so we use Lemmas 3.1.53(ii) and 3.1.55(ii) to prove that both injective and non-injective elements of $\mathrm{D}_{\alpha-1}$ are generated by the set $\mathrm{D}_\alpha \cup \{g\}$. ☐

Now, we have everything we need in order to calculate the rank of $\mathcal{PT}^a_{XY}$ in the remaining case.

Theorem 3.1.57. *Suppose* $1 \le \alpha = \operatorname{Rank} a = |X| < |Y| \le \aleph_0$. *Then*

$$\operatorname{rank}(\mathcal{PT}^a_{XY}) = \binom{|Y|}{\alpha} + \begin{cases} 0, & \text{if } a \text{ is neither full nor injective,} \\ 1, & \text{if } a \text{ is full or } [a \text{ is injective and } \alpha \le 2], \\ 2, & \text{if } a \text{ is injective and } \alpha \ge 3. \end{cases}$$

Proof. Since $\alpha = |X| < |Y| \le \aleph_0$, a is surjective, and we have $\operatorname{Rank} f < \alpha$ for all $f \in \mathcal{PT}_{XY}$. Thus, Corollary 3.1.45(i) implies that $\langle \mathrm{D}_{\alpha-1} \cup \mathrm{D}_\alpha \rangle = \mathcal{PT}_{XY}$. From Proposition 3.1.2(vi), we know that

$$\mathrm{J}_b = \mathrm{D}_b = \mathrm{D}_\alpha$$

is the maximal $\mathscr{J}$-class in $\mathcal{PT}^a_{XY}$. Furthermore, Lemma 3.1.1(iv) implies that a product resulting in an element of rank α cannot contain an element of a smaller rank. In other words, any generating set of $\mathcal{PT}^a_{XY}$ contains a generating set of $\langle \mathrm{D}_\alpha \rangle_a$. Therefore, if we denote $T = \langle \mathrm{D}_\alpha \rangle_a$, we have

$$\operatorname{rank}(\mathcal{PT}_{XY}) = \operatorname{rank}(\langle \mathrm{D}_{\alpha-1} \cup \mathrm{D}_\alpha \rangle_a) = \operatorname{rank}(T) + \operatorname{rank}(\mathcal{PT}^a_{XY} : T). \tag{3.9}$$

Now, Lemma 3.1.53(iii) gives $\operatorname{rank}(\mathcal{PT}^a_{XY} : T) = 0$, in the case when a is non-injective and non-full. Further, by Lemma 3.1.55, we have $\operatorname{rank}(\mathcal{PT}^a_{XY} : T) \ge 1$ if a is either injective or full. Thus, from Lemma 3.1.56 we may conclude that $\operatorname{rank}(\mathcal{PT}^a_{XY} : T) = 1$ in both of these cases.

We still need to calculate $\operatorname{rank}(T)$. Note that a is left-invertible by Lemma 3.0.2(ii), and $\mathcal{PT}_{XY} \cup a\mathcal{PT}_{XY}a \cup \mathcal{PT}_{XY}a$ is stable by Proposition 3.1.7(iii) (as it contains only finite-ranked maps). Hence, Proposition 2.6.4 applies. Here, note that (keeping the notation introduced in Section 2.6) by Corollary 3.1.4(ii) we have

$$|X_2| + |X_1| = |\,\mathrm{J}_b \,/\, \mathscr{H}\,| = |\,\mathrm{D}_b \,/\, \mathscr{H}\,| = |\,\mathrm{D}_\alpha \,/\, \mathscr{H}\,| = |\,\mathrm{D}_\alpha \,/\, \mathscr{L}\,| = \binom{|Y|}{\alpha}, \tag{3.10}$$

the penultimate equality following from the dual of Proposition 2.2.37(ii). Further, since $\mathcal{PT}$ is regular (by Proposition 3.0.1), Propositions 2.4.10(i), (iv) and 2.4.9(ii) give $\mathrm{J}^a_b = \mathrm{D}^a_b = \widehat{\mathrm{H}^a_b}$, while Theorem 3.1.26($v$) implies that $\mathrm{J}^a_b = \mathrm{D}^a_\alpha = \widehat{\mathrm{H}^a_b}$ is an $(\alpha+1)^\beta \times \Lambda_I = 1 \times \Lambda_I$ (as $\beta = |X \setminus \operatorname{im} a| = 0$) rectangular group over $\mathrm{H}^a_b \cong \mathrm{S}_\alpha$. In addition, recall that

$$\operatorname{rank}(\mathrm{S}_\alpha) = \begin{cases} 1, & \alpha = 1, 2; \\ 2, & \alpha \ge 3. \end{cases}$$

In order to use part (iii) of Proposition 2.6.4, we will identify the cases where $\operatorname{rank}(\mathrm{H}^a_b) \le |\,\mathrm{J}^a_b \,/\, \mathscr{H}^a\,|$. Clearly, if $\Lambda_I \ge 2$ or $\alpha \le 2$, we have

$$|\,\mathrm{J}^a_b \,/\, \mathscr{H}^a\,| = |\,\mathrm{J}^a_b \,/\, \mathscr{L}^a\,| = \Lambda_I \ge \operatorname{rank}(\mathrm{S}_\alpha) = \operatorname{rank}(\mathrm{H}^a_b).$$

This occurs only in the case when a is non-injective or $\alpha \le 2$. Under these assumptions, (3.10) and Proposition 2.6.4(*iii*) give $\operatorname{rank}(T) = |\,\mathrm{J}_b\,/\,\mathscr{H}\,| = \binom{|Y|}{\alpha}$.

In the remaining case, where a is injective and $\alpha \geq 3$, we have $|X_1| = |\,\mathrm{J}^a_b\,/\,\mathscr{H}^a\,| = 1 = \operatorname{rank}(\mathrm{S}_\alpha) - 1$, so in this case Proposition 2.6.4(*ii*) and (3.10) give

$$\operatorname{rank}(T) = |X_2| + \operatorname{rank}(\mathrm{S}_\alpha) = |X_2| + |X_1| + 1 = \binom{|Y|}{\alpha} + 1.$$

Therefore, the statement follows from (3.9). □

Remark 3.1.58. In particular, if $1 \leq \alpha = |X| < |Y| = \aleph_0$, then $\binom{|Y|}{\alpha} = |Y|^\alpha = \aleph_0$, so $\operatorname{rank}(\mathcal{PT}^a_{XY}) = \aleph_0 = |\,\mathcal{PT}^a_{XY}\,|$.

Remark 3.1.59. As we remarked a number of times, in the case that $\alpha = |X| < \aleph_0$, a is surjective, so $\mathcal{PT}^a_{XY} \cong \mathcal{PT}(Y,\sigma)$. Therefore, we have proved in the previous theorem the following: if $|Y| \leq \aleph_0$ and $|\pi_\sigma| < \aleph_0$, then

$$\operatorname{rank}(\mathcal{PT}(Y,\sigma)) = \binom{|Y|}{|\pi_\sigma|} + \begin{cases} 0, & \text{if } \mathrm{u}(\sigma) \neq Y \text{ and } \sigma \neq \Delta_{\mathrm{u}(\sigma)}, \\ 1, & \text{if } \mathrm{u}(\sigma) = Y \text{ or } [\sigma = \Delta_{\mathrm{u}(\sigma)} \text{ and } |\pi_\sigma| \leq 2], \\ 2, & \text{if } \sigma = \Delta_{\mathrm{u}(\sigma)} \text{ and } |\pi_\sigma| \geq 3, \end{cases}$$

where $\Delta_{\mathrm{u}(\sigma)} = \{(y,y) : y \in \mathrm{u}(\sigma)\}$ is the diagonal relation on $\mathrm{u}(\sigma)$.

3.1.6 Egg-box diagrams

This subsection is dedicated to the visual presentation of the structural results of the current section, via egg-box diagrams of different sandwich semigroups and their regular subsemigroups. The figures shown here were initially used in [34], and had been produced with the *Semigroups* package in GAP [98]. The author thanks Dr Attila Egri-Nagy and Dr James Mitchell for writing the code for creating them.

In the diagrams, we use the usual conventions for egg-box diagrams of $\mathscr{D}$-classes described in Section 1.3, with the addition of colouring the group $\mathscr{H}$-classes grey. Since all the sets in our examples are finite, Proposition 3.1.7(*iii*) implies that the sandwich element is stable in each of the examples, so in all of them holds $\mathscr{J}^a = \mathscr{D}^a$, by Proposition 2.2.25 (and Proposition 3.1.2(*vi*)). We illustrate the $\leq_{\mathscr{J}^a}$ order by connecting the pairs of related $\mathscr{J}^a$-classes by a line segment and placing each $\mathscr{J}^a$-class above all the $\mathscr{J}^a$-classes it covers. If one is reading the electronic version of this thesis, these connections and other details of the diagrams may be inspected by zooming in. Note that the sharpness of the images allows zooming in a long way.

In the examples, we assume $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$, and write $\mathcal{PT}_{mn}$ for the set $\mathcal{PT}_{\{1,2,\ldots,m\},\{1,2,\ldots,n\}}$. The sandwich elements are denoted in a form differing slightly from the one used above: we list all the elements $\{1, 2, \ldots, n\}$ in a row, and below each element we place its map or the symbol $-$, in the case when it does not have one (for example, $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & - & - & 1 & 3 \end{smallmatrix}\right) \in \mathcal{PT}_{54}$).

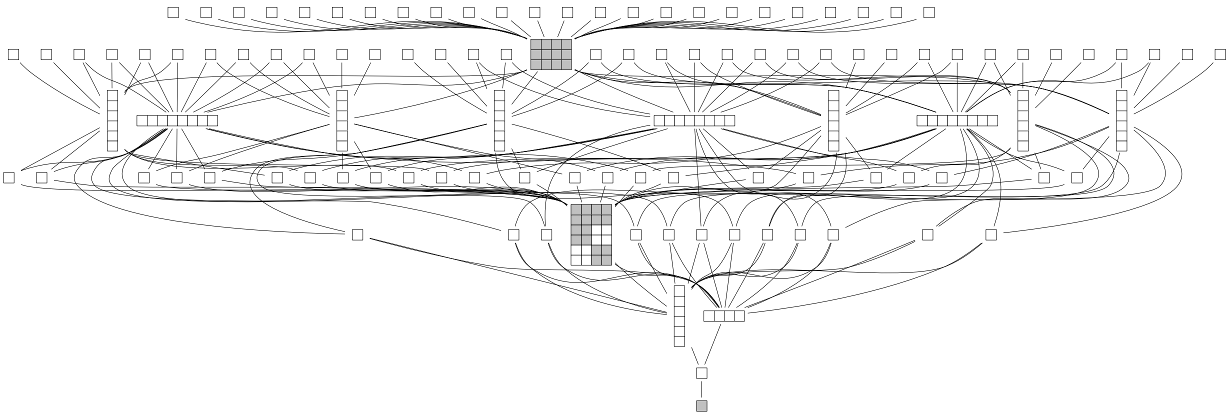


Figure 3.4: Egg-box diagram of the sandwich semigroup $\mathcal{PT}_{35}^{a}$, where $a = \left(\begin{smallmatrix}1&2&3&4&5\\1&1&2&2&-\end{smallmatrix}\right) \in \mathcal{PT}_{53}$. Note that a is non-full, non-injective and non-surjective.

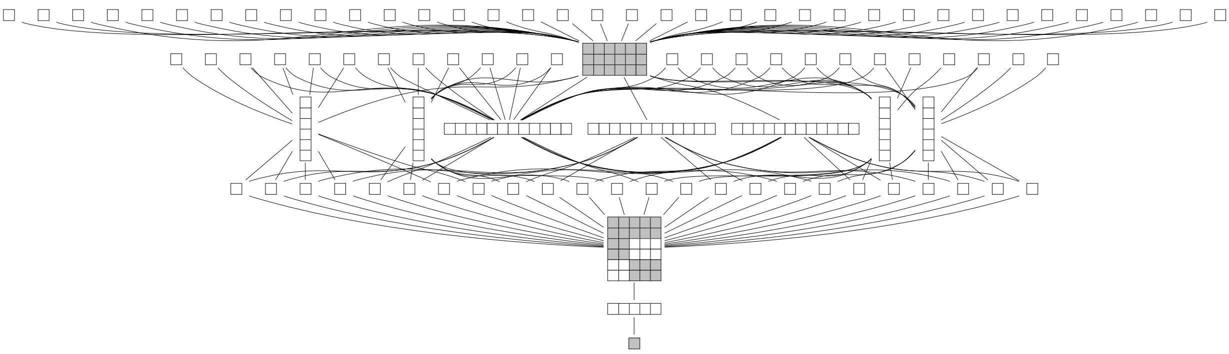


Figure 3.5: Egg-box diagram of the sandwich semigroup $\mathcal{PT}_{35}^{b}$, where $b = \left(\begin{smallmatrix}1&2&3&4&5\\1&1&2&2&2\end{smallmatrix}\right) \in \mathcal{PT}_{53}$. Note that b is full, non-injective and non-surjective.

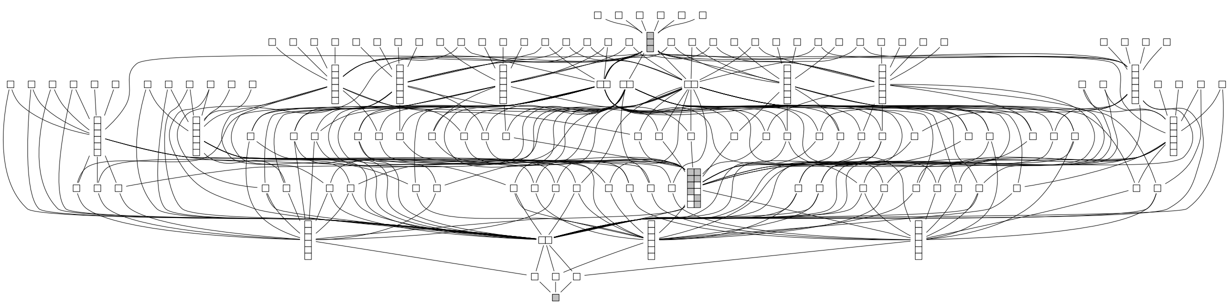


Figure 3.6: Egg-box diagram of the sandwich semigroup $\mathcal{PT}_{35}^{c}$, where $c = \left(\begin{smallmatrix}1&2&3&4&5\\1&2&-&-&-\end{smallmatrix}\right) \in \mathcal{PT}_{53}$. Note that c is non-full, injective and non-surjective.

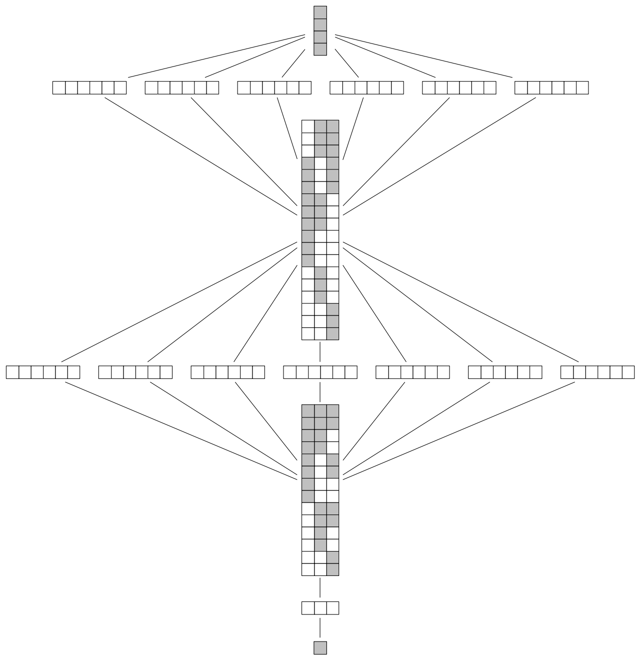


Figure 3.7: Egg-box diagram of the sandwich semigroup $\mathcal{PT}_{43}^{d}$, where $d = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right) \in \mathcal{PT}_{34}$. Note that d is full, injective and non-surjective.

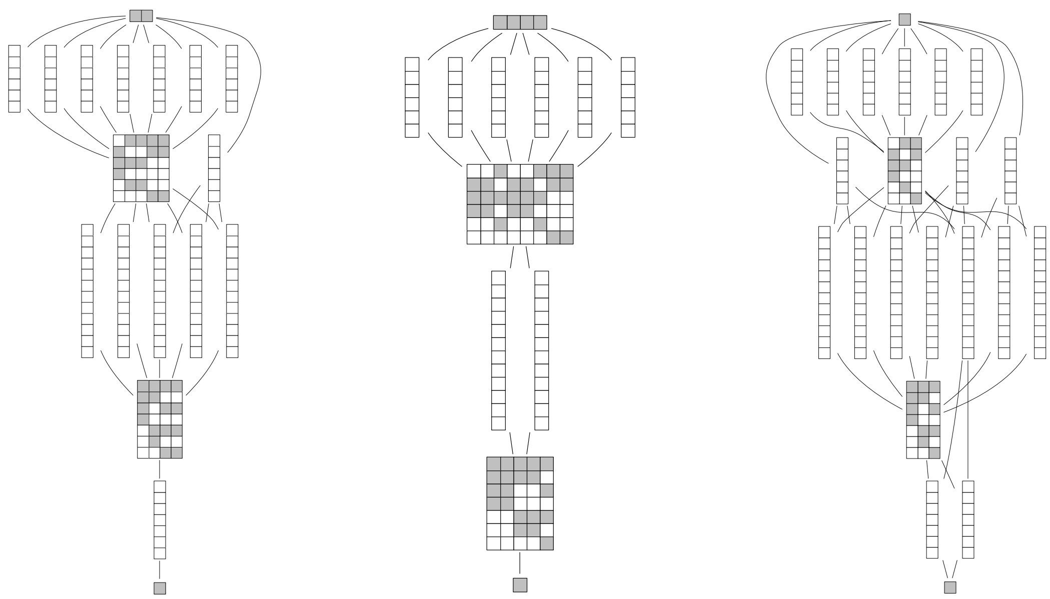


Figure 3.8: Left to right: egg-box diagrams of the sandwich semigroups $\mathcal{PT}_{35}^{e}$, $\mathcal{PT}_{35}^{f}$ and $\mathcal{PT}_{35}^{g}$, where $e = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & - \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 3 \end{smallmatrix}\right)$, and $g = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & - & - \end{smallmatrix}\right) \in \mathcal{PT}_{53}$. Note that e is non-full, non-injective and surjective, f is full, non-injective and surjective and g is non-full, injective and surjective.

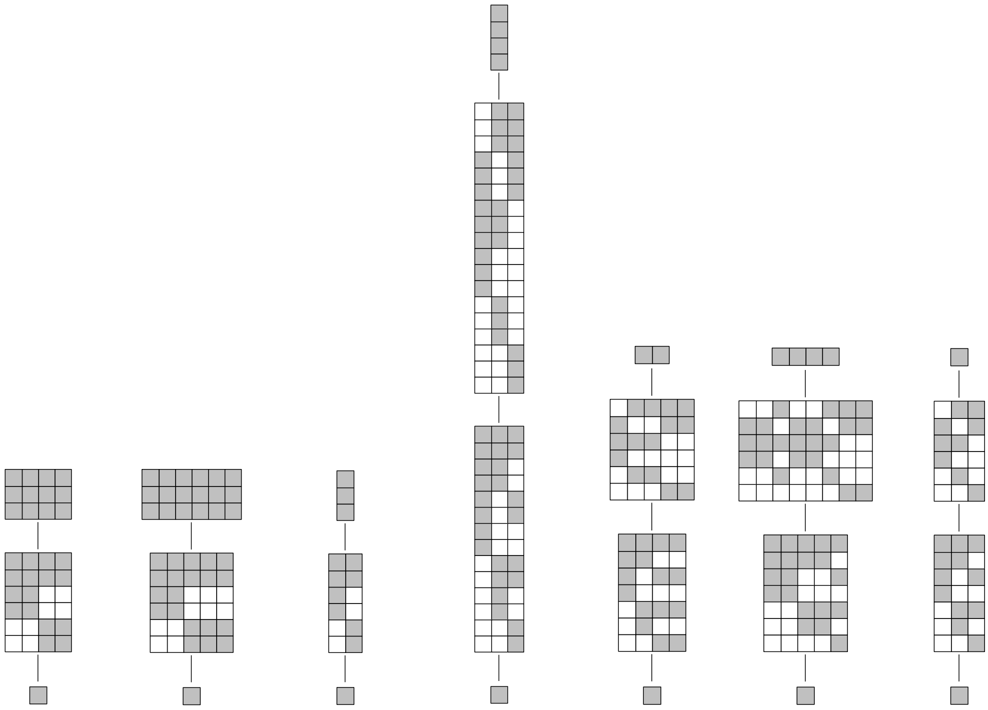


Figure 3.9: Left to right: egg-box diagrams of the regular sandwich semigroups $\mathrm{Reg}(\mathcal{PT}^a_{35})$, $\mathrm{Reg}(\mathcal{PT}^b_{35})$, $\mathrm{Reg}(\mathcal{PT}^c_{35})$, $\mathrm{Reg}(\mathcal{PT}^d_{43})$, $\mathrm{Reg}(\mathcal{PT}^e_{35})$, $\mathrm{Reg}(\mathcal{PT}^f_{35})$ and $\mathrm{Reg}(\mathcal{PT}^g_{35})$, where the sandwich elements a, b, c, d, e, f, and g are defined as in Figures 3.4–3.8.

By the theory in Subsection 2.3.4, the first three semigroups in Figure 3.9 are inflations of $\mathcal{PT}_2$, and the other four are inflations of $\mathcal{PT}_3$. Both $\mathcal{PT}_2$ and $\mathcal{PT}_3$ are shown below. Note also that $\mathrm{Reg}(\mathcal{PT}^g_{35}) \cong \mathcal{PT}_3$, since g is both injective and surjective (see Remark 3.1.28).

Figure 3.10: Egg-box diagrams of the partial transformation semigroups $\mathcal{PT}_2$ (left) and $\mathcal{PT}_3$ (right).

3.2 The category $\mathcal{T}$

Having conducted an in-depth investigation of the partial semigroup $\mathcal{PT}$ and the sandwich semigroups it contains, we turn to the partial semigroup $\mathcal{T}$ and the sandwich semigroups in it. Again, we point out that the results presented in this chapter are based on the investigation conducted in [34], and most of the results were originally published in that article. In a few instances, when that is not the case, we cite appropriately.

Since $\mathcal{T} = \{(A, f, B) : A, B \in \mathbf{Set}^+, f \in \mathbf{T}_{AB}\}$ is a regular and monoidal partial subsemigroup of $\mathcal{PT}$, we will be able to easily and efficiently prove the results concerning it. We always consider the corresponding statement for the case $\mathcal{PT}$ and its proof. In some cases, we simply adapt the proof, while in other cases the result for $\mathcal{T}$ is a direct consequence of the statement for $\mathcal{PT}$. If we are adapting the proof, we do not always give the new one in full details (the comprehensiveness depends on the number of changes made).

As mentioned at the beginning of this chapter, these results were first published in article [34]. However, many of the results of [29] are special cases of the results in this section, taking $|X| = |Y| < \aleph_0$. Unless significant for our study, these results for special cases will not be explicitly mentioned. For an extensive record of the results preceding the ones presented, see Section 1.1.

Note that we are now dealing with **full** transformations, which means that for any $A, B \in \mathbf{Set}^+$ and any $f \in \mathcal{T}_{AB}$, we have $\operatorname{dom} f = A$. This simplifies the statements. For example, the following lemma is the direct consequence of Lemma 3.1.1.

Lemma 3.2.1. *Let $A, B, C \in \mathbf{Set}^+$, $f \in \mathbf{T}_{AB}$, and $g \in \mathbf{T}_{BC}$. Then*

(i) $\operatorname{im}(fg) \subseteq \operatorname{im} g$, *with equality if and only if* $\operatorname{im} f$ *saturates* $\ker g$,

(ii) $\ker(fg) \supseteq \ker f$, *with equality if and only if* $\ker g$ *separates* $\operatorname{im} f$,

(iii) $\operatorname{Rank}(fg) \leq \min(\operatorname{Rank} f, \operatorname{Rank} g)$.

Thus, we have

Proposition 3.2.2. *Let $(A, f, B), (C, g, D) \in \mathcal{T}$. Then*

(i) $(A, f, B) \leq_{\mathscr{R}} (C, g, D) \Leftrightarrow A = C$ *and* $\ker f \supseteq \ker g$,

(ii) $(A, f, B) \leq_{\mathscr{L}} (C, g, D) \Leftrightarrow B = D$ *and* $\operatorname{im} f \subseteq \operatorname{im} g$,

(iii) $(A, f, B) \leq_{\mathscr{J}} (C, g, D) \Leftrightarrow \operatorname{Rank} f \leq \operatorname{Rank} g$,

(iv) $(A, f, B)\,\mathscr{R}\,(C, g, D) \Leftrightarrow A = C$ *and* $\ker f = \ker g$,

(v) $(A, f, B)\,\mathscr{L}\,(C, g, D) \Leftrightarrow B = D$ *and* $\operatorname{im} f = \operatorname{im} g$,

(vi) $(A, f, B)\,\mathscr{J}\,(C, g, D) \Leftrightarrow (A, f, B)\,\mathscr{D}\,(C, g, D) \Leftrightarrow \operatorname{Rank} f = \operatorname{Rank} g$.

Proof. Parts $(i) - (v)$ and the equivalence

$$(A, f, B) \mathscr{J} (C, g, D) \Leftrightarrow \operatorname{Rank} f = \operatorname{Rank} g$$

from (vi) are proved similarly as the corresponding parts of Proposition 3.1.2. For the direct implications, we use Lemma 3.2.1 instead of Lemma 3.1.1. For the converse implications, the key difference is the requirement for the maps to be full. However, we need not change the proofs substantially. When defining the auxiliary maps $h \in \mathcal{T}_{DB}$ in (i) and $q \in \mathcal{T}_{DB}$ in (iii), we simply choose any full map meeting the requirements (note that h in (ii) and (iii) are already full, since f is). The proof for $\mathscr{J} \subseteq \mathscr{D}$ is literally unchanged. □

As in the case of $\mathcal{PT}$, we define

$$\mathcal{T}_{AB} = \{(A, f, B) : f \in \mathbf{T}_{AB}\}, \qquad \text{for } A, B \in \mathbf{Set}^+$$

and conclude that the $\mathscr{J} = \mathscr{D}$-classes of $\mathcal{T}_{AB}$ are the sets

$$\mathrm{D}_\mu^{AB} = \mathrm{D}_\mu \cap \mathcal{T}_{AB} = \{(A, f, B) : f \in \mathbf{T}_{AB},\ \operatorname{Rank} f = \mu\},$$

for each cardinal $1 \leq \mu \leq \min(|A|, |B|)$. (Recall that $A \neq \emptyset$, so any map with domain A has a non-zero rank.) These $\mathscr{J}$-classes form a chain in $\mathcal{T}_{AB}$: $\mathrm{D}_\mu^{AB} \leq \mathrm{D}_\nu^{AB} \Leftrightarrow \mu \leq \nu$.

Furthermore, we may describe the combinatorial properties of D_μ^{AB}, using Proposition 3.2.2. If $|A| = \alpha$ and $|B| = \beta$, we have

$$|\,\mathrm{D}_\mu^{AB} / \mathscr{R}\,| = \mathcal{S}(\alpha, \mu) \quad \text{and} \quad |\,\mathrm{D}_\mu^{AB} / \mathscr{L}\,| = \binom{\beta}{\mu}$$

since, by fixing an $\mathscr{R}$- and an $\mathscr{L}$-class, we are partitioning A into μ-classes and choosing a μ-element set from B, respectively. Having fixed a kernel and an image, we may connect them in $\mu!$ ways, so each $\mathscr{H}$-class contains that many elements. Thus,

$$|\,\mathrm{D}_\mu^{AB} / \mathscr{H}\,| = \mathcal{S}(\alpha, \mu)\binom{\beta}{\mu} \quad \text{and} \quad |\,\mathrm{D}_\mu^{AB}\,| = \mu!\,\mathcal{S}(\alpha, \mu)\binom{\beta}{\mu}.$$

Summing the sizes of D_μ^{AB} for each possible rank μ, we enumerate the elements of $\mathcal{T}_{AB}$. Another way to do that is to calculate the number of ways to map each of the $|A|$ elements into any of $|B|$ elements of B. So,

$$|\,\mathcal{T}_{AB}\,| = \beta^\alpha = \sum_{\mu=1}^{\min(\alpha,\beta)} \mu!\,\mathcal{S}(\alpha, \mu)\binom{\beta}{\mu}.$$

Following the outline of Section 3.1, we investigate stability in $\mathcal{T}$. Note that the semigroup $\mathbf{T}_X^{\mathrm{fr}} = \{f \in \mathcal{T}_X : \operatorname{Rank} f < \aleph_0\}$ is periodic for each $X \in \mathbf{Set}^+$, the proof being the same as the proof of Lemma 3.1.6. Thus, we may prove an analogue of Proposition 3.1.7:

Proposition 3.2.3. *If* $(A, f, B) \in \mathcal{T}$*, then*

(i) (A, f, B) *is* $\mathscr{R}$*-stable* $\Leftrightarrow$ $[\operatorname{Rank} f < \aleph_0$ *or* f *is injective*$]$,

(ii) (A, f, B) *is* $\mathscr{L}$*-stable* $\Leftrightarrow$ $[\operatorname{Rank} f < \aleph_0$ *or* f *is surjective*$]$,

(iii) (A, f, B) *is stable* $\Leftrightarrow$ $[\operatorname{Rank} f < \aleph_0$ *or* f *is bijective*$]$.

Proof. The proof is conducted analogously as the proof of Proposition 3.1.7. The "compulsory" fullness of elements requires that we choose a slightly different element g in the proofs of the direct implications in (i) and (ii). It suffices to pick $g = \left(\begin{smallmatrix} F_j & F_i \setminus \{a\} & a \\ g_j & g_i & a \end{smallmatrix}\right)_{j \in I}$ for (i). In the proof of (ii), we may choose any full transformation g satisfying $\operatorname{im} g = \{f_i : i \in I\} \cup \{a\}$ and $g\restriction_{\operatorname{im}_g} = \operatorname{id}_{\operatorname{im}_g}$. □

Again, we identify the transformation $f \in \mathbf{T}_{CD}$ with the corresponding element (C, f, D) of $\mathcal{T}$, in cases when $C, D \in \mathbf{Set}^+$ are known or implied. As in Subsection 3.1.1, from now on we use $\mathcal{T}_{CD}$ instead of $\mathbf{T}_{CD}$.

3.2.1 Green's relations, regularity and stability in $\mathcal{T}^a_{XY}$

As in Subsection 3.1.1, we fix two nonempty sets $X, Y \in \mathbf{Set}^+$ and a transformation $a \in \mathcal{T}_{YX}$, in order to investigate the sandwich semigroup $\mathcal{T}^a_{XY}$. Since any domain-related notation and discussion is redundant, we use only the following:

$$a = \left(\begin{smallmatrix} A_i \\ a_i \end{smallmatrix}\right)_{i \in I}, \quad \sigma = \ker a, \quad A = \operatorname{im} a, \quad \alpha = \operatorname{Rank} a, \quad \xi = \min(|X|, |Y|),$$
$$\beta = |X \setminus \operatorname{im} a|, \qquad \lambda_i = |A_i| \text{ for } i \in I, \qquad \Lambda_J = \prod_{j \in J} \lambda_j \text{ for } J \subseteq I.$$

Also, we fix an element $b_i \in A_i$ for each $i \in I$, and we choose a partition $\{B_i : i \in I\}$ of the set X such that $a_i \in B_i$ for each $i \in I$. Then, the map

$$b = \left(\begin{smallmatrix} B_i \\ b_i \end{smallmatrix}\right)_{i \in I} \in \mathcal{T}_{XY}$$

satisfies $aba = a$ and $bab = b$, and we use it as a (partial) semigroup inverse.

From the definition of P-sets, Proposition 3.2.2 and Lemma 3.2.1, we have

(i) $\mathrm{P}^a_1 = \{f \in \mathcal{T}_{XY} \ : \ \ker(fa) = \ker f\}$
$\quad = \{f \in \mathcal{T}_{XY} \ : \ \ker a \text{ separates } \operatorname{im} f\}$,

(ii) $\mathrm{P}^a_2 = \{f \in \mathcal{T}_{XY} \ : \ \operatorname{im}(af) = \operatorname{im} f\}$
$\quad = \{f \in \mathcal{T}_{XY} \ : \ \operatorname{im} a \text{ saturates } \ker f\}$,

(iii) $\mathrm{P}^a = \{f \in \mathcal{T}_{XY} \ : \ \ker(fa) = \ker f, \ \operatorname{im}(af) = \operatorname{im} f\}$
$\quad = \{f \in \mathcal{T}_{XY} \ : \ \ker a \text{ separates } \operatorname{im} f, \ \operatorname{im} a \text{ saturates } \ker f\}$,

(iv) $\mathrm{P}^a_3 = \{f \in \mathcal{T}_{XY} \ : \ \operatorname{Rank}(afa) = \operatorname{Rank} f\}$.

We use this result, together with Theorem 2.2.3 to infer the description of Green's relations of $\mathcal{T}^a_{XY}$. A characterisation of these relations was first obtained in [96].

Although the following result has the same form as Theorem 3.1.10, we state it for the sake of completeness.

Theorem 3.2.4. *If $f \in \mathcal{T}_{XY}$ then in $\mathcal{T}^a_{XY}$ we have*

(i) $\mathrm{R}^a_f = \begin{cases} \mathrm{R}_f \cap \mathrm{P}^a_1, & f \in \mathrm{P}^a_1; \\ \{f\}, & f \notin \mathrm{P}^a_1. \end{cases}$

(ii) $\mathrm{L}^a_f = \begin{cases} \mathrm{L}_f \cap \mathrm{P}^a_2, & f \in \mathrm{P}^a_2; \\ \{f\}, & f \notin \mathrm{P}^a_2. \end{cases}$

(iii) $\mathrm{H}^a_f = \begin{cases} \mathrm{H}_f, & f \in \mathrm{P}^a; \\ \{f\}, & f \notin \mathrm{P}^a. \end{cases}$

(iv) $\mathrm{D}^a_f = \begin{cases} \mathrm{D}_f \cap \mathrm{P}^a, & f \in \mathrm{P}^a; \\ \mathrm{L}^a_f, & f \in \mathrm{P}^a_2 \setminus \mathrm{P}^a_1; \\ \mathrm{R}^a_f, & f \in \mathrm{P}^a_1 \setminus \mathrm{P}^a_2; \\ \{f\}, & f \notin (\mathrm{P}^a_1 \cup \mathrm{P}^a_2). \end{cases}$

(v) $\mathrm{J}^a_f = \begin{cases} \mathrm{J}_f \cap \mathrm{P}^a_3 \ (= \mathrm{D}_f \cap \mathrm{P}^a_3), & f \in \mathrm{P}^a_3; \\ \mathrm{D}^a_f, & f \notin \mathrm{P}^a_3. \end{cases}$

Further, if $f \notin \mathrm{P}^a$, then $\mathrm{H}^a_f = \{f\}$ is a non-group $\mathscr{H}^a$-class in $\mathcal{T}^a_{XY}$.

It is easily shown that Lemma 3.1.12 holds in the partial semigroup $\mathcal{T}$ as well. The proof is analogous to the original one, using the map $f = \begin{pmatrix} B_k & B_j \\ y & b_j \end{pmatrix}_{j\in J}$ for (i), and $f = \begin{pmatrix} X\setminus \mathrm{im}\, a & B_j \\ b_k & b_j \end{pmatrix}_{j\in J}$ for (ii). Thus, the equivalence

$$\mathscr{J}^a = \mathscr{D}^a \Leftrightarrow a \text{ is stable}$$

from Proposition 3.1.13 also holds in $\mathcal{T}^a_{XY}$. Furthermore, an analogue of the proof for Proposition 3.1.14 gives

Proposition 3.2.5. *We have $\mathrm{Reg}(\mathcal{T}^a_{XY}) = \mathrm{P}^a$. Moreover,*

(i) a is $\mathscr{R}$-stable $\Leftrightarrow \mathrm{P}^a_3 \subseteq \mathrm{P}^a_1$,

(ii) a is $\mathscr{L}$-stable $\Leftrightarrow \mathrm{P}^a_3 \subseteq \mathrm{P}^a_2$,

(iii) a is stable $\Leftrightarrow \mathrm{P}^a_3 = \mathrm{P}^a$.

It is important to note here that the first statement of the previous proposition may be deduced from Theorem 5.3 in [86].

Next, we state the parallels of Propositions 3.1.15 and 3.1.16. Recall that any element of $\mathcal{T}$ has a left- and a right-identity, since $\mathcal{T}$ is monoidal. Therefore, from Lemma 2.2.6 and Proposition 3.2.2, we may conclude that $\mathrm{J}^a_f \leq \mathrm{J}^a_g$ holds in $\mathcal{T}^a_{XY}$ if and only if one of the following is true:

(a) $f = g$,

(b) $\operatorname{Rank} f \leq \operatorname{Rank}(aga)$,

(c) $\operatorname{im} f \subseteq \operatorname{im}(ag)$,

(d) $\ker f \supseteq \ker(ga)$.

Moreover, from Proposition 2.2.7, for $f, g \in \mathcal{T}_{XY}$ we have

(i) if $f \in \mathrm{P}_1^a$, then

$$\mathrm{J}_f^a \leq \mathrm{J}_g^a \;\Leftrightarrow\; [\operatorname{Rank} f \leq \operatorname{Rank}(aga) \text{ or } \ker f \supseteq \ker(ga)];$$

(ii) if $f \in \mathrm{P}_2^a$, then $\mathrm{J}_f^a \leq \mathrm{J}_g^a \;\Leftrightarrow\; [\operatorname{Rank} f \leq \operatorname{Rank}(aga) \;\text{ or }\; \operatorname{im} f \subseteq \operatorname{im}(ag)]$;

(iii) if $f \in \mathrm{P}_3^a$, then $\mathrm{J}_f^a \leq \mathrm{J}_g^a \;\Leftrightarrow\; \operatorname{Rank} f \leq \operatorname{Rank}(aga)$;

(iv) if $g \in \mathrm{P}_1^a$, then

$$\mathrm{J}_f^a \leq \mathrm{J}_g^a \;\Leftrightarrow\; [\operatorname{Rank} f \leq \operatorname{Rank}(ag) \text{ or } \ker f \supseteq \ker g];$$

(v) if $g \in \mathrm{P}_2^a$, then $\mathrm{J}_f^a \leq \mathrm{J}_g^a \;\Leftrightarrow\; [\operatorname{Rank} f \leq \operatorname{Rank}(ga) \;\text{ or }\; \operatorname{im} f \subseteq \operatorname{im} g]$;

(vi) if $g \in \mathrm{P}_3^a$, then $\mathrm{J}_f^a \leq \mathrm{J}_g^a \;\Leftrightarrow\; \operatorname{Rank} f \leq \operatorname{Rank} g$.

We close the subsection with two vital results, which prove further similarities (and some differences) between $\mathcal{PT}_{XY}^a$ and $\mathcal{T}_{XY}^a$.

Proposition 3.2.6. *The regular $\mathscr{D}^a$-classes of $\mathcal{T}_{XY}^a$ are precisely the sets*

$$\mathrm{D}_\mu^a = \{f \in \mathrm{P}^a : \operatorname{Rank} f = \mu\}, \quad \textit{for each cardinal } 1 \leq \mu \leq \alpha = \operatorname{Rank} a.$$

Further, if $f \in \mathrm{P}^a$, then $\mathrm{D}_f^a = \mathrm{J}_f^a$ if and only if $\operatorname{Rank} f < \aleph_0$ or a is stable.

Proof. We use the same idea as in the proof of Proposition 3.1.18. In fact, the argument is virtually the same, the only differences being the use of the corresponding results from Section 3.2 instead of the results of Section 3.1, and the use of maps $f_J = \binom{B_j}{b_j}_{j \in J}$ instead of $f_J = \binom{a_j}{b_j}_{j \in J}$. □

Proposition 3.2.7.

(i) If $\alpha < \xi$, then the maximal $\mathscr{J}^a$-classes of $\mathcal{T}_{XY}^a$ are precisely the singleton sets $\{f\}$, for $f \in \mathcal{T}_{XY}$ with $\operatorname{Rank} f > \alpha$. Hence, all the maximal $\mathscr{J}^a$-classes of $\mathcal{T}_{XY}^a$ are trivial in this case.

(ii) If $\alpha = \xi$, then we have a single maximum $\mathscr{J}^a$-class in $\mathcal{T}_{XY}^a$, which is

$$\mathrm{J}_b^a = \{f \in \mathrm{P}_3^a : \operatorname{Rank} f = \alpha\}.$$

This maximal $\mathscr{J}^a$-class is clearly nontrivial.

Proof. Again, the process will be analogous to the original one (see the proof of Proposition 3.1.19). We only swap the auxiliary map h_2' with

$$h_2' = \begin{pmatrix} B_j & B_k \\ b_j & y_k \end{pmatrix}_{j\in J,\ k\in I\setminus J},$$

for some set $\{y_k : k \in I \setminus J\} \subseteq Y \setminus \operatorname{im} g$, where $y_l \neq y_t$ for $l \neq t$ (such a set exists because $\operatorname{Rank} g = |J| \leq \alpha < \min(|X|, |Y|)$). □

Before we continue, we need to consider the minimal $\mathscr{J}^a$-class in $\mathcal{T}_{XY}^a$. Let us enumerate the elements of rank 1 in $\mathcal{T}_{XY}$, using their images: for each $y \in Y$, let

$$f_y : X \to Y : x \mapsto y.$$

Fix an $y \in Y$. As $\operatorname{Rank} f_y = 1 < \aleph_0$, from Proposition 3.2.6 we may deduce $\mathrm{D}_{f_y}^a = \mathrm{J}_{f_y}^a$. Since the equivalence $\ker a$ trivially separates $\operatorname{im} f_y$, and the set $\operatorname{im} a$ trivially saturates $\ker f_y = \{(x, z) : x, z \in X\}$, we have

$$f_y \in \mathrm{P}_1^a \cap \mathrm{P}_2^a = \mathrm{P}^a.$$

Therefore, by Proposition 3.2.6 and by our characterisation of the relation $\leq_{\mathscr{J}^a}$, we have a minimal $\mathscr{J}^a$-class, which is not a singleton in general:

$$\mathrm{D}_1^a = \{f \in \mathrm{P}^a : \operatorname{Rank} f = 1\} = \{f \in \mathcal{T}_{XY} : \operatorname{Rank} f = 1\} = \{f_y : y \in Y\}.$$

Thus, unlike in $\mathcal{PT}_{XY}^a$, the minimum $\mathscr{J}^a$-class in $\mathcal{T}_{XY}^a$ it is not a singleton, unless Y is.

3.2.2 A structure theorem for $\operatorname{Reg}(\mathcal{T}_{XY}^a)$ and connections to (non-sandwich) semigroups of transformations

Following the outline of Section 3.1, in this subsection we describe the connections of $\mathcal{T}_{XY}^a$ and $\operatorname{Reg}(\mathcal{T}_{XY}^a)$ to the corresponding non-sandwich semigroups. Note that all elements of $\mathcal{T}$ are sandwich-regular, since $\mathcal{T}$ is a regular partial semigroup. Recall from Subsection 2.3.1 that the regular monoid $(a\mathcal{T}_{XY}a, \circledast)$ is a subsemigroup of $\mathcal{T}_{YX}^b$, where the map $\circledast = \star_b\restriction_{a\mathcal{T}_{XY}a}$ is independent of the choice of the inverse b. Moreover, we know that $\mathcal{T}_{XY}a = \mathcal{T}_X ba$ and $a\mathcal{T}_{XY} = ab\mathcal{T}_Y$, and we have an isomorphism

$$\eta : (a\mathcal{T}_{XY}a, \circledast) \to (ba\mathcal{T}_{XY}a, \cdot) : x \mapsto bx,$$

so $(a\mathcal{T}_{XY}a, \circledast) \cong (ba\mathcal{T}_X ba, \cdot)$ (the latter being the local monoid of $\mathcal{T}_X$ with respect to the idempotent ba). Since $ba = \binom{B_i}{a_i}_{i\in I} \in \mathcal{T}_X$, and $a_i \in B_i$ for each $i \in I$, any element in $ba\mathcal{T}_X ba$ corresponds to exactly one map in $\mathcal{T}_A$ and vice versa. Hence, the map

$$\upsilon : ba\mathcal{T}_X ba \to \mathcal{T}_A : f \mapsto f\restriction_A$$

is also an isomorphism. Consequently, here we use $\eta' = \eta \circ \upsilon$ just as we have used η in the case of $\mathcal{PT}_{XY}^a$.

We also introduce some notation used in earlier papers (see [44, 95, 111]):

$$\begin{aligned} \mathcal{T}(X, A) &= \{f \in \mathcal{T}_X : \operatorname{im} f \subseteq A\} \\ \mathcal{T}(Y, \sigma) &= \{f \in \mathcal{T}_Y : \text{ every } \ker f\text{-class is a union of } \sigma\text{-classes}\} \\ &= \{f \in \mathcal{T}_Y : \ker f \supseteq \sigma\}. \end{aligned}$$

The same arguments as the ones used in Subsection 3.1.2 prove that

$$\begin{aligned} \mathcal{T}_{XY}\, a = \mathcal{T}_X\, ba = \mathcal{T}(X, A) \quad \text{and} \\ a\,\mathcal{T}_{XY} = ab\,\mathcal{T}_Y = \mathcal{T}(Y, \sigma) \end{aligned}$$

are subsemigroups of $\mathcal{T}_X$ and $\mathcal{T}_Y$. More specifically, a principal left ideal of $\mathcal{T}_X$ and a principal right ideal of $\mathcal{T}_Y$, respectively.

Having examined these semigroups and their connections, we present the specialised forms of Diagrams 2.2 and 2.3 for the semigroup $\mathcal{T}^a_{XY}$ on Figure 3.11.

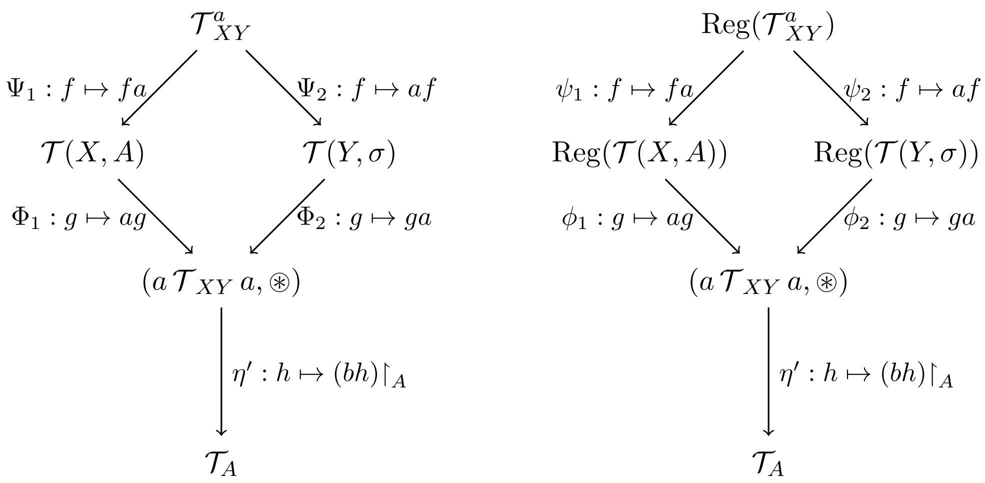


Figure 3.11: Diagrams illustrating the connections between $\mathcal{T}^a_{XY}$ and $(a\,\mathcal{T}_{XY}\, a, \circledast)$ (left) and between $\operatorname{Reg}(\mathcal{T}^a_{XY})$ and $(a\,\mathcal{T}_{XY}\, a, \circledast)$ (right).

Of course, all the maps in the figure are surmorphisms. Moreover, the conclusions of the Subsection 2.3.1 imply:

- ψ_1 and Ψ_1 are isomorphisms if and only if the implication (2.5) holds, which is true if and only if a is injective (see Lemma 3.0.2(i));
- ψ_2 and Ψ_2 are isomorphisms if and only if (2.6) holds, which is true if and only if a is surjective (by Lemma 3.0.2(ii)).

Thus, $\mathcal{T}(X, A)$ and $\mathcal{T}(Y, \sigma)$ arise as special cases of the $\mathcal{T}^a_{XY}$ construction, when a is injective or surjective, respectively. If a is a bijection, we have $A = X$, so $\mathcal{T}^a_{XY} \cong \mathcal{T}(X, A) = \mathcal{T}_X = \mathcal{T}_A$, and hence $\operatorname{Reg}(\mathcal{T}^a_{XY}) \cong \operatorname{Reg}(\mathcal{T}_X) = \mathcal{T}_X$.

Let us focus now on the semigroups on the right-hand side diagram. As the following lemma shows, the characterisations of $\operatorname{Reg}(\mathcal{T}(X, A))$ and $\operatorname{Reg}(\mathcal{T}(Y, \sigma))$

are a somewhat simplified version of the characterisations of $\mathrm{Reg}(\mathcal{PT}(X,A))$ and $\mathrm{Reg}(\mathcal{PT}(Y,\sigma))$, given in Lemma 3.1.21. However, the former are older of the two. The characterisation of $\mathrm{Reg}(\mathcal{T}(X,A))$ was first inferred in [111], while $\mathrm{Reg}(\mathcal{T}(Y,\sigma))$ was described in [95].

Lemma 3.2.8. *We have*

(i) $\mathrm{Reg}(\mathcal{T}(X,A)) = \{f \in \mathcal{T}(X,A) \ : \ \ker f$ *is saturated by* $A\}$,

(ii) $\mathrm{Reg}(\mathcal{T}(Y,\sigma)) = \{f \in \mathcal{T}(Y,\sigma) \ : \ \mathrm{im}\, f$ *is separated by* $\sigma\}$.

Proof. One can verify that in both cases the regular elements have the defining property of the right-hand side set - the argument is analogous to the one in the proof of Lemma 3.1.21. The proof in question also offers a "recipe" for showing the reverse containment. Keeping the same notation and assumptions, the only parts that need adjustment are the maps g. For (i), we choose $g = \binom{C_j}{c_j}_{j\in J} \in \mathcal{T}_X$, where $\{C_j : j \in J\}$ is any partition of the set X such that $f_j \in C_j$; for (ii), fix any $k \in J$ and let

$$g = \begin{pmatrix} A_{l_k} \cup R & A_{l_j} \\ w_k & w_j \end{pmatrix}_{j\in J\setminus\{k\}} \in \mathcal{T}_Y,$$

with $R = X \setminus \bigcup\{A_i : i \in I \setminus J\}$. It is easily seen that $f = fgf$ holds in both cases. □

As in Subsection 3.1.2, we state Theorem 2.3.8 in the form corresponding to the currently investigated sandwich semigroup.

Theorem 3.2.9. *The map*

$$\psi : \mathrm{Reg}(\mathcal{T}^a_{XY}) \to \mathrm{Reg}(\mathcal{T}(X,A)) \times \mathrm{Reg}(\mathcal{T}(Y,\sigma)) : f \mapsto (fa, af)$$

is injective, and

$$\mathrm{im}(\psi) = \{(g,h) \in \mathrm{Reg}(\mathcal{T}(X,A)) \times \mathrm{Reg}(\mathcal{T}(Y,\sigma)) : ag = ha\}.$$

In particular, $\mathrm{Reg}(\mathcal{T}^a_{XY})$ *is a pullback product of the regular semigroups* $\mathrm{Reg}(\mathcal{T}(X,A))$ *and* $\mathrm{Reg}(\mathcal{T}(Y,\sigma))$ *with respect to* $\mathcal{T}_A$.

3.2.3 The regular subsemigroup $\mathrm{P}^a = \mathrm{Reg}(\mathcal{T}^a_{XY})$

Continuing the analysis, here we provide the parallels of the results of Subsection 3.1.3 for the sandwich semigroup $\mathcal{T}^a_{XY}$.

Of course, we have $\mathrm{P}^a = \mathrm{Reg}(\mathcal{T}^a_{XY})$ (by Proposition 2.3.2(i)), since all the elements of $\mathcal{T}$ are sandwich-regular. Moreover, Lemma 2.3.3 implies that, for all $x \in \mathrm{P}^a$ and all $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}\}$, holds $\mathrm{K}^{\mathrm{P}^a}_x = \mathrm{K}^a_x$. As usual, for each $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}\}$ we write $\mathscr{K}^a$ for the corresponding Green's relation of P^a. Further, Lemma 2.3.4 gives $\mathscr{J}^{\mathrm{P}^a} = \mathscr{D}^{\mathrm{P}^a}$. The same approach works for the Proposition 3.1.24, so for $f \in \mathrm{P}^a$ we have:

(i) $\mathrm{R}_f^a = \mathrm{R}_f \cap \mathrm{P}^a = \{g \in \mathrm{P}^a \ : \ \ker g = \ker f\}$,

(ii) $\mathrm{L}_f^a = \mathrm{L}_f \cap \mathrm{P}^a = \{g \in \mathrm{P}^a \ : \ \operatorname{im} g = \operatorname{im} f\}$,

(iii) $\mathrm{H}_f^a = \mathrm{H}_f \cap \mathrm{P}^a = \{g \in \mathrm{P}^a \ : \ \ker g = \ker f, \ \operatorname{im} g = \operatorname{im} f\}$,

(iv) $\mathrm{D}_f^a = \mathrm{D}_f \cap \mathrm{P}^a = \{g \in \mathrm{P}^a \ : \ \operatorname{Rank} g = \operatorname{Rank} f\}$.

Also, the $\mathscr{J}^{\mathrm{P}^a} = \mathscr{D}^a$-classes of P^a are the sets

$$\mathrm{D}_\mu^a = \{g \in \mathrm{P}^a : \operatorname{Rank} g = \mu\} \quad \text{for each cardinal } 1 \leq \mu \leq \alpha = \operatorname{rank} a,$$

and these form a chain under the $\leq_{\mathscr{J}}$ ordering of $\mathscr{J}^{\mathrm{P}^a}$-classes:

$$\mathrm{D}_\mu^a \leq \mathrm{D}_\nu^a \Leftrightarrow \mu \leq \nu.$$

So, the minimum $\mathscr{J}^{\mathrm{P}^a} = \mathscr{D}^a$-class here differs from the one in the $\mathcal{PT}$-case, but the maximum class has the same from:

$$\mathrm{D}_1^a = \{f \in \mathcal{T}_{XY} : \operatorname{Rank} f = 1\} \ \text{ and } \ \mathrm{D}_\alpha^a = \{f \in \mathrm{P}^a \ : \ \operatorname{Rank} f = \alpha\}.$$

(Note that $|\,\mathrm{D}_1^a\,| = |Y|$.) If $\alpha = \xi = \max(X, Y)$, the latter is the maximal $\mathscr{J}^a$-class of $\mathcal{T}_{XY}^a$, as well (by Proposition 3.2.7). In Figure 3.14, we display the structure of the regular subsemigroups of several sandwich semigroups. The reader may check the egg-box diagrams of the original sandwich semigroups on Figures 3.12–3.13 to locate the maximal $\mathscr{J}^a$-classes.

Here, the role of the map ϕ (from the general theory) will be played by the map

$$\varphi = \psi_1\phi_1\eta' = \psi_2\phi_2\eta' : \mathrm{P}^a \to \mathcal{T}_A : f \mapsto (bafa){\restriction_A} = (fa){\restriction_A}.$$

Again, we write $\overline{f} = f\varphi = (fa){\restriction_A}$ for all $f \in \mathrm{P}^a$; furthermore, we define the relations $\widehat{\mathscr{K}^a}$ for each $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}\}$ in the same way as in the $\mathcal{PT}$-case.

When examining the map $\overline{f}$, in search for a suitable representation akin to that used in Subsection 3.1.3, we may conclude that the discussion preceding Lemma 3.1.25 applies. Namely, if $f = \binom{F_j}{f_j}_{j \in J}$ with $F_j \cap A = \{a_i : i \in I_j\}$, and we assume (without loss of generality) that $J \subseteq I$ and $f_j \in A_j$ for each $j \in J$, we have $\overline{f} = (fa){\restriction_A} = \binom{F_j \cap A}{a_j}_{j \in J}$. The main difference, as usual, concerns the domain: in this case, $\bigcup\{F_j : j \in J\} = X$ and thus $\bigcup\{F_j \cap A : j \in J\} = A$.

As in the case of sandwich semigroups of partial maps, we need some more information on the semigroup $\mathcal{T}_A$ in order to describe the inflation. From [45] we know: in $\mathcal{T}_A$, for a map h with $\operatorname{Rank} h = \mu$, holds

(i) $\mathrm{R}_h = \{g \in \mathcal{T}_A : \ker g = \ker h\}$;

(ii) $\mathrm{L}_h = \{g \in \mathcal{T}_A : \operatorname{im} g = \operatorname{im} h\}$;

(iii) $\mathrm{H}_h = \{g \in \mathcal{T}_A : \ker g = \ker h, \ \operatorname{im} g = \operatorname{im} h\}$;

(iv) $|\,\mathrm{H}_h\,| = \mu!$; furthermore, if H_h contains an idempotent, then $\mathrm{H}_h \cong \mathrm{S}_\mu$;

(v) $\mathrm{D}_h = \mathrm{J}_h = \{g \in \mathcal{T}_A : \operatorname{Rank} g = \operatorname{Rank} h = \mu\} = \mathrm{D}_\mu$.

(vi) If $\alpha = |A|$ is finite, then $\mathrm{D}_\alpha = \mathrm{H}_{\mathrm{id}_A} \cong \mathrm{S}_A$ and $\mathcal{T}_A \setminus \mathrm{D}_\alpha$ is an ideal of the semigroup $\mathcal{T}_A$.

Therefore, we may prove

Theorem 3.2.10. *Let* $f = \binom{F_j}{f_j}_{j\in J} \in \mathrm{P}^a$ *with* $\operatorname{Rank} f = \mu$. *Then*

(i) $\widehat{\mathrm{R}^a_f}$ *is the union of* μ^β $\mathscr{R}^a$*-classes of* P^a*;*

(ii) $\widehat{\mathrm{L}^a_f}$ *is the union of* Λ_J $\mathscr{L}^a$*-classes of* P^a*;*

(iii) $\widehat{\mathrm{H}^a_f}$ *is the union of* $\mu^\beta\Lambda_J$ $\mathscr{H}^a$*-classes of* P^a*, each of which has size* $\mu!$*;*

(iv) if $\mathrm{H}_{\overline{f}}$ *is a non-group* $\mathscr{H}$*-class of* $\mathcal{T}_A$*, then each* $\mathscr{H}^a$*-class of* P^a *contained in* $\widehat{\mathrm{H}^a_f}$ *is a non-group;*

(v) if $\mathrm{H}_{\overline{f}}$ *is a group* $\mathscr{H}$*-class of* $\mathcal{T}_A$*, then each* $\mathscr{H}^a$*-class of* P^a *contained in* $\widehat{\mathrm{H}^a_f}$ *is a group isomorphic to* S_μ*; further,* $\widehat{\mathrm{H}^a_f}$ *is a* $\mu^\beta \times \Lambda_J$ *rectangular group over* S_μ*, and its idempotents* $\mathrm{E}_a(\widehat{\mathrm{H}^a_f})$ *form a* $\mu^\beta \times \Lambda_J$ *rectangular band;*

(vi) $\widehat{\mathrm{D}^a_f} = \mathrm{D}^a_f = \mathrm{D}^a_\mu = \{g \in \mathrm{P}^a : \operatorname{Rank} g = \mu\}$ *is the union of:*

- *(a)* $\mu^\beta\, \mathcal{S}(\alpha,\mu)$ $\mathscr{R}^a$*-classes of* P^a*,*
- *(b)* $\sum_{\substack{K\subseteq I\\|K|=\mu}} \Lambda_K$ $\mathscr{L}^a$*-classes of* P^a*,*
- *(c)* $\mu^\beta\, \mathcal{S}(\alpha,\mu) \sum_{\substack{K\subseteq I\\|K|=\mu}} \Lambda_K$ $\mathscr{H}^a$*-classes of* P^a*.*

Proof. Unsurprisingly, the proof is very similar to the proof of Theorem 3.1.26. The reasoning is the same, and the references to the results of Section 3.1 are replaced with references to the corresponding results for $\mathcal{T}^a_{XY}$. The only part that requires additional commenting is (i). Of course, the main difference is the fact that all the domains are full, so all the elements of X have to belong to the domain of g, which gives μ^β $\mathscr{R}^a$-classes in $\widehat{\mathrm{D}^a_f}$. □

Due to the form of $\mathscr{D}^a$-classes in P^a and the cardinalities calculated in the previous theorem, we have

$$|\,\mathrm{D}^a_\mu\,| = \mu!\mu^\beta\, \mathcal{S}(\alpha,\mu) \sum_{\substack{K\subseteq I\\|K|=\mu}} \Lambda_K \qquad \text{and}$$

$$|\,\mathrm{P}^a\,| = \sum_{\mu=1}^{\alpha} |\,\mathrm{D}^a_\mu\,| = \sum_{\mu=1}^{\alpha} \mu!\mu^\beta\, \mathcal{S}(\alpha,\mu) \sum_{\substack{K\subseteq I\\|K|=\mu}} \Lambda_K.$$

As in Subsection 3.1.3, we discuss the possible simplifications of the last formula in special cases:

Proposition 3.2.11.

(i) If $\alpha \geq 2$ *and* $|X| \geq \aleph_0$ *then*

$$|\,\mathrm{P}^a\,| = 2^{|X|}\Lambda_I = \max(2^{|X|}, \Lambda_I).$$

(ii) If $\alpha \geq 2$ *and* $|X| < \aleph_0$ *and* $\lambda_i \geq \aleph_0$ *for some* $i \in I$*, then*

$$|\,\mathrm{P}^a\,| = \Lambda_I = \max_{i \in I} \lambda_i.$$

(iii) $|\,\mathrm{P}^a\,| < \aleph_0 \Leftrightarrow [\alpha = 1$ *and* $|Y| < \aleph_0]$
or $[\alpha \geq 2,\ |X| < \aleph_0$ *and* $\lambda_i < \aleph_0$ *for all* $i \in I]$.

(iv) $|\,\mathrm{P}^a\,| = \aleph_0 \Leftrightarrow [\alpha = 1$ *and* $|Y| = \aleph_0]$
or $[\alpha \geq 2,\ |X| < \aleph_0$ *and* $\max_{i \in I} \lambda_i = \aleph_0]$.

(v) $|\,\mathrm{P}^a\,| > \aleph_0 \Leftrightarrow [\alpha = 1$ *and* $|Y| > \aleph_0]$
or $\Big[\alpha \geq 2$ *and* $[|X| \geq \aleph_0$ *or* $\lambda_i > \aleph_0$ *for some* $i \in I]\Big]$.

Proof. For the first two parts, we slightly modify the proof of Proposition 3.1.30. The changes are minor: the smallest possible rank is 1 (not 0), and instead of terms $(\alpha + 1)^\beta$ and $\mathrm{S}(\alpha + 1, \mu + 1)$, we use α^β and $\mathrm{S}(\alpha, \mu)$, respectively. These make no difference in the argument, since the infinite values are not affected by a finite increase or decrease.

Note that each of the statements $(iii) - (v)$ has the form

$$A \Leftrightarrow [\alpha = 1 \wedge B] \vee [\alpha \geq 2 \wedge C].$$

Instead of proving the original statement, we will prove an equivalent one:

$$\Big[\alpha = 1 \Rightarrow [A \Leftrightarrow B]\Big] \wedge \Big[\alpha \geq 2 \Rightarrow [A \Leftrightarrow C]\Big]$$

(it is indeed equivalent, since $\alpha \geq 1$). Recall that $\alpha = 1$ implies $\mathrm{P}^a = \mathrm{D}_1^a = \{f \in \mathcal{T}_{XY} : \operatorname{rank} f = 1\}$, so in this case we have

$$|\,\mathrm{P}^a\,| = |Y|.$$

Thus, we may suppose $\alpha \geq 2$. In this case, we use reasoning analogous to the one in the proof of Proposition 3.1.30. □

Remark 3.2.12. Similarly as in the case of the semigroup $\mathcal{PT}^a_{XY}$, if $\alpha \geq \aleph_0$, we may suppose without loss of generality that the sequence $\langle \lambda_i : i \in I \rangle$ is nondecreasing and then Lemma 5.9 in [62] gives

$$\Lambda_I = (\sup_{i \in I} \lambda_i)^\alpha$$

After calculating the size of P^a, we tackle the problem of calculating its rank. As in the previous section, the term MI-domination is crucial here.

Proposition 3.2.13.

(i) The semigroup $\mathrm{P}^a = \mathrm{Reg}(\mathcal{T}^a_{XY})$ *is MI-dominated.*

(ii) The semigroup $\mathrm{P}^a = \mathrm{Reg}(\mathcal{T}^a_{XY})$ *is RP-dominated if and only if* $\mathrm{Rank}\, a < \aleph_0$.

Proof. (i) We apply the same argument as in the proof of Proposition 3.1.33, modifying only the auxiliary maps:

$$g = \begin{pmatrix} F_{j,k} \\ b_k \end{pmatrix}_{j \in J,\ k \in I_j} \quad \text{and} \quad h = \begin{pmatrix} B_j & B_m \\ f_j & b_m \end{pmatrix}_{j \in J,\ m \in I \setminus J}.$$

(ii) As in the proof of Proposition 3.1.33, P^a is RP-dominated if and only if W is factorisable. In other words, P^a is RP-dominated if and only if $\mathcal{T}_A$ is a factorisable semigroup, which occurs if and only if A is finite (by Lemma 3.1.32). □

In the next theorem, we omit the case when a is a bijection, because then we have $\mathcal{T}^a_{XY} \cong \mathcal{T}_A = \mathrm{Reg}(\mathcal{T}_A)$ (so $\mathrm{rank}(\mathrm{P}^a) = \mathrm{rank}(\mathcal{T}_A)$).

Theorem 3.2.14. *Suppose a is not a bijection.*

(i) If $|\,\mathrm{P}^a\,| \geq \aleph_0$, *then* $\mathrm{rank}(\mathrm{P}^a) = |\,\mathrm{P}^a\,|$.

(ii) If $|\,\mathrm{P}^a\,| < \aleph_0$, *then*

$$\mathrm{rank}(\mathrm{P}^a) = \begin{cases} |Y|, & \text{if } \alpha = 1; \\ 1 + \max(\alpha^\beta, \Lambda_I), & \text{if } \alpha \geq 2. \end{cases}$$

Proof. Recall that $\alpha = 1$ implies $\mathrm{P}^a = \mathrm{D}^a_1 = \{f \in \mathcal{T}_{XY} : \mathrm{Rank}\, f = 1\}$. Since in this case no subset of P^a can generate a map it does not contain, we have

$$\mathrm{rank}(\mathrm{P}^a) = |\,\mathrm{P}^a\,| = |Y|.$$

For the case $\alpha \geq 2$ we use essentially the same proof as for Theorem 3.1.34, swapping the references to the results of Section 3.1 for references to the corresponding results of this section (and swapping $\mathcal{PT}_A$ for $\mathcal{T}_A$). However, we need to discuss the subcase $|\,\mathrm{P}^a\,| \leq \aleph_0$ further. Since here $|\widehat{\mathrm{H}^a_b}/\mathscr{R}\,| = \alpha^\beta$, $|\widehat{\mathrm{H}^a_b}/\mathscr{L}\,| = \Lambda_I$, and $W \cong \mathcal{T}_A$ (so $G_W \cong \mathrm{S}_\alpha$), we may conclude that

$$\mathrm{rank}(\mathrm{P}^a) = \mathrm{rank}(\mathcal{T}_\alpha : \mathrm{S}_\alpha) + \max(\alpha^\beta, \Lambda_I, \mathrm{rank}(\mathrm{S}_\alpha)). \tag{3.11}$$

This suffices if $|\,\mathrm{P}^a\,| = \aleph_0$. In the case when $|\,\mathrm{P}^a\,| < \aleph_0$, we have $\alpha < \aleph_0$, so $\mathrm{S}_\alpha \le 2$. Since a is not a bijection, it is either non-surjective or non-injective. If the first is true, then $\beta = |X \setminus \operatorname{im} a| > 1$, so $\alpha^\beta \ge \operatorname{rank}(\mathrm{S}_\alpha)$. If the second is true, then $\Lambda_I \ge 2$, so $\Lambda_\alpha \ge \operatorname{rank}(\mathrm{S}_\alpha)$. Therefore, (3.11) implies

$$\operatorname{rank}(\mathrm{P}^a) = \operatorname{rank}(\mathcal{T}_\alpha : \mathrm{S}_\alpha) + \max(\alpha^\beta, \Lambda_I).$$

The final formula can now be deduced from the following two facts. Firstly, $\mathcal{T}_\alpha \setminus \mathrm{S}_\alpha$ is an ideal, and secondly, the set S_α, in union with any transformation of rank $\alpha - 1$, generates the whole semigroup $\mathcal{T}_\alpha$. □

Until now, in this subsection, we skipped the analysis of the cases when a is injective or surjective. Now we make up for that.

Remark 3.2.15.

- As we stated before, $\mathcal{T}^a_{XY} \cong \mathcal{T}(X, A)$ holds if and only if a is injective, which holds if and only if each class of $\ker a$ is a singleton. Therefore, injectivity implies $\Lambda_J = 1$ for all $J \subseteq I$. Hence, parts (ii) and (v) of Theorem 3.2.10 respectively imply that $\widehat{\mathscr{L}^a} = \mathscr{L}^a$ and that for each group $\mathscr{H}$-class $\mathrm{H}_{\overline{f}}$, $\widehat{\mathrm{H}^a_f}$ is an underlying set of a $\mu^\beta \times 1$ rectangular group over S_μ. Furthermore, Proposition 3.2.11 simplifies substantially, as the clauses featuring λ_i and Λ_I become redundant. Finally, from Theorem 3.2.14 we may deduce that, if $|X| < \aleph_0$, for each nontrivial subset A of X we have

$$\operatorname{rank}(\operatorname{Reg}(\mathcal{T}(X, A))) = \begin{cases} 1, & \text{if } |A| = 1; \\ 2 + |A|^{|X|-|A|}, & \text{if } |A| \ge 2. \end{cases}$$

- Now, we examine the case when a is surjective. This holds if and only if $\operatorname{im} a = X$, which is true if and only if $\mathcal{T}^a_{XY} \cong \mathcal{T}(Y, \sigma)$. Therefore, we have $\beta = |X \setminus \operatorname{im} a| = 0$, so $\mu^\beta = 1$. Furthermore, parts (i) and (v) of Theorem 3.2.10 respectively imply that $\widehat{\mathscr{R}^a} = \mathscr{R}^a$ and that for each group $\mathscr{H}$-class $\mathrm{H}_{\overline{f}}$, $\widehat{\mathrm{H}^a_f}$ is an underlying set of an $1 \times \Lambda_J$ rectangular group over S_μ. Again, from Theorem 3.2.14 we conclude that for any non-diagonal equivalence relation σ with $\mathrm{u}(\sigma) \subseteq Y$, we have

$$\operatorname{rank}(\operatorname{Reg}(\mathcal{T}(Y, \sigma))) = \begin{cases} |Y|, & \text{if } |\pi_\sigma| = 1; \\ 1 + \Lambda_I, & \text{if } |\pi_\sigma| \ge 2. \end{cases}$$

3.2.4 Idempotents and idempotent-generation

Using the same approach as in Subsection 3.1.4, here we investigate the set of idempotents and the idempotent-generated subsemigroup, respectively:

$$\mathrm{E}_a(\mathcal{T}^a_{XY}) = \{f \in \mathcal{T}_{XY} : f = f \star_a f\} \qquad (= \mathrm{E}_a(\mathrm{P}^a)), \text{ and}$$
$$\mathscr{E}^a_{XY} = \mathbb{E}_a(\mathcal{T}^a_{XY}) = \langle E_a(\mathcal{T}^a_{XY})\rangle_a \qquad (= \mathbb{E}_a(\mathrm{P}^a)).$$

Proposition 3.2.16.

(i) $\mathrm{E}_a(\mathcal{T}^a_{XY}) = \{f \in \mathcal{T}_{XY} : (af)\restriction_{\mathrm{im}\, f} = \mathrm{id}_{\mathrm{im}\, f}\}$.

(ii) If $|\,\mathrm{P}^a\,| \geq \aleph_0$, *then* $|\,\mathrm{E}_a(\mathcal{T}^a_{XY})| = |\,\mathrm{P}^a\,|$.

(iii) If $|\,\mathrm{P}^a\,| < \aleph_0$, *then*

$$|\,\mathrm{E}_a(\mathcal{T}^a_{XY})| = \sum_{\mu=1}^{\alpha} \mu^{|X|-\mu} \sum_{\substack{J\subseteq I\\ |J|=\mu}} \Lambda_J. \tag{3.12}$$

Proof. Part (i) is proved in a same manner as the corresponding part of Proposition 3.1.36. As for part (ii), the assumption $|\,\mathrm{P}^a\,| \geq \aleph_0$ implies, by Proposition 3.2.11 (parts (iv) and (v)), that exactly one of the following is true

– $\alpha = 1$ and $|Y| \geq \aleph_0$;

– $\alpha \geq 2$, $|X| < \aleph_0$, and $\lambda_i \geq \aleph_0$ for some $i \in I$;

– $\alpha \geq 2$ and $|X| \geq \aleph_0$.

Recall that $\alpha = 1$ guarantees $\mathrm{P}^a = \{f \in \mathcal{T}_{XY} : \mathrm{Rank}\, f = 1\}$. Furthermore, in this case we have $g \star_a g = g$ for any $g \in \mathcal{T}_{XY}$ with $\mathrm{Rank}\, g = 1$, so $\mathrm{E}_a(\mathcal{T}_{XY}) = \mathrm{E}_a(\mathrm{P}^a) = \mathrm{P}^a$. The remaining cases are handled in the same way as in the proof of Proposition 3.1.36 (applying the corresponding statements from this section), the only difference being the $2^{|X|}$ idempotents presented in the third case (because we obviously cannot use those). Here, since $\alpha \geq 2$, there exist $i_1, i_2 \in I$ such that $b_{i_1} \neq b_{i_2}$. Thus, for each partition $\{Q, W\}$ of the set $X \setminus \{a_1, a_2\}$, the maps

$$\begin{pmatrix} Q\cup\{a_1\} & W\cup\{a_2\} \\ b_1 & b_2 \end{pmatrix} \text{ and } \begin{pmatrix} W\cup\{a_1\} & Q\cup\{a_2\} \\ b_1 & b_2 \end{pmatrix}$$

are idempotents. Since the number of such partitions is $\mathcal{S}(|X| - 2, 2) = \mathcal{S}(|X|, 2) = 2^{|X|}$, we have presented $2 \cdot 2^{|X|} = 2^{|X|}$ different idempotents.

(iii) Suppose $|\,\mathrm{P}^a\,| < \aleph_0$. Proposition 3.2.11$(iii)$ implies that either $\alpha = 1$ with $|Y| < \aleph_0$, or $\alpha \geq 2$, $|X| < \aleph_0$, and $\lambda_i < \aleph_0$ for all $i \in I$. In the first case, the sum on the right-hand side of the equality (3.12) clearly equals to the size of Λ_I, i.e. the size of Y. As we showed above, this equals the number of idempotents if $\alpha = 1$. In the second case, we identify a set of properties which fully determine an idempotent, and then simply calculate the number of valid combinations. Let $f = \begin{pmatrix} F_j \\ f_j \end{pmatrix}_{j\in J}$ be an idempotent. Since $f \in \mathrm{P}^a$, we know that $\ker a$ separates $\mathrm{im}\, f$ and $\mathrm{im}\, a$ saturates $\ker f$. Moreover, we may assume without loss of generality that $J \subseteq I$ and $f_j \in A_j$ for all $j \in J$. The condition $(af)\restriction_{\mathrm{im}\, f} = \mathrm{id}_{\mathrm{im}\, f}$ now implies that $a_j \in F_j$ for all $j \in J$. Thus, the idempotent f is determined by

- the set $J \subseteq I$,

- its image $\mathrm{im}\, f$ (in other words, the choice of elements $f_j \in A_j$ for $j \in J$), and

- its kernel $\ker f$ (the choice of the set $F_j \setminus \{a_j\}$ for $j \in J$).

Hence, we define an idempotent of rank $1 \le \mu \le \alpha$ by choosing the pair $(J, \operatorname{im} f)$ in $\sum_{\substack{J\subseteq I \\ |J|=\mu}} \Lambda_J$ ways, and the kernel in $\mu^{|X|-\mu}$ ways. $\square$

Our next task is to give analogues of the statements of Theorem 3.1.39 and Lemma 3.1.41. As in the case of $\mathcal{PT}^a_{XY}$, in order to do that, we need some additional notation and information. Let $\mathrm{E}(\mathcal{T}_A)$ and $\mathbb{E}(\mathcal{T}_A)$ denote the set of idempotents and the idempotent-generated subsemigroup of $\mathcal{T}_A$, respectively. Also, for $f \in \mathcal{T}_A$ let $\operatorname{sh} f$, $\operatorname{codef} f$ and $\operatorname{coll} f$ be defined in the same way as for the elements of $\mathcal{PT}_A$ in Subsection 3.1.4. Then, from articles [48, 55, 57] we know the following:

Proposition 3.2.17.

(i) *If* $|A| < \aleph_0$, *then* $\mathbb{E}(\mathcal{T}_A) = \{\mathrm{id}_A\} \cup (\mathcal{T}_A \setminus \mathrm{S}_A)$, *and*

$$\operatorname{rank}(\mathbb{E}(\mathcal{T}_A)) = \operatorname{idrank}(\mathbb{E}(\mathcal{T}_A)) = \begin{cases} \binom{\alpha}{2} + 1, & \text{if } \alpha \neq 2; \\ 3, & \text{if } \alpha = 2. \end{cases}$$

(ii) *If* $|A| \ge \aleph_0$, *then*

$$\mathbb{E}(\mathcal{T}_A) = \{\mathrm{id}_A\} \cup \{f \in \mathcal{T}_A \setminus \mathrm{S}_A : \operatorname{sh} f < \aleph_0\} \\ \cup \{f \in \mathcal{T}_A : \operatorname{sh} f = \operatorname{coll} f = \operatorname{def} f \ge \aleph_0\}$$

and $\operatorname{rank}(\mathbb{E}(\mathcal{T}_A)) = \operatorname{idrank}(\mathbb{E}(\mathcal{T}_A)) = |\,\mathcal{T}_A\,| = 2^{|A|}$.

Finally, we are ready to prove

Theorem 3.2.18.

(i) $\mathscr{E}^a_{XY} = \mathbb{E}_a(\mathcal{T}^a_{XY}) = (\mathbb{E}(\mathcal{T}_A))\varphi^{-1}$,

(ii) $$\operatorname{rank}(\mathscr{E}^a_{XY}) = \begin{cases} |\mathscr{E}^a_{XY}| = |\,\mathrm{P}^a\,|, & |\,\mathrm{P}^a\,| \ge \aleph_0; \\ \binom{\alpha}{2} + \max(\alpha^\beta, \Lambda_I), & |\,\mathrm{P}^a\,| < \aleph_0 \text{ and } \alpha \neq 2. \\ 2 + \max(2^\beta, \Lambda_I), & |\,\mathrm{P}^a\,| < \aleph_0 \text{ and } \alpha = 2. \end{cases}$$

and $\operatorname{rank}(\mathscr{E}^a_{XY}) = \operatorname{idrank}(\mathscr{E}^a_{XY})$.

Proof. The proof is virtually the same as the proof of Theorem 3.1.39 (as always, instead of the results from Section 3.1, we reference the parallels from Section 3.2), but for *(ii)* we have to consider an additional case. Namely, if $\alpha = 1$, we have

$$\mathrm{P}^a = \{f \in \mathcal{T}_{XY} : \operatorname{Rank} f = 1\} = \mathrm{E}(\mathcal{T}^a_{XY}),$$

so $\mathrm{P}^a = \mathscr{E}^a_{XY}$. Since all the idempotents are constant maps, none of them can be generated by other idempotents, so $\operatorname{rank}(\mathscr{E}^a_{XY}) = \operatorname{idrank}(\mathscr{E}^a_{XY}) = |\mathscr{E}^a_{XY}|$. This corroborates the provided formula both in the case $|\,\mathrm{P}^a\,| \ge \aleph_0$ and in the case $|\,\mathrm{P}^a\,| < \aleph_0$, because $\alpha = 1$ implies $\ker a = Y \times Y$ and $\Lambda_I = |Y|$. $\square$

As in the analysis of $\mathcal{PT}^a_{XY}$, now we focus on the case where $\alpha < \aleph_0$. This assumption guarantees that $\mathcal{T}_A$ is a finite monoid with identity $\mathrm{id}_A = \overline{b}$, so we may use the proof of Lemma 3.1.41 to show that $\mathrm{J}^a_b = \mathrm{D}^a_\alpha = \widehat{\mathrm{H}^a_b}$. Furthermore, if $\alpha = \max(|X|, |Y|)$, J^a_b is the maximum $\mathscr{J}^a$-class of $\mathcal{T}^a_{XY}$. Therefore, we may prove an analogue of Theorem 3.1.42 using the same proof, but referencing the corresponding results for $\mathcal{T}^a_{XY}$:

Theorem 3.2.19. *If* $\alpha = \operatorname{rank} a < \aleph_0$, *then* $\mathscr{E}^a_{XY} = \mathbb{E}_a(\mathcal{T}^a_{XY}) = \mathrm{E}_a(\mathrm{D}^a_\alpha) \cup (\mathrm{P}^a \setminus \mathrm{D}^a_\alpha)$.

Remark 3.2.20. We close the subsection by describing the simplifications occurring in the above results in the special cases.

- If a is injective, in the semigroup $\mathcal{T}^a_{XY} \cong \mathcal{T}(X, A)$ we have $\Lambda_J = 1$ for all $J \subseteq I$, so Proposition 3.2.16 gives

$$|\,\mathrm{E}_a(\mathcal{T}(X, A))| = \sum_{\mu=1}^{\alpha} \mu^{|X|-\mu} \sum_{\substack{J \subseteq I \\ |J|=\mu}} 1 = \sum_{\mu=1}^{|A|} \mu^{|X|-\mu} \binom{|A|}{\mu}.$$

  Note that $\alpha = 1$ implies $|\,\mathrm{P}^a\,| = |Y| = 1$, since a is injective. Hence, Proposition 3.2.11 and Theorem 3.2.18 together imply

$$\begin{aligned}
\operatorname{rank}(\mathbb{E}(\mathcal{T}(X, A))) &= \operatorname{idrank}(\mathbb{E}(\mathcal{T}(X, A))) \\
&= \begin{cases}
|\mathbb{E}(\mathcal{T}(X, A))| = |\,\mathrm{P}^a\,| = 2^{|X|}, & |X| \geq \aleph_0; \\[4pt]
\binom{|A|}{2} + |A|^{|X|-|A|}, & |X| < \aleph_0 \text{ and } |A| \neq 2; \\[4pt]
2 + 2^{|X|-2}, & |X| < \aleph_0 \text{ and } |A| = 2.
\end{cases}
\end{aligned}$$

- If a is surjective, in the semigroup $\mathcal{T}^a_{XY} \cong \mathcal{T}(Y, \sigma)$ we have $\operatorname{im} a = X$, so $\alpha = |\pi_\sigma| = |X|$ and $\beta = |X \setminus \operatorname{im} a| = 0$. Therefore, Proposition 3.2.16 yields

$$|\,\mathrm{E}_a(\mathcal{T}(Y, \sigma))| = \sum_{\mu=1}^{|\pi_\sigma|} \mu^{|\pi_\sigma|-\mu} \sum_{\substack{J \subseteq I \\ |J|=\mu}} \Lambda_J,$$

  while Proposition 3.2.11 and Theorem 3.2.18 together give

$$\begin{aligned}
\operatorname{rank}(\mathbb{E}(\mathcal{T}(Y, \sigma))) &= \operatorname{idrank}(\mathbb{E}(\mathcal{T}(Y, \sigma))) \\
&= \begin{cases}
|\mathbb{E}(\mathcal{T}(Y, \sigma))| = |Y|, & |\pi_\sigma| = 1; \\[4pt]
|\mathbb{E}(\mathcal{T}(Y, \sigma))| = \max(2^{|\pi_\sigma|}, \Lambda_I), & |\pi_\sigma| \geq \aleph_0; \\[4pt]
|\mathbb{E}(\mathcal{T}(Y, \sigma))| = \max_{i \in I} \lambda_i, & 2 \leq |\pi_\sigma| < \aleph_0 \text{ and } |Y| \geq \aleph_0; \\[4pt]
\binom{|\pi_\sigma|}{2} + \Lambda_I, & 3 \leq |\pi_\sigma| < \aleph_0 \text{ and } |Y| < \aleph_0; \\[4pt]
2 + \Lambda_I, & |Y| < \aleph_0 \text{ and } |\pi_\sigma| = 2.
\end{cases}
\end{aligned}$$

3.2.5 The rank of a sandwich semigroup $\mathcal{T}^a_{XY}$

As in Section 3.1, the last problem we consider for the sandwich semigroup $\mathcal{T}^a_{XY}$ is that of calculating of its rank. Not surprisingly, we have a similar situation as in the $\mathcal{PT}$-case. Namely, after considering a few simple cases, we focus on the remaining one, and we infer three formulas: the first one for the case where $\alpha < \min(|X|, |Y|)$, and the other two for the cases where $\alpha = |X|$ and $\alpha = |Y|$, respectively.

- **Suppose $|\mathbf{X}| = \mathbf{1}$.** Clearly, $\mathcal{T}^a_{XY}$ is a right-zero semigroup of size $|Y|$ (the number of possible maps of rank 1), so $\mathrm{rank}(\mathcal{T}^a_{XY}) = |\,\mathcal{T}^a_{XY}\,| = |Y|$.
- **Suppose $|\mathbf{Y}| = \mathbf{1}$.** Since we have no choice, in any map of $\mathcal{T}_{XY}$ all the elements of X map to the single element of Y, so $\mathrm{rank}(\mathcal{T}^a_{XY}) = |\,\mathcal{T}^a_{XY}\,| = 1$.
- **Suppose that either $|\mathbf{Y}| \geq \mathbf{2}$ and $|\mathbf{X}| \geq \aleph_\mathbf{0}$, or $|\mathbf{Y}| > \aleph_\mathbf{0}$.** Obviously, this holds if and only if $|\,\mathcal{T}^a_{XY}\,| = |Y|^{|X|} > \aleph_0$; in such case $\mathrm{rank}(\mathcal{T}^a_{XY}) = |\,\mathcal{T}^a_{XY}\,|$.
- **Suppose that $|\mathbf{X}|, |\mathbf{Y}| \geq \mathbf{2}$, $|\mathbf{X}| < \aleph_\mathbf{0}$, $|\mathbf{Y}| \leq \aleph_\mathbf{0}$, and that a is a bijection.** In fact, these hold if and only if $2 \leq \alpha = |X| = |Y| < \aleph_0$. Furthermore, a being a bijection implies that $\mathcal{T}^a_{XY} \cong \mathcal{T}_A = \mathcal{T}_X$. Since the assumption is that $|X| < \aleph_0$, by Theorem 3.1.3 in [45] we have

$$\mathrm{rank}(\mathcal{T}^a_{XY}) = \mathrm{rank}(\mathcal{T}_{|X|}) = \begin{cases} |X|, & \text{if } |X| = 1, 2; \\ 3, & \text{if } |X| \geq 3. \end{cases} \tag{3.13}$$

Thus, for the remainder of this subsection, we assume that $\mathbf{2} \leq |\mathbf{X}| < \aleph_\mathbf{0}$, $\mathbf{2} \leq |\mathbf{Y}| \leq \aleph_\mathbf{0}$, and that a is either non-injective or non-surjective or both. Again, these assumptions imply that $\alpha \leq \xi = \min(|X|, |Y|) < \aleph_0$, and that a is stable (by Proposition 3.2.3), so we know that $\mathscr{J}^a = \mathscr{D}^a$ (by Lemma 2.2.19).

Since the "setting" is similar as in Subsection 3.1.5, we use the same notation and infer similar conclusions. Of course, we need to adapt the arguments slightly. Instead of referencing the results of Section 3.1, we reference the corresponding results of this section. Further, we disregard the cases which contain assumptions of non-full maps, and we make sure the auxiliary maps in the proofs are full. Following these instructions, one may easily prove the following results:

Lemma 3.2.21. *In $\mathcal{T}^a_{XY}$ holds* $\mathrm{D}_0 \cup \mathrm{D}_1 \cup \cdots \cup \mathrm{D}_\alpha = \langle \mathrm{D}_\alpha \rangle_a$.

Proof. We may prove an analogue of Lemma 3.1.44(i). Hence, we need to show that $\mathrm{D}_{\alpha-1} \subseteq \langle \mathrm{D}_\alpha \rangle_a$. If a is non-surjective, we duplicate the proof of Lemma 3.1.44(ii), and if it is non-injective, we prove the parallel of Lemma 3.1.53(i). □

Lemma 3.2.22. *Suppose $\alpha < \xi$ and let $f \in \mathrm{D}_\alpha$. Then $f \in \mathrm{D}_{\alpha+1} \star_a \mathrm{D}_{\alpha+1}$.*

Lemma 3.2.23.

(i) If $\alpha = |Y| < \aleph_0$, then $\mathrm{P}^a_1 = \mathcal{T}^a_{XY}$, $\mathrm{P}^a_2 = \mathrm{P}^a$, and $\mathscr{R}^a = \mathscr{R}$ on $\mathcal{T}^a_{XY}$.

(ii) If $\alpha = |X| < \aleph_0$, then $\mathrm{P}^a_2 = \mathcal{T}^a_{XY}$, $\mathrm{P}^a_1 = \mathrm{P}^a$, and $\mathscr{L}^a = \mathscr{L}$ on $\mathcal{T}^a_{XY}$.

Now, we may use these statements to calculate the rank of $\mathcal{T}_{XY}^a$. We provide a layout of our plan in the following table:

a injective?	a surjective?	Reference	Egg-box diagram
N	N	Theorem 3.2.24	Figure 3.12
Y	N	Theorem 3.2.25	Figure 3.13
N	Y	Theorem 3.2.26	Figure 3.13
Y	Y	see (3.13)	Figure 3.14

Suppose $\alpha < \xi$. Recall that any generating set of $\mathcal{T}_{XY}^a$ has to include elements from each maximal $\mathscr{J}^a$-class (see Section 2.6), and that the maximal $\mathscr{J}^a$-classes are exactly the singletons $\{f\}$, such that $\operatorname{Rank} f > \aleph_0$ (by Proposition 3.2.7(i)). Thus, from Lemmas 3.2.21 and 3.2.22 follows

Theorem 3.2.24. *Suppose $|X| < \aleph_0$, $|Y| \leq \aleph_0$, and that $1 \leq \alpha < \xi$ (hence, a is non-surjective and non-injective). We have*

$$\operatorname{rank}(\mathcal{T}_{XY}^a) = \sum_{\mu=\alpha+1}^{\xi} |\,\mathrm{D}_\mu\,| = \sum_{\mu=\alpha+1}^{\xi} \mu! \binom{|Y|}{\mu} \mathcal{S}(|X|, \mu).$$

If $\alpha = |Y|$, the process of calculating the rank is virtually identical to the proof of Theorem 3.1.51. In it, we use Lemma 3.2.21. We obtain

Theorem 3.2.25. *Suppose that $1 \leq \alpha = \operatorname{Rank} a = |Y| < |X| < \aleph_0$ (hence, a is injective and non-surjective). Then*

$$\operatorname{rank}(\mathcal{T}_{XY}^a) = \mathcal{S}(|X|, \alpha).$$

By a dual argument we may prove prove the following:

Theorem 3.2.26. *Suppose that $1 \leq \alpha = \operatorname{Rank} a = |X| < |Y| \leq \aleph_0$ (hence, a is surjective and non-injective). Then*

$$\operatorname{rank}(\mathcal{T}_{XY}^a) = \binom{|Y|}{\alpha}.$$

In the proof, we use the assumption that a is surjective and non-injective, i.e. that

$$\beta = |X \setminus \operatorname{im} a| = 0 \quad \text{and} \quad \Lambda_I \geq 2 \geq \operatorname{rank}(\mathrm{S}_\alpha).$$

Remark 3.2.27. For the previous two results, we provide alternative formulations concerning the non-sandwich semigroups $\mathcal{T}(X, A)$ and $\mathcal{T}(Y, \sigma)$.

- If a is injective and non-surjective, then $\mathcal{T}_{XY}^a \cong \mathcal{T}(X, A)$, so for any proper subset A of X, we have

$$\operatorname{rank}(\mathcal{T}(X, A)) = \mathcal{S}(|X|, |A|).$$

This is a result from [44] (Theorem 2.3).

- If a is surjective and non-injective, then $\mathcal{T}^a_{XY} \cong \mathcal{T}(Y, \sigma)$, so for any non-diagonal equivalence σ on Y, we have

$$\operatorname{rank}(\mathcal{T}(Y, \sigma)) = \binom{|Y|}{|\pi_\sigma|}.$$

  Unlike the previous one, this result was originally proved in [34].

3.2.6 Egg-box diagrams

As in the previous section, we provide several egg-box diagrams (they originally appeared in [34], and all were generated by GAP [98]) to illustrate the structural results for $\mathcal{T}^a_{XY}$. For more information on the figures, see the introduction to Subsection 3.1.6.

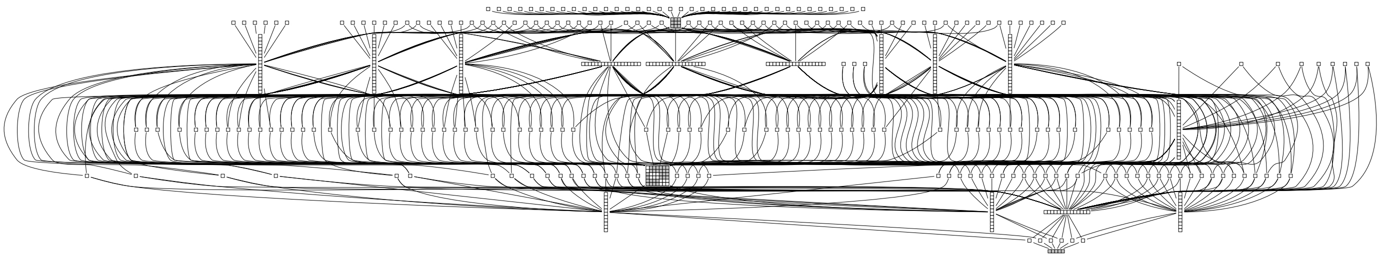


Figure 3.12: Egg-box diagram of the sandwich semigroup $\mathcal{T}_{45}^a$, where $a = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 3 \end{smallmatrix}\right) \in \mathcal{T}_{54}$. Note that a is non-injective and non-surjective.

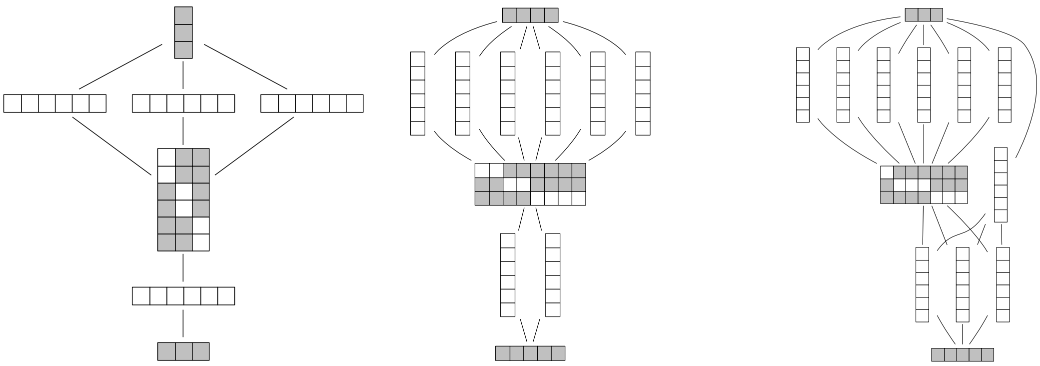


Figure 3.13: Left to right: egg-box diagrams of the sandwich semigroups $\mathcal{T}_{43}^b$, $\mathcal{T}_{35}^c$ and $\mathcal{T}_{35}^d$, where $b = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right) \in \mathcal{T}_{34}$, $c = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 3 \end{smallmatrix}\right) \in \mathcal{T}_{53}$, and $d = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 3 \end{smallmatrix}\right) \in \mathcal{T}_{53}$. Note that b is injective and non-surjective, while c and d are surjective and non-injective.

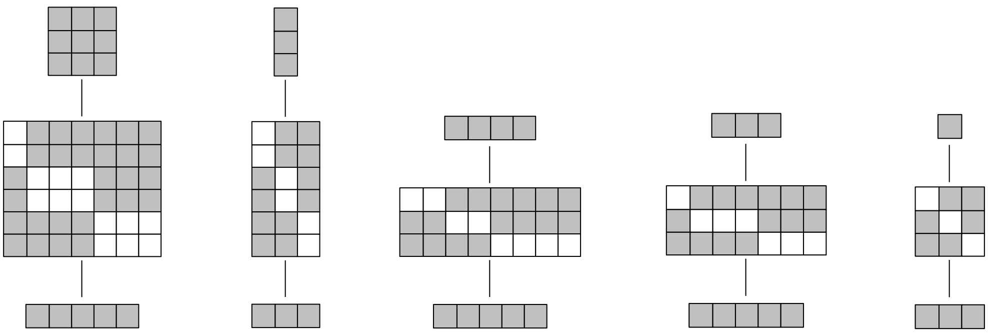


Figure 3.14: Left to right: egg-box diagrams of the regular sandwich semigroups $\text{Reg}(\mathcal{T}_{45}^a)$, $\text{Reg}(\mathcal{T}_{43}^b)$, $\text{Reg}(\mathcal{T}_{35}^c)$, $\text{Reg}(\mathcal{T}_{35}^d)$, and $\mathcal{T}_3$, where the sandwich elements a, b, c, and d are defined as in Figures 3.12 and 3.13. By the theory in Subsection 2.3.4, the first four semigroups are inflations of the fifth semigroup, $\mathcal{T}_3$.

3.3 The category $\mathcal{I}$

At the beginning of this chapter, we introduced three partial semigroups: $\mathcal{PT}$, $\mathcal{T}$ and $\mathcal{I}$. In the previous two sections, we have examined the first two and the sandwich semigroups they contain, so we focus now on the third one. In Proposition 3.0.3, we state that $\mathcal{I}$ can be expanded to an inverse partial semigroup. In other words, $\mathcal{I}$ is an inverse category, as defined in Definition 2.5.1 (see [67] and [23]).

Recall that in an inverse category, every element is uniquely sandwich-regular (see Proposition 3.0.3). Thus, the results of Section 2.5 apply in this case. Moreover, $\mathcal{I}$ is a regular and monoidal partial subsemigroup of $\mathcal{PT}$, so we may obtain some results on $\mathcal{I}$ by analysing the corresponding results for $\mathcal{PT}$ (and $\mathcal{T}$). In most cases, we skip the details and state only the major results, pointing out the differences in the proofs, if necessary.

The results stated here were either explicitly stated in [34], or are implied by the theory communicated in it. Thus, we cite this article as our source, if not stated otherwise.

Since we are dealing with injective maps, for any $A, B \in \mathbf{Set}$ and any $f \in \mathbf{I}_{AB}$, we have $\ker f = \{\{x\} : x \in \operatorname{dom} f\}$. Hence, from Lemma 3.1.1 directly follows

(i) $\operatorname{dom}(fg) \subseteq \operatorname{dom} f$, with equality if and only if $\operatorname{im} f \subseteq \operatorname{dom} g$,

(ii) $\operatorname{im}(fg) \subseteq \operatorname{im} g$, with equality if and only if $\operatorname{dom} g \subseteq \operatorname{im} f$,

(iii) $\operatorname{Rank}(fg) \leq \min(\operatorname{Rank} f, \operatorname{Rank} g)$.

Now, we may show

Proposition 3.3.1. *Let* $(A, f, B), (C, g, D) \in \mathcal{I}$. *Then*

(i) $(A, f, B) \leq_{\mathscr{R}} (C, g, D) \Leftrightarrow \; A = C$ *and* $\operatorname{dom} f \subseteq \operatorname{dom} g$,

(ii) $(A, f, B) \leq_{\mathscr{L}} (C, g, D) \Leftrightarrow B = D$ *and* $\operatorname{im} f \subseteq \operatorname{im} g$,

(iii) $(A, f, B) \leq_{\mathscr{J}} (C, g, D) \Leftrightarrow \operatorname{Rank} f \leq \operatorname{Rank} g$,

(iv) $(A, f, B) \mathscr{R} (C, g, D) \Leftrightarrow A = C$ *and* $\operatorname{dom} f = \operatorname{dom} g$,

(v) $(A, f, B) \mathscr{L} (C, g, D) \Leftrightarrow B = D$ *and* $\operatorname{im} f = \operatorname{im} g$,

(vi) $(A, f, B) \mathscr{J} (C, g, D) \Leftrightarrow (A, f, B) \mathscr{D} (C, g, D) \Leftrightarrow \operatorname{Rank} f = \operatorname{Rank} g$.

The proof is virtually identical to the proof of Proposition 3.1.2.

When restricted to the set $\mathcal{I}_{AB} = \{(A, f, B) : f \in \mathbf{I}_{AB}\}$ for some $A, B \in \mathbf{Set}$, these relations are called Green's relations of $\mathcal{I}_{AB}$. The $\mathscr{J} = \mathscr{D}$-classes of $\mathcal{I}_{AB}$ are the sets

$$\mathrm{D}_\mu^{AB} = \mathrm{D}_\mu \cap \mathcal{I}_{AB} = \{(A, f, B) : f \in \mathbf{I}_{AB}, \operatorname{Rank} f = \mu\},$$

for each cardinal $0 \leq \mu \leq \min(|A|, |B|)$. These $\mathscr{J}$-classes form a chain in $\mathcal{I}_{AB}$: $\mathrm{D}^{AB}_{\mu} \leq \mathrm{D}^{AB}_{\nu} \Leftrightarrow \mu \leq \nu$. Moreover, for a fixed cardinal μ, we may calculate the combinatorial structure of D^{AB}_{μ} (and $\mathcal{I}_{AB}$): if $|A| = \alpha$ and $|B| = \beta$, we have

$$|\,\mathrm{D}^{AB}_{\mu} \,/\, \mathscr{R}\,| = \binom{\alpha}{\mu}, \qquad |\,\mathrm{D}^{AB}_{\mu} \,/\, \mathscr{L}\,| = \binom{\beta}{\mu},$$

$$|\,\mathrm{D}^{AB}_{\mu} \,/\, \mathscr{H}\,| = \binom{\alpha}{\mu}\binom{\beta}{\mu}, \qquad |\,\mathrm{D}^{AB}_{\mu}\,| = \mu!\binom{\alpha}{\mu}\binom{\beta}{\mu},$$

$$|\,\mathcal{I}_{AB}\,| = \sum_{\mu=0}^{\min(\alpha,\beta)} \mu!\binom{\alpha}{\mu}\binom{\beta}{\mu}.$$

For the first two values, note that an $\mathscr{R}$-class in $\mathcal{I}$ is determined by the domain of its elements, and an $\mathscr{L}$-class in $\mathcal{I}$ is determined by the image of its elements. Since a fixed μ-element domain and a fixed μ-element image may be "connected" in $\mu!$ ways, each $\mathscr{H}$-class of D^{AB}_{μ} contains $\mu!$ elements. Therefore, the last three equalities follow. However, unlike in the cases of $|\,\mathcal{PT}_{AB}\,|$ and $|\,\mathcal{T}_{AB}\,|$, there does not exist a simpler formula for $|\,\mathcal{I}_{AB}\,|$ (as far the author is aware).

Again, we recycle the ideas used in Section 3.1, and use the same arguments (see the proofs of Lemma 3.1.6 and Proposition 3.1.7) to prove that the semigroup $\mathbf{I}^{\mathrm{fr}}_X = \{f \in \mathbf{I}_X : \operatorname{Rank} f < \aleph_0\}$ is periodic for each $X \in \mathbf{Set}$, and to show that

Proposition 3.3.2. *If $(A, f, B) \in \mathcal{I}$, then*

(i) (A, f, B) *is $\mathscr{R}$-stable* $\Leftrightarrow$ $[\operatorname{Rank} f < \aleph_0$ *or f is full*$]$,

(ii) (A, f, B) *is $\mathscr{L}$-stable* $\Leftrightarrow$ $[\operatorname{Rank} f < \aleph_0$ *or f is surjective*$]$,

(iii) (A, f, B) *is stable* $\Leftrightarrow$ $[\operatorname{Rank} f < \aleph_0$ *or f is full and surjective*$]$.

For simplicity, for any $C, D \in \mathbf{Set}$, we identify the transformation $f \in \mathbf{I}_{CD}$ with the corresponding element $(C, f, D) \in \mathcal{I}$, and hence we use $\mathcal{I}_{CD}$ instead of $\mathbf{I}_{CD}$.

3.3.1 Green's relations, regularity and stability in $\mathcal{I}^a_{XY}$

In this subsection, we aim to investigate sandwich semigroups in $\mathcal{I}$, so we fix two sets $X, Y \in \mathbf{Set}$, and a map $a \in \mathcal{I}_{YX}$, and we focus on the sandwich semigroup $\mathcal{I}^a_{XY}$. In order to describe it, we need the following notation:

$$a = \binom{b_i}{a_i}_{i\in I}, \qquad B = \operatorname{dom} a, \qquad A = \operatorname{im} a, \qquad \alpha = \operatorname{Rank} a, \qquad \beta = |X \setminus \operatorname{im} a|.$$

We also fix $b = a^{-1} = \binom{a_i}{b_i}_{i\in I} \in \mathcal{I}_{XY}$, noting that $aba = a$ and $bab = b$.

Furthermore, due to unique invertibility of the elements of $\mathcal{I}$, we have $(fg)^{-1} = g^{-1}f^{-1}$ for any $f, g \in \mathcal{I}$, so the map $\mathcal{I}^a_{XY} \to \mathcal{I}^b_{YX} : f \mapsto f^{-1}$ is an anti-isomorphism. This information will be vital in the following two subsections.

Now, we may characterise the P-sets of $\mathcal{I}^a_{XY}$, using the definition of P-sets and Proposition 3.3.1:

(i) $\mathrm{P}_1^a = \{f \in \mathcal{I}_{XY} \ : \ \mathrm{dom}(fa) = \mathrm{dom}\, f\}$
$= \{f \in \mathcal{I}_{XY} \ : \ \mathrm{im}\, f \subseteq \mathrm{dom}\, a\},$

(ii) $\mathrm{P}_2^a = \{f \in \mathcal{I}_{XY} \ : \ \mathrm{im}(af) = \mathrm{im}\, f\}$
$= \{f \in \mathcal{I}_{XY} \ : \ \mathrm{dom}\, f \subseteq \mathrm{im}\, a\},$

(iii) $\mathrm{P}^a = \{f \in \mathcal{I}_{XY} \ : \ \mathrm{dom}(fa) = \mathrm{dom}\, f,\ \mathrm{im}(af) = \mathrm{im}\, f\}$
$= \{f \in \mathcal{I}_{XY} \ : \ \mathrm{im}\, f \subseteq \mathrm{dom}\, a,\ \mathrm{dom}\, f \subseteq \mathrm{im}\, a\},$

(iv) $\mathrm{P}_3^a = \{f \in \mathcal{I}_{XY} \ : \ \mathrm{Rank}(afa) = \mathrm{Rank}\, f\}.$

Thus, Theorem 2.2.3 yields a characterisation of Green's relations. Again, the result is virtually identical to Theorems 3.1.10 and 3.2.4, but we state it for the sake of completeness.

Theorem 3.3.3. *If $f \in \mathcal{I}_{XY}$, then in $\mathcal{I}_{XY}^a$ we have*

(i) $\mathrm{R}_f^a = \begin{cases} \mathrm{R}_f \cap \mathrm{P}_1^a, & f \in \mathrm{P}_1^a; \\ \{f\}, & f \notin \mathrm{P}_1^a. \end{cases}$

(ii) $\mathrm{L}_f^a = \begin{cases} \mathrm{L}_f \cap \mathrm{P}_2^a, & f \in \mathrm{P}_2^a; \\ \{f\}, & f \notin \mathrm{P}_2^a. \end{cases}$

(iii) $\mathrm{H}_f^a = \begin{cases} \mathrm{H}_f, & f \in \mathrm{P}^a; \\ \{f\}, & f \notin \mathrm{P}^a. \end{cases}$

(iv) $\mathrm{D}_f^a = \begin{cases} \mathrm{D}_f \cap \mathrm{P}^a, & f \in \mathrm{P}^a; \\ \mathrm{L}_f^a, & f \in \mathrm{P}_2^a \setminus \mathrm{P}_1^a; \\ \mathrm{R}_f^a, & f \in \mathrm{P}_1^a \setminus \mathrm{P}_2^a; \\ \{f\}, & f \notin (\mathrm{P}_1^a \cup \mathrm{P}_2^a). \end{cases}$

(v) $\mathrm{J}_f^a = \begin{cases} \mathrm{J}_f \cap \mathrm{P}_3^a \ (= \mathrm{D}_f \cap \mathrm{P}_3^a), & f \in \mathrm{P}_3^a; \\ \mathrm{D}_f^a, & f \notin \mathrm{P}_3^a. \end{cases}$

Further, if $f \notin \mathrm{P}^a$, then $\mathrm{H}_f^a = \{f\}$ is a non-group $\mathscr{H}^a$-class in $\mathcal{I}_{XY}^a$.

Repeating our steps from Subsection 3.1.1, we may prove parallels of Lemma 3.1.12, and Propositions 3.1.13 and 3.1.14 for $\mathcal{I}_{XY}^a$. The argument is literally unaltered, we just cut the parts concerning non-injective maps, and swap the references for the ones corresponding the results of this section. Continuing in the same manner, we may conclude that, for $f, g \in \mathcal{I}_{XY}$, we have $\mathrm{J}_f^a \leq \mathrm{J}_g^a$ in $\mathcal{I}_{XY}^a$ (where $\leq$ denotes the relation $\leq_{\mathscr{J}}$) if and only if one of the following holds:

(a) $f = g$,

(b) $\mathrm{Rank}\, f \leq \mathrm{Rank}(aga)$,

(c) $\mathrm{im}\, f \subseteq \mathrm{im}(ag)$,

(d) $\mathrm{dom}\, f \subseteq \mathrm{dom}(ga)$.

This follows from Lemma 2.2.6 and the fact that for any $f \in \mathcal{I}$ there exists a left- and right-identity (since $\mathcal{I}$ is monoidal). Furthermore, from Propositions 2.2.7 and 3.3.1, we immediately obtain

Proposition 3.3.4. *Let* $f, g \in \mathcal{I}_{XY}$.

(i) If $f \in \mathrm{P}_1^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\mathrm{Rank}\, f \leq \mathrm{Rank}(aga)$ *or* $\mathrm{dom}\, f \subseteq \mathrm{dom}(ga)]$.

(ii) If $f \in \mathrm{P}_2^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\mathrm{Rank}\, f \leq \mathrm{Rank}(aga)$ *or* $\mathrm{im}\, f \subseteq \mathrm{im}(ag)]$.

(iii) If $f \in \mathrm{P}_3^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow \mathrm{Rank}\, f \leq \mathrm{Rank}(aga)$.

(iv) If $g \in \mathrm{P}_1^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\mathrm{Rank}\, f \leq \mathrm{Rank}(ag)$ *or* $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g]$.

(v) If $g \in \mathrm{P}_2^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow [\mathrm{Rank}\, f \leq \mathrm{Rank}(ga)$ *or* $\mathrm{im}\, f \subseteq \mathrm{im}\, g]$.

(vi) If $g \in \mathrm{P}_3^a$, *then* $\mathrm{J}_f^a \leq \mathrm{J}_g^a \Leftrightarrow \mathrm{Rank}\, f \leq \mathrm{Rank}\, g$.

Of course, parts (iii) and (vi) apply to elements of P^a, as well, since $\mathrm{P}^a \subseteq \mathrm{P}_3^a$, by Proposition 2.2.2$(i)$.

As in Subsection 3.1.1, we may prove that the regular $\mathscr{D}^a$-classes of $\mathcal{I}_{XY}^a$ are precisely the sets

$$\mathrm{D}_\mu^a = \{f \in \mathrm{P}^a : \mathrm{Rank}\, f = \mu\}, \quad \text{for each cardinal } 0 \leq \mu \leq \alpha = \mathrm{Rank}\, a.$$

Moreover, if $f \in \mathrm{P}^a$, then $\mathrm{D}_f^a = \mathrm{J}_f^a$ if and only if $\mathrm{Rank}\, f < \aleph_0$ or a is stable. The proof is virtually identical to that of Proposition 3.1.18.

Finally, copying the proof of Proposition 3.1.19, we may show the following statements (where $\xi = \min(|X|, |Y|)$):

(i) If $\alpha < \xi$, then the maximal $\mathscr{J}^a$-classes of $\mathcal{I}_{XY}^a$ are precisely the singleton sets $\{f\}$, for $f \in \mathcal{I}_{XY}$ with $\mathrm{Rank}\, f > \alpha$. Hence, all the maximal $\mathscr{J}^a$-classes of $\mathcal{I}_{XY}^a$ are trivial in this case.

(ii) If $\alpha = \xi$, then we have a single maximum $\mathscr{J}^a$-class in $\mathcal{I}_{XY}^a$, which is

$$\mathrm{J}_b^a = \{f \in \mathrm{P}_3^a : \mathrm{Rank}\, f = \alpha\}.$$

This maximal $\mathscr{J}^a$-class is clearly nontrivial.

3.3.2 The regular subsemigroup $\mathrm{Reg}(\mathcal{I}_{XY}^a)$

In this subsection, we focus on the semigroup

$$\mathrm{Reg}(\mathcal{I}_{XY}^a) = \mathrm{P}^a = \{f \in \mathcal{I}_{XY} : \mathrm{dom}\, f \subseteq \mathrm{im}\, a,\ \mathrm{im}\, a \subseteq \mathrm{dom}\, a\}.$$

The plan is to recreate the Diagrams 2.2 and 2.3 for the semigroup $S_{ij}^a \equiv \mathcal{I}_{XY}^a$. From the discussion in Subsection 2.3.1 it follows that

– $\mathcal{I}_{XY}\, a = \mathcal{I}_X(ba) = \{f \in \mathcal{I}_X : \mathrm{im}\, f \subseteq A\}$,

– $a\,\mathcal{I}_{XY} = (ab)\,\mathcal{I}_Y = \{f \in \mathcal{I}_Y : \operatorname{dom} f \subseteq B\}$, and

– $(a\,\mathcal{I}_{XY}\,a, \circledast) \cong (ba\,\mathcal{I}_{YX}\,ba, \cdot)$.

The first set is a principal left ideal (hence, an underlying set of a subsemigroup) of $\mathcal{I}_X$, and the corresponding semigroup is usually denoted $\mathcal{I}(X, A)$. It has been studied in [44]. Symmetrically, the second set is a principal right ideal and an underlying set of a subsemigroup of $\mathcal{I}_Y$. The corresponding semigroup is clearly anti-isomorphic to $\mathcal{I}(Y, B)$ via the map $f \mapsto f^{-1}$. We will denote it $\mathcal{I}(Y, B)^*$. Finally, the semigroup in the third line is a subsemigroup of $\mathcal{I}^b_{YX}$, and it is isomorphic to the local submonoid of $\mathcal{I}_X$ with respect to the idempotent $ba = \binom{a_i}{a_i}_{i\in I} \in \mathcal{I}_X$. Thus, it easy to see that it is isomorphic to $\mathcal{I}_A$. Similarly as in Section 3.1, we will identify the two semigroups. Thus, we have obtained

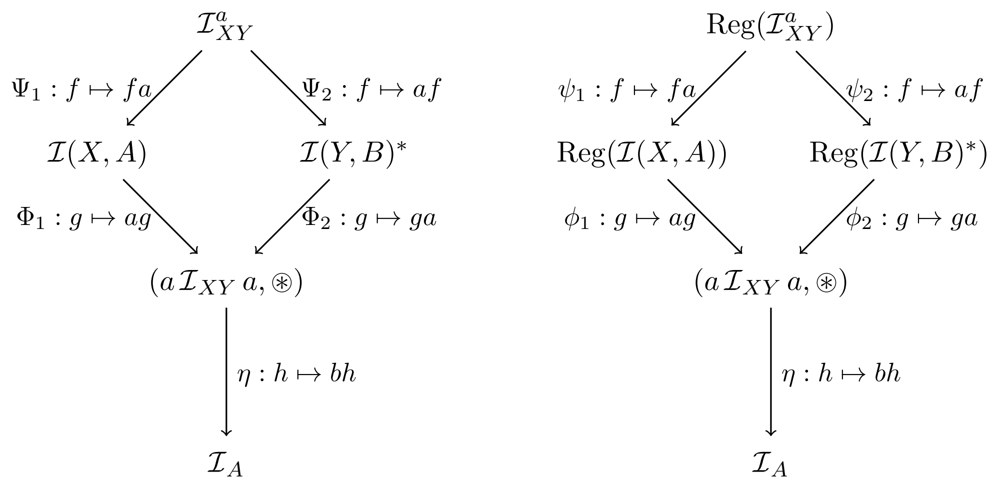


Figure 3.15: Diagrams illustrating the connections between $\mathcal{I}^a_{XY}$ and $(a\,\mathcal{I}_{XY}\,a, \circledast)$ (left) and between $\operatorname{Reg}(\mathcal{I}^a_{XY})$ and $(a\,\mathcal{I}_{XY}\,a, \circledast)$ (right).

This is where Proposition 2.5.2 comes in. Since the sandwich element a is uniquely sandwich-regular, it guarantees that all maps in the right-hand side diagram are isomorphisms and that all semigroups in it are inverse monoids. Thus,

$$\operatorname{Reg}(\mathcal{I}^a_{XY}) \cong \operatorname{Reg}(\mathcal{I}(X, A)) \cong \operatorname{Reg}(\mathcal{I}(Y, B)^*) \cong \mathcal{I}_A\,.$$

This means that the inflation described in Remark 2.3.13 is trivial. Moreover, in Theorem 3.1 of [44] was shown that $\operatorname{Reg}(\mathcal{I}(X, A)) = \mathcal{I}_A$, which also implies $\operatorname{Reg}(\mathcal{I}(Y, B)^*) = \mathcal{I}_B \cong \mathcal{I}_A$.

Since the results concerning $\mathcal{I}_A$ are well-known (see [45, 77, 81, 105]), we state them without proof: in $\mathcal{I}_A$, for a map h with $\operatorname{Rank} h = \mu$, we have

(i) $\mathrm{R}_h = \{g \in \mathcal{I}_A : \operatorname{dom} g = \operatorname{dom} h\}$.

(ii) $\mathrm{L}_h = \{g \in \mathcal{I}_A : \operatorname{im} g = \operatorname{im} h\}$.

(iii) $\mathrm{H}_h = \{g \in \mathcal{I}_A : \operatorname{dom} g = \operatorname{dom} h,\ \operatorname{im} g = \operatorname{im} h\}$.

(iv) $|\,\mathrm{H}_h\,| = \mu!$; furthermore, if H_h contains an idempotent, then $\mathrm{H}_h \cong \mathrm{S}_\mu$.

(v) $\mathrm{D}_h = \mathrm{J}_h = \{g \in \mathcal{I}_A : \operatorname{Rank} g = \operatorname{Rank} h = \mu\} = \mathrm{D}_\mu$.

(vi) If $\alpha = |A|$ is finite, then $\mathrm{D}_\alpha = \mathrm{H}_{\mathrm{id}_A} \cong \mathrm{S}_A$ and $\mathcal{I}_A \setminus \mathrm{D}_\alpha$ is an ideal of the semigroup $\mathcal{I}_A$.

(vii) $|\,\mathrm{E}(\mathcal{I}_A)| = 2^{|A|}$.

(viii) We have $\mathbb{E}(\mathcal{I}_A) = \mathrm{E}(\mathcal{I}_A)$. If $\alpha = |A| < \aleph_0$, then

$$\operatorname{rank}(\mathbb{E}(\mathcal{I}_A)) = \operatorname{idrank}(\mathbb{E}(\mathcal{I}_A)) = |\,\mathrm{E}(\mathrm{D}_{\alpha-1})| + 1 = \alpha + 1.$$

If $\alpha \geq \aleph_0$, then $\operatorname{rank}(\mathbb{E}(\mathcal{I}_A)) = \operatorname{idrank}(\mathbb{E}(\mathcal{I}_A)) = |\,\mathrm{E}(\mathcal{I}_A)| = 2^{|A|}$.

(ix) If $\alpha = |A| \geq 1$, then

$$\operatorname{rank}(\mathcal{I}_A) = \operatorname{rank}(\mathrm{S}_A) + 1 = \begin{cases} 2, & \alpha = 1, 2; \\ 3, & 3 \leq \alpha < \aleph_0; \\ 2^\alpha, & \alpha \geq \aleph_0. \end{cases} \tag{3.14}$$

If $|A| = 0$, then $\operatorname{rank}(\mathcal{I}_A) = |\,\mathcal{I}_A\,| = 1$.

Of course, these results apply to $\operatorname{Reg}(\mathcal{I}^a_{XY})$, as well. From Remark 2.5.3 we may infer some additional information. Namely, the map $\psi = (\psi_1, \psi_2) : \mathrm{P}^a \to \operatorname{Reg}(\mathcal{I}(X, A)) \times \operatorname{Reg}(\mathcal{I}(Y, B)^*) : f \mapsto (fa, af)$ is trivially an embedding, with

$$\operatorname{im}(\psi) = \{(f, g) \in T_1 \times T_2 : af = ga\} = \{(f, g) \in T_1 \times T_2 : f\phi_1 = g\phi_2\}.$$

Furthermore, P^a is always MI-dominated since it is a monoid.

Now, we fill in the gaps. Firstly, since

$$|\,\mathrm{P}^a\,| = |\,\mathcal{I}_A\,| = |\,\mathcal{I}_{AA}\,| = \sum_{\mu=0}^{\alpha} \binom{\alpha}{\mu}\binom{\alpha}{\mu}\mu! = \sum_{\mu=0}^{\alpha} \binom{\alpha}{\mu}^2 \mu!\ ,$$

we may conclude that $|\,\mathrm{P}^a\,| > \aleph_0$ if and only if $\alpha \geq \aleph_0$. In the other case, when $\alpha < \aleph_0$, $|\,\mathrm{P}^a\,|$ is obviously finite. Therefore, $|\,\mathrm{P}^a\,|$ cannot be countable.

Secondly, since P^a is MI-dominated and $\mathrm{MI}(\mathrm{P}^a) = V(a) = \{b\}$, Proposition 2.4.8 implies that P^a is RP-dominated if and only if the local monoid $b \star_a \mathrm{P}^a \star_a b$ is factorisable. Proposition 2.4.11 gives $ba\,\mathrm{P}^a\,ab \cong W$, so $ba\,\mathrm{P}^a\,ab \cong \mathcal{I}_A$. Therefore, Lemma 3.1.32, implies that P^a is RP-dominated if and only if A is finite.

We also give some more information on the idempotents of P^a. From Proposition 3.1.36(i) immediately follows that $f \in \mathrm{P}^a$ is an idempotent if and only if $(af)\restriction_{\operatorname{im} f} = \mathrm{id}_{\operatorname{im} f}$. Multiplying this equality by f^{-1} on the right (a revertible operation), we obtain $(aff^{-1})\restriction_{\operatorname{im} f} = f^{-1}$, i.e. $a\restriction_{\operatorname{im} f} = f^{-1}$. In other words, $f \in \mathrm{P}^a$ is an idempotent if and only if $f = a^{-1}\restriction_{\operatorname{dom} f}$. Given the explicit form of $a = \binom{b_i}{a_i}_{i\in I}$, we may conclude that idempotents of P^a have the form $\binom{a_j}{b_j}_{j\in J}$ for each $J \subseteq I$.

Finally, we examine the cases when a is either full, or surjective, or both, in order to get the full picture.

- If a is full, then $Y = \operatorname{dom} a$, $\mathcal{I}(Y,B)^* \equiv \mathcal{I}_Y$ and Ψ_1 is an isomorphism (because $ab = \mathrm{id}_Y$, so we have the implication 2.5). Thus, $\mathcal{I}^a_{XY} \cong \mathcal{I}(X,A)$, so the results for $\mathcal{I}^a_{XY}$ and $\operatorname{Reg}(\mathcal{I}^a_{XY})$ apply to $\mathcal{I}(X,A)$ and $\operatorname{Reg}(\mathcal{I}(X,A))$, respectively.
- Symmetrically, if a is surjective, then $X = \operatorname{im} a$, $\mathcal{I}(X,A) \equiv \mathcal{I}_X$, and Ψ_2 is an isomorphism (because $ba = \mathrm{id}_X$, so we have the implication 2.6). Thus, $\mathcal{I}^a_{XY} \cong \mathcal{I}(Y,B)^*$ in this case.
- Naturally, if a is both full and surjective, we have

$$\mathcal{I}^a_{XY} \cong \mathcal{I}(X,A) = \mathcal{I}_A\,.$$

3.3.3 The rank of a sandwich semigroup $\mathcal{I}^a_{XY}$

Due to the simplifications occurring in the sandwich semigroup of injective (partial) transformations, we were able to fast-forward to the problem of calculating its rank in just a few pages. Here, we follow the same "recipe" as in Sections 3.1 and 3.2. First, we discuss the simple cases. Before that, we need to make a few remarks. Write $\xi = \min(|X|,|Y|)$ and recall that

$$|\,\mathcal{I}_{XY}\,| = \sum_{\mu=0}^{\xi} \mu!\binom{|X|}{\mu}\binom{|Y|}{\mu}.$$

Since $\mathcal{I}^a_{XY}$ is anti-isomorphic to $\mathcal{I}^b_{YX}$, we may suppose without loss of generality that $|Y| \leq |X|$.

- **Suppose $|\mathbf{Y}| \geq \aleph_0$.** Then $\xi \geq \aleph_0$, so $|\,\mathcal{I}_{XY}\,| \geq \binom{\aleph_0}{\aleph_0} = 2^{\aleph_0}$. Thus, $\operatorname{rank}(\mathcal{I}^a_{XY}) = |\,\mathcal{I}_{XY}\,|$.
- **Suppose $|\mathbf{Y}| = \mathbf{0}$.** Then $\xi = 0$, so $\operatorname{rank}(\mathcal{I}^a_{XY}) = |\,\mathcal{I}_{XY}\,| = 1$.
- **Suppose $|\mathbf{X}| > \aleph_0$ and $|\mathbf{Y}| \neq \mathbf{0}$.** In this case, $|\,\mathcal{I}_{XY}\,| \geq \binom{|X|}{1} > \aleph_0$, so again $\operatorname{rank}(\mathcal{I}^a_{XY}) = |\,\mathcal{I}_{XY}\,|$.
- **Suppose that $|\mathbf{X}| \leq \aleph_0$, $\mathbf{0} < |\mathbf{Y}| < \aleph_0$ and that a is a full bijection.** Thus, we have $\alpha = |Y| = |X| < \aleph_0$, and $\operatorname{Reg}(\mathcal{I}^a_{XY}) \cong \mathcal{I}_A$. Therefore, in this case, the formula for $\operatorname{rank}(\mathcal{I}^a_{XY})$ is given in (3.14).

Therefore, for the remainder of this subsection, we assume that $|\mathbf{X}| \leq \aleph_0$, $\mathbf{0} < |\mathbf{Y}| < \aleph_0$, and that a is not a bijection. Here, too, we have $\alpha \leq \xi < \aleph_0$, so a is stable, by Proposition 3.3.2(*iii*). Hence, Lemma 2.2.19 implies $\mathscr{J}^a = \mathscr{D}^a$.

As in Subsection 3.2.5, we will list the results that can be proved by copying the proofs of the corresponding statements for $\mathcal{PT}$. Of course, we disregard the non-injective cases in that process. Additionally, we reference the results concerning $\mathcal{I}^a_{XY}$, instead of those concerning $\mathcal{PT}^a_{XY}$.

Recall that we have assumed $|Y| \leq |X|$ without loss of generality, so a is non-surjective. Then

Lemma 3.3.5. *In $\mathcal{I}^a_{XY}$ holds* $\mathrm{D}_0 \cup \mathrm{D}_1 \cup \cdots \cup \mathrm{D}_\alpha = \langle \mathrm{D}_\alpha \rangle_a$.

Lemma 3.3.6. *Suppose $\alpha < |Y| < \aleph_0$ and let $f \in \mathrm{D}_\alpha$. Then $f \in \mathrm{D}_{\alpha+1} \star_a \mathrm{D}_{\alpha+1}$.*

Lemma 3.3.7. *If $\alpha = |Y| < \aleph_0$, then $\mathrm{P}^a_1 = \mathcal{I}^a_{XY}$, $\mathrm{P}^a_2 = \mathrm{P}^a$, and $\mathscr{R}^a = \mathscr{R}$ on $\mathcal{I}^a_{XY}$.*

Of course, one could prove the corresponding, dual statements if the assumption was $|X| \leq |Y|$.

Suppose $\alpha < |Y| < \aleph_0$. Recall that any generating set of $\mathcal{I}^a_{XY}$ has to include elements from each maximal $\mathscr{J}^a$-class (see Section 2.6), and that the maximal $\mathscr{J}^a$-classes are exactly the singletons $\{f\}$, such that $\operatorname{Rank} f > \aleph_0$. Thus, from Lemmas 3.3.5 and 3.3.6 follows

Theorem 3.3.8. *Suppose $|X| \leq \aleph_0$, $|Y| < \aleph_0$, and that $1 \leq \alpha < |Y| \leq |X|$ (hence, a is non-surjective and non-full). We have*

$$\operatorname{rank}(\mathcal{I}^a_{XY}) = \sum_{\mu=\alpha+1}^{|Y|} |\,\mathrm{D}_\mu\,| = \sum_{\mu=\alpha+1}^{|Y|} \mu! \binom{|Y|}{\mu} \binom{|X|}{\mu}.$$

On the other hand, if $\alpha = |Y| < \aleph_0$, then $|Y| \neq |X|$ since a cannot be both full and surjective. Now, we have

Theorem 3.3.9. *Suppose that $1 \leq \alpha = \operatorname{Rank} a = |Y| < |X| < \aleph_0$ (hence, a is full and non-surjective). Then*

$$\operatorname{rank}(\mathcal{I}^a_{XY}) = \binom{|X|}{|Y|} + \begin{cases} 0, & \text{if } \alpha \leq 2; \\ 1, & \text{if } \alpha \geq 3; \end{cases}$$

Proof. We modify the proof of Theorem 3.1.51. Firstly, from Lemmas 3.3.5 and 3.3.6 we conclude that $\langle \mathrm{J}_b \rangle_a = \langle \mathrm{D}_\alpha \rangle_a = \mathcal{I}_{XY}$. Furthermore, by the argument from the proof of Theorem 3.1.51 we conclude that Proposition 2.6.3 applies and that $\mathrm{J}^a_b = \mathrm{D}^a_b = \widehat{\mathrm{H}^a_b} = \mathrm{H}^a_b \cong \mathrm{S}_\alpha$. Now, since

$$|\,\mathrm{J}^a_b \,/\, \mathscr{H}^a\,| = 1, \quad \text{and since} \quad \operatorname{rank}(\mathrm{S}_\alpha) = \begin{cases} 1, & \alpha = 1, 2; \\ 2, & \alpha \geq 3; \end{cases}$$

Proposition 2.6.3(ii) implies

$$\begin{aligned} \operatorname{rank}(\mathcal{I}_{XY}) &= |X_2| + \max(|X_1|, \operatorname{rank}(\mathrm{H}^a_b)) \\ &= |X_2| + \max(1, \operatorname{rank}(\mathrm{H}^a_b)) \\ &= |X_2| + |X_1| + \operatorname{rank}(\mathrm{H}^a_b) - 1. \end{aligned}$$

Thus, the statement follows from $|X_2| + |X_1| = |\,\mathrm{J}_b \,/\, \mathrm{H}\,| = |\,\mathrm{D}_\alpha \,/\, \mathscr{H}\,|$, and from the fact that D_α is an $\mathscr{L}$-class (by Proposition 2.2.37), so $|\,\mathrm{D}_\alpha \,/\, \mathscr{H}\,| = |\,\mathrm{D}_\alpha \,/\, \mathscr{R}\,| = \binom{|X|}{\alpha}$ (the size of $\mathrm{D}_\alpha \,/\, \mathscr{R}$ is calculated at the beginning of this section). □

In the proof, we use the assumption that a is non-surjective, so $\beta = |X \setminus \operatorname{im} a|$. Thus, if $\alpha \geq 1$ and that $|\,\mathrm{D}_\alpha \,/\, \mathscr{R}\,| = \binom{|X|}{\alpha}$

Remark 3.3.10. For the previous result, we provide an alternative formulation concerning the non-sandwich semigroup $\mathcal{I}(X, A)$. If a is full and non-surjective, then $\mathcal{I}^a_{XY} \cong \mathcal{I}(X, A)$, so for any proper subset A of X, we have

$$\operatorname{rank}(\mathcal{I}(X, A)) = \binom{|X|}{|A|} + \begin{cases} 0, & \text{if } |A| \leq 2; \\ 1, & \text{if } |A| \geq 3; \end{cases}$$

This is a result from [44].

3.3.4 Egg-box diagrams

As in the previous two sections, we provide several egg-box diagrams (they originally appeared in [34], and all were generated by GAP [98]) to illustrate the structural results for $\mathcal{I}_{XY}^a$. For more information on the figures, see the introduction to Subsection 3.1.6.

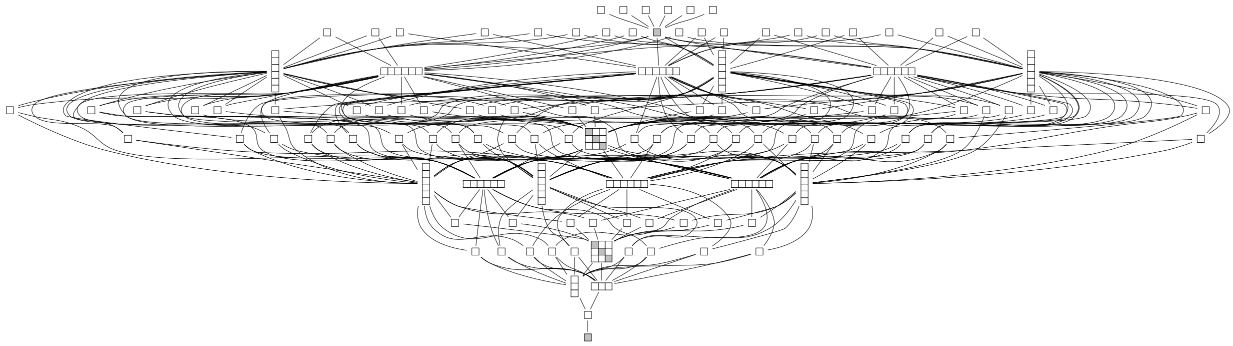


Figure 3.16: Egg-box diagram of the sandwich semigroup $\mathcal{I}_{44}^a$, where $a = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & - \end{smallmatrix}\right) \in \mathcal{PT}_{44}$. Note that a is non-full and non-surjective.

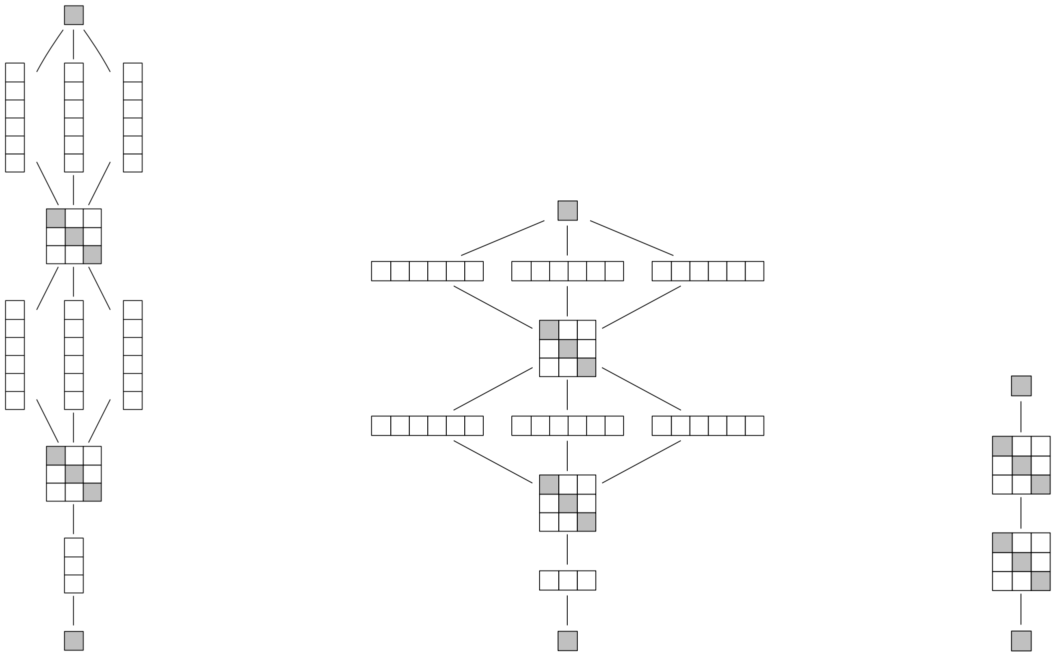


Figure 3.17: Left to right: egg-box diagrams of the sandwich semigroups $\mathcal{I}_{43}^b$ and $\mathcal{I}_{34}^c$ and $\mathcal{I}_3$, where $b = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right) \in \mathcal{I}_{34}$ and $c = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & - \end{smallmatrix}\right) \in \mathcal{I}_{43}$. Note that b is full and non-surjective, while c is surjective and non-full. By the theory presented in this section, the regular subsemigroups of the first two semigroups are isomorphic to the third semigroup, $\mathcal{I}_3$.

Chapter 4

Sandwich semigroups of matrices

In this chapter, we investigate the partial semigroup $\mathcal{M}(\mathbb{F})$ (as defined in Example 2.1.5) and the sandwich semigroups it contains. The reader will find that the results resemble the ones for the sandwich semigroups of transformations. However, there are significant differences and peculiarities which merit a detailed examination.

This investigation was originally conducted in [30]. However, the general results used there constitute only a portion of the results presented in Chapter 2 (since [30] preceded the article [34]). Thus, we refer to [30] as the source (unless stated otherwise), but we frequently take a different approach in proving the results.

Following the layout of Section 3.1, first we study the partial semigroup $\mathcal{M}(\mathbb{F})$ in terms of Green's relations, cardinalities within a hom-set, regularity and invertibility of its elements. Then, we focus on the linear sandwich semigroup $\mathcal{M}_{mn}^A(\mathbb{F})$: we describe its Green's classes and the relation $\leq_{\mathscr{J}^A}$, the links depicted in the commutative diagrams 2.2 and 2.3, as well as the connection of the semigroups $\operatorname{Reg}(\mathcal{M}_{mn}^A(\mathbb{F}))$ and $\mathcal{M}_{\operatorname{Rank} A}$. Furthermore, we examine the semigroup $\mathrm{P}^A = \operatorname{Reg}(\mathcal{M}_{mn}^A(\mathbb{F}))$ in detail, characterising its Green's relations and idempotents, proving MI-domination, describing the combinatorial structure and calculating its rank. Then, we classify the isomorphism classes of finite linear sandwich semigroups, enumerate idempotents and describe the idempotent-generated subsemigroup by characterising its elements and calculating its rank and idempotent rank. Finally, we calculate the rank of the linear sandwich semigroup $\mathcal{M}_{mn}^A(\mathbb{F})$ and present a number of egg-box diagrams, giving a visual presentation of the structural results presented in the chapter.

4.1 The category $\mathcal{M}(\mathbb{F})$

We need to introduce some notions and notation specific to the topic of this chapter. For $m, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and a field $\mathbb{F}$, let $\mathcal{M}_{mn}(\mathbb{F})$ denote the set of all $m \times n$ matrices over the field $\mathbb{F}$. Since there are q^{mn} ways to fill a $m \times n$ rectangular scheme with elements of a q-element set (field), we have $|\mathcal{M}_{mn}(\mathbb{F})| = |\mathbb{F}|^{mn}$. If $m = n$, then we write $\mathcal{M}_{mm}(\mathbb{F}) = \mathcal{M}_m(\mathbb{F})$. For convenience, we consider there to be a unique $m \times 0$ and $0 \times n$ matrix for any $m, n \geq 0$, which will be considered an empty matrix, and denoted $\emptyset$. Thus, we have $\mathcal{M}_{mn}(\mathbb{F}) = \{\emptyset\}$ if and only if $m = 0$ or $n = 0$. Furthermore, let

$$\mathcal{M}(\mathbb{F}) = \bigcup_{m,n \in \mathbb{N}} \mathcal{M}_{mn}(\mathbb{F})$$

be the set of all finite-dimensional matrices over $\mathbb{F}$. Note that, we do not consider $\emptyset$ to be an element of $\mathcal{M}(\mathbb{F})$. If the field is either known or not essential for our discussion, we shorten the notation to $\mathcal{M}_{mn}$, $\mathcal{M}_m$ and $\mathcal{M}$, respectively.

Recall that there exists an alternative way to think about matrices. Namely, the category of all finite-dimensional matrices over $\mathbb{F}$, $\mathcal{M}(\mathbb{F})$, is equivalent (but not isomorphic, see Section IV.4 in [83]) to the category of all finite-dimensional vector spaces over $\mathbb{F}$. To describe this connection, we need some more notation. Firstly, for vector spaces V and W, let $\mathrm{Hom}(V, W)$ denote the (hom-)set of all linear transformations from V to W. If $V = W$, the set $\mathrm{Hom}(V, W)$ is denoted $\mathrm{End}\, V$ and is an underlying set of the monoid of all endomorphisms of V. Secondly, for $m \geq 1$ and a fixed field $\mathbb{F}$, write $V_m = \mathbb{F}^m$ for the vector space of all $1 \times m$ row vectors over the field $\mathbb{F}$. Then, we identify $\mathcal{M}_{mn}$ with $\mathrm{Hom}(V_m, V_n)$ in the following manner: for a matrix $X \in \mathcal{M}_{mn}$, we define $\lambda_X : V_m \to V_n$ by $(v)\lambda_X = vX$ for all $v \in V_m$. If $m = n$, the map $X \to \lambda_X$ determines an isomorphism of monoids $\mathcal{M}_m \to \mathrm{End}\, V_m$. Thus, we may prove statements about $\mathcal{M}_{mn}$ by proving the equivalent statement about $\mathrm{Hom}(V_m, V_n)$, and vice versa. Hence, it will be beneficial to consider vector spaces and their properties.

Fix a field $\mathbb{F}$, and let 1 and 0 denote its identity and zero element, respectively. Let $\boldsymbol{\delta} : \mathcal{M} \to \mathbb{N}$ and $\boldsymbol{\rho} : \mathcal{M} \to \mathbb{N}$ be the maps giving the number of rows and the number of columns of a matrix, respectively. Clearly, the 5-tuple $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho})$, where $\cdot$ denotes the usual matrix multiplication, is a partial semigroup. Again, we abbreviate the notation for this partial semigroup to $\mathcal{M}$. Since for any $m \in \mathbb{N}$, the set $\mathcal{M}$ clearly contains the $m \times m$ identity matrix I_m (with 1's on the leading diagonal and 0's elsewhere), $\mathcal{M}$ is a monoidal partial semigroup. Furthermore, the i-th row of I_m is denoted e_{mi}, and the set $\{e_{m1}, \ldots, e_{mm}\}$ is the *standard basis* of V_m, in the sense that any element of V_m may be uniquely generated as a linear combination of the vectors from the basis. Let

$$W_{ms} = \mathrm{span}\{e_{m1}, \ldots, e_{ms}\}, \text{ for each } 1 \leq s \leq m$$

denote the (vector) subspace of V_m consisting of all linear combinations of vectors $e_{m1}, \ldots, e_{ms}$. Naturally, we have $\mathrm{span}\, \emptyset = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector.

Recall that $\mathcal{M}_m$ is the underlying set of a monoid with respect to the matrix multiplication, namely the *full linear monoid* of degree m (for some background, see [103]). In addition, the subset $\mathcal{G}_m = \mathcal{G}_m(\mathbb{F})$ of $\mathcal{M}_m$ consisting of invertible matrices is an underlying set of a group, namely the *general linear group* of degree m. It corresponds to the group of all automorphisms of V_m, denoted by $\mathrm{Aut}\, V_m$. Indeed, the restriction of the above map $X \to \lambda_X$ determines an isomorphism of groups $\mathcal{G}_m \to \operatorname{Aut} V_m$.

In order to characterise Green's preorders and relations, we need the 'vocabulary' to describe the properties of a single matrix X (linear transformation λ_X). For $X \in \mathcal{M}_{mn}$, and any $1 \le i \le m$ and $1 \le j \le n$, let $\mathfrak{r}_i(X)$ and $\mathfrak{c}_j(X)$ denote the ith row and jth column of X, respectively. These two intersect in coordinates (i, j), the corresponding element of X being denoted X_{ij}. Furthermore, let

$$\operatorname{Row} X = \operatorname{span}\{\mathfrak{r}_1(X), \dots, \mathfrak{r}_m(X)\}, \qquad \operatorname{Col} X = \operatorname{span}\{\mathfrak{c}_1(X), \dots, \mathfrak{c}_n(X)\},$$
$$\operatorname{Rank} X = \dim(\operatorname{Row} X) = \dim(\operatorname{Col} X)$$

denote the *row space*, *column space* and the *rank* of X, respectively. The notion of rank of a matrix corresponds (in a way) to the notion of rank of a map, since $\operatorname{Rank} X = \dim(\{(v)\lambda_X : v \in V_m\}) = \dim(\operatorname{im} \lambda_X)$ (thus, the rank of a product is not greater than the rank of any factor). In other words, $\operatorname{Rank} X$ measures the dimension (size) of a subspace of V_m defined by λ_X. For this reason, in this chapter, we use the notation $\operatorname{Rank} \lambda_X$ to mean the (space) dimension of $\operatorname{im} \lambda_X$, rather than its cardinality. Furthermore, we will modify the notion for the kernel of a linear transformation. Namely, for vector spaces V and W and $\alpha \in \operatorname{Hom}(V, W)$, we define

$$\operatorname{Ker} \alpha = \{v \in V : (v)\alpha = 0\}.$$

Thus, in this chapter, the kernel of a transformation is a subset of the domain, rather than a relation. However, $\operatorname{Ker} \alpha$ corresponds to $\ker \alpha$ in the following way: for $x, y \in V$, we have

$$x\alpha = y\alpha \quad \Leftrightarrow \quad (x, y) \in \ker \alpha \quad \Leftrightarrow \quad y \in x + \operatorname{Ker} \alpha.$$

Having introduced the necessary notions, we continue to investigate the partial semigroup $\mathcal{M}$. Recall that it has a dual partial semigroup $(\mathcal{M}, \bullet, \mathbb{N}, \boldsymbol{\rho}, \boldsymbol{\delta})$, where $A \bullet B = B \cdot A$. Interestingly, the operation of *transposition* $\mathcal{M} \to \mathcal{M} : A \to A^{\mathrm{T}}$ (turning rows into columns and vice versa) is a bijection, and we have $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$ for all $A, B \in \mathcal{M}$. Thus, $\mathcal{M}$ is in anti-isomorphic to its own dual (i.e. it is *self-dual*). This means that any result we prove about column spaces has a corresponding dual result about row spaces and vice versa. For this reason, it will occasionally be convenient to think of $\operatorname{Row} X$ and $\operatorname{Col} X$ as subspaces of $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively.

Our next goal is to describe Green's preorders and relations in $\mathcal{M}$. Recall that $\mathcal{M}$ being monoidal means that $\mathcal{M}^{(1)} = \mathcal{M}$, so the definitions of relations $\le_{\mathscr{R}}$, $\le_{\mathscr{L}}$ and $\le_{\mathscr{J}}$ simplify slightly. For example, $X \le_{\mathscr{R}} Y$ means that there exists $A \in \mathcal{M}$ such that $X = YA$. Therefore, as in [30], we may prove

Proposition 4.1.1. *Let $X, Y \in \mathcal{M}$. Then*

(i) $X \le_{\mathscr{R}} Y \Leftrightarrow \operatorname{Col} X \subseteq \operatorname{Col} Y$,

(ii) $X \le_{\mathscr{L}} Y \Leftrightarrow \operatorname{Row} X \subseteq \operatorname{Row} Y$,

(iii) $X \le_{\mathscr{J}} Y \Leftrightarrow \operatorname{Rank} X \le \operatorname{Rank} Y$,

(iv) $X \mathscr{R} Y \Leftrightarrow \operatorname{Col} X = \operatorname{Col} Y$,

(v) $X \mathscr{L} Y \Leftrightarrow \text{Row}\, X = \text{Row}\, Y$, *(vi)* $X \mathscr{J} Y \Leftrightarrow \text{Rank}\, X = \text{Rank}\, Y$.

Further, $\mathcal{M}$ is stable, so $\mathscr{J} = \mathscr{D}$.

Proof. Note that (*iv*), (*v*) and (*vi*) follow immediately from (*i*), (*ii*) and (*iii*), respectively. In addition, part (*ii*) follows from (*i*) by duality, so it suffices to prove only (*i*) and (*iii*).

(*i*) We have

$$\begin{aligned} X \leq_{\mathscr{R}} Y &\Leftrightarrow X = YA, \text{ for some } A \in \mathcal{M} \\ &\Leftrightarrow \mathfrak{c}_i(X) = \sum_{j=1}^{Y\boldsymbol{\rho}} \mathfrak{c}_j(Y) A_{ji} \text{ for each } 1 \leq i \leq X\boldsymbol{\rho}, (\text{for some } A_{ji} \in \mathbb{F}) \\ &\Leftrightarrow \text{Col}\, X \subseteq \text{Col}\, Y. \end{aligned}$$

(*iii*) Suppose that $X \leq_{\mathscr{J}} Y$. This means that $X = AYB$ for some $A, B \in \mathcal{M}$, so $\text{Rank}\, X \leq \text{Rank}\, Y$. Conversely, suppose that $\text{Rank}\, X \leq \text{Rank}\, Y$ for $X \in \mathcal{M}_{mn}$ and $Y \in \mathcal{M}_{kl}$. We need to show that $X = AYB$ for some $A \in \mathcal{M}_{mk}$ and $B \in \mathcal{M}_{ln}$. We show an equivalent statement instead: $\lambda_X = \alpha \circ \lambda_Y \circ \beta$, for some $\alpha \in \text{Hom}(V_m, V_k)$ and $\beta \in \text{Hom}(V_l, V_n)$. We will define the maps α and β by fixing their actions on the bases of V_m and V_l, respectively. Write $r = \text{Rank}\, X$ and $s = \text{Rank}\, Y$. In addition, let $\mathcal{B}_m = \{u_1, \ldots, u_m\}$ and $\mathcal{B}_k = \{v_1, \ldots, v_k\}$ be bases for V_m and V_k such that $\{u_{r+1}, \ldots, u_m\}$ and $\{v_{s+1}, \ldots, v_k\}$ are bases for $\text{Ker}\, \lambda_X$ and $\text{Ker}\, \lambda_Y$, respectively. We extend the linearly independent sets $\{u_1\lambda_X, \ldots, u_r\lambda_X\}$ and $\{v_1\lambda_Y, \ldots, v_s\lambda_Y\}$ arbitrarily to bases

$$\mathcal{B}_n = \{u_1\lambda_X, \ldots, u_r\lambda_X, w_{r+1}, \ldots, w_n\} \text{ and}$$
$$\mathcal{B}_l = \{v_1\lambda_Y, \ldots, v_s\lambda_Y, x_{s+1}, \ldots, x_l\}$$

for V_n and V_l, respectively. Since $r \leq s$, we may choose $\alpha \in \text{Hom}(V_m, V_k)$ and $\beta \in \text{Hom}(V_l, V_n)$ so that

$$u_i\alpha = v_i, \qquad u_j\alpha \in \text{span}\{v_{s+1}, \ldots, v_k\}$$

for all $1 \leq i \leq r$ and $r+1 \leq j \leq m$, and

$$(v_i\lambda_Y)\beta = u_i\lambda_X, \qquad (v_t\lambda_Y)\beta,\ x_j\beta \in \text{span}\{w_{r+1}, \ldots, w_m\}$$

for all $1 \leq i \leq r$, $r+1 \leq t \leq s$, and $s+1 \leq j \leq l$. It is easily seen that $\alpha \circ \lambda_Y \circ \beta = \lambda_X$ (by examining the actions of both sides on $\mathcal{B}_m$).

Recall from Lemma 2.2.19 that stability indeed implies $\mathscr{J} = \mathscr{D}$. To prove the last statement of the proposition, we need to show that $X \mathscr{J} XY \Rightarrow X \mathscr{R} XY$ and $X \mathscr{J} YX \Rightarrow X \mathscr{L} YX$. We show the first implication, and the second follows by duality. Suppose $X \mathscr{J} XY$. From (*vi*) we have that $\text{Rank}\, X = \text{Rank}\, XY$, i.e. $\dim(\text{Col}\, X) = \dim(\text{Col}(XY))$. Since $\text{Col}\, X \subseteq \text{Col}(XY)$ (by (*i*)), we have $\text{Col}\, X = \text{Col}(XY)$, so $X \mathscr{R} XY$. □

For any $0 \le s < \aleph_0$, let D_s denote the $\mathscr{J} = \mathscr{D}$-class of $\mathcal{M}$ containing all partial maps of rank s.

Let $m, n \in \mathbb{N}$. Green's relations of $\mathcal{M}$ define partitions of the set $\mathcal{M}_{mn}$, which determine Green's relations of $\mathcal{M}_{mn}$. For $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$ and any $X \in \mathcal{M}$, let $\mathrm{K}_X = \{Y \in \mathcal{M}_{mn} : X \mathscr{K} Y\}$ denote the $\mathscr{K}$-class of X in $\mathcal{M}_{mn}$, with an inherited partial order $\mathrm{K}_X \le_{\mathscr{K}} \mathrm{K}_Y \Leftrightarrow X \le_{\mathscr{K}} Y$ for $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{D} = \mathscr{J}\}$. From the previous proposition we may conclude that the $\mathscr{J} = \mathscr{D}$-classes of $\mathcal{M}_{mn}$ are the sets

$$\mathrm{D}_s^{mn} = \{X \in \mathcal{M}_{mn} : \mathrm{Rank}\, X = s\},$$

for each $0 \le s \le \min(|A|, |B|)$. These $\mathscr{J}$-classes form a chain in $\mathcal{M}_{mn}$: $\mathrm{D}_s^{mn} \le_{\mathscr{J}} \mathrm{D}_t^{mn} \Leftrightarrow s \le t$.

Furthermore, we may give an alternative description of Green's classes of a specified element of $\mathcal{M}_{mn}$, a result from [30].

Lemma 4.1.2. *Let $X \in \mathcal{M}_{mn}$. Then*

(i) $\mathrm{R}_X = \{Y \in \mathcal{M}_{mn} : \mathrm{Col}\, X = \mathrm{Col}\, Y\} = X\,\mathcal{G}_n$,

(ii) $\mathrm{L}_X = \{Y \in \mathcal{M}_{mn} : \mathrm{Row}\, X = \mathrm{Row}\, Y\} = \mathcal{G}_m\, X$,

(iii) $\mathrm{D}_X = \mathrm{J}_X = \{Y \in \mathcal{M}_{mn} : \mathrm{Rank}\, X = \mathrm{Rank}\, Y\} = \mathcal{G}_m\, X\,\mathcal{G}_n$,

Proof. It suffices to prove only (i), since (ii) is dual, and (iii) follows directly from the previous two because

$$\mathrm{J}_X = \mathrm{D}_X = \bigcup_{Y \in \mathrm{L}_X} \mathrm{R}_Y = \bigcup_{Y \in\ \mathrm{L}_X} Y\,\mathcal{G}_n = \bigcup_{Y \in\ \mathcal{G}_m\, X} Y\,\mathcal{G}_n = \mathcal{G}_m\, X\,\mathcal{G}_n$$

(the first equality following from Proposition 4.1.1).

Now, we focus on proving (i). The first equality follows from Proposition 4.1.1(iv). For the second, note that $X\,\mathcal{G}_n \subseteq \mathrm{R}_X$, by the definition of the relation $\mathscr{R}$. So, suppose that $Y \in \mathrm{R}_X$ and let us prove that $Y \in X\,\mathcal{G}_n$ (i.e. that $\lambda_Y = \lambda_X \gamma$ for some $\gamma \in \mathrm{Aut}(V_n)$). From Proposition 4.1.1(iv) we may conclude that $\mathrm{Col}(X) = \mathrm{Col}(Y)$ and $\boldsymbol{\rho}_X = \boldsymbol{\rho}_Y$. Thus, by definition, we have $\lambda_Y = \lambda_X \alpha$ for some $\alpha \in \mathrm{End}(V_n)$. Furthermore, since $\mathscr{R} \subseteq \mathscr{J}$, we infer that $\mathrm{Rank}\, X = \mathrm{Rank}\, Y$ and denote this value by r. Note that $\dim(\ker \lambda_X) = \dim(\ker \lambda_Y) = m - r$. Hence, we may choose a basis $\{v_1, \ldots, v_m\}$ for V_m so that $\{v_{r+1}, \ldots, v_m\}$ is a basis of $\ker \lambda_Y$. Then, $\{v_1\lambda_Y, \ldots, v_r\lambda_Y\} = \{(v_1\lambda_X)\alpha, \ldots, (v_r\lambda_X)\alpha\}$ is a basis for $\mathrm{im}\, \lambda_Y$, which may be extended to a basis $\{v_1\lambda_Y, \ldots, v_r\lambda_Y, q_{r+1}, \ldots, q_n\}$ of V_n. In particular, $\{(v_1\lambda_X)\alpha, \ldots, (v_r\lambda_X)\alpha\}$ is a linearly independent set. Thus, $\mathcal{B} = \{v_1\lambda_X, \ldots, v_r\lambda_X\}$ is a linearly independent set in V_n. Extend it arbitrarily to a basis of V_n:

$$\{v_1\lambda_X, \ldots, v_r\lambda_X, w_{r+1}, \ldots, w_n\}.$$

Now, let us define a map $\gamma \in \mathrm{Aut}(V_n)$ by

$$(v_i\lambda_X)\gamma = v_i\lambda_Y, \quad w_j\gamma = q_j, \quad \text{for all } 1 \le i \le r \text{ and } r+1 \le j \le n.$$

Clearly, $\gamma \in \mathrm{Aut}(V_n)$, as it maps a basis of V_n into a basis of V_n. Moreover, it is easily seen that $\lambda_Y = \lambda_X\gamma$. □

As in the previous chapter (and in [30]), we are going to examine the combinatorial structure of D_s^{mn}. To present these results, we need some additional notation and results. Firstly, for any integer $q \geq 2$, the q-factorials and q-binomial coefficients are defined by

$$[s]_q! = 1 \cdot (1+q)\cdots(1+q+\cdots+q^{s-1}) = \frac{(q-1)(q^2-1)\cdots(q^s-1)}{(q-1)^s}$$

and

$$\begin{aligned}\begin{bmatrix} m \\ s \end{bmatrix}_q &= \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{s-1})}{(q^s-1)(q^s-q)\cdots(q^s-q^{s-1})} \\ &= \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-s+1}-1)}{(q^s-1)(q^{s-1}-1)\cdots(q-1)},\end{aligned} \tag{4.1}$$

respectively. These are clearly well-defined if $m, s \in \mathbb{N}$ and $m \geq s$. In the case where $s = 0$ and $m \in \mathbb{N}$, define $[s]_q! = 1$ and $\left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right]_q = 1$. Further, if $q \geq \aleph_0$, $m \in \mathbb{N}$, and $s \in \mathbb{N}_0$ with $m \geq s$, we define

$$[s]_q! = \begin{cases} q, & s \geq 2; \\ 1, & s = 1, 0; \end{cases} \qquad \text{and} \qquad \begin{bmatrix} m \\ s \end{bmatrix}_q = \begin{cases} q, & m > s > 0; \\ 1, & m = s \text{ or } s = 0. \end{cases}$$

Secondly, for $0 \leq s \leq \min(m, n)$ let the matrix $J_{mns} \in \mathcal{M}_{mn}$ be defined by

$$J_{mns} = \begin{bmatrix} \mathrm{I}_s & \mathrm{O}_{s,n-s} \\ \mathrm{O}_{m-s,s} & \mathrm{O}_{m-s,n-s}, \end{bmatrix}$$

where I_s is the $s \times s$ identity matrix, and O_{kl} is the $k \times l$ zero matrix for all $k, l \in \mathbb{N}$ (if the size of the matrix is clear from the context, we write O). Of course, if $s = \min(m, n)$, then (at least) one of the values $m - s$ and $n - s$ is 0, so the O-matrices in J_{mns} having a zero-valued dimension are empty. Similarly, if $s = 0$, we have $J_{mns} = \mathrm{O}_{mn}$.

Lastly, we need the size of the group $\mathcal{G}_s \cong \mathrm{Aut}(V_s)$. Even though this is a well-known result, we prove it for the sake of completeness.

Lemma 4.1.3. *Suppose $|\mathbb{F}| = q$ and let $s \in \mathbb{N}$. Then*

$$|\,\mathcal{G}_s\,| = \begin{cases} q^{\binom{s}{2}}(q-1)^s[s]_q!, & q < \aleph_0; \\ q, & q \geq \aleph_0. \end{cases}$$

Proof. First, suppose $q = |\mathbb{F}| \geq \aleph_0$. We show that q is both an upper and a lower bound for $|\,\mathcal{G}_s\,|$. Note that the set $\mathcal{G}_s$ contains the matrix $\alpha\,\mathrm{I}_s$ for each $\alpha \in \mathbb{F}$. Hence $q \leq |\,\mathcal{G}_s\,| \leq \mathcal{M}_s = q^{s \cdot s} = q$, so $|\,\mathcal{G}_s\,| = q$. For a field with $|\mathbb{F}| = q < \aleph_0$, we calculate the number of matrices in $\mathcal{G}_s$ by choosing the rows one by one, so that each row forms a linearly independent set with the previously chosen rows. Since there are q^s $1 \times s$ vectors over $\mathbb{F}$, and there are q^i linear combinations of i vectors, we have

$$|\mathcal{G}_s| = (q^s - 1)(q^s - q)\cdots(q^s - q^{s-1}) = q^{\binom{s}{2}}(q-1)^s[s]_q!. \qquad \square$$

For convenience, we assume $\mathrm{Aut}(V_0)$ and $\mathcal{G}_0$ to be the groups containing only the empty map and empty matrix, respectively, so $|\mathcal{G}_0| = 1$.

Note that a similar argument shows that the q-binomial (4.1) is the number of s-dimensional subspaces of an m-dimensional vector space over a q-dimensional field. Let us elaborate on that. Clearly, an m-dimensional vector space over $\mathbb{F}$ has $|\mathbb{F}|^m$ elements. We want to calculate the number of s-dimensional subspaces. Let $\mathfrak{s}_s^m(q)$ denote this number. Obviously, $\mathfrak{s}_m^m(q) = 1$ and $\mathfrak{s}_0^m(q) = 1$ (the subspace containing only the zero vector $[0, 0, \ldots, 0]^{\mathrm{T}}$). Next, suppose $q \geq \aleph_0$, $m \geq 2$ and $1 \leq s \leq m-1$. Then, we have $\mathfrak{s}_s^m(q) \geq q$, because the s-dimensional subspaces of $\mathbb{F}^m$ generated by the vectors

$$\alpha e_{m1} + e_{m2}, \quad \alpha e_{m1} + e_{m3}, \quad \ldots, \quad \alpha e_{m1} + e_{m,s+1}, \qquad \text{for } \alpha \in \mathbb{F}$$

are all different. Thus, we have

$$q = \binom{|\mathcal{M}_{mn}|}{s} \geq \mathfrak{s}_s^m(q) \geq q,$$

so $\mathfrak{s}_s^m(q) = q$. Now, let $|\mathbb{F}| = q < \aleph_0$, $m \geq 2$ and $1 \leq s \leq m$. To fix an s-dimensional subspace of $\mathbb{F}^m$, we choose its basis, which may be any of the

$$(q^m - 1)(q^m - q)\cdots(q^m - q^{s-1})$$

s-element linearly independent subsets of $\mathbb{F}^m$. However, any such subspace has q^s elements, and therefore $(q^s - 1)(q^s - q)\cdots(q^s - q^{s-1})$ different bases. Thus, by dividing the two, we get the number of s-dimensional subspaces, q.e.d.

Now, as in [30], we may calculate the sizes of Green's classes of $\mathcal{M}_{mn}$. However, note that this result includes the case $q \geq \aleph_0$, as well.

Proposition 4.1.4. *Suppose $|\mathbb{F}| = q$, and let $0 \leq s \leq \min(m, n)$. Then*

(i) D_s^{mn} *contains* $\begin{bmatrix} m \\ s \end{bmatrix}_q$ $\mathscr{R}$*-classes,*

(ii) D_s^{mn} *contains* $\begin{bmatrix} n \\ s \end{bmatrix}_q$ $\mathscr{L}$*-classes,*

(iii) D_s^{mn} *contains* $\begin{bmatrix} m \\ s \end{bmatrix}_q \cdot \begin{bmatrix} n \\ s \end{bmatrix}_q$ $\mathscr{H}$*-classes, each of which has size* $|\mathcal{G}_s|$*,*

(iv) $|\mathrm{D}_s^{mn}| = \begin{bmatrix} m \\ s \end{bmatrix}_q \cdot \begin{bmatrix} n \\ s \end{bmatrix}_q \cdot |\mathcal{G}_s|$.

(The value $|\mathcal{G}_s|$ is calculated in Lemma 4.1.3.)

Proof. The first two statements follow directly from parts (*iv*) and (*v*) of Proposition 4.1.1 and the previous discussion. Moreover, from these we may immediately infer the number of $\mathscr{H}$-classes in D_s^{mn}. Now, we need to calculate the size of these classes. By Lemma 2.1.9(*iii*), all of them have the same cardinality, so we may pick a convenient one and enumerate its members. Let $H = \mathrm{H}_{J_{mns}}$ and $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in H$, with $A \in \mathcal{M}_s$, $B \in \mathcal{M}_{s,n-s}$, $C \in \mathcal{M}_{m-s,s}$ and $D \in \mathcal{M}_{m-s,n-s}$. By Proposition 4.1.1, we have $\operatorname{Row} J_{mns} = \operatorname{Row} H$ and $\operatorname{Col} J_{mns} = \operatorname{Col} H$. From the first equality we infer $B = \mathrm{O}_{s,n-s}$ and $D = \mathrm{O}_{m-s,n-s}$ (because the corresponding submatrices of J_{mns} are zero-matrices), and from the second we have $C = \mathrm{O}_{m-s,s}$ (for the same reason). Thus, $X = \left[\begin{smallmatrix} A & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right]$ with $\operatorname{Rank} X = s$, so $A \in \mathcal{G}_s$. Since each element of $\mathcal{G}_s$ corresponds to a single element of H, we have $|H| = |\,\mathcal{G}_s\,|$, so we proved (*iii*). Finally, part (*iv*) follows directly from (*iii*). □

If $q < \aleph_0$, we have the following as an immediate consequence of (*iv*):

$$q^{mn} = |\,\mathcal{M}_{mn}\,| = \sum_{s=0}^{\min(m,n)} \begin{bmatrix} m \\ s \end{bmatrix}_q \cdot \begin{bmatrix} n \\ s \end{bmatrix}_q \cdot q^{\binom{s}{2}}(q-1)^s [s]_q!.$$

Our next goal is to investigate regularity in $\mathcal{M}$ from different aspects. We present Propositions 4.1.5, 4.1.6, and 4.1.7 and Corollary 4.1.8, the first one being a result from [30], and the other three being new, as far the author is aware.

Proposition 4.1.5. *The linear partial semigroup $\mathcal{M}$ is regular.*

Proof. Let $X \in \mathcal{M}_{mn}$ and put $\operatorname{Rank} X = r$. We want to show that $X = XYX$ for some $Y \in \mathcal{M}_{nm}$. In other words, we will prove that $\lambda_X = \lambda_X \alpha \lambda_X$ for some $\alpha \in \operatorname{Hom}(V_n, V_m)$. Let $\{v_1, \ldots, v_m\}$ be a basis for V_m such that $\{v_{r+1}, \ldots, v_m\}$ is a basis for $\ker \lambda_X$. Then, $\{v_1\lambda_X, \ldots, v_r\lambda_X\}$ is a basis for $\operatorname{im} \lambda_X$. In particular, it is a linearly independent set that can be extended to a basis of V_n:

$$\{v_1\lambda_X, \ldots, v_r\lambda_X, w_{r+1}, \ldots, w_n\}.$$

If we choose α to be any linear transformation from $\operatorname{Hom}(V_n, V_m)$ satisfying $(v_i\lambda_X)\alpha = v_i$ for $1 \le i \le r$, then it is easily shown that $\lambda_X = \lambda_X \alpha \lambda_X$. □

Proposition 4.1.6. *The linear partial semigroup $\mathcal{M}$ can be expanded to a partial $*$-semigroup, but not to a regular partial $*$-semigroup.*

Proof. As in Example 2.1.5, we define $^* : \mathcal{M} \to \mathcal{M} : A \mapsto A^{\mathrm{T}}$ to be the operation of transposition (turning rows into columns and vice versa). Then, it is easily seen that $(A^*)\,\boldsymbol{\delta} = A\,\boldsymbol{\rho}$, $(A^*)\,\boldsymbol{\rho} = A\,\boldsymbol{\delta}$, $(A^*)^* = A$ and $(AB)^* = B^*A^*$. In other words, $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a partial $*$-semigroup.

However, we will prove that there does not exist an operation $^* : \mathcal{M} \to \mathcal{M}$ such that $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a regular partial $*$-semigroup. Suppose that $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a regular partial $*$-semigroup, for some $^* : \mathcal{M} \to \mathcal{M}$. Suppose that $\mathbb{F} = (\mathbb{F}, +, \odot)$, and suppose that 0 and 1 denote the neutral elements for $+$ and $\odot$, and -1 denotes the inverse of 1 with respect to $+$. Since for all $A \in \mathcal{M}$ we have $AA^*A = A$, any matrix $A \in \mathcal{M}_r$ having a group inverse $A^{-1} \in \mathcal{M}_r$, satisfies $A^* = A^{-1}$. First, we show that the involution * fixes the matrix $X = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Let $X^* = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Note that from $e_{21}XX^*X = e_{21}X$ and $e_{22}X^*XX^* = e_{22}X^*$ we have $\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$

and $\left[\begin{smallmatrix} c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & d \end{smallmatrix}\right]$, so $a = 1$ and $bc = d$. Furthermore, from $(XX)^* = X^*X^*$ we have

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2\right)^* = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}^2,$$

so

$$\begin{bmatrix} 1 & b \\ c & bc \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)^* = \begin{bmatrix} 1+bc & b+b^2c \\ c+bc^2 & bc+b^2c^2 \end{bmatrix}.$$

Thus, $bc = 0$. Since $\mathbb{F}$ is a field, we have $b = 0$ or $c = 0$. Thus, at most one of the elements b and c is non-zero. Suppose that $c \neq 0$ and $b = 0$. Then, from $(AB)^* = B^*A^*$ we have

$$\left(\begin{bmatrix} 1 & 0 \\ c & \frac{1}{c} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)^* = \begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ c & \frac{1}{c} \end{bmatrix}^{-1},$$

so

$$\left(\begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}\right)^* = \begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -c^2 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}.$$

This contradicts $X^{**} = X$. Similarly, the equality

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & \frac{1}{b} \end{bmatrix}\right)^* = \begin{bmatrix} 1 & -b^2 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$$

proves that the assumption $b \neq 0$ (which implies $c = 0$) leads to a contradiction, as well. Therefore, $X^* = X$.

Next, consider the matrix $Y = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. We will show that no matrix from $\mathcal{M}_2(\mathbb{F})$ can be Y^*. Let $Y^* = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$. Note that from $e_{21}YY^*Y = e_{21}Y$ and $e_{22}Y^*YY^* = e_{22}Y^*$ we have $\left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} r & s \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] = \left[\begin{smallmatrix} r & s \end{smallmatrix}\right]$, so $p + r = 1$ and $qr = s - sr$.

- First, suppose $r = 1$. Then $p = 0$ and $q = 0$, so $Y^* = (YY)^* = Y^*Y^*$ implies

$$\begin{bmatrix} 0 & 0 \\ 1 & s \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2\right)^* = \begin{bmatrix} 0 & 0 \\ 1 & s \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ s & s^2 \end{bmatrix}.$$

  Hence, $s = 1$ and $Y^* = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]$. But then, by $(AB)^* = B^*A^*$ we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^* = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\right)^* = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

  which contradicts $X^* = X$.

- Now, suppose $r \neq 1$. Then, $p = 1 - r$ and $s = \frac{qr}{1-r}$. Again, the rule $(AB)^* = B^*A^*$ gives that $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]^*$ equals

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\right)^* = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-r & q \\ r & \frac{qr}{1-r} \end{bmatrix} = \begin{bmatrix} 1 & q + \frac{qr}{1-r} \\ -r & \frac{-qr}{1-r} \end{bmatrix},$$

which implies $r = 0$ and $q = 0$. Thus, $Y^* = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right] = X^*$, so $Y = (Y^*)^* = (X^*)^* = X$. This is not true.

Therefore, neither case is possible and such a map $^* : \mathcal{M} \to \mathcal{M}$ does not exist. $\square$

Proposition 4.1.7. *Let $A \in \mathcal{M}_{mn}$. Then,*

(i) A is right-invertible in $\mathcal{M}_{nm}$ if and only if the system of linear equations $Ax_i^{\mathrm{T}} = e_{mi}^{\mathrm{T}}$ is solvable for each $1 \le i \le m$. In that case, the matrix $\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}^{\mathrm{T}}$ is a right-inverse of A.

(ii) A is left-invertible in $\mathcal{M}_{nm}$ if and only if the system of linear equations $x_iA = e_{ni}$ is solvable for each $1 \le i \le n$. In that case, the matrix $\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ is a left-inverse of A.

Proof. We prove only (i), as the second part is dual. Recall that, by definition, A is right-invertible in $\mathcal{M}$ if and only if there exists $B \in \mathcal{M}_{nm}$ such that $XAB = X$ for all $X \in \mathcal{M}_{nm}$. In particular, if we fix some $1 \le j \le m$ and let $X = \left[\begin{smallmatrix} e_{mj} \\ \mathrm{O}_{n-1,m} \end{smallmatrix}\right]$ (where $\mathrm{O}_{n-1,m}$ is the empty matrix if $n = 1$), then $XAB = X$ implies $\mathfrak{r}_j(AB) = e_{mj}$. Thus, if $B = [b_1^{\mathrm{T}} \cdots b_m^{\mathrm{T}}]$, then

$$\mathfrak{r}_j(A)b_i^{\mathrm{T}} = \mathrm{I}_m(j,i), \qquad \text{for all } 1 \le i \le m.$$

Since this holds for any $1 \le j \le m$, we obtain $Ab_i^{\mathrm{T}} = e_{mi}^{\mathrm{T}}$ for each $1 \le i \le m$. Hence, we have proved the direct implication. For the converse, suppose that the systems are solvable and choose a solution b_i^{T} for the system $Ax_i^{\mathrm{T}} = e_{mi}^{\mathrm{T}}$, for each $1 \le i \le m$. Then, we have $A\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}^{\mathrm{T}} = A[b_1^{\mathrm{T}} \cdots b_m^{\mathrm{T}}] = \mathrm{I}_m$, so $XAB = X$ for all $X \in \mathcal{M}_{nm}$. $\square$

Corollary 4.1.8. *Let $A \in \mathcal{M}_{mn}$. Then,*

(i) A is right-invertible in $\mathcal{M}_{nm}$ if and only if $\mathrm{Rank}(A) = m$.

(ii) A is right-invertible in $\mathcal{M}_{nm}$ if and only if $\mathrm{Rank}(A) = n$.

Proof. As in the previous lemma, we prove (i) and part (ii) is dual. Suppose that A is right-invertible in $\mathcal{M}_{nm}$. From the proof of Lemma 4.1.7, we conclude that there exists $B \in \mathcal{M}_{nm}$ such that $\mathfrak{r}_j(AB) = e_{mj}$ for all $1 \le j \le m$. In other words, $AB = \mathrm{I}_m$. Conversely, if $AB = \mathrm{I}_m$ for some $B \in \mathcal{M}_{nm}$, then $XAB = X$ for all $X \in \mathcal{M}_{nm}$, so A is right-invertible. Thus, A is right-invertible in $\mathcal{M}_{nm}$ if and only if there exists $B \in \mathcal{M}_{nm}$ such that $AB = \mathrm{I}_m$. Clearly, if $AB = \mathrm{I}_m$, then $\mathrm{Rank}(A) = m$ (because $\mathrm{Rank}(AB) \le \mathrm{Rank}(A) \le m$). Conversely, if $\mathrm{Rank}(A) = m \le n$, then the equation $Ax_i^{\mathrm{T}} = e_{mi}^{\mathrm{T}}$ is solvable for each $1 \le i \le m$ by the Rouché–Capelli theorem (because the coefficient matrix $A \in \mathcal{M}_{mn}$ has $\mathrm{Rank}(A) = \dim(\mathrm{Col}(A)) = m$, and so the augmented matrix also has m independent columns). $\square$

4.2 Linear sandwich semigroups

In the next step of our analysis, we use the gathered information on $\mathcal{M}$ to investigate sandwich semigroups of form $\mathcal{M}_{mn}^A$. Let us fix $m, n \in \mathbb{N}$ and an $n \times m$ matrix $A \in \mathcal{M}_{nm}$. Then, $\mathcal{M}_{mn}^A(\mathbb{F}) = \mathcal{M}_{mn}^A = (\mathcal{M}_{mn}, \star_A)$ denotes the sandwich semigroup of all $m \times n$ matrices over $\mathbb{F}$, the sandwich operation $\star_A$ being defined in the usual way:

$$X \star_A Y = XAY, \quad \text{for all } X, Y \in \mathcal{M}_{mn}.$$

If $m = n$, then the semigroup $\mathcal{M}_{mm}^A = \mathcal{M}_m^A$ is the variant of $\mathcal{M}_m$ with respect to the element $A \in \mathcal{M}_m$.

Our aim is to describe $\mathcal{M}_{mn}^A$ in detail, in the same manner as we did with sandwich semigroups of transformations. As it turns out, in this case, our task simplifies significantly, since sandwich elements of the same rank determine isomorphic semigroups over $\mathcal{M}_{mn}$. This is proved in the following Lemma, which is a result of [30].

Proposition 4.2.1.

(i) If $A \in \mathcal{M}_{nm}$, then the semigroups $\mathcal{M}_{mn}^A$ and $\mathcal{M}_{nm}^{A^{\mathrm{T}}}$ are anti-isomorphic.

(ii) If $A, B \in \mathcal{M}_{nm}$ are such that $\operatorname{Rank} A = \operatorname{Rank} B$, *then $\mathcal{M}_{mn}^A \cong \mathcal{M}_{mn}^B$.*

Proof. (i) From the properties of the operation of transposition $^{\mathrm{T}}$, it follows that the map $\mathcal{M}_{mn} \to \mathcal{M}_{nm} : X \mapsto X^{\mathrm{T}}$ is an anti-isomorphism of semigroups $\mathcal{M}_{mn}^A$ and $\mathcal{M}_{nm}^{A^{\mathrm{T}}}$.

(ii) Suppose $\operatorname{Rank} A = \operatorname{Rank} B$. Then, by Lemma 4.1.2$(iii)$, we have $A = UBV$, for some $U \in \mathcal{G}_n$ and $V \in \mathcal{G}_m$. Thus, define a map $\theta : \mathcal{M}_{mn}^A \to \mathcal{M}_{mn}^B : X \mapsto VXU$. Since V and U are both invertible, this map is clearly a bijection. Furthermore, for all $X, Y \in \mathcal{M}_{mn}$,

$$\begin{aligned}(X \star_A Y)\theta &= (XAY)\theta = VXAYU \\ &= VX(UBV)YU = (VXU)B(VYU) = (X)\theta \star_B (Y)\theta,\end{aligned}$$

so θ is an isomorphism. □

Put $\operatorname{Rank} A = r$. Instead of studying $\mathcal{M}_{mn}^A$, we may choose to study any sandwich semigroup $\mathcal{M}_{mn}^J$ with $J \in \mathcal{M}_{nm}$ and $\operatorname{Rank} J = r$. Naturally, we pick the "simplest" matrix possible,

$$J = J_{nmr} = \begin{bmatrix} \mathrm{I}_r & \mathrm{O}_{r,m-r} \\ \mathrm{O}_{n-r,r} & \mathrm{O}_{n-r,m-r} \end{bmatrix}.$$

So, from now on, we are investigating the sandwich semigroup $\mathcal{M}_{mn}^J$ with $J = J_{nmr}$. Note that, if $m = n = r$, we have $J = \mathrm{I}_m$, so $\mathcal{M}_{mn}^J \cong \mathcal{M}_m$, the full linear monoid of degree m. Since the properties of $\mathcal{M}_{mn}^J$ that we study are already known for $\mathcal{M}_m$, we will sometimes assume that $m = n = r$ does not hold. In these cases, we will provide a corresponding result for $\mathcal{M}_m$ and the reference for it. For background on the full linear monoid, we refer the reader to [103].

Due to the form of the matrix J, we introduce some new notation (in the same manner as it was done in [30]), which will enable easier calculating of products. Namely, if $k, l \geq r$ and if a matrix $X \in \mathcal{M}_{kl}$ is written in the 2×2 block form $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, we will be assuming that $A \in \mathcal{M}_r$, $B \in \mathcal{M}_{r,l-r}$, $C \in \mathcal{M}_{k-r,r}$ and $D \in \mathcal{M}_{k-r,l-r}$. For instance, we may write $J = \left[\begin{smallmatrix} I_r & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right]$. Thus, for $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \left[\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right] \in \mathcal{M}_{mn}$, we have

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \star_J \begin{bmatrix} E & F \\ G & H \end{bmatrix} &= \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} I & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{bmatrix} \right) \cdot \begin{bmatrix} E & F \\ G & H \end{bmatrix} \\ &= \begin{bmatrix} A & \mathrm{O} \\ C & \mathrm{O} \end{bmatrix} \cdot \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE & AF \\ CE & CF \end{bmatrix}. \end{aligned} \tag{4.2}$$

Similarly, for the same matrix $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{M}_{mn}$, we have $XJ = \left[\begin{smallmatrix} A & \mathrm{O} \\ C & \mathrm{O} \end{smallmatrix}\right]$, $JX = \left[\begin{smallmatrix} A & B \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right]$ and $JXJ = \left[\begin{smallmatrix} A & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right]$. In addition, for any $A \in \mathcal{M}_r$, $M \in \mathcal{M}_{m-r,r}$ and $N \in \mathcal{M}_{r,n-r}$, we define

$$[M, A, N] = \begin{bmatrix} A & AN \\ MA & MAN \end{bmatrix} \in \mathcal{M}_{mn}.$$

Using the calculation above, it is easily seen that $[M, A, N] \star_J [K, B, L] = [M, AB, L]$.

Remark 4.2.2. In [121], Thrall presented an alternative way to deal with sandwich semigroups. Let

$$\begin{aligned} M = \{X \in \mathcal{M}_{m+n-r} : \mathfrak{r}_1(X) = \mathfrak{r}_2(X) = \cdots = \mathfrak{r}_{n-r}(X) = \mathrm{O}, \\ \mathfrak{c}_{n+1}(X) = \mathfrak{c}_{n+2}(X) = \cdots = \mathfrak{c}_{m+n-r}(X) = \mathrm{O}\} \end{aligned}$$

and consider the map $\zeta : \mathcal{M}_{mn} \to M : \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \mathrm{O} & \mathrm{O} & \mathrm{O} \\ B & A & \mathrm{O} \\ D & C & \mathrm{O} \end{smallmatrix}\right]$, where the first matrix is in the above described 2×2 form. It is easily seen that

$$\left[\begin{smallmatrix} \mathrm{O} & \mathrm{O} & \mathrm{O} \\ B & A & \mathrm{O} \\ D & C & \mathrm{O} \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} \mathrm{O} & \mathrm{O} & \mathrm{O} \\ F & E & \mathrm{O} \\ H & G & \mathrm{O} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathrm{O} & \mathrm{O} & \mathrm{O} \\ AF & AE & \mathrm{O} \\ CF & CE & \mathrm{O} \end{smallmatrix}\right].$$

(cf. (4.2)), so ζ is clearly an isomorphism of semigroups $\mathcal{M}_{mn}^J$ and $(M, \cdot)$. Thus, instead of $\mathcal{M}_{mn}^J$, we may examine the (non-sandwich) semigroup $(M, \cdot)$. However, this approach does not seem to benefit or simplify our investigation in any way, so we do not pursue it any further.

4.2.1 Green's relations of linear sandwich semigroups

Finally, we are ready to describe the P-sets of $\mathcal{M}_{mn}^J$. The following Proposition is a result from [30].

Proposition 4.2.3. *In $\mathcal{M}_{mn}^J$ we have $\mathrm{P}^J = \mathrm{P}_3^J = \mathrm{Reg}(\mathcal{M}_{mn}^J)$. Further,*

(i) $\mathrm{P}_1^J = \{X \in \mathcal{M}_{mn} : \mathrm{Col}\, XJ = \mathrm{Col}\, X\}$
$= \{X \in \mathcal{M}_{mn} : \mathrm{Rank}\, XJ = \mathrm{Rank}\, X\}$,

(ii) $\mathrm{P}_2^J = \{X \in \mathcal{M}_{mn} : \operatorname{Row} JX = \operatorname{Row} X\}$
$= \{X \in \mathcal{M}_{mn} : \operatorname{Rank} JX = \operatorname{Rank} X\}$,

(iii) $\mathrm{P}_3^J = \{X \in \mathcal{M}_{mn} : \operatorname{Rank} JXJ = \operatorname{Rank} X\}$
$= \{[M, A, N] : A \in \mathcal{M}_r,\ M \in \mathcal{M}_{m-r,r},\ N \in \mathcal{M}_{r,n-r}\}$.

Proof. By Proposition 4.1.1, $\mathcal{M}$ is stable, so Proposition 2.2.23(iii) implies $\mathrm{P}_3^J = \mathrm{P}^J$. Furthermore, since $\mathcal{M}$ is regular (by Proposition 4.1.5), Proposition 2.2.29(iv) implies $\operatorname{Reg}(\mathcal{M}_{mn}^J) = \mathrm{P}^J$.

Now, we prove (i), and (ii) follows by a dual argument. The first equality follows from the definition of P_1^J and Proposition 4.1.1(iv). For the second one, note that the stability of $\mathcal{M}$ implies that $XJ \mathscr{R} X \Leftrightarrow XJ \mathscr{J} X$. Then, Proposition 4.1.1(vi) implies the statement.

Let us prove part (iii). Again, the first equality follows from the definition of P_3^J and Proposition 4.1.1(vi). For the second one, recall that $\mathrm{P}_3^J = \mathrm{P}^J$. Let $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{P}^J = \mathrm{P}_1^J \cap \mathrm{P}_2^J$. Then, we have

$$\begin{aligned} X \in \mathrm{P}_1^J &\Leftrightarrow \operatorname{Col} X = \operatorname{Col} XJ = \operatorname{Col}(\left[\begin{smallmatrix} A & O \\ C & O \end{smallmatrix}\right]) \\ &\Leftrightarrow \text{ each column of } \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] \text{ is in the span of the set of columns of } \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] \\ &\Leftrightarrow \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] = \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] N = \left[\begin{smallmatrix} AN \\ CN \end{smallmatrix}\right] \text{ for some } N \in \mathcal{M}_{r,n-r}\,. \end{aligned}$$

By a dual argument, we have

$$X \in \mathrm{P}_2^J \Leftrightarrow \left[\begin{smallmatrix} C & D \end{smallmatrix}\right] = M \left[\begin{smallmatrix} A & B \end{smallmatrix}\right] = \left[\begin{smallmatrix} MA & MB \end{smallmatrix}\right] \text{ for some } M \in \mathcal{M}_{m-r,r}\,.$$

Thus, we have $X \in \mathrm{P}^J$ if and only if $X = \left[\begin{smallmatrix} A & AN \\ MA & MAN \end{smallmatrix}\right] = [M, A, N]$ for some $A \in \mathcal{M}_r$, $M \in \mathcal{M}_{m-r,r}$ and $N \in \mathcal{M}_{r,n-r}$. □

Having described the P-sets, the next step is to characterise the Green's relations of $\mathcal{M}_{mn}^J$. We give the Theorem from [30], but note that the result has originally appeared (in a somewhat different form) in [20].

Theorem 4.2.4. *If $X \in \mathcal{M}_{mn}$, then in $\mathcal{M}_{mn}^J$ we have*

(i) $\mathrm{R}_X^J = \begin{cases} \mathrm{R}_X \cap \mathrm{P}_1^J, & X \in \mathrm{P}_1^J; \\ \{X\}, & X \notin \mathrm{P}_1^J. \end{cases}$

(ii) $\mathrm{L}_X^J = \begin{cases} \mathrm{L}_X \cap \mathrm{P}_2^J, & X \in \mathrm{P}_2^J; \\ \{X\}, & X \notin \mathrm{P}_2^J. \end{cases}$

(iii) $\mathrm{H}_X^J = \begin{cases} \mathrm{H}_X, & X \in \mathrm{P}^J; \\ \{X\}, & X \notin \mathrm{P}^J. \end{cases}$

(iv) $\mathrm{D}_X^J = \mathrm{J}_X^J = \begin{cases} \mathrm{D}_X \cap \mathrm{P}^J, & X \in \mathrm{P}^J; \\ \mathrm{L}_X^J, & X \in \mathrm{P}_2^J \setminus \mathrm{P}_1^J; \\ \mathrm{R}_X^J, & X \in \mathrm{P}_1^J \setminus \mathrm{P}_2^J; \\ \{X\}, & X \notin (\mathrm{P}_1^J \cup \mathrm{P}_2^J). \end{cases}$

Further, if $X \notin \mathrm{P}^J$, then $\mathrm{H}_X^J = \{X\}$ is a non-group $\mathscr{H}^J$-class in $\mathcal{M}_{mn}^J$.

Proof. Since $\mathcal{M}$ is a stable partial semigroup (by Proposition 4.1.1), Corollary 2.2.26 implies $\mathscr{D}^J = \mathscr{J}^J$. Hence, the theorem follows directly from Theorem 2.2.3. □

Note that $\mathcal{M}_{mn}^J$ is stable by Lemma 2.2.20, since $\mathcal{M}$ is stable. Applying the general theory from Chapter 2, we may describe the partial order $\leq_{\mathscr{J}^J}$. The following Proposition (from [30]) is the direct consequence of Lemma 2.2.6(*iii*) (recall that $\mathcal{M}$ is monoidal, so each element has a left- and right-identity in $\mathcal{M}$) and Proposition 4.1.1.

Proposition 4.2.5. *Let $X, Y \in \mathcal{M}_{mn}$. Then $\mathrm{J}_X^J \leq_{\mathscr{J}^J} \mathrm{J}_Y^J$ in $\mathcal{M}_{mn}^J$ if and only if one of the following holds:*

(i) $X = Y$,

(ii) $\operatorname{Rank} X \leq \operatorname{Rank} JYJ$,

(iii) $\operatorname{Row} X \subseteq \operatorname{Row} JY$,

(iv) $\operatorname{Col} X \subseteq \operatorname{Col} YJ$.

Moreover, Propositions 2.2.7 and 4.1.1 give

Proposition 4.2.6. *Let $X, Y \in \mathcal{M}_{mn}$.*

(i) If $X \in \mathrm{P}_1^J$, then

$$X \leq_{\mathscr{J}^J} Y \Leftrightarrow \operatorname{Rank} X \leq \operatorname{Rank} JYJ \ \text{ or } \ \operatorname{Col} X \subseteq \operatorname{Col} YJ.$$

(ii) If $X \in \mathrm{P}_2^J$, then

$$X \leq_{\mathscr{J}^J} Y \Leftrightarrow \operatorname{Rank} X \leq \operatorname{Rank} JYJ \ \text{ or } \ \operatorname{Row} X \subseteq \operatorname{Row} JY.$$

(iii) If $X \in \mathrm{P}^J$, then $X \leq_{\mathscr{J}^J} Y \Leftrightarrow \operatorname{Rank} X \leq \operatorname{Rank} JYJ$.

(iv) If $Y \in \mathrm{P}_1^J$, then $X \leq_{\mathscr{J}^J} Y \Leftrightarrow \operatorname{Rank} X \leq \operatorname{Rank} JY$ or $\operatorname{Col} X \subseteq \operatorname{Col} Y$.

(v) If $Y \in \mathrm{P}_2^J$, then $X \leq_{\mathscr{J}^J} Y \Leftrightarrow \operatorname{Rank} X \leq \operatorname{Rank} YJ$ or $\operatorname{Row} X \subseteq \operatorname{Row} Y$.

(vi) If $Y \in \mathrm{P}^J$, then $X \leq_{\mathscr{J}^J} Y \Leftrightarrow \operatorname{Rank} X \leq \operatorname{Rank} Y$.

The article [30] presents only parts (*iii*) and (*vi*), but the remaining ones are easily deduced from the results of the said paper.

Recall that $\operatorname{Reg}(\mathcal{M}_{mn}^J) = \mathrm{P}^J$ (by Proposition 4.2.3). The elements of $\operatorname{Reg}(\mathcal{M}_{mn}^J)$ were originally characterised in [71]. Here, we give the result from [30] describing the regular $\mathscr{D}^J$-classes and their relations.

Proposition 4.2.7. *The regular $\mathscr{D}^J$-classes of $\mathcal{M}_{mn}^J$ are precisely the sets*

$$\mathrm{D}_s^J = \{X \in \mathrm{P}^J : \operatorname{Rank} X = s\}, \quad \text{for each } 0 \leq s \leq r = \operatorname{Rank} J.$$

Proof. Firstly, recall from Theorem 4.2.4(iv) that for each $X \in \mathrm{P}^J$ we have $\mathrm{D}^J_X = \mathrm{D}_X \cap \mathrm{P}^J = \mathrm{D}^J_{\operatorname{Rank} X}$. Moreover, for any $0 \leq s \leq r = \operatorname{Rank} J$, the matrix $J_{mns} = \left[\begin{smallmatrix} J_{rrs} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right] \in \mathcal{M}_{mn}$ is in $\mathrm{P}^J \cap \mathrm{D}_s$, since $J_{mns} = [J_{rrs}, \mathrm{O}, \mathrm{O}]$ and $\operatorname{Rank} J_{mns} = s$. Therefore, the defined sets are all nonempty and describe regular $\mathscr{D}^J$-classes. □

Note that the class $\mathrm{D}^J_0 = \{\mathrm{O}_{mn}\}$ is a regular $\mathscr{D}^J$-class, and it is the minimal $\mathscr{D}^J = \mathscr{J}^J$-class in $\mathcal{M}^J_{mn}$, by Proposition 4.2.6(iii). In the following Proposition (a result of [30]), we discuss the maximal $\mathscr{D}^J$-classes. Here (and in the rest of this chapter), we will need a semigroup inverse of J. We take $K = J^{\mathrm{T}} = J_{mnr} \in \mathcal{M}_{mn}$, where $r = \operatorname{Rank} J$ (one can easily check that it is indeed an inverse using the discussion preceding Subsection 4.2.1).

Proposition 4.2.8.

(i) If $r < \min(m,n)$, then the maximal $\mathscr{J}^J = \mathscr{D}^J$-classes of $\mathcal{M}^J_{mn}$ are precisely the singleton sets $\{X\}$, for $X \in \mathcal{M}_{mn}$ with $\operatorname{Rank} X > r$. Hence, all the maximal $\mathscr{J}^J$-classes of $\mathcal{M}^J_{mn}$ are trivial in this case.

(ii) If $r = \min(m,n)$, then we have a single maximum $\mathscr{J}^J = \mathscr{D}^J$-class in $\mathcal{M}^J_{mn}$, which is

$$\mathrm{J}^J_K = \mathrm{D}^J_r = \{X \in \mathrm{P}^J : \operatorname{Rank} X = r\}.$$

This maximal $\mathscr{J}^J$-class is clearly nontrivial.

Proof. (i) Let $r < \min(m,n)$. The singletons described in the statement are indeed maximal $\mathscr{J}^J$-classes (by Proposition 2.2.10) since for $X \in \mathcal{M}_{mn}$ with $\operatorname{Rank} X > r$ we have $X \not\leq_{\mathscr{J}} J$ (by Proposition 4.1.1(iii)). Thus, it suffices to prove that the specified sets are the only maximal $\mathscr{J}^J$-classes. Suppose there exists $Y \in \mathcal{M}_{mn}$ with $\operatorname{Rank} Y \leq r$ such that J^J_Y is a maximal $\mathscr{J}^J$-class. Now, let $Z = \left[\begin{smallmatrix} \mathrm{I}_r & \mathrm{O} \\ \mathrm{O} & D \end{smallmatrix}\right] \in \mathcal{M}_{mn}$ with $D \neq \mathrm{O}$. By the previous discussion, $\mathrm{J}^J_Z = \{Z\}$ is a maximal $\mathscr{J}^J$-class with $Y \leq_{\mathscr{J}^J} Z$, different from J^J_Y (because $Y \neq Z$), which contradicts the maximality of J^J_Y.

(ii) Note that $J \cdot J^{\mathrm{T}} \cdot J = J$, so Proposition 4.2.3($iii$) implies $K \in \mathrm{P}^J_3 = \mathrm{P}^J$. Furthermore, from Theorem 4.2.4(iv) and Proposition 4.2.7, we have $\mathrm{J}^J_{J^{\mathrm{T}}} = \mathrm{D}^J_{J^{\mathrm{T}}} = \mathrm{D}^J_r$, and this $\mathscr{J}^J$-class is maximal (which follows from Proposition 4.2.5(vi)). □

4.2.2 A structure theorem for $\operatorname{Reg}(\mathcal{M}^J_{mn})$ and connections to (non-sandwich) matrix semigroups

Here, we present the results of Section 6 in [30], simplifying the arguments by applying the general theory from Chapter 2.

Keeping the previously introduced notation, we start by examining the diagrams 2.2 and 2.3 adjusted to the semigroup $\mathcal{M}^J_{mn}$:

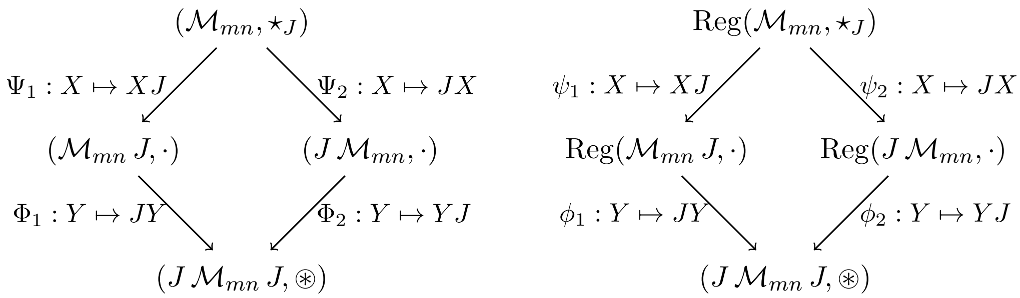


Figure 4.1: Diagrams illustrating the connections between $\mathcal{M}_{mn}^J$ and $(J\, \mathcal{M}_{mn}\, J, \circledast)$ (left) and between $\mathrm{Reg}(\mathcal{M}_{mn}^J)$ and $(J\, \mathcal{M}_{mn}\, J, \circledast)$ (right).

Of course, from general theory it follows that $J\, \mathcal{M}_{mn}\, J$ is a regular subsemigroup of $\mathcal{M}_{nm}^K$. Moreover, Lemma 4.2.1(i) implies that $\mathcal{M}_{mn}^J$ and $\mathcal{M}_{nm}^K$ are anti-isomorphic. Thus, $J\, \mathcal{M}_{mn}\, J$ is anti-isomorphic to the regular subsemigroup $K\, \mathcal{M}_{nm}\, K$ of $\mathcal{M}_{mn}^J$. In fact, as proved in [30],

Proposition 4.2.9. *We have*

$$(J\, \mathcal{M}_{mn}\, J, \star_K) = \mathrm{Reg}(J\, \mathcal{M}_{mn}\, J, \star_K) \cong (\mathcal{M}_r, \cdot).$$

Proof. The first equality follows from Proposition 2.3.2(iv). For the isomorphism, recall that for $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{M}_{mn}$ we have $J \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] J = \left[\begin{smallmatrix} A & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right]$. Now, consider the map $J\, \mathcal{M}_{mn}\, J \to \mathcal{M}_r : \left[\begin{smallmatrix} A & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right] \mapsto A$. It is clearly an isomorphism of semigroups $(J\, \mathcal{M}_{mn}\, J, \star_K)$ and $(\mathcal{M}_r, \cdot)$, since for $\left[\begin{smallmatrix} A & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right], \left[\begin{smallmatrix} E & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right] \in J\, \mathcal{M}_{mn}\, J$ we have

$$\begin{bmatrix} A & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{bmatrix} \cdot J \cdot \begin{bmatrix} E & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{bmatrix} = \begin{bmatrix} AE & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{bmatrix}. \qquad \square$$

Now, we describe the semigroups in the middle of the diagram 4.1. In order to do that, we introduce a new type of matrix semigroups: for $k \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{0\}$, let

$$\mathcal{C}_k(l) = \{X \in \mathcal{M}_k : \mathfrak{c}_{l+1}(X) = \cdots = \mathfrak{c}_k(X) = \mathrm{O}\} \text{ and}$$
$$\mathcal{R}_k(l) = \{X \in \mathcal{M}_k : \mathfrak{r}_{l+1}(X) = \cdots = \mathfrak{r}_k(X) = \mathrm{O}\}.$$

Note that, for any $X \in \mathcal{M}_k$, we have $X \in \mathcal{C}_k(l) \Leftrightarrow X^{\mathrm{T}} \in \mathcal{R}_k(l)$. Thus, $\mathcal{C}_k(l)$ and $\mathcal{R}_k(l)$ are anti-isomorphic. In previous articles, these semigroups have attracted interest due to their properties (see [101] and [117]), but ours is raised because of their connection to the sandwich semigroups, shown in [30]:

Proposition 4.2.10. *We have $\mathcal{M}_{mn}\, J = \mathcal{C}_m(r)$ and $J\, \mathcal{M}_{mn} = \mathcal{R}_n(r)$. Furthermore, $\mathcal{C}_m(r) \cong \mathcal{M}_{mr}^{J_1}$ and $\mathcal{R}_n(r) \cong \mathcal{M}_{rn}^{J_2}$, for $J_1 = J_{rmr} \in \mathcal{M}_{rm}$ and $J_2 = J_{nrr} \in \mathcal{M}_{nr}$.*

Proof. Recall that, for $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{M}_{mn}$, we have $XJ = \left[\begin{smallmatrix} A & \mathrm{O} \\ C & \mathrm{O} \end{smallmatrix}\right]$ and $JX = \left[\begin{smallmatrix} A & B \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right]$. Thus, the direct containment holds in both equalities of the first statement. The reverse containment is easy to check.

For the second statement, we prove only $\mathcal{C}_m(r) \cong \mathcal{M}_{mr}^{J_1}$, the other one being dual. From (4.2) we have

$$\begin{bmatrix} A & O \\ C & O \end{bmatrix} \star_J \begin{bmatrix} E & O \\ G & O \end{bmatrix} = \begin{bmatrix} AE & O \\ CE & O \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A \\ C \end{bmatrix} \star_{J_1} \begin{bmatrix} E \\ G \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} E \\ G \end{bmatrix} = \begin{bmatrix} AE \\ CE \end{bmatrix}.$$

So, the map $\mathcal{C}_m(r) \to \mathcal{M}_{mr} : \begin{bmatrix} A & O \\ C & O \end{bmatrix} \mapsto \begin{bmatrix} A \\ C \end{bmatrix}$ is clearly an isomorphism. □

Hence, we may adjust the left diagram on Figure 4.1:

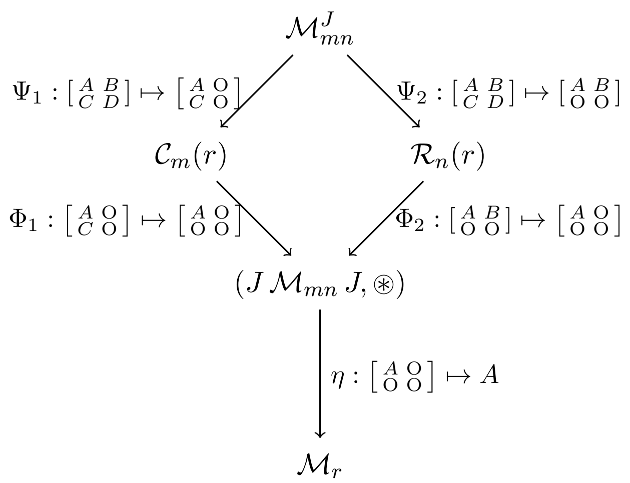


Figure 4.2: Diagram illustrating the connections between $\mathcal{M}_{mn}^J$ and $\mathcal{M}_r$.

Moreover, we may characterise the regular elements of these semigroups, which was originally done in [101]. However, we will use an alternative description, given in [30].

Proposition 4.2.11. *We have*

$$\operatorname{Reg}(\mathcal{C}_m(r)) = \operatorname{Reg}(\mathcal{M}_{mn}\,J) = \mathrm{P}^J\,J = \{X \in \mathcal{C}_m(r) : \operatorname{Rank} JX = \operatorname{Rank} X\},$$
$$\operatorname{Reg}(\mathcal{R}_m(r)) = \operatorname{Reg}(J\,\mathcal{M}_{mn}) = J\,\mathrm{P}^J = \{X \in \mathcal{R}_m(r) : \operatorname{Rank} XJ = \operatorname{Rank} X\}.$$

Proof. The first two equalities in both cases follow from Propositions 4.2.10 and 2.3.2, respectively. We will show the third equality for $\operatorname{Reg}(\mathcal{C}_m(r))$, and the other statement follows by a dual argument. Let $X = YJ$ for some $Y \in \mathrm{P}^J$. From Proposition 4.2.3(*iii*), it follows that $Y = \begin{bmatrix} A & AN \\ MA & MAN \end{bmatrix}$ for some $A \in \mathcal{M}_r$, $M \in \mathcal{M}_{m-r,r}$ and $N \in \mathcal{M}_{r,n-r}$, so

$$X = YJ = \begin{bmatrix} A & O \\ MA & O \end{bmatrix}.$$

Thus, $JX = \begin{bmatrix} A & O \\ O & O \end{bmatrix}$ and $\operatorname{Rank} X = \operatorname{Rank} JX$. Conversely, if $X = \begin{bmatrix} A & O \\ C & O \end{bmatrix} \in \mathcal{C}_m(r)$ with $\operatorname{Rank} JX = \operatorname{Rank} X$, then $JX \mathrel{\mathscr{J}} X$, so $JX \mathrel{\mathscr{L}} X$ (by the stability of $\mathcal{M}$). Therefore, $\operatorname{Row}(\begin{bmatrix} A & O \\ C & O \end{bmatrix}) = \operatorname{Row}(\begin{bmatrix} A & O \\ O & O \end{bmatrix})$, which implies that $C = MA$ for some $M \in$

$\mathcal{M}_{m-r,r}$ (i.e. each row of C is a linear combination of the rows of A). Hence, $X = [M, A, \mathrm{O}] \cdot J \in \mathrm{P}^J J$. □

Thus, we obtain the following diagram:

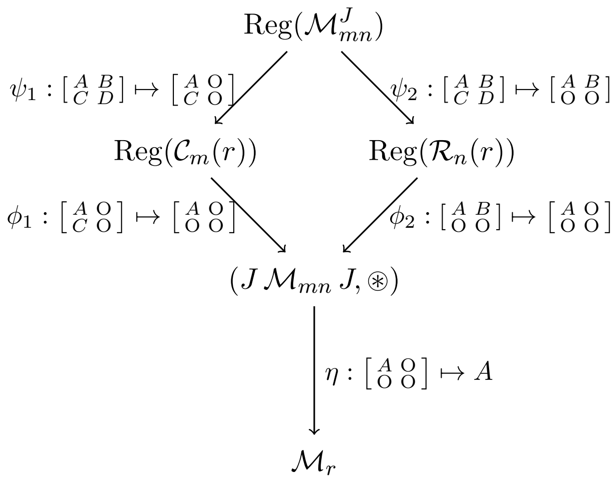


Figure 4.3: Diagram illustrating the connections between $\mathrm{Reg}(\mathcal{M}_{mn}^J)$ and $\mathcal{M}_r$.

Of course, as in Chapter 3, we have some special cases: as we proved in Proposition 4.2.10,

- if $J = J_{nmm}$, then $\mathcal{M}_{mn}^J \cong \mathcal{R}_n(m)$, and
- if $J = J_{nmn}$, then $\mathcal{M}_{mn}^J \cong \mathcal{C}_n(m)$.

Hence, we will be able to apply the results we obtain for $\mathcal{M}_{mn}^J$ to get results for $\mathcal{R}_n(m)$ and $\mathcal{C}_n(m)$, as well.

We close this subsection by stating Theorem 2.3.8 for the sandwich semigroup $\mathcal{M}_{mn}^J$. This result was originally proved in [30].

Theorem 4.2.12. *The map*

$$\psi = (\psi_1, \psi_2) : \mathrm{P}^J \to \mathrm{Reg}(\mathcal{C}_m(r)) \times \mathrm{Reg}(\mathcal{R}_n(r)) : X \mapsto (XJ, JX)$$

is injective, and

$$\begin{aligned} \mathrm{im}(\psi) &= \{(Y, Z) \in \mathrm{Reg}(\mathcal{C}_m(r)) \times \mathrm{Reg}(\mathcal{R}_n(r)) : JY = ZJ\} \\ &= \{(Y, Z) \in \mathrm{Reg}(\mathcal{C}_m(r)) \times \mathrm{Reg}(\mathcal{R}_n(r)) : Y\phi_1 = Z\phi_2\}. \end{aligned}$$

In particular, P^J *is a pullback product of* $\mathrm{Reg}(\mathcal{C}_m(r))$ *and* $\mathrm{Reg}(\mathcal{R}_n(r))$ *with respect to* $\mathcal{M}_r$.

4.2.3 The regular subsemigroup $\mathrm{P}^J = \mathrm{Reg}(\mathcal{M}^J_{mn})$

We continue our study, following the same outline as in Section 3.1. This subsection is dedicated to the regular subsemigroup of $\mathcal{M}^J_{mn}$.

Firstly, we examine the Green's relations of $\mathrm{P}^J = \mathrm{Reg}(\mathcal{M}^J_{mn})$. For $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$ and $X \in \mathrm{P}^J$, let $\mathscr{K}^{\mathrm{P}^J}$ and $\mathrm{K}^{\mathrm{P}^J}_X$ denote the $\mathscr{K}$-relation of P^J and its class containing the matrix X, respectively. Then, from Lemma 2.3.3, Theorem 4.2.4, Proposition 4.1.1, and Proposition 2.3.4 we have

Proposition 4.2.13. *Let $X \in \mathrm{P}^J = \mathrm{Reg}(\mathcal{M}^J_{mn})$. Then*

(i) $\mathrm{R}^{\mathrm{P}^J}_X = \mathrm{R}_X \cap \mathrm{P}^J = \{Y \in \mathrm{P}^J \ : \ \mathrm{Col}\, Y = \mathrm{Col}\, X\}$,

(ii) $\mathrm{L}^{\mathrm{P}^J}_X = \mathrm{L}_X \cap \mathrm{P}^J = \{Y \in \mathrm{P}^J \ : \ \mathrm{Row}\, Y = \mathrm{Row}\, X\}$,

(iii) $\mathrm{H}^{\mathrm{P}^J}_X = \mathrm{H}_X \cap \mathrm{P}^J = \mathrm{H}_X$
$= \{Y \in \mathrm{P}^J \ : \ \mathrm{Col}\, Y = \mathrm{Col}\, X, \ \mathrm{Row}\, Y = \mathrm{Row}\, X\}$,

(iv) $\mathrm{J}^{\mathrm{P}^J}_X = \mathrm{D}^{\mathrm{P}^J}_X = \mathrm{D}_X \cap \mathrm{P}^J = \{Y \in \mathrm{P}^J \ : \ \mathrm{Rank}\, Y = \mathrm{Rank}\, X\}$.

The $\mathscr{J}^{\mathrm{P}^J} = \mathscr{D}^{\mathrm{P}^J}$-classes of P^J are the sets

$$\mathrm{D}^J_s = \{Y \in \mathrm{P}^J : \mathrm{Rank}\, Y = s\} \qquad \text{for each } 0 \leq s \leq r = \mathrm{Rank}\, J,$$

and these form a chain under the ordering $\leq_{\mathscr{J}}$ on the $\mathscr{J}^{\mathrm{P}^J}$-classes:

$$\mathrm{D}^J_0 < \mathrm{D}^J_1 < \cdots < \mathrm{D}^J_r\,.$$

Parts $(i)-(iv)$ of this result were proved in [20], but we presented here Corollary 6.1 from [30].

Therefore, for $X \in \mathrm{P}^J$ and any $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}, \mathrm{J}\}$, we have $\mathrm{K}^{\mathrm{P}^J}_X = \mathrm{K}^J_X$. Hence, we will denote the Green's relations and their classes of P^J the same way as we did in $\mathcal{M}^J_{mn}$. Furthermore, note that Proposition 2.2.42 implies the stability of P^J, since it is a regular subsemigroup of a stable semigroup $\mathcal{M}^J_{mn}$.

Our next task is to study the inflation described in Remark 2.3.13, in the semigroup $\mathrm{P}^J\,\mathrm{Reg}(\mathcal{M}^J_{mn})$. First, consider the map

$$\varphi = \psi_1\phi_1\eta = \psi_2\phi_2\eta : \mathrm{P}^J \to \mathcal{M}_r,$$

defined via the surmorphisms from Diagram 4.3. To shorten the notation, for $X \in \mathrm{P}^J$, let $\overline{X}$ denote the map $X\varphi$ of the element X. Furthermore, for $S \subseteq \mathrm{P}^J$, let $\overline{S} = \{\overline{X} : X \in S\}$. Clearly, if $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{P}^J$, then $\overline{X} = A$.

As in Chapter 2, for each $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$, we introduce the relation

$$X \widehat{\mathscr{K}^J} Y \Leftrightarrow \overline{X}\, \mathscr{K}\, \overline{Y} \qquad (\text{in } \mathcal{M}_r)$$

on P^J. As usual, for $X \in \mathrm{P}^J$, $\widehat{\mathrm{K}^J_X}$ denotes the $\widehat{\mathscr{K}^J}$-class of the element X. Further, let $\mathrm{E}_J(\mathrm{Reg}(\mathcal{M}^J_{mn})) = \mathrm{E}_J(\mathcal{M}^J_{mn})$ denote the set consisting of all idempotents in

$\mathrm{Reg}(\mathcal{M}_{mn}^J)$ (naturally, these are all the idempotents of $\mathcal{M}_{mn}^J$, as well). For any subset $S \subseteq \mathcal{M}_{mn}^J$, let $\mathrm{E}_J(S)$ denote the set of all the idempotents contained in this subset.

Here, we need some information on the full linear monoid $\mathcal{M}_r$ and the general linear group $\mathcal{G}_r$. In the following lemma, we state those. For the non-referenced statements, we refer the reader to the monograph [103].

Lemma 4.2.14. *Let $X \in \mathcal{M}_r$ with $\mathrm{Rank}\, X = s$. In $\mathcal{M}_r$, we have*

(i) $\mathrm{R}_X = \{Y \in \mathcal{M}_r : \mathrm{Col}\, Y = \mathrm{Col}\, X\}$*;*

(ii) $\mathrm{L}_X = \{Y \in \mathcal{M}_r : \mathrm{Row}\, Y = \mathrm{Row}\, X\}$*;*

(iii) $\mathrm{H}_X = \{Y \in \mathcal{M}_r : \mathrm{Col}\, Y = \mathrm{Col}\, X,\ \mathrm{Row}\, Y = \mathrm{Row}\, X\}$*;*

(iv) $|\,\mathrm{H}_X\,| = |\,\mathcal{G}_s\,|$ *(see Lemma 4.1.3); furthermore, if H_X contains an idempotent, then $\mathrm{H}_X \cong \mathcal{G}_s$;*

(v) $\mathrm{D}_X = \mathrm{J}_X = \{Y \in \mathcal{M}_r : \mathrm{Rank}\, Y = \mathrm{Rank}\, X\}$*;*

(vi) we have $\mathrm{D}_{\mathrm{I}_r} = \mathrm{H}_{\mathrm{I}_r} \cong \mathcal{G}_r$, and $\mathcal{M}_r \setminus \mathcal{G}_r$ is an ideal of the semigroup $\mathcal{M}_r$.

*(vii) (*Erdos, [42]*) Each matrix $X \in \mathcal{M}_r$ with $\mathrm{Rank}\, X < r$ may be presented as a product of idempotents.*

*(viii) (*Waterhouse, [127]*) If $|\mathbb{F}| < \aleph_0$, then*

(a) $\mathrm{rank}(\mathcal{G}_1) = 1$*, and* $\mathrm{rank}(\mathcal{G}_r) = 2$ *if* $r \geq 2$*.*

(b) $\mathcal{M}_r = \langle \mathcal{G}_r \cup \{X\} \rangle$ *for any* $X \in \mathrm{D}_{r-1}(\mathcal{M}_r)$

(c) $\mathrm{rank}(\mathcal{M}_1) = 2$*, and* $\mathrm{rank}(\mathcal{M}_r) = 3$ *if* $r \geq 2$*.*

Note that parts $(i) - (iii)$ and $(v) - (vi)$ follow from Lemma 4.1.2 and the fact that $\mathrm{Rank}(AB) \leq \min(\mathrm{Rank}(A), \mathrm{Rank}(B))$.

Next, we aim to show the parallel of Theorem 3.1.26 for the semigroup $\mathcal{M}_{mn}^J$. In [30], this has been done for the case $|\mathbb{F}| = q < \aleph_0$. Here, we use a different argument and prove the result for any field $\mathbb{F}$. The results concerning the case $|\mathbb{F}| \geq \aleph_0$ are new, as far as the author is aware.

Theorem 4.2.15. *Let $X \in \mathrm{P}^J$ with $\mathrm{Rank}\, X = s$. Then*

(i) $\widehat{\mathrm{R}_X^J}$ *is the union of* $q^{s(m-r)}$ $\mathscr{R}^J$*-classes of* P^J*;*

(ii) $\widehat{\mathrm{L}_X^J}$ *is the union of* $q^{s(n-r)}$ $\mathscr{L}^J$*-classes of* P^J*;*

(iii) $\widehat{\mathrm{H}_X^J}$ *is the union of* $q^{s(m+n-2r)}$ $\mathscr{H}^J$*-classes of* P^J*, each of which has size* $|\,\mathcal{G}_s\,|$*;*

(iv) if $\mathrm{H}_{\overline{X}}$ is a non-group $\mathscr{H}$-class of $\mathcal{M}_r$, then each $\mathscr{H}^J$-class of P^J contained in $\widehat{\mathrm{H}_J^J}$ is a non-group;

(v) *if* $\mathrm{H}_{\overline{X}}$ *is a group* $\mathscr{H}$*-class of* $\mathcal{M}_r$*, then each* $\mathscr{H}^J$*-class of* P^J *contained in* $\widehat{\mathrm{H}_X^J}$ *is a group isomorphic to* $\mathcal{G}_s$*; further,* $\widehat{\mathrm{H}_X^J}$ *is a* $q^{s(m-r)} \times q^{s(n-r)}$ *rectangular group over* $\mathcal{G}_s$*, and its idempotents* $\mathrm{E}_J(\widehat{\mathrm{H}_X^J})$ *form a* $q^{s(m-r)} \times q^{s(n-r)}$ *rectangular band.*

(vi) $\widehat{\mathrm{D}_X^J} = \mathrm{D}_X^J = \mathrm{D}_s^J = \{Y \in \mathrm{P}^J : \mathrm{Rank}\, Y = s\}$ *is the union of:*

(a) $q^{s(m-r)} \begin{bmatrix} r \\ s \end{bmatrix}_q$ $\mathscr{R}^J$*-classes of* P^J*,*

(b) $q^{s(n-r)} \begin{bmatrix} r \\ s \end{bmatrix}_q$ $\mathscr{L}^J$*-classes of* P^J*,*

(c) $q^{s(m+n-2r)} \left(\begin{bmatrix} r \\ s \end{bmatrix}_q\right)^2$ $\mathscr{H}^J$*-classes of* P^J*.*

Proof. (i) Firstly, note that

$$|\widehat{\mathrm{H}_Y^J}/\mathscr{R}^J| = |(\widehat{\mathrm{R}_Y^J} \cap \widehat{\mathrm{L}_Y^J})/\mathscr{R}^J| = |\widehat{\mathrm{R}_Y^J}/\mathscr{R}^J|, \tag{4.3}$$

for any $Y \in \mathrm{P}^J$. Secondly, choose any $Q = \left[\begin{smallmatrix} A_c & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right] \in \mathcal{M}_{mn}$, where $A_c \in \mathcal{M}_r$ with $\mathfrak{c}_{s+1}(A_c) = \cdots = \mathfrak{c}_r(A_c) = \mathrm{O}$ and $\mathrm{Col}(A_c) = \mathrm{Col}(\overline{X})$. Note that $Q \in \widehat{\mathrm{R}_X^J}$ (by Lemma 4.2.14(i)), so equality (4.3) and Theorem 2.3.12(i) imply that it suffices to calculate the number of $\mathscr{R}^J$-classes containing an element Z with $\overline{Z} = \overline{Q}$. Such an element is of the form

$$\begin{bmatrix} A_c & A_cN \\ MA_c & MA_cN \end{bmatrix},$$

for some $M \in \mathcal{M}_{m-r,r}$ and $N \in \mathcal{M}_{r,n-r}$. Since each column of $\left[\begin{smallmatrix} A_cN \\ MA_cN \end{smallmatrix}\right]$ is a linear combination of the columns of the matrix $\left[\begin{smallmatrix} A_c \\ MA_c \end{smallmatrix}\right]$, the latter sub-matrix determines the $\mathscr{R}^J$-class of the whole matrix Z. Due to the properties of A_c, we have $MA_c \in \mathcal{M}_{m-r,r}$ with $\mathfrak{c}_{s+1}(MA_c) = \cdots = \mathfrak{c}_r(MA_c) = \mathrm{O}$. Since the number of matrices satisfying these conditions is $q^{(m-r)s}$, this is an upper bound for the number of $\mathscr{R}^J$-classes in $\widehat{\mathrm{H}_Q^J}$. Moreover, it turns out to be the exact value! We prove this in two steps:

- First, we show that any matrix $T \in \mathcal{M}_{m-r,r}$ with $\mathfrak{c}_{s+1}(T) = \cdots = \mathfrak{c}_r(T) = \mathrm{O}$ may be generated as the product MA_c, for some $M \in \mathcal{M}_{m-r,r}$.
- Then, we show that $\mathrm{Col}\left(\left[\begin{smallmatrix} A_c \\ M_1A_c \end{smallmatrix}\right]\right) \neq \mathrm{Col}\left(\left[\begin{smallmatrix} A_c \\ M_2A_c \end{smallmatrix}\right]\right)$ if and only if $M_1A_c \neq M_2A_c$.

Recall that each row of MA_c is a linear combination of the rows of A_c, and vice versa, each matrix $P \in \mathcal{M}_{m-r,r}$ whose rows are linear combinations of the rows of A_c, may be presented as a product MA_c for some $M \in \mathcal{M}_{m-r,r}$. Since $\mathrm{Rank}\, A_c = s$ and the last $r - s$ columns are zero-columns, we have $\mathrm{Row}(A_c) = W_{rs}$; hence, by adjusting the auxiliary matrix M, any matrix $P \in \mathcal{M}_{m-r,r}$, whose rows belong to W_{rs}, may be obtained as a product MA_c. This completes the proof for the first step. Let us prove the second. Note that the direct implication is obvious. For the reverse, assume that $M_1A_c \neq M_2A_c$. Then, the two matrices differ in at least one "coordinate", say (i, j). However, then $\mathfrak{c}_j(\left[\begin{smallmatrix} A_c \\ M_1 A_c \end{smallmatrix}\right]) \notin \mathrm{Col}(\left[\begin{smallmatrix} A_c \\ M_2 A_c \end{smallmatrix}\right])$. Let us elaborate: we have $\mathfrak{c}_j(\left[\begin{smallmatrix} A_c \\ M_1 A_c \end{smallmatrix}\right]) - \mathfrak{c}_j(\left[\begin{smallmatrix} A_c \\ M_2 A_c \end{smallmatrix}\right]) = w$, where $w \in V_m$ has 0's in the first r coordinates and a non-zero element in the $r+i$-th coordinate; since the columns of A_c are linearly independent, the columns of $\left[\begin{smallmatrix} A_c \\ M_2 A_c \end{smallmatrix}\right]$ cannot generate a non-zero vector having 0's in the first r positions.

Part (ii) is dual. Since $\widehat{\mathrm{H}^J_Y} = \widehat{\mathrm{R}^J_Y} \cap \widehat{\mathrm{L}^J_Y}$, (iii) follows directly from (i), (ii), Theorem 2.3.12(i) and Lemma 4.2.14(iv).

(iv) and (v). In the proofs of (i) and (ii), we showed that

$$r = |\widehat{\mathrm{H}^J_Y} / \mathscr{R}^J\,| = q^{s(m-r)} \text{ and } l = |\widehat{\mathrm{H}^J_Y} / \mathscr{L}^J\,| = q^{s(n-r)}.$$

Moreover, by Lemma 4.2.14(iv), each group $\mathscr{H}$-class of rank s in $\mathcal{M}_r$ is isomorphic to $\mathcal{G}_s$. Thus, Theorem 2.3.12 implies the statements.

(vi). From Lemma 2.3.9(iv) it follows that $\widehat{\mathrm{D}^J_X} = \mathrm{D}^J_X$, so the characterisation of the $\widehat{\mathscr{D}^J}$-classes follows from Proposition 4.2.13. We need to prove the rest. It suffices to show only (a), because (b) is dual, and (c) follows directly from (a) and (b). From (i), it follows that each $\widehat{\mathscr{R}^J}$-class in $\widehat{\mathrm{D}^J_X}$ contains $q^{s(m-r)}$ $\mathscr{R}^J$-classes. Furthermore, by Remark 2.3.13 and Proposition 4.1.4(i), we have

$$|\widehat{\mathrm{D}^J_X} / \widehat{\mathscr{R}^J}| = |\,\mathrm{D}_s(\mathcal{M}_r) / \mathscr{R}\,| = \begin{bmatrix} r \\ s \end{bmatrix}_q.$$

Thus, the product of the two values is the number of $\mathscr{R}^J$-classes in $\widehat{\mathrm{D}^J_X}$. □

Remark 4.2.16. As promised, Theorem 4.2.15 applies even when $\mathbb{F}$ is infinite. In that case, we calculate the values using the laws of calculating with infinite cardinals and the notions defined in Sections 3.1 and 4.1. For instance, since s, m and r are finite, $\widehat{\mathrm{R}^J_X}$ is the union of $q^{s(m-r)} = q$ $\mathscr{R}^J$-classes of P^J.

Directly from Theorem 4.2.15, we may conclude the following:

$$|\,\mathrm{D}^J_s\,| = \begin{cases} q^{s(m+n-2r)}(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]_q)^2 \cdot q^{\binom{s}{2}}(q-1)^s[s]_q!, & q < \aleph_0; \\ q, & q \geq \aleph_0 \text{ and } s \geq 1; \\ 1, & q \geq \aleph_0 \text{ and } s = 0. \end{cases}$$

and

$$\begin{aligned} |\,\mathrm{P}^J\,| &= \sum_{s=0}^{r} |\,\mathrm{D}^J_s\,| \\ &= \begin{cases} \sum_{s=0}^{r} q^{s(m+n-2r)}(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]_q)^2 \cdot q^{\binom{s}{2}}(q-1)^s[s]_q!, & q < \aleph_0; \\ q, & q \geq \aleph_0 \text{ and } r \neq 0. \\ 1, & q \geq \aleph_0 \text{ and } r = 0. \end{cases} \end{aligned}$$

Thus, $|\,\mathrm{P}^J\,|$ is infinite if and only if $|\mathbb{F}| = q \geq \aleph_0$ and $\mathrm{Rank}\, J > 0$. In that case, the two cardinalities are equal.

As in the previous chapter, now we turn to the problem of calculating the rank of P^J. In order to apply Theorem 2.4.16, we need to show that P^J is MI-dominated. Thus, we prove a lemma from [30] characterising the mid-identities, regularity preserving elements, and the idempotents of P^J, and then we show that each idempotent is $\preceq$-below a mid-identity.

Lemma 4.2.17.

(i) $\mathrm{E}_J(\mathcal{M}^J_{mn}) = \mathrm{E}_J(\mathrm{P}^J)$
$$= \{[M, A, N] : A \in \mathrm{E}(\mathcal{M}_r), M \in \mathcal{M}_{m-r,r}, N \in \mathcal{M}_{r,n-r}\}.$$

(ii) $\mathrm{MI}(\mathrm{Reg}(\mathcal{M}^J_{mn})) = \{[M, \mathrm{I}_r, N] : M \in \mathcal{M}_{m-r,r}, N \in \mathcal{M}_{r,n-r}\}$.

(iii) $\mathrm{RP}(\mathrm{Reg}(\mathcal{M}^J_{mn})) = \mathrm{D}^J_r$.

Proof. For (i), note that Lemma 2.3.11 gives $\mathrm{E}_J(\mathcal{M}^J_{mn}) = (\mathrm{E}(\mathcal{M}_r))\varphi^{-1}$, so the statement follows.

Since $\mathcal{M}$ is a stable and regular (by Propositions 4.1.1 and 4.1.5) partial semigroup, from Proposition 2.4.10 follows that $\mathrm{MI}(\mathrm{P}^J) = \mathrm{E}_J(\mathrm{J}^J_K)$ and $\mathrm{RP}(\mathrm{P}^J) = \mathrm{J}^J_K$. Since $\mathrm{J}^J_K = \mathrm{D}^J_K$, we immediately obtain (iii). To show (ii), note that $\mathrm{Rank}[M, A, N] = \mathrm{Rank}\, A$; hence, part (i) and Lemma 4.2.14(vi) imply

$$\begin{aligned}\mathrm{MI}(\mathrm{P}^J) = \mathrm{E}_J(\mathrm{D}^J_K) &= \{[M, A, N] : A \in \mathrm{E}_J(\mathrm{D}_r), M \in \mathcal{M}_{m-r,r}, N \in \mathcal{M}_{r,n-r}\}\\ &= \{[M, \mathrm{I}_r, N] : M \in \mathcal{M}_{m-r,r}, N \in \mathcal{M}_{r,n-r}\}.\end{aligned} \qquad \square$$

The following Proposition was not stated explicitly in [30]. However, one may discern an implicit proof of MI-domination in the proof of Theorem 6.10.

Proposition 4.2.18.

(i) The semigroup $\mathrm{P}^J = \mathrm{Reg}(\mathcal{M}^J_{mn})$ *is MI-dominated.*

(ii) The semigroup $\mathrm{P}^J = \mathrm{Reg}(\mathcal{M}^J_{mn})$ *is RP-dominated.*

Proof. (i) By Proposition 2.4.5(iv), it suffices to prove that each element of P^J belongs to

$$E \star_J \mathrm{P}^J \star_J E = E \cdot J\, \mathrm{P}^J\, J \cdot E, \qquad \text{for some } E \in \mathrm{MI}(\mathrm{P}^J).$$

Let $X \in \mathrm{P}^J$ be arbitrary. Proposition 4.2.3(iii) implies that $X = [M, A, N] = \left[\begin{smallmatrix} A & AN \\ MA & MAN \end{smallmatrix}\right]$ for some $A \in \mathcal{M}_r$, $M \in \mathcal{M}_{m-r,r}$ and $N \in \mathcal{M}_{r,n-r}$. Let $E = [M, \mathrm{I}_r, N]$. Clearly, $E \in \mathrm{MI}(\mathrm{P}^J)$ by Lemma 4.2.17(i). Then,

$$EJXJE = [M, \mathrm{I}_r, N] \star_J [M, A, N] \star_J [M, \mathrm{I}_r, N] = [M, \mathrm{I}_r\, A\, \mathrm{I}_r, N] = [M, A, N].$$

(ii) Having proved part (i), we may apply Proposition 2.4.8 for the second part. First, we recall that for each $E \in \mathrm{MI}(\mathrm{P}^J)$, the local monoid $E \star_J \mathrm{P}^J \star_J E$ is isomorphic to the semigroup $W \cong \mathcal{M}_r$ (see Proposition 2.4.11 and the discussion following it). Thus, it suffices to prove that $\mathcal{M}_r = \mathcal{G}_r \cdot \mathrm{E}(\mathcal{M}_r)$. Let $X \in \mathcal{M}_r$ and let $\mathcal{B} = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_r\}$ be a basis for V_r such that $\{u_{k+1}, \ldots, u_r\}$ is a basis for $\ker(\lambda_X)$. Then, extend the linearly independent set $\{u_1\lambda_X, \ldots, u_k\lambda_X\}$ arbitrarily to a basis $\{u_1\lambda_X, \ldots, u_k\lambda_X, v_{k+1}, \ldots, v_r\}$ for V_r. Now, define $\alpha \in \mathrm{Aut}(V_r)$ as the linear transformation satisfying

$$u_i\alpha = u_i\lambda_X, \quad \text{and} \quad u_j\alpha = v_j$$

for all $1 \leq i \leq k$ and $r+1 \leq j \leq r$. Also, we define $\beta \in \mathrm{End}(V_r)$ as the linear transformation satisfying

$$u_i\lambda_X\beta = u_i\lambda_X, \quad \text{and} \quad v_j\beta = \mathbf{0}$$

for all $1 \leq i \leq k$ and $r+1 \leq j \leq r$. Thus, the matrix corresponding α has rank r, the matrix corresponding β is an idempotent, and we have $\alpha\beta = \lambda_X$. □

Finally, we are ready to calculate the rank of P^J. This result was proved in [30] as Theorem 6.10, under the assumption that $q \leq \aleph_0$. Here, we apply Theorem 2.4.16 and include the case $q \geq \aleph_0$, as well.

Theorem 4.2.19. *Suppose $r \geq 1$.*

(i) If $q > \aleph_0$, then $\mathrm{rank}(\mathrm{P}^J) = |\,\mathrm{P}^J\,| = q$.

(ii) If $q \leq \aleph_0$, $L = \max(m,n)$, and $m = n = r$ does not hold, then

$$\mathrm{rank}(\mathrm{P}^J) = q^{r(L-r)} + 1.$$

Proof. If $q > \aleph_0$ and $r \geq 1$, then $|\,\mathrm{P}^J\,| = q$ (as discussed below Remark 4.2.16), so P^J cannot be generated by a set of smaller cardinality. Now, suppose $q \leq \aleph_0$. Recall that $\mathcal{M}$ is regular (Proposition 4.1.5), P^J is MI-dominated (Proposition 4.2.18(i)), and $\mathcal{M}_r \setminus \mathcal{G}_r$ is an ideal of $\mathcal{M}_r$ (Lemma 4.2.14(vi)); therefore, Theorem 2.4.16 gives

$$\mathrm{rank}(\mathrm{P}^J) = \mathrm{rank}(\mathcal{M}_r : \mathcal{G}_r) + \max(|\widehat{\mathrm{H}^J_Y} / \mathscr{R}^J\,|, |\widehat{\mathrm{H}^J_Y} / \mathscr{L}^J\,|, \mathrm{rank}(\mathcal{G}_r)).$$

Here, we have $|\widehat{\mathrm{H}^J_Y} / \mathscr{R}^J\,| = q^{s(m-r)}$ and $|\widehat{\mathrm{H}^J_Y} / \mathscr{L}^J\,| = q^{s(n-r)}$, as calculated in the proof for parts (i) and (ii) of Theorem 4.2.15. Now, Lemma 4.2.14($viii$) implies $\mathrm{rank}(\mathcal{M}_r : \mathcal{G}_r) = 1$ and $\mathrm{rank}(\mathcal{G}_r) \leq 2 \leq q$ for all $r \in \mathbb{N}$. Since $m = n = r$ does not hold, we have $r < \max(m,n) = L$, so

$$\mathrm{rank}(\mathrm{P}^J) = \mathrm{rank}(\mathcal{M}_r : \mathcal{G}_r) + \max(q^{s(m-r)}, q^{s(n-r)}, \mathrm{rank}(\mathcal{G}_r)) = 1 + q^{r(L-r)}. \quad □$$

Remark 4.2.20. In the case $m = n = r$, we have $\mathcal{M}^J_{mn} \cong \mathcal{M}_r$, and Proposition 4.2.3 gives $\mathrm{P}^J = \mathrm{P}^J_3 = \mathcal{M}_r$. Hence, $\mathrm{rank}(\mathrm{P}^J) = \mathrm{rank}(\mathcal{M}_r)$, which is stated in Lemma 4.2.14 ($viii$). The only case remaining is $r = 0$. Then, we have $\mathrm{P}^J = \{\mathrm{O}_{mn}\}$, so $\mathrm{rank}(\mathrm{P}^J) = 1$.

In Proposition 4.2.1(ii), we have given a sufficient condition for two sandwich semigroups to be isomorphic. Now, we have gathered enough information to classify the isomorphism classes of finite linear sandwich semigroups, as in [30].

Theorem 4.2.21. *Let* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *be two finite fields with* $|\mathbb{F}_1| = q_1$ *and* $|\mathbb{F}_2| = q_2$*. Further, let* $m, n, k, l \in \mathbb{N}$ *and let* $A \in \mathrm{D}_r(\mathcal{M}_{mn})$ *and* $B \in \mathrm{D}_s(\mathcal{M}_{lk})$*. Then, the following are equivalent*

- *(i)* $\mathcal{M}^A_{mn}(\mathbb{F}_1) \cong \mathcal{M}^B_{kl}(\mathbb{F}_2)$,
- *(ii) one of the following holds*
  - *(a)* $r = s = 0$ *and* $q_1^{mn} = q_2^{kl}$*, or*
  - *(b)* $r = s \geq 1$*,* $(m, n) = (k, l)$*, and* $q_1 = q_2$*.*

Further, if $r \geq 1$*, then* $\mathcal{M}^A_{mn}(\mathbb{F}_1) \cong \mathcal{M}^B_{kl}(\mathbb{F}_2)$ *if and only if* $\mathrm{Reg}(\mathcal{M}^A_{mn}(\mathbb{F}_1)) \cong \mathrm{Reg}(\mathcal{M}^B_{kl}(\mathbb{F}_2))$*.*

Proof. Note that, in the case where $r \neq s$, we have $\mathcal{M}^A_{mn}(\mathbb{F}_1) \ncong \mathcal{M}^B_{kl}(\mathbb{F}_2)$, because (by Proposition 4.2.7) the first semigroup has $r+1$ regular $\mathscr{D}^A$-classes, whereas the second has $s+1$ regular $\mathscr{D}^B$-classes. Thus, in this case the two sandwich semigroups cannot be isomorphic. So, suppose $r = s$. If $r = s = 0$, then $\mathcal{M}^A_{mn}(\mathbb{F}_1)$ and $\mathcal{M}^B_{kl}(\mathbb{F}_2)$ are both zero-semigroups (for any two elements, the product is always the zero-matrix). Clearly, two such semigroups are isomorphic if and only if their sizes are equal. Of course, we have $|\mathcal{M}^A_{mn}(\mathbb{F}_1)| = q_1^{mn}$ and $|\mathcal{M}^B_{kl}(\mathbb{F}_2)| = q_2^{kl}$, so $(a) \Rightarrow (i)$.

Now, suppose $r = s \geq 1$. We assume there exists an isomorphism, examine the structure of the regular subsemigroups P^A and P^B and draw conclusions. Firstly, from Theorem 4.2.15(v) follows that any group $\mathscr{H}^A$-class ($\mathscr{H}^B$-class) in P^A (P^B) is isomorphic to the group $\mathcal{G}_1(\mathbb{F}_1) \cong \mathbb{F}_1^\times$ (resp. $\mathcal{G}_1(\mathbb{F}_2) \cong \mathbb{F}_2^\times$) of cardinality $q_1 - 1$ ($q_2 - 1$). Thus, the two sandwich semigroups can be isomorphic only if $q_1 = q_2$, so we write $q = q_1 = q_2$. Secondly, part (vi) of the same Theorem implies that the class D_1^A (D_1^B) in P^A (P^B) contains $q^{m-r}\begin{bmatrix} r \\ 1 \end{bmatrix}_q$ ($q^{k-r}\begin{bmatrix} r \\ 1 \end{bmatrix}_q$) $\mathscr{R}^A$-classes ($\mathscr{R}^B$-classes). Hence, the equality $m = k$ is necessary for the two sandwich semigroups to be isomorphic. By a dual argument, so is $n = l$. Therefore, we have proved that $(i) \Rightarrow (a) \vee (b)$.

Since $(b) \Rightarrow (i)$ follows from Proposition 4.2.1(ii), the equivalence of (i) and (ii) is proved. In the last statement, it is obvious that $\mathcal{M}^A_{mn}(\mathbb{F}_1) \cong \mathcal{M}^B_{kl}(\mathbb{F}_2)$ implies $\mathrm{Reg}(\mathcal{M}^A_{mn}(\mathbb{F}_1)) \cong \mathrm{Reg}(\mathcal{M}^B_{kl}(\mathbb{F}_2))$. For the converse, we use the contrapositive: if $\mathcal{M}^A_{mn}(\mathbb{F}_1) \ncong \mathcal{M}^B_{kl}(\mathbb{F}_2)$ and $r \geq 1$, then the equivalence proved above gives $\neg(b)$; if we negate either of the two equalities, then we have $\mathrm{Reg}(\mathcal{M}^A_{mn}(\mathbb{F}_1)) \ncong \mathrm{Reg}(\mathcal{M}^B_{kl}(\mathbb{F}_2))$, by the discussion in the previous paragraph. □

Remark 4.2.22. As expected, the infinite case is rather more complicated. The reasoning above does not work since the fact that two fields have isomorphic multiplicative groups in this case does not imply their being isomorphic. Take, for instance, the fields $\mathbb{Q}$ and $\mathbb{Z}_3(x)$. They are clearly non-isomorphic, having different characteristics. However, we will show that their multiplicative groups are isomorphic. First, let P denote the set of all primes, and, for any $q \in \mathbb{Q}^\times$, consider the following (unique) decomposition

$$q = s \cdot \prod_{p \in P} p^{n_p},$$

where $s \in \{-1, +1\}$ and $n_p \in \mathbb{Z}$ (only a finite number of them being non-zero) for all $p \in \mathbb{P}$. Since the set of primes is countably infinite, this decomposition establishes an isomorphism

$$\mathbb{Q}^\times \to \mathbb{Z}_2 \oplus \bigoplus_{i \in \mathbb{N}} \mathbb{Z} \quad (\cong \mathbb{Z}_2 \oplus F),$$

where F is a free abelian group of countably infinite rank. On the other hand, if A is the set of all irreducible polynomials in $\mathbb{Z}_3[x]$ with the leading coefficient 1, then A is countably infinite and every element $f \in \mathbb{Z}_3(x)^\times$ may be written in a unique manner as

$$f(x) = w \cdot \prod_{p \in A} p(x)^{n_p},$$

where $w \in \mathbb{Z}_2$ and $n_p \in \mathbb{Z}$ (only a finite number of them being non-zero) for all $p \in A$. Again, this establishes an isomorphism

$$\mathbb{Z}_3(x)^\times \to \mathbb{Z}_2 \oplus \bigoplus_{i \in \mathbb{N}} \mathbb{Z} \quad (\cong \mathbb{Z}_2 \oplus F).$$

Therefore, $\mathbb{Q}^\times \cong \mathbb{Z}_3(x)^\times$.

Thus, the case $q \geq \aleph_0$ requires a different approach. Since the general theory in Chapter 2 does not advance our knowledge on this front, we leave this problem open, as did the authors of [30]. However, we present some conclusions made in that article:

1. If $m, n, k, l \in \mathbb{N}$, $|\mathbb{F}_1| = |\mathbb{F}_2|$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = 0$, then $\mathcal{M}^A_{mn}(\mathbb{F}_1) \cong \mathcal{M}^B_{kl}(\mathbb{F}_2)$ since both are zero semigroups of size $|\mathbb{F}_1| = |\mathbb{F}_2|$.

2. If $\mathbb{F}_1^\times \cong \mathbb{F}_2^\times$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = 1$, then $J = J_{nm1} = \left[\begin{smallmatrix} 1 & O \\ O & O \end{smallmatrix}\right]$, so

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \star_J \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}a_{m1} & a_{m2}b_{m2} & \cdots & a_{mn}b_{mn} \end{bmatrix}$$

   and hence $\mathcal{M}^A_{mn}(\mathbb{F}_1) \cong \mathcal{M}^B_{mn}(\mathbb{F}_2)$.

3. If $\mathcal{M}^A_{mn}(\mathbb{F}_1) \cong \mathcal{M}^B_{kl}(\mathbb{F}_2)$ then $\operatorname{Rank}(A) = \operatorname{Rank}(B)$ (as in the proof of Theorem 4.2.21).

4. If $\mathcal{M}^A_{mn}(\mathbb{F}_1) \cong \mathcal{M}^B_{kl}(\mathbb{F}_2)$ and $\operatorname{Rank}(A) = \operatorname{Rank}(B) = r \geq 2$, then $\mathbb{F}_1 \cong \mathbb{F}_2$. Namely, Theorem 4.2.15$(v)$ states that the maximal subgroups of $\mathcal{M}^A_{mn}$ are isomorphic to $\mathcal{G}_s(\mathbb{F}_1)$ for $0 \leq s \leq r$, and in [27] it has been proved that $\mathcal{G}_s(\mathbb{F}_1) \cong \mathcal{G}_s(\mathbb{F}_1)$ implies $\mathbb{F}_1 \cong \mathbb{F}_2$ for $s \geq 2$.

Remark 4.2.23. Note that, regardless of the cardinality of the field $\mathbb{F}$, the following is true: for $A, B \in \mathcal{M}_{nm}$, we have $\mathcal{M}^A_{mn} \cong \mathcal{M}^B_{mn}$ if and only if $\operatorname{Rank} A = \operatorname{Rank} B$. This is an earlier result, proved in [66].

We close this subsection by discussing the simplifications occurring in the case that $r = \min(m, n)$.

Remark 4.2.24.

- If $J = J_{nmm}$, then $\mathcal{M}^J_{mn} \cong \mathcal{R}_n(m)$. Since $r = m$, Theorem 4.2.15(i) gives $\widehat{\mathscr{R}^J} = \mathscr{R}^J$; furthermore, part ($v$) of the same theorem implies that, if $X \in \mathcal{M}_{mn}$ is a matrix with $\operatorname{Rank} X = s$ and $\mathrm{H}_{\overline{X}}$ is a group $\mathscr{H}$-class of $\mathcal{M}_r$, then $\widehat{\mathrm{H}^J_X}$ is a $1 \times q^{s(n-m)}$ rectangular group over $\mathcal{G}_s$. Of course, the equality $r = m$ simplifies somewhat the rest of the formulas, as well. Most significantly, from Theorem 4.2.19, we conclude that

$$\operatorname{rank}(\operatorname{Reg}(\mathcal{R}_n(m))) = \begin{cases} q, & \text{if } q \geq \aleph_0; \\ q^{m(n-m)} + 1, & \text{if } q < \aleph_0 \text{ and } n \neq m. \end{cases}$$

- If $J = J_{nmn}$, then $\mathcal{M}^J_{mn} \cong \mathcal{C}_m(n)$. Naturally, the results are dual to the ones in the previous case.

4.2.4 Idempotents and idempotent-generation

In Lemma 4.2.17(i), we characterised the idempotents of $\mathcal{M}^J_{mn}$. Here, we enumerate them, describe the idempotent-generated subsemigroup of $\mathcal{M}^J_{mn}$ and calculate its rank.

In order to present these results, we will need some more information on the idempotents and the idempotent-generated subsemigroup of $\mathcal{M}_r$. The first statement of the following proposition was proved in [30], and the second one may be easily deduced from that proof. Similarly, part (iv) was proved in [25], and from that proof (more specifically, from Lemma 2.4 of [25]) we may infer (v).

Let $\mathbb{E}(\mathcal{M}_r)$ and $\mathbb{E}_J(\mathcal{M}^J_{mn})$ denote the idempotent-generated subsemigroups of $\mathcal{M}_r$ and $\mathcal{M}^J_{mn}$, respectively.

Proposition 4.2.25.

(i) If $|\mathbb{F}| = q < \aleph_0$, *then*

$$|\operatorname{E}(\mathcal{M}_r)| = \sum_{s=0}^{r} |\operatorname{E}(\mathrm{D}_s(\mathcal{M}_r))| = \sum_{s=0}^{r} q^{s(r-s)} \begin{bmatrix} r \\ s \end{bmatrix}_q . \tag{4.4}$$

(ii) If $|\mathbb{F}| = q \geq \aleph_0$, *then* $|\operatorname{E}(\mathcal{M}_r)| = \begin{cases} q, & \text{if } r \geq 1; \\ 2, & \text{if } r = 1; \\ 1, & \text{if } r = 0. \end{cases}$

*(iii) (*Erdos, [42]*)* $\mathbb{E}(\mathcal{M}_r) = \langle \operatorname{E}(\mathcal{M}_r) \rangle = (\mathcal{M}_r \setminus \mathcal{G}_r) \cup \{\mathrm{I}_r\}$.

(iv) (Dawlings, [25]) *If* $|\mathbb{F}| = q < \aleph_0$ *and* $r \geq 1$*, then*

$$\operatorname{rank}(\mathcal{M}_r \setminus \mathcal{G}_r) = \operatorname{idrank}(\mathcal{M}_r \setminus \mathcal{G}_r) = \frac{q^r - 1}{q - 1},$$

so $\operatorname{rank}(\mathbb{E}(\mathcal{M}_r)) = \operatorname{idrank}(\mathbb{E}(\mathcal{M}_r)) = \frac{q^r-1}{q-1} + 1$.

(v) Suppose $|\mathbb{F}| = q \geq \aleph_0$.

- *If* $r = 1$*, then* $\operatorname{rank}(\mathbb{E}(\mathcal{M}_r)) = \operatorname{idrank}(\mathbb{E}(\mathcal{M}_r)) = 2$.
- *If* $r \geq 2$*, then* $\operatorname{rank}(\mathcal{M}_r \setminus \mathcal{G}_r) = q$*, so* $\operatorname{rank}(\mathbb{E}(\mathcal{M}_r)) = \operatorname{idrank}(\mathbb{E}(\mathcal{M}_r)) = q + 1 = q$.

Proof. We prove only parts (i), (ii) and (v) since the other two were stated together with the corresponding references.

(i) and (ii). To calculate the number of idempotents in $\mathcal{M}_r$, we enumerate the idempotent endomorphisms of V_r. From Proposition 3.2.16(i), (by fixing $X = Y = V_r$ and $a = \operatorname{id}_X$) we deduce that

$$\operatorname{E}(V_r) = \{\alpha \in \operatorname{End}(V_r) : \alpha\restriction_{\operatorname{im}\alpha} = \operatorname{id}_{\operatorname{im}\alpha}\}.$$

To fix an idempotent of rank s, first we specify its image of rank s, i.e. an s-dimensional subspace W of V_r. By the discussion preceding Proposition 4.1.4, this may be done in $\begin{bmatrix} r \\ s \end{bmatrix}_q$ ways. Let $\mathcal{B} = \{v_1, \ldots, v_r\}$ be a basis for V_r such that $\mathcal{B}_1 = \{v_1, \ldots, v_s\}$ is a basis for W. Having fixed the image W, we know that the idempotent must map $\mathcal{B}_1$ identically, and we need to define how it maps the elements $v_{s+1}, \ldots, v_r$. Of course, the images of these elements have to be in W. Thus, any of the q^s elements of the space W (i.e. linear combinations of the elements of $\mathcal{B}_1$) is a possible image. Since the rank of an idempotent in $\operatorname{End}(V_r)$ may be any integer from 0 to r, we have proved (4.4) regardless of $|\mathbb{F}|$ being finite or infinite, and both statements follow directly.

(v) In the case $r = 1$, we clearly have $\operatorname{E}(\mathcal{M}_r) = \operatorname{E}(q) = \{0, 1\}$ (the only solutions of the equation $x^2 - x = 0$ in any field) so $\mathbb{E}(\mathcal{M}_r) = \operatorname{E}(\mathcal{M}_r)$ and $\operatorname{rank}(\mathbb{E}(\mathcal{M}_r)) = \operatorname{idrank}(\mathbb{E}(\mathcal{M}_r)) = 2$. Now, suppose $r \geq 2$. In Lemma 2.4 of [25], Dawlings proved that any generating set $E' \subseteq \operatorname{E}(\mathcal{M}_r)$ of $\mathcal{M}_r \setminus \mathcal{G}_r$ necessarily covers the principal factor PF_{r-1} of $\mathcal{M}_r$ containing the maps of rank $r - 1$. The semigroup $\operatorname{PF}_{r-1} = (\operatorname{D}_{r-1} \cup \{0\}, \cdot)$ is defined in the following way: for all $s, t \in \operatorname{D}_{r-1}$, let $s \cdot 0 = 0 \cdot s = 0$, and

$$st = \begin{cases} st, & st \in \operatorname{D}_{r-1}; \\ 0, & st \notin \operatorname{D}_{r-1}. \end{cases}$$

(for the background on principal factors, see [58]). The term "E' covers PF_{r-1}" means that E' has a non-empty intersection with each non-zero $\mathscr{L}$-class and each non-zero $\mathscr{R}$-class of PF_{r-1}. However, since the Green's relations (of non-zero elements) of PF_{r-1} clearly hold in $\mathcal{M}_r$, as well, we have

$$\operatorname{rank}(\mathcal{M}_r \setminus \mathcal{G}_r) \geq |\operatorname{PF}_{r-1} / \mathscr{R}| - 1 \geq |\operatorname{D}_{r-1} / \mathscr{R}| = q,$$

the last equality following from $q \geq \aleph_0$ and Proposition 4.1.4(i). □

Recall that, in the case where $r = \operatorname{Rank} J = 0$, we have $J = \mathrm{O}_{nm}$. Hence, $\mathrm{P}^J = \mathrm{E}_J(\mathrm{P}^J) = \{\mathrm{O}_{mn}\}$, rendering any further investigation of idempotents and the idempotent-generated subsemigroup redundant. Similarly, if $m = n = r$, we have $\mathcal{M}_{mn}^J = \mathcal{M}_{mn}^{\mathrm{I}_r} \cong \mathcal{M}_r$, so Proposition 4.2.25 describes the idempotents and the idempotent-generated subsemigroup in this case. For these reasons, for the rest of this subsection, we assume that $m = n = r$ is not the case and that $r > 0$.

In the following result, we calculate the size of $\mathrm{E}_J(\mathcal{M}_{mn}^J)$ using Proposition 4.2.25(i) and (ii). The first part of the proposition was proved in [30], and the second part is a new addition.

Proposition 4.2.26.

(i) If $|\mathbb{F}| = q < \aleph_0$, *then*

$$|\,\mathrm{E}_J(\mathcal{M}_{mn}^J)| = \sum_{s=0}^{r} q^{s(m+n-r-s)} \begin{bmatrix} r \\ s \end{bmatrix}_q . \tag{4.5}$$

(ii) If $|\mathbb{F}| = q \geq \aleph_0$, $r \geq 1$, *and* $m = n = r$ *is not the case, then* $|\,\mathrm{E}_J(\mathcal{M}_{mn}^J)| = q$.

Proof. First, we calculate the number of idempotents of rank s in P^J, and then we sum over all the possible values of s (we have $0 \leq s \leq r$). Recall that $\operatorname{Rank} A = \operatorname{Rank} \overline{A}$ for any $A \in \mathrm{P}^J$. Thus, each idempotent of rank s in P^J corresponds to some idempotent of the same rank in $\mathcal{M}_r$. More precisely, Theorem 4.2.15(v) implies that for each idempotent from the set $\mathrm{E}(\mathrm{D}_s(\mathcal{M}_r))$ there exist $q^{s(m-r)} \cdot q^{s(n-r)}$ idempotents mapping to it. In the process of proving Proposition 4.2.25(i), we showed that (4.4) holds regardless of $q = |\mathbb{F}|$ being finite or infinite. Hence, (4.5) holds regardless of $q = |\mathbb{F}|$ being finite or infinite. Thus, we proved (i), and (ii) follows from the fact that $m + n - 2r > 0$ (which holds because $m = n = r$ is not the case). □

Next, we characterise the idempotent-generated subsemigroup and calculate its rank and idempotent rank. Both results were proved in [30]. Here, we consider the case $q \geq \aleph_0$, as well, and provide simplified proofs, applying the general theory presented in Chapter 2.

Theorem 4.2.27. *In* $\mathcal{M}_{mn}^J$, *we have*

$$\mathbb{E}_J(\mathcal{M}_{mn}^J) = \langle \mathrm{E}_J(\mathcal{M}_{mn}^J)\rangle_J = (\mathrm{P}^J \setminus \mathrm{D}_r^J) \cup \mathrm{E}_J(\mathrm{D}_r^J).$$

Proof. From Theorem 2.3.15 and Proposition 4.2.25(iii) it follows that

$$\begin{aligned}\mathbb{E}_J(\mathcal{M}_{mn}^J) &= (\mathbb{E}(\mathcal{M}_r))\varphi^{-1} = \big((\mathcal{M}_r \setminus \mathcal{G}_r) \cup \{\mathrm{I}_r\}\big)\varphi^{-1} \\ &= (\mathcal{M}_r \setminus \mathcal{G}_r)\varphi^{-1} \cup (\mathrm{I}_r)\varphi^{-1} = (\mathrm{P}^J \setminus \mathrm{D}_r^J) \cup \mathrm{E}_J(\mathrm{D}_r^J).\end{aligned}$$

□

Theorem 4.2.28. *Suppose $r \geq 1$.*

(i) If $|\mathbb{F}| = q < \aleph_0$, and $L = \max(m, n)$, then

$$\operatorname{rank}(\mathbb{E}_J(\mathcal{M}_{mn}^J)) = \operatorname{idrank}(\mathbb{E}_J(\mathcal{M}_{mn}^J)) = q^{r(L-r)} + \frac{q^r - 1}{q - 1}.$$

(ii) If $|\mathbb{F}| = q \geq \aleph_0$ and $m = n = r$ is not the case, then

$$\operatorname{rank}(\mathbb{E}_J(\mathcal{M}_{mn}^J)) = \operatorname{idrank}(\mathbb{E}_J(\mathcal{M}_{mn}^J)) = q.$$

Proof. Recall that P^J is MI-dominated (by Proposition 4.2.18(i)) and that Theorem 4.2.15 implies $|\widehat{\mathrm{H}_K^J}/\mathscr{R}^J| = q^{r(m-r)}$ and $|\widehat{\mathrm{H}_K^J}/\mathscr{L}^J| = q^{r(n-r)}$. Hence, Theorem 2.4.17 gives

$$\operatorname{rank}(\mathbb{E}_J(\mathrm{P}^J)) = \operatorname{rank}(\mathbb{E}(\mathcal{M}_r)) + \max(q^{r(m-r)}, q^{r(n-r)}) - 1 \quad \text{and}$$
$$\operatorname{idrank}(\mathbb{E}_J(\mathrm{P}^J)) = \operatorname{idrank}(\mathbb{E}(\mathcal{M}_r)) + \max(q^{r(m-r)}, q^{r(n-r)}) - 1.$$

Therefore, parts (i) and (ii) follow immediately from Proposition 4.2.25(iv) and (v). □

Naturally, we may apply the results of this subsection to obtain information on the idempotents and the idempotent-generated subsemigroup of $\mathcal{R}_n(m)$ and $\mathcal{C}_m(n)$.

Remark 4.2.29.

- If $J = J_{nmm}$, then $\mathcal{M}_{mn}^J \cong \mathcal{R}_n(m)$. Since $r = m$, Proposition 4.2.26 gives

$$|\,\mathrm{E}_J(\mathcal{R}_n(m))| = \begin{cases} \sum_s^m q^{s(n-s)} \begin{bmatrix} m \\ s \end{bmatrix}_q, & q < \aleph_0; \\ q, & q \geq \aleph_0 \text{ and } m \neq n. \end{cases}$$

  Moreover, Theorem 4.2.27 applies as well, and Theorem 4.2.28 gives

$$\operatorname{rank}(\mathbb{E}_J(\mathcal{R}_n(m))) = \begin{cases} q^{m(n-m)} + \frac{q^m-1}{q-1}, & q < \aleph_0; \\ q, & q \geq \aleph_0 \text{ and } m \neq n. \end{cases}$$

- If $J = J_{nmn}$, then $\mathcal{M}_{mn}^J \cong \mathcal{C}_m(n)$, and the results are dual.

4.2.5 The rank of a sandwich semigroup $\mathcal{M}_{mn}^J$

Finally, we turn to the problem of calculating the rank of the semigroup $\mathcal{M}_{mn}^J$. Not surprisingly, the results and techniques used here evoke those of Subsection 3.2.5.

As always, we start with the simpler cases.

- **Suppose $\mathbf{r = 0}$.** Then, $A \star_J B = \mathrm{O}_{mn}$ for all $A, B \in \mathcal{M}_{mn}$, so

$$\operatorname{rank}(\mathcal{M}_{mn}^J) = |\,\mathcal{M}_{mn} \setminus \{\mathrm{O}_{mn}\}| = \begin{cases} q^{mn} - 1, & q < \aleph_0; \\ q, & q \geq \aleph_0. \end{cases}$$

- **Suppose $\mathbf{r \geq 1}$ and $\mathbf{q > \aleph_0}$.** Clearly, $|\,\mathcal{M}^J_{mn}\,| = q$, so $\mathrm{rank}(\mathcal{M}^J_{mn}) = q$ (since an uncountable set cannot be generated by a set of smaller cardinality).

- **Suppose $\mathbf{r \geq 1}$, $\mathbf{q < \aleph_0}$, and $\mathbf{m = n = r}$.** In this case, $\mathcal{M}^J_{mn} \cong \mathcal{M}_r$, so the value $\mathrm{rank}(\mathcal{M}^J_{mn})$ is stated in Lemma 4.2.14$(viii)(c)$.

- **Suppose $\mathbf{r \geq 1}$, $\mathbf{q = \aleph_0}$, and $\mathbf{m = n = r}$.** Again, $\mathcal{M}^J_{mn} \cong \mathcal{M}_r$. Since $|\,\mathcal{M}_r\,| = q^{mn} = \aleph_0$, we have $\mathrm{rank}(\mathcal{M}_r) \leq \aleph_0$. Let us show that the value cannot be finite. Suppose that $\mathcal{M}_r(\mathbb{F})$ is finitely generated. Then, so is the multiplicative group $\mathbb{F}^\times$ of the field $\mathbb{F}$ (consider the determinants of the generators for $\mathcal{M}_r(\mathbb{F})$). It is well-known that any subgroup of a finitely generated Abelian group is finitely generated (for instance, see Exercise 10.7.(ii) in [109]). We have the following cases:

  - Suppose $\mathbb{F}$ is a field of characteristic 0. Then, $\mathbb{F}^\times$ contains a copy of $\mathbb{Q}^\times$, which is not finitely generated. However, since any subgroup of a finitely generated Abelian group is finitely generated, this contradicts the conclusion that $\mathbb{F}^\times$ is finitely generated.
  - Suppose $\mathbb{F}$ is a field of characteristic p. Thus, it is either an algebraic or a transcendental field extension over $\mathbb{F}_p$ (the finite field of cardinality p). In the first case, $\mathbb{F}$ is a finite field, which contradicts the assumption. In the second case, there exists a transcendental element $X \in \mathbb{F}$ over the field $\mathbb{F}_p$. Thus, $\mathbb{F}^\times$ contains a copy of the multiplicative group $\mathbb{F}_p(X)^\times$, which is not finitely generated (if it were, the extension would be algebraic). Again, this contradicts the earlier conclusion that $\mathbb{F}^\times$ is finitely generated.

  Since in both cases we arrive at a contradiction, $\mathcal{M}_r(\mathbb{F})$ cannot be finitely generated.

Therefore, for the remainder of this subsection, we assume that $\mathbf{r \geq 1}$, $\mathbf{q \leq \aleph_0}$, and that $\mathbf{m = n = r}$ is not the case.

Recall the notation for $\mathscr{D} = \mathscr{J}$-classes of $\mathcal{M}_{mn}$ (note that these are not the $\mathscr{D}^J = \mathscr{J}^J$ classes of $\mathcal{M}^J_{mn}$):

$$\mathrm{D}^{mn}_s = \{M \in \mathcal{M}_{mn} : \mathrm{Rank}\, M = s\}.$$

First, we present a mechanism for generating the "lower" $\mathscr{D}$-classes (a result from [30]), which enables us to immediately calculate $\mathrm{rank}(\mathcal{M}^J_{mn})$ in the case when $r < \min(m, n)$.

Lemma 4.2.30. *Put $l = \min(m, n)$. If $0 \leq s \leq \min(l-1, r)$ and $m = n = r$ is not the case, then $\mathrm{D}^{mn}_s \subseteq \mathrm{D}^{mn}_{s+1} \star_J \mathrm{D}^{mn}_l$.*

Proof. Suppose $m = n = r$ is not the case. Let $X \in \mathrm{D}_s(\mathcal{M}_{mn})$, where $0 \leq s \leq \min(l-1, r)$. Then, $\lambda_X \in \mathrm{Hom}(V_m, V_n)$ and we need to define maps $\alpha, \beta \in \mathrm{Hom}(V_m, V_n)$ such that

$$\alpha \circ \lambda_J \circ \beta = \lambda_X. \tag{4.6}$$

Let $\mathscr{B} = \{v_1, \dots, v_m\}$ be a basis of V_m, where $\{v_{s+1}, \dots, v_m\}$ is a basis of $\ker(\lambda_X)$. We define the map $\alpha \in \mathrm{Hom}(V_m, V_n)$ in the following way

$$(v_i)\alpha = \begin{cases} e_{ni}, & \text{if } 1 \leq i \leq s; \\ 0, & \text{if } s < i < m. \\ e_{nn}, & \text{if } i = m. \end{cases}$$

Clearly, $\mathrm{Rank}(\alpha) = s + 1$. Since $(e_{ni})\lambda_J = e_{mi}$ for $1 \leq i \leq r$, and $\ker(\lambda_J) = \mathrm{span}\{e_{r+1}, \dots, e_n\}$, our argument differs for the cases $r < m$ and $r = m$.

Suppose $\mathbf{r < m}$. It suffices to choose $\beta \in \mathrm{Hom}(V_m, V_n)$ to be any linear transformation of rank l satisfying $e_{mi} \mapsto (v_i)\lambda_X$ for all $1 \leq i \leq s$ (such a transformation exists, since $\mathrm{Rank}\, X = s < l = \min(m, n)$).

Suppose $\mathbf{r = m}$. By our assumption, $r = m = n$ is not the case, so we have $r = m < n$. In this case, β may be any linear transformation from $\mathrm{Hom}(V_m, V_n)$ of rank l satisfying $e_{mi} \mapsto (v_i)\lambda_X$ for all $1 \leq i \leq s$, and $e_{rn} = e_{mn} \mapsto 0$ (this condition is needed because $(v_m)\alpha\lambda_J = e_{mn}$).

One may easily check that the equality (4.6) holds in both cases. □

Applying a simple (reverse) induction, one may show the following:

Corollary 4.2.31.

(i) If $r < \min(m, n) = l$, then $\langle \mathrm{D}^{mn}_{r+1} \cup \mathrm{D}^{mn}_{r+2} \cup \dots \cup \mathrm{D}^{mn}_{l} \rangle_J = \mathcal{M}_{mn}$.

(ii) If $r = \min(m, n) = l$, then $\langle \mathrm{D}^{mn}_{l} \rangle_J = \mathcal{M}_{mn}$.

We are now ready to calculate the rank of $\mathcal{M}^J_{mn}$ in the case that $r < \min(m, n)$. This result was proved in [30] for $q < \aleph_0$, and we expand it to include the case $q = \aleph_0$.

Theorem 4.2.32. *Suppose $r < l = \min(m, n)$. Then, $\mathcal{M}^J_{mn} = \langle \Omega \rangle_J$, where $\Omega = \{X \in \mathcal{M}_{mn} : \mathrm{Rank}\, X > r\}$. Further, any generating set for $\mathcal{M}^J_{mn}$ contains Ω, so it follows that*

$$\mathrm{rank}(\mathcal{M}^J_{mn}) = |\Omega| = \begin{cases} \sum_{s=r+1}^{l} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q q^{\binom{s}{2}} (q-1)^s [s]_q!, & \text{if } q < \aleph_0; \\ q, & \text{if } q = \aleph_0. \end{cases}$$

Proof. By the discussion in Section 2.6, any generating set of $\mathcal{M}^J_{mn}$ must include elements from every maximal $\mathscr{J}^J$-class. Under the assumptions of the theorem, Proposition 4.2.8(i) guarantees that the maximal $\mathscr{J}^J$-classes are precisely the singletons $\{X\}$ such that $\mathrm{Rank}\, X > r$ (hence, the possible value ranges from $r+1$ to $\min(m, n) = l$). Therefore, any generating set contains all such elements, and $\mathrm{rank}(\mathcal{M}^J_{mn}) \geq \sum_{i=r+1}^{l} |\mathrm{D}^s_{mn}|$. In fact, from Corollary 4.2.31(i) follows that this value is a lower bound, as well. The size of D^s_{mn} is calculated in Proposition 4.1.4(iv), so the statement follows. □

Next, we aim to calculate the rank in the case $r = \min(m, n)$. We include an auxiliary results, which appeared in [30].

Lemma 4.2.33.

(i) If $r = m < n$, then $\mathrm{P}_2^J = \mathcal{M}_{mn}^J$, $\mathrm{P}_1^J = \mathrm{P}^J$, and $\mathscr{L}^J = \mathscr{L}$ on $\mathcal{M}_{mn}^J$.

(ii) If $r = n < m$, then $\mathrm{P}_1^J = \mathcal{M}_{mn}^J$, $\mathrm{P}_2^J = \mathrm{P}^J$, and $\mathscr{R}^J = \mathscr{R}$ on $\mathcal{M}_{mn}^J$.

Proof. We prove only part (i), since (ii) follows by a dual argument. Since $r = m < n$, J is left-invertible in $\mathcal{M}_{mn}$ by Corollary 4.1.8(ii). Thus, the dual of Lemma 2.2.38 implies the statement. □

Finally, we present a theorem stating the rank of $\mathcal{M}_{mn}^J$ with $\operatorname{Rank} J = r = \min(m, n)$ in the case $q < \aleph_0$ [30], and in the case $q = \aleph_0$, as well.

Theorem 4.2.34. *Suppose $r = \min(m, n)$ and $m \neq n$. Then,*

$$\operatorname{rank}(\mathcal{M}_{mn}^J) = \begin{cases} \left[{L \atop l}\right]_q, & \textit{if } q < \aleph_0; \\ q, & \textit{if } q = \aleph_0; \end{cases}$$

where $l = \min(m, n)$ and $L = \max(m, n)$.

Proof. Without loss of generality, we may suppose that $r = m < n$ (because the case $r = n < m$ is dual). Thus, Lemma 4.2.33(i) (and its proof) applies. In particular, J is left-invertible. Recall that the partial semigroup $\mathcal{M}$ is stable (by Proposition 4.1.1). In addition, Proposition 4.1.1 and Corollary 4.2.31(ii) give $\langle \mathrm{J}_K \rangle_J = \langle \mathrm{D}_m^{mn} \rangle_J = \mathcal{M}_{mn}^J$. Thus, we may apply Proposition 2.6.4 in order to calculate $\operatorname{rank}(\mathcal{M}_{mn}^J)$. Firstly, note that $n > m = r \geq 1$ and $q \geq 2$, so

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(q^n - 1) \cdots (q^{n-r+1} - 1)}{(q^r - 1) \cdots (q - 1)} \geq \frac{q^{n-r+1} - 1}{q - 1} = q^{n-r} + q^{n-r-1} \cdots + 1 \geq 2.$$

Therefore, Theorem 4.2.15(v), Lemma 4.2.14$(viii)(a)$ and Proposition 4.1.4(ii) imply

$$\operatorname{rank}(\mathrm{H}_K^J) = \operatorname{rank}(\mathcal{G}_r) \leq 2 \leq \begin{bmatrix} n \\ r \end{bmatrix}_q = |\, \mathrm{D}_b \,/\, \mathscr{L} \,| = |\, \mathrm{J}_b \,/\, \mathscr{H} \,|,$$

the last equality following from Proposition 4.1.1 and the dual of Proposition 2.2.37(ii). Finally, Proposition 2.6.4(iii) implies

$$\operatorname{rank}(\mathcal{M}_{mn}^J) = \operatorname{rank}(T) = |\, \mathrm{J}_b \,/\, \mathscr{H} \,| = \begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{bmatrix} L \\ l \end{bmatrix}_q. \qquad \square$$

Corollary 4.2.35. *From the previous theorem, the reader may readily conclude that*

$$\operatorname{rank}(\mathcal{R}_n(m)) = \begin{bmatrix} n \\ m \end{bmatrix}_q \quad \textit{and} \quad \operatorname{rank}(\mathcal{C}_m(n)) = \begin{bmatrix} m \\ n \end{bmatrix}_q.$$

4.2.6 Egg-box diagrams

As in the previous chapter, we provide several egg-box diagrams (they originally appeared in [30], and all were generated by GAP [98]) to illustrate the structural results for $\mathcal{M}_{mn}^J$. For more information on egg-box diagrams, see the introduction to Subsection 3.1.6.

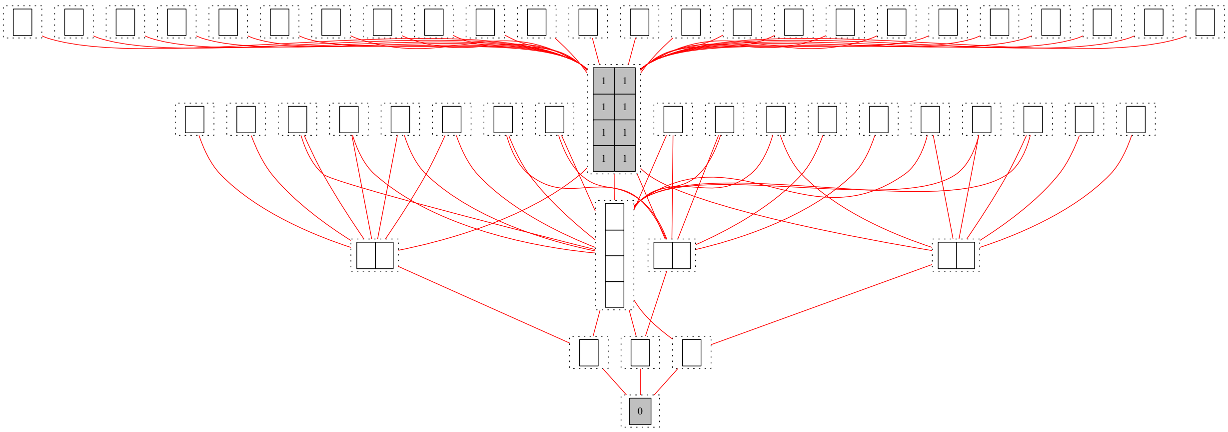


Figure 4.4: Egg-box diagram of the linear sandwich semigroup $\mathcal{M}_{32}^{J_{231}}(\mathbb{Z}_2)$.

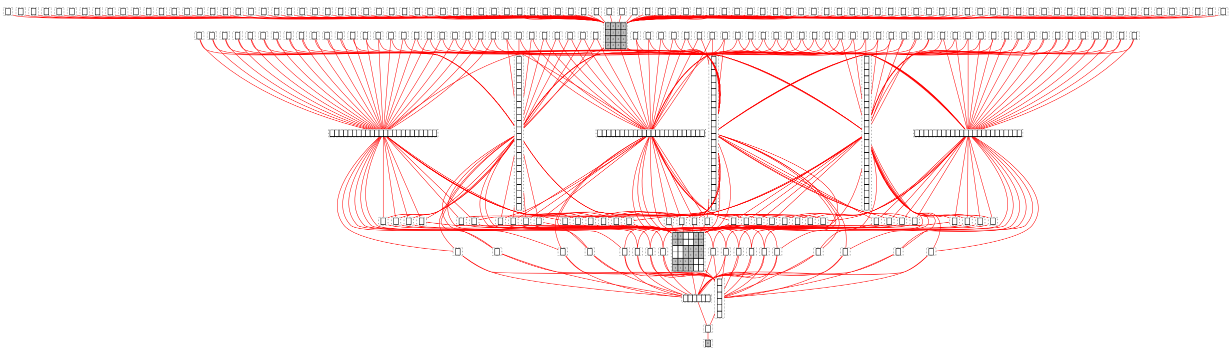


Figure 4.5: Egg-box diagram of the linear sandwich semigroup $\mathcal{M}_{33}^{J_{332}}(\mathbb{Z}_2)$.

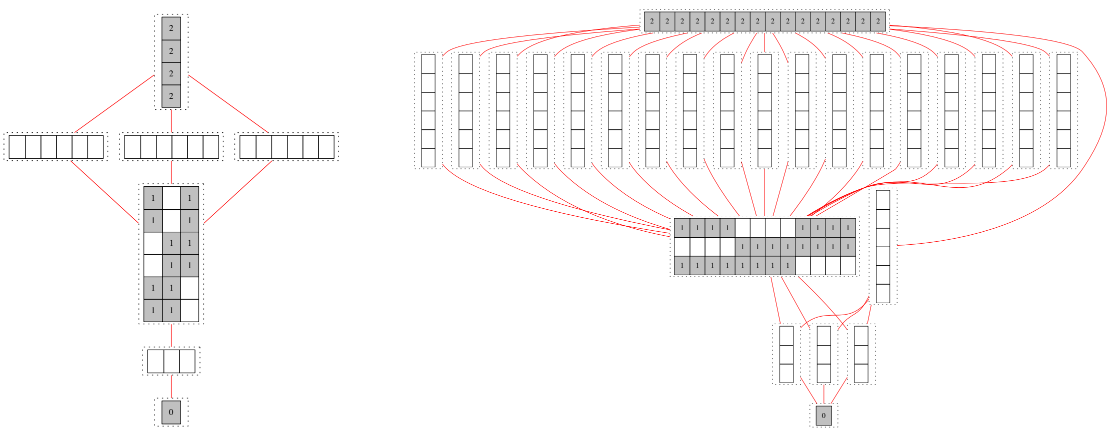


Figure 4.6: Egg-box diagrams of the linear sandwich semigroups $\mathcal{M}_{32}^{J_{232}}(\mathbb{Z}_2)$ and $\mathcal{M}_{24}^{J_{422}}(\mathbb{Z}_2)$ (left and right, respectively).

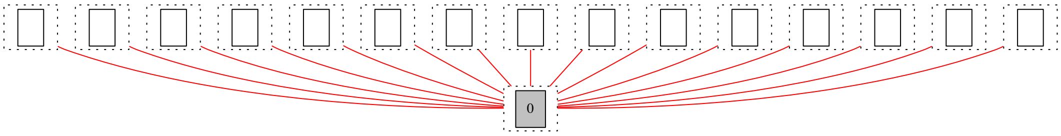


Figure 4.7: Egg-box diagram of the linear sandwich semigroup $\mathcal{M}_{22}^{O_{22}}(\mathbb{Z}_2)$ or, equivalently, $\mathcal{M}_{21}^{O_{12}}(\mathbb{F}_4)$.

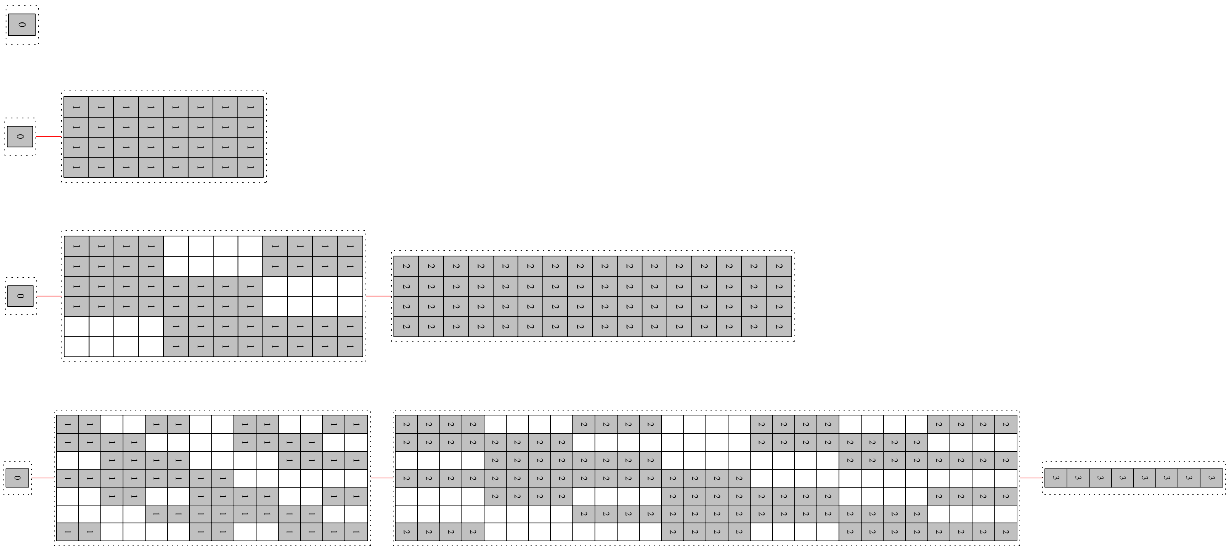


Figure 4.8: Egg-box diagrams (drawn sideways) of the regular linear sandwich semigroups $\text{Reg}(\mathcal{M}_{43}^{J}(\mathbb{Z}_2))$ where $\text{Rank}(J) = 0, 1, 2, 3$ (top to bottom).

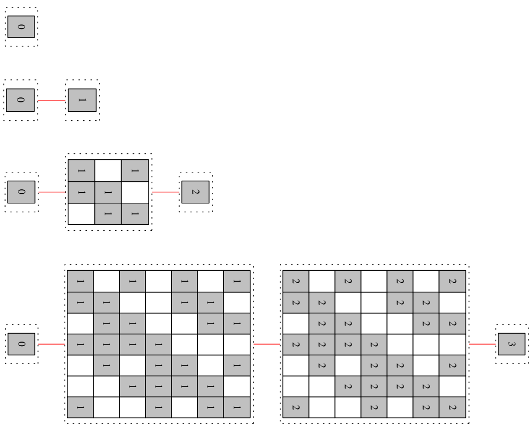


Figure 4.9: Egg-box diagrams (drawn sideways) of the full linear semigroups $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, all over $\mathbb{Z}_2$ (top to bottom). By the theory in Subsection 2.3.4, the regular semigroups in Figure 4.8 are inflations of these semigroups, top to bottom respectively.

Chapter 5

Sandwich semigroups of partitions

In this chapter, we embark on the task of investigating sandwich semigroups in several types of diagram categories. Namely, we study the partition category $\mathcal{P}$, the planar partition category $\mathscr{P}\mathcal{P}$, the Brauer category $\mathcal{B}$, the Temperley-Lieb category $\mathcal{TL}$, the partial Brauer category $\mathcal{PB}$, and the Motzkin category $\mathscr{M}$. First, we define the corresponding partial semigroups and examine their properties, and then we investigate the sandwich semigroups they contain. Unlike sandwich semigroups of transformations, these have not been studied in the past. However, the idea is fully justified, since diagram categories and diagram algebras play a significant role in representation theory [52,90], classical groups [10], knot theory [63,64,69,70,115], invariant theory [78,79], statistical mechanics [65,68,89,119], theoretical physics [90] et al. Furthermore, each category that we study attracted considerable scientific interest in the past (for instance, $\mathcal{P}$ in [65,89], $\mathscr{P}\mathcal{P}$ in [52,65], $\mathcal{B}$ in [10,79], $\mathcal{PB}$ in [91,92], $\mathcal{TL}$ in [115,119], $\mathscr{M}$ in [7]) and is therefore worth investigating.

This chapter is entirely based on [28], and here we cite this paper as the source of the results unless otherwise stated.

Again, we follow the layout of Section 3.1. After introducing the necessary definitions and notions, we formally define the partial semigroups we will be studying. Then, we describe their Green's relations, characterise and enumerate Green's classes, and investigate the topic of regularity. Next, we focus on the sandwich semigroup $\mathcal{K}^{\sigma}_{mn}$ (where $\mathcal{K}$ is one of $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathcal{TL}$ or $\mathscr{M}$). We conduct the usual investigation up to a point; we investigate Green's relations and classes, maximal $\mathscr{J}^{\sigma}$-classes, connections to other non-sandwich diagram semigroups, the regular subsemigroup P^{σ} and the inflation from Subsection 2.3.4, idempotents, and the idempotent-generated subsemigroup. For most of the diagram categories, we are not able to follow through with our further "program", because it turns out that $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathcal{TL}$ and $\mathscr{M}$ do not have the properties needed for the combinatorial part of the investigation (i.e. MI-domination of the regular subsemigroup). However, the category $\mathcal{B}$ turns out to be much more amenable to analysis via our techniques. So, we are able to give necessary and sufficient conditions for two sandwich semigroups in $\mathcal{B}$ to be isomorphic, to describe the combinatorial structure of P^{σ}, apply the formulae holding in the case of MI-domination and calculate the rank of $\mathcal{B}^{\sigma}_{mn}$.

5.1 Partial semigroups of partitions

As in the previous cases, our first task is to define the partial semigroups corresponding to our categories. We start with the category $\mathcal{P}$ since all of the others are its subcategories. For any positive integer $n \in \mathbb{N}$, we define $[n] = \{1, 2, \ldots, n\}$. In addition, we assume $[0] = \emptyset$. Furthermore, for any $A \subseteq \mathbb{N}_0$, let $A' = \{a' : a \in A\}$ and $A'' = \{a'' : a \in A\}$. Now, for $m, n \in \mathbb{N}_0$, let $\mathcal{P}_{mn}$ denote the set of all partitions of the set $[m] \cup [n]'$. Then,

$$\mathcal{P} = \bigcup_{m,n \in \mathbb{N}_0} \mathcal{P}_{mn}$$

is the set of all such set partitions. For a partition $\sigma \in \mathcal{P}$, let ε_σ denote the corresponding equivalence.

Let $m, n \in \mathbb{N}_0$ and fix any partition $\sigma \in \mathcal{P}_{mn}$. We depict it in a specific manner: we create a graph with $m+n$ vertices in the plane $\mathbb{R}^2$, respecting the following rules

- each element $a \in [m]$ is assigned to the vertex $(a, 1)$;
- each element $b' \in [n]'$ is assigned to the vertex $(b, 0)$;
- for each equivalence class S of σ, the vertices corresponding to the elements of S constitute a (connected) component of the graph;
- each edge of the graph is drawn inside the rectangle $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \max(m, n),\ 0 \leq y \leq 1\}$.

In Figure 5.1, we present such a graph for the partition

$$\{\{1,5,6\},\{2\},\{3,4,2'\},\{7,8'\},\{1',6'\},\{3',4'\},\{5'\},\{7'\}\} \in \mathcal{P}_{78}\,. \tag{5.1}$$

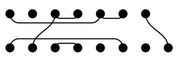
$\alpha =$ 

Figure 5.1: An example of a diagram corresponding the partition (5.1).

The reader will immediately realise that, in general, there exist multiple graphs corresponding to the same partition. We identify the partition with any such diagram. Therefore, the properties of diagram $\alpha \in \mathrm{P}_{mn}$ we are interested in are its components. Those containing both upper and lower vertices (i.e. elements of both $[m]$ and $[n]'$) are called *transversals*. The number of transversals is the *rank* of the partition α. The components containing only upper vertices (elements of $[m]$) are *upper nontransversals*. The *lower nontransversals* are defined dually.

Since the transversals, together with the upper and the lower nontransversals, precisely determine the partition containing them, we may present that partition via the following scheme

$$\left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ \cline{4-6} B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{array} \right),$$

where $A_i \cup B_i'$ $(1 \leq i \leq r)$ are the transversals, C_i $(1 \leq i \leq s)$ are the upper nontransversals, and D_i' $(1 \leq i \leq t)$ are the lower nontransversals (if any of these sets is a singleton, we omit the brackets). For instance, the partition (5.1) may be presented as

$$\left(\begin{array}{c|c|c|c|c|c} \{3,4\} & 7 & \{1,5,6\} & 2 & \multicolumn{2}{c}{} \\ \cline{3-6} 2 & 8 & \{1,6\} & \{3,4\} & 5 & 7 \end{array} \right).$$

Note that any of the numbers r, s, t can be zero. Moreover, for the partition $\emptyset \in \mathcal{P}_{00}$, all three are simultaneously zero.

Having introduced the notion of a diagram, we may define a partial operation of multiplication on $\mathcal{P}$. For partitions $\alpha \in \mathcal{P}_{mn}$ and $\beta \in \mathcal{P}_{kl}$, the product $\alpha\beta$ will be defined if and only if $n = k$; in that case, we use any two diagrams representing α and β to define the *product diagram* $\Pi(\alpha, \beta)$ in the following way:

- modifying the diagram representing $\alpha \in \mathcal{P}_{mn}$, we create the graph $\alpha_\downarrow$, by renaming each (lower) vertex $x' \in [n]'$ to x'' (hence obtaining a graph on the vertex set $[m] \cup [n]''$);

- modifying the diagram representing $\beta \in \mathcal{P}_{nl}$, we create the graph $\beta^\uparrow$, by renaming each (upper) vertex $x \in [n]$ to x'' (hence obtaining a graph on the vertex set $[n]'' \cup [l]$);

- by identifying the vertices of the set $[n]''$ in $\alpha_\downarrow$ with the corresponding vertices of $[n]''$ in $\beta^\uparrow$, we obtain the graph $\Pi(\alpha, \beta)$.

(In future, we will sometimes talk about the product diagram of more than two diagrams, which is constructed accordingly.) Using the product diagram $\Pi(\alpha, \beta)$, we define the product partition $\alpha \cdot \beta = \alpha\beta$ on the set $[m] \cup [k]'$, by

$$(r, s) \in \alpha\beta \Leftrightarrow r \text{ and } s \text{ belong to the same component of } \Pi(\alpha, \beta),$$

for $r, s \in [m] \cup [k]'$. In other words, we obtain $\varepsilon_{\alpha\beta}$ by taking the smallest equivalence relation containing $\varepsilon_{\alpha_\downarrow} \cup \varepsilon_{\beta^\uparrow}$, and removing any pair containing an element of $[n]''$. Via this approach, one may easily prove that

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

for all $\alpha, \beta, \gamma \in \mathcal{P}$ such that $\alpha\beta$ and $\beta\gamma$ are defined.

In Figure 5.2, we provide an example illustrating the process of multiplication of partitions (diagrams).

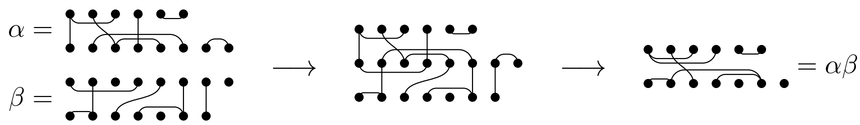


Figure 5.2: Multiplication of partitions α and β via the product diagram $\Pi(\alpha, \beta)$.

In addition, we define a unary operation on $\mathcal{P}$, which serves as a tool for "inversion" of elements; namely, the involution $^* : \mathcal{P} \to \mathcal{P}$, which maps

$$\left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ \cline{4-6} B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{array} \right) \quad \text{to} \quad \left(\begin{array}{c|c|c|c|c|c} B_1 & \cdots & B_r & D_1 & \cdots & D_t \\ \cline{4-6} A_1 & \cdots & A_r & C_1 & \cdots & C_s \end{array} \right),$$

may be interpreted as reflecting the diagram (representing the partition) in a horizontal axis. It is easily seen that, for any $\alpha \in \mathcal{P}$, $(\alpha^*)^* = \alpha$ and α^* is an inverse of α. Furthermore, by analysing the example in Figure 5.2, one may easily conclude that

$$(\alpha\beta)^* = \beta^*\alpha^*,$$

for any $\alpha, \beta \in \mathcal{P}$ such that the product $\alpha\beta$ is defined (we just reflect all the diagrams in the process in a horizontal axis). Thus, the map $\mathcal{P} \to \mathcal{P} : \alpha \mapsto \alpha^*$ is an anti-isomorphism.

Finally, for $m, n \in \mathbb{N}_0$ and $\alpha \in \mathcal{P}_{mn}$, we define

$$\alpha\,\boldsymbol{\delta} = m \qquad \text{and} \qquad \alpha\,\boldsymbol{\rho} = n.$$

Thus, for any $\alpha, \beta \in \mathcal{P}$, the product $\alpha\beta$ is defined if and only if $\alpha\,\boldsymbol{\rho} = \beta\,\boldsymbol{\delta}$. Furthermore, $\alpha^*\,\boldsymbol{\rho} = \alpha\,\boldsymbol{\delta}$ and $\alpha^*\,\boldsymbol{\delta} = \alpha\,\boldsymbol{\rho}$. Therefore, we may conclude that $(\mathcal{P}, \cdot, \mathbb{N}_0, \boldsymbol{\delta}, \boldsymbol{\rho})$ is a partial semigroup (as defined in [90]), and $(\mathcal{P}, \cdot, \mathbb{N}_0, \boldsymbol{\delta}, \boldsymbol{\rho}, ^*)$ is a regular partial $*$-semigroup. Moreover, $\mathcal{P}$ is monoidal, since, for $m \in \mathbb{N}_0$, the partition $\iota_m = \{\{x, x'\} : 0 \leq x \leq \min(m)\}$ is the left identity of $\mathcal{P}_{nm}$ and the left identity of $\mathcal{P}_{mn}$, for any $n \in \mathbb{N}_0$.

Now, we introduce the partial subsemigroups of $\mathcal{P}$ we are interested in. Firstly, let

$$\mathcal{B} = \{\alpha \in \mathcal{P} : \text{ each block of } \alpha \text{ has exactly two elements}\},$$
$$\mathcal{PB} = \{\alpha \in \mathcal{P} : \text{ each block of } \alpha \text{ has at most two elements}\}.$$

Clearly, both subsets are closed for involution. Moreover, this holds for multiplication, as well. Let us elaborate on this conclusion. In the process of multiplication, any merging of blocks happens in the middle row of the product diagram. Since the (maximal) number of elements per block is 2, any new block is either a loop in the middle row, or a path containing (at most) two transversals, one at each end. Thus, the new block contains at most two elements in the resulting partition. Hence, we may conclude that $\mathcal{B}$ and $\mathcal{PB}$ are subcategories of $\mathcal{P}$ (they are the Brauer and partial

Brauer category, respectively), and that

$$(\mathcal{B},\ \cdot\restriction_{\mathcal{B}\times\mathcal{B}},\ \boldsymbol{\delta}\restriction_{\mathcal{B}},\ \boldsymbol{\rho}\restriction_{\mathcal{B}},\ {}^*\restriction_{\mathcal{B}}) \quad \text{and} \quad (\mathcal{PB},\ \cdot\restriction_{\mathcal{PB}\times\mathcal{PB}},\ \boldsymbol{\delta}\restriction_{\mathcal{PB}},\ \boldsymbol{\rho}\restriction_{\mathcal{PB}},\ {}^*\restriction_{\mathcal{PB}})$$

are both regular partial $*$-semigroups.

Secondly, let

$$\mathscr{P}\mathcal{P} = \{\alpha \in \mathcal{P} : \alpha \text{ may be presented by a planar diagram}\},$$

(recall that diagrams are always drawn respecting the rules on page 198). As in graph theory, a diagram is planar if it has no intersecting edges. In Figure 5.3, the reader may inspect two diagrams representing the same partition, one of them non-planar, and the other planar. Clearly, the partition itself is planar, because some planar graph represents it.

Figure 5.3: A non-planar (on the left) and a planar (on the right) representation of the same partition from $\mathscr{P}\mathcal{P}_{67}$.

The set $\mathscr{P}\mathcal{P}$ is by definition closed for the operation of involution, since reflecting a planar diagram in a horizontal axis produces a planar diagram. Multiplication, however, is a bit more complicated. In order to discuss it, we further explore the problem of representing a partition. In [40], the authors introduced the canonical graph of a planar partition. We apply the same construction, but generalise the notion to include all partitions. In order to do that, we introduce additional notation. Suppose $m, n \in \mathbb{N}$ (we discuss the case $\min(m, n) = 0$ below). Let $1 \le k \le m$ and $1 \le l \le n$, and let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_l\}$ be subsets of $[m]$ and $[n]$, respectively, such that $a_1 < \dots < a_k$ and $b_1 < \dots < b_l$. Then, we define graphs Γ_A, $\Gamma_{B'}$, and $\Gamma_{A\cup B'}$, by $V(\Gamma_A) = V(\Gamma_{B'}) = V(\Gamma_{A\cup B'}) = [m] \cup [n]'$, and

$$E(\Gamma_A) = \{\{a_i, a_{i+1}\} : 1 \le i \le k-1\}, \quad E(\Gamma_{B'}) = \{\{b'_i, b'_{i+1}\} : 1 \le i \le l-1\}$$
$$E(\Gamma_{A\cup B'}) = E(\Gamma_A) \cup E(\Gamma_{B'}) \cup \{\{a_1, b'_1\}, \{a_k, b'_k\}\}.$$

(Here, $V(G)$ and $E(G)$ denote the vertex and edge set of a graph G.) Note that a block of a partition is always a nonempty set, so there is no need to define $\Gamma_\emptyset$. Now, the *canonical diagram of* a partition $\alpha \in \mathcal{P}_{mn}$ is the graph Γ_α (drawn respecting the "rules" for diagrams), with

$$V(\Gamma_\alpha) = [m] \cup [n]' \ \text{ and } \ E(\Gamma_\alpha) = \bigcup_{X\in\alpha} E(\Gamma_X),$$

where the union is over all blocks X of α. The reader may easily check that the right-hand side diagram in Figure 5.3 is the canonical graph of the partition

$\{\{1,2,3,4\},\{5,6,3',4',7'\},\{1',2'\},\{5',6'\}\}$.

We claim: a partition $\alpha \in \mathrm{P}$ may be presented by a planar diagram if and only if its canonical diagram Γ_α may be drawn in planar fashion. In the following result (Lemma 7.1 of [40]), we show the direct implication (the reverse being clear). For the statement, we need some additional definitions from [40]. Again, let $l, k \in \mathbb{N}$ and let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ be nonempty subsets of $\mathbb{N}$ such that $a_1 < \ldots < a_k$ and $b_1 < \ldots < b_l$. We introduce the following terms.

- A and B are *separated* if $a_k < b_1$ or $b_l < a_1$; in these cases, we write $A < B$ or $B < A$, respectively.
- A is *nested by* B if there exists some $1 \le i < l$ such that $b_i < a_1$ and $a_k < b_{i+1}$.
- A and B are *nested* if A is nested by B or vice versa.

Now, we may prove

Lemma 5.1.1. *Let* $\alpha = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ \cline{4-6} B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{array}\right) \in \mathscr{P}\mathcal{P}_{mn}$*, with* $\min(A_1) < \cdots < \min(A_r)$*. Then*

(i) $A_1 < \cdots < A_r$ *and* $B_1 < \cdots < B_r$*,*

(ii) for all $1 \le i < j \le s$*,* C_i *and* C_j *are either nested or separated,*

(iii) for all $1 \le i < j \le t$*,* D_i *and* D_j *are either nested or separated,*

(iv) for all $1 \le i \le r$ *and* $1 \le j \le s$*, either* A_i *and* C_j *are separated or else* C_j *is nested by* A_i*,*

(v) for all $1 \le i \le r$ *and* $1 \le j \le t$*, either* B_i *and* D_j *are separated or else* D_j *is nested by* B_i*.*

Consequently, the canonical diagram Γ_α *may be drawn in planar fashion.*

Proof. Note that the canonical diagram of a partition is constructed in such a way that the edges of the same block do not intersect. Thus, the canonical diagram Γ_α is planar if the edges of different blocks (components) do not intersect. That clearly holds if the statements $(i)-(v)$ are true.

Now, we need to prove that α being planar implies $(i)-(v)$. Suppose $\min(A_1) < \cdots < \min(A_r)$. Note that, for all $1 \le i \le r$, $\min(A_i)$ and $\min(B_i)'$ are connected by a path inside the rectangle $\{(x,y) \in \mathbb{R}^2 : 0 \le x \le \max(m,n),\ 0 \le y \le 1\}$ in any graph representing α. Thus, if there existed $j < k$ so that $\min(B_j) > \min(B_k)$, the above-mentioned paths for j and k would necessarily intersect. Therefore, we have $\min(B_1) < \cdots < \min(B_r)$, and it suffices to show that $A_1 < \cdots < A_r$, (ii), and (iv) hold, as the remaining parts follow by duality. For all three statements, we prove the contrapositive. Suppose that $A_1 < \cdots < A_r$ is not true. Then, there exist $1 \le i < j \le r$ such that $x > y$ for some $x \in A_i$ and $y \in A_j$. Hence,

$$\min(A_i) < \min(A_j) < y < x. \tag{5.2}$$

Since $\alpha \in \mathscr{P}\mathcal{P}$, there exists a planar diagram D representing it. However, (5.2) implies that the path connecting $\min(A_j)$ and $\min(B_j)$ intersects the path connecting $\min(A_i)$ and x, which contradicts the planarity of D. Similarly, if we suppose that (ii) is false, there exist $1 \leq i < j \leq s$ such that for some $x \in C_i$

$$\min(C_i) < \min(C_j) < x < \max(C_j).$$

If (iv) is false, there exist $1 \leq i \leq r$ and $1 \leq j \leq s$ such that either

- $\min(C_j) < \min(A_i) < x < \max(A_i)$, for some $x \in C_j$, or
- $\min(A_i) < \min(C_j) < y < \max(C_j)$ for some $y \in A_i$.

In both cases, any diagram representing α has intersections of paths belonging to two components, so it cannot be planar. $\square$

Now, we prove that $\mathscr{P}\mathcal{P}$ is closed for multiplication. First, note that a product diagram (see Figure 5.2) obtained by composing two planar diagrams is necessarily planar. Now, suppose that $\alpha, \beta \in \mathscr{P}\mathcal{P}$ and $\alpha\beta \notin \mathscr{P}\mathcal{P}$. Thus, for the product $\alpha\beta = \left(\begin{smallmatrix} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{smallmatrix}\right)$, one of the statements $(i) - (v)$ from Lemma 5.1.1 is false. Then, by the proof of the same lemma, in $\alpha\beta$ there exist two components X, Y and vertices $u, v \in V(X)$ and $q, w \in V(Y)$, such that, in any diagram representing $\alpha\beta$, a path connecting u and v intersects a path connecting q and w. Note that any product diagram $\Pi(\alpha, \beta)$ may be considered such a diagram, if we "forget" the vertices in the middle row and keep the rest of the diagram intact. Thus, (for any diagram representations of α and β,) the product diagram $\Pi(\alpha, \beta)$ is non-planar, contradicting the first assertion.

At last, we may conclude that $(\mathscr{P}\mathcal{P},\ \cdot\restriction_{\mathscr{P}\mathcal{P} \times \mathscr{P}\mathcal{P}},\ \boldsymbol{\delta}\restriction_{\mathscr{P}\mathcal{P}},\ \boldsymbol{\rho}\restriction_{\mathscr{P}\mathcal{P}},\ {}^*\restriction_{\mathscr{P}\mathcal{P}})$ is a regular monoidal ($\iota_m \in \mathscr{P}\mathcal{P}$, for $m \in \mathbb{N}$) partial $*$-semigroup. Moreover, if we define

$$\mathcal{TL} = \mathcal{B} \cap \mathscr{P}\mathcal{P} \quad \text{and} \quad \mathscr{M} = \mathcal{PB} \cap \mathscr{P}\mathcal{P},$$

the corresponding partial subsemigroups of $\mathcal{P}$,

$$(\mathcal{TL},\ \cdot\restriction_{\mathcal{TL} \times \mathcal{TL}},\ \boldsymbol{\delta}\restriction_{\mathcal{TL}},\ \boldsymbol{\rho}\restriction_{\mathcal{TL}},\ {}^*\restriction_{\mathcal{TL}}) \text{ and } (\mathscr{M},\ \cdot\restriction_{\mathscr{M} \times \mathscr{M}},\ \boldsymbol{\delta}\restriction_{\mathscr{M}},\ \boldsymbol{\rho}\restriction_{\mathscr{M}},\ {}^*\restriction_{\mathscr{M}}),$$

are clearly regular monoidal partial $*$-semigroups, as well. They are the Temperley-Lieb and Motzkin category, respectively. Figure 5.4 (from [28]) illustrates the relations among the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathcal{B}$, $\mathscr{M}$ and $\mathcal{TL}$, and gives a diagram representative of an element of each of them.

Before continuing the investigation, we need to explore an interesting connection between categories $\mathscr{P}\mathcal{P}$ and $\mathcal{TL}$. Let

$$\mathcal{TL}^{\text{even}} = \bigcup_{m,n \in \mathbb{N}_0} \mathcal{TL}_{2m,2n}\,.$$

Clearly, this set is closed for involution and multiplication, so it defines a new subcategory (partial subsemigroup) of $\mathcal{TL}$ (and $\mathcal{P}$). As it turns out, the categories $\mathscr{P}\mathcal{P}$

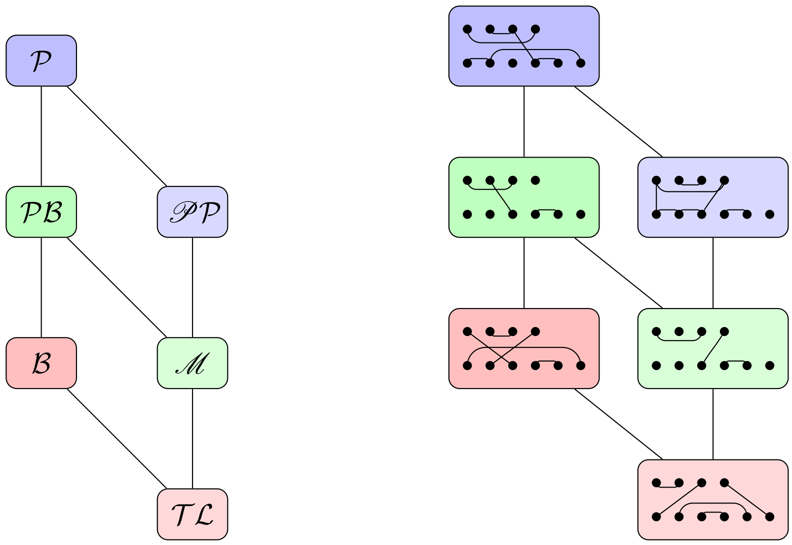


Figure 5.4: Subcategories of $\mathcal{P}$ (left) and representative elements from each (right).

and $\mathcal{TL}^{\text{even}}$ are closely related! For $\alpha \in \mathscr{P}\mathcal{P}_{mn}$, draw the canonical diagram of the partition α, and then construct $\tilde{\alpha} \in \mathcal{TL}_{2m,2n}$ by "tracing around" the blocks of α, as in Figure 5.5 (from [28]). This is not a new idea; it was applied in Section 1 of [52] to prove some relations between the planar partition $(\mathscr{P}\mathcal{P}_n, \cdot)$ and Temperley-Lieb monoid $(\mathcal{TL}_{2n}, \cdot)$.

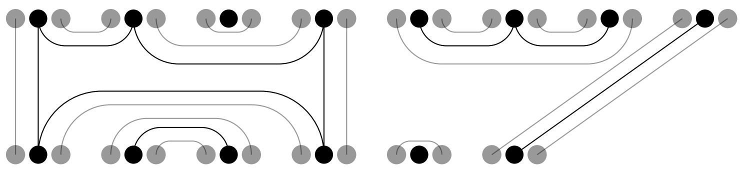


Figure 5.5: A planar partition α from $\mathscr{P}\mathcal{P}_{8,6}$ (black), with its corresponding Temperley-Lieb partition $\tilde{\alpha}$ from $\mathcal{TL}_{16,12}$ (grey).

Intuitively, it is clear that this map is an isomorphism. We leave it at that and skip the proof of the previous statement, because the formal definition and proof are rather technical and lengthy, but do not seem to benefit our investigation.

Having given a detailed introduction to each of the partial semigroups we are interested in, we discuss the endomorphism monoids in each them. These are the partition monoids $\mathcal{P}_m$, planar partition monoids $\mathscr{P}\mathcal{P}_m$, Brauer monoids $\mathcal{B}_m$, partial Brauer monoids $\mathcal{PB}_m$, Motzkin monoids $\mathscr{M}_m$ and Temperley-Lieb monoids $\mathcal{TL}_m$ (also known as Jones monoids $\mathcal{J}_m$), for $m \in \mathbb{N}$. Note that the partition ι_m is the corresponding identity in each case. Furthermore, the invertible elements of $\mathcal{P}_m$ are the partitions

$$\{\{x, (x\pi)'\} : x \in [m]\}, \qquad \text{for } \pi \in \mathrm{S}_m\,.$$

Thus, the automorphism groups of $\mathscr{P}\mathcal{P}_m$, $\mathscr{M}_m$, and $\mathcal{TL}_m$ are trivial, and the automorphism groups of $\mathcal{P}_m$, $\mathcal{B}_m$, and $\mathcal{PB}_m$ may be identified with the symmetric group S_m.

Our next task is to characterise Green's relations in the partial semigroups $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{M}$ and $\mathcal{TL}$. In order to do that, we introduce some additional notation. For $\alpha \in \mathcal{P}$,

$$\begin{aligned}
\mathrm{dom}(\alpha) &= \{x \in [m] : x \text{ belongs to a transversal of } \alpha\},\\
\mathrm{codom}(\alpha) &= \{x \in [n] : x' \text{ belongs to a transversal of } \alpha\},\\
\ker(\alpha) &= \{(x,y) \in [m]\times[m] : (x,y) \in \varepsilon_\alpha\},\\
\mathrm{coker}(\alpha) &= \{(x,y) \in [n]\times[n] : (x',y') \in \varepsilon_\alpha\},\\
\mathrm{N_U}(\alpha) &= \{X \in \alpha : X \text{ is an upper nontransversal of } \alpha\},\\
\mathrm{N_L}(\alpha) &= \{X \in \alpha : X' \text{ is a lower nontransversal of } \alpha\},
\end{aligned}$$

are the *domain*, *codomain*, *kernel*, *cokernel*, and the sets of upper and lower nontransversals of α, respectively. Recall that $\mathrm{Rank}(\alpha)$ is the number of transversals of α. Hence, for $\alpha = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{array}\right)$, we have $\mathrm{Rank}(\alpha) = r$, and

$$\begin{aligned}
\mathrm{dom}(\alpha) &= \bigcup_{i=1}^{r} A_i, \quad \mathrm{N_U}(\alpha) = \{C_i : 1 \le i \le s\},\\
\mathrm{codom}(\alpha) &= \bigcup_{i=1}^{r} B_i, \quad \mathrm{N_L}(\alpha) = \{D_i : 1 \le i \le t\},\\
[m]/\ker(\alpha) &= \{A_i : 1 \le i \le r\} \cup \{C_i : 1 \le i \le s\},\\
[n]/\mathrm{coker}(\alpha) &= \{B_i : 1 \le i \le r\} \cup \{D_i : 1 \le i \le t\}.
\end{aligned}$$

Furthermore, it is easily seen that, for partitions $\alpha, \beta \in \mathcal{P}$ with $\alpha\boldsymbol{\rho} = \beta\boldsymbol{\delta}$ the following relations (and their duals) hold

$$\begin{array}{ccc}
\mathrm{dom}(\alpha\beta) \subseteq \mathrm{dom}(\alpha), & \ker(\alpha\beta) \supseteq \ker(\alpha), & \mathrm{N_U}(\alpha\beta) \supseteq \mathrm{N_U}(\alpha),\\
\mathrm{dom}(\alpha) = \mathrm{codom}(\alpha^*), & \ker(\alpha^*) = \mathrm{coker}(\alpha), & \mathrm{N_U}(\alpha^*) = \mathrm{N_L}(\alpha),\\
\mathrm{Rank}(\alpha) = \mathrm{Rank}(\alpha^*), & \multicolumn{2}{c}{\mathrm{Rank}(\alpha\beta) \le \min(\mathrm{Rank}(\alpha), \mathrm{Rank}(\beta)).}
\end{array} \tag{5.3}$$

Before continuing, we need to point out that, due to the defining properties of the categories $\mathcal{B}$ and $\mathcal{TL}$, for all $m, n \in \mathbb{N}$, we have

$$\mathcal{B}_{mn} \neq \emptyset \quad \Leftrightarrow \quad \mathcal{TL}_{mn} \neq \emptyset \quad \Leftrightarrow \quad m \equiv n \pmod 2. \tag{5.4}$$

Finally, we are ready to characterize Green's relations of the partial semigroups $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{M}$ and $\mathcal{TL}$, as in [28].

Proposition 5.1.2. *Let $\mathcal{K}$ denote any of the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{M}$ or $\mathcal{TL}$. If $\alpha, \beta \in \mathcal{K}$, then in the category $\mathcal{K}$ we have*

(i) $\alpha \leq_{\mathscr{R}} \beta \;\Leftrightarrow\; \alpha\,\boldsymbol{\delta} = \beta\,\boldsymbol{\delta},\ \ker(\alpha) \supseteq \ker(\beta),$ *and* $\mathrm{N_U}(\alpha) \supseteq \mathrm{N_U}(\beta)$;

(ii) $\alpha \leq_{\mathscr{L}} \beta \;\Leftrightarrow\; \alpha\,\boldsymbol{\rho} = \beta\,\boldsymbol{\rho},\ \mathrm{coker}(\alpha) \supseteq \mathrm{coker}(\beta),$ *and* $\mathrm{N_L}(\alpha) \supseteq \mathrm{N_L}(\beta)$;

(iii) $$\alpha \leq_{\mathscr{J}} \beta \;\Leftrightarrow\; \begin{cases} \mathrm{Rank}\,\alpha \leq \mathrm{Rank}\,\beta, & \text{if } (a), \\ \mathrm{Rank}\,\alpha \leq \mathrm{Rank}\,\beta \text{ and } \mathrm{Rank}\,\alpha \equiv \mathrm{Rank}\,\beta \pmod 2, & \text{if } (b), \end{cases}$$

where (a) and (b) are the cases $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathscr{M}, \mathcal{PB}\}$ and $\mathcal{K} \in \{\mathcal{B}, \mathcal{TL}\}$, respectively;

(iv) $\alpha\,\mathscr{R}\,\beta \;\Leftrightarrow\; \ker(\alpha) = \ker(\beta)$ *and* $\mathrm{N_U}(\alpha) = \mathrm{N_U}(\beta)$
$\Leftrightarrow\; \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ *and* $\ker(\alpha) = \ker(\beta)$;

(v) $\alpha\,\mathscr{L}\,\beta \;\Leftrightarrow\; \mathrm{coker}(\alpha) = \mathrm{coker}(\beta)$ *and* $\mathrm{N_L}(\alpha) = \mathrm{N_L}(\beta)$
$\Leftrightarrow\; \mathrm{codom}(\alpha) = \mathrm{codom}(\beta)$ *and* $\mathrm{coker}(\alpha) = \mathrm{coker}(\beta)$;

(vi) $\alpha\,\mathscr{J}\,\beta \;\Leftrightarrow\; \mathrm{Rank}\,\alpha = \mathrm{Rank}\,\beta.$

Furthermore, the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{M}$, and $\mathcal{TL}$ are all stable, so $\mathscr{J} = \mathscr{D}$ in each of these categories.

Proof. To keep the argument concise, we write $\alpha = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{array} \right) \in \mathcal{P}_{mn}$ and $\beta = \left(\begin{array}{c|c|c|c|c|c} E_1 & \cdots & E_q & G_1 & \cdots & G_u \\ F_1 & \cdots & F_q & H_1 & \cdots & H_v \end{array} \right) \in \mathcal{P}_{kl}$, and we assume $\min(A_1) < \ldots < \min(A_r)$ and $\min(E_1) < \ldots < \min(E_r)$.

(*i*) Suppose $\alpha \leq_{\mathscr{R}} \beta$. Then, $\alpha = \beta\gamma$ for some $\gamma \in \mathcal{K}$, so (5.3) implies $\alpha\,\boldsymbol{\delta} = \beta\,\boldsymbol{\delta}$, $\ker(\alpha) \supseteq \ker(\beta)$, and $\mathrm{N_U}(\alpha) \supseteq \mathrm{N_U}(\beta)$. Conversely, suppose that $\alpha\,\boldsymbol{\delta} = \beta\,\boldsymbol{\delta}$, $\ker(\alpha) \supseteq \ker(\beta)$, and $\mathrm{N_U}(\alpha) \supseteq \mathrm{N_U}(\beta)$. From the third assumption, we have $s \geq u$, and we may suppose without loss of generality that $G_i = C_i$ for all $1 \leq i \leq u$. Then, the second assumption implies that each of the remaining blocks of α may be presented in the form $\bigcup_{i \in J} E_i$, where $\emptyset \neq J \subseteq [q]$. Thus, the equality $\beta\beta^* = \left(\begin{array}{c|c|c|c|c|c} E_1 & \cdots & E_q & G_1 & \cdots & G_u \\ E_1 & \cdots & E_q & G_1 & \cdots & G_u \end{array} \right)$ gives $\alpha = \beta\beta^*\alpha \leq_{\mathscr{R}} \beta$.

(*iv*) follows immediately from (*i*) (having $\ker(\alpha)$, one may determine $\mathrm{dom}(\alpha)$ from $\mathrm{N_U}(\alpha)$, and vice versa). Furthermore, (*ii*) and (*v*) are duals of (*i*) and (*iv*), respectively.

(*iii*) If $\alpha \leq_{\mathscr{J}} \beta$, then $\alpha = \gamma_1\beta\gamma_2$, for some $\gamma_1, \gamma_2 \in \mathcal{K}$. Hence, $\mathrm{Rank}\,\alpha \leq \mathrm{Rank}\,\beta$ (the additional condition in the case (b) following from (5.4)). Conversely, suppose $\mathrm{Rank}\,\alpha \leq \mathrm{Rank}\,\beta$ (and $\mathrm{Rank}\,\alpha \equiv \mathrm{Rank}\,\beta \pmod 2$ if (*b*)). Then, $r \leq q$. If

$$\gamma_1 = \left(\begin{array}{c|c|c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & & & C_s \\ E_1 & \cdots & E_r & E_{r+1} \cup E_{r+2} & \cdots & E_{q-1} \cup E_q & G_1 \;|\; \cdots & G_u \end{array} \right),$$

and

$$\gamma_2 = \left(\begin{array}{c|c|c|c|c|c|c|c} F_1 & \cdots & F_r & F_{r+1} \cup F_{r+2} & \cdots & F_{q-1} \cup F_q & H_1 \;|\; \cdots & H_v \\ B_1 & \cdots & B_r & D_1 & \cdots & & & D_t \end{array} \right),$$

we clearly have $\alpha = \gamma_1\beta\gamma_2$. Moreover, $\gamma_1, \gamma_2 \in \mathcal{K}$. Let us elaborate on this. Firstly, if α and β are planar, so are γ_1 and γ_2, because their blocks satisfy $(i) - (v)$ of Lemma 5.1.1 (any two "union blocks" are separated, because $E_{r+1}, \ldots, E_q$ are parts of transversal blocks in $\ker(\beta)$ and $E_{r+1} < \ldots < E_q$). Secondly, if $\alpha, \beta \in \mathcal{B}$ (or $\mathcal{PB}$), then $\gamma_1, \gamma_2 \in \mathcal{B}$ (or $\mathcal{PB}$), as the E_i's and F_j's are singletons in this case.

Part (vi) follows directly from (iii). For the last statement, recall that, by Lemma 2.2.19, stability implies $\mathscr{J} = \mathscr{D}$. Thus, it suffices to prove stability. For any $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathcal{B}, \mathcal{PB}, \mathscr{M}, \mathcal{TL}\}$, any $i, j \in \mathbb{N}_0$, and any $\alpha \in \mathcal{K}_{ji}$, consider the semigroups $\alpha\mathcal{K}_{ij}$ and $\mathcal{K}_{ij}\alpha$. Clearly, both are finite, and hence periodic. Then, α is stable, by Lemma 2.2.27. Since α was chosen arbitrarily, the whole partial semigroup $\mathcal{K}$ is stable. □

Fix $m, n \in \mathbb{N}_0$. Recall that, for any $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathcal{B}, \mathcal{PB}, \mathscr{M}, \mathcal{TL}\}$ and any $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$, Green's relation $\mathscr{K}$ of $\mathcal{K}_{mn}$ is the relation $\mathscr{K} \cap (\mathcal{K}_{mn} \cap \mathcal{K}_{mn})$. As in [28], we may immediately conclude the following:

Corollary 5.1.3. *Let $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathcal{B}, \mathcal{PB}, \mathscr{M}, \mathcal{TL}\}$, $m, n \in \mathbb{N}_0$, and suppose $m \equiv n \pmod 2$ if $\mathcal{K} \in \{\mathcal{B}, \mathcal{TL}\}$. Then the $\mathscr{J} = \mathscr{D}$-classes of $\mathcal{K}_{mn}$ are the sets*

$$\mathrm{D}_r(\mathcal{K}_{mn}) = \{\alpha \in \mathcal{K}_{mn} : \mathrm{Rank}\,\alpha = r\} \qquad \text{for each } 0 \le r \le \min(m,n), \text{ where } r \equiv m \equiv n \pmod 2 \text{ if } \mathcal{K} \in \{\mathcal{B}, \mathcal{TL}\}.$$

These classes form a chain: $\mathrm{D}_r(\mathcal{K}_{mn}) \le_{\mathscr{J}} \mathrm{D}_s(\mathcal{K}_{mn}) \Leftrightarrow r \le s$.

Now, we turn to the combinatorial side of the story. In order to present it, we need to introduce some combinatorial notions we have not mentioned previously. For $n \in \mathbb{N}_0$,

$\mathrm{B}(n)$ is the number of partitions of an n-element set (if $n = 0$, we define $\mathrm{B}(0) = 1$), known as the nth Bell number (see [1], A000110). It can be calculated via the well-known formula $\mathrm{B}(n) = \sum_{k=1}^{n} \mathcal{S}(n,k)$, where $\mathcal{S}(n,k)$ is the Stirling number of the second kind (from Section 3.1).

$n!!$ is known as the double factorial ([1], A123023). We define it by

$$n!! = \begin{cases} n \cdot (n-2) \cdot \dots \cdot 1, & \text{if } n \in \mathbb{N}_0 \text{ is odd;} \\ 0, & \text{if } n \in \mathbb{N}_0 \text{ is even;} \\ 1, & \text{if } n = -1. \end{cases}$$

Usually, if n is even, $n!!$ is defined by $n \cdot (n-2) \cdot \dots \cdot 2$. However, it will be more convenient to use our definition, because then, for $n \in \mathbb{N}_0$, the value $(n-1)!!$ equals the number of partitions of an n-element set into blocks of size 2.

$\mathrm{a}(n)$ is defined by the recurrence

$$\mathrm{a}(n) = \mathrm{a}(n-1) + (n-1)\,\mathrm{a}(n-2) \;\; \text{for} \;\; n \ge 2, \qquad \text{and} \qquad \mathrm{a}(0) = \mathrm{a}(1) = 1.$$

(See [1], A000085). One may easily prove that $a(n)$ is the number of partitions of an n-element set into blocks of size at most 2 (choose an element and consider the block containing it).

$C(n)$ is the nth Catalan number ([1], A000108), defined by $C(n) = \frac{1}{n+1}\binom{2n}{n}$ for $n \in \mathbb{N}_0$. It is well-known that Catalan numbers obey the following recurrence:

$$C(0) = 1,\ C(n) = \sum_{i=1}^{n} C(i-1)\, C(n-i) \text{ for } n \geq 1. \tag{5.5}$$

We will show that $C(n)$ is the number of non-crossing partitions of the set $[n]$. Here, by a *non-crossing partition* of the set $[n]$, we mean a partition having the following property: if each element $1 \leq i \leq n$ is assigned to $(1, \frac{2\pi i}{n})$ (in polar coordinates) and the elements of the same partition are connected by an edge drawn within the circle $((0,0),1)$, then the edges connecting different blocks do not intersect. We define $f(n)$ to be the number of such partitions (of $[n]$), and we want to show that $f(n) = C(n)$. Firstly, note that $\emptyset$ has a single non-crossing partition $\emptyset$, so $f(0) = 1$. Furthermore, if $n \geq 1$, then consider the block containing the element n, and suppose l is its minimal element. Clearly, $1 \leq l \leq n$, and the sets $[l-1]$ and $\{l, l+1, \ldots, n-1, n\}$ both form a non-crossing partition; also note that the partition of $\{l, l+1, \ldots, n-1, n\}$ may be identified with one of the $f(n-l)$ non-crossing partitions of $[n-l]$ (since elements n and l are connected, we identify the two). Thus, $f(n)$ is described by the recurrence 5.5, and hence $f(n) = C(n)$.

Finally, note that any planar partition $\alpha \in \mathscr{P}\mathcal{P}_{mn}$ corresponds to a non-crossing partition of $[m+n]$, via the map

$$i \mapsto m+1-i, \quad j' \mapsto m+j, \quad \text{for } i \in [m] \text{ and } j \in [n]$$

(see Figure 5.6). It is easily seen that this correspondence is a bijection, therefore $|\mathscr{P}\mathcal{P}_{mn}| = C(m+n)$.

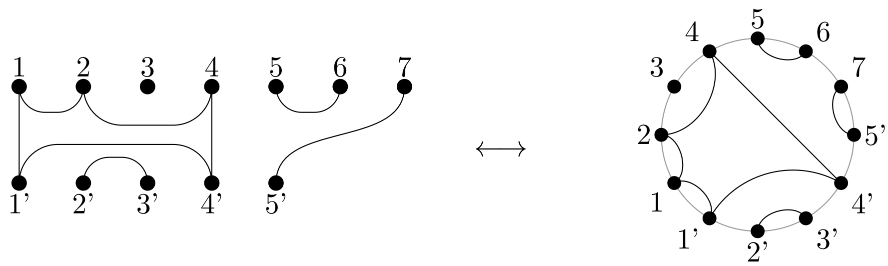


Figure 5.6: A planar partition α from $\mathscr{P}\mathcal{P}_{7,5}$, with its corresponding non-crossing partition of [12].

In addition, if $x \notin \mathbb{N}_0$, we define $C(x) = 0$.

$\mu(n,k)$ is defined by the recurrence

$$\mu(n,k) = \mu(n-1,k-1) + \mu(n-1,k) + \mu(n-1,k+1) \quad \text{if } n \geq 1 \text{ and } 0 \leq k \leq n, \quad \mu(0,0) = 1, \quad \mu(n,k) = 0 \text{ if } n < k \text{ or } k < 0. \tag{5.6}$$

These are the Motzkin triangle numbers (see [1], A064189).

$\mu(n)$ is the nth Motzkin number $\mu(n,0)$ ([1], A001006). It is well-known that the Motzkin numbers satisfy the following recurrence

$$\mu(0) = 1, \quad \mu(n) = \mu(n-1) + \sum_{i=1}^{n-1} \mu(i-1)\mu(n-i-1) \quad \text{for } n \geq 1 \tag{5.7}$$

(e.g., see [2]). Let $g(n)$ denote the number of non-crossing partitions of the set $[n]$ into blocks of size ≤ 2. We will prove that $g(n)$ satisfies the recurrence (5.7), so $g(n) = \mu(n)$. Clearly, the empty set can be partitioned in only one way. Suppose $n \geq 1$ and consider the block B containing the element n: it can be a singleton, in which case there are $g(n-1)$ ways to partition $n-1$; otherwise, $B = \{n,i\}$ for some $1 \leq i \leq n-1$, so the elements of the sets $[i-1]$ and $\{i+1,\ldots,n-1\}$ can be partitioned in $g(i-1)$ and $g(n-1-i)$ ways, respectively. Thus, $g(n)$ satisfies (5.7).

Recall from the discussion concerning Catalan numbers that each planar partition from P_{mn} corresponds to a non-crossing partition of $[m+n]$. Thus, it is easily seen that $g(m+n) = \mu(m+n)$ is the number of elements in $\mathscr{M}_{mn}$.

$\mathrm{p}(n,k)$ is defined by the recurrence

$$\begin{aligned} \mathrm{p}(n,k) &= \mathrm{p}(n-1,k-1) + \mathrm{p}(n-1,k+1), \\ &\qquad \text{if } 0 \leq k \leq n \text{ and } n \equiv k \pmod 2, \\ \mathrm{p}(n,n) &= 1 \text{ for all } n \geq 0, \\ \mathrm{p}(n,0) &= 1 \text{ if } n \equiv 0 \pmod 2 \text{ and } n \geq 0 \\ \mathrm{p}(n,k) &= 0, \text{ if } n < k, \text{ or } k < 0 \text{ or } n \not\equiv k \pmod 2. \end{aligned} \tag{5.8}$$

It is easily seen that the numbers

$$\frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}$$

satisfy the above recurrence. These numbers correspond to the number of subdiagonal rectangular lattice paths from $(0,0)$ to $(\frac{n+k}{2}, \frac{n-k}{2})$ (see [13, page 303]).

For more information on these number sequences, we refer the reader to the Online Encyclopedia of Integer Sequences [1].

Now, we prove

Proposition 5.1.4. *If $m, n \in \mathbb{N}_0$, then*

(i) $|\,\mathcal{P}_{mn}\,| = \mathrm{B}(m+n)$,

(ii) $|\,\mathcal{PB}_{mn}\,| = \mathrm{a}(m+n)$,

(iii) $|\,\mathcal{B}_{mn}\,| = (m+n+1)!!$,

(iv) $|\,\mathscr{P}\mathcal{P}_{mn}\,| = \mathrm{C}(m+n)$,

(v) $|\,\mathscr{M}_{mn}\,| = \mu(m+n)$,

(vi) $|\,\mathcal{TL}_{mn}\,| = \mathrm{C}(\frac{m+n}{2})$,

Proof. Parts $(i)-(v)$ follow from the above discussion, so we prove only (vi) (note that, for m, n even the statement follows from $\mathcal{TL}^{even} \cong \mathscr{P}\mathcal{P}$). For $k \in \mathbb{N}_0$, let $h(k)$ denote the number of non-crossing partitions of the set $[2k]$ into blocks of size 2. We may enumerate these partitions in the following way: if $k \in \mathbb{N}$, the block containing the element $2k$ is $\{2k, 2i-1\}$ for some $1 \leq i \leq k$ ($2k$ cannot be connected to an even vertex, because it would not be possible to create a non-crossing matching); hence, the elements of the sets $[2i-2]$ and $\{2i, \ldots, 2k-1\}$ may be connected in $h(\frac{2i-2}{2})$ and $h(\frac{2k-1-2i+1}{2})$ ways. Since $h(0) = 1$, the sequence $h(i) : i \in \mathbb{N}_0$ satisfies the recurrence (5.5). Thus, $h(k) = \mathrm{C}(k)$ for all $k \in \mathbb{N}_0$. From the discussion concerning Catalan numbers and non-crossing partitions, we conclude that $|\,\mathcal{TL}_{mn}\,| = h(\frac{m+n}{2}) = \mathrm{C}(\frac{m+n}{2})$. □

Finally, we may calculate the combinatorial properties of the hom-set $\mathcal{K}_{mn}$ for $\mathcal{K} = \mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{B}$, $\mathcal{TL}$. Part (i) is crucial, and it is implied by earlier results: for $\mathcal{P}$, $\mathcal{B}$, and $\mathcal{TL}$, the formulae follow from Theorems 7.5, 8.4 and 9.5 in [38], respectively; for $\mathcal{PB}$ and $\mathscr{M}$, the formulae follow from Propositions 2.7 and 2.8 in [32], respectively.

Proposition 5.1.5. *Let $\mathcal{K}$ denote any of the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{B}$ or $\mathcal{TL}$. Let $m, n \in \mathbb{N}_0$, fix some $0 \leq r \leq \min(m, n)$, and suppose $r \equiv m \equiv n \pmod 2$ if $\mathcal{K}$ is $\mathcal{B}$ or $\mathcal{TL}$.*

(i) The number of $\mathscr{R}$-classes contained in $\mathrm{D}_r(\mathcal{K}_{mn})$ is given by

$$|\,\mathrm{D}_r(\mathcal{K}_{mn})/\mathscr{R}\,| = \begin{cases} \sum_{i=r}^{m} \binom{i}{r} \mathcal{S}(m,i), & \text{if } \mathcal{K} = \mathcal{P}, \\ \frac{2r+1}{2m+1}\binom{2m+1}{m-r}, & \text{if } \mathcal{K} = \mathscr{P}\mathcal{P}, \\ \binom{m}{r}\mathrm{a}(m-r), & \text{if } \mathcal{K} = \mathcal{PB}, \\ \mu(m,r), & \text{if } \mathcal{K} = \mathscr{M}, \\ \binom{m}{r}(m-r-1)!!, & \text{if } \mathcal{K} = \mathcal{B}, \\ \frac{r+1}{m+1}\binom{m+1}{(m-r)/2}, & \text{if } \mathcal{K} = \mathcal{TL}. \end{cases}$$

(ii) *The number of $\mathscr{L}$-classes contained in* $\mathrm{D}_r(\mathcal{K}_{mn})$ *is given by*

$$|\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{L}\,| = \begin{cases} \sum_{i=r}^{n} \binom{i}{r} \mathcal{S}(n,i), & \text{if } \mathcal{K} = \mathcal{P}, \\ \frac{2r+1}{2n+1}\binom{2n+1}{n-r}, & \text{if } \mathcal{K} = \mathscr{P}\mathcal{P}, \\ \binom{n}{r}\,\mathrm{a}(n-r), & \text{if } \mathcal{K} = \mathcal{PB}, \\ \mu(n,r), & \text{if } \mathcal{K} = \mathscr{M}, \\ \binom{n}{r}(n-r-1)!!, & \text{if } \mathcal{K} = \mathcal{B}, \\ \frac{r+1}{n+1}\binom{n+1}{(n-r)/2}, & \text{if } \mathcal{K} = \mathcal{TL}. \end{cases}$$

(iii) $|\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{H}\,| = |\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{R}\,| \cdot |\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{L}\,|$
(parts (i) and (ii) give $|\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{R}\,|$ *and* $|\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{L}\,|$*).*

(iv) *The size of any $\mathscr{H}$-class H in* $\mathrm{D}_r(\mathcal{K}_{mn})$ *is given by*

$$|H| = \begin{cases} r!, & \text{if } \mathcal{K} \text{ is one of } \mathcal{P}, \mathcal{PB}, \mathcal{B}; \\ 1, & \text{if } \mathcal{K} \text{ is one of } \mathscr{P}\mathcal{P}, \mathscr{M}, \mathcal{TL}. \end{cases}$$

(v) $$|\,\mathrm{D}_r(\mathcal{K}_{mn})| = \begin{cases} |\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{H}\,| \cdot r!, & \text{if } \mathcal{K} \text{ is one of } \mathcal{P}, \mathcal{PB}, \mathcal{B}; \\ |\,\mathrm{D}_r(\mathcal{K}_{mn})/\,\mathscr{H}\,|, & \text{if } \mathcal{K} \text{ is one of } \mathscr{P}\mathcal{P}, \mathscr{M}, \mathcal{TL}. \end{cases}$$

Proof. (*i*) By Proposition 5.1.2, $\mathrm{D}_r(\mathcal{K}_{mn}) = \{\alpha \in \mathcal{K}_{mn} : \mathrm{Rank}(\alpha) = r\}$ and the $\mathscr{R}$-class of a diagram is completely determined by its domain and kernel. Thus, in each case, it suffices to calculate the number of all possible domain-kernel combinations of rank r in $\mathcal{K}_{mn}$.

- Suppose $\mathcal{K} = \mathcal{P}$. First, we choose the number of blocks of the kernel $r \leq i \leq m$ (the lower bound is the required rank, and the upper is the number of elements of $[m]$). Such a kernel may be chosen in $\mathcal{S}(m,i)$ ways. In it, any r classes may constitute the domain.

- Suppose $\mathcal{K} = \mathcal{PB}$. In this case, the blocks of the kernel containing the elements of the domain are all singletons, so the domain may be chosen in $\binom{m}{r}$ ways. Thus, we need to partition the rest of the set $[m]$ into blocks of size ≤ 2. This may be done in $\mathrm{a}(m-r)$ ways.

- Suppose $\mathcal{K} = \mathscr{M}$. For $i, j \in \mathbb{N}_0$, let $q(i,j)$ denote the number of partitions of the set $[i]$ into blocks of size ≤ 2, with at least j classes that are both singletons and unnested in the kernel (these will be the domain classes). Note that $|\,\mathrm{D}_r(\mathscr{M}_{mn})/\,\mathscr{R}\,| = q(m,r)$. Let us find a recurrence describing $q(i,j)$. We clearly have $q(0,0) = 1$, and $q(i,j) = 0$ if $i < j$, $j < 0$ or $i < 0$. Suppose $i \geq j \geq 0$. To calculate $q(i,j)$, consider the block containing the element i.

- It may be a domain block, which means that $\{i\}$ is an unnested block of the kernel. In this case, we need to partition the set $[i-1]$ (into blocks of size ≤ 2) so that we have $j-1$ domain (singleton) blocks; this can be done in $q(i-1, j-1)$ ways.
- It may be a non-domain singleton block. Then, we need to partition the set $[i-1]$, and we still need j domain (singleton) blocks. There are $q(i-1, j)$ such partitions.
- It may be a non-domain, non-singleton block. In this case, we need to partition $[i-1]$ (into blocks of size ≤ 2) so that we have j domain singleton blocks and one more unnested singleton block which will contain i's pair in the kernel. Equivalently, we may choose any of $q(i-1, j+1)$ partitions of $[i-1]$ into blocks of size ≤ 2 with $j+1$ domain classes (the domain class containing the biggest domain element will be i-s pair since otherwise we lose the planarity property).

Therefore, the numbers $q(i, j)$ satisfy the recurrence (5.6), so $q(m, r) = \mu(m, r)$.

- Suppose $\mathcal{K} = \mathcal{B}$. As in the proof for $\mathcal{PB}$, the elements of the domain belong to singleton sets in the kernel, so the domain may be chosen in $\binom{m}{r}$ ways. The remaining $m-r$ elements of $[m]$ need to be partitioned into blocks of size 2, which may be done in $(m-r-1)!!$ ways.

- Suppose $\mathcal{K} = \mathcal{TL}$. For $i, j \in \mathbb{N}_0$ with $i \equiv j \pmod 2$, let $q(i, j)$ denote the number of partitions of the set $[i]$ into blocks of size ≤ 2, with exactly j singleton classes and $\frac{i-j}{2}$ two-element classes, such that the singletons are unnested in the kernel. Note that $|\,\mathrm{D}_r(\mathcal{TL}_{mn})/\mathscr{R}\,| = q(m, r)$. Let us find a recurrence describing $q(i, j)$. We clearly have $q(i, i) = 1$ for all $i \geq 0$, and $q(i, 0) = 1$ for even i's. Furthermore, $q(i, j) = 0$ if $i < j$, or $j < 0$, or $i \not\equiv j (\mathrm{mod}\, 2)$. Suppose $i \geq j \geq 0$ and $i \equiv j (\mathrm{mod}\, 2)$. To calculate $q(i, j)$, consider the block containing the element i.

  - It may be a domain block, which means that $\{i\}$ is an unnested block of the kernel. In this case, we need to partition the set $[i-1]$ (into blocks of size ≤ 2) so that we have $j-1$ domain (singleton) blocks; this can be done in $q(i-1, j-1)$ ways.
  - It may be a non-domain block. In this case, we need to partition $[i-1]$ (into blocks of size ≤ 2) so that we have j domain singleton blocks and one more unnested singleton block which will contain i's pair in the kernel. Equivalently, we may choose any of $q(i-1, j+1)$ partitions of $[i-1]$ into blocks of size ≤ 2 with $j+1$ domain classes (the domain class containing the biggest domain element will be i-s pair since otherwise we lose the planarity property).

  Therefore, the numbers $q(i, j)$ satisfy the recurrence (5.8), so $q(m, r) = \mathrm{p}(m, r)$.

- Suppose $\mathcal{K} = \mathscr{P}\mathcal{P}$. We have concluded that the partial semigroup $\mathscr{P}\mathcal{P}$ is isomorphic to $\mathcal{TL}^{\text{even}}$ (see page 204). More precisely, each partition of rank r in $\mathscr{P}\mathcal{P}_{mn}$ corresponds to a partition of rank $2r$ in $\mathcal{TL}_{2m,2n}$. Hence,

$$|\,\mathrm{D}_r(\mathscr{P}\mathcal{P}_{mn})/\mathscr{R}\,| = |\,\mathrm{D}_{2r}(\mathcal{TL}_{2m,2n})/\mathscr{R}\,| = \frac{2r+1}{2m+1}\binom{2m+1}{(2m-2r)/2}.$$

Part (ii) follows by a dual argument, (iii) is a direct consequence of (i) and (ii), and (v) follows immediately from (iii) and (iv) (proved below).

(iv) If we fix an $\mathscr{R}$-class and an $\mathscr{L}$-class of $\mathrm{D}_r(\mathcal{K}_{mn})$, the number of elements in their intersection equals the number of ways to pair the r transversal classes of the kernel with r transversal classes of the cokernel. If $\mathcal{K} \subseteq \mathscr{P}\mathcal{P}$, there is only one such pairing, while non-planar partitions may be created in $r!$ ways. □

As in the previous chapters, here we answer the remaining questions concerning different aspects of regularity in our partial semigroups. These results are new, as far as the author is aware.

Proposition 5.1.6. *Neither of the partial semigroups $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{B}$, and $\mathcal{TL}$ can be expanded to an inverse partial semigroup.*

Proof. Consider the partition

$$\alpha = \left(\begin{array}{c|c} 1 & \{2,3\} \\ \hline 1 & \{2,3\} \end{array}\right) \in \mathcal{P}_3 \cap \mathscr{P}\mathcal{P}_3 \cap \mathcal{B}_3 \cap \mathcal{PB}_3 \cap \mathcal{TL}_3 \cap \mathscr{M}_3\,.$$

Clearly, α is an idempotent, and hence a self-inverse element. However, the partition

$$\beta = \left(\begin{array}{c|c} \{1,2\} & 3 \\ \hline \{1,2\} & 3 \end{array}\right) \in \mathcal{P}_3 \cap \mathscr{P}\mathcal{P}_3 \cap \mathcal{B}_3 \cap \mathcal{PB}_3 \cap \mathcal{TL}_3 \cap \mathscr{M}_3\,.$$

is also a semigroup inverse of α. Therefore, α does not have a unique inverse in either of the partial semigroups $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathcal{TL}$ and $\mathscr{M}$. □

Proposition 5.1.7. *Let $\mathcal{K}$ denote any of the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{B}$, or $\mathcal{TL}$. Let $m, n \in \mathbb{N}_0$ (and $m \equiv n \pmod 2$ if $\mathcal{K}$ is $\mathcal{B}$ or $\mathcal{TL}$), and let $\alpha \in \mathcal{K}_{mn}$. Then,*

(i) α is right-invertible in $\mathcal{K}_{nm}$ if and only if $\ker(\alpha) = \{\{x\} : x \in [m]\}$ and $\mathrm{dom}(\alpha) = [m]$. In that case, α^* is a right inverse of α.

(ii) α is left-invertible in $\mathcal{K}_{nm}$ if and only if $\mathrm{coker}(\alpha) = \{\{x\} : x \in [n]\}$ and $\mathrm{codom}(\alpha) = [n]$. In that case, α^* is a left inverse of α.

Proof. We prove only the first part since the second is dual. Suppose $\ker(\alpha) = \{\{x\} : x \in [m]\}$ and $\mathrm{dom}(\alpha) = [m]$. Then, $\alpha\alpha^* = \iota_m$, so $\beta\alpha\alpha^* = \beta$ for any $\beta \in \mathcal{K}_{nm}$. Thus, α is right-invertible. Conversely, suppose α is right-invertible. In other words, there exists $\beta \in \mathcal{K}_{nm}$ such that $\zeta\alpha\beta = \zeta$ for each $\zeta \in \mathcal{K}_{nm}$. Suppose the opposite: either $\ker(\alpha)$ has a non-singleton class C, or there exists an element $1 \le i \le m$ such that $i \notin \mathrm{dom}(\alpha)$. In the first case, there exist $a, b \in C$ with $a \neq b$. If we choose a partition $\zeta \in \mathcal{K}_{nm}$ such that a' and b' belong to different transversal blocks, then these blocks become connected in the product graph $\Pi(\zeta, \alpha, \beta)$, for any $\beta \in \mathcal{K}_{nm}$. Thus, $\zeta\alpha\beta$ cannot equal ζ (for any $\beta \in \mathcal{K}_{nm}$), which contradicts the assumption of right-invertibility. In the second case, we choose a partition $\zeta \in \mathcal{K}_{nm}$ such that i' belongs to a transversal block. By the above discussion, we may assume that $\{i\}$ is a class of $\ker(\alpha)$, so a similar argument leads to a contradiction. □

Corollary 5.1.8.

(i) If $\mathcal{K}_{mn}$ contains a right-invertible element, then $m \leq n$.

(ii) If $\mathcal{K}_{mn}$ contains a left-invertible element, then $n \leq m$.

Proof. We prove only the first statement, as the second follows by a dual argument. From Proposition 5.1.7(i), we have that a right-invertible element $\alpha \in \mathcal{K}_{mn}$ has rank m, and we know that $\mathrm{Rank}(\alpha) \leq \min(m, n)$. Thus, $m \leq n$. □

5.2 Sandwich semigroups in diagram categories

Having studied the partial semigroups $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PB}$ and $\mathscr{M}$ in detail, we may now focus on the sandwich semigroups in them. Therefore, we consider a partition $\sigma \in \mathcal{K}_{nm}$ (where $\mathcal{K}$ denotes any of the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PB}$ and $\mathscr{M}$) with $m, n \in \mathbb{N}_0$ (and $m \equiv n \pmod 2$ if $\mathcal{K}$ is $\mathcal{B}$ or $\mathcal{TL}$) and we let $r = \mathrm{Rank}\,\sigma$. We aim to investigate the sandwich semigroup

$$\mathcal{K}_{mn}^{\sigma} = (\mathcal{K}_{mn}, \star_{\sigma}).$$

Recall that $\mathcal{K} \to \mathcal{K} : \alpha \mapsto \alpha^*$ maps $\mathcal{K}_{mn}$ to $\mathcal{K}_{nm}$ and that it is an anti-isomorphism for all $\mathcal{K} \in \{\mathrm{P}, \mathscr{P}\mathcal{P}, \mathscr{M}, \mathcal{PB}, \mathcal{B}, \mathcal{TL}\}$, because $(\alpha\beta)^* = \beta^*\alpha^*$ and $(\alpha^*)^* = \alpha$ (see the previous section), so it is easily seen that

Lemma 5.2.1. *Let $\mathcal{K}$ be any of the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PB}$ and $\mathscr{M}$. Let $m, n \in \mathbb{N}_0$, $\sigma \in \mathcal{K}_{mn}$ and suppose $m \equiv n \pmod 2$. Then, $\mathcal{K}_{mn}^{\sigma}$ is anti-isomorphic to $\mathcal{K}_{nm}^{\sigma^*}$.*

Thus, we may assume without loss of generality that $n \leq m$, which implies $r = \mathrm{Rank}\,\sigma \leq n \leq m$. Hence, we write

$$\sigma = \left(\begin{array}{c|c|c|c|c|c} X_1 & \cdots & X_r & U_1 & \cdots & U_s \\ \cline{4-6} Y_1 & \cdots & Y_r & V_1 & \cdots & V_t \end{array} \right).$$

Furthermore, if $\mathcal{K}$ is one of $\mathscr{P}\mathcal{P}$, $\mathcal{TL}$ or $\mathscr{M}$, we assume $\min X_1 < \ldots < \min X_r$. These assumptions apply to our investigation for the rest of the chapter, unless otherwise stated.

Again, we point out that the results presented in this chapter were published in [28].

5.2.1 Green's relations and regularity in $\mathcal{K}^\sigma_{mn}$

As in the previous chapters, we examine P-sets, Green's relations and their classes, as well as the maximal and minimal $\mathscr{J}^\sigma$-classes of $\mathcal{K}^\sigma_{mn}$.

First, we describe P-sets in $\mathcal{K}^\sigma_{mn}$. However, we do not give combinatorial criteria for membership in these sets, because they are quite cumbersome, while not very useful. For instance, a partition $\alpha \in \mathcal{K}_{mn}$ belongs to P^σ_1 if and only if the restriction of $\mathrm{coker}(\alpha) \vee \ker(\sigma)$ on $\mathrm{codom}(\alpha)$ is the relation $\mathrm{coker}(\alpha) \cap (\mathrm{codom}(\alpha) \times \mathrm{codom}(\alpha))$, and each class of $\mathrm{coker}(\alpha) \vee \ker(\sigma)$ containing an element of $\mathrm{codom}(\alpha)$ also contains an element of $\mathrm{dom}(\sigma)$. Hence, we simply state the characterisation following from the definition:

Proposition 5.2.2. *We have* $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{K}^\sigma_{mn})$ *and*

$$\begin{aligned}
\mathrm{P}^\sigma_1 &= \{\alpha \in \mathcal{K}_{mn} : \mathrm{Rank}(\alpha\sigma) = \mathrm{Rank}(\alpha)\},\\
\mathrm{P}^\sigma_2 &= \{\alpha \in \mathcal{K}_{mn} : \mathrm{Rank}(\sigma\alpha) = \mathrm{Rank}(\alpha)\},\\
\mathrm{P}^\sigma_3 &= \mathrm{P}^\sigma = \{\alpha \in \mathcal{K}_{mn} : \mathrm{Rank}(\alpha\sigma) = \mathrm{Rank}(\sigma\alpha) = \mathrm{Rank}(\alpha)\}\\
&\qquad\quad = \{\alpha \in \mathcal{K}_{mn} : \mathrm{Rank}(\sigma\alpha\sigma) = \mathrm{Rank}(\alpha)\}.
\end{aligned}$$

Proof. By Proposition 5.1.2, $\mathcal{K}$ is a stable category, so Proposition 2.2.23(*iii*) implies $\mathrm{P}^\sigma_3 = \mathrm{P}^\sigma$. Moreover, since $\mathcal{K}$ is a regular partial $*$-semigroup, Proposition 2.2.29(*iv*) gives $\mathrm{Reg}(\mathcal{K}^\sigma_{mn}) = \mathrm{P}^\sigma$. Thus, the first two statements follow from the definition of P-sets, Proposition 5.1.2(*vi*) and from $\alpha\sigma \mathrel{\mathscr{R}} \alpha \Leftrightarrow \alpha\sigma \mathrel{\mathscr{J}} \alpha$ and $\sigma\alpha \mathrel{\mathscr{L}} \alpha \Leftrightarrow \sigma\alpha \mathrel{\mathscr{J}} \alpha$ (both hold by stability), respectively. The third statement follows directly from the definition of P^σ_3 and Proposition 5.1.2(*vi*). $\square$

Remark 5.2.3. Though complex in general, the combinatorial characterisations simplify in some cases. For example, if $\mathcal{K} = \mathcal{B}$, Proposition 5.3.8 provides an elegant and useful description of the P-sets in $\mathcal{B}^\sigma_{mn}$.

Having introduced P-sets, we may describe Green's relations of $\mathcal{K}^\sigma_{mn}$. Recall that $\mathcal{K}$ is a stable semigroup (Proposition 5.1.2). Thus, Corollary 2.2.26 implies that $\mathscr{D}^\sigma = \mathscr{J}^\sigma$ in $\mathcal{K}^\sigma_{mn}$. Therefore, Theorem 2.2.3 gives

Theorem 5.2.4. *Let $\mathcal{K}$ denote any of the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{B}$, or $\mathcal{TL}$. Suppose $\alpha \in \mathcal{K}_{mn}$. Then, in $\mathcal{K}^\sigma_{mn}$ we have*

(i) $\mathrm{R}^\sigma_\alpha = \begin{cases} \mathrm{R}_\alpha \cap \mathrm{P}^\sigma_1, & \alpha \in \mathrm{P}^\sigma_1;\\ \{\alpha\}, & \alpha \notin \mathrm{P}^\sigma_1.\end{cases}$

(ii) $\mathrm{L}^\sigma_\alpha = \begin{cases} \mathrm{L}_\alpha \cap \mathrm{P}^\sigma_2, & \alpha \in \mathrm{P}^\sigma_2;\\ \{\alpha\}, & \alpha \notin \mathrm{P}^\sigma_2.\end{cases}$

(iii) $\mathrm{H}^\sigma_\alpha = \begin{cases} \mathrm{H}_\alpha, & \alpha \in \mathrm{P}^\sigma;\\ \{\alpha\}, & \alpha \notin \mathrm{P}^\sigma.\end{cases}$

(iv) $\mathrm{D}^\sigma_\alpha = \mathrm{J}^\sigma_\alpha = \begin{cases} \mathrm{D}_\alpha \cap \mathrm{P}^\sigma, & \alpha \in \mathrm{P}^\sigma; \\ \mathrm{L}^\sigma_\alpha, & \alpha \in \mathrm{P}^\sigma_2 \setminus \mathrm{P}^\sigma_1; \\ \mathrm{R}^\sigma_\alpha, & \alpha \in \mathrm{P}^\sigma_1 \setminus \mathrm{P}^\sigma_2; \\ \{\alpha\}, & \alpha \notin (\mathrm{P}^\sigma_1 \cup \mathrm{P}^\sigma_2). \end{cases}$

Further, if $\alpha \notin \mathrm{P}^\sigma$*, then* $\mathrm{H}^\sigma_\alpha = \{\alpha\}$ *is a non-group* $\mathscr{H}^\sigma$*-class in* $\mathcal{K}^\sigma_{mn}$*.*

Moreover, since $\mathcal{K}$ is monoidal, σ has a left- and right-identity in $\mathcal{K}$, so Lemma 2.2.6 and Proposition 2.2.7 apply as well. Furthermore, from Theorem 5.2.4 and Propositions 2.2.7 and 5.1.2 we have

Corollary 5.2.5. *The regular* $\mathscr{J}^\sigma = \mathscr{D}^\sigma$*-classes of* $\mathcal{K}^\sigma_{mn}$ *are precisely the sets*

$$\mathrm{D}^\sigma_q = \mathrm{D}^\sigma_q(\mathcal{K}^\sigma_{mn}) = \mathrm{D}_q \cap \mathrm{P}^\sigma = \{\alpha \in \mathrm{P}^\sigma : \mathrm{Rank}(\alpha) = q\}$$

for each $0 \leq q \leq r$*, and where* $q \equiv r \pmod 2$ *if* $\mathcal{K}$ *is* $\mathcal{B}$ *or* $\mathcal{TL}$*. These form a chain under the usual ordering of* $\mathscr{J}^\sigma$*-classes:* $\mathrm{D}^\sigma_p \leq \mathrm{D}^\sigma_q \Leftrightarrow p \leq q$*.*

Of course, we are especially interested in the maximal and minimal $\mathscr{J}^\sigma = \mathscr{D}^\sigma$-classes of $\mathscr{K}^\sigma_{mn}$. They are described in the following two results from [28].

Proposition 5.2.6. *Suppose* $\mathcal{K}$ *is one of* $\mathcal{P}$*,* $\mathscr{P}\mathcal{P}$*,* $\mathcal{PB}$*,* $\mathscr{M}$*,* $\mathcal{B}$ *or* $\mathcal{TL}$*. Further, suppose* $m \equiv n \pmod 2$*, if* $\mathcal{K} = \mathcal{B}$ *or* $\mathcal{K} = \mathcal{TL}$*.*

(i) If $r < \min(m,n)$*, then the trivial maximal* $\mathscr{J}^\sigma$*-classes of* $\mathcal{K}^\sigma_{mn}$ *are the singleton sets* $\{\alpha\}$ *for* $\alpha \in \mathcal{K}_{mn}$ *with* $\mathrm{Rank}(\alpha) > r$*. If* $\mathcal{K}$ *is one of* $\mathcal{P}$*,* $\mathcal{PB}$ *or* $\mathcal{B}$*, then* $\mathcal{K}^\sigma_{mn}$ *has no nontrivial maximal* $\mathscr{J}^\sigma$*-classes. If* $\mathcal{K}$ *is one of* $\mathscr{P}\mathcal{P}$*,* $\mathscr{M}$ *or* $\mathcal{TL}$*, the following are equivalent:*

(a) $\mathcal{K}^\sigma_{mn}$ *has a nontrivial maximal* $\mathscr{J}^\sigma$*-class,*

(b) $\mathrm{Pre}(\sigma) \subseteq \mathrm{D}_r(\mathcal{K}_{mn})$*,*

(c) $\mathrm{Pre}(\sigma) = \mathrm{V}(\sigma)$*,*

in which case the nontrivial maximal $\mathscr{J}^\sigma$*-class is the set* $\mathrm{D}^\sigma_r = \{\alpha \in \mathrm{P}^\sigma : \mathrm{Rank}(\alpha) = r\}$*.*

(ii) If $r = \min(m,n)$*, then the set* $\mathrm{D}^\sigma_r = \{\alpha \in \mathrm{P}^\sigma : \mathrm{Rank}(\alpha) = r\}$ *is the maximum* $\mathscr{J}^\sigma$*-class of* $\mathcal{K}^\sigma_{mn}$*. This maximal* $\mathscr{J}^\sigma$*-class is clearly nontrivial.*

Proof. Recall that we assumed without loss of generality that $m \geq n$.

(*ii*). If $r = n$, then $\ker(\sigma) = \{\{x\} : x \in [n]\}$ and $\mathrm{dom}(\sigma) = [n]$. Thus, σ is right-invertible by Proposition 5.1.7(*i*), so Proposition 2.2.35 implies that $\mathcal{K}^\sigma_{mn}$ has a maximum $\mathscr{J}^\sigma$-class, and it contains α^*. Therefore, $\mathrm{J}^\sigma_{\alpha^*} = \mathrm{D}^\sigma_{\alpha^*}$ is the maximum $\mathscr{J}^\sigma$-class, and from Theorem 5.2.4(*iv*) and Corollary 5.2.5 follows $\mathrm{D}^\sigma_{\alpha^*} = \mathrm{P}^\sigma \cap \mathrm{D}_{\alpha^*} = \mathrm{D}^\sigma_r$ (since $\alpha^* \in \mathrm{P}^\sigma$).

(*i*). Suppose $\alpha \in \mathcal{K}_{mn}$ with $r < \mathrm{Rank}(\alpha)$. By Proposition 5.1.2(*iii*), we have $\alpha \not\leq_{\mathscr{J}} \sigma$, so Lemma 2.2.10 implies that $\{\alpha\}$ is a trivial maximal $\mathscr{J}^\sigma$-class.

Suppose that $\mathcal{K}$ is one of $\mathcal{P}$, $\mathcal{PB}$ or $\mathcal{B}$. It suffices to show that $\mathcal{K}^\sigma_{mn}$ does not contain a nontrivial maximal $\mathscr{J}^\sigma$-class. By Proposition 2.2.13, this holds if and only if there exists an element $\alpha \in \mathcal{K}_{mn}$ such that $(\alpha, \sigma) \notin \mathscr{J}$ and $(\sigma, \sigma\alpha\sigma) \in \mathscr{J}$. We analyse the following cases (and use Proposition 5.1.2(vi) throughout the proof):

$\mathbf{s > 0}$ and $\mathbf{t > 0}$. If $\mathcal{K} = \mathcal{P}$, put

$$\alpha = \left(\begin{array}{c|c|c|c|c|c|c} Y_1 & \cdots & Y_r & V_t & V_1 & \cdots & V_{t-1} \\ X_1 & \cdots & X_r & U_s & U_1 & \cdots & U_{s-1} \end{array}\right) \tag{5.9}$$

Now, clearly $\sigma\alpha\sigma = \sigma$, so $\sigma\alpha\sigma \mathrel{\mathscr{J}} \sigma$. Furthermore, $\text{Rank}(\alpha) = r + 1 \neq r = \text{Rank}(\alpha)$, so we have $(\alpha, \sigma) \notin \mathscr{J}$, as required.

If $\mathcal{K}$ is one of $\mathcal{PB}$ or $\mathcal{B}$, the partition (5.9) may not belong to $\mathcal{K}$ (depending on the classes V_t and U_s), so we may have to modify it.

- If $|U_s| = |V_t| = 1$, then $\alpha \in \mathcal{K}$, so we need no changes.
- If $|U_s| = 2$ and $|V_t| = 1$, we have $U_s = \{u_1', u_2'\}$ and $V_t = \{v\}$ for some $u_1, u_2 \in [n]$ and $v \in [m]$. Hence, we replace the transversal $V_t \cup U_s$ of α by the pair of blocks $\{v, u_1'\}$ and $\{u_2'\}$. Again, $\sigma\alpha\sigma = \sigma$ and $(\sigma, \alpha) \notin \mathscr{J}$.
- The case with $|U_s| = 2$ and $|V_t| = 1$ is dual.
- If $|U_s| = 2$ and $|V_t| = 2$, we have $U_s = \{u_1', u_2'\}$ and $V_t = \{v_1, v_2\}$ for some $u_1, u_2 \in [n]$ and $v_1, v_2 \in [m]$. In this case, we replace the transversal $V_t \cup U_s'$ of α by the pair of blocks $\{v_1, u_1'\}$ and $\{v_2, u_2'\}$. Similarly, we have $\sigma\alpha\sigma = \sigma$ and $(\sigma, \alpha) \notin \mathscr{J}$.

Since $r < n \leq m$, if $\mathcal{K}$ is one of $\mathcal{PB}$ or $\mathcal{B}$, then we have $s, t > 0$, so this is the only case possible. Hence, in the remaining cases we will assume $\mathcal{K} = \mathcal{P}$.

$\mathbf{s = 0}$ and $\mathbf{t > 0}$. Since $s = 0$ and $r < n \leq m$, we have $[n] = \bigcup_{i=1}^r X_i$ and we may assume (without loss of generality) that $|X_r| \geq 2$. Fix some partition $\{Z, W\}$ of the set X_r, and put

$$\alpha = \left(\begin{array}{c|c|c|c|c|c|c|c} Y_1 & \cdots & Y_{r-1} & Y_r & V_t & V_1 & \cdots & V_{t-1} \\ X_1 & \cdots & X_{r-1} & Z & W & & & \end{array}\right)$$

It is easily seen that $\sigma\alpha\sigma = \sigma$ and $(\sigma, \alpha) \notin \mathscr{J}$.

$\mathbf{s > 0}$ and $\mathbf{t = 0}$. This case follows by a dual argument.

$\mathbf{s = t = 0}$. It follows that $[n] = \bigcup_{i=1}^r X_i$ and $[m] = \bigcup_{i=1}^r Y_i$. Since $r < n \leq m$, we may assume without loss of generality that $|X_r| \geq 2$. Fix some $x \in X_r$, and let $U = X_r \setminus \{x\}$. We consider the following cases:

- $|Y_r| \geq 2$. In this case, we fix some $y \in Y_r$ and let $V = Y_r \setminus \{y\}$. Then, the partition $\alpha = \left(\begin{array}{c|c|c|c|c} Y_1 & \cdots & Y_{r-1} & y & V \\ X_1 & \cdots & X_{r-1} & x & U \end{array}\right) \in \mathcal{K}_{mn}$ satisfies $\sigma = \sigma\alpha\sigma$ and $(\sigma, \alpha) \notin \mathscr{J}$.

- $|Y_r| = 1$, in which case we may assume $|Y_{r-1}| \geq 2$. We fix $z \in Y_{r-1}$ and $W = Y_{r-1} \setminus \{z\}$. If $\alpha = \left(\begin{array}{c|c|c|c|c|c} Y_1 & \cdots & Y_{r-2} & Y_r & z & W \\ X_1 & \cdots & X_{r-2} & X_{r-1} & x & U \end{array} \right) \in \mathcal{K}_{mn}$, then

$$\sigma\alpha\sigma = \left(\begin{array}{c|c|c|c|c} X_1 & \cdots & X_{r-2} & X_{r-1} & X_r \\ Y_1 & \cdots & Y_{r-2} & Y_r & Y_{r-1} \end{array} \right),$$

so $\sigma\alpha\sigma \mathrel{\mathscr{J}} \alpha$ and $(\alpha, \sigma) \notin \mathscr{J}$.

Now, suppose that $\mathcal{K}$ is one of $\mathscr{P}\mathcal{P}$, $\mathscr{M}$ or $\mathcal{TL}$. Since $\mathcal{K}$ is stable (by Proposition 5.1.2), regular and $\mathscr{H}$-trivial (which is easily deduced from parts (iv) and (v) of Proposition 5.1.2 and Lemma 5.1.1(i)), Proposition 2.2.17 implies the statement concerning the existence of a nontrivial maximal $\mathscr{J}^\sigma$-class (for (b), recall that $\mathrm{Pre}(\sigma) \subseteq \mathcal{K}_{mn}$ and $\mathscr{D} = \mathscr{J}$ in $\mathcal{K}$). Finally, from Lemma 2.2.12(ii) it follows that, if the nontrivial maximal $\mathscr{J}^\sigma$-class exists, then it is the class containing σ^*, i.e.

$$\mathrm{J}^\sigma_{\sigma^*} = \mathrm{D}^\sigma_{\sigma^*} = \mathrm{D}^\sigma_r . \qquad \square$$

Now, we consider the minimal $\mathscr{J}^\sigma$-classes. Note that they coincide with the minimal ideals of the semigroup $\mathcal{K}^\sigma_{mn}$. As in [28], we prove

Proposition 5.2.7. *Let z be the smallest possible rank of partitions from $\mathcal{K}_{mn}$. Then, the minimal ideal of $\mathcal{K}^\sigma_{mn}$ is the set $\mathrm{D}_z = \mathrm{D}^\sigma_z$. Further, we have $\mathrm{D}_z / \mathscr{R}^\sigma = \mathrm{D}_z / \mathscr{R}$ and $\mathrm{D}_z / \mathscr{L}^\sigma = \mathrm{D}_z / \mathscr{L}$.*

Proof. Since z is the smallest possible rank of partitions from $\mathcal{K}_{mn}$, for any $\alpha \in \mathrm{D}_z = \{\zeta \in \mathcal{K}_{mn} : \mathrm{Rank}(\zeta) = z\}$ we have

$$z \leq \mathrm{Rank}(\sigma\alpha\sigma) \leq \mathrm{Rank}(\alpha) = z,$$

so $\mathrm{Rank}(\sigma\alpha\sigma) = \mathrm{Rank}(\alpha)$. Thus, Proposition 5.2.2 implies $\mathrm{D}_z \subseteq \mathrm{P}^\sigma$, which gives

$$\mathrm{D}_z = \mathrm{D}_z \cap \mathrm{P}^\sigma = \mathrm{D}^\sigma_z,$$

the last equality following from Proposition 5.2.5. Now, Propositions 2.2.7(iii) and 5.1.2(iii) imply that D^σ_z is the minimal $\mathscr{J}^\sigma = \mathscr{D}^\sigma$-class in $\mathcal{K}_{mn}$.

For the last statement, we prove only the first part, as the second is dual. Suppose $\alpha \in \mathrm{D}_z$. From the discussion above we have $\alpha \in \mathrm{P}^\sigma$, so

$$\mathrm{R}_\alpha \subseteq \mathrm{D}_\alpha = \mathrm{D}_z \subseteq \mathrm{P}^\sigma \subseteq \mathrm{P}^\sigma_1 .$$

Hence, Theorem 5.2.4(i) gives $\mathrm{R}^\sigma_\alpha = \mathrm{R}_\alpha \cap \mathrm{P}^\sigma_1 = \mathrm{R}_\alpha$. $\square$

5.2.2 A structure theorem for $\mathcal{K}^\sigma_{mn}$ and connections to (non-sandwich) partition semigroups

Following the outline of previous chapters, here we consider the diagrams in Figures 2.2 and 2.3, applying the results proved in this chapter, and infer further conclusions.

Recall that $\sigma^* \in \mathrm{V}(\sigma)$ and

$$\sigma\sigma^* = \left(\begin{array}{c|c|c|c|c|c} X_1 & \cdots & X_r & U_1 & \cdots & U_s \\ \hline X_1 & \cdots & X_r & U_1 & \cdots & U_s \end{array} \right) \in \mathcal{K}_n \quad \text{and} \quad \sigma^*\sigma = \left(\begin{array}{c|c|c|c|c|c} Y_1 & \cdots & Y_r & V_1 & \cdots & V_s \\ \hline Y_1 & \cdots & Y_r & V_1 & \cdots & V_s \end{array} \right) \in \mathcal{K}_m \,.$$

Furthermore, we introduce an additional partition

$$\tau = \left(\begin{array}{c|c|c|ccc} 1 & \cdots & r & & & \\ \cline{4-6} X_1 & \cdots & X_r & U_1 & \cdots & U_s \end{array} \right) \in \mathcal{K}_{rn} \,.$$

Recall that we assume $\min(X_1) < \cdots < \min(X_r)$ if $\mathcal{K}$ is one of $\mathscr{PP}$, $\mathscr{M}$ and $\mathcal{TL}$. Hence, τ is planar if σ is (by Lemma 5.1.1).

Now, consider the semigroups in Figures 2.2 and 2.3. Recall that

$$\mathcal{K}_{mn}\, \sigma = \mathcal{K}_m\, \sigma^*\sigma, \quad \text{and} \quad \sigma\, \mathcal{K}_{mn} = \sigma\sigma^*\, \mathcal{K}_n \,.$$

(see Subsection 2.3.1). For the sake of convenience, instead of examining the semigroup $(\sigma\, \mathcal{K}_{mn}\, \sigma, \circledast)$, we will deal with the isomorphic semigroup $(\sigma\, \mathcal{K}_{mn}\, \sigma\sigma^*, \cdot)$ (see page 50). Let $\alpha \in \sigma\, \mathcal{K}_{mn}\, \sigma\sigma^* = \sigma\sigma^*\, \mathcal{K}_n\, \sigma\sigma^*$. Then, the blocks of α have the following form: for each $1 \le i \le s$, the sets U_i and U_i' are nontransversals; any other block (whether transversal or nontransversal) is of the form $\bigcup_{i\in I} X_i \,\cup\, \bigcup_{j\in J} X_j'$ for some subsets $I, J \subseteq [r]$, with at least one of I, J nonempty. Therefore, we may define a map

$$\sigma\sigma^*\, \mathcal{K}_n\, \sigma\sigma^* \to \mathcal{K}_r : \alpha \mapsto \alpha^\natural \tag{5.10}$$

in the following way: for each block of α of the form $B = \bigcup_{i\in I} X_i \,\cup\, \bigcup_{j\in J} X_j'$, in $\alpha^\natural$ we include the block $I \cup J'$. It is easily seen that the map is well-defined and that $\alpha^\natural = \tau\alpha\tau^*$. Since $\tau^*\tau = \sigma\sigma^*$, we may infer

$$\tau^*\alpha^\natural\tau = \tau^*\tau\alpha\tau^*\tau = \sigma\sigma^*\alpha\sigma\sigma^* = \alpha,$$

the last equality following from the fact that $\alpha = \sigma\beta\sigma\sigma^*$ for some $\beta \in \mathcal{K}_{mn}$. Therefore, the map (5.10) is an isomorphism.

Thus, we may transform slightly the Diagrams 2.2 and 2.3, arriving at

We close the subsection by applying Theorem 2.3.8 to the sandwich semigroup $\mathcal{K}_{mn}^\sigma$.

Theorem 5.2.8. *The map*

$$\psi : \mathrm{P}^\sigma \to \mathrm{Reg}(\mathcal{K}_m\, \sigma^*\sigma) \times \mathrm{Reg}(\sigma\sigma^*\, \mathcal{K}_n) : \alpha \mapsto (\alpha\sigma, \sigma\alpha)$$

is injective, and

$$\mathrm{im}(\psi) = \{(\beta, \gamma) \in \mathrm{Reg}(\mathcal{K}_m\, \sigma^*\sigma) \times \mathrm{Reg}(\sigma\sigma^*\, \mathcal{K}_n) : \sigma\beta = \gamma\sigma\}.$$

In particular, P^σ *is a pullback product of the regular semigroups* $\mathrm{Reg}(\mathcal{K}_m\, \sigma^*\sigma)$ *and* $\mathrm{Reg}(\sigma\sigma^*\, \mathcal{K}_n)$ *with respect to* $\mathcal{K}_r$.

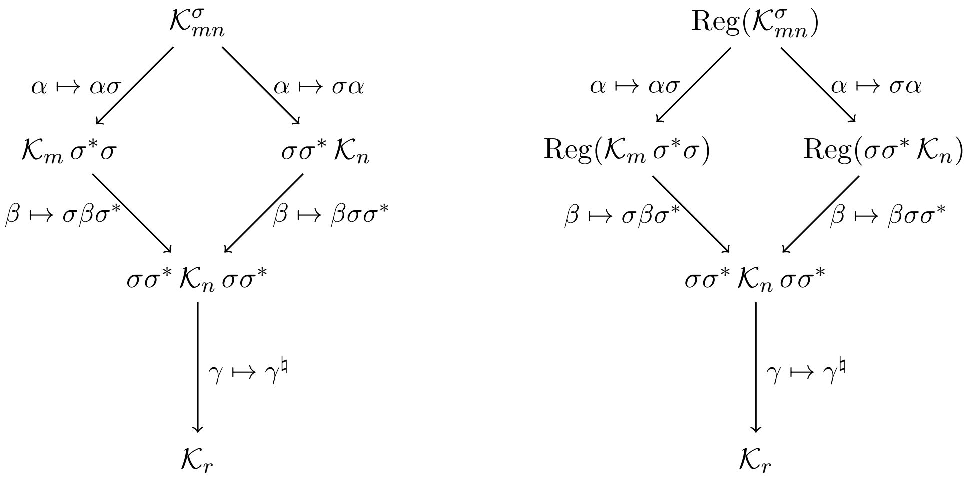


Figure 5.7: Diagrams illustrating the connections between $\mathcal{K}^\sigma_{mn}$ and $\sigma\sigma^*\,\mathcal{K}_n\,\sigma\sigma^*$ (left) and between $\operatorname{Reg}(\mathcal{K}^\sigma_{mn})$ and $\sigma\sigma^*\,\mathcal{K}_n\,\sigma\sigma^*$ (right).

5.2.3 The regular subsemigroup $\mathrm{P}^\sigma = \operatorname{Reg}(\mathcal{K}^\sigma_{mn})$

As we are about to see, the situation in categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$ and $\mathcal{TL}$ turns out to be much more complex than in $\mathcal{B}$ or in the categories of transformations. In this subsection, we present those properties of the subsemigroup $\operatorname{Reg}(\mathcal{K}^\sigma_{mn})$, which we are able to prove in general. Namely, we explore some details concerning the inflation described in Subsection 2.3.4 and characterise Green's relations of $\operatorname{Reg}(\mathcal{K}^\sigma_{mn})$.

In order to do that, we define a surmorphism

$$\varphi : \operatorname{Reg}(\mathcal{K}^\sigma_{mn}) \to \mathcal{K}_r : \alpha \to (\sigma\alpha\sigma\sigma^*)^\natural,$$

which corresponds to the map $\phi : \mathrm{P}^a \to W$ in the general theory. Thus, the properties of φ are similar to the properties of ϕ, and are shown by analogous arguments. Here, we point out the most important ones. Firstly, suppose $\alpha \in \mathrm{P}^\sigma$. Since $\tau^*\tau = \sigma\sigma^*$, Proposition 5.2.2 gives

$$\alpha \mathrel{\mathscr{J}} \sigma\alpha\sigma = (\sigma\sigma^*\sigma)\alpha(\sigma\sigma^*\sigma) = \tau^*\tau\sigma\alpha\tau^*\tau\sigma \leq_{\mathscr{J}} \tau\sigma\alpha\tau^* \leq_{\mathscr{J}} \alpha,$$

so $\alpha \mathrel{\mathscr{J}} \tau\sigma\alpha\tau^* = \tau\sigma\alpha\tau^*\tau\tau^* = \tau\sigma\alpha\sigma\sigma^*\tau^* = \alpha\varphi$. Hence, we proved that

$$\operatorname{Rank}(\alpha) = \operatorname{Rank}(\alpha\varphi) \quad \text{for all } \alpha \in \mathrm{P}^\sigma\,.$$

Secondly, the proof for Theorem 2.3.12 may be adjusted to our case, so we have

Theorem 5.2.9. *Let $\alpha \in \mathrm{P}^\sigma$ and put $k = |\widehat{\mathrm{H}^\sigma_\alpha}/\mathscr{R}^\sigma|$ and $l = |\widehat{\mathrm{H}^\sigma_\alpha}/\mathscr{L}^\sigma|$. Then*

(i) the restriction of the map φ to the set H^σ_α, $\varphi\restriction_{\mathrm{H}^\sigma_\alpha} : \mathrm{H}^\sigma_\alpha \to \mathrm{H}_{\alpha\varphi}$ is a bijection,

(ii) H^σ_α is a group if and only if $\mathrm{H}_{\alpha\varphi}$ is a group, in which case these groups are isomorphic,

(iii) if H^σ_α *is a group, then* $\widehat{\mathrm{H}^\sigma_\alpha}$ *is a* $k \times l$ *rectangular group over* $\mathrm{H}_{\alpha\varphi}$*,*

(iv) if H^σ_α *is a group, then* $\mathrm{E}_\sigma(\widehat{\mathrm{H}^\sigma_\alpha})$ *is a* $k \times l$ *rectangular band.*

(Recall that $\widehat{\mathrm{H}^\sigma_\alpha} = \bigcup_{x \in \mathrm{H}_{\alpha\varphi}} x\varphi^{-1}$ *and that* $\mathrm{E}_\sigma(S)$ *denotes the set* $\{x \in S : x\sigma x = x\}$*, for all* $S \subseteq \mathcal{K}_{mn}$*.)*

Moreover, part (iii) directly implies

Corollary 5.2.10. *Suppose* $q \leq \mathrm{Rank}(\sigma)$*, (and* $q \equiv \mathrm{Rank}(\sigma) \pmod 2$ *if* $\mathcal{K} = \mathcal{B}$ *or* $\mathcal{TL}$*). Then, in the class* D^σ_q *of* $\mathcal{K}^\sigma_{mn}$*, the group* $\mathscr{H}^\sigma$*-classes are*

- *isomorphic to the symmetric group* S_q *if* $\mathcal{K}$ *is one of* $\mathcal{P}$*,* $\mathcal{PB}$ *or* $\mathcal{B}$*,*
- *trivial if* $\mathcal{K}$ *is one of* $\mathscr{P}\mathcal{P}$*,* $\mathscr{M}$ *or* $\mathcal{TL}$*.*

Proof. As suggested above, by Theorem 5.2.9(ii), it suffices to consider the group $\mathscr{H}$-classes of the monoid $\mathcal{K}_r$. Let $\theta = \iota_r$. Then, $\mathcal{K}^\theta_{rr} \cong \mathcal{K}_r$ and Proposition 5.2.2 gives $\mathrm{P}^\sigma = \mathcal{K}_r$, so Theorem 5.2.4 implies that $\mathscr{H}^\sigma = \mathscr{H}$. Hence, it suffices to consider an $\mathscr{H}$-class in $\mathcal{K}$ of an arbitrary idempotent of rank q, which is easily shown (see Proposition 5.1.2(iii) and (iv)) to be isomorphic to S_q (if $\mathcal{K}$ is one of $\mathcal{P}$, $\mathcal{PB}$ or $\mathcal{B}$) or the trivial group (if $\mathcal{K}$ is one of $\mathscr{P}\mathcal{P}$, $\mathscr{M}$ or $\mathcal{TL}$). □

Applying this result to the case when $r = n$, we may describe the maximal $\mathscr{J}^\sigma$-class (D^σ_r) from Proposition 5.2.6(ii) in more detail. We give the result for the case $m \geq n$, but its dual holds as well.

Proposition 5.2.11. *Suppose that* $m \geq n = r$*.*

(i) If $\mathcal{K}$ *is one of* $\mathcal{P}$*,* $\mathcal{PB}$ *or* $\mathcal{B}$*, the class* D^σ_r *is a left-group over* S_r*.*

(ii) If $\mathcal{K}$ *is one of* $\mathscr{P}\mathcal{P}$*,* $\mathcal{B}$ *or* $\mathcal{TL}$*, the class* D^σ_r *is a left-zero semigroup.*

Proof. Since $\sigma \in \mathcal{K}_{nm}$ and $\mathrm{Rank}(\sigma) = r = n$, the partition σ is right-invertible (by Proposition 5.1.7(i)). Further, $\mathcal{K}$ is stable (see Proposition 5.1.2), so Proposition 2.2.35(ii) implies that the maximum $\mathscr{J}^\sigma$-class of $\mathcal{K}^\sigma_{mn}$ is an $\mathscr{L}^\sigma$-class, and a left-group over $\mathrm{H}^\sigma_{\sigma^*}$. In Proposition 5.2.6$(ii)$, we characterised the maximum $\mathscr{J}^\sigma$-class, and in Corollary 5.2.10, we proved that $\mathrm{H}^\sigma_{\sigma^*}$ is either isomorphic to S_r (if $\mathcal{K}$ is one of $\mathcal{P}$, $\mathcal{PB}$ or $\mathcal{B}$) or trivial (if $\mathcal{K}$ is one of $\mathscr{P}\mathcal{P}$, $\mathscr{M}$ or $\mathcal{TL}$), so the statement follows. □

Our next step is to describe Green's relations of the regular subsemigroup P^σ. From Lemma 2.3.3, Theorem 5.2.4, and Proposition 5.1.2 we have

Proposition 5.2.12. *Suppose* $\mathcal{K}$ *is one of* $\mathcal{P}$*,* $\mathscr{P}\mathcal{P}$*,* $\mathcal{PB}$*,* $\mathscr{M}$*,* $\mathcal{B}$ *or* $\mathcal{TL}$*, and let* $\alpha \in \mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{K}^\sigma_{mn})$*. Then*

(i) $\mathrm{R}^{\mathrm{P}^\sigma}_\alpha = \mathrm{R}_\alpha \cap \mathrm{P}^\sigma = \{\beta \in \mathrm{P}^\sigma : \ker(\beta) = \ker(\alpha),\ \mathrm{N_U}(\beta) = \mathrm{N_U}(\alpha)\}$,

(ii) $\mathrm{L}^{\mathrm{P}^\sigma}_\alpha = \mathrm{L}_\alpha \cap \mathrm{P}^\sigma = \{\beta \in \mathrm{P}^\sigma : \mathrm{coker}(\beta) = \mathrm{coker}(\alpha),\ \mathrm{N_L}(\beta) = \mathrm{N_L}(\alpha)\}$,

(iii) $\mathrm{H}_\alpha^{\mathrm{P}^\sigma} = \mathrm{H}_\alpha \cap \mathrm{P}^\sigma = \{\beta \in \mathrm{P}^\sigma : \ker(\beta) = \ker(\alpha),\ \mathrm{N_U}(\beta) = \mathrm{N_U}(\alpha),$
$\mathrm{coker}(\beta) = \mathrm{coker}(\alpha),\ \mathrm{N_L}(\beta) = \mathrm{N_L}(\alpha)\}$,

(iv) $\mathrm{J}_\alpha^{\mathrm{P}^\sigma} = \mathrm{D}_\alpha^{\mathrm{P}^\sigma} = \mathrm{D}_\alpha \cap \mathrm{P}^\sigma = \{\beta \in \mathrm{P}^\sigma : \mathrm{Rank}(\beta) = \mathrm{Rank}(\alpha)\}$.

In general, sandwich semigroups of partitions have a far more complex structure than sandwich semigroups of transformations (because transformations are, by definition, more restricted than partitions). In particular, difficulties arise in the investigation of the combinatorial structure (for this reason, we did not give combinatorial criteria for membership in P-sets), so we do not pursue the investigation in this direction (in the general case) any further. That means that we are unable to infer results concerning cardinalities and ranks. Furthermore, it turns out that, if $\mathcal{K}$ is one of $\mathcal{P}$, $\mathscr{PP}$, $\mathcal{PB}$, $\mathscr{M}$, or $\mathcal{TL}$, then P^σ is not MI-dominated (see Remark 5.3.19). Note that in these cases P^σ is not even RP-dominated (see Proposition 2.4.4), since $\sigma^* \in \mathrm{V}(\sigma)$ is a mid-identity. Thus, we are also unable to apply the results of Subsection 2.4.3 in these cases.

Interestingly, in the case of the Brauer category, we have a much nicer situation. The structure of $\mathcal{B}_{mn}^\sigma$ (and $\mathrm{Reg}(\mathcal{B}_{mn}^\sigma)$) is simpler, so we are able to give succinct and elegant characterisations and combinatorial descriptions. Most importantly, we have MI-domination. For this reason, the sandwich semigroup $\mathcal{B}_{mn}^\sigma$ and its regular subsemigroup will be investigated separately in the Section 5.3.

5.2.4 Idempotents and idempotent-generation

Here, we give some general results concerning idempotents, with as much detail as we were able to deduce in spite of the above-mentioned "irregular" properties of the semigroup $\mathcal{K}_{mn}^\sigma$ in general. In particular, we will characterise the sets $\mathrm{E}_\sigma(\mathcal{K}_{mn}^\sigma) = \mathrm{E}_\sigma(\mathrm{P}^\sigma)$, $\mathrm{MI}(\mathrm{P}^\sigma)$ and $\mathrm{RP}(\mathrm{P}^\sigma)$, which contain the idempotents, mid-identities and regularity-preserving elements of P^σ, respectively. Furthermore, we will describe the idempotent-generated subsemigroup $\mathbb{E}_\sigma(\mathcal{K}_{mn}^\sigma) = \mathbb{E}_\sigma(\mathrm{P}^\sigma)$.

Proposition 5.2.13.

(i) $\mathrm{E}_\sigma(\mathcal{K}_{mn}^\sigma) = \{\alpha \in \mathcal{K}_{mn} : \alpha\sigma\alpha = \alpha\}$.

(ii) $\mathrm{MI}(\mathrm{P}^\sigma) = \mathrm{E}_\sigma(\mathrm{D}_r^\sigma)$.

(iii) $\mathrm{RP}(\mathrm{P}^\sigma) = \mathrm{D}_r^\sigma$.

Proof. The first statement is obvious, while the rest follow from Proposition 2.4.10(*iv*) and Theorem 5.2.4(*iv*), as $\mathcal{K}$ is stable (Proposition 5.1.2) and regular in all cases. □

Of course, it is possible to give a combinatorial criterion for an element to be idempotent, but (as in the case with P-sets) it does not give a great deal of additional insight. Instead, we state an alternative description of the set of idempotents proved in Lemma 2.3.11,

$$\mathrm{E}_\sigma(\mathcal{K}_{mn}^\sigma) = (\mathrm{E}(\mathcal{K}_r))\varphi^{-1}.$$

We will also need its counterpart for the idempotent-generated subsemigroup (from Theorem 2.3.15)

$$\mathbb{E}_\sigma(\mathcal{K}^\sigma_{mn}) = (\mathbb{E}(\mathcal{K}_r))\varphi^{-1}. \tag{5.11}$$

Evidently, in order to characterise the members of the idempotent-generated subsemigroup, we need some information on idempotent-generated subsemigroup $\mathbb{E}(\mathcal{K}_r)$ of the semigroup $\mathcal{K}_r$. We give these in the following proposition. Note that the partitions from $\mathcal{K}_r$ of rank r are identified with the corresponding permutations from S_r (e.g. ι_r is identified with id_r).

Proposition 5.2.14.

(i) *(follows from Proposition 16 in [36])* $\mathbb{E}(\mathcal{P}_r) = \{\iota_r\} \cup (\mathcal{P}_r \setminus \mathrm{S}_r)$;

(ii) *(follows from Theorem 1.11(b) in [52])* $\mathbb{E}(\mathscr{P}\mathcal{P}_r) = \mathscr{P}\mathcal{P}_r$

(iii) *(follows from Proposition 2 in [88])* $\mathbb{E}(\mathcal{B}_r) = \{\iota_r\} \cup (\mathcal{B}_r \setminus \mathrm{S}_r)$;

(iv) *(follows from Theorem 1.11(a) in [52]; also see [9])* $\mathbb{E}(\mathcal{TL}_r) = \mathcal{TL}_r$;

(v) *(Theorem 3.18 in [32])*

$$\mathbb{E}(\mathcal{PB}_r) = \mathrm{E}\left(\mathrm{D}_r(\mathcal{PB}_r) \cup \mathrm{D}_{r-1}(\mathcal{PB}_r)\right) \;\cup\; \bigcup_{q=0}^{r-2} \mathrm{D}_q(\mathcal{PB}_r);$$

(vi) *(Theorem 4.17 in [32])*

$$\begin{aligned}\mathbb{E}(\mathscr{M}_r) =& \{\iota_A : A \subseteq [r] \text{ is cosparse}\} \cup \\ & \{\alpha \in \mathcal{M}_r : \mathrm{dom}(\alpha) \text{ and } \mathrm{codom}(\alpha) \text{ are non-cosparse}\},\end{aligned}$$

where $A \subseteq [r]$ is cosparse if the set $B = [r] \setminus A$ satisfies the following: for all $i \in [r]$, $i \in B \Rightarrow i+1 \notin B$.

Now, we may prove

Theorem 5.2.15.

(i) $\mathbb{E}_\sigma(\mathcal{P}^\sigma_{mn}) = \mathrm{V}(\sigma) \cup (\mathrm{P}^\sigma \setminus \mathrm{D}^\sigma_r)$;

(ii) $\mathbb{E}_\sigma(\mathscr{P}\mathcal{P}^\sigma_{mn}) = \mathrm{P}^\sigma = \mathrm{Reg}(\mathscr{P}\mathcal{P}^\sigma_{mn})$;

(iii) $\mathbb{E}_\sigma(\mathcal{B}^\sigma_{mn}) = \mathrm{V}(\sigma) \cup (\mathrm{P}^\sigma \setminus \mathrm{D}^\sigma_r)$;

(iv) $\mathbb{E}_\sigma(\mathcal{TL}^\sigma_{mn}) = \mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{TL}^\sigma_{mn})$;

(v) $\mathbb{E}_\sigma(\mathcal{PB}^\sigma_{mn}) = \mathrm{E}_\sigma(\mathrm{D}^\sigma_r \cup \mathrm{D}^\sigma_{r-1}) \;\cup \bigcup_{q=0}^{r-2} \mathrm{D}^\sigma_q$.

Proof. Parts *(ii)* and *(iv)* follow directly from (5.11), the corresponding parts of Theorem 5.2.14 and the fact that φ is surjective. Similarly, parts *(i)* and *(iii)* may be inferred from (5.11) and the corresponding parts of Theorem 5.2.14 in the following way

$$\begin{aligned}\mathbb{E}(\mathcal{K}^\sigma_{mn}) = (\mathbb{E}(\mathcal{K}_r))\varphi^{-1} &= (\{\iota_r\} \cup (\mathcal{K}_r \setminus \mathrm{S}_r))\varphi^{-1}\\ &= (\iota_r)\varphi^{-1} \cup ((\mathcal{K}_r)\varphi^{-1} \setminus (\mathrm{S}_r)\varphi^{-1})\\ &= (\mathrm{E}(\mathrm{D}_r(\mathcal{K}_r)))\varphi^{-1} \cup ((\mathcal{K}_r)\varphi^{-1} \setminus (\mathrm{S}_r)\varphi^{-1})\\ &= \mathrm{E}_\sigma(\mathrm{D}^\sigma_r) \cup (\mathrm{P}^\sigma \setminus \mathrm{D}^\sigma_r)\\ &= \mathrm{E}_\sigma(\mathrm{J}^\sigma_{\sigma^*}) \cup (\mathrm{P}^\sigma \setminus \mathrm{D}^\sigma_r) = \mathrm{V}(\sigma) \cup (\mathrm{P}^\sigma \setminus \mathrm{D}^\sigma_r),\end{aligned}$$

the last three equalities following from the fact that φ preserves rank and idempotence, Corollary 5.2.5 and Proposition 2.4.10(*ii*), respectively. Finally, an analogous argument shows part (*v*). □

One immediately observes that the semigroup $\mathbb{E}_\sigma(\mathscr{M}^\sigma_{mn})$ is missing in the previous theorem. This omission was made because the task of describing $(\mathbb{E}(\mathscr{M}_r))\varphi^{-1}$ requires additional investigation since the criterion for membership in $\mathbb{E}(\mathscr{M}_r)$ is far more complex.

5.3 The category $\mathcal{B}$

Now, we focus on the category $\mathcal{B}$. As alluded in the previous section, its defining properties make the sandwich semigroups in it more amenable to analysis via our techniques:

- Note that, for any $\alpha \in \mathcal{B}$, all the diagrams representing it have the same set of edges. Therefore, from now on, we will refer to the unique diagram representing α.
- If sandwich elements have equal ranks, we may prove they are isomorphic (see Lemma 5.3.1); with some additional analysis, we give a sufficient and necessary condition for semigroups $\mathcal{B}^\sigma_{mn}$ and $\mathcal{B}^\tau_{kl}$ to be isomorphic (Theorem 5.3.4).
- We are able to analyse the product of partitions (see the introduction of Subsection 5.2.3) and to infer succinct descriptions of the P-sets (Proposition 5.3.8). Furthermore, in Subsection 5.3.2 we discuss the equivalences which correspond to kernels and cokernels of Brauer diagrams, and using these results, we are able to describe the combinatorial structure of P^σ (Theorem 5.3.11) and enumerate its elements and idempotents (Corollary 5.3.13 and Theorem 5.3.14).
- We prove that P^σ is MI-dominated (Proposition 5.3.17), and then we obtain the formulae for the ranks of the regular subsemigroup $\mathrm{Reg}(\mathcal{B}^\sigma_{mn}) = \mathrm{P}^\sigma$ (Theorem 5.3.20) and the idempotent-generated subsemigroup $\mathbb{E}(\mathcal{B}^\sigma_{mn})$ (Theorem 5.3.21).
- Finally, we are able to prove that in $\mathcal{B}^\sigma_{mn}$, we may apply the "generating downwards" technique (see Corollary 5.3.23). Then, we infer the formulae for the rank (Theorems 5.3.24 and 5.3.25).

Again, we remind the reader that the results presented in this chapter are based on the investigation conducted in [34], and most of the results were originally published in that article. In a few instances, when that is not the case, we cite appropriately.

5.3.1 Isomorphism of sandwich semigroups in $\mathcal{B}$

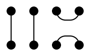
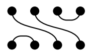
Our first step in this part of the investigation will be answering the question: Under which circumstances are two sandwich semigroups of Brauer partitions isomorphic?

Lemma 5.3.1. *Let $m, n \in \mathbb{N}_0$ and $\sigma, \tau \in \mathcal{B}_{nm}$. If* $\operatorname{Rank}(\sigma) = \operatorname{Rank}(\tau)$, *then* $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{mn}$.

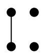
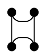
Proof. By Proposition 5.1.2(iii), from $\operatorname{Rank}(\sigma) = \operatorname{Rank}(\tau)$ we have $\sigma = \gamma_1\tau\gamma_2$ for some $\gamma_1, \gamma_2 \in \mathcal{B}$. Moreover, since $\gamma_1 \in \mathcal{B}_n$ and $\gamma_2 \in \mathcal{B}_m$, we may modify both by breaking all the nontransversals and creating transversals instead. (For instance, by breaking the $\frac{n-\operatorname{Rank}(\gamma_1)}{2}$ upper and $\frac{n-\operatorname{Rank}(\gamma_1)}{2}$ lower nontransversals of γ_1, we obtain $n - \operatorname{Rank}(\gamma_1)$ upper and $n - \operatorname{Rank}(\gamma_1)$ lower elements, to be paired in the modified partition π_1 so that the elements of the upper nontransversals of σ are connected in the product diagram $\Pi(\pi_1, \tau)$.) Hence, there exist $\pi_1 \in \mathrm{S}_n$ and $\pi_2 \in \mathrm{S}_m$ with $\sigma = \pi_1\tau\pi_2$. Therefore, the map $\mathcal{B}_{mn} \to \mathcal{B}_{mn} : \alpha \mapsto \pi_2\alpha\pi_1$ is a isomorphism (since π_1 and π_2 are invertible with respect to ι_n and ι_m), so $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{mn}$. $\square$

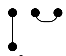
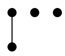
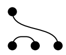
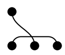
Remark 5.3.2. Lemma 5.3.1 does not hold in any of the categories $\mathcal{TL}$, $\mathscr{PP}$, $\mathscr{M}$, $\mathcal{PB}$ or $\mathcal{P}$.

For the first three, consider the partitions

$\alpha =$  and $\beta =$  .

It is easily seen that $\iota_4 \in \operatorname{Pre}(\alpha) \setminus \operatorname{V}(\alpha)$, and it may be shown (via some calculation) that $\operatorname{Pre}(\beta) = \operatorname{V}(\beta)$ in $\mathcal{TL}$, $\mathscr{M}$ and $\mathscr{PP}$. By Proposition 5.2.6(i), if $\mathcal{K}$ is one of those three, the semigroup $\mathcal{K}^{\beta}_4$ has a nontrivial maximal $\mathscr{J}^{\alpha}$-class, and $\mathcal{K}^{\alpha}_4$ does not.

In $\mathcal{P}$, we consider $\alpha =$  and $\beta =$  . Then, it is easily seen that $|\,\mathrm{D}^{\alpha}_1(\mathcal{P}^{\alpha}_2)| = 4$ and $|\,\mathrm{D}^{\beta}_1(\mathcal{P}^{\beta}_2)| = 9$.

If $\mathcal{K} = \mathcal{PB}$, consider $\alpha =$  and $\beta =$  . Then, it is easy to check that $|\operatorname{Reg}(\mathcal{PB}^{\alpha}_{13})| = 8$ and $|\operatorname{Reg}(\mathcal{PB}^{\beta}_{13})| = 6$ (the difference being  and ).

In addition, we have:

Lemma 5.3.3. *If $q \in \mathbb{N}$, then*

(i) *$\mathcal{B}^{\sigma}_{2q,0}$ is a left-zero semigroup of size $(2q-1)!!$ for any $\sigma \in \mathcal{B}_{0,2q}$,*

(ii) *$\mathcal{B}^{\sigma}_{2q-1,1}$ is a left-zero semigroup of size $(2q-1)!!$ for any $\sigma \in \mathcal{B}_{1,2q-1}$,*

Proof. Let us prove (i). Let $\sigma \in \mathcal{B}_{0,2q}$. Then, for $\alpha, \beta \in \mathcal{B}_{2q,0}$ we have $\alpha\sigma\beta = \alpha$. Therefore, $\mathcal{B}^{\sigma}_{2q,0}$ is a left-zero semigroup, and $|\,\mathcal{B}_{2q,0}\,| = (2q+0-1)!!$, by Proposition 5.1.4($iii$). Part ($ii$) is proved analogously. $\square$

Of course, the dual statement holds as well, but we do not state it.

It turns out that these two results cover nearly all the cases when isomorphism occurs. We prove that in the following theorem.

Theorem 5.3.4. *Let* $m, n, k, l \in \mathbb{N}_0$ *with* $m \equiv n \pmod 2$ *and* $k \equiv l \pmod 2$. *Further, let* $\sigma \in \mathcal{B}_{nm}$ *and* $\tau \in \mathcal{B}_{lk}$ *with* $r = \mathrm{Rank}(\sigma)$ *and* $s = \mathrm{Rank}(\tau)$. *Then* $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{kl}$ *if and only if one of the following holds:*

(a) $(m, n, r) = (k, l, s)$,

(b) $m + n \leq 2$ *and* $k + l \leq 2$,

(c) renaming if necessary, $(m, n, r) = (2q, 0, 0)$ *and* $(k, l, s) = (2q - 1, 1, 1)$ *for some* $q \in \mathbb{N}$,

(d) renaming if necessary, $(m, n, r) = (0, 2q, 0)$ *and* $(k, l, s) = (1, 2q - 1, 1)$ *for some* $q \in \mathbb{N}$.

Proof. First, we prove that any of $(a) - (d)$ implies $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{kl}$. By Lemma 5.3.1, $(a) \Rightarrow \mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{kl}$. Further, by Lemma 5.3.3 and its dual, $(c) \vee (d) \Rightarrow \mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{kl}$. Finally, if $m+n \leq 2$ and $k+l \leq 2$, then $|\,\mathcal{B}_{mn}\,| = |\,\mathcal{B}_{kl}\,| = 1$ (since $(-1)!! = 1!! = 1$), so the two semigroups are isomorphic.

Now, we prove that $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{kl}$ implies that one of $(a) - (d)$ holds. Suppose that $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{kl}$ and (b) is not the case. Then, Proposition 5.1.4(iii) gives

$$(m + n - 1)!! = |\,\mathcal{B}^{\sigma}_{mn}\,| = |\,\mathcal{B}^{\tau}_{kl}\,| = (k + l - 1)!!.$$

Note that $x!!$ is strictly increasing for odd $x \geq 1$. Furthermore, from $\neg(b)$ we know that $m+n$ and $k+l$ are not both ≤ 2, so $m+n = k+l$. Thus, we have the following cases

$\mathbf{(m, n) = (k, l)}$. Thus, $r \equiv m \equiv k \equiv s \pmod 2$. Recall from Corollary 5.2.10 that the group $\mathscr{H}$-classes (more precisely, $\mathscr{H}^{\sigma}$-classes and $\mathscr{H}^{\tau}$-classes) of maximal rank in $\mathcal{B}^{\sigma}_{mn}$ and $\mathcal{B}^{\tau}_{kl}$ are isomorphic to S_r and S_s, respectively (and their sizes are $r!$ and $s!$, respectively). Then, $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}^{\tau}_{kl}$ implies $r! = s!$. Since $r \equiv s \pmod 2$, and $x!$ is strictly increasing for odd $x \geq 1$ and also for even $x \geq 0$, we may conclude that $r = s$, so we have (a).

$\mathbf{(m, n) \neq (k, l)}$. Without loss of generality we may assume $m > k$. (Thus, from $m + n = k + l$ we have $n < l$.) Recall from Proposition 5.2.7 that the minimal $\mathscr{J}^{\sigma} = \mathscr{D}^{\sigma}$-class of $\mathcal{B}^{\sigma}_{mn}$ is $\mathrm{D}_z(\mathcal{B}_{mn})$, where $z \in \{0, 1\}$ and $z \equiv m \pmod 2$. Furthermore, Propositions 5.2.7 and 5.1.5(i) give

$$|\,\mathrm{D}_z(\mathcal{B}_{mn})/\mathscr{R}^{\sigma}\,| = |\,\mathrm{D}_z(\mathcal{B}_{mn})/\mathscr{R}\,| = (m)f \quad \text{and}$$
$$|\,\mathrm{D}_z(\mathcal{B}_{mn})/\mathscr{L}^{\sigma}\,| = |\,\mathrm{D}_z(\mathcal{B}_{mn})/\mathscr{L}\,| = (n)f,$$

where $f : \mathbb{N}_0 \to \mathbb{N}_0$ is defined by

$$(x)f = \begin{cases} \binom{x}{1} \cdot (x - 1 - 1)!! = x!!, & \text{if } x \text{ is odd,} \\ \binom{x}{0} \cdot (x - 0 - 1)!! = (x - 1)!!, & \text{if } x \text{ is even.} \end{cases}$$

Similarly, the minimal $\mathscr{J}^\tau = \mathscr{D}^\tau$-class of $\mathcal{B}^\tau_{kl}$ is $\mathrm{D}_w(\mathcal{B}_{kl})$, where $w \in \{0,1\}$ and $w \equiv k \pmod 2$, and

$$|\,\mathrm{D}_w(\mathcal{B}_{kl})/\mathscr{R}^\tau\,| = (k)f \quad \text{and} \quad |\,\mathrm{D}_w(\mathcal{B}_{kl})/\mathscr{L}^\tau\,| = (l)f.$$

Since $\mathcal{B}^\sigma_{mn} \cong \mathcal{B}^\tau_{kl}$, we have $(m)f = (k)f$ and $(n)f = (l)f$. From the definition of f it follows that, for $x, y \in \mathbb{N}_0$ with $x < y$,

$$(x)f = (y)f \Leftrightarrow (x,y) = (0,1) \text{ or } (x,y) = (2q-1, 2q) \text{ for some } q \in \mathbb{N}.$$

Recall that $m > k$ and $n < l$, so

- $(m,k) = (1,0)$ or $(2q, 2q-1)$ for some $q \in \mathbb{N}$, and
- $(n,l) = (0,1)$ or $(2p-1, 2p)$ for some $p \in \mathbb{N}$.

Moreover, from $m \equiv n \pmod 2$ and $k \equiv l \pmod 2$ follows that

- $(m,k) = (1,0)$ and $(n,l) = (2p-1, 2p)$ for some $p \in \mathbb{N}$, and
- $(m,k) = (2q, 2q-1)$ or $(n,l) = (0,1)$ for some $q \in \mathbb{N}$.

Note that, in both cases, the rank of the sandwich elements is determined by their smaller coordinate. We proved that either (*c*) or (*d*) is true in this case. □

Remark 5.3.5. Since $\mathcal{B}^\sigma_{mn}$ and $\mathcal{B}^{\sigma^*}_{nm}$ are anti-isomorphic (see page 200), from the above, one may easily infer the classification up to anti-isomorphism, as well.

5.3.2 A combinatorial digression

In order to describe the regular subsemigroup $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{B}^\sigma_{mn})$, we need to investigate kernels, cokernels, and the way they interact in the product of diagrams. Therefore, we introduce new combinatorial notions and describe their properties.

Let ε be an equivalence relation on a set X, and π_ε the corresponding partition of X. Then, ε is a

- *2-equivalence*, if each class of π_ε has size 2.
- *1-2-equivalence*, if each class of π_ε has size ≤ 2. In this case, the number of singleton classes of π_ε is the *rank* of ε, denoted by $\mathrm{Rank}(\varepsilon)$.

Note that, in the case where ε is an 1-2-equivalence and $|X|$ is finite, we have $\mathrm{Rank}(\varepsilon) \equiv |X| \pmod 2$.

Clearly, 1-2-equivalences on $[m]$ are the kernels of elements of $\mathcal{B}_{mn}$. Moreover, if $\alpha \in \mathcal{B}_{mn}$, then $\mathrm{Rank}(\ker(\alpha)) = \mathrm{Rank}(\mathrm{coker}(\alpha)) = \mathrm{Rank}(\alpha)$ since all the singleton classes of $\ker(\alpha)$ are elements of transversal classes.

Firstly, we are interested in the number of these equivalences. If $|X| = m$ is finite, there are $(m-1)!!$ 2-equivalences on X (see the comments on $n!!$, on page

207). For $q \in \mathbb{N}_0$ with $q \leq m$ and $q \equiv m \pmod 2$, the number of 1-2-equivalences on the set X with rank q is

$$\kappa(m, q) = \binom{m}{q}(m-q-1)!!. \tag{5.12}$$

(We may choose q elements for the q singleton classes in $\binom{m}{q}$ ways, and the remaining $m-q$ elements may be paired in $(m-q-1)!!$ ways.)

Secondly, we will investigate the join of 1-2-equivalences. Recall that, for partitions $\alpha, \beta \in \mathcal{P}$ with $\beta\,\boldsymbol{\delta} = \alpha\,\boldsymbol{\rho}$, the equivalence $\operatorname{coker}(\alpha) \vee \ker(\beta)$ (that is, the transitive closure of $\operatorname{coker}(\alpha) \circ \ker(\beta)$) describes the connections among the elements of the middle row in the product diagram $\Pi(\alpha, \beta)$. In particular, the classes of $\operatorname{coker}(\alpha) \vee \ker(\beta)$ of odd size determine the transversals in the product (diagram) $\Pi(\alpha, \beta)$. Thus, we will be interested in the join of 1-2 equivalences which has a specified number of classes of odd size: suppose $m, r, q \in \mathbb{N}_0$ are such that $q \leq r \leq m$ and $q \equiv r \equiv m \pmod 2$, and fix a set X with $|X| = m$ and a 1-2-equivalence υ on X with $\operatorname{Rank}(\upsilon) = r$; then, $\kappa(m, r, q)$ denotes the number of 1-2-equivalences ε on X such that

$$\operatorname{Rank}(\varepsilon) = q \quad \text{and} \quad \varepsilon \vee \upsilon \text{ has precisely } q \text{ classes of odd size.}$$

Note that the value $\kappa(m, r, q)$ does not depend on the choice of the set X or the choice of equivalence υ (since we do not require planarity), as long as they have the required properties.

We will visualise the join $\varepsilon \vee \upsilon$ of 1-2-equivalences (on a set X) ε and υ as a cut-out from the middle row of a product diagram: the vertices corresponding the elements of X will be placed in a horizontal row, and the connections within the non-singleton classes of ε and υ will be indicated by an edge drawn below and above the row of vertices, respectively.

Now, we may prove

Lemma 5.3.6. *If $m, r \in \mathbb{N}_0$ are such that $r \leq m$ and $r \equiv m \pmod 2$, then $\kappa(m, r, r) = \frac{(m+r-1)!!}{(2r-1)!!}$.*

Proof. Firstly, we define the numbers $\lambda(m, r)$ for $m, r \in \mathbb{N}_0$ with $r \leq m$ and $r \equiv m \pmod 2$ in the following way:

(1) $\lambda(m, r) = (m-1)!!$ if $r = 0$,

(2) $\lambda(m, r) = 1$ if $m = r$,

(3) $\lambda(m, r) = \lambda(m-1, r-1) + (m-r)\lambda(m-2, r)$ if $0 < r < m$.

It is easily shown that $\frac{(m+r-1)!!}{(2r-1)!!}$ satisfies the above recurrence. Therefore, it suffices to prove the same for $\kappa(m, r, r)$. By the above discussion, we may assume, without loss of generality, that $X = \{1, \ldots, m\}$ and that

$$\pi_\upsilon = \Big\{\{1\}, \ldots, \{r\}, \{r+1, r+2\}, \ldots, \{m-1, m\}\Big\}.$$

Here, $r = \text{Rank}(\upsilon)$.

Consider $\kappa(m, 0, 0)$. Since $r = q = 0$, all the classes of ε need to be non-singleton, so any of $(m-1)!!$ 2-equivalences of $[m]$ fits. Thus, (1) holds. Next, consider $\kappa(m, m, m)$. Since $q = r = m$, we have $\upsilon = \Delta_{[m]}$ and none of its classes are joined in $\varepsilon \vee \upsilon$. The only 1-2-equivalence satisfying that is $\varepsilon = \upsilon$, so (2) holds as well. For (3), suppose $0 < r < m$. To calculate the number of possible choices for ε, we consider the possible forms of class A containing the element 1. We have two cases:

$$A = \{1\}, \quad \text{or} \quad A = \{1, a\}, \text{ for some } a \in \{r+1, \ldots, m\}.$$

(Since $\{1\}$ is a class of π_υ, the element 1 cannot be connected to an element from $\{2, \ldots r\}$; otherwise, $\varepsilon \vee \upsilon$ would contain at most $r - 1$ odd-sized components.) In the first case, the class A remains intact in $\varepsilon \vee \upsilon$, so we need to partition $\{2, \ldots, m\}$ so that we obtain $r - 1$ classes of odd size. If we recall the form of the partition υ, it is easily seen that this may be done in $\kappa(m-1, r-1, r-1)$ ways. In the second case, we may choose the element a in $m - r$ ways. Then, the component of $\varepsilon \vee \upsilon$ containing 1 also contains a and b, where $b \in \{a-1, a+1\}$. Note that, in ε, the element b cannot be connected to any of the elements of $\{2, \ldots, r\}$ because its $\varepsilon \vee \upsilon$-class needs to be an odd-sized component. Thus, we may identify the newly created group $\{1, a, b\}$ with the element 1, and eliminate a and b (see Figure 5.8). Hence, the number of ways to connect the remaining elements (including the "artificial" 1) and obtain r odd-sized components in $\varepsilon \vee \upsilon$ is $\kappa(m-2, r, r)$. ☐

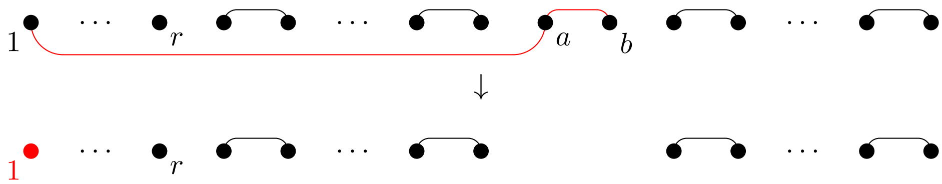


Figure 5.8: A visual aid for the proof of Lemma 5.3.6.

Further, we show

Lemma 5.3.7. *If* $m, r, q \in \mathbb{N}_0$ *are such that* $q \le r \le m$ *and* $q \equiv r \equiv m \pmod 2$, *then*

$$\kappa(m, r, q) = \binom{r}{q} \frac{(r-q-1)!!(m+q-1)!!}{(r+q-1)!!}.$$

Proof. We use a similar approach as in the proof of Lemma 5.3.6. We define the numbers $\lambda(m, r, q)$ for $m, r, q \in \mathbb{N}_0$ with $q \le r \le m$ and $q \equiv r \equiv m \pmod 2$ in the following way:

(1) $\lambda(m, r, q) = (m-1)!!$ if $q = 0$,

(2) $\lambda(m, r, q) = \binom{m}{q}(m-q-1)!!$ if $m = r$,

(3) $\lambda(m,r,q) = \frac{(m+r-1)!!}{(2r-1)!!}$ if $r = q$,

(4) $\lambda(m,r,q) = \lambda(m-1,r-1,q-1) + (r-1)\lambda(m-2,r-2,q) + (m-r)\lambda(m-2,r,q)$

if $0 < q < r < m$.

Again, it may be shown that $\binom{r}{q}\frac{(r-q-1)!!(m+q-1)!!}{(r+q-1)!!}$ satisfies the above recurrence, so it suffices to prove the same for $\kappa(m,r,q)$. Here too we assume without loss of generality that $X = [m]$ and

$$\pi_\upsilon = \Big\{\{1\},\dots,\{r\},\{r+1,r+2\},\dots,\{m-1,m\}\Big\}.$$

(1) and (2). Note that in the case $r = m$ we have $\upsilon = \{(x,x) : x \in [m]\}$. Suppose that either $q = 0$ or $r = m$. Then, for every 1-2-equivalence ε with $\mathrm{Rank}(\varepsilon) = q$, the relation $\varepsilon \vee \kappa$ has q odd-sized blocks. Thus, we may choose any of the $\kappa(m,q) = \binom{m}{q}(m-q-1)!!$ 1-2-equivalences of rank q (see (5.12)).

(3) was shown in Lemma 5.3.6.

(4). Suppose $0 < q < r < m$. To calculate the number of possible choices for ε, we consider the possible forms of class A containing the element 1. We have three cases:

$$A = \{1\} \quad \text{or} \quad A = \{1,a\}, \text{ with } 1 < a \leq r \quad \text{or} \quad r < a \leq m.$$

The second case is possible since $q < r$ (cf. proof of Lemma 5.3.6). Now, the partitions corresponding to the first and third case are enumerated in the same way as in the proof of Lemma 5.3.6. Similarly, for the second case we may choose a in $r-1$ ways and hence we create an even-sized component. Thus, we need to partition the remaining $m-2$ elements to create q odd-sized components in $\varepsilon \vee \upsilon$ (and we have "spent" 2 singleton components of υ), which may be done in $\kappa(m-2,r-2,q)$ ways. □

5.3.3 The regular subsemigroup $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{B}^\sigma_{mn})$

Now, we return to the topic of our investigation, the sandwich semigroup $\mathcal{B}^\sigma_{mn}$. The crucial step is to infer a concise characterisation for each of its P-sets. Then, applying the results of the previous subsection, we describe the combinatorial structure of its regular subsemigroup P^σ and enumerate its regular elements and idempotents.

Again, we fix $m,n \in \mathbb{N}_0$ such that $m \equiv n \pmod 2$. Further, we continue to fix some $\sigma \in \mathcal{B}_{nm}$ with $\mathrm{Rank}(\sigma) = r$. By Lemma 5.3.1, we may assume without loss of generality that

$$\sigma = \left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & r+1,r+2 & \cdots & n-1,n \\ \hline 1 & \cdots & r & r+1,r+2 & \cdots & m-1,m \end{array}\right), \tag{5.13}$$

which gives τ (as defined in Subsection 5.2.2) of the form

$$\left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & \multicolumn{3}{c}{} \\ 1 & \cdots & r & r+1, r+2 & \cdots & n-1, n \end{array}\right).$$

First, we aim to describe the elements of the set P_1^σ in $\mathcal{B}_{mn}^\sigma$. Thus, let $\alpha \in \mathcal{B}_{mn}$ and consider the product diagram $\Pi(\alpha, \sigma)$. Note that each component is either a path or a loop. More precisely, each component is of one of the following forms:

(C1) $x \overset{\alpha}{\longleftrightarrow} z$, for some $x, z \in [m]$,

(C2) $x' \overset{\sigma}{\longleftrightarrow} z'$, for some $x, z \in [m]$,

(C3) $x \overset{\alpha}{\longleftrightarrow} y_1'' \overset{\sigma}{\longleftrightarrow} y_2'' \overset{\alpha}{\longleftrightarrow} \cdots \overset{\sigma}{\longleftrightarrow} y_{2k}'' \overset{\alpha}{\longleftrightarrow} z$, for some $x, z \in [m]$, $k \in \mathbb{N}$ and $y_1, y_2, \ldots, y_{2k} \in [n]$,

(C4) $x' \overset{\sigma}{\longleftrightarrow} y_1'' \overset{\alpha}{\longleftrightarrow} y_2'' \overset{\sigma}{\longleftrightarrow} \cdots \overset{\alpha}{\longleftrightarrow} y_{2k}'' \overset{\sigma}{\longleftrightarrow} z'$, for some $x, z \in [m]$, $k \in \mathbb{N}$ and $y_1, y_2, \ldots, y_{2k} \in [n]$,

(C5) $y_1'' \overset{\alpha}{\longleftrightarrow} y_2'' \overset{\sigma}{\longleftrightarrow} \cdots \overset{\alpha}{\longleftrightarrow} y_{2k}'' \overset{\sigma}{\longleftrightarrow} y_1''$, for some $k \in \mathbb{N}$ and $y_1, y_2, \ldots, y_{2k} \in [n]$,

(C6) $x \overset{\alpha}{\longleftrightarrow} y_1'' \overset{\sigma}{\longleftrightarrow} y_2'' \overset{\alpha}{\longleftrightarrow} \cdots \overset{\alpha}{\longleftrightarrow} y_{2k-1}'' \overset{\sigma}{\longleftrightarrow} z'$, for some $x, z \in [m]$, $k \in \mathbb{N}$ and $y_1, y_2, \ldots, y_{2k-1} \in [n]$,

(where $x \overset{\alpha}{\longleftrightarrow} z$ means that elements x and z are connected by an edge in the diagram α). In $\alpha\sigma$, these components result in upper nontransversals (in the cases (C1) and (C3)), lower nontransversals (in the cases (C2) and (C4)) or transversals (in the case (C6)). The components of the form (C5) have no effect since they form loops contained in the middle row. Hence, the rank of $\alpha\sigma$ equals the number of components of type (C6) in $\Pi(\alpha, \sigma)$. Moreover, we may conclude that every equivalence class of $\mathrm{coker}(\alpha) \vee \ker(\sigma)$ (describing the connections in the middle row) is of the form

- $\{y_1, \ldots, y_{2k}\}$, for some component of $\Pi(\alpha, \sigma)$ of type (C3), (C4) or (C5), or
- $\{y_1, \ldots, y_{2k-1}\}$ for some component of $\Pi(\alpha, \sigma)$ of type (C6).

Therefore, if $q = \mathrm{Rank}(\alpha)$, from Proposition 5.2.2 we have

$$\begin{aligned} \alpha \in \mathrm{P}_1^\sigma \Leftrightarrow \mathrm{Rank}(\alpha\sigma) = q &\Leftrightarrow \Pi(\alpha, \sigma) \text{ has } q \text{ components of type (C6)} \\ &\Leftrightarrow \mathrm{coker}(\alpha) \vee \ker(\sigma) \text{ has } q \text{ classes of odd size} \\ &\Leftrightarrow \mathrm{coker}(\alpha) \vee \ker(\sigma) \text{ separates } \mathrm{codom}(\alpha), \end{aligned}$$

the last equivalence following from the fact that, in $\mathrm{coker}(\alpha) \vee \ker(\sigma)$ classes of odd size contain exactly one element of $\mathrm{codom}(\alpha)$.

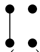
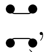
From the duality of P_1^σ and P_2^σ and Proposition 5.2.2, we immediately obtain:

Proposition 5.3.8. *In $\mathcal{B}^{\sigma}_{mn}$, we have*

(i) $\mathrm{P}^{\sigma}_1 = \{\alpha \in \mathcal{B}_{mn} : \mathrm{coker}(\alpha) \vee \ker(\sigma)$ *separates* $\mathrm{codom}(\alpha)\}$,

(ii) $\mathrm{P}^{\sigma}_2 = \{\alpha \in \mathcal{B}_{mn} : \ker(\alpha) \vee \mathrm{coker}(\sigma)$ *separates* $\mathrm{dom}(\alpha)\}$,

(iii) $\mathrm{P}^{\sigma} = \mathrm{P}^{\sigma}_3 = \{\alpha \in \mathcal{B}_{mn} : \mathrm{coker}(\alpha) \vee \ker(\sigma)$ *separates* $\mathrm{codom}(\alpha)$ *and* $\ker(\alpha) \vee \mathrm{coker}(\sigma)$ *separates* $\mathrm{dom}(\alpha)\}$.

Remark 5.3.9. Note that the previous proposition also follows from Proposition 2.2.43. Furthermore, the same proposition clearly holds in $\mathcal{TL}$, as well. However, in $\mathcal{P}$, $\mathscr{PP}$, $\mathcal{PB}$ or $\mathscr{M}$, we do not have restrictions on the parity of classes, so we cannot draw conclusions similar to the ones above. For a case in point, consider $\alpha =$  and $\sigma =$ , both from $\mathcal{M}_2 \subseteq \mathcal{PB}_2 \cap \mathscr{PP}_2 \cap \mathcal{P}_2$. Clearly, $\mathrm{Rank}(\alpha\sigma) \neq \mathrm{Rank}(\alpha) \neq \mathrm{Rank}(\sigma\alpha)$, even though $\mathrm{coker}(\alpha) \vee \ker(\sigma) = \ker(\alpha) \vee \mathrm{coker}(\sigma)$ separates $\mathrm{dom}(\alpha) = \mathrm{codom}(\alpha)$.

Finally, we are ready to continue where we left off in Subsection 5.2.3. We need to describe the inflation from Theorem 5.2.9 in combinatorial terms. Recall the map

$$\varphi : \mathrm{P}^{\sigma} \to \mathcal{B}_r : \alpha \to (\sigma\alpha\sigma\sigma^*)^{\natural} = \tau\sigma\alpha\tau^*,$$

and let $\overline{\alpha}$ denote the partition $\alpha\varphi$ for $\alpha \in \mathcal{P}^{\sigma}$. Furthermore, recall that, for $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$ and $\alpha, \beta \in \mathrm{P}^{\sigma}$, we define $\alpha \widehat{\mathscr{K}^{\sigma}} \beta \Leftrightarrow \overline{\alpha} \mathscr{K} \overline{\beta}$, and $\widehat{\mathrm{K}^{\sigma}_{\alpha}}$ denotes the $\widehat{\mathscr{K}^{\sigma}}$-class α.

Now, we give characterisations of Green's classes in the Brauer monoid $\mathcal{B}_r$, which were obtained in [92]. From these, it is easy to calculate the size of $\mathscr{H}$-classes and to prove that the group $\mathscr{H}$-classes are isomorphic to S_q. Then, the statement (vi) follows from the fact that $\mathrm{Rank}(\alpha\beta) \leq \min(\mathrm{Rank}(\alpha), \mathrm{Rank}(\beta))$.

Lemma 5.3.10. *Let $\alpha \in \mathcal{B}_r$ with $\mathrm{Rank}(\alpha) = q$. In $\mathcal{B}_r$, we have*

(i) $\mathrm{R}_{\alpha} = \{\beta \in \mathcal{B}_r : \ker(\beta) = \ker(\alpha)\}$,

(ii) $\mathrm{L}_{\alpha} = \{\beta \in \mathcal{B}_r : \mathrm{coker}(\beta) = \mathrm{coker}(\alpha)\}$,

(iii) $\mathrm{H}_{\alpha} = \{\beta \in \mathcal{B}_r : \ker(\beta) = \ker(\alpha),\ \mathrm{coker}(\beta) = \mathrm{coker}(\alpha)\}$,

(iv) $|\,\mathrm{H}_{\alpha}\,| = q!$*; furthermore, if* H_{α} *contains an idempotent, then* $\mathrm{H}_{\alpha} \cong \mathrm{S}_q$*;*

(v) $\mathrm{D}_{\alpha} = \mathrm{J}_{\alpha} = \{\beta \in \mathcal{B}_r : \mathrm{Rank}(\beta) = q\}$*;*

(vi) we have $\mathrm{D}_{\iota_r} = \mathrm{H}_{\iota_r} \cong \mathrm{S}_r$ *and* $\mathcal{B}_r \setminus \mathcal{S}_r$ *is an ideal of the semigroup* $\mathcal{B}_r$*.*

(vii) (Lemma 2.1 in [93])

$$\mathrm{Rank}(\mathcal{B}_r) = \begin{cases} 1, & \text{if } r \in \{0,1\}, \\ 2, & \text{if } r = 2, \\ 3, & \text{if } r \geq 3, \end{cases} \qquad \text{and} \quad \mathrm{Rank}(\mathcal{B}_r : \mathrm{S}_r) = \begin{cases} 1, & \text{if } r \geq 2, \\ 0, & \text{if } r \leq 1, \end{cases}$$

(viii) *(Proposition 2 in [88])* $\mathrm{Rank}(\mathbb{E}(\mathcal{B}_r)) = \mathrm{idrank}(\mathbb{E}(\mathcal{B}_r)) = 1 + \binom{r}{2}$.

We use the information obtained above to describe the inflation.

Theorem 5.3.11. *Let* $0 \le q \le r$ *with* $q \equiv r \pmod 2$.

(i) D_q^σ *contains* $\binom{r}{q}(r-q-1)!!$ $\widehat{\mathscr{R}^\sigma}$*-classes, each of which contains* $\frac{(m+q-1)!!}{(r+q-1)!!}$ $\mathscr{R}^\sigma$*-classes.*

(ii) D_q^σ *contains* $\binom{r}{q}(r-q-1)!!$ $\widehat{\mathscr{L}^\sigma}$*-classes, each of which contains* $\frac{(n+q-1)!!}{(r+q-1)!!}$ $\mathscr{L}^\sigma$*-classes.*

(iii) D_q^σ *contains* $\binom{r}{q}^2(r-q-1)!!^2$ $\widehat{\mathscr{H}^\sigma}$*-classes, each of which contains exactly* $\frac{(m+q-1)!!(n+q-1)!!}{(r+q-1)!!^2}$ $\mathscr{H}^\sigma$*-classes.*

(iv) *Each* $\mathscr{H}^\sigma$*-class in* D_q^σ *has size* $q!$, *and group* $\mathscr{H}^\sigma$*-classes in* D_q^σ *are isomorphic to the symmetric group* S_q.

(v) *An* $\mathscr{H}^\sigma$*-class* $\mathrm{H}_\alpha^\sigma \subseteq \mathrm{D}_q^\sigma$ *is a group if and only if* $\mathrm{H}_{\overline{\alpha}} \subseteq \mathrm{D}_q(\mathcal{B}_r)$ *is a group* $\mathscr{H}$*-class of* $\mathcal{B}_r$, *in which case* $\widehat{\mathrm{H}_\alpha^\sigma}$ *is a* $\frac{(m+q-1)!!}{(r+q-1)!!} \times \frac{(n+q-1)!!}{(r+q-1)!!}$ *rectangular group over* S_q.

Proof. Recall that φ maps the $\widehat{\mathscr{K}^\sigma}$-classes of P^σ to $\mathscr{K}$-classes of $\mathcal{B}_r$ (preserving the ranks of elements), and the correspondence is bijective. Thus, D_q^σ contains $|\,\mathrm{D}_q(\mathcal{B}_r)/\mathscr{R}\,| = \binom{r}{q}(r-q-1)!!$ $\widehat{\mathscr{R}^\sigma}$-classes (by Proposition 5.1.5). A dual statement can be made for $\widehat{\mathscr{L}^\sigma}$-classes, and we immediately obtain the number of $\widehat{\mathscr{H}^\sigma}$-classes, as well.

(i). We prove the second assertion. First, we calculate the number of $\mathscr{R}^\sigma$-classes in D_q^σ. From Proposition 5.2.12(i) it follows that such an $\mathscr{R}^\sigma$-class is uniquely determined by its kernel (because the upper nontransversals are precisely the non-singleton classes of the kernel). Thus, it suffices to calculate the number of 1-2-equivalences which may be kernels of a regular element of rank q. By Proposition 5.3.8(iii), for any such equivalence α, $\alpha \vee \mathrm{coker}(\sigma)$ separates $\mathrm{dom}(\alpha)$ (equivalently, $\alpha \vee \mathrm{coker}(\sigma)$ has q classes of odd size). Hence, we need the number of 1-2 equivalences on $[m]$ of rank q, such that $\alpha \vee \mathrm{coker}(\sigma)$ has q odd-sized classes, which is $\kappa(m,r,q) = \binom{r}{q}\frac{(r-q-1)!!(m+q-1)!!}{(r+q-1)!!}$ (by Lemma 5.3.7).

Since D_q^σ contains $\binom{r}{q}(r-q-1)!!$ $\widehat{\mathscr{R}^\sigma}$-classes, it suffices to prove that all $\widehat{\mathscr{R}^\sigma}$-classes in D_q^σ contain the same number of $\mathscr{R}^\sigma$-classes. Thus, suppose $\alpha, \beta \in \mathrm{D}_q^\sigma$ and consider $\widehat{\mathrm{R}_\alpha^\sigma}$ and $\widehat{\mathrm{R}_\beta^\sigma}$. Since all $\mathscr{R}^\sigma$-classes in the same $\mathscr{D}^\sigma$-class have the same size (Lemma 1.3.4(iii)), it is enough to show that these $\widehat{\mathscr{R}^\sigma}$-classes have the same size.

From $\alpha\,\mathscr{D}^\sigma\,\beta$ it follows that $\alpha\,\mathscr{R}^\sigma\,\gamma\,\mathscr{L}^\sigma\,\beta$ for some $\gamma \in \mathrm{D}_q^\sigma$. Since α and γ are $\mathscr{R}^\sigma$-related, we have $\widehat{\mathrm{R}_\alpha^\sigma} = \widehat{\mathrm{R}_\gamma^\sigma}$. Thus, we may assume without loss of generality that $\alpha\,\mathscr{L}^\sigma\,\beta$, which implies $\overline{\alpha}\,\mathscr{L}\,\overline{\beta}$. Then, $\mathrm{coker}(\overline{\alpha}) = \mathrm{coker}(\overline{\beta})$ (by Lemma 5.3.10(ii)) and $\mathrm{Rank}(\overline{\alpha}) = \mathrm{Rank}(\overline{\beta}) = q$. We claim that $\overline{\beta} = \pi\overline{\alpha}$ for some permutation $\pi \in \mathrm{S}_r$: suppose $\mathrm{dom}(\overline{\beta}) = \{c_1, \ldots, c_q\}$ and $\mathrm{dom}(\overline{\alpha}) = \{d_1, \ldots, d_q\}$ with $c_i < c_j$ and $d_i < d_j$ for $i < j$, and fix a permutation $\pi \in \mathrm{S}_r$ such that

- $c_i\pi = d_i$ for each $1 \le i \le q$, and
- for any two-element class G of $\mathrm{dom}(\overline{\beta})$ there exists a two element-class H of $\mathrm{dom}(\overline{\alpha})$ such that π maps G to H.

Let us define permutations

$$\varrho = \left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & r+1 & \cdots & m \\ 1\pi & \cdots & r\pi & r+1 & \cdots & m \end{array} \right) \in \mathrm{S}_m \quad \text{and} \quad \varsigma = \left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & r+1 & \cdots & n \\ 1\pi & \cdots & r\pi & r+1 & \cdots & n \end{array} \right) \in \mathrm{S}_n,$$

and note that $\tau\sigma \cdot \varrho = \pi \cdot \tau\sigma$ and $\sigma\varrho = \varsigma\sigma$ (see (5.13)).

We will prove that

$$\theta : \widehat{\mathrm{R}^\sigma_\alpha} \to \widehat{\mathrm{R}^\sigma_\beta} : \gamma \mapsto \varrho\gamma$$

is a well-defined map. Let $\gamma \in \widehat{\mathrm{R}^\sigma_\alpha}$. Then, $\gamma\theta \in \widehat{\mathrm{R}^\sigma_\beta}$ if the following two statements are true:

(a) $\gamma\theta \in \mathrm{P}^\sigma$, and (b) $\gamma\theta \ \widehat{\mathscr{R}^\sigma} \ \beta$ (i.e. $\overline{\gamma\theta}\,\mathscr{R}\,\overline{\beta}$).

For the first one, note that from $\sigma\varrho = \varsigma\sigma$ we have

$$\begin{aligned} \mathrm{Rank}(\sigma(\gamma\theta)\sigma) = \mathrm{Rank}(\sigma\varrho\gamma\sigma) &= \mathrm{Rank}(\varsigma\sigma\gamma\sigma) \\ &= \mathrm{Rank}(\sigma\gamma\sigma) = \mathrm{Rank}(\gamma) = \mathrm{Rank}(\varrho\gamma) = \mathrm{Rank}(\gamma\theta) \end{aligned}$$

since ς and ϱ are permutations, and since $\gamma \in \mathrm{P}^\sigma$. To prove (b), note that $\tau\sigma \cdot \varrho = \pi \cdot \tau\sigma$ gives

$$\overline{\gamma\theta} = \overline{\varrho\gamma} = \tau\sigma(\varrho\gamma)\tau^* = \pi(\tau\sigma\gamma\tau^*) = \pi\overline{\gamma}\,\mathscr{R}\,\pi\overline{\alpha} = \overline{\beta},$$

where $\pi\overline{\gamma}\,\mathscr{R}\,\pi\overline{\alpha}$ follows from $\gamma \in \widehat{\mathrm{R}^\sigma_\alpha}$ and the fact that $\mathscr{R}$ is a left congruence. Thus, we proved that θ is well-defined. Since ϱ is a permutation, θ is injective, so $|\widehat{\mathrm{R}^\sigma_\alpha}| \le |\widehat{\mathrm{R}^\sigma_\beta}|$. The reverse inequality follows by symmetry, and so $|\widehat{\mathrm{R}^\sigma_\alpha}| = |\widehat{\mathrm{R}^\sigma_\beta}|$.

Part (ii) is dual to (i), and (iii) follows immediately from (i) and (ii). Finally, parts (iv) and (v) follow from Theorem 5.2.9, Lemma 5.3.10 and parts (i) and (ii) (because for any $\alpha \in \mathrm{P}^\sigma$ we have $|\widehat{\mathrm{H}^\sigma_\alpha}/\mathscr{R}^\sigma| = |(\widehat{\mathrm{R}^\sigma_\alpha} \cap \widehat{\mathrm{L}^\sigma_\alpha})/\mathscr{R}^\sigma| = |\widehat{\mathrm{R}^\sigma_\alpha}/\mathscr{R}^\sigma|$ and similarly $|\widehat{\mathrm{H}^\sigma_\alpha}/\mathscr{L}^\sigma| = |\widehat{\mathrm{L}^\sigma_\alpha}/\mathscr{L}^\sigma|$). □

Remark 5.3.12. In Figure 5.13 the reader may inspect egg-box diagrams for the regular semigroups $\mathrm{Reg}(\mathcal{B}^{\sigma_1}_{66})$ and $\mathrm{Reg}(\mathcal{B}^{\sigma_2}_{64})$, and the Brauer monoid $\mathcal{B}_4$, where $\sigma_1 \in \mathcal{B}_{66}$ with $\mathrm{Rank}(\sigma_1) = 4$, and $\sigma_2 \in \mathcal{B}_{46}$ with $\mathrm{Rank}(\sigma_2) = 4$. By comparing the diagrams, the reader may verify that all the $\widehat{\mathscr{R}}$-classes ($\widehat{\mathscr{L}}$-classes) in a common $\mathscr{D}^\sigma$-class have the same number of $\mathscr{R}^\sigma$-classes ($\mathscr{L}^\sigma$-classes). However, in general, this does not hold in other diagram categories (see Figure 5.12).

From Theorem 5.3.11(iii) and (iv) we may immediately infer the size of the regular class D^σ_q. Summing over the possible ranks (see Corollary 5.2.5), we obtain

Corollary 5.3.13. *The size of the regular subsemigroup* $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{B}^\sigma_{mn})$ *is given by*

$$|\,\mathrm{P}^\sigma\,| = \sum_{\substack{0\le q\le r\\ q\equiv r \pmod 2}} \binom{r}{q}^2 \frac{(r-q-1)!!^2(m+q-1)!!(n+q-1)!!}{(r+q-1)!!^2}\cdot q!$$

Of course, we are also interested in the rank of this semigroup. We will be able to obtain it from Theorem 2.4.16, if we prove that P^σ is MI-dominated. Since the topic of MI-domination in $\mathcal{B}$ merits a separate subsection, we postpone this part of the investigation. Instead, we enumerate the idempotents of $\mathcal{B}^\sigma_{mn}$.

For this, we need to know more about the idempotents in the Brauer monoid $\mathcal{B}_r$. In [31], the authors give several formulae for the number of idempotents of rank $0\le q\le r$ in $\mathcal{B}_r$ (denoted $|\,\mathrm{E}(\mathrm{D}_q(\mathcal{B}_r))|$). Here, we use the one from Theorem 30: for $r\in\mathbb{N}_0$ and $0\le q\le r$ with $r\equiv q \pmod 2$,

$$|\,\mathrm{E}(\mathrm{D}_q(\mathcal{B}_r))| = \binom{r}{q}(r-q-1)!!\cdot a_{rq},\quad \text{where } a_{rq} \text{ is defined by the recurrence}$$

$$\begin{aligned} a_{rr} &= 1 && \text{for all } r,\\ a_{r0} &= (r-1)!! && \text{if } r \text{ is even},\\ a_{rq} &= a_{r-1,q-1} + (r-q)a_{r-2,q} && \text{if } 0<q\le r-2. \end{aligned}$$

Note that the recurrence for the numbers a_{rq} is the same as the recurrence for $\lambda(r,q)$ in the proof of Lemma 5.3.6. Thus, from the proof of Lemma 5.3.6 it follows that $a_{rq} = \kappa(r,q,q) = \frac{(r+q-1)!!}{(2q-1)!!}$, so we have

$$|\,\mathrm{E}(\mathrm{D}_q(\mathcal{B}_r))| = \binom{r}{q}\frac{(r-q-1)!!(r+q-1)!!}{(2q-1)!!}. \tag{5.14}$$

Now, we may prove

Theorem 5.3.14. *The number of idempotents of* $\mathcal{B}^\sigma_{mn}$ *is given by*

$$|\,\mathrm{E}_\sigma(\mathcal{B}^\sigma_{mn})| = \sum_{\substack{0\le q\le r\\ q\equiv r \pmod 2}} \binom{r}{q}\frac{(r-q-1)!!(m+q-1)!!(n+q-1)!!}{(r+q-1)!!(2q-1)!!}.$$

The q*th term in the above sum gives the number of idempotents from* D^σ_q*.*

Proof. First, we enumerate the idempotents in the regular class D^σ_q. Suppose $0\le q\le r$ with $q\equiv r \pmod 2$. By Theorem 5.3.11(v), for each group $\mathscr{H}$-class (i.e for each idempotent) in $\mathrm{D}_q(\mathcal{B}_r)$, the idempotents mapping to it form a $\frac{(m+q-1)!!}{(r+q-1)!!}\times\frac{(n+q-1)!!}{(r+q-1)!!}$ rectangular band. Therefore, (5.14) implies

$$|\,\mathrm{E}_\sigma(\mathrm{D}^\sigma_q)| = \binom{r}{q}\frac{(r-q-1)!!(m+q-1)!!(n+q-1)!!}{(r+q-1)!!(2q-1)!!}.$$

We obtain the number of idempotents of $\mathcal{B}^\sigma_{mn}$ by summing over appropriate q. □

Remark 5.3.15. From (5.14) we may conclude that

$$|\,\mathrm{E}(\mathcal{B}_r)| = \sum_{\substack{0\leq q\leq r \\ q\equiv r \pmod 2}} \binom{r}{q} \frac{(r-q-1)!!(r+q-1)!!}{(2q-1)!!},$$

which simplifies the formula from [31] (an alternative formula may be found in Proposition 4.10 in [76]).

5.3.4 MI-domination and the ranks of $\mathrm{Reg}(\mathcal{B}^\sigma_{mn})$ and $\mathbb{E}(\mathcal{B}^\sigma_{mn})$

As promised, in this subsection we prove that the regular subsemigroup $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{B}^\sigma_{mn})$ of the sandwich semigroup $\mathcal{B}^\sigma_{mn}$ is MI-dominated, and we apply Theorems 2.4.16 and 2.4.17 in order to calculate the ranks of the semigroups P^σ and $\mathbb{E}(\mathcal{B}^\sigma_{mn})$. We keep the assumptions from the previous subsections (in particular, we assume that σ is of the form stated in (5.13)).

As in [28], first, we give a technical lemma, whose proof encompasses the core of the argument proving MI-domination. Note that it has an obvious dual, but we do not state it.

Lemma 5.3.16. *If $\alpha \in \mathrm{P}^\sigma_2$, then $\alpha = \lambda \star_\sigma \alpha$ for some $\lambda \in \mathrm{MI}(\mathrm{P}^\sigma)$.*

Proof. First, we analyse α and $\sigma\alpha$ under the stated assumptions. Write

$$\alpha = \left(\begin{array}{c|c|c|c|c|c} a_1 & \cdots & a_q & C_1 & \cdots & C_s \\ \hline b_1 & \cdots & b_q & D_1 & \cdots & D_t \end{array} \right) \in \mathrm{P}^\sigma_2 .$$

By Proposition 5.2.2, $\alpha \in \mathrm{P}^\sigma_2$ implies that $\mathrm{codom}(\sigma\alpha) = \mathrm{codom}(\alpha) = \{b_1, \ldots, b_q\}$. Since $\mathrm{dom}(\sigma\alpha) \subseteq \mathrm{dom}(\sigma) \subseteq [r]$, the q transversals of $\sigma\alpha$ are $\{x_1, b_1'\}, \ldots, \{x_q, b_q'\}$ for some $x_1, \ldots, x_q \in [r]$. As for the nontransversals, we have three types:

- The lower nontransversals $D_1', \ldots, D_t'$ of α are preserved in the product $\sigma\alpha$. Since $\mathrm{Rank}(\sigma\alpha) = q$, these are all the lower nontransversals of $\sigma\alpha$.
- The upper nontransversals $\{r+1, r+2\}, \ldots, \{n-1, n\}$ of σ are also preserved in the product $\sigma\alpha$. If $q = r$, these are all the upper nontransversals of $\sigma\alpha$.
- If $q < r$, the remaining $k = \frac{r-q}{2}$ upper nontransversals of $\sigma\alpha$ are contained in $[r]$. Suppose these are $\{y_1, z_1\}, \ldots, \{y_k, z_k\}$.

Putting these together, we have

$$\sigma\alpha = \left(\begin{array}{c|c|c|c|c|c|c|c|c} x_1 & \cdots & x_q & y_1, z_1 & \cdots & y_k, z_k & r+1, r+2 & \cdots & n-1, n \\ \hline b_1 & \cdots & b_q & D_1 & & \cdots & \cdots & & D_t \end{array} \right).$$

Next, we construct the corresponding λ in four stages. Since we want the result of the product $\lambda\sigma\alpha$ to be α,

(1) $\{a_1, x_1'\}, \ldots, \{a_q, x_q'\}$ will all be transversals of λ.

Further, by Proposition 2.4.9(i) $\mathrm{MI}(\mathrm{P}^\sigma) = \mathrm{V}(\sigma)$, so we want λ to be an inverse of σ. Thus, we will define λ so that $\mathrm{dom}(\lambda) = \mathrm{codom}(\lambda) = [r]$ (note that $\{a_1, \ldots, a_q\} \subseteq [r]$, since $\mathrm{Rank}(\sigma\alpha) = \mathrm{Rank}(\alpha)$), and

(2) $\{r+1, r+2\}', \ldots, \{n-1, n\}'$ will be all the lower nontransversals of λ.

Now, we need to construct $r - q = 2k$ further transversals and $\frac{m-r}{2}$ upper nontransversals, so that in the product diagram $\Pi(\lambda, \sigma\alpha)$ there exists a path giving rise to the upper nontransversal C_i, for all $1 \le i \le s$. In order to do this, we analyse the nontransversals of $\sigma\alpha$. Consider some $1 \le j \le k$. Since $\{y_j, z_j\}$ is a nontransversal of $\sigma\alpha$ and since $\{y_j, y_j'\}$ and $\{z_j, z_j'\}$ are transversals of σ, the product graph $\Pi(\sigma, \alpha)$ contains a path from y_j'' to z_j''. The first edge in this path is clearly the upper nontransversal of α containing the element y_j. Write $\{y_j, w_j\}$ for this nontransversal (note that $w_j \in [r] \setminus \{x_1, \ldots, x_q\}$; further, if the above mentioned path has length 1, then $w_j = z_j$). Renaming if necessary, we may assume $C_j = \{y_j, w_j\}$ (note that, for $l \neq p$ we have $\{y_l, w_l\} \neq \{y_p, w_p\}$, since they belong to different components of $\Pi(\sigma, \alpha)$). So, we want

(3) $\{y_1, y_1'\}, \ldots, \{y_k, y_k'\}$ and $\{w_1, z_1'\}, \ldots, \{w_k, z_k'\}$ to be transversals of λ, and

(4) $C_{k+1}, \ldots, C_s$ to be upper nontransversals of λ.

It is easily seen that the blocks listed in (1)$-$(4) are disjoint. Further,

$$\{x_1, \ldots, x_q\} \cup \{y_1, \ldots, y_k\} \cup \{z_1, \ldots, z_k\} \cup \{r+1, \ldots, n\} = [n], \quad \text{and}$$
$$\{a_1, \ldots, a_q\} \cup \{y_1, \ldots, y_k\} \cup \{w_1, \ldots, w_k\} \cup C_{k+1} \cup \ldots \cup C_s = [m],$$

so the partition

$$\lambda = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} a_1 & \cdots & a_q & y_1 & \cdots & y_k & w_1 & \ldots & w_k & C_{k+1} & \ldots & C_s \\ x_1 & \cdots & x_q & y_1 & \cdots & y_k & z_1 & \ldots & z_k & r+1, r+2 & \ldots & n-1, n \end{array} \right)$$

is well-defined.

Now, we show that $\lambda\sigma\alpha = \alpha$ and that $\lambda \in \mathrm{MI}(\mathrm{P}^\sigma) = \mathrm{V}(\sigma)$. The first one is easily verified. For the second, we need to prove that $\lambda\sigma\lambda = \lambda$ and $\sigma\lambda\sigma = \sigma$. Since $\mathrm{Rank}(\lambda) = \mathrm{Rank}(\sigma)$ (so $\lambda \mathscr{J} \sigma$), by Lemma 2.2.15 it suffices to prove the latter. For this, it is enough to show that $\sigma\lambda$ contains the transversals

$$\{x_1, x_1'\}, \ldots, \{x_q, x_q'\}, \{y_1, y_1'\}, \ldots, \{y_k, y_k'\}, \{z_1, z_1'\}, \ldots, \{z_k, z_k'\}.$$

Firstly, since $y_1, \ldots, y_k \in [r]$, the sets $\{y_1, y_1'\}, \ldots, \{y_k, y_k'\}$ are all transversals in both σ and λ, and so in $\sigma\lambda$ as well.

Secondly, suppose $1 \le i \le q$ and consider the element x_i. Since $\{x_i, b_i'\}$ is a transversal of $\sigma\alpha$ and since $\{x_i, x_i'\}$ and $\{a_i, b_i'\}$ are transversals of σ and α respectively, there is a path in the product graph $\Pi(\sigma, \alpha)$ of the form

$$x_i \overset{\sigma}{\longleftrightarrow} x_i'' \overset{\alpha}{\longleftrightarrow} u_1'' \overset{\sigma}{\longleftrightarrow} u_2'' \overset{\alpha}{\longleftrightarrow} \cdots \overset{\sigma}{\longleftrightarrow} u_{2l}'' = a_i'' \overset{\alpha}{\longleftrightarrow} b_i' \tag{5.15}$$

for some $l \in \mathbb{N}_0$ and some $u_1, \ldots, u_{2l} \in [m]$ (if $l = 0$, then $x_i'' = a_i''$). Clearly, all the edges in this path coming from σ are edges in $\Pi(\sigma, \lambda)$ as well. As for the edges coming from α, note that they are upper nontransversals in α, except for the last edge. Obviously, the only upper nontransversals of α that are not blocks of λ are $C_1, \ldots, C_k$. However, from the construction of λ we know that these are all involved in components of type (C3), as enumerated at the beginning of Subsection 5.3.3, so they cannot belong to the path (5.15), it being a component of type (C6). Thus, all the edges in (5.15) coming from α, apart from the last one, are also in the product graph $\Pi(\sigma, \lambda)$. Since $\{a_i, x_i'\}$ is a transversal of λ, the product diagram $\Pi(\sigma, \lambda)$ contains the path

$$x_i \overset{\sigma}{\longleftrightarrow} x_i'' \overset{\lambda}{\longleftrightarrow} u_1'' \overset{\sigma}{\longleftrightarrow} u_2'' \overset{\lambda}{\longleftrightarrow} \cdots \overset{\sigma}{\longleftrightarrow} u_{2l}'' = a_i'' \overset{\lambda}{\longleftrightarrow} x_i'.$$

Thus, $\{x_1, x_1'\}, \ldots, \{x_q, x_q'\}$ are all transversals of $\sigma\lambda$.

Finally, we suppose $1 \leq j \leq k$ and consider the element z_j. As in the previous case, $\{z_j, y_j\}$ is a nontransversal of $\sigma\alpha$. From the analysis below step (2), we know that the product graph $\Pi(\sigma, \alpha)$ contains a path of the form

$$y_j \overset{\sigma}{\longleftrightarrow} y_j'' \overset{\alpha}{\longleftrightarrow} w_j'' \overset{\sigma}{\longleftrightarrow} v_1'' \overset{\alpha}{\longleftrightarrow} v_2'' \overset{\sigma}{\longleftrightarrow} \cdots \overset{\alpha}{\longleftrightarrow} v_{2l}'' = z_j'' \overset{\sigma}{\longleftrightarrow} z_j \tag{5.16}$$

for some $l \in \mathbb{N}_0$ and $v_1, \ldots, v_{2l} \in [m]$ (see Figure 5.9). Since this path contains the edge $\{y_j'', w_j''\}$ corresponding to the component C_j of α, it cannot contain the edge coming from C_p, for any $p \in [k]$ with $p \neq j$ (because such an edge belongs to the component connecting y_p and z_p). Thus, all the edges of the path (5.16), apart from the second, belong to the product graph $\Pi(\sigma, \lambda)$ as well. As λ contains the transversal $\{w_j, z_j'\}$, the product diagram $\Pi(\sigma, \lambda)$ contains the path

$$z_j' \overset{\lambda}{\longleftrightarrow} w_j'' \overset{\sigma}{\longleftrightarrow} v_1'' \overset{\lambda}{\longleftrightarrow} v_2'' \overset{\sigma}{\longleftrightarrow} \cdots \overset{\lambda}{\longleftrightarrow} v_{2l}'' = z_j'' \overset{\sigma}{\longleftrightarrow} z_j,$$

(see Figure 5.9) so $\{z_j, z_j'\}$ is a transversal of $\sigma\lambda$. As noted above, this completes the proof. □

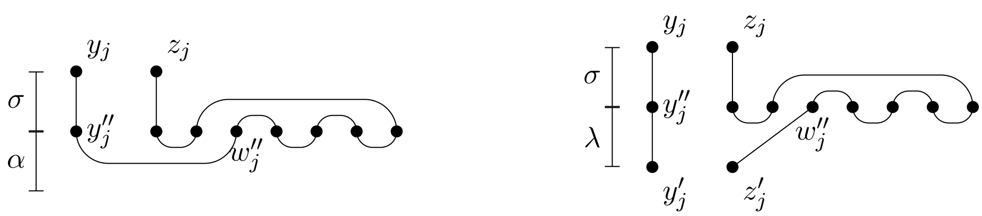


Figure 5.9: (from [28]) Left: a component of type (5.16) in the product graph $\Pi(\sigma, \alpha)$. Right: the corresponding two components of $\Pi(\sigma, \lambda)$.

Now, we are ready now to prove that P^σ is MI-dominated.

Proposition 5.3.17. *The semigroup* $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{B}^\sigma_{mn})$ *is MI-dominated.*

Proof. Suppose $\alpha \in \mathrm{E}_\sigma(\mathrm{P}^\sigma)$. We need to show that α is $\preceq$-below a mididentity, i.e. that $\alpha = \varepsilon \star_\sigma \alpha \star_\sigma \varepsilon$ for some $\varepsilon \in \mathrm{MI}(\mathrm{P}^\alpha)$.

Note that Lemma 5.3.16 and its dual tell us that $\alpha = \lambda \star_\sigma \alpha \star_\sigma \varrho$ for some $\lambda, \varrho \in \mathrm{MI}(\mathrm{P}^\sigma)$. Since $\mathrm{MI}(\mathrm{P}^\sigma)$ is a subsemigroup, $\lambda \star_\sigma \varrho \in \mathrm{MI}(\mathrm{P}^\sigma)$. Then, for $\varepsilon = \lambda \star_\sigma \varrho$ we have

$$\varepsilon \star_\sigma \alpha \star_\sigma \varepsilon = \lambda \star_\sigma \varrho \star_\sigma \alpha \star_\sigma \lambda \star_\sigma \varrho = \lambda \star_\sigma \alpha \star_\sigma \varrho = \alpha,$$

the penultimate equality following from the fact that $\lambda, \varrho \in \mathrm{MI}(\mathrm{P}^\sigma)$. □

In addition, we may prove

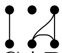
Proposition 5.3.18. *The semigroup* $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{B}^\sigma_{mn})$ *is RP-dominated.*

Proof. Propositions 2.4.8 and 5.3.17 imply that P^σ is RP-dominated if and only if the local monoid $\varepsilon \star_\sigma \mathrm{P}^\sigma \star_\sigma \varepsilon$ is factorisable for each $\varepsilon \in \mathrm{MI}(\mathrm{P}^\sigma)$. From Proposition 2.4.9(i), we have $\mathrm{MI}(\mathrm{P}^\sigma) = \mathrm{V}(\sigma)$, so Proposition 2.4.11 implies that $\varepsilon \star_\sigma \mathrm{P}^\sigma \star_\sigma \varepsilon \cong (\sigma \mathrm{P}^\sigma \sigma, \circledast) \cong \mathcal{B}_r$ for each $\varepsilon \in \mathrm{MI}(\mathrm{P}^\sigma)$. Thus, the semigroup P^σ is RP-dominated if and only if $\mathcal{B}_r$ is factorisable. It suffices to show that $\mathcal{B}_r = \mathrm{S}_r \cdot \mathrm{E}(\mathcal{B}_r)$. Suppose $\alpha = \left(\begin{array}{c|c|c|c|c|c} a_1 & \cdots & a_q & c_1, c_2 & \cdots & c_{2s-1}, c_{2s} \\ b_1 & \cdots & b_q & d_1, d_2 & \cdots & d_{2s-1}, d_{2s} \end{array}\right) \in \mathcal{B}_r$. Let

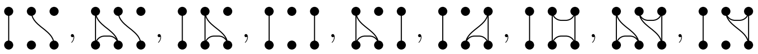
$\beta = \left(\begin{array}{c|c|c|c|c|c|c|c} a_1 & \cdots & a_q & c_1 & c_2 & \cdots & c_{2s-1} & c_{2s} \\ b_1 & \cdots & b_q & d_1 & d_2 & \cdots & d_{2s-1} & d_{2s} \end{array}\right)$ and $\mu = \left(\begin{array}{c|c|c|c|c|c} b_1 & \cdots & b_q & d_1, d_2 & \cdots & d_{2s-1}, d_{2s} \\ b_1 & \cdots & b_q & d_1, d_2 & \cdots & d_{2s-1}, d_{2s} \end{array}\right)$.

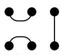
It is easily seen that $\beta \in \mathrm{S}_r$, $\mu \in \mathrm{E}(\mathcal{B}_r)$ and $\beta\mu = \alpha$. Therefore, $\mathcal{B}_r \subseteq \mathrm{S}_r \cdot \mathrm{E}(\mathcal{B}_r)$. As reverse containment is clear, the result follows. □

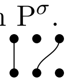
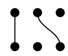
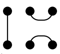
Remark 5.3.19. As we already mentioned in the previous section, if $\mathcal{K}$ is any of the categories $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$ and $\mathcal{TL}$, then the regular subsemigroup $\mathrm{Reg}(\mathcal{K}^\sigma_{mn})$ is not MI-dominated in general.

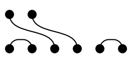
If $\mathcal{K} = \mathcal{P}$ or $\mathcal{K} = \mathscr{P}\mathcal{P}$, consider $\sigma =$  $\in \mathscr{P}\mathcal{P}_3 \subseteq \mathcal{P}_3$. Then, applying Proposition 2.4.9(i), one may calculate (via GAP [98] or by hand) that

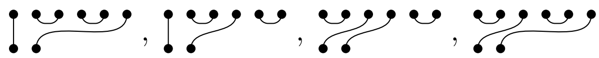
$$\mathrm{MI}(\mathrm{Reg}(\mathcal{P}^\sigma_3)) = \mathrm{MI}(\mathrm{Reg}(\mathscr{P}\mathcal{P}^\sigma_3)) = \mathrm{V}(\sigma)$$

$= \{$  $\}$

Now, consider $\alpha =$  $\in \mathscr{P}\mathcal{P}_3 \subseteq \mathcal{P}_3$. It is easily seen that $\alpha\sigma\alpha = \alpha$, so $\alpha \in \mathrm{E}(\mathrm{P}^\sigma)$. However, by Proposition 5.1.2(i), α is not $\leq_{\mathscr{R}}$-below any of the above mid-identities, so it follows that α is not $\preceq$-below any mid-identity in P^σ.

Similarly, if $\mathcal{K} = \mathscr{M}$ or $\mathcal{K} = \mathcal{PB}$, consider $\sigma =$  $\in \mathscr{M}_3 \subseteq \mathcal{PB}_3$. Then, it is easy to see that $\mathrm{MI}(\mathrm{Reg}(\mathscr{M}^\sigma_3)) = \mathrm{MI}(\mathrm{Reg}(\mathcal{PB}^\sigma_3)) = \mathrm{V}(\sigma) = \{$  $\}$. Note that $\alpha =$  $\in \mathscr{M}_3 \subseteq \mathcal{PB}_3$ satisfies $\alpha\sigma\alpha = \alpha$, but α is not $\leq_{\mathscr{R}}$-below the above mid-identity.

Finally, if $\mathcal{K} = \mathcal{TL}$, consider $\sigma =$  $\in \mathcal{TL}_{26}$. Then, it is easy to see that

$\mathrm{MI}(\mathrm{Reg}(\mathcal{TL}^\sigma_{6,2})) = \mathrm{V}(\sigma) = \{$  $\}$.

Note that $\alpha =$  $\in \mathcal{TL}_{6,2}$ satisfies $\alpha\sigma\alpha = \alpha$, but α is not $\leq_{\mathscr{R}}$-below any mid-identity in P^σ.

Now, we may calculate $\mathrm{Rank}(\mathrm{P}^\sigma)$. Recall that, if $r = m = n$, then $\sigma = \iota_r$, so $\mathcal{B}^\sigma_{mn} = \mathrm{P}^\sigma \cong \mathcal{B}_r$. Thus, we may suppose $r = m = n$ is not the case. Furthermore, since $\mathcal{B}^\sigma_{mn}$ and $\mathcal{B}^{\sigma^*}_{nm}$ are anti-isomorphic (see page 200), we may suppose $m \geq n$.

Theorem 5.3.20. *If $m \geq n$, and if $r = m = n$ does not hold, then the rank of the regular semigroup $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{B}^\sigma_{mn})$ is given by*

$$\mathrm{Rank}(\mathrm{P}^\sigma) = \frac{(m+r-1)!!}{(2r-1)!!} + \begin{cases} 1, & \text{if } r \geq 2, \\ 0, & \text{if } r \leq 1. \end{cases}$$

Proof. From Theorem 5.3.11(*i*) and (*ii*), it follows that $|\widehat{\mathrm{H}^\sigma_{\sigma^*}}/\mathscr{R}^\sigma| = |\widehat{\mathrm{R}^\sigma_{\sigma^*}}/\mathscr{R}^\sigma| = \frac{(m+r-1)!!}{(2r-1)!!}$ and $|\widehat{\mathrm{H}^\sigma_{\sigma^*}}/\mathscr{L}^\sigma| = |\widehat{\mathrm{L}^\sigma_{\sigma^*}}/\mathscr{L}^\sigma| = \frac{(n+r-1)!!}{(2r-1)!!}$. Since $\mathcal{B}_r \setminus \mathrm{S}_r$ is an ideal of $\mathcal{B}_r$ (by Lemma 5.3.10(*vi*)) and since P^σ is MI-dominated, Theorem 2.4.16 gives

$$\begin{aligned}\mathrm{Rank}(\mathrm{P}^\sigma) &= \mathrm{Rank}(\mathcal{B}_r : \mathrm{S}_r) + \max\left(\frac{(m+r-1)!!}{(2r-1)!!}, \frac{(n+r-1)!!}{(2r-1)!!}, \mathrm{Rank}(\mathrm{S}_r)\right), \\ &= \mathrm{Rank}(\mathcal{B}_r : \mathrm{S}_r) + \max\left(\frac{(m+r-1)!!}{(2r-1)!!}, \mathrm{Rank}(\mathrm{S}_r)\right)\end{aligned}$$

so the result follows from Lemma 5.3.10(*vii*) and the formula for $\mathrm{Rank}(\mathrm{S}_r)$ from page 110. □

Next, we calculate the rank of the idempotent-generated subsemigroup. Again, we assume that $m \geq n$, but we do not need to exclude the case $r = m = n$.

Theorem 5.3.21. *If $m \geq n$, then the rank and the idempotent rank of the idempotent-generated semigroup $\mathbb{E}_\sigma(\mathcal{B}^\sigma_{mn})$ are given by*

$$\mathrm{Rank}(\mathbb{E}_\sigma(\mathcal{B}^\sigma_{mn})) = \mathrm{idrank}(\mathbb{E}_\sigma(\mathcal{B}^\sigma_{mn})) = \frac{(m+r-1)!!}{(2r-1)!!} + \binom{r}{2}.$$

Proof. Keeping in mind the sizes calculated in the proof of Theorem 5.3.20 and the fact that P^σ is MI-dominated, from Theorem 2.4.17 we deduce

$$\begin{aligned}\mathrm{Rank}(\mathbb{E}_\sigma(\mathrm{P}^\sigma)) &= \mathrm{Rank}(\mathbb{E}(\mathcal{B}_r)) + \frac{(m+r-1)!!}{(2r-1)!!} - 1 \quad \text{and} \\ \mathrm{idrank}(\mathbb{E}_\sigma(\mathrm{P}^\sigma)) &= \mathrm{idrank}(\mathbb{E}(\mathcal{B}_r)) + \frac{(m+r-1)!!}{(2r-1)!!} - 1.\end{aligned}$$

Thus, the result follows from Lemma 5.3.10(*viii*). □

5.3.5 The rank of a sandwich semigroup $\mathcal{B}^\sigma_{mn}$

In the penultimate subsection of the thesis, we calculate the rank of a sandwich semigroup in the Brauer category. Again, we fix $m, n \in \mathbb{N}_0$ and $\sigma \in \mathcal{B}_{nm}$, and we write $r = \mathrm{Rank}(\sigma)$. By Lemma 5.3.1, we may suppose without loss of generality that σ is of form (5.13), and since $\mathcal{B}^{\sigma}_{mn}$ and $\mathcal{B}^{\sigma^*}_{nm}$ are anti-isomorphic, we may assume that $m \geq n$. Furthermore, as $r = n = m$ implies $\mathcal{B}^{\sigma}_{mn} \cong \mathcal{B}_r$ (because $\sigma = \iota_r$) and the rank of $\mathcal{B}_r$ is well-known (see 5.3.10(vii)), we may exclude the case $m = n = r$.

Keeping in mind these assumptions, we prove a lemma which will be the base for our technique of "downwards generating". For simplicity, we denote $\mathrm{D}_q = \mathrm{D}_q(\mathcal{B}_{mn})$ for each $0 \leq q \leq n$ with $q \equiv n \pmod 2$.

Lemma 5.3.22. *If $\alpha \in \mathrm{D}_q$, where $q \leq r$ and $q < n$, then $\alpha = \beta \star_\sigma \gamma$ for some $\beta, \gamma \in \mathrm{D}_{q+2}$.*

Proof. Write $\alpha = \left(\begin{array}{c|c|c|c|c|c} a_1 & \cdots & a_q & c_1, d_1 & \cdots & c_s, d_s \\ b_1 & \cdots & b_q & e_1, f_1 & \cdots & e_t, f_t \end{array}\right) \in \mathrm{D}_q$. From $q < n$, we may conclude that $s, t \geq 1$. Now, let

$$\beta = \left(\begin{array}{c|c|c|c|c|c|c|c} a_1 & \cdots & a_q & c_s & d_s & c_1, d_1 & \cdots & c_{s-1}, d_{s-1} \\ 1 & \cdots & q & n-1 & n & q+1, q+2 & \cdots & n-3, n-2 \end{array}\right) \quad \text{and}$$

$$\gamma = \left(\begin{array}{c|c|c|c|c|c|c|c} 1 & \cdots & q & m-1 & m & q+1, q+2 & \cdots & m-3, m-2 \\ b_1 & \cdots & b_q & e_t & f_t & e_1, f_1 & \cdots & e_{t-1}, f_{t-1} \end{array}\right).$$

It is easily seen that $\beta, \gamma \in \mathrm{D}_{q+2}$ and that $\beta \star_\sigma \gamma = \alpha$ (the cases $r = n < m$ and $r < n \leq m$ need to be considered separately). □

Therefore, it may be proved by descending induction that $\mathrm{D}_q \subseteq \langle \mathrm{D}_r \rangle_\sigma$ for all $0 \leq q \leq r$ with $q \equiv n \pmod 2$, and that $\mathrm{D}_r \subseteq \langle D_{r+2} \rangle_\sigma$ if $r < n$. We may immediately conclude that

Corollary 5.3.23.

(i) If $r < n \leq m$, then $\langle \mathrm{D}_{r+2} \cup \ldots \cup \mathrm{D}_n \rangle_\sigma = \mathcal{B}^{\sigma}_{mn}$, and

(ii) If $r = n < m$, then $\langle \mathrm{D}_r \rangle_\sigma = \mathcal{B}^{\sigma}_{mn}$.

Observe that the cases $r < n$ and $r = n$ differ, because in the former $\mathcal{B}^{\sigma}_{mn}$ has trivial maximal $\mathscr{J}^{\sigma}$-classes, while in the latter it has a unique maximal $\mathscr{J}^{\sigma}$-class, which is nontrivial (Proposition 5.2.6). Thus, we treat them separately:

Theorem 5.3.24. *If $r < n \leq m$, then $\mathcal{B}^{\sigma}_{mn} = \langle \Omega \rangle_\sigma$, where $\Omega = \{\alpha \in \mathcal{B}_{mn} : \mathrm{Rank}(\alpha) > r\}$. Furthermore, every generating set for $\mathcal{B}^{\sigma}_{mn}$ contains Ω, and so*

$$\mathrm{Rank}(\mathcal{B}^{\sigma}_{mn}) = |\Omega| = \sum_{\substack{r<q\leq n \\ r\equiv n \pmod 2}} \binom{m}{q}\binom{n}{q}(m-q-1)!!(n-q-1)!! \cdot q!.$$

Proof. Since $\Omega = \mathrm{D}_{r+2} \cup \mathrm{D}_{r+4} \cup \ldots \cup \mathrm{D}_n$, the formula for $|\Omega|$ follows from Proposition 5.1.5(iii) and (v). By Proposition 5.2.6(i), each element of Ω determines a trivial maximal $\mathscr{J}^{\sigma}$-class. Then, by the discussion at the beginning of Section 2.6, every generating set for $\mathcal{B}^{\sigma}_{mn}$ contains Ω. Since Corollary 5.3.23(i) implies that Ω generates $\mathcal{B}^{\sigma}_{mn}$ and the size of Ω is given by Proposition 5.1.5, the result follows. □

Theorem 5.3.25. *If $r = n < m$, then* $\mathrm{Rank}(\mathcal{B}^{\sigma}_{mn}) = \binom{m}{n}(m-n-1)!!$.

Proof. By Corollary 5.3.23(ii), we have $\langle \mathrm{D}_r \rangle_\sigma = \mathcal{B}^\sigma_{mn}$. Since $\mathrm{D}_r = \mathrm{J}_{\sigma^*}$ is the maximum $\mathscr{J}$-class in the hom-set $\mathcal{B}_{mn}$ (by Proposition 5.1.2(iii)), and since σ is right-invertible in (the stable partial semigroup) $\mathcal{B}$ with $\sigma^* \in \mathrm{RI}(\sigma)$ (by Proposition 5.1.7(i)), we wish to apply Proposition 2.6.3(ii). By the discussion preceding that proposition, we have $\mathrm{J}_{\sigma^*} = \mathrm{L}_{\sigma^*}$; further, $\mathrm{D}^\sigma_{\sigma^*} = \mathrm{J}^\sigma_{\sigma^*} = \mathrm{L}^\sigma_{\sigma^*}$ is the maximum $\mathscr{J}^\sigma$-class of $\mathcal{B}^\sigma_r$ and is a left-group over $\mathrm{H}^\sigma_{\sigma^*} \cong \mathrm{S}_r$ (cf. Corollary 5.2.10). We need to prove that $\mathrm{Rank}(\mathrm{S}_r) \leq |\, \mathrm{D}^\sigma_{\sigma^*} \,/\, \mathscr{H}^\sigma \,| = |\, \mathrm{D}^\sigma_r \,/\, \mathscr{H}^\sigma \,|$. Consider the partition

$$\beta = \left(\begin{array}{c|c|c|c|c|c|c} 1 & \cdots & n-1 & m & n, n+1 & \cdots & m-2, m-1 \\ 1 & \cdots & n-1 & n & & & \end{array} \right) \in \mathcal{B}_{mn};$$

one may easily prove that $\sigma\beta = \iota_n$, so $\beta \in \mathrm{E}_\sigma(\mathrm{D}^\sigma_r)$. Thus, D^σ_r contains at least two idempotents, and so $|\, \mathrm{D}^\sigma_r \,/\, \mathscr{H}^\sigma \,| \geq 2 \geq \mathrm{Rank}(\mathrm{S}_r)$. Keeping in mind that $\mathrm{J}_{\sigma^*} = \mathrm{L}_{\sigma^*}$, by Propositions 2.6.3($ii$), 5.1.2 and 5.1.5($i$), we have

$$\mathrm{Rank}(\mathcal{B}^\sigma_{mn}) = |\, \mathrm{J}_{\sigma^*} \,/\, \mathscr{H} \,| = |\, \mathrm{D}_{\sigma^*} \,/\, \mathscr{H} \,| = |\, \mathrm{D}_{\sigma^*} \,/\, \mathscr{R} \,| = \binom{m}{n}(m-n-1)!!. \qquad \square$$

Remark 5.3.26. If $\mathcal{K}$ is one of $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$ or $\mathcal{TL}$, we cannot prove similar results since, in general, Lemma 5.3.22 does not have an analogue in these categories, i.e.

- in the case $r < n \leq m$, the semigroup $\mathcal{K}^\sigma_{mn}$ is not generated by its maximal $\mathscr{J}^\sigma$-classes, and
- in the case $r = n < m$, the semigroup $\mathcal{K}^\sigma_{mn}$ is not generated by D_n.

All of this may be verified with GAP [98].

5.3.6 Egg-box diagrams

As in the previous chapters, we provide several egg-box diagrams (they originally appeared in [28], and all were generated by GAP [98]) to illustrate the structural results for $\mathcal{B}^\sigma_{mn}$. For more information on egg-box diagrams, see the introduction to Subsection 3.1.6.

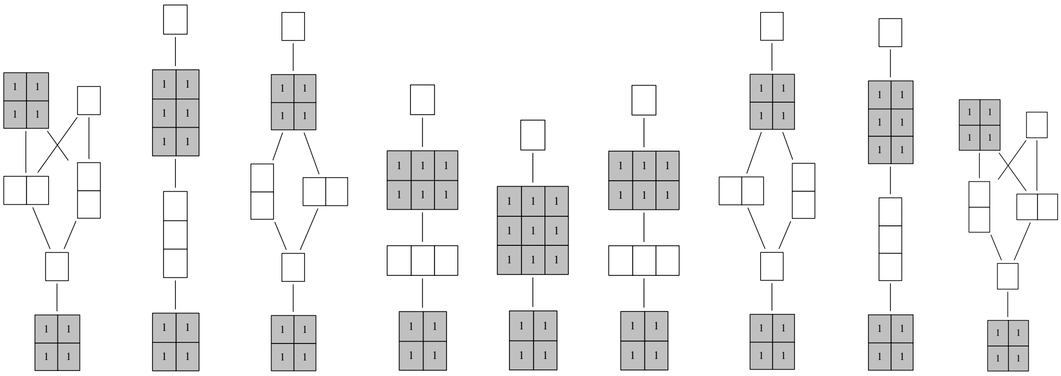
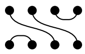
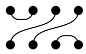


Figure 5.10: The variants $\mathcal{TL}_4^\sigma$ for each $\sigma \in \mathrm{D}_2(\mathcal{TL}_4)$.

Note that only the rightmost and leftmost variants have nontrivial maximal $\mathscr{J}^\sigma$-classes (also, the single trivial maximal class in all of these variants is $\{\iota_4\}$). These two variants correspond to sandwich elements  and .

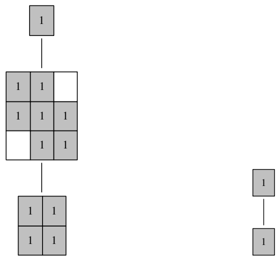


Figure 5.11: Egg-box diagrams of the Temperley-Lieb monoids $\mathcal{TL}_4$ (left) and $\mathcal{TL}_2$ (right). The regular subsemigroups of the variants in Figure 5.10 are inflations of $\mathcal{TL}_2$.

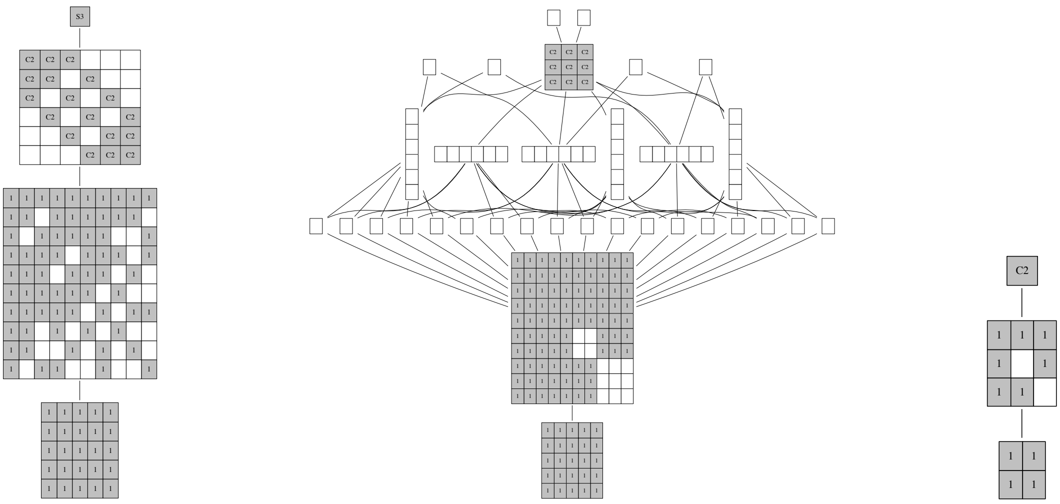
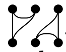


Figure 5.12: Egg-box diagrams of the partition monoids $\mathcal{P}_3$ (left) and $\mathcal{P}_2$ (right), and the variant $\mathcal{P}_3^\sigma$ (see the comment below), whose regular subsemigroup is an inflation of $\mathcal{P}_2$.

In the previous figure, the sandwich element of the variant is $\sigma =$ . Observe that $\mathcal{P}_3^\sigma$ has only trivial maximal $\mathscr{J}^\sigma$-classes, but σ is $\mathscr{J}$-related to each of its pre-inverses (since none of the elements of S_3 are its pre-inverses). This shows that the converse of Corollary 2.2.14 is not true in general. Moreover, note that $\mathcal{P}$ is stable, but not $\mathscr{H}$-trivial (for instance, in the monoid $\mathcal{P}_3 = \mathcal{P}_{33}^{\iota_3}$ we have $\mathscr{H}^{\iota_3} = \mathscr{H}$, but $\mathrm{H}_{\iota_3} = \mathrm{S}_3$), so this example also shows that Proposition 2.2.17 need not hold if S is not $\mathscr{H}$-trivial.

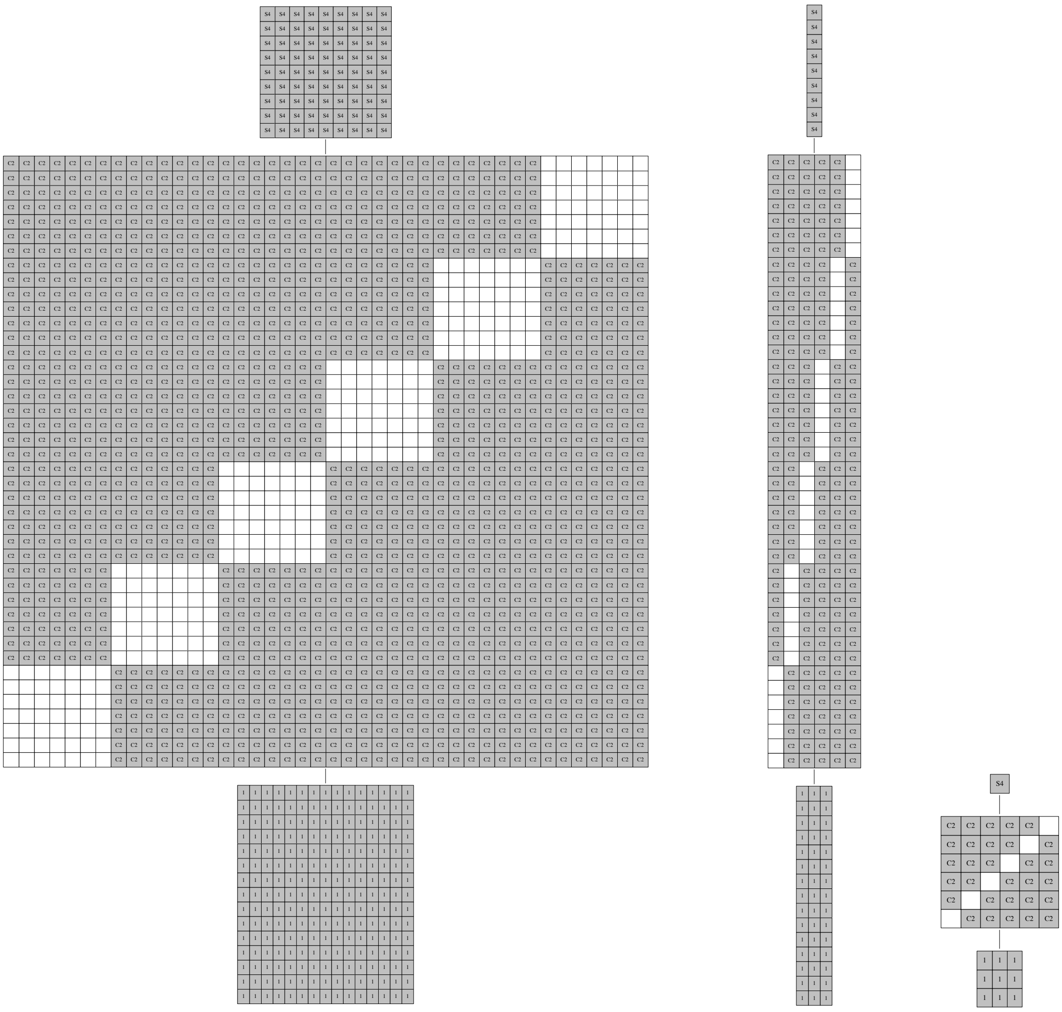


Figure 5.13: Left to right: egg-box diagrams of the regular sandwich semigroups $\mathrm{Reg}(\mathcal{B}_{66}^{\sigma_1})$, $\mathrm{Reg}(\mathcal{B}_{64}^{\sigma_2})$ and $\mathcal{B}_4$, where $\sigma_1 \in \mathcal{B}_{66}$ and $\sigma_2 \in \mathcal{B}_{46}$ both have rank 4. The first two are inflations of the last semigroup, $\mathcal{B}_4$.

The previous figure illustrates Theorem 5.3.11 (in particular, statements (i) and (ii)). Note that in diagram categories other than $\mathcal{B}$, $\widehat{\mathscr{R}^\sigma}$-classes in the same $\mathscr{D}^\sigma$-class do not necessarily contain the same number of $\mathscr{R}$-classes.

Conclusion

In this thesis, we have investigated sandwich semigroups in a locally small category. In this process, we introduced the notions of a partial semigroup, sandwich-regularity and MI-domination. We studied structural and combinatorial properties of these semigroups and provided results under various assumptions such as right-invertibility, (sandwich-)regularity, stability, or having a right-identity, for certain elements. The obtained results provide a solid framework for investigating a sandwich semigroup, and (under certain assumptions) its regular subsemigroup and idempotent-generated subsemigroup. In Chapters 3–5, we have applied these results and built on them, thereby thoroughly describing the sandwich semigroups in $\mathcal{PT}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{M}(\mathbb{F})$, $\mathcal{P}$, $\mathscr{PP}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{TL}$ and $\mathcal{B}$. By comparing the obtained results, the reader will see the big picture. In particular, one should consider

- the relationships between the properties of the sandwich semigroups and the whole category (for instance, consider Green's relations and stability in the category $\mathcal{PT}$, along with the same properties in the sandwich semigroups within),
- the way in which the choice of the sandwich element affects the features of the sandwich semigroup (e.g. consider the formulae for the rank of the sandwich semigroups in Subsections 3.1.5, 4.2.5, and 5.3.5), and
- the way in which the nature of the elements of the category shapes the specificities of the sandwich semigroups in it (for instance, compare the idempotent-generated subsemigroups of the sandwich semigroups in the diagram categories, in Theorem 5.2.15).

Hopefully, the reader is well acquainted with sandwich semigroups by now. If that is the case, the author has achieved one of her goals. If, in addition, the reader is tempted to experiment with various sandwich semigroups or partial semigroups, we could not wish for more. We list some directions of investigation and open problems worth exploring.

- It would be interesting to import and "translate" further terms (and results) related to category theory into our theory of partial semigroups. A good starting point would be to incorporate the notions of functors and products of categories. This merging of languages and techniques of two fields would possibly lead to significant advancements in both of them.

- Since our knowledge on sandwich semigroups advanced immensely each time we considered a new "family" of categories, we are convinced that the subject would benefit from consideration of sandwich categories in new (here unexplored) categories. Of course, such an investigation would also improve our knowledge of those categories. This is especially true for the fields where sandwich operations arise naturally (see page 3). One might consider looking into the category **Rel** (objects: sets, morphisms: binary relations) and the categories **Grp** (objects: groups, morphisms: group homomorphisms) and **Ring** (objects: rings, morphisms: ring homomorphisms).

- Of course, there is still room for investigation, even in the categories studied here. For instance, the following problems were not considered here and remain open problems for now:

  - Give a characterisation of the ideals in the regular subsemigroup P^a (of a sandwich semigroup) in the categories $\mathcal{PT}$, $\mathcal{T}$ and $\mathcal{I}$. Of course, it would also be interesting to try and obtain some results in the general case (i.e. for $\mathrm{Reg}(S_{ij}^a)$, where a is sandwich-regular). Here, we point out that the ideals of the regular subsemigroups of sandwich semigroups in $\mathcal{M}$ and $\mathcal{B}$ were described in [30] and [28], respectively.
  - Give a complete classification of the isomorphism classes of linear sandwich semigroups over infinite fields.
  - In $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, and $\mathcal{TL}$: classify the isomorphism classes of sandwich semigroups; calculate the ranks of sandwich semigroups; describe the combinatorial structure of and enumerate the idempotents in the regular subsemigroup $\mathrm{Reg}(\mathcal{K}_{mn}^\sigma)$; and calculate the (idempotent) ranks of $\mathrm{Reg}(\mathcal{K}_{mn}^\sigma)$ and the idempotent-generated subsemigroup $\mathbb{E}(\mathcal{K}_{mn}^\sigma)$.
  - Look at linear and diagram categories with infinite spaces or sets included.

Prošireni izvod

Dajemo skraćeni pregled rezulatata teze na srpskom. Pretpostavljamo da je čitalac upoznat sa osnovama teorije polugrupa (videti [58]) i osnovama teorije kategorija (videti [83]). Sva tvrđenja, definicije i napomene su numerisane isto kao i njihove verzije na engleskom.

U disertaciji izlažemo sadržaj radova [33], [34] i [28]. Autorka teze je koautor na ovim radovima i oni su nastali u okviru istraživanja za ovu tezu. Uz to, u četvrtoj glavi prikazujemo rezultate rada [30]. Na njemu autorka nije učestvovala, no rezultati u njemu se mogu izvesti iz opštih rezulatata (kasnije) dokazanih u [33].

Tema našeg istraživanja su sendvič polugrupe u lokalno malim kategorijama.

Definicija 2.0.1. Neka je S lokalno mala kategorija sa klasom objekata I. Neka su $i, j \in I$ dva fiksirana objekta i neka je a fiksiran morfizam $j \to i$. Ako S_{ij} označava skup svih morfizama $i \to j$, i na tom skupu definišemo operaciju

$$x \star_a y = xay, \qquad \text{za sve } x, y \in S_{ij},$$

onda je $S_{ij}^a = (S_{ij}, \star_a)$ polugrupa . Nazivamo je *sendvič polugrupa* nad S_{ij} koja odgovara a.

Da bismo bolje razumeli i lakše opisali ove polugrupe, definišemo pojam parcijalne polugrupe iz [30].

Definicija 2.1.1. *Parcijalna polugrupa* je uređena petorka $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ koja se sastoji od klase S, parcijalnog binarnog preslikavanja $(x, y) \mapsto x \cdot y$ (definisanog na nekom podskupu $S \times S$), klase "koordinata" I i funkcija $\boldsymbol{\delta}, \boldsymbol{\rho} : S \to I$, koje određuju leve i desne koordinate elemenata iz S, redom. Pri tome, moraju biti zadovoljena sledeća četiri uslova: za sve $x, y, z \in S$

(i) $x \cdot y$ je definisano ako i samo ako je $x\,\boldsymbol{\rho} = y\,\boldsymbol{\delta}$;

(ii) ako je $x \cdot y$ definisano, onda je $(x \cdot y)\,\boldsymbol{\delta} = x\,\boldsymbol{\delta}$ i $(x \cdot y)\,\boldsymbol{\rho} = y\,\boldsymbol{\rho}$;

(iii) ako su $x \cdot y$ i $y \cdot z$ definisani, onda je $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

(iv) za sve $i, j \in I$, klasa $S_{ij} = \{x \in S : x\,\boldsymbol{\delta} = i,\ x\,\boldsymbol{\rho} = j\}$ je skup.

Štaviše, parcijalna polugrupa $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ je *monoidalna* ako zadovoljava sledeće

(v) postoji preslikavanje $I \to S : i \mapsto e_i$ takvo da za sve $x \in S$ imamo $x \cdot e_{x\,\boldsymbol{\rho}} = x = e_{x\,\boldsymbol{\delta}} \cdot x$

Da bismo pojednostavili i skratili zapis, parcijalnu polugrupu $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ poistovećujemo sa njenim nosačem S, ukoliko su ostale komponente poznate ili nisu bitne za našu diskusiju. Takođe, umesto $x \cdot y$ skraćujemo na xy.

Lako se uviđa da se svaka monoidalna parcijalna polugrupa može interpretirati kao lokalno mala kategorija, i obratno. Shodno tome, ispitivaćemo (monoidalne) parcijalne polugrupe i sendvič polugrupe u njima.

Parcijalna polugrupa $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ je *regularna*, ako su svi njeni elementi (fon Nojman) regularni (tj. za svako $x \in S$ postoji $y \in S$ tako da je $xyx = x$). Ukoliko, pored toga, svaki element ima jedinstven inverz (tj. za svako $x \in S$ postoji jedinstveno $y \in S$ tako da je $xyx = x$ i $yxy = y$), onda je ta parcijalna polugrupa *inverzna*. Dalje, ukoliko se za parcijalnu polugrupu $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ može definisati preslikavanje ${}^* : S \to S : x \mapsto x^*$ tako da za sve $x, y \in S$ važi

(a) $(x^*)\,\boldsymbol{\delta} = x\,\boldsymbol{\rho}$, $(x^*)\,\boldsymbol{\rho} = x\,\boldsymbol{\delta}$, $(x^*)^* = x$ i

(b) ako je $x \cdot y$ definisano, onda je $(x \cdot y)^* = y^* x^*$,

onda je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho}, {}^*)$ *parcijalna $*$-polugrupa*. Ukoliko u njoj važi i $xx^*x = x$ za sve $x \in S$, onda je u pitanju *regularna parcijalna $*$-polugrupa*.

Za parcijalnu polugrupu S definišemo monoidalnu parcijalnu polugrupu $S^{(1)}$ na sledeći način: za svaku koordinatu $i \in I$ u skup S_{ii} dodajemo element e_{ii} (ukoliko S_{ii} već ne poseduje takav element) koji se ponaša kao neutralni element u svim slučajevima kada može da se pomnoži sa nekim elementom.

Sada možemo definisati Grinove poretke i relacije u parcijalnoj polugrupi S. Za $x, y \in S$ definišemo

$$\begin{aligned} x \leq_{\mathscr{R}} y \quad &\Leftrightarrow \quad \text{postoji } s \in S^{(1)} \text{ tako da je } x = ys, \\ x \leq_{\mathscr{L}} y \quad &\Leftrightarrow \quad \text{postoji } s \in S^{(1)} \text{ tako da je } x = sy, \\ x \leq_{\mathscr{H}} y \quad &\Leftrightarrow \quad x \leq_{\mathscr{L}} y \text{ i } x \leq_{\mathscr{R}} y, \\ x \leq_{\mathscr{J}} y \quad &\Leftrightarrow \quad \text{postoji } s, t \in S^{(1)} \text{ tako da je } x = syt. \end{aligned}$$

Dalje, za sve $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{J}\}$ uvodimo relaciju $\mathscr{K} = \leq_{\mathscr{K}} \cap \geq_{\mathscr{K}}$. Peta relacija je $\mathscr{D} = \mathscr{R} \circ \mathscr{L}$. Može se pokazati da je $\mathscr{D}$ najmanja relacija ekvivalencije nad S koja sadrži i $\mathscr{R}$ i $\mathscr{L}$. Uz to, u glavnom tekstu dokazujemo direktnu paralelu Grinove leme za parcijalne polugrupe (Lema 2.1.8, preuzeta iz [30]). Ovde ćemo navesti verziju te leme koja se odnosi na hom-setove (takođe iz [30]). Ako je $x \in S_{ij}$, za sve $\mathrm{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{J}, \mathscr{D}\}$ uvedimo oznaku $\mathrm{K}_x = \{y \in S_{ij} : x \,\mathscr{K}\, y\}$.

Lema 2.1.9. Neka je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ parcijalna polugrupa sa $i, j \in I$ i neka su x, y proizvoljni elementi skupa $S_{ij} = \{z \in S : \; z\,\boldsymbol{\delta} = i, \; z\,\boldsymbol{\rho} = j\}$.

(i) Ako je $x \mathscr{R} y$ i elementi $s, t \in S^{(1)}$ zadovoljavaju $x = ys$ i $y = xt$, onda su funkcije $\mathrm{L}_x \to \mathrm{L}_y : w \to wt$ i $\mathrm{L}_y \to \mathrm{L}_x : w \to ws$ uzajamno inverzne bijekcije. Restrikcije ovih preslikavanja na H_x i H_y redom su takođe uzajamno inverzne bijekcije.

(ii) Ako je $x \mathscr{L} y$ i elementi $s, t \in S^{(1)}$ zadovoljavaju $x = sy$ i $y = tx$, onda su funkcije $\mathrm{R}_x \to \mathrm{R}_y : w \to tw$ i $\mathrm{R}_y \to \mathrm{R}_x : w \to sw$ uzajamno inverzne bijekcije. Restrikcije ovih preslikavanja na H_x i H_y redom su takođe uzajamno inverzne bijekcije.

(iii) Ako je $x \mathscr{D} y$, onda važi $|\,\mathrm{R}_x\,| = |\,\mathrm{R}_y\,|$, $|\,\mathrm{L}_x\,| = |\,\mathrm{L}_y\,|$ i $|\,\mathrm{H}_x\,| = |\,\mathrm{H}_y\,|$.

Struktura sendvič polugrupe

U nastavku ćemo koristiti do sada definisane pojmove da opišemo osobine sendvič polugrupe. Pošto se malo šta može reći u opštem slučaju, u većini tvrđenja ćemo postaviti određene (manje ili više stroge) pretpostavke za parcijalnu polugrupu ili za sendvič element.

Grinove relacije sendvič polugrupe ćemo opisati preko takozvanih *P-skupova*.

$$\mathrm{P}_1^a = \{x \in S_{ij} : xa \mathscr{R} x\}, \qquad \mathrm{P}_2^a = \{x \in S_{ij} : ax \mathscr{L} x\},$$
$$\mathrm{P}_3^a = \{x \in S_{ij} : axa \mathscr{J} x\}, \qquad \mathrm{P}^a = \mathrm{P}_1^a \cap \mathrm{P}_2^a,$$

Neka je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ parcijalna polugrupa i S_{ij}^a sendvič polugrupa sadržana u njoj (gde $i, j \in I$ i $a \in S_{ji}$). Da bismo izbegli zabunu, Grinove relacije u parcijalnoj polugrupi označavamo sa $\mathscr{K}$, a u sendvič polugrupi sa $\mathscr{K}^\alpha$. Dalje, za $x \in S_{ij}$, klasa Grinove relacije $\mathscr{K}^a$ koja ga sadrži označava se sa K_x^a (za sve $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}, \mathrm{J}\}$).

Prvo, navodimo teoremu iz [30].

Teorema 2.2.3. Neka je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ parcijalna polugrupa gde $i, j \in I$ i $a \in S_{ji}$. Ako je $x \in S_{ij}$, onda

(i) $\mathrm{R}_x^a = \begin{cases} \mathrm{R}_x \cap \mathrm{P}_1^a, & \text{ako } x \in \mathrm{P}_1^a \\ \{x\}, & \text{ako } x \in S_{ij} \setminus \mathrm{P}_1^a, \end{cases}$

(ii) $\mathrm{L}_x^a = \begin{cases} \mathrm{L}_x \cap \mathrm{P}_2^a, & \text{ako } x \in \mathrm{P}_2^a \\ \{x\}, & \text{ako } x \in S_{ij} \setminus \mathrm{P}_2^a, \end{cases}$

(iii) $\mathrm{H}_x^a = \begin{cases} \mathrm{H}_x, & \text{ako } x \in \mathrm{P}^a \\ \{x\}, & \text{ako } x \in S_{ij} \setminus \mathrm{P}^a, \end{cases}$

(iv) $\mathrm{D}_x^a = \begin{cases} \mathrm{D}_x \cap \mathrm{P}^a, & \text{ako } x \in \mathrm{P}^a \\ \mathrm{L}_x^a, & \text{ako } x \in \mathrm{P}_2^a \setminus \mathrm{P}_1^a \\ \mathrm{R}_x^a, & \text{ako } x \in \mathrm{P}_1^a \setminus \mathrm{P}_2^a \\ \{x\}, & \text{ako } x \in S_{ij} \setminus (\mathrm{P}_1^a \cup \mathrm{P}_2^a), \end{cases}$

(v) $\mathrm{J}_x^a = \begin{cases} \mathrm{J}_x \cap \mathrm{P}_3^a, & \text{ako } x \in \mathrm{P}_3^a \\ \mathrm{D}_x^a, & \text{ako } x \in S_{ij} \setminus \mathrm{P}_3^a. \end{cases}$

Ukoliko $x \in S_{ij} \setminus \mathrm{P}^a$, onda je $\mathrm{H}_x^a = \{x\}$ negrupna $\mathscr{H}^a$-klasa u S_{ij}^a.

Sledeći rezultat (iz [28]) nam daje informacije o parcijalnim uređenjima $\leq_{\mathscr{R}^a}$, $\leq_{\mathscr{L}^a}$ i $\leq_{\mathscr{J}^a}$ u S^a_{ij}. Podsećamo da element $x \in S$ ima levu (desnu) jedinicu ako postoji $y \in S$ tako da je $yx = x$ ($xy = x$).

Lema 2.2.6. Neka $a \in S_{ji}$ ima levu i desnu jedinicu u S. Ako $x, y \in S_{ij}$, onda važi

(i) $x \leq_{\mathscr{R}^a} y \Leftrightarrow x = y$ ili $x \leq_{\mathscr{R}} ya$,

(ii) $x \leq_{\mathscr{L}^a} y \Leftrightarrow x = y$ ili $x \leq_{\mathscr{L}} ay$,

(iii) $x \leq_{\mathscr{J}^a} y \Leftrightarrow x = y$ ili $x \leq_{\mathscr{R}} ya$ ili $x \leq_{\mathscr{L}} ay$ ili $x \leq_{\mathscr{J}} aya$.

Naravno, relacija $\leq_{\mathscr{J}^a}$ definiše poredak nad $\mathscr{J}^a$-klasama, a nama su posebno značajne maksimalne klase u odnosu na taj poredak. Sledeća dva rezultata (iz [28]) se bave prirodom maksimalnih $\mathscr{J}^a$-klasa.

Lema 2.2.10. Ako je $x \in S_{ij}$ takav da u S važi $x \not\leq_{\mathscr{J}} a$, onda je $\{x\}$ maksimalna $\mathscr{J}^a$-klasa u S^a_{ij}; pored toga, $\{x\}$ je neregularna $\mathscr{D}^a$-klasa.

Maksimalne $\mathscr{J}^a$-klase ovog tipa ćemo nazivati *trivijalnim*, a sve ostale *netrivijalnim*. U narednoj lemi otkrivamo više o drugoj vrsti.

Lema 2.2.12. Neka je $a \in S_{ji}$ regularan element.

(i) Polugrupa S^a_{ij} sadrži najviše jednu $\mathscr{J}^a$-klasu.

(ii) Ako postoji netrivijalna maksimalna $\mathscr{J}^a$-klasa, onda ona sadrži $\mathrm{Pre}(a) = \{x \in S : axa = a\}$.

(iii) Ako postoji netrivijalna maksimalna $\mathscr{J}^a$-klasa, i ako je ona istovremeno i $\mathscr{D}^a$-klasa, onda je regularna.

U Propoziciji 2.2.17 (iz [28]) dajemo ekvivalentne uslove za postojanje netrivijalne maksimalne $\mathscr{J}^a$-klase u S^a_{ij}.

Osim Grinovih relacija i parcijalnih uređenja, zanima nas i stabilnost u parcijalnoj polugrupi i u sendvič polugrupi. U oba slučaja koristimo standardnu definiciju stabilnosti iz teorije polugrupa (videti npr. [108]): element $a \in S$ je

- *$\mathscr{R}$-stabilan* ako za sve $x \in S$ važi $xa \mathrel{\mathscr{J}} x \Rightarrow xa \mathrel{\mathscr{R}} x$,
- *$\mathscr{L}$-stabilan* ako za sve $x \in S$ važi $ax \mathrel{\mathscr{J}} x \Rightarrow ax \mathrel{\mathscr{L}} x$,
- *stabilan*, ako je i $\mathscr{R}$-stabilan i $\mathscr{L}$-stabilan.

Prvi rezultat je lema iz [30] koja opisuje odnos stabilnosti u dve strukture.

Lema 2.2.20. Neka je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ stabilna parcijalna polugrupa. Tada je S^a_{ij} stabilna za sve $i, j \in I$ i sve $a \in S_{ji}$.

Naravno, prirodno se postavlja pitanje značaja stabilnosti, tj. prednosti koju donosi stabilnost u sendvič polugrupi. Na to pitanje odgovara naredna lema iz [33], kao i Propozicija 2.2.25 i Posledica 2.2.26.

Propozicija 2.2.23. Neka je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ parcijalna polugrupa sa $i, j \in I$ i $a \in S_{ji}$. Ako je

(i) a $\mathscr{R}$-stabilan, onda važi $\mathrm{P}_3^a \subseteq \mathrm{P}_1^a$,

(ii) a $\mathscr{L}$-stabilan, onda važi $\mathrm{P}_3^a \subseteq \mathrm{P}_2^a$,

(iii) a je stabilan, onda važi $\mathrm{P}_3^a = \mathrm{P}^a$.

U Lemi 2.2.27 navodimo neke dovoljne uslove za $\mathscr{R}$- i $\mathscr{L}$-stabilnost.

Naravno, zanima nas i regularnost elemenata u senvič polugrupi. U narednoj propoziciji iz [33] navodimo zaključke koji se mogu izvesti u opštem slučaju.

Propozicija 2.2.29. Neka je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ parcijalna polugrupa sa $i, j \in I$ i $a \in S_{ji}$. Tada je

(i) P_1^a levi ideal polugrupe S_{ij}^a,

(ii) P_2^a desni ideal polugrupe S_{ij}^a,

(iii) P^a potpolugrupa u S_{ij}^a,

(iv) $\mathrm{Reg}(S_{ij}^a) = \mathrm{P}^a \cap \mathrm{Reg}(S)$,

(v) $\mathrm{Reg}(S_{ij}^a) = \mathrm{P}^a \Leftrightarrow \mathrm{P}^a \subseteq \mathrm{Reg}(S)$.

Primetimo da delovi *(iii)* i *(iv)* impliciraju da je u regularnoj parcijalnoj polugrupi S regularni deo $\mathrm{Reg}(S_{ij}^a)$ sendvič polugrupe S_{ij}^a regularna potpolugrupa.

Za kraj ovog bloka tvrđenja o strukturi sendvič polugrupe, ispitujemo posledice izbora desno-invertibilnog (i simetrično, levo-invertibilnog) sendvič elementa ($a \in S_{ji}$ je desno-invertibilan element ako postoji $b \in S_{ij}$ tako da je $xab = x$ za sve $x \in S_{ij}$). Navodimo dva tvrđenja iz [28].

Propozicija 2.2.35. Neka je $a \in S_{ji}$ desno-invertibilan element.

(i) Sendvič polugrupa S_{ij}^a ima jedinstvenu maksimalnu $\mathscr{J}^a$-klasu, i ona sadrži skup svih desnih inverza elementa a, $\mathrm{RI}(a)$.

(ii) Ako je S_{ij}^a stabilna, onda je ta maksimalna $\mathscr{J}^a$-klasa u stvari $\mathscr{L}^a$-klasa, i u pitanju je leva grupa (tj. direktan proizvod grupe i polugrupe levih nula) sa skupom idempotenata $\mathrm{RI}(a)$.

Bitno je naglasiti da analogno tvrđenje (Propozicija 2.2.37) važi i u hom-setu S_{ij}.

Lema 2.2.38. Neka je $(S, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ parcijalna polugrupa sa $i, j \in I$ i $a \in S_{ji}$. Ako je a desno-invertibilan u S_{ij}, tada je $\mathrm{P}_1^a = S_{ij}$, $\mathrm{P}^a = \mathrm{P}_2^a$ i $\mathscr{R}^a = \mathscr{R}$ nad S_{ij}^a.

U nastavku navodimo tri rezultata iz [33] vezana za osobine parcijalne polugrupe koje se prenose na njene parcijalne potpolugrupe. Naravno, posebnu pažnju ćemo posvetiti odnosu osobina sendvič polugrupa sadržanih u njima.

Propozicija 2.2.40. Neka je T parcijalna potpolugrupa parcijalne polugrupe S, i neka $x, y \in T$. Tada za sve $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}\}$ važi

(i) ako $y \in \mathrm{Reg}(T)$, onda $x \leq_{\mathscr{K}^S} y \Leftrightarrow x \leq_{\mathscr{K}^T} y$;

(ii) ako $x, y \in \mathrm{Reg}(T)$, onda $x \mathscr{K}^S y \Leftrightarrow x \mathscr{K}^T y$.

(Gde $\mathscr{K}^S$ i $\mathscr{K}^T$ označavaju relaciju $\mathscr{K}$ u S i T, redom.)

Slično, u Propoziciji 2.2.42 pokazujemo da se svojstvo stabilnosti elementa nasleđuje u regularnoj potpolugrupi.

Dalje, imamo

Propozicija 2.2.43. Neka je a element iz T_{ji} u parcijalnoj polugrupi $(T, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ gde $i, j \in I$, i neka je T parcijalna potpolugrupa u S. Tada je

(i) $\mathrm{P}_1^a(T) \subseteq \mathrm{P}_1^a(S) \cap T$, gde važi jednakost ako je $T_{ij} \cup T_{ij}a \subseteq \mathrm{Reg}(T)$,

(ii) $\mathrm{P}_2^a(T) \subseteq \mathrm{P}_2^a(S) \cap T$, gde važi jednakost ako je $T_{ij} \cup aT_{ij} \subseteq \mathrm{Reg}(T)$,

(iii) $\mathrm{P}^a(T) \subseteq \mathrm{P}^a(S) \cap T$, gde važi jednakost ako je $T_{ij} \cup T_{ij}a \cup aT_{ij} \subseteq \mathrm{Reg}(T)$,

(iv) $\mathrm{P}_3^a(T) \subseteq \mathrm{P}_3^a(S) \cap T$, gde važi jednakost ako je a stabilna u S i važi $T_{ij} \cup T_{ij}a \cup aT_{ij} \subseteq \mathrm{Reg}(T)$.

(Ovde, $\mathrm{P}_l^a(S)$ i $\mathrm{P}_l^a(T)$ označavaju skupove P_l^a u S i T, redom.)

Propozicija 2.2.44. Neka je a element iz T_{ji} u parcijalnoj polugrupi $(T, \cdot, I, \boldsymbol{\delta}, \boldsymbol{\rho})$ gde $i, j \in I$, i neka je T parcijalna potpolugrupa u S. Tada je

(i) $\mathscr{R}^a(T) \subseteq \mathscr{R}^a(S) \cap (T \times T)$, gde važi jednakost ako $T_{ij} \cup T_{ij}a \subseteq \mathrm{Reg}(T)$,

(ii) $\mathscr{L}^a(T) \subseteq \mathscr{L}^a(S) \cap (T \times T)$, gde važi jednakost ako $T_{ij} \cup aT_{ij} \subseteq \mathrm{Reg}(T)$,

(iii) $\mathscr{H}^a(T) \subseteq \mathscr{H}^a(S) \cap (T \times T)$, gde važi jednakost ako $T_{ij} \cup T_{ij}a \cup aT_{ij} \subseteq \mathrm{Reg}(T)$,

(Gde $\mathscr{K}^a(S)$ i $\mathscr{K}^a(T)$ označavaju relaciju $\mathscr{K}^a$ u S_{ij}^a i T_{ij}^a, redom.)

Struktura $\mathrm{Reg}(S_{ij}^a)$

U nastavku, istražujemo strukturu regularnog dela sendvič polugrupe S_{ij}^a. Pošto regularni elementi u opštem slučaju ne moraju da čine potpolugrupu, uvešćemo dodatnu pretpostavku: neka je $a \in S_{ji}$ *sendvič-regularan*, tj. element takav da je $\{a\} \cup aS_{ij}a \subseteq \mathrm{Reg}(S)$ (gde je $aS_{ij}a = \{asa : s \in S_{ij}\}$). Pod ovom pretpostavkom, iz teorije polugrupa znamo da je skup $\mathrm{V}(a) = \{x \in S : axa = a, xax = x\}$ neprazan, pa možemo fiksirati nekog predstavnika $b \in \mathrm{V}(a)$. Tada su skupovi $S_{ij}a$ i aS_{ij} nosači podgrupa u $(S_{ii}, \cdot)$ i $(S_{jj}, \cdot)$, redom. Dalje, $(aS_{ij}a, \star_b)$ je potpolugrupa u S_{ji}^b, i ne zavisi od izbora inverza b, tj. za ma koji element $c \in \mathrm{V}(a)$, imamo $\star_c\restriction_{aS_{ij}a} = \star_b\restriction_{aS_{ij}a}$. Da bismo to istakli, operaciju u polugrupi $(aS_{ij}a, \star_b)$ obeležavamo sa $\circledast$. Najzad, primetimo da su operacije

$$(aS_{ij}a, \circledast) \to (baS_{ij}a, \cdot) : x \mapsto bx \quad \text{i} \quad (aS_{ij}a, \circledast) \to (aS_{ij}ab, \cdot) : x \mapsto xb$$

izomorfizmi.

Takođe, pokazujemo sledeći rezultat (iz [33]):

Propozicija 2.3.2. Neka je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S i neka je $b \in \mathrm{V}(a)$. Tada je

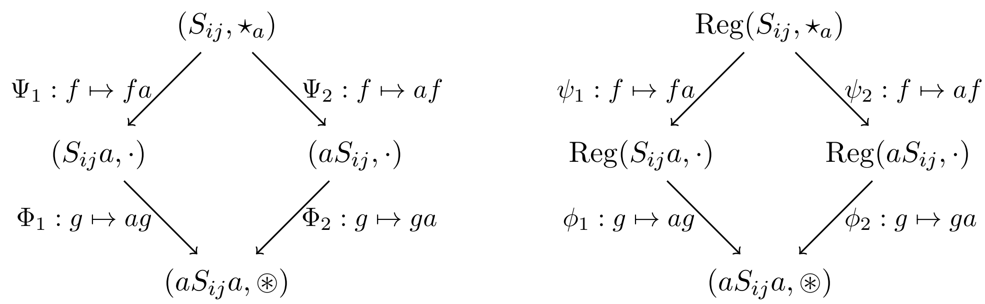
(i) $\mathrm{Reg}(S_{ij}^a) = \mathrm{P}^a$ regularna potpolugrupa u S_{ij}^a,

(ii) $\mathrm{Reg}(S_{ij}a, \cdot) = \mathrm{P}^a\, a = \mathrm{P}_2^a\, a$ regularna potpolugrupa u $(S_{ij}a, \cdot)$,

(iii) $\mathrm{Reg}(aS_{ij}, \cdot) = a\, \mathrm{P}^a = a\, \mathrm{P}_1^a$ regularna potpolugrupa u $(aS_{ij}, \cdot)$,

(iv) $aS_{ij}a = \mathrm{Reg}(aS_{ij}a, \circledast) = a\, \mathrm{P}^a\, a = a\, \mathrm{P}_1^a\, a = a\, \mathrm{P}_2^a\, a$ regularna potpolugrupa u S_{ji}^b.

Odnosi opisani ovde su slikovito prikazani na Slici 2.14 iz [33]. Primetimo da su sva preslikavanja na slici sirjektivni homomorfizmi.



Slika 2.14: Dijagrami koji prikazuju odnose između S_{ij}^a i $(aS_{ij}a, \circledast)$ (levo) i između $\mathrm{Reg}(S_{ij}^a)$ i $(aS_{ij}a, \circledast)$ (desno).

Pošto ćemo strukture iz desnog dijagrama koristiti i u nastavku, pojednostavićemo njihove oznake. Neka je $\mathrm{P}^a = \mathrm{Reg}(S_{ij}, \star_a)$, $T_1 = \mathrm{Reg}(S_{ij}a, \cdot) = \mathrm{P}^a\, a$, $T_2 = \mathrm{Reg}(aS_{ij}, \cdot) = a\, \mathrm{P}^a$ i $W = (aS_{ij}a, \circledast) = a\, \mathrm{P}^a\, a$.

Kao i u [33], definišemo funkciju ψ i pokazujemo njene osobine:

$$\psi = (\psi_1, \psi_2) : \mathrm{P}^a \to T_1 \times T_2 : x \mapsto (xa, ax).$$

Teorema 2.3.8. Ako je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S, onda je

(i) ψ injektivna i

(ii) $\mathrm{im}(\psi) = \{(s,t) \in T_1 \times T_2 : as = ta\} = \{(s,t) \in T_1 \times T_2 : s\phi_1 = t\phi_2\}$.

Pored toga, P^a je pullback proizvod T_1 i T_2 u odnosu na W i epimorfizme ψ_1 i ψ_2.

Za detalje vezane za pojam pullback proizvoda, videti [14] ili neki sličan izvor na temu osnova Univerzalne algebre.

Dalje, primenjujući opštiji rezultat u [58] i prethodne rezultate u ovom radu, dokazujemo sledeće (kombinacija Lema 2.3.3 i 2.3.4):

Lema 2.3.3. Neka je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S u kojoj važi $\mathscr{J} = \mathscr{D}$. Ako je $\mathscr{K}^{\mathrm{P}^a}$ bilo koja Grinova relacija u $\mathrm{Reg}(S^a_{ij}) = \mathrm{P}^a$, tada je $\mathscr{K}^{\mathrm{P}^a} = \mathscr{K}^a \cap (\mathrm{P}^a \times \mathrm{P}^a)$. Štaviše, za sve $x \in \mathrm{P}^a$ važi $\mathrm{K}^{\mathrm{P}^a}_x = \mathrm{K}^a_x$.

Da bismo bolje razumeli strukturu $\mathrm{Reg}(S^a_{ij})$, posmatramo preslikavanje $\phi : \psi_1\phi_1 = \psi_2\phi_2 = \mathrm{P}^a \to W : x \mapsto axa$. Takođe, za svako $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$ definišemo relaciju na P^a na sledeći način: $x\widehat{\mathscr{K}^a}y$ ako u W važi $\overline{x}\,\mathscr{K}^{\circledast}\,\overline{y}$ (gde $\overline{x}$ označava sliku $x\phi$ elementa x). Lako se uviđa da su u pitanju relacije ekvivalencije. Za $x \in \mathrm{P}^a$, klasu relacije $\widehat{\mathscr{K}^a}$ koja ga sadrži označavamo sa $\widehat{\mathrm{K}^a_x}$.

Nakon podrobnog ispitivanja, u [33] smo došli do ključnih rezultata vezanih za $\mathrm{Reg}(S^a_{ij})$ koje izlažemo u naredna četiri tvrđenja.

Lema 2.3.11. Ako je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S, važi

$$\mathrm{E}_a(\mathrm{P}^a) = \mathrm{E}_a(S^a_{ij}) = (\mathrm{E}_b(W))\phi^{-1}.$$

(Gde $\mathrm{E}_a(\mathrm{P}^a)$ i $\mathrm{E}_b(W)$ označavaju skupove svih idempotenata u P^a i W redom.)

Podsećamo da je $r \times l$ *pravougaona traka* polugrupa izomorfna sa $(I \times J, \cdot)$ gde je $|I| = r$, $|J| = l$ i operacija je definisana sa $(i,j)\cdot(k,l) = (i,l)$ za sve $(i,j),(k,l) \in I \times J$. Dalje, $r \times l$ *pravougaona grupa* nad grupom G je direktan proizvod $r \times l$ pravougaone trake i grupe G.

Teorema 2.3.12. Neka je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S. Ako je $x \in \mathrm{P}^a$ tako da je $r = |\widehat{\mathrm{H}^a_x}/\mathscr{R}^a|$ i $l = |\widehat{\mathrm{H}^a_x}/\mathscr{L}^a|$, onda važi

(i) restrikcija preslikavanja $\phi : \mathrm{P}^a \to W$ na skup H^a_x, $\phi\restriction_{\mathrm{H}^a_x} : \mathrm{H}^a_x \to \mathrm{H}^{\circledast}_{\overline{x}}$, je bijekcija;

(ii) H^a_x je grupa ako i samo ako je $\mathrm{H}^{\circledast}_{\overline{x}}$ grupa, u kom slučaju su te grupe izomorfne;

(iii) ako je H^a_x grupa, onda je $\widehat{\mathrm{H}^a_x}$ $r \times l$ pravougaona grupa nad $\mathrm{H}^{\circledast}_{\overline{x}}$;

(iv) ako je H^a_x grupa, onda je $\mathrm{E}_a(\widehat{\mathrm{H}^a_x})$ $r \times l$ pravougaona traka.

Napomena 2.3.13. Kada posmatramo strukturu Grinovih klasa, polugrupa $\mathrm{P}^a = \mathrm{Reg}(S^a_{ij})$ je "proširenje" polugrupe $W = (aS_{ij}a, \circledast)$. Naime,

- proizvoljna $\widehat{\mathscr{J}^a}$-klasa $\widehat{\mathrm{J}^a_x}$ u P^a sadrži samo jednu $\mathscr{J}^{\mathrm{P}^a}$-klasu, $\mathrm{J}^{\mathrm{P}^a}_x$, i ona odgovara $\mathscr{J}^{\circledast}$-klasi $\mathrm{J}^{\circledast}_{\overline{x}}$ u W; štaviše, parcijalna uređenja $(\mathrm{P}^a/\mathscr{J}^{\mathrm{P}^a}, \leq_{\mathscr{J}^{\mathrm{P}^a}})$ i $(W/\mathscr{J}^{\circledast}, \leq_{\mathscr{J}^{\circledast}})$ su izomorfna;

- proizvoljna $\widehat{\mathscr{D}^a}$-klasa $\widehat{\mathrm{D}^a_x}$ u P^a sadrži samo jednu $\mathscr{D}^a$-klasu, D^a_x, i ona odgovara $\mathscr{D}^{\circledast}$-klasi $\mathrm{D}^{\circledast}_{\overline{x}}$ u W; ova korespondencija je "1-1" i "na", što znači da svaka $\mathscr{D}^{\circledast}$-klasa korespondira tačno jednoj $\widehat{\mathscr{D}^a}$-klasi;

- svaka $\widehat{\mathscr{K}^a}$-klasa (gde je $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}\}$) u P^a je unija $\mathscr{K}^a$-klasa;

- struktura proizvoljne $\mathscr{D}^{\circledast}$-klase $\mathrm{D}_{\overline{x}}^{\circledast}$ u odnosu na relacije $\mathscr{R}^{\circledast}$, $\mathscr{L}^{\circledast}$ i $\mathscr{H}^{\circledast}$, je ista kao struktura $\widehat{\mathscr{D}_x^a}$ u odnosu na relacije $\widehat{\mathscr{R}^a}$, $\widehat{\mathscr{L}^a}$ i $\widehat{\mathscr{H}^a}$, redom, u smislu da svaka $\mathscr{K}^{\circledast}$-klasa $\mathrm{K}_{\overline{x}}^{\circledast}$ odgovara tačno jednoj $\widehat{\mathscr{K}^a}$-klasi, $\widehat{\mathrm{K}_x^a}$;

- proizvoljna $\widehat{\mathscr{H}^a}$-klasa $\widehat{\mathrm{H}_x^a}$ je unija $\mathscr{H}^a$-klasa, i one su ili sve negrupne $\mathscr{H}^a$-klase (ako je $\mathrm{H}_{\overline{x}}^{\circledast} = \mathrm{H}_{\overline{x}}^b$ negrupna $\mathscr{H}^{\circledast}$-klasa u W) ili sve grupe (ako je $\mathrm{H}_{\overline{x}}^{\circledast} = \mathrm{H}_{\overline{x}}^b$ grupa); u drugom slučaju, $\widehat{\mathrm{H}_x^a}$ je pravougaona grupa.

Poslednje tri obzervacije su ilustovane na Slici 2.4, u obliku egg-box dijagrama izdvojene $\mathscr{D}^a$-klase u P^a i njene odgovarajuće $\mathscr{D}^{\circledast}$-klase u W. Grupne $\mathscr{H}^a$- i $\mathscr{H}^{\circledast}$-klase su osenčene, a deblje linije na levom egg-box dijagramu označavaju granice između $\widehat{\mathscr{R}^a}$-klasa i između $\widehat{\mathscr{L}^a}$-klasa.

Teorema 2.3.15. Ako je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S, važi

$$\mathbb{E}_a(S_{ij}^a) = \mathbb{E}_a(\mathrm{P}^a) = (\mathbb{E}_b(W))\phi^{-1}.$$

(Gde $\mathbb{E}_a(\mathrm{P}^a)$ i $\mathbb{E}_b(W)$ označavaju idempotentno-generisane potpolugrupe u P^a i W redom.)

MI-dominacija u $\mathrm{Reg}(S_{ij}^a)$

U narednoj fazi istraživanja uvodimo nove pojmove koji se odnose na regularnu potpolugrupu $\mathrm{Reg}(S_{ij}^a)$, a pomoći će nam u ispitivanju ranga te polugrupe i idempotentno-generisane potpolugrupe.

Za početak, uvodimo neophodne definicije (primetimo da se one odnose na regularne polugrupe uopšte, ne samo u sendvič polugrupama). U regularnoj polugrupi T element $u \in T$ je *međujedinica* ako je $xuy = xy$ za sve $x, y \in T$. Sa druge strane, element u *očuvava regularnost* ako je polugrupa $(T, \star_u)$ regularna. Skupove svih međujedinica u T i svih elemenata iz T koji očuvavaju regularnost označavamo sa $\mathrm{MI}(T)$ i $\mathrm{RP}(T)$, redom.

Iz teorije polugrupa nam je poznato da se regularne polugrupe mogu parcijalno urediti: $x \preceq y$ ako i samo ako je $x = ey = yf$ za neke idempotente $e, f \in \mathrm{E}(S)$. Koristeći to uređenje, uvodimo:

Definicija 2.4.2. Regularna polugrupa T je

- *RP-dominirana* ako je svaki element u T ispod nekog elementa iz $\mathrm{RP}(T)$ u odnosu na relaciju $\preceq$;

- *MI-dominirana* ako je svaki *idempotent* u T ispod nekog elementa iz $\mathrm{MI}(T)$ u odnosu na relaciju $\preceq$.

U Poglavlju 2.4.1 detaljno razrađujemo teoriju razvijenu u [33], vezanu za ove pojmove. Kao najznačajnije rezulatate izdvajamo Propozicije 2.4.4, 2.4.5 i 2.4.8.

Zahvaljujući ovim rezultatima, pokazujemo kombinovan rezultat iz [33] i [28]:

Propozicija 2.4.10. Neka je $a \in S_{ji}$ sendvič-regularan i $b \in \mathrm{V}(a)$. Tada imamo sledeće:

(i) $\mathrm{MI}(\mathrm{P}^a) = \mathrm{E}_a(\widehat{\mathrm{H}_b^a}) = \mathrm{V}(a) \subseteq \mathrm{Max}_{\preceq}(\mathrm{P}^a)$.

(ii) $\mathrm{RP}(\mathrm{P}^a) = \widehat{\mathrm{H}_b^a}$.

(iii) Ako je S stabilna, onda je $\mathrm{J}_b^a = \mathrm{D}_b^a$.

(iv) Ako je S i stabilna i regularna, onda je $\mathrm{RP}(\mathrm{P}^a) = \mathrm{J}_b^a$ i $\mathrm{MI}(\mathrm{P}^a) = \mathrm{E}_a(\mathrm{J}_b^a)$.

Uz to, u [33] smo dokazali i

Propozicija 2.4.11. Neka je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S. Za proizvoljan element $e \in \mathrm{V}(a)$, restrikcija preslikavanja ϕ na lokalni monoid $W_e = \{e \star_a x \star_a e : x \in \mathrm{P}^a\}$ je izomorfizam $\phi\restriction_{W_e} : W_e \to W$.

Koristeći ove rezultate dokazujemo ključan rezultat iz [33], Propoziciju 2.4.14. Odatle izvodimo

Teorema 2.4.16. Neka je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S. Dalje, neka je $r = |\widehat{\mathrm{H}_b^a}/\mathscr{R}^a|$ i $l = |\widehat{\mathrm{H}_b^a}/\mathscr{L}^a|$, i pretpostavimo da je $W \setminus G_W$ ideal polugrupe W. Tada je

$$\mathrm{rank}(\mathrm{P}^a) \geq \mathrm{rank}(W : G_W) + \max(r, l, \mathrm{rank}(G_W)),$$

a jednakost važi ako je P^a MI-dominirana.

Teorema 2.4.17. Neka je $a \in S_{ji}$ sendvič-regularan element parcijalne polugrupe S. Dalje, neka je $r = |\widehat{\mathrm{H}_b^a}/\mathscr{R}^a|$ i $l = |\widehat{\mathrm{H}_b^a}/\mathscr{L}^a|$. Tada je

$$\mathrm{rank}(\mathbb{E}_a(\mathrm{P}^a)) \geq \mathrm{rank}(\mathbb{E}_b(W)) + \max(r, l) - 1$$

i

$$\mathrm{idrank}(\mathbb{E}_a(\mathrm{P}^a)) \geq \mathrm{idrank}(\mathbb{E}_b(W)) + \max(r, l) - 1,$$

a u oba izraza važi jednakost ako je P^a MI-dominirana.

Pretposlednja sekcija u drugoj glavi je posvećena slučaju kada je kategorija sa kojom radimo inverzna, što je opisano u narednoj definiciji.

Definicija 2.5.1. Kategorija X je *inverzna kategorija* ako za svaki morfizam $f : A \to B$ postoji jedinstveni morfizam $g : B \to A$ takav da je $fgf = f$ i $gfg = g$.

U tom slučaju imamo pojednostavljenu situaciju koja je opisana u narednom rezultatu iz [33] i njegovim posledicama:

Propozicija 2.5.2. Neka je S kategorija u kojoj $a \in S_{ji}$ i svi elementi iz $aS_{ij}a$ imaju jedinstvene inverze i $\mathrm{V}(a) = \{b\}$. Tada su sva preslikavanja na dijagramu 2.14 izomorfizmi (i stoga je preslikavanje $\phi : \mathrm{P}^a \to W$ izomorfizam), i sve polugrupe na njemu su inverzni monoidi.

Propozicija 2.5.2 ima niz posledica, koje su značajno pojednostavljene verzije tvrđenja koja se odnose na opšti slučaj. Dajemo kratak pregled najvažnijih:

- preslikavanje $\psi = (\psi_1, \psi_2)$ iz Teoreme 2.3.8 je trivijalno injektivno;
- pošto je ϕ izomorfizam, relacije $\widehat{\mathscr{K}^a}$ su identične relacijama $\mathscr{K}^a$, pa su pravougaone grupe iz Teoreme 2.3.12 u stvari grupe;
- iz istog razloga, Teorema 2.3.15 je trivijalno tačna;
- Propozicija 2.4.10 kaže da se $\mathrm{MI}(\mathrm{P}^a) = \{b\}$ i $\mathrm{RP}(\mathrm{P}^a) = \mathrm{H}_b^a$ sastoje samo od jedinice i invertibilnih elemenata, redom, što važi u svakom monoidu;
- jasno, P^a je MI-dominirana, pa se Teorema 2.4.16 svodi na "$\mathrm{rank}(\mathrm{P}^a) = \mathrm{rank}(W : G_W) + \mathrm{rank}(G_W)$ ako je $W \setminus G_W$ ideal u W";
- iz istog razloga, Teorema 2.4.17 se svodi na "$\mathrm{rank}(\mathbb{E}_a(\mathrm{P}^a)) = \mathrm{rank}(\mathbb{E}_b(W))$ i $\mathrm{idrank}(\mathbb{E}_a(\mathrm{P}^a)) = \mathrm{idrank}(\mathbb{E}_b(W))$".

Najzad, u poslednjoj sekciji navodimo rezultate vezane za rang sendvič polugrupe. Ovde ćemo navesti njeno najznačajnije tvrđenje, koje je izmenjena verzija Propozicije 3.26 iz [28].

Propozicija 2.6.3. Neka je S parcijalna polugrupa takva da je $a \in S_{ji}$ desno invertibilan. Dalje, pretpostavimo da je svaki element iz $S_{ij} \cup aS_{ij}a$ stabilan i da je svaki element iz aS_{ij} $\mathscr{R}$-stabilan. Uvedimo oznake $X_1 = |\,\mathrm{J}_b^a \,/\, \mathscr{H}^a\,|$, $X_2 = |(\mathrm{J}_b \setminus \mathrm{J}_b^a)/\,\mathscr{H}\,|$ i $T = \langle \mathrm{J}_b \rangle_a$.

(i) Tada važi $T = \langle \mathrm{J}_b^a \cup X_2 \rangle_a$ i

(ii) $\mathrm{rank}(T) = |X_2| + \max(|X_1|, \mathrm{rank}(\mathrm{H}_b^a))$.

(iii) Ako je $\mathrm{rank}(\mathrm{H}_b^a) \leq |\,\mathrm{J}_b^a \,/\, \mathscr{H}^a\,|$, onda je $\mathrm{rank}(T) = |\,\mathrm{J}_b \,/\, \mathscr{H}\,|$.

Ovaj rezultat ima prirodan dual koji ćemo izostaviti.

Sendvič polugrupe transformacija

Nakon što smo ispitali sendvič polugrupe u opštem slučaju, primenjujemo dobijene rezultate u konkretnim kategorijama. Glava 3 je u celosti posvećena rezultatima rada [34] vezanim za sendvič polugrupe transformacija, od kojih ovde navodimo samo odabrane.

Neka **Set** označava klasu svih skupova i neka je $\mathbf{Set}^+ = \mathbf{Set} \setminus \emptyset$. Za $A, B \in \mathbf{Set}$, definišimo

$$\begin{aligned}\mathbf{T}_{AB} &= \{f : f \text{ je preslikavanje } A \to B\},\\ \mathbf{PT}_{AB} &= \{f : f \text{ je preslikavanje } C \to B,\ \text{za neko } C \subseteq A\},\\ \mathbf{I}_{AB} &= \{f : f \text{ je injektivno preslikavanje } C \to B,\ \text{za neko } C \subseteq A\}.\end{aligned}$$

Dalje, neka je

$$\begin{aligned}\mathcal{PT} &= \{(A,f,B) : A,B \in \mathbf{Set},\ f \in \mathbf{PT}_{AB}\},\\ \mathcal{T} &= \{(A,f,B) : A,B \in \mathbf{Set}^+,\ f \in \mathbf{T}_{AB}\},\\ \mathcal{I} &= \{(A,f,B) : A,B \in \mathbf{Set},\ f \in \mathbf{I}_{AB}\}.\end{aligned}$$

Jasno, u $\mathcal{PT}$ možemo definisati parcijalnu binarnu operaciju

$$(A,f,B)\cdot(C,g,D) = \begin{cases} (A, f\circ g, D), & \text{ako je } B = C;\\ \text{nije definisano}, & \text{inače}, \end{cases}$$

a $\mathcal{T}$ i $\mathcal{I}$ su njene potklase zatvorene za tu operaciju. Neka je $\boldsymbol{\delta} : \mathcal{PT} \to \mathbf{Set} : (A,f,B) \mapsto A$ i $\boldsymbol{\rho} : \mathcal{PT} \to \mathbf{Set} : (A,f,B) \mapsto B$. Tada se lako pokazuje da su $(\mathcal{PT},\cdot,\mathbf{Set},\boldsymbol{\delta},\boldsymbol{\rho})$, $(\mathcal{T},\cdot\restriction_{\mathcal{T}},\mathbf{Set}^+,\boldsymbol{\delta}\restriction_{\mathcal{T}},\boldsymbol{\rho}\restriction_{\mathcal{T}})$ i $(\mathcal{I},\cdot\restriction_{\mathcal{I}},\mathbf{Set},\boldsymbol{\delta}\restriction_{\mathcal{I}},\boldsymbol{\rho}\restriction_{\mathcal{I}})$ monoidalne parcijalne polugrupe. U disertaciji smo pokazali da su sve tri regularne (rezultat iz [34]), kao i :

Lema 3.0.2. Ako je $\mathcal{Z}$ jedna od parcijalnih polugrupa $\mathcal{PT}$, $\mathcal{T}$ i $\mathcal{I}$, tada je funkcija

(i) $a \in \mathcal{Z}_{XY}$ desno invertibilna u $\mathcal{Z}_{YX}$ ako i samo ako je puna i injektivna;

(ii) $a \in \mathcal{Z}_{XY}$ levo invertibilna u $\mathcal{Z}_{YX}$ ako i samo ako je sirjektivna.

Propozicija 3.0.3. Parcijalna polugrupa $\mathcal{I}$ može biti proširena do parcijalne regularne $*$-polugrupe, koja je inverzna parcijalna polugrupa. Sa druge strane, ni $\mathcal{PT}$, ni $\mathcal{T}$ ne mogu biti proširene do parcijalne $*$-polugrupe.

Takođe, u rezultatu iz [34], okarakterisali smo Grinove relacije u ovim parcijalnim polugrupama. U tvrđenju, dom, im i ker označavaju redom domen, sliku i jezgro preslikavanja, dok je njegov rang, Rank, kardinalnost njegove slike.

Propozicija 3.1.2. Neka je $\mathcal{Z}$ jedna od parcijalnih polugrupa $\mathcal{PT}$, $\mathcal{T}$ i $\mathcal{I}$, i neka je $(A,f,B),(C,g,D) \in \mathcal{Z}$. Tada važi

(i) $(A,f,B) \leq_{\mathscr{R}} (C,g,D) \Leftrightarrow$
$$A = C,\ \operatorname{dom} f \subseteq \operatorname{dom} g \text{ i } \ker f \supseteq (\ker g)\restriction_{\operatorname{dom} f},$$

(ii) $(A,f,B) \leq_{\mathscr{L}} (C,g,D) \Leftrightarrow B = D$ i $\operatorname{im} f \subseteq \operatorname{im} g$,

(iii) $(A,f,B) \leq_{\mathscr{J}} (C,g,D) \Leftrightarrow \operatorname{Rank} f \leq \operatorname{Rank} g$,

(iv) $(A,f,B)\,\mathscr{R}(C,g,D) \Leftrightarrow A = C$, $\operatorname{dom} f = \operatorname{dom} g$ i $\ker f = \ker g$,

(v) $(A,f,B)\,\mathscr{L}(C,g,D) \Leftrightarrow B = D$ i $\operatorname{im} f = \operatorname{im} g$,

(vi) $(A,f,B)\,\mathscr{J}(C,g,D) \Leftrightarrow (A,f,B)\,\mathscr{D}(C,g,D) \Leftrightarrow \operatorname{Rank} f = \operatorname{Rank} g$.

U slučaju $\mathcal{T}$ i $\mathcal{I}$ uslovi se mogu pojednostaviti, no te formulacije preskačemo.

Za proizvoljne skupove A i B uvedimo oznake $\mathcal{PT}_{AB} = \{(A,f,B) : f \in \mathbf{PT}_{AB}\}$, $\mathcal{T}_{AB} = \{(A,f,B) : f \in \mathbf{T}_{AB}\}$ (ovde pretpostavljamo $A, B \neq \emptyset$) i $\mathcal{I}_{AB} = \{(A,f,B) : f \in \mathbf{I}_{AB}\}$. Kao u [34], u disertaciji smo kombinatorno opisali strukturu skupova

$\mathcal{PT}_{AB}$, $\mathcal{T}_{AB}$ i $\mathcal{I}_{AB}$, i njegovih preseka sa $\mathscr{D}$-klasama (preciznije, izračunali smo kardinalnost tih skupova, kao i broj $\mathscr{R}$-, $\mathscr{L}$- i $\mathscr{H}$-klasa). Uz to, ustanovili smo ekvivalentne uslove za stabilnost elementa u svakoj od parcijalnih polugrupa.

Propozicija 3.1.7. Neka je $\mathcal{Z}$ jedna od parcijalnih polugrupa $\mathcal{PT}$, $\mathcal{T}$ i $\mathcal{I}$, i neka je $(A, f, B) \in \mathcal{Z}$. Tada važi

(i) (A, f, B) je $\mathscr{R}$-stabilan $\Leftrightarrow$ [$\operatorname{Rank} f < \aleph_0$ ili je f puna i injektivna],

(ii) (A, f, B) je $\mathscr{L}$-stabilan $\Leftrightarrow$ [$\operatorname{Rank} f < \aleph_0$ ili je f sirjektivna],

(iii) (A, f, B) je stabilan $\Leftrightarrow$ [$\operatorname{Rank} f < \aleph_0$ ili je f puna i bijektivna].

Fiksirajmo skupove $X, Y \in \mathbf{Set}$ i preslikavanje $a \in \mathbf{PT}_{YX}$ (za kategoriju $\mathcal{T}$ i neprazne X, Y uzimamo $a \in \mathbf{T}_{YX}$, a za kategoriju $\mathcal{I}$ uzimamo $a \in \mathbf{I}_{YX}$), kao i oznake

$$a = \binom{A_i}{a_i}_{i\in I}, \quad B = \operatorname{dom} a, \quad \sigma = \ker a, \quad A = \operatorname{im} a, \quad \alpha = \operatorname{Rank} a.$$
$$\beta = |X \setminus \operatorname{im} a|, \quad \lambda_i = |A_i| \text{ za } i \in I, \quad \Lambda_J = \prod_{j\in J} \lambda_j \text{ za } J \subseteq I.$$

Ispitivaćemo sendvič polugrupu $\mathcal{PT}^a_{XY}$ (odnosno $\mathcal{T}^a_{XY}$ i $\mathcal{I}^a_{XY}$). U tezi (i u [34]) smo opisali P-skupove naše polugrupe (Propozicija 3.1.8), a direktno iz opšte teorije dobijamo karakterizaciju Grinovih relacija, kao i tvrđenja vezana za poredak $\mathscr{J}^a$-klasa. Uz to, dokazujemo i sledeća tvrđenja iz [34].

Propozicija 3.1.18. Regularne $\mathscr{D}^a$-klase u $\mathcal{PT}^a_{XY}$ su tačno skupovi

$$\mathrm{D}^a_\mu = \{f \in \mathrm{P}^a : \operatorname{Rank} f = \mu\}, \quad \text{za svaki kardinal } 0 \le \mu \le \alpha = \operatorname{Rank} a.$$

Dalje, ako je $f \in \mathrm{P}^a$, onda $\mathrm{D}^a_f = \mathrm{J}^a_f$ važi ako i samo ako je $\operatorname{Rank} f < \aleph_0$ ili je a stabilna.

Prethodno tvrđenje je identično za $\mathcal{I}^a_{XY}$, a u slučaju $\mathcal{T}^a_{XY}$ je jedina razlika što nemamo preslikavanja ranga 0.

Propozicija 3.1.19.

(i) Ako je $\alpha < \min(|X|, |Y|)$, onda su maksimalne $\mathscr{J}^a$-klase polugrupe $\mathcal{PT}^a_{XY}$ ($\mathcal{T}^a_{XY}$ i $\mathcal{I}^a_{XY}$) tačno oni singltoni $\{f\}$, za $f \in \mathcal{PT}_{XY}$ ($\mathcal{T}_{XY}$ i $\mathcal{I}_{XY}$) za koje je $\operatorname{Rank} f > \alpha$. Dakle, sve maksimalne $\mathscr{J}^a$-klase u $\mathcal{PT}^a_{XY}$ ($\mathcal{T}^a_{XY}$ i $\mathcal{I}^a_{XY}$) su u ovom slučaju trivijalne.

(ii) Ako je $\alpha = \min(|X|, |Y|)$, onda imamo jedinstvenu maksimalnu $\mathscr{J}^a$-klasu u $\mathcal{PT}^a_{XY}$ ($\mathcal{T}^a_{XY}$ i $\mathcal{I}^a_{XY}$), i to je $\mathrm{J}^a_b = \{f \in \mathrm{P}^a_3 : \operatorname{Rank} f = \alpha\}$. Ova maksimalna $\mathscr{J}^a$-klasa je netrivijalna.

U narednom poglavlju, uz pomoć Dijagrama 2.14 određujemo odnos između $\operatorname{Reg}(\mathcal{PT}^a_{XY})$ i $(a\mathcal{PT}_{XY}a, \circledast)$. Prvo, uvodimo oznake

$$\mathcal{PT}(X, A) = \{f \in \mathcal{PT}_X \;:\; \operatorname{im} f \subseteq A\}$$
$$\mathcal{PT}(Y, \sigma) = \{f \in \mathcal{PT}_Y \;:\; \text{svaka } \ker f\text{-klasa je unija } \sigma\text{-klasa}\}.$$

Dalje, dokazujemo da je funkcija $\eta : (a\,\mathcal{PT}_{XY}\,a, \circledast) \to (ba\,\mathcal{PT}_{XY}\,a, \cdot) : x \mapsto bx$ izomorfizam, pri čemu važi $(ba\,\mathcal{PT}_{XY}\,a, \cdot) = (ba\,\mathcal{PT}_{X}\,ba, \cdot) \equiv \mathcal{PT}_A$. Na isti način kao u [34], naše analize su rezultovale dijagramima 3.2 i 3.3. Slično, u slučaju $\mathrm{Reg}(\mathcal{T}^a_{XY})$ i $\mathrm{Reg}(\mathcal{I}^a_{XY})$ dobijamo redom Dijagrame 3.11 i 3.15, gde su polugrupe $\mathcal{T}(X, A)$, $\mathcal{I}(X, A)$ i $\mathcal{T}(Y, \sigma)$ definisane analogno, a $\mathcal{I}(Y, B)^* = \{f^{-1} : f \in \mathcal{I}(Y, B)\}$ (gde f^{-1} označava inverznu funkciju za injekciju f).

Dalje, navodimo verzije Teoreme 2.3.8 za $\mathrm{Reg}(\mathcal{PT}^a_{XY})$, $\mathrm{Reg}(\mathcal{T}^a_{XY})$ i $\mathrm{Reg}(\mathcal{I}^a_{XY})$ i pokazujemo da se Grinove relacije na njima poklapaju sa Grinovim relacijama na odgovarajućim sendvič polugrupama. Primenjujući Teoremu 2.3.12, opisujemo strukturu polugrupe P^a u vidu prirode $\mathscr{H}^a$-klasa, broja $\mathscr{K}^a$-klasa u fiksiranoj $\widehat{\mathscr{K}^a}$-klasi, kao i u odgovarajućoj $\mathscr{D}^a = \widehat{\mathscr{D}^a}$-klasi. To nam omogućava da izračunamo odgovarajuće kardinalnosti i da damo ekvivalentne uslove koji obezbeđuju da je P^a konačan, prebrojivo beskonačan ili neprebrojiv.

Najzad, u sva tri slučaja pokazujemo da je polugrupa P^a MI-dominirana uvek, a RP-dominirana ako i samo ako je a konačnog ranga. Štaviše, kao u [34], pokazali smo da

Teorema 3.1.34.

(i) Ako je $|\,\mathrm{P}^a\,| \geq \aleph_0$, onda je $\mathrm{rank}(\mathrm{P}^a) = |\,\mathrm{P}^a\,|$.

(ii) Ako je $|\,\mathrm{P}^a\,| < \aleph_0$, onda je

$$\mathrm{rank}(\mathrm{P}^a) = \begin{cases} 1, & \text{if } \alpha = 0; \\ 1 + \max(2^\beta, \Lambda_I), & \text{if } \alpha = 1; \\ 2 + \max(3^\beta, \Lambda_I), & \text{if } \alpha = 2; \\ 2 + \max((\alpha+1)^\beta, \Lambda_I, 2), & \text{if } \alpha \geq 3. \end{cases}$$

Uz to, opisali smo idempotente, izračunali njihov broj i pokazali

Teorema 3.1.39.

$$\mathrm{rank}(\mathcal{E}^a_{XY}) = \mathrm{idrank}(\mathcal{E}^a_{XY}) = \begin{cases} |\mathcal{E}^a_{XY}| = |\,\mathrm{P}^a\,|, & |\,\mathrm{P}^a\,| \geq \aleph_0; \\ \binom{\alpha+1}{2} + \max((\alpha+1)^\beta, \Lambda_I), & |\,\mathrm{P}^a\,| < \aleph_0. \end{cases}$$

Naravno, dokazali smo i verzije prethodna dva rezultata za kategorije $\mathcal{T}$ i $\mathcal{I}$, ali ih ovde preskačemo. Za kraj, računamo rang sendvič polugrupe transformacija. Ovde ćemo, radi uštede prostora, komentarisati rezultate samo za $\mathcal{PT}^a_{XY}$, no bitno je napomenuti da su rangovi izračunati i za $\mathcal{T}^a_{XY}$ i $\mathcal{I}^a_{XY}$.

Prvo, analiziramo jednostavnije slučajeve. Tu spadaju slučajevi kada je neki od skupova X ili Y prazan (polugrupa je jednoelementna), kada je rang sendvič elementa 0 (tada množenjem ne možemo generisati nove elemente), kada je X prebrojivo beskonačan ili je $|Y|$ neprebrojiv (tada polugrupa ima neprebrojivo mnogo elemenata) i kada je a puna bijekcija (tada imamo polugrupu izomorfnu sa $\mathcal{PT}_A$). U preostalim slučajevima smo, uz brojna pomoćna tvrđenja, dokazali i Teoreme 3.3.8, 3.1.51 i 3.1.57. Za kraj, u Poglavlju 3.1.6 dajemo egg-box dijagrame raznih sendvič polugrupa transformacija kao ilustraciju naših rezultata. Još jednom napominjemo da su prikazani rezultati objavljeni u radu [34].

Sendvič polugrupe matrica

U narednoj glavi prezentujemo i nadograđujemo rezultate iz [30] u svetlu naših opštih rezultata. Ovde ćemo spomenuti samo ona tvrđenja koja su prvi put objavljena u ovoj tezi i ona tvrđenja koja su proširena u odnosu na originalna iz [30].

Za prirodne brojeve $m, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ i polje $\mathbb{F}$, neka $\mathcal{M}_{mn}(\mathbb{F})$ označava skup svih matrica dimenzije $m \times n$ nad poljem $\mathbb{F}$ (ako je $m = n$, pišemo $\mathcal{M}_{mn}(\mathbb{F}) = \mathcal{M}_m(\mathbb{F})$) i neka je $\mathcal{M}(\mathbb{F}) = \bigcup_{m,n\in\mathbb{N}} \mathcal{M}_{mn}(\mathbb{F})$. Ako je polje $\mathbb{F}$ poznato ili njegov izbor ne pravi razliku u našoj diskusiji, koristićemo oznake $\mathcal{M}_{mn}$, $\mathcal{M}_m$ i $\mathcal{M}$.

Fiksirajmo polje $\mathbb{F}$ i definišimo preslikavanja $\boldsymbol{\delta} : \mathcal{M} \to \mathbb{N}$ i $\boldsymbol{\rho} : \mathcal{M} \to \mathbb{N}$. Tada je petorka $(\mathcal{M}, \cdot, \mathbb{N}, \boldsymbol{\delta}, \boldsymbol{\rho})$, gde je $\cdot$ uobičajeno množenje matrica, regularna monoidalna parcijalna polugrupa. U ovoj glavi ispitujemo tu parcijalnu polugrupu i sendvič polugrupe u njoj, i pri tom pratimo program istraživanja korišćen u prethodnoj glavi.

Značajan doprinos disertacije u ovoj glavi je proširivanje rezultata iz [30] na beskonačna polja, omogućeno opštom teorijom izloženom u drugoj glavi. Konkretno, u Lemi 4.1.3 računamo (opšte poznatu) veličinu grupe automorfizama nad $V_s(\mathbb{F})$, što nam uz proširenu definiciju q-binomnog koeficijenta (na strani 166) daje podlogu za dokazivanje Propozicije 4.1.4 u kojoj prikazujemo kombinatornu strukturu homseta $\mathcal{M}_{mn}$. Slično, Teorema 4.2.15 je unapređena verzija Teoreme 6.4 iz [30], jer se odnosi i na matrice nad beskonačnim poljima. Dalje, u Teoremama 4.2.19, 4.2.27, 4.2.28 i 4.2.34 izračunavamo redom rang skupa P^a, opisujemo njegove idempotente, i računamo rang idempotentno-generisane potpolugrupe i rang čitave sendvič polugrupe. U tim tvrđenjima je takođe uključen slučaj $|\mathbb{F}| \geq \aleph_0$. Osim toga, dokazujemo i naredne (originalne) rezulatate u ovoj glavi:

Propozicija 4.1.6. Parcijalna polugrupa $\mathcal{M}$ može da bude proširena do parcijalne $*$-polugrupe, ali ne i do regularne parcijalne $*$-polugrupe.

Posledica 4.1.8. Neka je $A \in \mathcal{M}_{mn}$. Tada važi

(i) A je desno invertibilna $\mathcal{M}_{nm}$ ako i samo ako je $\mathrm{Rank}(A) = m$.

(ii) A je levo invertibilna $\mathcal{M}_{nm}$ ako i samo ako je $\mathrm{Rank}(A) = n$.

Propozicija 4.2.18.

(i) Polugrupa $\mathrm{P}^J = \mathrm{Reg}(\mathcal{M}^J_{mn})$ je MI-dominirana.

(ii) Polugrupa $\mathrm{P}^J = \mathrm{Reg}(\mathcal{M}^J_{mn})$ je RP-dominirana.

Sendvič polugrupe particija

U poslednjoj glavi sprovodimo analizu iz prethodnih glava za kategorije particija koje definišemo u nastavku. Rezultati su objavljeni u radu [28] Za prirodan broj $n \in \mathbb{N}$, definišimo $[n] = \{1, 2, \ldots, n\}$. Radi potpunosti, definišemo $[0] = \emptyset$. Dalje, za svaki skup $A \subseteq \mathbb{N}_0$, definišemo $A' = \{a' : a \in A\}$ i $A'' = \{a'' : a \in A\}$. Sada, za proizvoljne nenegativne brojeve $m, n \in \mathbb{N}_0$, neka $\mathcal{P}_{mn}$ označava skup svih particija skupa $[m] \cup [n]'$. Dalje, neka je $\mathcal{P} = \bigcup_{m,n \in \mathbb{N}_0} \mathcal{P}_{mn}$ skup svih takvih particija.

Fiksirajmo $m, n \in \mathbb{N}_0$ i odaberimo proizvoljnu particiju $\sigma \in \mathcal{P}_{mn}$. Možemo slikovito da je prikažemo na sledeći način: pravimo graf sa $m + n$ čvorova u $\mathbb{R}^2$, uz poštovanje pravila u nastavku

- svaki element $a \in [m]$ je pridružen čvoru $(a, 1)$;
- svaki element $b' \in [n]'$ je pridružen čvoru $(b, 0)$;
- za svaki blok S particije σ, čvorovi koji odgovaraju elementima skupa S čine (povezanu) komponentu grafa;
- svaka grana grafa je smeštena u unutrašnjosti pravougaonika $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \max(m, n),\ 0 \leq y \leq 1\}$.

Takav prikaz nazivamo dijagramom particije. Primer dijagrama se može videti na Slici 5.1. Naravno, u opštem slučaju postoji više dijagrama koji odgovaraju istoj particiji. Pošto nas zanimaju komponente grafa, particiju identifikujemo sa bilo kojim takvim dijagramom. Neka je $\alpha \in \mathrm{P}_{mn}$. Blokove koji sadrže i gornje i donje čvorove (tj. elemente iz skupa $[m]$ i iz $[n]'$) nazivamo *transverzale*. Broj transverzala je *rang* particije α. Blokovi koje sadrže samo gornje čvorove (elemente iz $[m]$) su *gornje netransverzale*. *Donje netransverzale* definišemo dualno. Pošto transverzale, zajedno sa gornjim i donjim netranverzalama, određuju particiju, možemo da je prikažemo preko sledeće šeme

$$\left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ \cline{4-6} B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{array} \right),$$

gde su $A_i \cup B_i'$ $(1 \leq i \leq r)$ transverzale, C_i $(1 \leq i \leq s)$ gornje netransverzale, i D_i' $(1 \leq i \leq t)$ donje netransverzale (ako je bilo koji od ovih skupova singlton, izostavljamo zagrade).

Za particije $\alpha \in \mathcal{P}_{mn}$ i $\beta \in \mathcal{P}_{kl}$, proizvod $\alpha\beta$ će biti definisan ako i samo ako je $n = k$; u tom slučaju, koristimo dva proizvoljna dijagrama koji predstavljaju α i β redom i definišemo *produkt-dijagram* $\Pi(\alpha, \beta)$ na sledeći način:

- modifikujemo dijagram koji predstavlja $\alpha \in \mathcal{P}_{mn}$ i kreiramo graf $\alpha_{\downarrow}$, tako što svaki (donji) čvor $x' \in [n]'$ preimenujemo u x'';
- modifikujemo dijagram koji predstavlja $\beta \in \mathcal{P}_{nl}$, i kreiramo graf $\beta^{\uparrow}$, tako što svaki (gornji) čvor $x \in [n]$ preimenujemo u x'';

- identifikujemo čvorove iz skupa $[n]''$ u $\alpha_\downarrow$ sa istoimenim čvorovima skupa $[n]''$ u $\beta^\uparrow$, i dobijamo graf $\Pi(\alpha,\beta)$.

Koristeći taj dijagram $\Pi(\alpha,\beta)$, određujemo particiju koja odgovara proizvodu $\alpha\cdot\beta = \alpha\beta$ na skupu čvorova $[m]\cup[k]'$, tako da

$$(r,s)\in\alpha\beta \Leftrightarrow r \text{ i } s \text{ pripadaju istoj komponenti u } \Pi(\alpha,\beta),$$

za $r,s\in[m]\cup[k]'$. Osim toga, definišemo i unarnu operaciju $^* : \mathcal{P}\to\mathcal{P}$ koja "obrće"particiju po x-osi (samo posmatramo dijagram "u ogledalu").

Najzad, definišemo preslikavanja $\boldsymbol{\delta},\boldsymbol{\rho} : \mathrm{P}\to\mathbb{N}_0$ tako da za sve prirodne brojeve $m,n\in\mathbb{N}_0$ i svaku particiju $\alpha\in\mathcal{P}_{mn}$ važi $\alpha\boldsymbol{\delta} = m$ i $\alpha\boldsymbol{\rho} = n$. Tada je $(\mathcal{P},\cdot,\mathbb{N}_0,\boldsymbol{\delta},\boldsymbol{\rho})$ parcijalna polugrupa, a $(\mathcal{P},\cdot,\mathbb{N}_0,\boldsymbol{\delta},\boldsymbol{\rho},^*)$ je regularna monoidalna parcijalna $*$-polugrupa.

Dalje, definišimo

$$\begin{aligned}\mathcal{B} &= \{\alpha\in\mathcal{P} : \text{ svaki blok u } \alpha \text{ ima tačno dva elementa}\}, \quad \text{i}\\ \mathcal{PB} &= \{\alpha\in\mathcal{P} : \text{ svaki blok u } \alpha \text{ ima najviše dva elementa}\}.\end{aligned}$$

$\mathcal{B}$ and $\mathcal{PB}$ su potkategorije u $\mathcal{P}$ (Brauerova i parcijalna Brauerova kategorija, redom), a

$$(\mathcal{B},\ \cdot\restriction_{\mathcal{B}\times\mathcal{B}},\ \boldsymbol{\delta}\restriction_{\mathcal{B}},\ \boldsymbol{\rho}\restriction_{\mathcal{B}},\ ^*\restriction_{\mathcal{B}}) \quad \text{i} \quad (\mathcal{PB},\ \cdot\restriction_{\mathcal{PB}\times\mathcal{PB}},\ \boldsymbol{\delta}\restriction_{\mathcal{PB}},\ \boldsymbol{\rho}\restriction_{\mathcal{PB}},\ ^*\restriction_{\mathcal{PB}})$$

su obe regularne monoidalne parcijalne $*$-polugrupe.

Sledeći skup particija koje posmatramo je

$$\mathscr{P}\mathcal{P} = \{\alpha\in\mathcal{P} : \alpha \text{ može da bude prikazan planarnim dijagramom}\},$$

(poštujući gore navedena pravila za dijagrame). Kao u teoriji grafova, dijagram je planaran ako mu se nikoji par grana ne seče. Tada je podstruktura

$$(\mathscr{P}\mathcal{P},\ \cdot\restriction_{\mathscr{P}\mathcal{P}\times\mathscr{P}\mathcal{P}},\ \boldsymbol{\delta}\restriction_{\mathscr{P}\mathcal{P}},\ \boldsymbol{\rho}\restriction_{\mathscr{P}\mathcal{P}},\ ^*\restriction_{\mathscr{P}\mathcal{P}})$$

regularna monoidalna parcijalna $*$-polugrupa. Štaviše, ako je $\mathcal{TL} = \mathcal{B}\cap\mathscr{P}\mathcal{P}$ i $\mathscr{M} = \mathcal{PB}\cap\mathscr{P}\mathcal{P}$, onda su

$$(\mathcal{TL},\ \cdot\restriction_{\mathcal{TL}\times\mathcal{TL}},\ \boldsymbol{\delta}\restriction_{\mathcal{TL}},\ \boldsymbol{\rho}\restriction_{\mathcal{TL}},\ ^*\restriction_{\mathcal{TL}}) \ \text{ i } \ (\mathscr{M},\ \cdot\restriction_{\mathscr{M}\times\mathscr{M}},\ \boldsymbol{\delta}\restriction_{\mathscr{M}},\ \boldsymbol{\rho}\restriction_{\mathscr{M}},\ ^*\restriction_{\mathscr{M}}),$$

regularne monoidalne parcijalne $*$-polugrupe, takođe. U pitanju su Temperli-Lib kategorija i Mockinova kategorija, redom.

U petoj glavi smo istraživali osobine $\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{P}\mathcal{P}$, $\mathcal{TL}$ i $\mathscr{M}$, kao i sendvič polugrupa koje one sadrže. Izloženi rezultati su objavljeni u [28]. Naš prvi cilj je karakterizacija Grinovih relacija u tim parcijalnim polugrupama. U tu svrhu uvodimo dodatnu notaciju. Za $\alpha\in\mathcal{P}_{mn}$, neka ε_α označava odgovarajuću relaciju ekvivalencije i definišimo

$$\begin{aligned}
\operatorname{dom}(\alpha) &= \{x \in [m] : x \text{ pripada transverzali u } \alpha\},\\
\operatorname{codom}(\alpha) &= \{x \in [n] : x' \text{ pripada transverzali u } \alpha\},\\
\ker(\alpha) &= \{(x,y) \in [m]\times[m] : (x,y) \in \varepsilon_\alpha\},\\
\operatorname{coker}(\alpha) &= \{(x,y) \in [n]\times[n] : (x',y') \in \varepsilon_\alpha\},\\
\mathrm{N_U}(\alpha) &= \{X \in \alpha : X \text{ je gornja netransverzala u } \alpha\},\\
\mathrm{N_L}(\alpha) &= \{X \in \alpha : X' \text{ je donja netransverzala u } \alpha\},
\end{aligned}$$

su *domen*, *kodomen*, *jezgro*, *kojezgro*,i skupovi gornjih i donjih netransverzala u α, redom.

Propozicija 5.1.2. Neka $\mathcal{K}$ označava bilo koju od kategorija $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{M}$ i $\mathcal{TL}$. Ako je $\alpha,\beta \in \mathcal{K}$, onda u kategoriji $\mathcal{K}$ imamo

(i) $\alpha \leq_{\mathscr{R}} \beta \;\Leftrightarrow\; \alpha\,\boldsymbol{\delta} = \beta\,\boldsymbol{\delta},\ \ker(\alpha) \supseteq \ker(\beta),\ \text{i}\ \mathrm{N_U}(\alpha) \supseteq \mathrm{N_U}(\beta)$;

(ii) $\alpha \leq_{\mathscr{L}} \beta \;\Leftrightarrow\; \alpha\,\boldsymbol{\rho} = \beta\,\boldsymbol{\rho},\ \operatorname{coker}(\alpha) \supseteq \operatorname{coker}(\beta),\ \text{i}\ \mathrm{N_L}(\alpha) \supseteq \mathrm{N_L}(\beta)$;

(iii) $\alpha \leq_{\mathscr{J}} \beta \;\Leftrightarrow\; \begin{cases} \operatorname{Rank}\alpha \leq \operatorname{Rank}\beta, & \text{ako važi } (a),\\ \operatorname{Rank}\alpha \leq \operatorname{Rank}\beta \text{ i } \operatorname{Rank}\alpha \equiv \operatorname{Rank}\beta \pmod 2, & \text{ako važi } (b), \end{cases}$

gde su (a) i (b) slučajevi $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathscr{M}, \mathcal{PB}\}$ i $\mathcal{K} \in \{\mathcal{B}, \mathcal{TL}\}$, redom;

(iv) $\alpha\,\mathscr{R}\,\beta \;\Leftrightarrow\; \ker(\alpha) = \ker(\beta) \text{ i } \mathrm{N_U}(\alpha) = \mathrm{N_U}(\beta)$
$\;\Leftrightarrow\; \operatorname{dom}(\alpha) = \operatorname{dom}(\beta) \text{ i } \ker(\alpha) = \ker(\beta)$;

(v) $\alpha\,\mathscr{L}\,\beta \;\Leftrightarrow\; \operatorname{coker}(\alpha) = \operatorname{coker}(\beta) \text{ i } \mathrm{N_L}(\alpha) = \mathrm{N_L}(\beta)$
$\;\Leftrightarrow\; \operatorname{codom}(\alpha) = \operatorname{codom}(\beta) \text{ i } \operatorname{coker}(\alpha) = \operatorname{coker}(\beta)$;

(vi) $\alpha\,\mathscr{J}\,\beta \;\Leftrightarrow\; \operatorname{Rank}\alpha = \operatorname{Rank}\beta$.

Uz to, kategorije $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{B}$, $\mathcal{PB}$, $\mathscr{M}$ i $\mathcal{TL}$ su sve stabilne, pa u svakoj od tih kategorija važi $\mathscr{J} = \mathscr{D}$.

Nakon toga, za $m,n \in \mathbb{N}_0$ računamo kombinatornu strukturu hom-seta $\mathcal{K}_{mn}$ za sve $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathcal{B}, \mathcal{PB}, \mathscr{M}, \mathcal{TL}\}$ (broj preseka sa $\mathscr{R}$-, $\mathscr{L}$- i $\mathscr{H}$-klasama, veličinu $\mathscr{D}$-klasa i čitavog hom-seta). Dalje, u Propoziciji 5.1.6 pokazujemo da se nijedna od navedenih šest parcijalnih polugrupa ne može proširiti do inverzne parcijalne polugrupe i u Propoziciji 5.1.7 dajemo karakterizacije desno- i levo-invertibilnih elemenata za svaku od navedenih parcijalnih polugrupa.

U nastavku ispitujemo sendvič polugrupe u navedenim kategorijama. U tu svrhu, neka su $m,n \in \mathbb{N}_0$, neka je $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathcal{B}, \mathcal{PB}, \mathscr{M}, \mathcal{TL}\}$ i fiksirajmo $\sigma \in \mathcal{K}_{nm}$ (ako je $\mathcal{K} \in \{\mathcal{B}, \mathcal{TL}\}$, pretpostavljamo $m \equiv n (\operatorname{mod} 2)$). Prvo, koristeći opštu teoriju, opisujemo Grinove relacije sendvič polugrupe $\mathcal{K}^\sigma_{mn}$. Dalje, izvodimo opis regularnih $\mathscr{J}^\sigma = \mathscr{D}^\sigma$-klasa, i dajemo tvrđenje o maksimalnim klasama među njima:

Propozicija 5.2.6. Neka je $\mathcal{K}$ bilo koja od $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{B}$ ili $\mathcal{TL}$. Dalje, neka je $m \equiv n \pmod 2$, ako je $\mathcal{K} = \mathcal{B}$ ili $\mathcal{K} = \mathcal{TL}$. Uz to, označimo $\mathrm{Pre}(\sigma) = \{\alpha \in \mathcal{K}_{mn} : \sigma\alpha\sigma = \sigma\}$.

(i) Ako je $r < \min(m, n)$, onda su trivijalne maksimalne $\mathscr{J}^\sigma$-klase u $\mathcal{K}^\sigma_{mn}$ singltoni $\{\alpha\}$ za $\alpha \in \mathcal{K}_{mn}$ sa $\mathrm{Rank}(\alpha) > r$. Ako je $\mathcal{K}$ jedna od $\mathcal{P}$, $\mathcal{PB}$ ili $\mathcal{B}$, onda $\mathcal{K}^\sigma_{mn}$ nema netrivijalnih maksimalnih $\mathscr{J}^\sigma$-klasa. Ako je $\mathcal{K}$ jedna od $\mathscr{P}\mathcal{P}$, $\mathscr{M}$ ili $\mathcal{TL}$, sledeći uslovi su ekvivalentni:

  (a) $\mathcal{K}^\sigma_{mn}$ ima netrivijalnu maksimalnu $\mathscr{J}^\sigma$-klasu,
  (b) $\mathrm{Pre}(\sigma) \subseteq \mathrm{D}_r(\mathcal{K}_{mn})$,
  (c) $\mathrm{Pre}(\sigma) = \mathrm{V}(\sigma)$,

  u kom slučaju je netrivijalna maksimalna $\mathscr{J}^\sigma$-klasa skup $\mathrm{D}^\sigma_r = \{\alpha \in \mathrm{P}^\sigma : \mathrm{Rank}(\alpha) = r\}$.

(ii) Ako je $r = \min(m, n)$, onda je skup $\mathrm{D}^\sigma_r = \{\alpha \in \mathrm{P}^\sigma : \mathrm{Rank}(\alpha) = r\}$ maksimalna $\mathscr{J}^\sigma$-klasa u $\mathcal{K}^\sigma_{mn}$. Ta maksimalna $\mathscr{J}^\sigma$-klasa je netrivijalna.

Kroz isti proces kao u prethodnim glavama, analiziramo dijagrame 2.2 i 2.3. Neka je $\sigma = \begin{pmatrix} X_1 & \cdots & X_r & U_1 & \cdots & U_s \\ Y_1 & \cdots & Y_r & V_1 & \cdots & V_t \end{pmatrix}$. Kao paralelu preslikavanju η iz Glava 3 i 4, definišemo funkciju $\sigma\sigma^* \mathcal{K}_n \sigma\sigma^* \to \mathcal{K}_r : \alpha \mapsto \alpha^\natural$, gde je $r = \operatorname{rank}\sigma$, na sledeći način: za svaki blok u α oblika $B = \bigcup_{i\in I} X_i \cup \bigcup_{j\in J} X'_j$, u $\alpha^\natural$ uvrštavamo blok $I \cup J'$. Naša analiza je rezultovala dijagramima sa Slike 5.7.

U nastavku ispitujemo regularnu potpolugrupu $\mathrm{P}^\sigma = \mathrm{Reg}(\mathcal{K}^\sigma_{mn})$ naše sendvič polugrupe. No, ispostavlja se da, u slučaju kada je $\mathcal{K}$ jedna od kategorija $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$ i $\mathcal{TL}$, nismo u mogućnosti da na osnovu prethodnih rezultata izračunamo kombinatorne aspekte odnosa P^σ i $\mathcal{K}_r$ (preciznije, odnose $\widehat{\mathscr{K}^a}$-klasa u P^σ i $\mathscr{K}$ klasa u $\mathcal{K}_r$). Uz to, ispostavlja se da u tim regularnim potpolugrupama nemamo MI-dominaciju. Stoga, pokazujemo naredni rezultat koji preciznije opisuje maksimalnu regularnu $\mathscr{D}^\sigma$-klasu, ispitujemo idempotente i opisujemo idempotentno-generisanu potpolugrupu, i time završavamo diskusiju o opštem slučaju.

Propozicija 5.2.11. Pretpostavimo da je $m \geq n = r$.

(i) Ako je $\mathcal{K}$ jedna od $\mathcal{P}$, $\mathcal{PB}$ ili $\mathcal{B}$, klasa D^σ_r je leva grupa nad S_r (tj. direktan proizvod te grupe i neke polugrupe levih nula).

(ii) Ako je $\mathcal{K}$ jedna od $\mathscr{P}\mathcal{P}$, $\mathcal{B}$ ili $\mathcal{TL}$, klasa D^σ_r je polugrupa levih nula.

Sendvič polugrupe u $\mathcal{B}$

Od svih navedenih kategorija particija, najpogodnija za ispitivanje nam je kategorija $\mathcal{B}$, pošto njene sendvič polugrupe imaju posebne osobine, od kojih je najbitnija MI-dominirana regularna potpolugrupa. Stoga, ovde posmatramo sendvič polugrupu $\mathcal{B}^\sigma_{mn}$.

Prvo, izvodimo klasifikaciju sendvič polugrupa u $\mathcal{B}$ do na izomorfizam. Zatim, nakon opširne kombinatorne analize, izvodimo kombinatorni opis P-skupova u $\mathcal{B}^\sigma_{mn}$:

Propozicija 5.3.8. U polugrupi $\mathcal{B}_{mn}^{\sigma}$, važi

(i) $\mathrm{P}_1^{\sigma} = \{\alpha \in \mathcal{B}_{mn} : \mathrm{coker}(\alpha) \vee \ker(\sigma) \text{ razdvaja } \mathrm{codom}(\alpha)\}$,

(ii) $\mathrm{P}_2^{\sigma} = \{\alpha \in \mathcal{B}_{mn} : \ker(\alpha) \vee \mathrm{coker}(\sigma) \text{ razdvaja } \mathrm{dom}(\alpha)\}$,

(iii) $\mathrm{P}^{\sigma} = \mathrm{P}_3^{\sigma} = \{\alpha \in \mathcal{B}_{mn} : \mathrm{coker}(\alpha) \vee \ker(\sigma) \text{ razdvaja } \mathrm{codom}(\alpha) \text{ i } \ker(\alpha) \vee \mathrm{coker}(\sigma) \text{ razdvaja } \mathrm{dom}(\alpha)\}$.

(Ovde, relacija ekvivalencije razdvaja elemente skupa ako nikoja dva elementa nisu u istoj klasi ekvivalencije.)

Zahvaljujući ovom rezultatu, uspevamo da izračunamo odnose $\widehat{\mathscr{K}^a}$-klasa u P^{σ} i $\mathscr{K}$ klasa u $\mathcal{B}_r$ i da izvršimo enumeraciju idempotenata. Dalje, dokazujemo Lemu 5.3.16, koja je ključna za dokazivanje narednog tvrđenja:

Propozicija 5.3.17. Polugrupa $\mathrm{P}^{\sigma} = \mathrm{Reg}(\mathcal{B}_{mn}^{\sigma})$ je MI-dominirana.

Pored toga, pokazujemo da je ona i RP-dominirana, a da MI-dominacija ne važi u regularnim delovima sendvič polugrupa u $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$ i $\mathcal{TL}$.

Ovi rezultati omogućavaju izračunavanja ranga regularne potpolugrupe P^{σ} i idempotentno-generisane potpolugrupe $\mathbb{E}(\mathcal{B}_{mn}^{\sigma})$. Najzad, značajno jednostavnija struktura particija u $\mathcal{B}$ nam omogućava da izračunamo i rang čitave sendvič-polugrupe.

Na kraju svake glave dajemo egg-box dijagrame (koji su uobičajena tehnika za prikazivanje polugrupa) raznih sendvič polugrupa iz date kategorije, kao ilustraciju dobijenih rezultata.

Zaključak

Ova teza se bavi sendvič polugrupama u lokalno malim kategorijama. U procesu istraživanja smo uveli pojmove parcijalne polugrupe, sendvič-regularnosti i MI-dominacije. To nam je omogućilo da ispitujemo strukturne i kombinatorne osobine ovih polugrupa i da dokažemo rezultate pod različitim pretpostavkama, kao što su desna invertibilnost, (sendvič-)regularnost, stabilnost ili postojanje desne jedinice za određene elemente. Dobijeni rezultati daju osnovu za istraživanje konkretne sendvič polugrupe i (pod određenim uslovima) njene regularne potpolugrupe i idempotentno-generisane potpolugrupe. U glavama 3-5 primenjujemo te rezultate da detaljno ispitamo sendvič polugrupe u $\mathcal{PT}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{M}(\mathbb{F})$, $\mathcal{P}$, $\mathscr{P}\mathcal{P}$, $\mathcal{PB}$, $\mathscr{M}$, $\mathcal{TL}$ i $\mathcal{B}$. Pri tome, pokazujemo dodatne rezultate gde je to moguće.

Naravno, u našoj temi ostaje prostora za dalja istraživanja.

- Bilo bi interesantno "prevesti" dodatne pojmove iz teorije kategorija u teoriju parcijalnih polugrupa. To ukrštanje "jezika" i tehnika bi potencijalno dovelo do napretka u obe oblasti.
- Pošto je naše znanje o sendvič polugrupama u opštem slučaju napredovalo svaki put kada smo posmatrali novu "familiju" kategorija, smisleno je očekivati da bi novo ispitivanje takve prirode takođe donelo napredak.
- Naravno, i u kategorijama koje smo ovde istraživali ostaje prostora za dalje istraživanje.

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Index

Index of Notation

General

$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	the set of natural numbers with zero
$\min X$	the minimum of X
$\max X$	the maximum of X
$\bmod n$	congruence modulo n
$f{\restriction_X}$	the restriction of f on X
$X \setminus Y$	the set difference of X and Y
x^{-1}	the group/semigroup inverse element of x in a group/inverse semigroup
S_X	the symmetric group on X, page 4
$X \times Y$	the direct product of X and Y, page 5
$[x]_\sigma$	the equivalence class of σ containing x, page 5
Δ_X	the diagonal relation on X, page 6
xf	the map of x under f, page 6
Xf	the direct image of X under f, page 6
Xf^{-1}	the inverse image of X under f, page 6
$\mathrm{End}\, G$	the set of all endomorphisms on G, page 7
$\ker f$	the kernel of f, page 6
$G \cong H$	G is isomorphic to H, page 7
$\mathrm{Aut}\, G$	the set of all automorphisms of G, page 7
$G/\ker f$	the quotient (factor) algebra of G with respect to f, page 7
$V(G)$	the vertex set of G, page 201
$E(G)$	the edge set of G, page 201

Semigroups

S^1	the semigroup S with adjoined identity, page 8
XY	the set consisting of all elements of the form xy, where $x \in X$ and $y \in Y$; if either set is a singleton, we omit the braces, page 8
$\mathrm{R}_x, \mathrm{L}_x, \mathrm{H}_x, \mathrm{D}_x, \mathrm{J}_x$	the $\mathscr{K}$-class containing x, page 10

$\mathrm{E}(S)$	the set of idempotents of S, page 11
$\mathrm{Pre}(a)$	the set of pre-inverses of a, page 13
$\mathrm{Post}(a)$	the set of post-inverses of a, page 13
$\mathrm{V}(a)$	the set of semigroup inverses of a, page 13
$\mathrm{Reg}(S)$	the set of regular elements of S, page 13
$\preceq$	the natural partial order in a regular semigroup, page 13
$\langle X \rangle$	the subsemigroup generated by the elements from X, page 17
$\mathrm{rank}(S)$	the rank of the semigroup S, page 17
$\mathrm{idrank}(S)$	the idempotent rank of the semigroup S, page 17
$\mathbb{E}(S)$	the idempotent-generated subsemigroup of S, page 17
$\mathrm{rank}(S : A)$	the relative rank of a semigroup S with respect to the subset $A \subseteq S$, page 17
$\mathrm{idrank}(S : A)$	the relative idempotent rank of a semigroup S with respect to the subset $A \subseteq S$, page 17

Sandwich semigroups

$a\,\boldsymbol{\delta}$	the domain of the morphism a, page 7
$a\,\boldsymbol{\rho}$	the range of the morphism a, page 7
$\star_a$	the sandwich operation corresponding the element a, page 20
S_{ij}	the hom-set of S consisting of all morphisms $i \to j$, page 21
e_i	the identity corresponding the object i in a partial semigroup, page 21
S_{ij}^a	the sandwich semigroup of S_{ij} with respect to a, page 22
S_i	the hom-set S_{ii}, page 22
x^*	the semigroup inverse of x in a (partial) $*$-semigroup or in a regular (partial) $*$-semigroup, page 23
$S^{(1)}$	the monoidal partial semigroup obtained from the partial semigroup S, page 23
$\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$, $\leq_{\mathscr{J}}$	Green's preorders of a (partial) semigroup, page 24
$\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, $\mathscr{J}$	Green's relations of a (partial) semigroup, page 24
$\mathscr{K}^a$	(for $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$) Green's relation of a sandwich semigroup, page 25
K_x^a	(for $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}, \mathrm{J}\}$) the $\mathscr{K}^a$-class of a sandwich semigroup containing x, page 25
K_x	(for $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}, \mathrm{J}\}$) the $\mathscr{K}$-class of a hom-set containing x, page 25
$\mathrm{P}_1^a, \mathrm{P}_2^a, \mathrm{P}_3^a, \mathrm{P}^a$	P-sets of a sandwich semigroup, page 27
$\mathrm{RI}(x)$	the set of right-inverses of x, page 43
$\mathrm{E}_a(U)$	the set of idempotents in U with respect to the sandwich multiplication $\star_a$, page 44
$\circledast$	the operation $\star_b\restriction_{aS_{ij}a}$ where $b \in \mathrm{V}(a)$, page 49

Ψ_1, ψ_1	the surmorphism $(S_{ij}, \star_a) \to (S_{ij}a, \cdot) : x \mapsto xa$ and its restriction to P^a, respectively, page 49
Ψ_2, ψ_2	the surmorphism $(S_{ij}, \star_a) \to (aS_{ij}, \cdot) : x \mapsto ax$ and its restriction to P^a, respectively, page 49
Φ_1, ϕ_1	the surmorphism $(S_{ij}a, \cdot) \to (aS_{ij}a, \circledast) : y \mapsto ay$ and its restriction to T_1, respectively, page 49
Φ_2, ϕ_2	the surmorphism $(aS_{ij}, \cdot) \to (aS_{ij}a, \circledast) : y \mapsto ya$ and its restriction to T_2, respectively, page 49
T_1	the semigroup $\mathrm{Reg}(S_{ij}a, \cdot)$, page 52
T_2	the semigroup $\mathrm{Reg}(aS_{ij}, \cdot)$, page 52
W	the semigroup $(aS_{ij}a, \circledast)$, page 52
$\mathscr{K}^{\circledast}$	(for $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$) Green's relations of the semigroup W, page 53
$\mathrm{K}_x^{\circledast}$	(for $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}, \mathrm{J}\}$) the $\mathscr{K}^{\circledast}$-class of W containing x, page 53
ϕ	the homomorphism $\mathrm{P}^a \to W : x \mapsto axa$, page 54
ψ	the homomorphism $\mathrm{P}^a \to T_1 \times T_2 : x \mapsto (xa, ax)$, page 54
$\overline{x}$	the map $x\phi$ of $x \in \mathrm{P}^a$, page 55
$\overline{X}$	the set $\{\overline{x} : x \in X\}$, page 55
$\widehat{\mathscr{K}^a}$	(for $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$) the ϕ-preimage in P^a of the relation $\mathscr{K}$ in W , page 55
$\widehat{\mathrm{K}_x^a}$	(for $\mathrm{K} \in \{\mathrm{R}, \mathrm{L}, \mathrm{H}, \mathrm{D}, \mathrm{J}\}$) the ϕ-preimage of the class $\mathrm{K}_{\overline{x}}^{\circledast}$, page 55
$\mathrm{V}_a(x)$	the set of semigroup inverses of x with respect to the sandwich multiplication $\star_a$, page 56
$\langle X \rangle_a$	the $\star_a$-subsemigroup generated by X, page 61
$\mathbb{E}_a(U)$	the idempotent-generated subsemigroup of U with respect to the sandwich multiplication $\star_a$, page 62
$\mathrm{MI}(T)$	the set of mid-identities of T, page 63
$\mathrm{RP}(T)$	the set of regularity-preserving elements of T, page 63
$\mathrm{Max}_{\preceq}(T)$	the set of all $\preceq$-maximal idempotents of T, page 64

Transformations

$\mathrm{dom}\, f$	the domain of a map f, page 6
$\mathrm{im}\, f$	the image of a map f, page 6
$\mathrm{Rank}\, f$	the rank of a map f, page 6
$\begin{pmatrix} F_i \\ f_i \end{pmatrix}_{i \in I}$	the description of a map where the elements of F_i map to f_i for each $i \in I$, page 6
$\mathcal{T}_X$	the full transformation semigroup over X, page 19
$\mathcal{PT}_X$	the class of all partial transformations over X, page 19
$\mathcal{I}_X$	the symmetric inverse semigroup over X, page 19
Set	the class of all sets, page 82

$\mathbf{Set}^+$	the class of all nonempty sets, page 82
$\mathbf{T}_{AB}$, $\mathcal{T}_{AB}$	the set of all maps $A \to B$, page 82
$\mathbf{PT}_{AB}$, $\mathcal{PT}_{AB}$	the set of all partial maps $A \to B$, page 82
$\mathbf{I}_{AB}$, $\mathcal{I}_{AB}$	the set of all injective (partial) maps $A \to B$, page 82
$\mathcal{PT}$	the class of all partial transformations, page 82
$\mathcal{T}$	the class of all full transformations, page 82
$\mathcal{I}$	the class of all injective partial transformations, page 82
id_X	the identity map on X, page 83
D_μ^{AB}	the $\mathscr{J} = \mathscr{D}$-class of $\mathcal{PT}_{AB}$ consisting of maps of rank μ, page 87
$\mathcal{S}(\mu, \kappa)$	Stirling number of the second kind/the number of ways to partition a κ-element set into μ blocks , page 87
$\kappa!$	factorial of κ/the size of the symmetric group over a set of size κ, page 87
$\binom{\kappa}{\mu}$	binomial coefficient/the number of μ-element subsets of a κ-element set, page 87
S^{fr}	the set of all finite-rank elements of S, page 89
η	the map $(a\,\mathcal{PT}_{XY}\,a, \circledast) \to (ba\,\mathcal{PT}_{XY}\,a, \cdot) : x \mapsto bx$, page 97
$\mathcal{PT}(X, A)$	the set of all partial transformations on X with image restricted by A, page 97
$\mathcal{PT}(X, \sigma)$	the set of all partial transformations on X with kernel restricted by σ, page 98
$\mathrm{u}(\theta)$	the underlying set of an equivalence relation θ, page 99
π_θ	the partition corresponding to an equivalence relation θ, page 99
$\mathrm{sh}\, f$	the shift of a map f, page 113
$\mathrm{def}\, f$	the defect of a map f, page 113
$\mathrm{coll}\, f$	the collapse of a map f, page 113
$\mathrm{codef}\, f$	the codefect of a map f, page 113
$\mathcal{T}(X, A)$	the set of all full transformations on X with image restricted by A, page 137
$\mathcal{T}(X, \sigma)$	the set of all full transformations on X with kernel restricted by σ, page 137
$\mathcal{I}(X, A)$	the set of all partial injective transformations on X with image restricted by A, page 155
$\mathcal{I}(X, A)^*$	the set $\{f^{-1} : f \in \mathcal{I}(X, A)\}$, page 155

Linear algebra

$\mathcal{M}(\mathbb{F})$	the set of all finite-dimensional matrices over $\mathbb{F}$, page 162
$\mathrm{Hom}(V, W)$	the hom-set of linear transformations $V \to W$, page 162
V_m	the vector space of $1 \times m$ row vectors over the field $\mathbb{F}$, page 162
λ_X	the linear transformation corresponding the matrix X, page 162

I_m	the $m \times m$ identity matrix, page 162
$\{e_{m1}, \ldots, e_{mm}\}$	the standard basis of V_m, page 162
W_{ms}	the (vector) subspace of V_m consisting of all linear combinations of vectors $e_{m1}, \ldots, e_{ms}$, page 162
$\operatorname{span} X$	the set of linear combinations of the elements of X, page 162
$\mathbf{0}$	the zero vector, page 162
$\mathcal{G}_m(\mathbb{F})$	the set of all invertible matrices over $\mathbb{F}$, page 162
$\mathfrak{r}_i(X)$	the ith row of a matrix X, page 163
$\mathfrak{c}_i(X)$	the ith column of a matrix X, page 163
$\operatorname{Row} X$	the row space of a matrix X, page 163
$\operatorname{Col} X$	the column space of a matrix X, page 163
$\operatorname{Rank} X$	the rank of a matrix X, page 163
$\operatorname{Ker} \alpha$	the kernel of a linear transformation α, page 163
A^{T}	the transposition of a matrix X, page 163
$[s]_q!$	q-factorial, page 166
$\begin{bmatrix} m \\ s \end{bmatrix}_q$	q-binomial coefficient, page 166
J_{mns}	the $m \times n$ matrix containing I_s in its upper left corner, and 0's elsewhere, page 166
O_{kl}	the $k \times l$ zero matrix, page 166
$\mathfrak{s}_s^m(q)$	the number of s-dimensional subspaces of an m-dimensional vector space over a q-dimensional field, page 167
$[M, A, N]$	the matrix $\begin{bmatrix} A & AN \\ MA & MAN \end{bmatrix}$, page 172
$\mathbb{F}^{\times}$	the multiplicative group of a field $\mathbb{F}$, page 185

Partitions

$[n]$	the set $\{1, 2, \ldots, n\}$, page 198
$\mathcal{P}$	the set of all set partitions, page 198
ε_σ	the equivalence corresponding the partition σ, page 198
A'	the set $\{a' : a \in A\}$, page 198
A''	the set $\{a'' : a \in A\}$, page 198
$\Pi(\alpha, \beta)$	the product diagram of α and β, page 199
ι_m	the identity partition corresponding id_m, page 200
$\mathcal{B}$	the set of all Brauer partitions, page 200
$\mathcal{PB}$	the set of all partial Brauer partitions, page 200
$\mathscr{P}\mathcal{P}$	the set of all planar partitions, page 201
$\mathcal{TL}$	the set of all Temperley-Lieb partitions, page 203
$\mathscr{M}$	the set of all Motzkin partitions, page 203
$\mathrm{dom}(\alpha)$	the domain of a partition α, page 205
$\mathrm{codom}(\alpha)$	the codomain of a partition α, page 205

$\ker(\alpha)$	the kernel of a partition α, page 205
$\operatorname{coker}(\alpha)$	the cokernel of a partition α, page 205
$\mathrm{N_U}(\alpha)$	the set of upper nontransversals of α, page 205
$\mathrm{N_L}(\alpha)$	the set of lower nontransversals of α, page 205
$\mathrm{Rank}(\alpha)$	the number of transversals of a partition α, page 205
$\mathrm{B}(n)$	the nth Bell number; the number of partitions of an n-element set, page 207
$n!!$	the double factorial, page 207
$\mathrm{a}(n)$	the number of partitions of an n-element set into blocks of size at most 2, page 208
$\mathrm{C}(n)$	the nth Catalan number, page 208
$\mu(n,k)$	Motzkin triangle number, page 209
$\mu(n)$	the nth Motzkin number $\mu(n,0)$, page 209
$\mathrm{p}(n,k)$	the number of subdiagonal rectangular lattice paths from $(0,0)$ to $(\frac{n+k}{2},\frac{n-k}{2})$, page 209
$\kappa(m,q)$	the number of 1-2-equivalences with rank q on an m-element set, page 228
$\kappa(m,r,q)$	the number of certain 1-2-equivalences on an m-element set, see page 228

Short biography

Ivana Đurđev was born in Sombor, on March 6th, 1991. In 2013, she obtained a Bachelor's degree in Mathematics at the Faculty of Sciences of the University of Novi Sad. For her performance, she received the University Award for Academic Excellence. In 2015, she earned a Master's degree in Theoretical Mathematics at the same faculty. She received the award for the Best Master's Thesis in the Area of Mathematics for the same year. After that, she enrolled PhD studies of Mathematics at the same faculty. During her studies, Ivana participated in various international algebraic conferences and seminars. She is a co-author of three scientific articles.

Between 2015 and 2016, Ivana worked as a Graduate Student Instructor at the Faculty of Sciences of the University of Novi Sad. In 2017, she was employed as a Junior Researcher at the Mathematical Institute of the Serbian Academy of Sciences and Arts. She was promoted to the position of Research Assistant in January 2019.

Novi Sad, September 2nd, 2020.

Ivana Đurđev

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Abstract: Let S be a locally small category, and fix two (not necessarily distinct) objects i, j in S. Let S_{ij} and S_{ji} denote the set of all morphisms $i \to j$ and $j \to i$, respectively. Fix $a \in S_{ji}$ and define $(S_{ij}, \star_a)$, where $x \star_a y = xay$ for $x, y \in S_{ij}$. Then, $(S_{ij}, \star_a)$ is a semigroup, known as a *sandwich semigroup*, and denoted by S_{ij}^a. In this thesis, we conduct a thorough investigation of sandwich semigroups (in locally small categories) in general, and then apply these results to infer detailed descriptions of sandwich semigroups in a number of categories.

Firstly, we introduce the notion of a partial semigroup, and establish a framework for describing a category in "semigroup language". Then, we prove various results describing Green's relations and preorders, stability and regularity of S_{ij}^a. In particular, we emphasize the relationships between the properties of the sandwich semigroup and the properties of the category containing it. Also, we highlight the significance of the properties of the sandwich element a. In this process, we determine a natural condition on a called *sandwich regularity* which guarantees that the regular elements of S_{ij}^a form a subsemigroup tightly connected to certain non-sandwich semigroups. We explore these connections in detail and infer major structural results on $\mathrm{Reg}(S_{ij}^a)$ and the generation mechanisms in it. Finally, we investigate ranks and idempotent ranks of the regular subsemigroup $\mathrm{Reg}(S_{ij}^a)$ and idempotent-generated subsemigroup $\mathbb{E}(S_{ij}^a)$ of S_{ij}^a. In general, we are able to infer expressions for lower bounds for these values. However, we show that in the case when $\mathrm{Reg}(S_{ij}^a)$ is MI-dominated (a property which has to do with the "covering power" of certain local monoids), the mentioned lower bounds are sharp.

We apply the general theory to sandwich semigroups in various transformation categories (partial maps $\mathcal{PT}$, injective maps $\mathcal{I}$, totally defined maps $\mathcal{T}$, and matrices $\mathcal{M}(\mathbb{F})$ – corresponding to linear transformations of vector spaces over a field $\mathbb{F}$) and diagram categories (partition $\mathcal{P}$, planar partition $\mathscr{PP}$, Brauer $\mathcal{B}$, partial Brauer $\mathcal{PB}$, Motzkin $\mathscr{M}$, and Temperley-Lieb $\mathcal{TL}$ categories), one at a time. In each case, we investigate the partial semigroup itself in terms of Green's relations and regularity and then focus on a sandwich semigroup in it. We apply the general results to thoroughly describe its structural and combinatorial properties. Furthermore, since in each category that we consider all elements are sandwich-regular, we may apply the theory concerning the regular subsemigroup in all of these cases. In particular, $\mathrm{Reg}(S_{ij}^a)$ turns out to be tightly connected to a certain non-sandwich monoid for each category S we consider, and we are able to describe $\mathrm{Reg}(S_{ij}^a)$ and $\mathbb{E}(S_{ij}^a)$. However, we conduct the combinatorial part of the investigation only for the sandwich semigroups in transformation categories ($\mathcal{PT}$, $\mathcal{I}$, $\mathcal{T}$, and $\mathcal{M}(\mathbb{F})$) and sandwich semigroups in the Brauer category $\mathcal{B}$ since only these have MI-dominated regular subsemigroups (and some other properties that make them more amenable to investigation). For these sandwich semigroups, we enumerate regular Green's classes and idempotents, and we calculate the ranks (and idempotent ranks, where appropriate) of $\mathrm{Reg}(S_{ij}^a)$, $\mathbb{E}(S_{ij}^a)$ and S_{ij}^a.
AB

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Izvod: Neka je S lokalno mala kategorija. Fiksirajmo proizvoljne (ne nužno različite) objekte i i j iz S. Neka S_{ij} i S_{ji} označavaju skupove svih morfizama $i \to j$ i $j \to i$, redom. Fiksirajmo morfizam $a \in S_{ji}$ i definišimo strukturu $(S_{ij}, \star_a)$, gde je $x \star_a y = xay$ za sve $x, y \in S_{ij}$. Tada je $(S_{ij}, \star_a)$ *sendvič polugrupa*, koju označavamo sa S_{ij}^a. U tezi ćemo sprovesti detaljno ispitivanje sendvič polugrupa (u lokalno maloj kategoriji) u opštem slučaju, a zatim ćemo primeniti dobijene rezultate u cilju opisivanja sendvič polugrupa u konkretnim kategorijama.

Najpre uvodimo pojam parcijalne polugrupe i postavljamo osnovu koja nam omogućava da opišemo kategoriju na "jeziku polugrupa". Zatim slede brojni rezultati koji opisuju Grinove relacije i poretke, kao i stabilnost i regularnost polugrupe $(S_{ij}, \star_a)$. Tu posebno ističemo veze između osobina sendvič polugrupe i parcijalne polugrupe koja je sadrži. Takođe, posebnu pažnju posvećujemo uticaju sendvič elementa a na osobine sendvič polugrupe $(S_{ij}, \star_a)$. Kao najbitniji primer se izdvaja osobina *sendvič-regularnosti*; naime, dokazujemo da, ako je a sendvič-regularan, onda regularni elementi iz S_{ij}^a formiraju podgrupu koja je usko povezana sa određenim "ne-sendvič" polugrupama. U tezi detaljno ispitujemo te veze i dobijamo važne rezultate o strukturi polugrupe $\mathrm{Reg}(S_{ij}, \star_a)$ i mehanizmima generisanja u njoj. Za kraj, ispitujemo rangove i idempotentne rangove regularne potpolugrupe $\mathrm{Reg}(S_{ij}, \star_a)$ i idempotentno-generisane potpolugrupe $\mathbb{E}(S_{ij}, \star_a)$. U opštem slučaju možemo dati donja ograničenja za ove vrednosti. Međutim, u slučaju kada je regularna polugrupa $\mathrm{Reg}(S_{ij}, \star_a)$ MI-dominirana (što znači da je određeni lokalni monoidi pokrivaju), ta donja ograničenja su dostignuta.

U ostatku teze, primenjujemo opštu teoriju na sendvič polugrupe u brojnim kategorijama transformacija (parcijalne funkcije $\mathcal{PT}$, injektivne parcijalne funkcije $\mathcal{I}$, potpuno definisane funkcije $\mathcal{T}$ i matrice $\mathcal{M}(\mathbb{F})$, koje predstavljaju linearne transformacije vektorskih prostora nad poljem $\mathbb{F}$) i kategorijama dijagrama (particije $\mathcal{P}$, planarne particije $\mathscr{P}\mathcal{P}$, Brauerove $\mathcal{B}$, parcijalne Brauerove $\mathcal{PB}$, Mockinove $\mathscr{M}$, i Temperli-Lib $\mathcal{TL}$ particije). U svakom od ovih slučajeva, prvo istražujemo parcijalnu polugrupu iz aspekta Grinovih relacija i regularnosti, a zatim se fokusiramo na (proizvoljnu) sendvič polugrupu u njoj. Pri tome, primenjujemo opšte rezultate da bismo detaljno opisali njenu strukturu i kombinatorne osobine. Osim toga, u svim slučajevima primenjujemo i teoriju vezanu za regularnu potpolugrupu, pošto su svi elementi u našim kategorijama sendvič-regularni. To znači da je u svakoj kategoriji S koju razmatramo, $\mathrm{Reg}(S_{ij}, \star_a)$ usko povezana sa određenim monoidom, i preko te veze možemo opisati polugrupe $\mathrm{Reg}(S_{ij}, \star_a)$ i $\mathbb{E}(S_{ij}, \star_a)$. Ipak, kombinatorni deo ispitivanja sprovodimo samo za sendvič polugrupe u kategorijama transformacija ($\mathcal{PT}$, $\mathcal{I}$, $\mathcal{T}$ i $\mathcal{M}(\mathbb{F})$) i sendvič polugrupe u Brauerovoj kategoriji $\mathcal{B}$, pošto samo one imaju MI-dominirane regularne potpolugrupe (i još neke osobine koje ih čine pogodnijim za ispitivanje). U ovim

sendvič polugrupama računamo broj regularnih Grinovih klasa i idempotenata, i izračunavamo rangove (i idempotentne rangove, ako postoje) polugrupa $\mathrm{Reg}(S_{ij}, \star_a)$, $\mathbb{E}(S_{ij}, \star_a)$ i S_{ij}^a.
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Овај Образац чини саставни део докторске дисертације, односно докторског уметничког пројекта који се брани на Универзитету у Новом Саду. Попуњен Образац укоричити иза текста докторске дисертације, односно докторског уметничког пројекта.

План третмана података

Назив пројекта/истраживања
Сендвич полугрупе у локално малим категоријама
Назив институције/институција у оквиру којих се спроводи истраживање
а) Математички институт Српске академије наука и уметности б) Природно - математички факултет, Универзитет у Новом Саду в)
Назив програма у оквиру ког се реализује истраживање
-
1. Опис података
1.1 Врста студије *Укратко описати тип студије у оквиру које се подаци прикупљају* **Пошто је истраживање искључиво теоријског карактера, није вршено никакво прикупљање података. Из тог разлога се остатак обрасца не односи на њега, те је подразумевани одговор у свакој рубрици: није вршено прикупљање података.** 1.2 Врсте података а) квантитативни

б) квалитативни

1.3. Начин прикупљања података

а) анкете, упитници, тестови

б) клиничке процене, медицински записи, електронски здравствени записи

в) генотипови: навести врсту ___________________________________

г) административни подаци: навести врсту _________________________

д) узорци ткива: навести врсту____________________________________

ђ) снимци, фотографије: навести врсту_______________________________

е) текст, навести врсту ____

ж) мапа, навести врсту ____

з) остало: описати ____

1.3 Формат података, употребљене скале, количина података

1.3.1 Употребљени софтвер и формат датотеке:

a) Excel фајл, датотека ___________________

b) SPSS фајл, датотека ___________________

c) PDF фајл, датотека _____________________

d) Текст фајл, датотека ___________________

e) JPG фајл, датотека _____________________

f) Остало, датотека ______________________

1.3.2. Број записа (код квантитативних података)

а) број варијабли ____

б) број мерења (испитаника, процена, снимака и сл.) _______________________

1.3.3. Поновљена мерења

а) да

б) не

Уколико је одговор да, одговорити на следећа питања:

а) временски размак између поновљених мера је ________________________________

б) варијабле које се више пута мере односе се на __________________________________

в) нове верзије фајлова који садрже поновљена мерења су именоване као _____________

Напомене: ____

Да ли формати и софтвер омогућавају дељење и дугорочну валидност података?

а) Да

б) Не

Ако је одговор не, образложити ____

2. Прикупљање података

2.1 Методологија за прикупљање/генерисање података

2.1.1. У оквиру ког истраживачког нацрта су подаци прикупљени?

а) експеримент, навести тип ____

б) корелационо истраживање, навести тип ____

ц) анализа текста, навести тип ____

д) остало, навести шта ____

2.1.2 Навести врсте мерних инструмената или стандарде података специфичних за одређену научну дисциплину (ако постоје).

____

____

2.2 Квалитет података и стандарди

2.2.1. Третман недостајућих података

а) Да ли матрица садржи недостајуће податке? Да Не

Ако је одговор да, одговорити на следећа питања:

а) Колики је број недостајућих података? ___________________________

б) Да ли се кориснику матрице препоручује замена недостајућих података? Да Не

в) Ако је одговор да, навести сугестије за третман замене недостајућих података

____

2.2.2. На који начин је контролисан квалитет података? Описати

____

____

2.2.3. На који начин је извршена контрола уноса података у матрицу?

____

____

3. Третман података и пратећа документација

3.1. Третман и чување података

3.1.1. Подаци ће бити депоновани у ____________________________________ *репозиторијум.*

3.1.2. URL адреса ____

3.1.3. DOI ____

3.1.4. Да ли ће подаци бити у отвореном приступу?

а) Да

б) Да, али после ембарга који ће трајати до ____________________________________

в) Не

Ако је одговор не, навести разлог ____

3.1.5. Подаци неће бити депоновани у репозиторијум, али ће бити чувани.

Образложење

____

____

3.2 Метаподаци и документација података

3.2.1. Који стандард за метаподатке ће бити примењен? ____________________________________

3.2.1. Навести метаподатке на основу којих су подаци депоновани у репозиторијум.

____

____

Ако је потребно, навести методе које се користе за преузимање података, аналитичке и процедуралне информације, њихово кодирање, детаљне описе варијабли, записа итд.

____

____

____

____

3.3 Стратегија и стандарди за чување података

3.3.1. До ког периода ће подаци бити чувани у репозиторијуму? _________________________

3.3.2. Да ли ће подаци бити депоновани под шифром? Да Не

3.3.3. Да ли ће шифра бити доступна одређеном кругу истраживача? Да Не

3.3.4. Да ли се подаци морају уклонити из отвореног приступа после извесног времена?

Да Не

Образложити

____

____

4. Безбедност података и заштита поверљивих информација

Овај одељак МОРА бити попуњен ако ваши подаци укључују личне податке који се односе на учеснике у истраживању. За друга истраживања треба такође размотрити заштиту и сигурност података.

4.1 Формални стандарди за сигурност информација/података

Истраживачи који спроводе испитивања с људима морају да се придржавају Закона о заштити података о личности *(https://www.paragraf.rs/propisi/zakon_o_zastiti_podataka_o_licnosti.html)* и одговарајућег институционалног кодекса о академском интегритету.

4.1.2. Да ли је истраживање одобрено од стране етичке комисије? Да Не

Ако је одговор Да, навести датум и назив етичке комисије која је одобрила истраживање

____

4.1.2. Да ли подаци укључују личне податке учесника у истраживању? Да Не

Ако је одговор да, наведите на који начин сте осигурали поверљивост и сигурност информација везаних за испитанике:

а) Подаци нису у отвореном приступу

б) Подаци су анонимизирани

ц) Остало, навести шта

____

____

5. Доступност података

5.1. Подаци ће бити

а) јавно доступни

б) доступни само уском кругу истраживача у одређеној научној области *ц) затворени*

Ако су подаци доступни само уском кругу истраживача, навести под којим условима могу да их користе:

____

____

Ако су подаци доступни само уском кругу истраживача, навести на који начин могу приступити подацима:

____

____

5.4. Навести лиценцу под којом ће прикупљени подаци бити архивирани.

____

6. Улоге и одговорност

6.1. Навести име и презиме и мејл адресу власника (аутора) података

____

6.2. Навести име и презиме и мејл адресу особе која одржава матрицу с подацима

____

6.3. Навести име и презиме и мејл адресу особе која омогућује приступ подацима другим истраживачима

____