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On Integral Transforms and Convolution Equations on the Spaces of Tempered Ultradistributions

Ph D thesis
1992

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I wish to express my gratitude to my advisor prof. Stevan Pilipović, who supervise my study and scientific work. Prof. Pilipović thoroughly reviewed the manuscript of the thesis, making numerous criticism and suggestions. Advices of academician prof. Bogoljub Stanković were a valuable aid during the final revisions. Seminars which I attended during summer semester 1989/90 at the Institute of Mathematics and Geometry of University of Innsbruck, as well as interesting conversations with prof. Norbert Ortner and prof. Michael Oberguggenberger, encouraged me to start to study the theory of ultradistributions. A result of this work is this doctoral thesis.

May 30, 1992 *Dušanka Kovačević*

Contents

Chapter 1

Introduction

In the thesis we introduce and investigate the spaces which are natural generalizations of the space $\mathcal{S}'$, of Schwartz's tempered distributions [56], in Denjoy-Carleman-Komatsu's theory of ultradistributions. Our aim was to obtain spaces which are "lager" then $\mathcal{S}'$ and preserve all its good properties. Among others, a remarkable one, that the Fourier transform does not take us outside of that space, which allows us to employ in the theory of ultradistributions the most effective way of solving problems in mathematical physics, the Fourier transform method.

What does "a ultradistribution theory" means? We accepted the concept of Ciorănescu and Zsido ([14]). Let $\mathcal{G}$ be a family of parameters. Assume that every $\sigma \in \mathcal{G}$ is associated with a locally convex topological vector space $\mathcal{D}_\sigma$ of infinitely differentiable functions $\varphi : \mathbf{R} \to \mathbf{C}$, with a compact support such that,

1. $\mathcal{D}_\sigma$ is inductive limit of a sequence of Frécet spaces;
2. The topology of $\mathcal{D}_\sigma$ is stronger than the topology of pointwise convergence;
3. $\mathcal{D}_\sigma$ is algebra under pointwise multiplication;
4. For every compact set $K \subset \mathbf{R}$ and open set $D \subset K$, there exists $\varphi \in \mathcal{D}_\sigma$, such that $0 \leq \varphi \leq 1$, $\varphi(s) = 1$, for $s \in K$, and supp$\varphi \subset D$.

5. The vector space $\mathcal{E}_\sigma$, of all functions $\psi : \mathbf{R} \to \mathbf{C}$, such that $\varphi\psi \in \mathcal{D}_\sigma$, for each $\varphi \in \mathcal{D}_\sigma$, which is equipped with the projective limit topology defined by the linear mappings,

$$\mathcal{E}_\sigma \to \mathcal{D}_\sigma, \qquad \psi \mapsto \varphi\psi, \quad \varphi \in \mathcal{D}_\sigma,$$

has as a dense subspace, the linear space $\mathcal{A}$ of all complex functions on $\mathbf{R}$, which can be extended analytically on some complex neighborhood of $\mathbf{R}$.

If the above assumptions hold we say that $(\mathcal{D}_\sigma)_{\sigma\in\mathcal{G}}$ is a *theory of ultradistributions.*

Theories of ultradistributions can be compared. *The ultradistribution theory* $(\mathcal{D}_\tau)_{\tau\in\mathcal{T}}$ *is larger than* $(\mathcal{D}_\sigma)_{\sigma\in\mathcal{G}}$ if and only if for every $\sigma \in \mathcal{G}$ there exists $\tau \in \mathcal{T}$ such that one of the equivalent inclusions $\mathcal{D}_\tau \subset \mathcal{D}_\sigma$ and $\mathcal{E}_\tau \subset \mathcal{E}_\sigma$ hold, which imply that the inclusions are continuous and have dense ranges. So in that case $\mathcal{D}'_\sigma \subset \mathcal{D}'_\tau$ and $\mathcal{E}'_\sigma \subset \mathcal{E}'_\tau$, where the inclusions are continuous and have dense ranges. If $(\mathcal{D}_\tau)_{\tau\in\mathcal{T}}$ is larger than $(\mathcal{D}_\sigma)_{\sigma\in\mathcal{G}}$ and conversely, then we say that $(\mathcal{D}_\tau)_{\tau\in\mathcal{T}}$ and $(\mathcal{D}_\sigma)_{\sigma\in\mathcal{G}}$ are *equivalent ultradistribution theories.* Several theories of ultradistributions are developed.

Denjoy-Carleman-Komatsu's theory (see [53], [41], [12], [33], [34], [35], [36]): Let $\mathcal{M}$ denotes the set of all positive sequences $(M_p)_{p\in\mathbf{N}}$ which satisfy:

$(M.1)$ $\qquad M_p^2 \le M_{p-1}\, M_{p+1},\; p \in \mathbf{N},$

$(M.3)'$ $\qquad \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$

For each $(M_p)_{p\in\mathbf{N}}$ from $\mathcal{M}$,

$$\mathcal{D}^{\{M_p\}} = \{f \in \mathcal{D}(\mathbf{R}),\ \text{there exists } h > 0 \text{ such that } \sup_{\substack{p\in\mathbf{N}_0\\ x\in\mathbf{R}}} \frac{|\,\phi^{(\alpha)}(x)\,|}{h^\alpha M_\alpha} < \infty\},$$

$$\mathcal{D}^{(M_p)} = \{f \in \mathcal{D}(\mathbf{R}),\ \text{for each } h > 0 \text{ such that } \sup_{\substack{\alpha\in\mathbf{N}_0\\ x\in\mathbf{R}}} \frac{|\,\phi^{(\alpha)}(x)\,|}{h^\alpha M_\alpha} < \infty\}.$$

Beurling-Björck's theory (see [3], [6]) Let $\mathcal{A}$ denotes the set of all functions $\omega : [0,\infty) \to [0,\infty)$ which have the following properties:

$$\omega(0) = 0; \qquad \omega(t+s) \le \omega(t) + \omega(s), \quad \forall t, s > 0;$$

$$\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2} dt < \infty; \quad \log t = O(\omega(t))$$

For $\omega \in \mathcal{A}$,

$$\mathcal{D}_\omega = \left\{ f \in \mathcal{D}(\mathbf{R}), \int_{-\infty}^{\infty} |\hat{f}(t)| e^{\lambda\omega(t)} dt < \infty \text{ for all } \lambda > 0 \right\}.$$

Cioranescu-Zsidó's theory (see [14]) Let (t_n) be a sequence of positive numbers such that $\sum_n 1/t_n < \infty$. The function

$$\omega_{\{t_n\}}(\zeta) = \prod_{n=1}^{\infty} (1 + \frac{i\zeta}{t_n}), \quad \zeta \in \mathbf{C},$$

is an entire function of exponential type zero. Ω denotes the set of all these functions. For $\omega \in \Omega$,

$$\mathcal{D}_\omega = \left\{ f \in \mathcal{D}(\mathbf{R}), \int_{-\infty}^{\infty} |\hat{f}(t)\omega(Lt)^n| dt < \infty \text{ for all } L > 0 \text{ and } n \in \mathbf{N} \right\}.$$

Braun-Meise-Taylor's theory (see [4]) This an modification of Beurling's approach. A function $\omega : [0,\infty) \to [0,\infty)$ belongs to the set of weighted functions $\mathcal{W}$ if it is continuous and satisfies:

(α) there exists $C \ge 1$ such that $\omega(2t) \le (1 + \omega(t))$, $t \ge 0$;

(β) $\int_1^\infty \frac{\omega(t)}{1+t^2} dt < \infty$;

(γ) $\log(1+t) = o(\omega(t))$, $t \to \infty$;

(δ) $t \mapsto \omega(e^t)$ is a convex function.

For $\omega \in \mathcal{W}$,

$$\mathcal{D}_{(\omega)} = \left\{ f \in \mathcal{D}(\mathbf{R}), \int_{-\infty}^{\infty} |\hat{f}(t)| e^{\lambda\omega(t)} dt < \infty \text{ for all } \lambda > 0 \right\},$$

$$\mathcal{D}_{\{\omega\}} = \left\{ f \in \mathcal{D}(\mathbf{R}), \int_{-\infty}^{\infty} |\hat{f}(t)| e^{\lambda\omega(t)} dt < \infty \text{ for some } \lambda > 0 \right\}.$$

The theories $\{\mathcal{D}_\omega\}_{\omega\in\Omega}$, $\{\mathcal{D}^{(M_p)}\}_{(M_p)\in\mathcal{M}}$, $\{\mathcal{D}^{\{M_p\}}\}_{(M_p)\in\mathcal{M}}$, $\{\mathcal{D}_{(\omega)}\}_{\omega\in\mathcal{W}}$ and $\{\mathcal{D}_{\{\omega\}}\}_{\omega\in\mathcal{W}}$ are equivalent and strictly larger than $\{\mathcal{D}_\omega\}_{\omega\in\mathcal{A}}$ (see [14] and [4]).

Following Denjoy-Carleman-Komatsu's approach to the theory of ultradistributions we introduce the spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ of tempered ultradistributions, which are subspaces of the ultradistribution spaces $\mathcal{D}'^{(M_p)}$ and $\mathcal{D}'^{\{M_p\}}$ of Beurling and Roumieu type, study the elementary operations and the various integral transforms on them, the convolution and the ultratempered convolution of ultradistributions and determine a necessary and sufficient conditions for a convolutor of a space of tempered ultradistributions to be hypoelliptic in a space of integrable ultradistributions. In the special case when (M_p) is a Gevrey's sequence $(p^{\alpha p})_{p\in \mathrm{N}}$, $\alpha > 1$, the space $\mathcal{S}'^{(M_p)}$ is the space Σ'_α, which was investigated by Pilipović ([45]), and the test space $\mathcal{S}^{\{M_p\}}$ of the space $\mathcal{S}'^{\{M_p\}}$ is Gelfand-Shilov space $\mathcal{S}^\alpha_\alpha$([20] [18] [7], [32], [17]). We generalized results known in the case when (M_p) is a Gevrey's sequence for a wider class of sequences. The proofs which are trivial generalizations of the proofs of known results are omitted. The proof of the theorem which determines explicitly the space of multipliers of the space $\mathcal{S}'^{(M_p)}$ is simpler then the proof of analogous assertion for Σ'_α. Moreover, results which were not known even in the case of Gevrey's sequences were given. We determined explicitly the space of multipliers, characterized the test spaces by the Fourier transformation, Wigner distribution and Bargmann transformation, gave the boundary value representation of the space $\mathcal{S}'^{\{M_p\}}$, investigated the Fourier and Laplace transformations on $\mathcal{S}'^{\{M_p\}}$ and the Hermit expansion of its elements, the Hilbert transformation on $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$, proved the equivalence of definitions of convolution and of ultratempered convolution of ultradistributions of Beurling type and determined a necessary and sufficient conditions for a convolutor to be hypoelliptic in a space of integrable ultradistributions. The definitions and obtained results are given in one dimensional case (with exception of the fifth chapter) but they can be easily generalized for more dimensional case.

In the second chapter we define spaces of ultrarapidly decreasing ultradifferentiable functions and their duals, spaces of tempered ultradistributions of Beurling and Roumieu type. We investigate their topological properties, relations with the known distribution and ultradistribution spaces, structural properties and Hermite expansion and the boundary value representation of their elements. Elementary operations (translation, differentiation, ultradifferentiation and multiplication) on $\mathcal{S}^*$ and $\mathcal{S}'^*$ are investigated in third chapter. The space $\mathcal{O}_M^{(M_p)}$ of multipliers of the spaces $\mathcal{S}^*$ and $\mathcal{S}'^*$ is determined explicitly. The fourth chapter, which results are obtained in cooperation with prof. Pilipović, is devoted to the investigations of various integral transforms on the spaces $\mathcal{S}^*$ and $\mathcal{S}'^*$. We use results about Hermite expansion to obtain results for the Fourier and Laplace transform, characterize $\mathcal{S}^*$ by the Fourier transform, Wigner distribution and Bargmann transform. In the last section of the chapter we study the Hilbert transform on $\mathcal{S}'^*$, which is a generalization of the corresponding one on the space of tempered distributions, defined by Ishikawa ([26]). In the fifth chapter, which results are obtained in cooperation with prof. Pilipović and prof. Kamiński, we investigate in details the equivalence of several definitions of the convolution of Beurling type ultradistributions. Also, we introduce several definitions of ultratempered convolutions of Beurling type ultradistributions and prove their equivalence. In the last chapter we study hypoelliptic convolution equations in the Beurling and Roumieu ultradistribution spaces $\mathcal{D}'^*_{L^q}$, $q \in [1, \infty]$.

We remark that Björk ([6]) and Gruzdinski ([22]) studied the spaces $\mathcal{S}'_\omega$ of "ω-tempered distributions", which are generalizations of the space of Schwartz's tempered distributions, Beurling-Björk's theory of ultradistributions, and did not studied the problems which we consider. Relations between the spaces $\mathcal{S}'_\omega$ and $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ are discussed in the second chapter.

Jenssen and van Eijndhoven ([27]) studied Gelfand-Shilov type spaces $W_M^{M^\times}$ ([21]), where $M^\times$ is the Young conjugate of a suitable function M. In their approach M tends to infinity faster than x and slower than x^2. In the special case: $M(x) = \alpha x^{1/\alpha}$, $x > 0$, $1/2 \leq \alpha < 1$, $W_M^{M^\times}$ is the space $\mathcal{S}_\alpha^\alpha$. $W_M^{M^\times}$ is characterized in [27] by the Hermite expansion, Fourier transform, Wigner distribution and Bargmann transform. We obtained analogous characterizations of the spaces $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$, but in our approach the role of M has the function associated to the sequence (M_p), which is increasing and tends to infinity slower than x. For example if $M_p = p!^\alpha$, $\alpha > 1$, then $M(x) \sim Cx^{1/\alpha}$ and the Young's conjugate for such a function does not exist at all. It is easy to see that the natures of the spaces $W_M^{M^\times}$, $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are different, because of that our methods are quite different.

Notation and Notions

The sets of nonnegative integers, natural, real, complex and complex numbers with positive imaginary part are denoted by $\mathbf{N}_0$, $\mathbf{N}$, $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{C}_+$. Throughout the thesis the letter C (without super- or subscript) will denote a positive constant, not necessarily the same at each occurrence;

$$\langle x\rangle^\beta = (1+|x|^2)^{\beta/2}, \quad \beta \in \mathbf{N}_0,\ x \in \mathbf{R},$$

$$D = \frac{1}{i}\frac{\partial}{\partial x}, \qquad i = \sqrt{-1},$$

The sequence of Hermite functions h_n is

$$h_n(x) = \frac{(-1)^n}{\sqrt[4]{\pi}\sqrt{2^n n!}}\, e^{x^2/2}\,(e^{-x^2})^{(n)}, \qquad n \in \mathbf{N}_0,\ x \in \mathbf{R}.$$

and the sequence of the Hermite functions of the second kind $\tilde{h}_n$ ([64]) is

$$\tilde{h}_n(\zeta) = \begin{cases} \sum_{m\geq n} h_n(\zeta)h_m^{-1}(\zeta)h_{m+1}^{-1}(\zeta)((m+1)/2)^{-1/2}, & \zeta \in \mathbf{C},\ \mathrm{Im}\zeta \neq 0, \\ \frac{1}{2}(\tilde{h}_n(\xi+i0)+\tilde{h}_n(\xi-i0)), & \zeta = \xi + i\eta \in \mathbf{C},\ \mathrm{Im}\zeta = 0, \end{cases}$$

see [64]. The norm in the space $L^r = L^r(\mathbf{R})$, $r \in [1,\infty]$, is denoted by $\|\cdot\|_r$. The Fourier transform, Wigner distribution and Bargmann transform are defined respectively by

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbf{R}} e^{-ix\xi}\varphi(x)dx, \quad \xi \in \mathbf{R},\ \varphi \in L^1,$$

$$\mathbf{W}(x,y;f) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \exp(-iyt)f(x+t/2)\overline{f(y-t/2)}dt, \quad f \in L^2,\ x,y \in \mathbf{R},$$

$$(\mathbf{A}f)(\zeta) = \pi^{-1/4}\int_{\mathbf{R}} \exp(-1/2(\zeta^2+x^2)+\sqrt{2}\zeta x)f(x)\,dx, \quad f \in L^2,\ \zeta \in \mathbf{C},$$

(see [7], [28] and [29]).

A locally convex topological vector space is ($\text{F}\bar{\text{S}}$)-space (resp. (LS)-space) if it is a projective limit of countable, compact specter of spaces. If the mentioned specter is also nuclear the space is (FN)-space (resp. (LN)-space), see [19].

(M_p) is a sequence of positive numbers which satisfies some of the following conditions, see [33]:

(M.1) (*logarithmic convexity*)

$$M_p^2 \leq M_{p-1}\, M_{p+1}, \;\; p \in \mathbb{N};$$

(M.2)' (*stability under differential operators*)

$$M_{p+1} \leq AH^p M_p, \;\; p \in \mathbb{N}_0, \;\; \text{for some } A, H \geq 0;$$

(M.2) (*stability under ultradifferential operators*)

$$M_p \leq AH^p \min_{0 \leq q \leq p} M_{p-q} M_q, \;\; p, q \in \mathbb{N}_0, \;\; \text{for some } A, H \geq 0;$$

(M.3)' (*non-quasi-analyticity*)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty;$$

(M.3) (*strong non-quasi-analyticity*)

$$\sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq Aq \frac{M_q}{M_{q+1}}, \;\; q \in \mathbb{N}.$$

Throughout the thesis we will assume (M.1), (M.3)' and $M_0 = 1$. In some assertions we will suppose (M.2)', (M.2) and (M.3), as well. The letter H will always denote the constant mentioned in (M.2)' or (M.2).

The so-called associated functions for the sequence (M_p) are defined by

$$M(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p}{M_p}, \qquad \tilde{M}(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p p!}{M_p}, \qquad \rho > 0.$$

We denote by $\mathcal{R}$ a family of positive sequences which increases to infinity. This set is partially ordered and directed by the relation $(r_p) \preceq (s_p)$ defined by $r_p \leq s_p$, $p > p_0$, for some $p_0 \in \mathbb{N}$. The associated functions for the sequence $N_p = M_p(\prod_{k=1}^{p} a_k)$, $(a_p) \in \mathcal{R}$, are denoted by N_{a_p} and $\tilde{N}_{a_p}$.

Let us recall the definitions of Beurling and Roumieau spaces of ultradifferentiable functions ([33]). If K is a regular compact subset of R and $h > 0$, the space $\mathcal{E}^{M_p}_{K,h}$ is the space of functions ϕ from C^∞ such that

$$||\phi||_{K,h} = \sup_{\substack{\alpha\in N_0 \\ x\in K}} \frac{| \phi^{(\alpha)}(x) |}{h^\alpha M_\alpha} < \infty, \tag{1.1}$$

and $\mathcal{D}^{M_p}_{K,h}$ is the space of all φ from C^∞ with support in K which satisfy (1.1). The basic spaces of functions of class (M_p) and of class $\{M_p\}$ are defined by

$$\mathcal{E}^{(M_p)}_K = \text{proj}\lim_{h\to 0}\mathcal{E}^{M_p}_{K,h}, \qquad \mathcal{E}^{\{M_p\}}_K = \text{ind}\lim_{h\to\infty}\mathcal{E}^{M_p}_{K,h},$$

$$\mathcal{E}^{(M_p)} = \text{proj}\lim_{K\subset\subset R}\mathcal{E}^{(M_p)}_K, \qquad \mathcal{E}^{\{M_p\}} = \text{proj}\lim_{K\subset\subset R}\mathcal{D}^{\{M_p\}}_K.$$

$$\mathcal{D}^{(M_p)}_K = \text{proj}\lim_{h\to 0}\mathcal{D}^{M_p}_{K,h}, \qquad \mathcal{D}^{\{M_p\}}_K = \text{ind}\lim_{h\to\infty}\mathcal{D}^{M_p}_{K,h},$$

$$\mathcal{D}^{(M_p)} = \text{ind}\lim_{K\subset\subset R}\mathcal{D}^{(M_p)}_K, \qquad \mathcal{D}^{\{M_p\}} = \text{ind}\lim_{K\subset\subset R}\mathcal{D}^{\{M_p\}}_K.$$

The notation $K \subset\subset$ R means that K is compact and "grows" up to R.

Let $(a_p) \in \mathcal{R}$ and K be a compact set in R. $\mathcal{D}^{M_p}_{K,a_p}$ is the space of smooth functions φ on R supported by K such that

$$||\varphi||_{K,a_p} = \sup_{\substack{\alpha\in N_0 \\ x\in K}} \frac{| \partial^\alpha \varphi(x) |}{N_\alpha} < \infty, \tag{1.2}$$

where $N_p = M_p R_p$ and $R_p = \prod_{i=1}^p a_i, p \in N_0$. It is shown in [36] that

$$\mathcal{D}^{\{M_\alpha\}}_K = \text{proj}\lim_{(r_p)\in\mathcal{R}}\mathcal{D}^{M_p}_{K,r_p}.$$

The common notation for the symbols (M_p) and $\{M_p\}$ will be $*$.

The strong duals of $\mathcal{D}^{(M_p)}$ and $\mathcal{D}^{\{M_p\}}$, denoted by $\mathcal{D}'^{(M_p)}$ and $\mathcal{D}'^{\{M_p\}}$, are called Beurling and Roumieu spaces of ultradistributions. It is said that a locally convex space F is a *space of ultradistributions* if and only if F is algebraic subspace of $\mathcal{D}'^*$, the inclusion mapping $F \to \mathcal{D}'^*$ is continuous and $\mathcal{D}^*$ is dense in the space F.

If $s \in [1, \infty]$, as in [48] we define

$$\mathcal{D}^{(M_p)}_{L^s} = \text{proj}\lim_{h\to 0}\mathcal{D}^{M_p}_{L^s,h} \qquad \mathcal{D}^{\{M_p\}}_{L^s} = \text{ind}\lim_{h\to\infty}\mathcal{D}^{M_p}_{L^s,h},$$

where $\mathcal{D}^{M_p}_{L^s,h}$ is the space of functions ϕ from C^∞ for which,

$$\|\phi\|_{L^s,h} = \sup_{\alpha\in\mathbf{N}_0} \frac{\|\partial^\alpha\phi\|_s}{h^\alpha M_\alpha} < \infty, \tag{1.3}$$

equipped with the norm $\|\cdot\|_{L^s,h}$. The corresponding strong duals of $\mathcal{D}^*_{L^s}$, $\mathcal{D}'^*_{L^t}$, $t = s/(s-1)$, are subspaces of Beurling and Roumieau spaces of ultradistribution. We denote by $\dot{\mathcal{B}}^*$ the completion of $\mathcal{D}^*$ in $\mathcal{D}^*_{L^\infty}$. The strong dual of $\dot{\mathcal{B}}^*$ is denoted by $\mathcal{D}'^*_{L^1}$. Let

$$\tilde{\mathcal{D}}^{\{M_p\}}_{L^r} = \operatorname{proj}\lim_{\ell_p\in\mathcal{R}} \mathcal{D}^{M_p}_{L^r,\ell_p},$$

where $\mathcal{D}^{M_p}_{L^r,\ell_p}$, $\ell_p \in \mathcal{R}$, is the space of functions ϕ from C^∞, for which

$$\|\phi\|_{L^r,\ell_p} = \sup_{\alpha\in\mathbf{N}_0} \frac{\|\partial^\alpha\phi\|_r}{\left(\prod\limits_{1\le\beta\le\alpha} h_\beta\right) M_\alpha} < \infty, \tag{1.4}$$

and let us denote the completion of $\mathcal{D}^{\{M_p\}}$ in $\tilde{\mathcal{D}}^{\{M_p\}}$ by $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$. The strong dual of $\tilde{\mathcal{D}}^{\{M_p\}}_{L^r}$ are denoted by $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^q}$, $q = r/(r-1)$, and the strong dual of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ is denoted by $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. From [50, Lemma 3.(i),(ii)] it follows that in the set theoretical sense $\tilde{\mathcal{D}}^{\{M_p\}}_{L^r} = \mathcal{D}^{\{M_p\}}_{L^r}$, $r \in (1,\infty)$, and $\dot{\tilde{\mathcal{B}}}^{\{M_p\}} = \dot{\mathcal{B}}^{\{M_p\}}$, and that the inclusion mappings $i : \mathcal{D}^{\{M_p\}}_{L^r} \to \tilde{\mathcal{D}}^{\{M_p\}}_{L^r}$ and $i : \dot{\mathcal{B}}^{\{M_p\}} \to \dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ are continuous. Hence, $\tilde{\mathcal{D}}'^*_{L^q}$ is a topological subspace of $\mathcal{D}'^*_{L^q}$.

Let $a > 0$, $(a_p) \in \mathcal{R}$

$$P_a(\zeta) = (1+\zeta^2)\prod_{p\in\mathbf{N}}\left(1 + \frac{\zeta^2}{a^2 m_p^2}\right), \quad \zeta \in \mathbf{C},$$

and

$$P_{a_p}(\zeta) = (1+\zeta^2)\prod_{p\in\mathbf{N}}\left(1 + \frac{\zeta^2}{a_p^2 m_p^2}\right), \quad \zeta \in \mathbf{C},$$

If (M.1), (M.2) and (M.3) hold, an ultradistribution T is in $\mathcal{D}'^{(M_p)}_{L^q}$ (resp. $\mathcal{D}'^{\{M_p\}}_{L^q}$), $q \in [1,\infty]$, if and only if there are $b > 0$ (resp. $(b_p) \in \mathcal{R}$) such that

$$f = P_b(D)F_1 + F_2 \quad \left(\text{resp. } f = P_{b_p}(D)F_1 + F_2\right), \quad D = \frac{1}{i}\frac{\partial}{\partial x},$$

where $F_1, F_2 \in L^q$ ([50, Theorem 1]).

By $\ell^2(b_k)$, where $b_k = (b_{1,k}, b_{2,k}, \ldots)$ is a sequence of real numbers, we denote a Köthe space of sequences $x = (x_1, x_2, \ldots)$ of complex numbers with the norm,

$$|||x||| = \left(\sum_{n \in \mathbb{N}} |x_n|^2 (b_{n,k})^2 \right)^{1/2}.$$

It is said that a formal series

$$P(\xi) = \sum_{\alpha \in \mathbb{N}_0} a_\alpha \xi^\alpha, \xi \in \mathbb{R},$$

defines an ultrapolynomial of class (M_p) (resp. $\{M_p\}$) whenever the coefficients a_α satisfy the estimate

$$|a_\alpha| \leq C L^\alpha M_\alpha, \quad \alpha \in \mathbb{N}_0,$$

for some $L > 0$ and C (resp. for every $L > 0$ and some C). The corresponding operator $P(D) = \sum_\alpha a_\alpha D^\alpha$ is an ultradifferential operator of class (M_p) (resp. $\{M_p\}$). Conditions (M.1), (M.2) and (M.3) imply that P_a (resp. P_{a_p}) is an ultradifferential operator of the class (M_p) (resp. $\{M_p\}$) ([33]). We say that function f is of ultrapolynomial growth of class $*$, if and only if there is ultrapolynomial P of class $*$, such that

$$|f(x)| \leq P(|x|), \quad x \in \mathbb{R}.$$

Note, if (M.2)' is fulfilled function f is of ultrapolynomial growth of class (M_α) (resp. $\{M_\alpha\}$) if and only if for some $m > 0$ and some C (resp. for every $m > 0$ there exists C such that),

$$|f(x)| \leq C \exp M(m|x|), \quad x \in \mathbb{R}.$$

Chapter 2

Spaces $\mathcal{S}^*$ and $\mathcal{S}'^*$

In the chapter we define spaces of ultrarapidly decreasing ultradifferentiable functions and their duals, spaces of tempered ultradistributions of Beurling and Roumieu type. We investigate their topological properties, relations with the known distribution and ultradistribution spaces, structural properties and Hermite expansion and the boundary value representation of their elements.

2.1 Topological properties

Let $m > 0$ and $r \in [1, \infty)$ be given.

Definition 2.1 *$\mathcal{S}_r^{M_p,m}$ and $\mathcal{S}_\infty^{M_p,m}$ are the spaces of all the smooth functions φ on* $\mathbf{R}$ *which satisfy that*

$$\sigma_{m,r}(\varphi) = \left(\sum_{\alpha,\beta \in \mathrm{N}_0} \int_{\mathrm{R}} \mid \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \langle x \rangle^\beta \varphi^{(\alpha)}(x) \mid^r dx \right)^{1/r} < \infty$$

and

$$\sigma_{m,\infty}(\varphi) = \sup_{\alpha,\beta \in \mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \| \langle x \rangle^\beta \varphi^{(\alpha)} \|_\infty < \infty,$$

respectively, equipped with the topologies induced by the norms $\sigma_{m,r}$ and $\sigma_{m,\infty}$, respectively.

By $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ we denote the projective ($m \to \infty$) and the inductive ($m \to 0$) limits of the spaces $\mathcal{S}_2^{M_p,m}$ respectively.

The space $\mathcal{S}'^*$ is the strong dual of $\mathcal{S}^*$.

Note, $\mathcal{S}_r^{M_p,m}$ is a special case of the space $\ell^r(m,F)$ (see [66]). Using the analogous idea as in [66] one can prove that the space $\mathcal{S}_r^{M_p,m}$ is a a Banach space, and especially, that $\mathcal{S}_2^{M_p,m}$ is a Hilbert space where the scalar product of $\phi, \psi \in \mathcal{S}_2^{M_p,m}$ is defined by

$$(\phi,\psi) = \sum_{\alpha,\beta\in\mathbf{N}_0} \int_{\mathbf{R}} \left(\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\right)^2 \langle x\rangle^{2\beta}\phi^{(\alpha)}(x)\,\overline{\psi^{(\alpha)}}(x)\,dx.$$

Under the assumptions (M.1) and (M.3)' the space $\mathcal{S}^*$ is non-trivial, since the space $\mathcal{D}^*$ is non-trivial ([33]) and $\mathcal{D}^* \subset \mathcal{S}^*$. Moreover, $\mathcal{D}^*$ is a proper subset of $\mathcal{S}^*$. If $\rho \in \mathcal{D}^*$, $\rho \geq 0$, $supp\rho \subset [-1,1]$, $\rho(x) = 1$ for $|x| \leq 1/2$, and (x_j) is a sequence of elements of $\mathbf{R}$ such that $|\,x_j\,| + 2 \leq |x_{j+1}\,|$, $j \in \mathbf{N}$, the function

$$\phi(x) = \sum_{j\in\mathbf{N}} \frac{\rho(x-x_j)}{\langle x_j\rangle^j}, \quad x \in \mathbf{R} \tag{2.1}$$

is an example of a function which belongs to $\mathcal{S}^*$ and does not belong to $\mathcal{D}^*$. It will be proved that if (M.2)' holds, $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) can be represented as the projective (resp. inductive) limit of the spaces $\mathcal{S}_r^{M_p,m}$, $r \in [1,\infty]$, when $m \to \infty$ (resp. $m \to 0$). In [50] Pilipović proved the next theorem.

Theorem 2.2 *([50]) If (a_p), $(b_p) \in \mathcal{R}$, and if $\mathcal{S}^{M_p}_{a_p,b_p}$ is the space of smooth functions ϕ on $\mathbf{R}$ which satisfy,*

$$\wp_{a_p,b_p}(\phi) = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{\|\langle x\rangle^\beta\phi^{(\alpha)}\|_\infty}{M_\alpha\left(\prod_{\gamma=1}^{\alpha} a_\gamma\right)\, M_\beta\left(\prod_{\gamma=1}^{\beta} b_\gamma\right)} < \infty,$$

equipped with the topology induced by the norm $\wp_{(a_p),(b_p)}$. Then

$$\mathcal{S}^{\{M_p\}} = \operatorname{proj\,lim}_{(a_p),(b_p)\in\mathcal{R}} \mathcal{S}^{M_p}_{a_p,b_p}.$$

A non-trivial example of an element of the space $\mathcal{S}'^*$ is

$$\langle f,\varphi\rangle = \int_{\mathbf{R}} f\varphi dx, \quad \varphi \in \mathcal{S}^*,$$

where f is a locally integrable function of ultrapolynomial growth of class $*$. It will be shown that if (M.1), (M.2) and (M.3) are fulfilled each element of $\mathcal{S}'^*$ can be represented as an ultraderivative of the class $*$ of a continuous function of ultrapolynomial growth of class $*$.

Theorem 2.3 *1. $\{\sigma_{m,\infty}, m > 0\}$ (resp. $\{\wp_{a_p,b_p,\infty}, (a_p),(b_p) \in \mathcal{R}\}$) and $\{s_{m,\infty}, m > 0\}$ (resp. $\{S_{a_p,b_p,\infty}, (a_p),(b_p) \in \mathcal{R}\}$), are equivalent families of norms on the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$), where*

$$s_{m,\infty}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^{(\alpha)}\|_\infty,$$

$$\Big(\textit{resp. } S_{a_p,b_p,\infty}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{\|x^\beta\varphi^{(\alpha)}\|_\infty}{(\prod_{p=1}^{\alpha} a_p)M_\alpha(\prod_{p=1}^{\beta} b_p)M_\beta}\Big).$$

2. If (M.2)' holds then the families of norms $\{\sigma_{m,r};\ m > 0\}$, $r \in [1,\infty]$, $\{s_{m,r}; m > 0\}$, $r \in [1,\infty]$, and $\{\varsigma_m;\ m > 0\}$ (resp. $\{\wp_{a_p,b_p,r}, (a_p),(b_p) \in \mathcal{R}\}$, $\{S_{a_p,b_p,r}, (a_p),(b_p) \in \mathcal{R}\}$, $r \in [1,\infty]$, and $\{\Lambda_{(a_p),(b_p),r}, (a_p),(b_p) \in \mathcal{R}\}$) are mutually equivalent on the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$), where

$$s_{m,p}(\varphi) = \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta\varphi^{(\alpha)}\|_p,$$

$$\varsigma_m(\varphi) = \sup_{\alpha\in\mathbf{N}_0} \frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)} \exp[M(m|\cdot|)]\|_\infty,$$

$$\Big(\textit{resp. } \wp_{(a_p),(b_p),r}(\varphi) = \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{\|\langle x\rangle^\beta\varphi^{(\alpha)}\|_r}{(\prod_{p=1}^{\infty} a_p)M_\alpha(\prod_{p=1}^{\infty} b_p)M_\beta},$$

$$S_{a_p,b_p,r}(\varphi) = \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{\|x^\beta\varphi^{(\alpha)}\|_r}{(\prod_{p=1}^{\infty} a_p)M_\alpha(\prod_{p=1}^{\infty} b_p)M_\beta},$$

$$\Lambda_m(\varphi) = \sup_{\alpha\in\mathbf{N}_0} \frac{1}{(\prod_{p=1}^{\alpha} a_p)M_\alpha} \|\varphi^{(\alpha)} \exp[N_{b_p}(|\cdot|)]\|_\infty\Big).$$

3. If (M.2) holds, $\{s_{m,2},\ m > 0\}$ (resp. $\{S_{a_p,b_p,2}, (a_p),(b_p) \in \mathcal{R}\}$) is equivalent to any of the families of norms $\{\theta_\delta, \delta > 0\}$ and $\{\bar{s}_{m,2}, m > 0\}$ (resp. $\{\Theta_{a_p},(a_p) \in \mathcal{R}\}$ and $\{\bar{S}_{a_p},(a_p) \in \mathcal{R}\}$) on the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$), where

$$\theta_\delta(\varphi) = \sum_{n\in\mathbf{N}_0} |\ a_n\ |^2 \exp[2M(\delta\sqrt{2n+1})],\ \ \varphi \overset{L^2}{=} \sum_{n\in\mathbf{N}_0} a_n h_n,$$

$$\bar{s}_{m,2}(\varphi) = \sum_{\alpha,\beta\in\mathbf{N}} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|(x^\beta\varphi)^{(\alpha)}\|_2$$

$$\Big(resp.\ \Theta_{a_p}(\varphi) = \sum_{n\in\mathbf{N}_0} | a_n |^2 \exp[2N_{a_p}(\sqrt{2n+1})],\ \ \varphi \overset{L^2}{=} \sum_{n\in\mathbf{N}_0} a_n h_n,$$

$$\bar{S}_{a_p,b_p}(\varphi) = \sum_{\alpha,\beta\in\mathbf{N}} \frac{\|(x^\beta\varphi)^{(\alpha)}\|_2}{(\prod_{p=1}^{\alpha} a_p)M_\alpha(\prod_{p=1}^{\beta} b_p)M_\beta}$$

4. *Let (M.2) hold and let φ be a smooth function on* $\mathbf{R}$. *If for each (resp. some)* $\ell > 0$ *and each* $\beta \in \mathbf{N}_0$

$$\sup_{\alpha\in\mathbf{N}_0} \frac{\|x^\beta\varphi^{(\alpha)}\|_2}{\ell^\alpha M_\alpha} < \infty, \tag{2.2}$$

and for each (resp. some) $\ell > 0$ *and each* $\alpha \in \mathbf{N}_0$

$$\sup_{\beta\in\mathbf{N}_0} \frac{\|x^\beta\varphi^{(\alpha)}\|_2}{\ell^\beta M_\beta} < \infty, \tag{2.3}$$

then for each (resp. some) $\ell > 0$

$$\sum_{\alpha,\beta\in\mathbf{N}_0} \frac{\|x^\beta\varphi^{(\alpha)}\|_2}{\ell^{\alpha+\beta} M_\alpha M_\beta} < \infty. \tag{2.4}$$

Note,

- If (M.2)' is fulfilled, in the definition of $\mathcal{S}^*$ the space $\mathcal{S}_2^{M_p,m}$ can be replaced by $\mathcal{S}_p^{M_p,m}$, $p \in [1,\infty]$.
- The last part of the theorem is an analog of Kashpirovski's result: $\mathcal{S}_\beta^\alpha = \mathcal{S}^\alpha \cap \mathcal{S}_\beta$ ([32] see also [17]).

In order to prove the assertion we need the following estimations, which are proved in [32] (see also [2]).

For every $m, n \in \mathbf{N}$ we have

$$x^m h_n(x) = 2^{-m/2} \sum_{k=0}^{m} \alpha_{k,m}^{(n)}\, h_{n-m+2k}(x),\ \ x \in \mathbf{R}, \tag{2.5}$$

$$0 \le |\alpha_{k,m}^{(n)}| \le \binom{m}{k} \left((2n+1)^{m/2} + m^{m/2}\right). \tag{2.6}$$

If $n - m + 2k < 0$, we take by definition $h_{n-m+2k} = 0$.

Let $\Re^0$ be the identity operator and $\Re^k = (x^2 - d^2/dx^2)^k$, $k \in \mathbf{N}$. $\Re$ is formally a self-adjoint operator and $\Re h_n = (2n+1)h_n$. If $\varphi \in \mathcal{S}$ and $k \in \mathbf{N}$, then

$$\Re^k \varphi = \sum_{\substack{0 \le n \le k \\ p+q=2n}} C_{p,q}^{(k)} x^p \varphi^{(q)}, \quad |C_{p,q}^{(k)}| \le 10^k\, k^{k-\frac{p+q}{2}}. \tag{2.7}$$

Proof of Theorem 2.3: 1. Obviously, for each smooth function φ and $m > 0$, $s_{m,\infty}(\varphi) \le \sigma_{m,\infty}(\varphi)$. Since for each $L > 0$ from (M.3)',

$$\frac{L^k k!}{M_k} \longrightarrow 0 \text{ as } k \to \infty, \tag{2.8}$$

see [33, (4.5)], and since

$$\langle x \rangle^\beta \le 2^{\beta/2} max\left(1, |x|^\beta\right), \; x \in \mathbf{R}, \beta \in \mathbf{N}_0,$$

for each $m > 0$, there exists C, such that for each smooth function φ, and $\alpha, \beta \in \mathbf{N}_0$,

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x \rangle^\beta \varphi^{(\alpha)}\|_\infty \le \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} 2^\beta \max\left(\|\varphi^{(\alpha)}\|_\infty, \|x^\beta \varphi^{(\alpha)}\|_\infty\right)$$

$$\le \max\left(C \frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)}\|_\infty, \frac{(2m)^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^{(\alpha)}\|_\infty\right)$$

$$\le C \sup_{\beta \in \mathbf{N}_0} \frac{(2m)^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^\alpha\|_\infty = C s_{m,\infty}(\varphi).$$

Therefore for each $m > 0$ there exists C such that for each smooth function φ, $\sigma_{m,\infty}(\varphi) \le C s_{m,\infty}(\varphi)$.

2. Let $t \in (1, \infty)$ and $\gamma = [1/t] + 1$. Applying (M.2)' we get that for each $m > 0$ there exists C such that for each smooth function φ,

$$s_{m,t}(\varphi) \le \sum_{\alpha,\beta \in \mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \left(\sup_{|x| \le 1} |x^\beta \varphi^{(\alpha)}(x)| + \right. \tag{2.9}$$

$$\left. + \sup_{|x|>1} |x^{\beta+\gamma} \varphi^{(\alpha)}| \int_{|x|>1} |x^{-\gamma}| dx \right)$$

$$\leq \sum_{\alpha,\beta\in \mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|x^\beta\varphi^{(\alpha)}\|_\infty + \mathcal{C}\sum_{\alpha,\beta\in \mathrm{N}_0} \frac{m^{\alpha+\beta}H^{\gamma\beta}}{M_\alpha M_{\beta+\gamma}}\|x^{\beta+\gamma}\varphi^{(\alpha)}\|_\infty$$

$$\leq \mathcal{C}s_{m(1+H^\gamma),\infty}(\varphi).$$

The inequality

$$|x^\beta\varphi^{(\alpha)}(x)| \leq \beta\int_{\mathrm{R}}|t^\beta\varphi^{(\alpha)}(t)|dt + \int_{\mathrm{R}}|t^\beta\varphi^{(\alpha+1)}(t)|dt, \quad x\in \mathrm{R},$$

which holds for each smooth function φ and $\alpha,\beta\in \mathrm{N}_0$ and condition (M.2)' imply that for each $m>0$ there exists $\mathcal{C}$, such that for each smooth function φ,

$$s_{m,\infty}(\varphi) \leq \sup_{\alpha,\beta\in \mathrm{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left(\beta\int_{\mathrm{R}}|t^\beta\varphi^{(\alpha)}(t)|dt + \right. \tag{2.10}$$

$$\left. + \int_{\mathrm{R}}|t^\beta\varphi^{(\alpha+1)}(t)|dt\right)$$

$$\leq \mathcal{C}\sup_{\alpha,\beta\in \mathrm{N}_0}\left(\frac{2^\beta m^{\alpha+\beta}}{M_\alpha M_\beta}\int_{\mathrm{R}}|t^\beta\varphi^{(\alpha)}(t)|dt + \frac{(Hm)^{\alpha+1}m^{\alpha+\beta+1}}{M_{\alpha+1}M_\beta}\int_{\mathrm{R}}|t^\beta\varphi^{(\alpha+1)}(t)|dt\right)$$

$$\leq \mathcal{C}s_{2m(1+H),1}(\varphi).$$

Let $t\in(1,\infty)$, $q=t/(t-1)$ and $\gamma=[1/q]+1$. The Hölder inequality, (2.8) and (M.2)' imply that for each $m>0$ there exists $\mathcal{C}$ such that for each smooth function φ,

$$s_{m,1}(\varphi) = \sum_{\alpha,\beta\in \mathrm{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left(\int_{|x|\leq 1}|\varphi^{(\alpha)}(x)|dx + \right. \tag{2.11}$$

$$\left. + \int_{|x|>1}|x^\beta\varphi^{(\alpha)}(x)|dx\right)$$

$$\leq \sum_{\alpha,\beta\in \mathrm{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left(\mathcal{C}\left(\int_{|x|\leq 1}|\varphi^{(\alpha)}(x)|^t dx\right)^{1/t} + \right.$$

$$+\left(\left(\int_{|x|>1}|x^{\beta+\gamma}\varphi^{(\alpha)}(x)|^t dx\right)^{1/t}\left(\int_{|x|>1}|x|^{-\gamma q}dx\right)^{1/q}\right)$$

$$\leq C\sum_{\alpha,\beta\in\mathrm{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left(\|\varphi^{(\alpha)}\|_t+\|x^{\beta+\gamma}\varphi^{(\alpha)}\|_t\right)$$

$$\leq C\left(\sum_{\alpha\in\mathrm{N}_0}\frac{m^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\|_t+\sum_{\alpha,\beta\in\mathrm{N}_0}\frac{m^{\alpha+\beta}H^{\gamma\beta}}{M_\alpha M_{\beta+\gamma}}\|x^{\beta+\gamma}\varphi^{(\alpha)}\|_t\right)$$

$$\leq C s_{m(1+H^\gamma),t}(\varphi).$$

The equivalence of $\{s_{m,r},\ m>0\}$ and $\{s_{m,p},\ m>0\}$, $r,p\in[1,\infty]$, follows from (2.9), (2.10) and (2.11). The proof of the equivalence of $\{\sigma_{m,p},m>0\}$ and $\{\sigma_{m,r},m>0\}$, where $r,p\in[1,\infty]$, is analogous.

The condition (M.2)' implies that for each $\varphi\in\mathcal{S}^{(M_p)}$ and each $m>0$ there exists C such that for each $\alpha,\beta\in\mathrm{N}_0$ and for $|x|>k>1$

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\,|x^\beta\varphi^{(\alpha)}(x)|\ \leq C\,\frac{m^\alpha(mH)^{\beta+1}}{M_\alpha M_{\beta+1}}\,|x^\beta\varphi^{(\alpha)}(x)|$$

$$\leq\ \frac{C}{k}\,\frac{m^\alpha(mH)^{\beta+1}}{M_\alpha M_{\beta+1}}\,|x^{\beta+1}\varphi^{(\alpha)}(x)|\ \leq\ \frac{C}{k}.$$

Therefore *for each $m>0$ and $\varphi\in\mathcal{S}^{(M_p)}$, $(m^{\alpha+\beta}/(M_\alpha M_\beta))\,|x^\beta\varphi^{(\alpha)}(x)|$ converges uniformly in $\alpha,\beta\in\mathrm{N}_0$ to zero as $|x|$ tends to infinity.* The definition of the space $\mathcal{S}^{(M_p)}$ implies that $(m^{\alpha+\beta}/(M_\alpha M_\beta))\,|x^\beta\varphi^{(\alpha)}(x)|$, $m>0$, converges to zero uniformly in $x\in\mathbf{R}$ as $(\alpha+\beta)$ tends to infinity. Hence, for given element φ of $\mathcal{S}^{(M_p)}$ and each $m>0$ there are $\alpha_0,\beta_0\in\mathrm{N}_0$ and $x_0\in\mathbf{R}$ such that

$$\sup_{\alpha,\beta\in\mathrm{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|x^\beta\varphi^{(\alpha)}\|_\infty=\frac{m^{\alpha_0+\beta_0}}{M_{\beta_0}M_{\alpha_0}}|x_0^{\beta_0}\varphi^{(\alpha_0)}(x_0)|$$

$$=\sup_{\alpha,\beta\in\mathrm{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|x^\beta\varphi^{(\alpha)}\|_\infty=\|\sup_{\beta\in\mathrm{N}_0}\left(\sup_{\alpha\in\mathrm{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|x^\beta\varphi^{(\alpha)}|\right)\|_\infty$$

$$= \| \sup_{\alpha\in\mathrm{N}_0} \left(\sup_{\beta\in\mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}| \right) \|_\infty = \sup_{\alpha\in\mathrm{N}_0} \left(\| \sup_{\beta\in\mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}| \|_\infty \right)$$

$$= \sup_{\alpha\in\mathrm{N}_0} \left(\frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)} \exp[M(m\|\cdot\|)]\|_\infty \right).$$

3. Let us prove the equivalence of the systems $\{s_{m,2},\, m > 0\}$ and $\{\theta_\delta,\, \delta > 0\}$, which together with the fact that

$$\bar{s}_{m,2}(\varphi) = \frac{1}{\sqrt{2\pi}} s_{m,2}(\mathcal{F}\varphi) \text{ and } \theta_\delta(\mathcal{F}\varphi) = \theta_\delta(\varphi), \quad \varphi \in \mathcal{S}^*,$$

imply the equivalence of $\{\bar{s}_{m,2},\, m > 0\}$ and $\{\theta_\delta,\, \delta > 0\}$. It is enough to prove that if (2.4) holds then the estimation

$$\sum_{n\in\mathrm{N}_0} |a_n|^2 \exp[2M(\delta\sqrt{2n+1})] < \infty, \tag{2.12}$$

holds for $\delta = (\sqrt{20e}(1+H)^4\ell)^{-1}$, and conversely that if (2.12) holds then (2.4) holds for $\ell = H\sqrt{8/\delta}$.

Suppose that (2.4) holds. In the estimations which are to follow we shall use (2.7), (M.1), Stirling's formula, (M.2) and the fact that for each $n \in \mathrm{N}_0$ and $L > 0$,

$$L^{n-k} \frac{k!}{n!} \frac{M_n}{M_k} \to 0 \quad \text{as } k \to \infty, \tag{2.13}$$

which follows from (M.3)' since

$$\frac{k!}{n!} \frac{M_k}{M_n} = \frac{k}{Lm_k} \cdot \frac{k-1}{Lm_{k-1}} \cdot \dots \cdot \frac{n+1}{Lm_{n+1}} \to 0 \text{ as } k \to \infty,$$

where $m_n = M_n/M_{n-1}$, $n = 1, 2, \dots$ (see [33, (4.5)]).

There exists C such that for each $k \in \mathrm{N}_0$,

$$\left(\sum_{n\in\mathrm{N}_0} |a_n|^2 (2n+1)^{2k} \right)^{1/2} = \|\Re^k \varphi\|_2 \le \sum_{\substack{0\le n\le k \\ p+q=2n}} C_{p,q}^{(k)} \|x^p \varphi^{(q)}\|_2$$

$$\le C \sum_{\substack{0\le n\le k \\ p+q=2n}} 10^k\, k^{k-\frac{p+q}{2}}\, \ell^{p+q}\, M_p M_q$$

$$\leq C \sum_{\substack{0\leq n\leq k\\ p+q=2n}} 10^k \frac{k^k}{(\frac{p+q}{2})^{\frac{p+q}{2}}} \ell^{p+q} M_{p+q} \leq C \sum_{0\leq n\leq k} \frac{k! e^k 10^k}{n!} \ell^{2n} \frac{M_{2n}}{M_{2k}} M_{2k}$$

$$\leq C\, e^k\, 20^k\, H^{2k}\, \ell^{2k} \sum_{0\leq n\leq k} \frac{1}{2^n} \frac{k!}{n!} \frac{M_n M_n}{(Hl)^{2(k-n)} M_k M_k} M_{2k}$$

$$\leq C\, e^k\, 20^k\, H^{2k}\, \ell^{2k}\, M_{2k}.$$

Moreover, from above and (M.2)' we have that for some C,

$$\left(\sum_{n\in\mathbf{N}_0} |a_n|^2 (2n+1)^{2k-1}\right)^{1/2} \leq C\, e^k\, 20^k\, H^{4k}\, \ell^{2k}\, M_{2k-1}.$$

It follows that there exists C such that for each $\alpha, n \in \mathbf{N}_0$,

$$|a_n|^2 (2n+1)^\alpha \leq C\, (\sqrt{20e}\,(1+H)^2\, \ell)^{2\alpha}\, M_\alpha^2.$$

By putting $\alpha + 2$ instead of α in the above inequality and by using (M.2)' we get that for each $\alpha, n \in \mathbf{N}_0$ and $\delta = (\sqrt{20e}\,(1+H)^4\,\ell)^{-1}$ there exists C, such that

$$\frac{|a_n|^2 \delta^{2\alpha} (2n+1)^\alpha}{M_\alpha^2} \leq \frac{C}{(2n+1)^2},$$

which implies

$$|a_n|^2 \exp[2M(\delta\sqrt{2n+1})] = |a_n|^2 \sup_{\alpha\in\mathbf{N}_0} \frac{\delta^{2\alpha}(2n+1)^\alpha}{M_\alpha^2} \leq \frac{C}{(2n+1)^2}.$$

Therefore,

$$\sum_{n\in\mathbf{N}_0} |a_n|^2 \exp[2M(\delta\sqrt{2n+1})] < \infty.$$

Suppose that for some $\delta > 0$ inequality (2.12) holds. Applying (2.5) and the Cauchy-Schwartz inequality we get that there exists C such that for each $m \in \mathbf{N}_0$,

$$\|x^m \varphi\|_2 \leq 2^{-m/2} \|\sum_{n\in\mathbf{N}_0} a_n \left(\sum_{k\leq m} \alpha_{k,m}^{(n)} h_{n-m+2k}\right)\|_2$$

$$\leq 2^{-m/2} \sum_{n\in\mathbf{N}_0} |a_n| \left(\sum_{k\leq m} \binom{m}{k} \left((2n+1)^{m/2} + m^{m/2}\right) \right)$$

штампарске грешке!!!

$$\leq 2^{m/2} \sum_{n\in\mathbf{N}_0} |a_n| \exp[M(\delta(2n+1)) - M(\delta(2n+1))] \left((2n+1)^{m/2} + m^{m/2}\right)$$

$$\leq 2^{m/2} \left(\sum_{n\in\mathbf{N}_0} |a_n|^2 \exp[2M(\delta(2n+1))] \right)^{1/2} \cdot$$

$$\cdot \left(\sum_{n\in\mathbf{N}_0} \exp[-2M(\delta(2n+1))] \left((2n+1)^{m/2} + m^{m/2}\right)^2 \right)^{1/2}$$

$$\leq \mathcal{C}\, 2^{m/2} \left(\sum_{n\in\mathbf{N}_0} \exp[-2M(\delta(2n+1))] \left((2n+1)^{m/2} + m^{m/2}\right)^2 \right)^{1/2}$$

$$\leq \mathcal{C}\, 2^{m/2} \sup_{n\in\mathbf{N}_0} \left(\left((2n+1)^{m/2} + m^{m/2}\right) \exp[-\frac{1}{2}M(\delta(2n+1))] \right) \cdot$$

$$\cdot \left(\sum_{n\in\mathbf{N}_0} \exp[-M(\delta(2n+1))] \right)^{1/2}$$

$$\leq \mathcal{C} \left(\frac{2}{\delta}\right)^{m/2} M_m \left(\delta^{m/2} \sup_n \frac{(2n+1)^{m/2} \exp[-\frac{1}{2}M(\delta(2n+1))]}{M_m} \right) + \frac{\delta^{m/2} m^{m/2}}{M_m}.$$

Since

$$\frac{\delta^{m/2} m^{m/2}}{M_m} \leq \frac{m! e^m \delta^{m/2}}{M_m} \to 0 \text{ as } m \to \infty,$$

and

$$\sup_{n\in\mathbf{N}_0} \frac{\delta^{m/2}(2n+1)^{m/2} \exp[\frac{-1}{2}M(\delta(2n+1))]}{M_m} = \frac{1}{M_m} \sup_{n\in\mathbf{N}_0} \left(\frac{\delta^m (2n+1)^m}{\exp[M(\delta(2n+1))}] \right)^{1/2}$$

$$= \frac{\sqrt{M_m}}{M_m} \to 0, \text{ as } m \to \infty$$

which follow from [33, (3.3)], we have that

$$\|x^m\varphi\|_2 \le C\left(\sqrt{2/\delta}\right)^m M_m. \tag{2.14}$$

By the Fourier transform we obtain that for each $n \in \mathrm{N}_0$,

$$\|\varphi^{(n)}\|_2 = \frac{1}{\sqrt{2\pi}}\|\mathcal{F}(\varphi^{(n)})\|_2 = \frac{1}{\sqrt{2\pi}}\|x^n\mathcal{F}(\varphi)\|_2 \tag{2.15}$$

$$= \frac{1}{\sqrt{2\pi}}\|x^n\sum_{k\in\mathrm{N}_0}a_kh_k\|_2 = \frac{1}{\sqrt{2\pi}}\|x^n\varphi\|_2 \le C\left(\sqrt{\frac{2}{\delta}}\right)^n M_n. \tag{2.16}$$

If $\alpha,\beta \in \mathrm{N}_0$ and $\gamma = \min(\alpha, 2\beta)$ by using (2.14), (2.15), (M.1), (M.3)' and (M.2) we get

$$\left(\|x^\beta\varphi^{(\alpha)}\|_2\right)^2 = (x^\beta\varphi^{(\alpha)}, x^\beta\varphi^{(\alpha)})_{L^2} = |((x^{2\beta}\varphi^{(\alpha)})^{(\alpha)},\varphi)_{L^2}| \tag{2.17}$$

$$\le \left|\sum_{\kappa=0}^{\gamma}\binom{\alpha}{\kappa}\frac{(2\beta)!}{(2\beta-\kappa)!}(x^{2\beta-\kappa}\varphi^{(2\alpha-\kappa)},\varphi)_{L^2}\right|$$

$$\le \sum_{\kappa=0}^{\gamma}\binom{\alpha}{\kappa}\binom{2\beta}{\kappa}\kappa!\,\|x^{2\beta-\kappa}\varphi\|_2\,\|\varphi^{(2\alpha-\kappa)}\|_2$$

$$\le C\sum_{\kappa=0}^{\gamma}\binom{\alpha}{\kappa}\binom{2\beta}{\kappa}\kappa!\,(2/\delta)^{(\alpha+\beta-\kappa)}\,\frac{M_\kappa^2}{M_\kappa^2}\,M_{2\alpha-\kappa}M_{2\beta-\kappa}$$

$$\le C\sum_{\kappa=0}^{\gamma}\binom{\alpha}{\kappa}\binom{2\beta}{\kappa}(2/\delta)^{(\alpha+\beta-\kappa)}\,\frac{\kappa!}{M_\kappa^2}M_{2\alpha}M_{2\beta}$$

$$\le C\,H^{2(\alpha+\beta)}\,M_\alpha^2\,M_\beta^2\sum_{\kappa=0}^{\gamma}\binom{\alpha}{\kappa}\binom{2\beta}{\kappa}(2/\delta)^{\alpha+\beta}$$

$$\le C\,8^{\alpha+\beta}\,H^{2(\alpha+\beta)}\,\delta^{-(\alpha+\beta)}M_\alpha^2M_\beta^2,$$

which imply that (2.4) holds for $\ell = H\sqrt{8/\delta}$.

Applying the analogous reason as in (2.17) one can prove the last part of the theorem. □

Theorem 2.4 *1. The spaces $\mathcal{S}^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ are ($F\bar{S}$)-spaces $\mathcal{S}^{\{M_p\}}$ and $\mathcal{S}'^{(M_p)}$ are (LS)-spaces.*

2. If (M.2) is fulfilled $\mathcal{S}^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ are (FN)-spaces $\mathcal{S}^{\{M_p\}}$ and $\mathcal{S}'^{(M_p)}$ are (LN)-spaces respectively.

3. If (M.2)' is fulfilled then

$$\mathcal{D}^* \hookrightarrow \mathcal{S}^* \hookrightarrow \mathcal{E}^*, \quad \mathcal{S}^* \hookrightarrow \mathcal{S}.$$

$$\mathcal{E}'^* \hookrightarrow \mathcal{S}'^* \hookrightarrow \mathcal{D}'^*, \quad \mathcal{S}' \hookrightarrow \mathcal{S}'^*,$$

where "$A \hookrightarrow B$" means that the inclusion mapping of the space A into the space B is continuous and that A is dense in B.

Proof: 1. We will prove that $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are (F$\bar{\text{S}}$) and (LS) spaces respectively. Since the dual an (F$\bar{\text{S}}$)-space is an (LS)-space and vice versa, the rest of the assertion will follow. In order to prove 1., we will prove that the inclusion mapping

$$i : \mathcal{S}_2^{M_p,\bar{m}} \longrightarrow \mathcal{S}_2^{M_p,m}, \quad m < \bar{m},$$

is compact. Since $\mathcal{S}_2^{M_p,\bar{m}}$ and $\mathcal{S}_2^{M_p,m}$ are Banach spaces, it is enough to prove that the unit ball B of the space $\mathcal{S}_2^{M_p,\bar{m}}$ is a relatively compact set in $\mathcal{S}_2^{M_p,m}$. Using the analogous idea as in the proof of [66, p.29, Satz 1] one can prove the next assertion.

A set B is relatively compact in $\mathcal{S}_2^{M_p,m}$ if and only if

(i) *for each α , $\beta \in \mathrm{N}_0$ the set $B^\alpha_\beta = \{\langle x\rangle^\beta \varphi^{(\alpha)}, \varphi \in B\}$ is a relatively compact set in L^2, and*

(ii) *the sum $\displaystyle\sum_{\alpha,\beta\in \mathrm{N}_0} \int_{\mathrm{R}} |\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2 dx$ converges uniformly for all $\varphi \in B$.*

Let us prove that B fulfills (i) by checking whether B^α_β, $\alpha, \beta \in \mathrm{N}_0$, fulfills the assumptions of Kolmogoroff's Theorem ([19]). It is obvious that for each $\alpha, \beta \in \mathrm{N}_0$ the set $B^\beta_\alpha = \{\langle x\rangle^\beta \varphi^{(\alpha)}, \varphi \in B\}$ is bounded in the space L^2.

Applying the Hölder inequality and the Fubini-Tonelli theorem we get that for $\varphi \in B$ and $\alpha, \beta \in \mathrm{N}_0$

$$\int_{\mathrm{R}} |\langle x+h\rangle^\beta \varphi^{(\alpha)}(x+h) - \langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2 dx$$

$$\leq \int_{\mathrm{R}} \left(\int_0^1 |\frac{d}{dt}(\langle x+th\rangle^\beta \varphi^{(\alpha)}(x+th))|\, dt \right)^2 dx$$

$$\leq \int_{\mathrm{R}} \left(\int_0^1 |\frac{d}{dt}(\langle x+th\rangle^\beta \varphi^{(\alpha)}(x+th))|^2 dt \right) dx$$

$$\leq \beta^2 h^2 \int_0^1 \left(\int_{\mathrm{R}} |\langle x+th\rangle^\beta \varphi^{(\alpha)}(x+th))|^2 dx \right) dt$$
$$+ h^2 \int_0^1 (\int_{\mathrm{R}} |\langle x+th\rangle^\beta \varphi^{(\alpha+1)}(x+th))|^2 dx)\, dt$$

$$\leq \beta^2 h^2 \left(\int_{\mathrm{R}} |\langle \xi\rangle^\beta \varphi^{(\alpha)}(\xi))|^2 d\xi \right) + h^2 \left(\int_{\mathrm{R}} |\langle \xi\rangle^\beta \varphi^{(\alpha+1)}(\xi))|^2 d\xi \right)$$

$$\leq h^2 \left(\beta^2 \frac{M_\alpha M_\beta}{\bar{m}^{\alpha+\beta}} + \frac{M_{\alpha+1} M_\beta}{\bar{m}^{\alpha+\beta+1}} \right).$$

Hence, $\int_{\mathrm{R}} |\langle x+h\rangle^\beta \varphi^{(\alpha)}(x+h) - \langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2 dx$ converges to zero uniformly for $\varphi \in B$ as h tends to zero.

For each $\varphi \in B$ and $k > 0$

$$\langle k\rangle^2 \int_{\mathrm{R}\setminus[-k,k]} |\langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2 dx \leq \int_{\mathrm{R}\setminus[-k,k]} |\langle x\rangle^{\beta+1} \varphi^{(\alpha)}(x)|^2 dx \leq \frac{M_\alpha M_{\beta+1}}{\bar{m}^{\alpha+\beta+1}}.$$

Therefore,

$$\int_{\mathrm{R}\setminus[-k,k]} |\langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2 dx \leq \langle k\rangle^{-2} \frac{M_\alpha M_{\beta+1}}{\bar{m}^{\alpha+\beta+1}}, \quad \varphi \in B.$$

According to the theorem of Kolmogoroff, it follows that the set B_α^β, $\alpha, \beta \in \mathrm{N}_0$, is relative compact in L^2.

Let us prove that B fulfills condition (ii). For each $\varepsilon > 0$ there exists $\mu \in \mathbf{N}_0$ such that $m^\alpha \leq \varepsilon \tilde{m}^\alpha$ for all $\alpha \geq \mu$. Hence, for each $\varphi \in B$,

$$\sum_{\substack{\alpha \geq \mu \\ \beta \in \mathbf{N}_0}} \int_{\mathbf{R}} \left|\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\langle x\rangle^\beta \varphi^{(\alpha)}(x)\right|^2 dx$$

$$\leq \varepsilon^2 \sum_{\substack{\alpha \geq \mu \\ \beta \in \mathbf{N}_0}} \int_{\mathbf{R}} \left|\frac{\tilde{m}^{\alpha+\beta}}{M_\alpha M_\beta}\langle x\rangle^\beta \varphi^{(\alpha)}(x)\right|^2 dx \leq \varepsilon^2.$$

2. If (M.2) is fulfilled $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) is isomorphic to the space of projective (resp. inductive) limit of Köthe space $\ell^2(b_k)$ (resp. $\ell^2(c_k)$) (see [19]), where

$$b_k = (b_{1,k}, b_{2,k}, \ldots), \quad b_{n,k} = \exp[M(k\sqrt{2n+1})],$$

$$(\textit{resp. } c_k = (c_{1,k}, c_{2,k}, \ldots), \quad c_{n,k} = \exp[M((1/k)\sqrt{2n+1})]), \qquad n, k \in \mathbf{N},$$

respectively. The isomorphism is given by

$$\varphi \mapsto (a_n) \quad \text{where} \quad \varphi = \sum_{n=0}^{\infty} a_n h_n \quad (\text{ see Theorem 2.3 3.}).$$

In order to prove the assertion it is enough to prove that for some $\ell > k$,

$$\sum_{n \in \mathbf{N}_0} b_{n,k}/b_{n,\ell} < \infty \quad (\text{resp.} \quad \sum_{n \in \mathbf{N}_0} c_{n,k}/c_{n,\ell} < \infty) \text{ (see [19, p.112, 4.3.])}.$$

The inequalities

$$M(k\rho) + M(\rho) \leq 2M((k+1)\rho), \quad \rho > 0$$

$$2M(\rho) \leq M(H\rho) + \log A, \quad \rho > 0, \quad ([33, \text{Proposition } 3.6.]),$$

imply that for $\ell > H(k+1)$,

$$\sum_{n \in \mathbf{N}_0} \frac{b_{n,k}}{b_{n,\ell}} \leq \sum_{n \in \mathbf{N}_0} \exp[-M(\sqrt{2n+1})] < \infty.$$

3. Since the proofs of the assertion in the cases $* = (M_p)$ and $* = \{M_p\}$ are analogous we will prove the assertion in the first case. Let $\varphi \in \mathcal{D}^{(M_p)}$ and

$supp\varphi \subset [-k,k]$, $k > 1$. The condition (M.3)' implies that for each $m > 0$ there exists C, such that

$$\sup_{\alpha,\beta\in\mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\alpha} \|\langle x\rangle^\beta \varphi^{(\alpha)}\|_\infty = \sup_{\alpha,\beta\in\mathrm{N}_0} \frac{(mk)^\beta m^\alpha}{M_\beta M_\alpha} \|\varphi^{(\alpha)}\|_\infty$$

$$\leq C \sup_{\alpha\in\mathrm{N}_0} \frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)}\|_\infty.$$

It follows that the inclusion mapping $i : \mathcal{D}^{(M_p)} \longrightarrow \mathcal{S}^{(M_p)}$ is continuous.

The sequence $(\varphi_j)_j$, where $\varphi_j(x) = \rho(x/j)\rho(x)$ and ρ is a function defined by (2.1) converges to φ in the space $\mathcal{S}^{(M_p)}$, since for fixed $\varphi \in \mathcal{S}^{(M_p)}$ and $m > 0$, $\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|x^\beta \varphi^{(\alpha)}(x)|$ converges uniformly in $\alpha, \beta \in \mathrm{N}_0$ as $|x|$ tends to infinity (see the proof of Theorem 2.3.). It follows that $\mathcal{D}^{(M_p)}$ is dense in $\mathcal{S}^{(M_p)}$. □

It follows from the above that the space $\mathcal{S}^*$ and its dual space $\mathcal{S}'^*$ are complete, bornologic Montel spaces, that $\mathcal{S}^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ are Freche spaces and if (M.2) is fulfilled $\mathcal{S}^*$ and $\mathcal{S}'^*$ are nuclear and separable. (see [19] and [52]).

Let us compare spaces $\mathcal{S}'^{\{M_p\}}$, $(M_p)_{p\in\mathcal{M}}$, of tempered distributions of Roumieu type, which are defined in the thesis, and the spaces $\mathcal{S}'_\omega$ of ω-tempered distributions, which are investigated by Björk and Gruzdinski. Applying [14, Theorem 1.8] one can easily conclude that foe each $\omega \in \mathcal{A}$ there exists a sequence $(M_p) \in \mathcal{M}$, such that its associated function satisfy $\omega(\rho) \leq M(\rho)$, $\rho > 0$. This implies that $\mathcal{S}_\omega \subset \mathcal{S}^{\{M_p\}}$ and by the closed graph theorem the inclusion is continuous. If we suppose that ω is weighted function in Braun-Meise-Taylor sense not only that there exists a sequence $(M_p) \in \mathcal{M}$, such that $\mathcal{S}_\omega \subset \mathcal{S}^{\{M_p\}}$ and that the inclusion is continuous, but also that for sequence $(M_p) \in \mathcal{M}$ which satisfy (M.2), and there exists $k \in \mathrm{N}$, such that $\liminf_{j\to\infty} m_{jk}/m_j > 1$, its associated function M is equivalent to an element of $\mathcal{W}$, hence $\mathcal{S}^{\{M_p\}} = \mathcal{S}_M$.

2.2 Structural Properties

Theorem 2.5 *Let $r \in (1,\infty]$ and $f \in \mathcal{D}'^{(M_p)}$ (resp. $f \in \mathcal{D}'^{\{M_p\}}$) .*

1. (FIRST STRUCTURAL THEOREM) *$f \in \mathcal{S}'^{(M_p)}$ (resp. $f \in \mathcal{S}'^{\{M_p\}}$) if and only if f is of the form*

$$f = \sum_{\alpha,\beta\in\mathbf{N}_0} (\langle x\rangle^{\beta} F_{\alpha,\beta})^{(\alpha)}, \tag{2.18}$$

in the sense of convergence in $\mathcal{S}'^{(M_p)}$ (resp. $\mathcal{S}'^{\{M_p\}}$), where $(F_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}_0}$ is a sequence of elements from L^r, such that for some (resp. each) $m > 0$,

$$\begin{cases} \left(\displaystyle\sum_{\alpha,\beta\in\mathbf{N}_0} \int_{\mathbf{R}} \left|\frac{M_\alpha M_\beta}{m^{\alpha+\beta}} F_{\alpha,\beta}(x)\right|^r\right)^{1/r} < \infty, & r \in (1,\infty), \\ \displaystyle\sup_{\substack{\alpha,\beta\in\mathbf{N}_0\\ x\in\mathbf{R}}} \left(\frac{M_\alpha M_\beta}{m^{\alpha+\beta}} |F_{\alpha,\beta}(x)|\right) < \infty, & r = \infty. \end{cases} \tag{2.19}$$

2. (SECOND STRUCTURAL THEOREM [50]) *Let (M.2) and (M.3) be fulfilled. $f \in \mathcal{S}'^*$ if and only if f is of the form*

$$f = P(D)F, \tag{2.20}$$

where P is an ultradifferentiable operator of class $*$, and F is a continuous function on $\mathbf{R}$ of ultrapolynomial growth of class $*$.

3. (HERMIT EXPANSION) *Let (M.2) be fulfilled. $f \in \mathcal{S}'^{(M_p)}$(resp. $f \in \mathcal{S}'^{\{M_p\}}$) if and only if in $\mathcal{S}'^{(M_p)}$ (resp. $\mathcal{S}'^{\{M_p\}}$)*

$$f(x) = \sum_{n\in\mathbf{N}_0} a_n h_n(x),\ x \in \mathbf{R}, \tag{2.21}$$

and for some (resp. each) $\delta > 0$

$$\sum_{n\in\mathbf{N}_0} |\, a_n\,|^2 \exp[-2M(\delta\sqrt{2n+1})] < \infty. \tag{2.22}$$

Note, the second structural theorem is proved by Pilipović in [50].

Proof: 1. (case (M_p)) The proof of the assertion 1 in the case (M_p) is analogous to the proof of [45, Theorem 5.2.]. It follows easily that (2.18) determines an element of the space $\mathcal{S}^{(M_p)}$ let us prove the converse. Let $q = r/(r-1)$. Note, $q \in [1, \infty)$. Since $\mathcal{S}'^{(M_p)}$ is a strict $(F\bar{S})$-space, we have

$$\mathcal{S}'^{(M_p)} = \text{ind}\lim_{m\to\infty}\left(\overline{\mathcal{S}_q^{M_p,m}}\right)',$$

in the sense of strong topologies, where $\overline{\mathcal{S}_q^{M_p,m}}$ is the closure of $\mathcal{S}^{(M_p)}$ in the space $\mathcal{S}_q^{M_p,m}$, with the topology induced by the space $\mathcal{S}_q^{M_p,m}$.

If $f \in \mathcal{S}'^{(M_p)}$, there exists $m > 0$, such that f has a continuous linear extension on $\overline{\mathcal{S}_q^{M_p,m}}$. The Hahn-Banach theorem implies that f has a continuous, linear extension on $\mathcal{S}_q^{M_p,m}$ with the same dual norm. We denote this extension again by f. Let $T_p(m)$ be the space of sequences $(\psi_{\alpha,\beta})_{\alpha,\beta\in\mathrm{N}_0}$ from $L^r(\mathrm{R})$ equipped with the norm

$$\|(\psi_{\alpha,\beta})_{\alpha,\beta}\| = \left(\sum_{\alpha,\beta\in\mathrm{N}_0}\int_{\mathrm{R}}|\frac{m^{\alpha+\beta}}{M_\beta M_\alpha}\psi_{\alpha,\beta}|^q dx\right)^{1/q} < \infty.$$

The mapping

$$i : \mathcal{S}_q^{M_p m} \to T_q(m) \quad i : \varphi \mapsto ((-1)^\alpha \langle x\rangle^\beta \varphi^{(\alpha)})_{\alpha,\beta}$$

is an isometry of $\mathcal{S}_q^{M_p m}$ onto $G_p(m) = i\left(\mathcal{S}_q^{M_p m}\right) \subset T_q(m)$. We define a continuous linear functional $\tilde{f}$ on $G_q(m)$ by

$$\langle \tilde{f}, (\psi_{\alpha,\beta})_{\alpha,\beta}\rangle = \langle f, i^{-1}((\psi_{\alpha,\beta})_{\alpha,\beta})\rangle,\ (\psi_{\alpha,\beta})_{\alpha,\beta} \in G_q(m).$$

Again by the Hahn-Banach theorem we extended $\tilde{f}$ linearly and continuously on $T_q(m)$ with the same dual norm, and denote this extension by F.

It is known (see [66, p.29, Hilfsatz 2.]) that the fact $F \in (T_q(m))'$ implies the existence of a sequence $(F_{\alpha,\beta})_{\alpha,\beta\in\mathrm{N}_0}$ from L^r such that F has a form

$$\langle F, (\psi_{\alpha,\beta})_{\alpha,\beta}\rangle = \sum_{\alpha,\beta\in\mathrm{N}_0}\int_{\mathrm{R}} F_{\alpha,\beta}(x)\psi_{\alpha,\beta}(x)dx,\ ((\psi_{\alpha,\beta})_{\alpha,\beta}) \in T_q(m),$$

and the norm of F is given by

$$\|F\| = \begin{cases} \left(\sum_{\alpha,\beta\in\mathbf{N}_0} \int_{\mathbf{R}} \left| \frac{M_\alpha M_\beta}{m^{\alpha+\beta}} F_{\alpha,\beta}(x) \right|^r \right)^{1/r} < \infty, & r \in (1,\infty), \\ \sup_{\substack{\alpha,\beta\in\mathbf{N}_0 \\ x\in\mathbf{R}}} \frac{M_\alpha M_\beta}{m^{\alpha+\beta}} |F_{\alpha,\beta}(x)| < \infty, & r = \infty. \end{cases}$$

Thus $\|F\| = \|f\| < \infty$ and for each $\varphi \in \mathcal{S}^{(M_p)}$ we have,

$$\langle f,\varphi\rangle = \langle \tilde{f}, ((-1)^\alpha \langle x\rangle^\beta \varphi^{(\alpha)}))_{\alpha,\beta}\rangle \ = \ \langle F, ((-1)^\alpha \langle x\rangle^\beta \varphi^{(\alpha)})_{\alpha,\beta}\rangle$$

$$= \sum_{\alpha,\beta\in\mathbf{N}_0} (-1)^\alpha \int_{\mathbf{R}} F_{\alpha,\beta}(x)\langle x\rangle^\beta \varphi^{(\alpha)}(x)dx = \sum_{\alpha,\beta\in\mathbf{N}_0} \langle ((\langle x\rangle^\alpha \ F_{\alpha,\beta})^{(\beta)}, \varphi\rangle,$$

which implies 1. in the case (M_p).

1. (case $\{M_p\}$) It follows easily that (2.18) determines an element of $\mathcal{S}'^{\{M_p\}}$. To prove the converse we will use the dual Mittag-Leffler lemma ([33, Lemma 1.4]) similarly as in the proof of [33, Proposition 8.6].

Let $X_m = \mathcal{S}_q^{M_p,m}$ and let $Y_m = \{(\varphi_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}_0}, \|\varphi\|_{Y_m} < \infty\}$, where $q = r/(r-1)$, and

$$\|\varphi\|_{Y_m} = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\varphi_{\alpha,\beta}\|_p.$$

The space Y_m is reflexive Banach space. According to Banach-Alaoglu's theorem ([54]), bounded set in Y_m is weakly compact Y_m. Therefore the inclusion mapping $i : Y_{m'} \to Y_m$, $m' > m$ is weakly compact. We will identify X_m with a closed subspace of Y_m in which is X_m is mapped by the mapping

$$X_m \to Y_m, \ \langle x\rangle^\beta D^\alpha : \varphi \mapsto (\langle x\rangle^\beta \varphi^{(\alpha)})_{\alpha,\beta}.$$

Clearly (X_m) and (Y_m) are injective sequences of Banach spaces and if $m' > m$ than $X_{m'} \cap Y_m = X_m$. It follows that the quotient space $Z_m = Y_m/X_m$ (with the quotient topology) is also an injective weakly compact sequence of Banach spaces. It follows from the dual Mittag-Leffler lemma that

$$0 \leftarrow \lim\operatorname{proj}_{m\to 0} X'_m \overset{\sum(-1)^\alpha D^\alpha \langle x\rangle^\beta}{\longleftarrow} \lim\operatorname{proj}_{m\to 0} Y'_m$$

is topologically exact (see [33]). The above and the facts: $\lim\text{proj}_{m\to 0}X'_m$ $=(\lim\text{ind}_{m\to 0}X_m)'$ and $\lim\text{proj}_{m\to 0}Y'_m$ $=(\lim\text{ind}_{m\to 0}Y_m)'$ imply that the space $\lim\text{ind}_{m\to 0}X_m$ has the same strong dual as $\lim\text{ind}_{m\to 0}X_m$, which is a closed subspace of $\lim\text{ind}_{m\to 0}Y_m$. Since Y'_m is the Banach space of all $F=(F_{\alpha,\beta})$, $F_{\alpha,\beta}\in L^r$, such that

$$\|f\|_{Y'_m} = \begin{cases} \left(\sum_{\alpha,\beta\in\mathbf{N}_0}\int_{\mathbf{R}} |\frac{M_\alpha M_\beta}{m^{\alpha+\beta}}F_{\alpha,\beta}(x)|^r\right)^{1/r}, & r\in(1,\infty), \\ \sup_{\substack{\alpha,\beta\in\mathbf{N}_0 \\ x\in\mathbf{R}}}\left(\frac{M_\alpha M_\beta}{m^{\alpha+\beta}}|F_{\alpha,\beta}(x)|\right), & r=\infty. \end{cases}$$

The assertion is proved.

3. Note $h_n\in\mathcal{S}^*$. Clearly, if f is of the form (2.21) and (2.22) holds, then it belongs to $\mathcal{S}^*$, and $a_n=\langle f,h_n\rangle$, $n\in\mathbf{N}_0$.

Assume that $f\in\mathcal{S}'^*$. Let $a_n=\langle f,h_n\rangle$, $n\in\mathbf{N}_0$. For each $\varphi\overset{L^2}{=}\sum_n b_nh_n$, which is element of $\mathcal{S}^*$ we have,

$$\langle f,\varphi\rangle=\sum_{n\in\mathbf{N}_0}a_n\bar{b}_n.$$

From the theory of Köthe space and the fact that for some $\delta>0$ (resp. for every $\delta>0$), $\sum|b_n|^2\exp[2M(\delta\sqrt(2n+1)]<\infty$, it follows that (2.22) holds for the sequence (a_n).

Put $f_n=\sum_{k\le n}a_kh_k$. One can easily prove that for every $\varphi\in\mathcal{S}'^*$ the sequence $(\langle f-f_n,\varphi\rangle)$ tends to zero when $n\to\infty$. □

2.3 Boundary value representation

Theorem 2.6 *Let (M.2) and (M.3) be fulfilled and let f be a continuous ultradifferentiable function of class $*$ of ultrapolynomial growth of class $*$ with Hermite series expansion $\sum_n a_nh_n$.*

1. *The sum $\sum_n(ia_n/(2\pi))\tilde{h}_n(\zeta)$ (see 2.) converges uniformly in compact subsets of eider the upper or lower open half plane.*

2. *The sum* $\sum_n (ia_n/(2\pi))(\tilde{h}_n(\zeta) - \tilde{h}_n(\bar{\zeta}))$ *converges in the upper half plane to a real harmonic function* $u(\zeta)$.

3. *The function* $u(\xi + i\eta)$ *converges to* $f(\xi)$ *as* $\eta \to 0^+$ *uniformly on compact subsets of* R.

Proof: 0. Let us first prove next assertion. *If (M.2)' is fulfilled,* $K \subset \mathbf{R}$ *is a compact set, such that* $K \subset (-\infty, 0)$ *or* $(0, \infty)$ *and* $\zeta \in \mathbf{R} + iK$, *then the function*

$$\varphi_\zeta : \mathbf{R} \to \mathbf{C}, \ \ t \mapsto \varphi_\zeta(t) = \frac{\exp(-t^2/2)}{2\pi i(t - \zeta)}, \tag{2.23}$$

is an element of $\mathcal{S}^*$, *and the family* $\{\varphi_\zeta, \zeta \in \mathbf{R} + iK\}$ *is uniformly bounded in* $\mathcal{S}^*$.

In [47] is proved that

$$|t^\beta \varphi_\zeta^{(\alpha)}(t)| \le C\beta^{\beta/2}\alpha! \sum_{\gamma=0}^{\alpha} |\eta|^{-\alpha+\gamma-1}, \quad \zeta = \xi + i\eta \in \mathbf{R} + iK.$$

Applying the Stirling formula, the fact that

$$\sup_{\eta \in K} \left(\sum_{\gamma=0}^{\alpha} |\eta|^{-\alpha+\gamma-1} \right) \le (\alpha + 1) \sup_{0 \le \gamma \le \alpha} \left(\inf_{\eta \in K} |\eta| \right)^{-\alpha+\gamma-1},$$

and the conditions (M.2)' and (M.3)' we get that there is C which depends on K, such that

$$|t^\beta \varphi_\zeta^{(\alpha)}(t)| \le C \frac{m^\alpha H^\alpha (\alpha+1)!\, m^\beta e^\beta \beta!}{M_{\alpha+1} M_\beta} \frac{M_\alpha M_\beta}{m^\alpha m^\beta} \le C \frac{M_\alpha M_\beta}{m^\alpha m^\beta}.$$

This implies the assertion.

1. Let K be a compact subset of $(0, \infty)$ or $(-\infty, 0)$ and let B be bounded subset of R. Since

$$\tilde{h}_n(\zeta) h_m(\zeta) = -\int_{\mathbf{R}} \frac{h_n(t) h_m(t)}{t - \zeta}\, dt, \ \ \mathrm{Im}\zeta \ne 0, \ m \le n,$$

we have for $\zeta \in B + iK$

$$\pi^{-1/4} \exp(-\zeta^2/2) \tilde{h}_n(\zeta) = \int_{\mathbf{R}} \frac{h_n(t) \exp(-t^2/2)}{\zeta - t} dt = \langle \varphi_\zeta(t), h_n(t) \rangle,$$

where φ_ζ denotes the function defined by (2.23). Therefore,

$$\sum_{n\in\mathbf{N}_0}\left|\frac{ia_n}{2\pi}\tilde{h}_n(\zeta)\right| \le |\exp(\zeta^2/2)| \sum_{n\in\mathbf{N}_0}|a_n|\,|\langle\varphi_\zeta,h_n\rangle|$$

$$\le |\exp(\zeta^2/2)| \sum_{n\in\mathbf{N}_0}|a_n|^2\exp[2M(\delta\sqrt{2n+1})]\cdot$$

$$\cdot\sum_{n\in\mathbf{N}_0}|\langle\varphi_\zeta,h_n\rangle|^2\exp[-2M(\delta\sqrt{2n+1})] < \infty, \quad \zeta\in B+iK.$$

2. The convergence of the given series follows from 1.. The limit function is harmonic, since for $\zeta\in\mathbf{C}_+$,

$$\sum_{n\in\mathbf{N}_0}(ic_n/(2\pi))(\tilde{h}_n(\zeta)-\tilde{h}_n(\bar{\zeta}))$$

$$=\pi^{1/4}\sum_{n\in\mathbf{N}_0}\frac{ia_n}{2\pi}\int_{\mathbf{R}}h_n(t)\exp(-t^2/2)\left(\frac{\exp(\zeta^2/2)}{\zeta-t}-\frac{\exp(\bar{\zeta}^2/2)}{\bar{\zeta}-t}\right)dt,$$

the expression in the brackets is purely imaginary and $\sum_n(ic_n/(2\pi))(\tilde{h}_n(\zeta)-\tilde{h}_n(\bar{\zeta}))$ is real.

3. For $\zeta\in\mathbf{C}_+$ and $n\in\mathbf{N}_0$

$$\sum_{n\in\mathbf{N}_0}\frac{ic_n}{2\pi}(\tilde{h}_n(\zeta)-\tilde{h}_n(\bar{\zeta}))$$

$$=\Big(\exp(-\bar{\zeta}^2/2)-\exp(\bar{\zeta}^2/2)\Big)\sum_{n\in\mathbf{N}_0}\frac{ic_n}{2\pi}\int_{\mathbf{R}}\frac{h_n(t)e^{-t^2/2}}{\bar{\zeta}-t}\,dt-$$

$$-\exp(\zeta^2/2)\sum_{n\in\mathbf{N}_0}\frac{ic_n}{2\pi}h_n(t)\exp(-t^2/2)\left(\frac{1}{\zeta-t}-\frac{1}{\bar{\zeta}-t}\right)dt.$$

Using the assertion proved in part 0. of this proof, the fact that f is a regular element of $\mathcal{S}'^*$ and that $f=\sum c_nh_n$ in the weak sense in $\mathcal{S}'^*$, we get

$$\sum_{n\in\mathbf{N}_0}\frac{ic_n}{2\pi}h_n(t)\exp(-t^2/2)\left(\frac{1}{\zeta-t}-\frac{1}{\bar{\zeta}-t}\right)dt$$

$$=\sum_{n\in\mathbf{N}_0}\frac{ic_n}{2\pi}h_n(t)\Big(\varphi_\zeta(t)-\varphi_{\bar{\zeta}}(t)\Big)\,dt$$

$$= \sum_{n\in\mathbb{N}_0} \frac{ic_n}{2\pi}\langle h_n, \varphi_\zeta - \varphi_{\bar\zeta}\rangle = \frac{i}{2\pi}\langle f, \varphi_\zeta - \varphi_{\bar\zeta}\rangle$$

$$= \frac{i}{2\pi}\int_{\mathbb{R}} f(t)\exp(-t^2/2)\left(\frac{1}{\zeta - t} - \frac{1}{\bar\zeta - t}\right) dt.$$

Therefore for $\zeta = \xi + i\eta$,

$$\lim_{\eta\to 0^+}\left(-\exp(\zeta^2/2)\sum_{n\in\mathbb{N}_0}\frac{ic_n}{2\pi}\int_{\mathbb{R}} h_n(t)\exp(-t^2/2)\left(\frac{1}{\zeta - t} - \frac{1}{\bar\zeta - t}\right) dt\right)$$

$$= \exp(\xi^2/2)\frac{1}{2\pi i}\lim_{\eta\to 0^+}\int_{\mathbb{R}} f(t)\exp(-t^2/2)\left(\frac{1}{\zeta - t} - \frac{1}{\bar\zeta - t}\right) dt.$$

Since the right-hand side of the above equality is the Poisson integral representation of the function $f\exp(-t^2/2)$ at $\xi \in \mathbb{R}$, it follows that

$$\lim_{\eta\to 0^+}\left(-\exp(\zeta^2/2)\sum_{n\in\mathbb{N}_0}\frac{ic_n}{2\pi}\int_{\mathbb{R}} h_n(t)\exp(-t^2/2)\left(\frac{1}{\zeta - t} - \frac{1}{\bar\zeta - t}\right) dt\right) = f(t)$$

and the convergence is uniform on K.

Since f is a regular element of $\mathcal{S}'^*$, we have

$$\int_{\mathbb{R}} f(t)\frac{e^{-t^2/2}}{t - \bar\zeta}\, dt = \sum_{n\in\mathbb{N}_0} c_n \int_{\mathbb{R}} h_n(t)\frac{e^{-t^2/2}}{t - \bar\zeta}\, dt.$$

The integral on the left-hand side is convergent for almost all ξ as $\eta \to 0^+$ (see [60, Theorem 105]) and thus for almost all ξ, we have

$$\lim_{\eta\to 0^+}(\exp(\bar\zeta^2/2) - \exp(\zeta^2/2)\sum_{n\in\mathbb{N}_0}\frac{ic_n}{2\pi}\int_{\mathbb{R}}\frac{h_n(t)\exp(-t^2/2)}{t - \bar\zeta}dt = 0.$$

Since $\exp(i\eta\xi) - \exp(-i\eta\xi) \to 0$, as $\eta \to 0^+$, uniformly on K. In order to prove that the above convergence is uniform on K it is enough to prove that the integral $\int_{\mathbb{R}} f(t)e^{-t^2/2}(t - \bar\zeta)^{-1}\, dt$ is uniformly bounded for $\xi \in K$, $\eta \in (0, \varepsilon)$. Let $F = f\exp(-x^2/2)$ and $H = h\exp(-x^2/2)$, where h is a smooth function such that in the case $* = (M_p)$ for some $L > 0$ and C and in the case $* = \{M_p\}$ for each L there exists C, such that

$$|f(x)| \le h(x) \le \sup(|h(x)|, |h'(x)|) \le C\exp[M(L|x|)], \quad x \in \mathbb{R}.$$

The function $h = (\omega * \tilde{h})$, where $\omega \in \mathcal{D}^*$ is such that $\omega \geq 0$, supp$\omega \subset [-1,1]$, $\int_{[-1,1]} \omega(t)dt = 1$ and $\tilde{h}(x) = \sup_{|u|\leq|x|+2} |f(u)|$ fulfills the above condition. Let us prove that. For almost all $x \in \mathbf{R}$

$$h(x) = \int_{-1}^{1} \tilde{h}(x-t)\omega(t)dt \geq \max\left(\tilde{h}(|x|-1), \tilde{h}(|x|)\right) \geq |f(x)|.$$

$L_1 = 40L$. Since f is of ultrapolynomial growth of class $*$

$$h(x) \leq h(|x|+1) \leq C \sup_{|t|\leq|x|+3} \exp[M(L|x|)] \leq C\exp[M(L_1|x|)],\ x \in \mathbf{R}.$$

Since

$$|h'(x)| \leq \tilde{h}(|x|+1)\int_{\mathbf{R}} |\omega'(t)|dt,\ \ x \in \mathbf{R},$$

by the same argument we finish the proof of the assertion.

From the estimation

$$\left|\int_{\mathbf{R}} \frac{F(t)}{t-\zeta}dt\right| \leq \left|\int_{\mathbf{R}} \frac{tF(t+\xi)}{t^2+\eta^2}dt\right| + \sup_{t\in\mathbf{R}}|F(t)|\,\eta\int_{\mathbf{R}} \frac{dt}{(t-\xi)^2+\eta^2},$$

it follows that we have to prove only that the first integral on the right-hand side of the above inequality is uniformly bounded for $\xi \in K$ and $\eta \in (0,\varepsilon)$. For each $\xi \in K$, we have

$$\left|\int_{\mathbf{R}} \frac{tF(t+\xi)}{t^2+\eta^2}dt\right| \leq \int_{\mathbf{R}} \frac{tH(t+\xi)}{t^2+\eta^2}dt$$

$$\leq \int_0^{\infty} \frac{t(H(\xi+t)-H(\xi-t))}{t^2+\eta^2}dt$$

$$\leq \int_0^{\infty}\left|\int_{-1}^{1}\frac{\partial}{\partial u}H(\xi+tu)du\right|dt \leq 2\int_{\mathbf{R}} H'(t)dt < \infty.\Box$$

One can analogously as in [47] deduce a boundary value representation of $\mathcal{S}'^*$.

Let us recall the following assertion.

Lemma 2.7 *[64, Lemma 1] If $t \geq 0$ and $n \in \mathbf{N}_0$,*

1. *$\tilde{h}_n(it)/\tilde{h}_n(0)$ is real, positive and monotonically decreasing;*

2. $\tilde{h}_n(0)/\tilde{h}_{n-1}(0)$ *is bounded.*

Theorem 2.8 *Let* $f = (\sum_n a_n h_n) \in \mathcal{S}'^*$ *and let*

$$u(\xi,\eta) = \sum_{n\in\mathbf{N}_0} a_n \frac{h_n(\xi)\tilde{h}_n(\eta)}{\tilde{h}_n(0)},\ v(\xi,\eta) = i\sum_{n\in\mathbf{N}_0} a_{n-1}\frac{h_n(\xi)\tilde{h}_n(\eta)}{\tilde{h}_{n-1}(0)},\ \xi+i\eta \in \mathbf{C}_+.$$

Than u and v are real valued smooth functions for (ξ,η) in the open upper half plane; $u(\cdot,\eta)$ converges to f in $\mathcal{S}'^*$ and $v(\cdot,\eta)$ converges to

$$\tilde{f} = i\sum_{n=1}^{\infty} a_{n-1}\frac{\tilde{h}_n(0)}{\tilde{h}_{n-1}(0)}h_n,$$

in $\mathcal{S}'^*$ as $\eta \to 0^+$.

Proof: Let $\varphi = \sum_n d_n h_n \in \mathcal{S}^*$ and $\eta \geq 0$ than

$$|\langle u(\cdot,\eta),\varphi\rangle - \langle f,\varphi\rangle| = \left|a_n\bar{d}_n(1-\tilde{h}_n(i\eta))\tilde{h}_n^{-1}(0))\right| \tag{2.24}$$

and

$$|\langle v(\cdot,\eta),\varphi\rangle - \langle \tilde{f},\varphi\rangle| = \left|\sum_{n\in\mathbf{N}_0} c_{n-1}\bar{d}_n\frac{\tilde{h}_n(i\eta)-\tilde{h}_n}{\tilde{h}_{n-1}(0)}\right|. \tag{2.25}$$

The fact that the series $(\sum c_n\bar{d}_n)$ and $(\sum c_{n-1}\bar{d}_n)$ converge and the properties of the hermit functions of the second kind, which are given in the previous lemma, imply that the right-hand side of (2.24) and (2.25) converge uniformly for all $\eta \geq 0$. We can therefore take the limit $\eta \to 0^+$ termwiese and conclude the desired result. □

Chapter 3

Elementary Operations

Elementary operations (translation, differentiation, ultradifferentiation and multiplication) on $\mathcal{S}^*$ and $\mathcal{S}'^*$ are investigated in this chapter. The space $\mathcal{O}_M^{(M_p)}$ of multipliers of the spaces $\mathcal{S}^*$ and $\mathcal{S}'^*$ is determined explicitly.

Let

$$P^{(M_p)}(x,D) = \textstyle\sum_{\mu,\nu\in\mathrm{N}_0}\ a_{\mu,\nu}\,(-1)^\nu D^\nu x^\mu,$$
$$\Big(\text{resp.}\ \ P^{\{M_p\}}(x,D) = \textstyle\sum_{\mu,\nu\in\mathrm{N}_0}\ a_{\mu,\nu}\,(-1)^\nu D^\nu x^\mu\Big),\quad x\in\mathrm{R}, \tag{3.1}$$

where $a_{\mu,\nu}$ are complex numbers such that there exist $L>0$ and $\mathcal{C}$ (resp. every $L>0$ there exists $\mathcal{C}$) and

$$|a_{\mu,\nu}| \le \mathcal{C}\frac{L^{\mu+\nu}}{M_\mu M_\nu},\ \ \mu,\nu\in\mathrm{N}_0. \tag{3.2}$$

The formal adjoint operator $(\sum_{\mu,\nu} a_{\mu,\nu}x^\nu D^\mu)$ of $P^*(x,D)$ will be denoted by $Q^*(x,D)$. Note, for each fixed $x\in\mathrm{R}$, $P^*(x,D)$ is an ultradifferencial operator of class $*$.

Theorem 3.1 *1. Let $h_0>0$. The family of translation operators*

$$\tau_h:\mathcal{S}^*\longrightarrow\mathcal{S}^*,\ \ \tau_h:\varphi(\cdot)\mapsto\varphi(\cdot-h),\ \ |h|\le h_0,$$

is uniformly continuous.

2. If (M.2)' is fulfilled the mappings

$$(-1)^{\nu}D^{\nu} : \mathcal{S}^* \longrightarrow \mathcal{S}^*, \;\; \varphi \mapsto (-1)^{\nu}D^{\nu}\varphi, \;\; \nu \in \mathrm{N}, \tag{3.3}$$

$$P^*(x,D) : \mathcal{S}^* \;\longrightarrow\; \mathcal{S}, \;\; \varphi \mapsto P^*(x,D)\varphi, \tag{3.4}$$

and their adjoint mappings

$$D^{\mu} : \mathcal{S}'^* \;\longrightarrow\; \mathcal{S}'^*, \quad \nu \in \mathrm{N}, \tag{3.5}$$

$$Q^*(x,D) : \; \mathcal{S}' \;\longrightarrow\; \mathcal{S}'^*, \tag{3.6}$$

are continuous. For each $f \in \mathcal{S}'$ *(tempered distribution) we have*

$$Q^*(x,D)f = \sum_{\mu,\nu \in \mathrm{N}_0} a_{\mu,\nu}\, x^{\mu} D^{\nu} f, \tag{3.7}$$

where the series on the right hand side converge absolutely in $\mathcal{S}'^*$.

3. If (M.2) is fulfilled the mapping

$$P^*(x,D) : \; \mathcal{S}^* \;\longrightarrow\; \mathcal{S}^*, \;\; \varphi \mapsto P^*(x,D)\varphi, \tag{3.8}$$

and its adjoint

$$Q^*(x,D) : \mathcal{S}'^* \;\longrightarrow\; \mathcal{S}'^*, \tag{3.9}$$

is continuous and for each $f \in \mathcal{S}'^*$ *and (3.7) holds.*

Proof: We will prove the theorem only in the case $* = (M_p)$, using the definition of the space $\mathcal{S}^{(M_p)}$ and Theorem 2.3. Analogously applying Theorem 2.2 and Theorem 2.3 the assertion can be proved in the case $* = \{M_p\}$.

1. If $m > 0$, $\varphi \in \mathcal{S}^*$ and $|h| \leq h_0$, we have

$$\sup_{\alpha,\beta \in \mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x\rangle^{\beta} (\tau_h \varphi)^{(\alpha)}\|_\infty \leq$$

$$\leq \sup_{\alpha,\beta \in \mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \sup_{x \in \mathrm{R}} |\langle x-h\rangle^{\beta} \varphi^{(\alpha)}(x)| \leq \sup_{\alpha,\beta \in \mathrm{N}_0} \frac{(2\langle h_0\rangle m)^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x\rangle^{\beta} \varphi^{(\alpha)}\|_\infty .$$

2. We will prove only the continuity of the mappings (3.4) and (3.6) since the proof of continuity of (3.3) and (3.5) is similar and simpler. Let us prove that (3.4) is a continuous mapping. Applying (3.2), (M.2)', (M.1), and (M.3)', we get that for $\varphi \in \mathcal{S}^*$ and $\alpha, \beta \in \mathrm{N}_0$,

$$\|x^\beta (P(x,D)\varphi)^{(\alpha)}\|_\infty \leq$$

$$\leq \mathcal{C} \sum_{\mu,\nu \in \mathrm{N}_0} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \frac{L^{\nu+\mu}}{M_\nu M_\mu}, \|((x^\beta)^{(k)} x^\mu \varphi)^{(\alpha+\nu-k)}\|_\infty \leq$$

$$\leq \mathcal{C} \sum_{\mu,\nu \in \mathrm{N}_0} \left(\sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \binom{\beta}{k} k! \cdot \right.$$

$$\left. \cdot \frac{H^{\alpha\nu} H^{\beta\mu} L^{\nu+\mu}}{M_{\nu+\alpha} M_{\mu+\beta}} \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty \right)$$

$$\leq \mathcal{C} \sum_{\mu,\nu \in \mathrm{N}_0} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \binom{\beta}{k} \frac{1}{4^{\mu+\nu}} \frac{((1+4L)(1+H^\alpha)(1+H^\beta))^{2k} k!}{M_k} \cdot$$

$$\cdot \frac{((1+4L)(1+H^\alpha)(1+H^\beta))^{\mu+\nu+\alpha+\beta-2k}}{M_{\nu+\alpha-k}\, M_{\mu+\beta-k}} \|(x^{\mu+\beta-k}\, \varphi)^{(\nu+\alpha-k)}\|_\infty$$

$$\leq \mathcal{C} \sup_{\vartheta,\eta} \frac{((1+4L)(1+H^\alpha)(1+H^\beta))^{\vartheta+\eta}}{M_\vartheta M_\eta} \|(x^\eta \varphi)^{(\vartheta)}\|_\infty .$$

This implies the continuity of (3.4).

Taking into account that the image of a bounded set under a continuous linear mapping is a bounded set, the continuity of *3.* Let us prove that (3.8) is a continuous mapping. Applying respectively (M.2), (M.1) and (M.3)' we get that for each $m > 0$ there exists $\mathcal{C}$ such that for each $\varphi \in \mathcal{S}^{(M_p)}$,

$$\sup_{\alpha,\beta \in \mathrm{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta (P^*(x,D)\varphi)^{(\alpha)}\|_\infty \leq$$

$$\leq \mathcal{C} \sup_{\alpha,\beta \in \mathrm{N}_0} \sum_{\mu,\nu \in \mathrm{N}_0} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \binom{\beta}{k} k! \frac{H^{\nu+\alpha} H^{\mu+\beta} m^{\beta+\alpha}}{M_{\nu+\alpha} M_{\mu+\beta}} L^{\mu+\nu}.$$

$$\cdot \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty \leq$$

$$\leq C \sup_{\alpha,\beta\in \mathbf{N}_0} \sum_{\mu,\nu\in\mathbf{N}_0} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \frac{1}{8^{\alpha+\beta+\mu+\nu}} \binom{\alpha+\nu}{k}\binom{\beta}{k} \cdot$$

$$\cdot \frac{k!(8mL(1+H))^{2k}}{M_k M_k} \frac{(8mL(1+H))^{\alpha+\beta+\mu+\nu-2k}}{M_{\nu+\alpha-k}M_{\mu+\beta-k}} \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty$$

$$\leq C \sup_{\alpha,\beta\in \mathbf{N}_0} \sum_{\mu,\nu\in\mathbf{N}_0} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \frac{1}{8^{\alpha+\beta+\mu+\nu}} \frac{(16mL(1+H))^{\alpha+\beta+\mu+\nu-2k}}{M_{\nu+\alpha-k}M_{\mu+\beta-k}} \cdot$$

$$\cdot \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty \leq$$

$$\leq C \sup_{\alpha,\beta\in \mathbf{N}_0} \frac{(16mL(1+H))^{\beta+\alpha}}{M_\beta M_\alpha} \|(x^\beta\varphi)^{(\alpha)}\|_\infty,$$

which imply the continuity of (3.8).

Suppose (M.2)' (resp. (M.2)). Let $f \in \mathcal{S}'$ (resp. $f \in \mathcal{S}'^*$). Since for each $\varphi \in \mathcal{S}^*$

$$\langle f, \sum_{\mu,\nu\leq n} a_{\mu,\nu}(-1)^\nu D^\nu x^\mu \varphi\rangle = \langle \sum_{\mu,\nu\leq n} a_{\mu,\nu} x^\mu D^\nu f, \varphi\rangle$$

converges to

$$\langle f, P(x,D)\varphi\rangle = \langle \sum_{\mu,\nu\leq n} a_{\mu,\nu} x^\mu D^\nu f, \varphi\rangle,$$

as $n \to \infty$, (3.7) is a continuous mapping.

Applying similar arguments as in the proof of the continuity of (3.6), one can prove the continuity of (3.9). □

3.1 The spaces of multipliers

Definition 3.2 *$\mathcal{O}_M^*$ is the space of all $\varphi \in \mathcal{E}^*$ such that for all $\psi \in \mathcal{S}^*$ the pointwiese product $\varphi\cdot\psi$ belongs to $\mathcal{S}^*$. The topology on $\mathcal{O}_M^*$ is coarsest topology such that for each $\phi \in \mathcal{S}^*$ the mapping $\mathcal{O}_M^* \to \mathcal{S}^*$ defined by $\psi \mapsto \psi\phi$ is continuous.*

The inclusion mappings $\mathcal{S}^* \longrightarrow \mathcal{O}_M^* \longrightarrow \mathcal{S}'^*$ are continuous. Moreover, $\mathcal{S}^*$ is dense in $\mathcal{O}_M^*$.

Theorem 3.3 *Let $\varphi \in \mathcal{E}^*$.*

1. *The condition*

   (a) *for all $\psi \in \mathcal{S}^{(M_p)}$ (resp. $\psi \in \mathcal{S}^{\{M_p\}}$), the pointwiese product $\varphi\psi$ belongs to $\mathcal{S}^{(M_p)}$(resp. $\mathcal{S}^{\{M_p\}}$) ;*

   implies the next one

   (b) *for every $m > 0$ there exist $\ell > 0$ and C (resp. for some $m > 0$ and every $\ell > 0$ there is C) such that*

$$\sup_{\alpha \in \mathrm{N}_0} \frac{m^\alpha}{M_\alpha} |\varphi^{(\alpha)}(x)| \leq C \sum_{\beta \in \mathrm{N}_0} \frac{\ell^\beta}{M_\beta} \langle x \rangle^\beta, \; x \in \mathrm{R}. \tag{3.10}$$

2. *If (M.2) is fulfilled the above conditions are equivalent.*

Proof: Let us assume that $\varphi \in \mathcal{E}^{(M_p)}$ (resp. $\varphi \in \mathcal{E}^{\{M_p\}}$), that (1a) is and (1b) is not fulfilled. For some (resp. for each) $m > 0$ there exists a sequence $(x_j)_j$ such that $|x_j|$ tends to infinity as $j \to \infty$, and

$$\sup_{\alpha \in \mathrm{N}_0} \frac{m^\alpha}{M_\alpha} |\varphi^{(\alpha)}(x_j)| > M_j \sum_{\beta \in \mathrm{N}_0} \frac{1}{M_\beta} \langle x_j \rangle^\beta. \tag{3.11}$$

Without the loss of generality we may suppose that $|x_j| + 2 \leq |x_{j+1}|$, $j \in \mathrm{N}$. Consider the function $\phi \in \mathcal{S}^{(M_p)}$ (resp. $\phi \in \mathcal{S}^{\{M_p\}}$), defined by (2.1). The conditions (M.1) and (M.3)' imply that for some (resp. for each) $m > 0$

$$\sup_{\alpha \in \mathrm{N}_0} \frac{m^\alpha}{M_\alpha} |(\phi\varphi)^{(\alpha)}(x_j)| = \sup_{\alpha \in \mathrm{N}_0} \frac{m^\alpha}{M_\alpha} \sum_{k=0}^{\alpha} \binom{\alpha}{k} |\frac{\rho^{(k)}(0)}{\langle x_j \rangle^j} \varphi^{(\alpha-k)}(x_j)|$$

$$= \sup_{\alpha \in \mathrm{N}_0} \frac{m^\alpha}{M_\alpha} | \frac{\rho(0)}{\langle x_j \rangle^j} \varphi^{(\alpha)}(x_j)| > M_j \sum_{\beta \geq j} \frac{1}{M_\beta} \langle x_j \rangle^{\beta - j}$$

$$\geq M_j \sum_{\beta \geq j} \frac{1}{M_{\beta-j} M_j} \langle x_j \rangle^{\beta-j} = \sum_{\beta \in \mathrm{N}_0} \frac{1}{M_\beta} \langle x_j \rangle^\beta > 1.$$

Hence, for some (resp. for each) $m > 0$, $(\sup_\alpha (m^\alpha / M_\alpha) |(\phi\varphi)^\alpha(x_j)|)$ does not converge to zero as $|x_j| \to \infty$, which is a contradiction (see the proof of Theorem 2.3, p. 17).

Let (1b) and (M.2) be fulfilled and let $\psi \in \mathcal{S}^{(M_p)}$ (resp. $\psi \in \mathcal{S}^{\{M_p\}}$). We will prove that $\varphi\psi \in \mathcal{S}^{(M_p)}$ (resp. $\varphi\psi \in \mathcal{S}^{\{M_p\}}$). The conditions (M.1), (1b) and (M.2) imply that for each $m > 0$ there exist $\ell > 1$ and C (resp. for some $m > 0$ and every $\ell > 0$ there is C), such that

$$\sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|\langle x\rangle^\beta(\psi\varphi)^{(\alpha)}\|_\infty \leq \sup_{\alpha,\beta\in\mathbf{N}_0} \sum_{k\leq\alpha} \binom{\alpha}{k} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x\rangle^\beta \psi^{(k)}\varphi^{(\alpha-k)}\|_\infty$$

$$\leq \sup_{\alpha,\beta\in\mathbf{N}_0} \sum_{k\leq\alpha} \binom{\alpha}{k} \frac{m^{k+\beta}}{M_k M_\beta} \|\langle x\rangle^\beta \psi^{(k)} \frac{(4m)^{\alpha-k}}{M_{\alpha-k}} \varphi^{(\alpha-k)}\|_\infty$$

$$\leq \sup_{\alpha,\beta\in\mathbf{N}_0} \sum_{k\leq\alpha} \binom{\alpha}{k} \frac{1}{4^\alpha} \frac{(4m)^{k+\beta}}{M_k M_\beta} \|\sum_\gamma \frac{\ell^\gamma}{M_\gamma}\langle x\rangle^{\beta+\gamma}\psi^{(k)}\|_\infty$$

$$\leq \sup_{\alpha,\beta\in\mathbf{N}_0} \sup_{k\in\mathbf{N}_0} \frac{(4m)^{k+\beta}}{M_k M_\beta} \|\sum_{\gamma\in\mathbf{N}_0} \frac{\ell^\gamma}{M_\gamma}\langle x\rangle^{\beta+\gamma}\psi^{(k)}\|_\infty \sum_{k\in\mathbf{N}_0} \binom{\alpha}{k} \frac{1}{4^\alpha}$$

$$\leq C \sup_{\alpha,\beta\in\mathbf{N}_0} \sum_{\gamma\in\mathbf{N}_0} \frac{(4m)^{\alpha+\beta}\ell^\gamma}{M_\alpha M_\beta M_\gamma} \|\langle x\rangle^{\beta+\gamma}\varphi^{(\alpha)}\|_\infty$$

$$\leq C \sup_{\alpha,\beta\in\mathbf{N}_0} \sum_{\gamma\in\mathbf{N}_0} \frac{1}{2^\gamma} \frac{(4m)^{\alpha+\beta}(2\ell)^\gamma H^{\beta+\gamma}}{M_\alpha M_{\beta+\gamma}} \|\langle x\rangle^{\beta+\gamma}\varphi^{(\alpha)}\|_\infty$$

$$\leq C \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{(4m\ell(1+H))^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x\rangle^\beta\varphi^{(\alpha)}\|_\infty < \infty. \square$$

Theorem 3.4 *1. The mapping*

$$\mathcal{O}_M^* \longrightarrow \mathcal{S}'^*, \ \ \varphi \mapsto f\varphi, \ \ f \in \mathcal{S}'^*,$$

is continuous.

2. Suppose (M.2). The pointwiese multiplication

$$\mathcal{S}^* \times \mathcal{O}_M^* \longrightarrow \mathcal{S}^*, \ (\psi, \varphi) \mapsto \psi\varphi, \tag{3.12}$$

$$\mathcal{S}'^* \times \mathcal{O}_M^* \longrightarrow \mathcal{S}'^*, \ (f, \varphi) \mapsto f\varphi, \tag{3.13}$$

are separately continuous mappings.

Proof: 1. The assertion follows immediately the following facts: by the definition of $\mathcal{O}_M^*$ the mapping

$$\mathcal{O}_M^* \longrightarrow \mathcal{S}^*, \qquad \varphi \mapsto \psi\varphi, \qquad \psi \in \mathcal{S}^*,$$

is continuous $\langle f\varphi, \phi\rangle = \langle f, \varphi\phi\rangle$, for each $\psi \in \mathcal{S}^*$.

2. From the first part of the theorem and the proof of Theorem 3.1, it follows that (3.12) is separately continuous. Since, for each $\psi \in \mathcal{S}^*$, $\langle f\varphi, \phi\rangle = \langle f, \varphi\phi\rangle$, and since (3.12) is separately continuous, (3.13) has the same property.

Theorem 3.5 *If $\phi \in \mathcal{E}^*$ and for all $f \in \mathcal{S}'^*$, the product ϕf belongs to $\mathcal{S}'^*$, then ϕ belongs to $\mathcal{O}_M^*$.*

Proof: Our assumption implies that for every $\varphi \in \mathcal{S}^*$ the mapping

$$f \mapsto \langle \phi f, \varphi\rangle,$$

is continuous linear functional on $\mathcal{S}'^*$. Since $\mathcal{S}'^*$ is a reflexive space (since it is Montel), there is $\psi \in \mathcal{S}^*$ such that for each $f \in \mathcal{S}'^*$,

$$\langle \phi f, \varphi\rangle = \langle f, \psi\rangle.$$

In particular, for each $\rho \in \mathcal{D}^*$, we have

$$\langle \phi\rho, \varphi\rangle = \langle \rho, \psi\rangle,$$

which implies that

$$\langle \rho, \phi\varphi\rangle = \langle \rho, \psi\rangle.$$

Hence for all $\varphi \in \mathcal{S}^*$ we have $\phi\varphi = \psi \in \mathcal{S}^*$. It follows $\phi \in \mathcal{O}_M^*$. □

Chapter 4

Integral transforms

This chapter, which results are obtained in cooperation with prof. Pilipović, is devoted to the investigations of various integral transforms on the spaces $\mathcal{S}^*$ and $\mathcal{S}'^*$. One of such integral transform is already studied in Chapter 3.. Namely, the Hermite expansion of elements of the basic spaces and their duals, can be regarded as a generalized integral transform in Zemanian's sense ([68, Chapter IX]). We use results about Hermite expansion to obtain results for the Fourier and Laplace transform, following an analogous idea to the Pilipović's one, for the space Σ'_α. Moreover, we characterize $\mathcal{S}^*$ by the Fourier transform, Wigner distribution and Bargmann transform, and obtain analogous results to Jenssen and von Eijndhoven's for Gelfand-Shilov space $W_M^{M^\times}$ ([27]). Let us remark once again, that the natures of the spaces $W_M^{M^\times}$ and $\mathcal{S}^*$ are different, and therefore our methods are different. In the last section of the chapter we study the Hilbert transform on $\mathcal{S}'^*$, which is a generalization of the corresponding one on the space of tempered distributions, defined by Ishikawa ([26]). Structural properties of the basic spaces imply that the Hilbert transform of a tempered ultradistribution is defined uniquely up to an entire function of ultrapolynomial growth.

4.1 Fourier Transform and Integral Characterizations

In this section we suppose that the conditions (M.1), (M.2) and (M.3)' are fulfilled.

From Theorem 2.3, Parseval's formula and the property

$$\mathcal{F}(D^\alpha\varphi)(\xi) = \xi^\alpha(\mathcal{F}\varphi)(\xi), \quad \mathcal{F}(x^\alpha\varphi)(\xi) = (-D)^\alpha(\mathcal{F}\varphi)(\xi), \qquad \varphi \in \mathcal{S}, \tag{4.1}$$

it follows easily that the Fourier transform is an isomorphism of $\mathcal{S}^*$ onto itself. As usual, we define the Fourier transform of $f \in \mathcal{S}'^*$ by

$$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle, \quad \varphi \in \mathcal{S}^*.$$

If $P^*(x, D)$ is an operator defined by (3.1), then from (4.1) and the continuity of $P^*(x, D)$, it follows that for each $f \in \mathcal{S}'^*$,

$$\mathcal{F}(P^*(\cdot, D)f(\cdot))(\xi) = P^*(-D, \xi)(\mathcal{F}f)(\xi), \qquad \xi \in \mathrm{R}.$$

In the next theorem we give characterizations of the space $\mathcal{S}^*$, by the Fourier transform, Wigner distribution and Bargmann transform.

Theorem 4.1 *1.* [CHARACTERIZATION BASED ON THE FOURIER TRANSFORM] *A function φ belongs to $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) if and only if it is square integrable and for each (resp. some) $h > 0$,*

$$\varphi(\cdot) = \mathcal{O}(\exp[-M(h\,|\cdot|)]) \text{ and } (\mathcal{F}\varphi)(\cdot) = \mathcal{O}(\exp[-M(h\,|\cdot|)]).$$

2. [CHARACTERIZATION BASED ON THE WIGNER DISTRIBUTION] *A function $\varphi \in \mathcal{S}^{(M_p)}$ (resp. $\varphi \in \mathcal{S}^{\{M_p\}}$) if and only if for each (resp. some) $\lambda > 0$*

$$\mathrm{W}(x, y; \varphi) = \mathcal{O}(\exp[-M(\lambda(x^2 + y^2)^{1/2})]).$$

3. [CHARACTERIZATION BASED ON BARGMANN TRANSFORM] *A function $\varphi \in \mathcal{S}^{(M_p)}$ (resp. $\varphi \in \mathcal{S}^{\{M_p\}}$) if and only if for each (resp. some) $\lambda > 0$ there exists C, such that*

$$|(\mathrm{A}\varphi)(\zeta)| \le C \exp[\frac{1}{2}|\zeta|^2 - M(\lambda|\zeta|)], \qquad \zeta \in \mathrm{C}.$$

Proof: Parts 1. and 3. of Theorem 2.3 imply that 1. holds. From this and parts 2. and 4. of Theorem 2.3 and the calculation based on the properties of a function M parts 2. and 3. follow. □

4.2 Laplace transform

In this section we will assume that the conditions (M.1), (M.2) and (M.3) are fulfilled. By $\mathcal{S}'^*_+$ we denote the subspace of $\mathcal{S}'^*$, consisting of elements supported by $[0,\infty)$.

Let $g \in \mathcal{S}'^*_+$. For fixed $\eta > 0$ we define "$g\exp(-\eta\cdot)$", as an element of $\mathcal{S}'^*$ by

$$\langle g\exp(-y\cdot), \varphi\rangle = \langle g, \theta\exp(-y\cdot)\varphi\rangle, \qquad \varphi \in \mathcal{S}^*,$$

where θ is an element of $\mathcal{E}^*$ such that for some $\varepsilon > 0$, $\theta(x) = 1$, $x \in (-\varepsilon,\infty)$, $\theta(x) = 0$, $x \in (-\infty, -2\varepsilon)$. It is easy to see that the definition does not depend on the choice of θ. As usual (see for example [62]) we define the Laplace transform of $g \in \mathcal{S}'^*_+$ by

$$(\mathcal{L}g)(\zeta) = \mathcal{F}(g\exp(-\eta\cdot))(\xi), \;\; \zeta = \xi + i\eta \in \mathbf{C}_+.$$

Clearly, for fixed $\eta > 0$ it is an element of $\mathcal{S}'^*$.

Let

$$G(\zeta) = \langle g, \theta\exp(i\zeta\cdot)\rangle, \quad \zeta = \xi + i\eta \in \mathbf{C}_+, \tag{4.2}$$

where θ is as above. The function G is holomorphic on $\mathbf{C}_+$ and does not depend on θ.

Following an analogous idea as in [47] (see also [64]) one can prove the next assertion.

Theorem 4.2 *Let $g \in \mathcal{S}'^*_+$ and G be defined by (4.2).*

1. *For every $\varepsilon > 0$ there are C and $k > 0$ such that*

$$|G(\zeta)| \le C\exp\left[\varepsilon\eta + \left(M(k|\xi|) + \tilde{M}\left(k|\eta|^{-1}\right)\right)\right], \quad \zeta = \xi + i\eta \in \mathbf{C}_+.$$

2. *For fixed $\eta > 0$, $(\mathcal{L}g)(\xi + i\eta) = G(\xi + i\eta)$, $\xi \in \mathbf{R}$.*

3. *There is* $G(\cdot + i0) \in \mathcal{S}'^*$ *such that in the sense of convergence in* $\mathcal{S}'^*$

$$G(\xi + i\eta) \to G(\xi + i0), \text{ as } \eta \to 0^+ \quad \text{and} \quad G(\xi + i0) = (\mathcal{F}g)(\xi),\ \xi \in \mathbf{R}.$$

4. *If* $G_k(\zeta) = (\mathcal{L}g_k)(\zeta)$, $\zeta \in \mathbf{C}_+$, $k = 1, 2$, *and* $G_1(\xi + i0) = G_2(\xi + i0)$, $\xi \in \mathbf{R}$, *then* $g_1 = g_2$.

4.3 Hilbert transform

In order to define the Hilbert transform on $\mathcal{S}'^*$ we follow Ishikawa's ideas for tempered distributions ([26]), and represent $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) as the projective limit of appropriate spaces $\mathcal{D}_a^{M_p}$, $a > 0$ (resp. $\mathcal{D}_{a_p}^{M_p}$, $a_p \in \mathcal{R}$). But in the contrary to the case of tempered distributions we do not have that $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) is dense in the space $\mathcal{D}_a^{M_\alpha}$ (resp. $\mathcal{D}_{a_p}^{M_p}$). We overcome this difficulty by parts 4. and 5. of Theorem 4.8.

Let $b > 0$ (resp. $b_p \in \mathcal{R}$) be given and let P_b (resp. P_{b_p}) be an entire function such that for some constants $L > 0$ and C,

$$|P_b(\zeta)| \leq C \exp[M(L|\zeta|)] \Big(\text{resp. } |P_{b_p}(\zeta)| \leq C \exp[N_{b_p}(L|\zeta|)]\Big),\ \zeta \in \mathbf{C}, \tag{4.3}$$

$$\left.\begin{array}{c} \exp[M(b|\zeta|)] \leq P_b(\zeta) \Big(\text{resp. } \exp[N_{b_p}(|\zeta|)] \leq P_{b_p}(\zeta)\Big), \\ \zeta = \xi + i\eta \in \mathbf{C}, \quad \xi^2 \geq \eta^2. \end{array}\right\} \tag{4.4}$$

In the case when (M.1), (M.2) and (M.3) hold an example of such an entire function is

$$P_b(\zeta) = \prod_{\alpha=1}^{\infty}(1 + \frac{\zeta^2}{b^2 m_\alpha^2}) \Big(\text{resp. } P_{b_p}(\zeta) = \prod_{\alpha=1}^{\infty}(1 + \frac{\zeta^2}{{b_\alpha}^2 m_\alpha^2})\Big),\ \zeta \in \mathbf{C}.$$

From [33, p.91] it follows that this entire function fulfills conditions (4.3) and (4.4) since for $\zeta = \xi + i\eta \in \mathbf{C}$ and $\xi^2 \geq \eta^2$,

$$\left|\prod_{\alpha=1}^{\infty}(1 + \frac{\zeta^2}{b_\alpha^2 m_\alpha^2})\right| \geq \sup_{\beta \in \mathbf{N}} \prod_{\alpha=1}^{\beta}\left|1 + \frac{\zeta^2}{b_\alpha^2 m_\alpha^2}\right| \geq \sup_{\beta \in \mathbf{N}} \prod_{\alpha=1}^{\beta}\left|\frac{\zeta^2}{b_\alpha^2 m_\alpha^2}\right| = \exp[2N_{b_p}(|\zeta|)].$$

It follows from (4.3) that $P_b(D)$ (resp. $P_{b_p}(D)$) is an ultradifferential operator of class (M_p) (resp. $\{M_p\}$) (see [33, Proposition 4.5.]).

Let us now give the structural characterization of basic spaces adopted for the investigations of the Hilbert transform.

Definition 4.3 *Let a, $b > 0$ and (a_p), $(b_p) \in \mathcal{R}$. $\mathcal{D}^{M_p}_{a,b}$, $\mathcal{D}^{M_p,a}_{b}$, $\mathcal{D}^{M_p}_{a_p,b_p}$ and $\mathcal{D}^{M_p,a_p}_{b_p}$ are respectively the spaces of smooth functions φ on $\mathbf{R}$ such that*

$$p_{a,b}(\varphi) = \sup_{\alpha\in\mathbf{N}_0} \frac{a^\alpha}{M_\alpha}\|(P_b\varphi)^{(\alpha)}\|_\infty < \infty,$$

$$q_{a,b}(\varphi) = \sup_{\alpha\in\mathbf{N}_0} \frac{a^\alpha}{M_\alpha}\|P_b\varphi^{(\alpha)}\|_\infty < \infty,$$

$$p_{a_p,b_p}(\varphi) = \sup_{\alpha\in\mathbf{N}_0} \frac{\|(P_{b_p}\varphi)^{(\alpha)}\|_\infty}{(\prod_{p=1}^{\alpha} a_p)M_\alpha} < \infty,$$

$$q_{a_p,b_p}(\varphi) = \sup_{\alpha\in\mathbf{N}_0} \frac{\|P_{b_p}\varphi^{(\alpha)}\|_\infty}{(\prod_{p=1}^{\alpha} a_p)M_\alpha} < \infty,$$

equipped respectively with the topologies induced by the norms $p_{a,b}$, $q_{a,b}$, p_{a_p,b_p} and q_{a_p,b_p} respectively.

$$\mathcal{D}^{M_p}_{a} = \operatorname{proj}\lim_{b>0}\mathcal{D}^{M_p}_{a,b}, \qquad \mathcal{D}^{M_p,a} = \operatorname{proj}\lim_{b>0}\mathcal{D}^{M_p,a}_{b},$$

$$\mathcal{D}^{M_p}_{a_p} = \operatorname{proj}\lim_{(b_p)\in\mathcal{R}}\mathcal{D}^{M_p}_{a_p,b_p}, \qquad \mathcal{D}^{M_p,a_p} = \operatorname{proj}\lim_{(b_p)\in\mathcal{R}}\mathcal{D}^{M_p,a_p}_{b_p}$$

In the sequel we need some estimates of derivatives of P_b (resp. P_{b_p}) and $1/P_b$ (resp. $1/P_{b_p}$).

Lemma 4.4 *If P_b (resp. P_{b_p}) fulfills (4.3) and (4.4) then*

1. *For every $r > 0$ there is C, such that*

$$|\left(P_{b_p}(\zeta)\right)^{(\gamma)}| \leq C\frac{\gamma!}{r^\gamma}|P_{b_p/2}(\zeta))|, \quad \zeta\in\mathbf{C},\ \gamma\in\mathbf{N}_0. \tag{4.5}$$

2. *There exist $r > 0$ and C such that*

$$|\left(\frac{1}{P_{b_p}(\xi)}\right)^{(\gamma)}| \leq C\frac{\gamma!}{r^\gamma}\exp[-N_{2b_p}(|\xi|)],\ \xi\in\mathbf{R},\ \gamma\in\mathbf{N}_0. \tag{4.6}$$

The corresponding inequalities hold for P_b.

Proof 1. Applying Cauchy's formula and (4.3) one can easily obtain (4.5).

2. Since $P_{b_p}(0) \neq 0$, there exist $r > 0$ and C such that $|P_{b_p}(\zeta)| \geq C$, for $|\zeta| \leq 2r$. By the Cauchy formula for $\xi \in \mathbf{R}$ and $|\xi| \leq r$,

$$|\left(\frac{1}{P_{b_p}(\xi)}\right)^{(\gamma)}| = |\frac{\gamma!}{2\pi i}\int_{|\zeta-\xi|=r}\frac{d\zeta}{P_{b_p}(\zeta)(\zeta-\xi)^{\gamma+1}}| \leq \frac{\gamma!}{r^\gamma}. \tag{4.7}$$

Let $\xi \in \mathbf{R}$, $|\xi| > r$ and K_ξ be a circle with the radius $|\xi|/\sqrt{2}$ and the center in ξ. By applying Cauchy's formula and (4.4) we obtain that for every $\gamma \in \mathbf{N}_0$,

$$|\left(\frac{1}{P_{b_p}(\xi)}\right)^{(\gamma)}| = |\frac{\gamma!}{2\pi i}\int_{K_\xi}\frac{1}{P_{b_p}(\zeta)(\zeta-\xi)^{\gamma+1}}d\zeta|$$

$$\leq \frac{\gamma!2^{\gamma/2}}{|\xi|^\gamma}\sup_{\Theta\in[0,2\pi]}\exp[-N_{b_p}(|\xi+\frac{|\xi|}{\sqrt{2}}e^{\Theta i}|)]$$

$$\leq \frac{\gamma!2^{\gamma/2}}{r^\gamma}\exp[-N_{b_p}(||\xi|-\frac{|\xi|}{\sqrt{2}}|)] \leq C\,\frac{\gamma!2^{\gamma/2}}{r^\gamma}\exp[-N_{2b_p}(|\xi|)].$$

This and (4.7) imply (4.6).

Theorem 4.5 *If (M.2)' holds then,*

$$\mathcal{S}^{(M_p)} = \operatorname{proj}\lim_{a>0}\mathcal{D}_a^{M_p} = \operatorname{proj}\lim_{a>0}\mathcal{D}^{M_p,a},$$

$$\mathcal{S}^{\{M_p\}} = \operatorname{proj}\lim_{(a_p)\in\mathcal{R}}\mathcal{D}_{a_p}^{M_p} = \operatorname{proj}\lim_{(a_p)\in\mathcal{R}}\mathcal{D}^{M_p,a_p}.$$

Proof: We shall prove the assertion in the case $* = \{M_p\}$ since it is more complicated than the other one and the ideas for both cases are similar.

First we prove that there exists C such that for each $\varphi \in \mathbf{C}^\infty$, $\wp_{a_p,b_p}(\varphi) \leq Cq_{a_p,b_p/\sqrt{2}}(\varphi)$. Condition (M.3)' imply

$$\wp_{a_p,b_p}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}_0}\frac{\|\langle x\rangle^\beta\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^\alpha a_p\right)M_\alpha\left(\prod_{p=1}^\beta b_p\right)M_\beta}$$

$$\leq \sup_{\alpha,\beta\in\mathbf{N}_0}\frac{2^{\beta/2}\|\max(1,|x|^\beta)\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^\alpha a_p\right)M_\alpha\left(\prod_{p=1}^\beta b_p\right)M_\beta}$$

$$\leq C\left(\sup_{\alpha,\beta\in\mathbf{N}_0}\frac{2^{\beta/2}\|\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^\alpha a_p\right)M_\alpha\left(\prod_{p=1}^\beta b_p\right)M_\beta} + \sup_{\alpha,\beta\in\mathbf{N}_0}\frac{2^{\beta/2}\|x^\beta\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^\alpha a_p\right)M_\alpha\left(\prod_{p=1}^\beta b_p\right)M_\beta}\right)$$

$$\leq \mathcal{C}\left(\sup_{\alpha\in\mathbb{N}_0}\frac{\|\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha}+\sup_{\alpha\in\mathbb{N}_0}\frac{\|\exp[N_{b_p/\sqrt{2}}]\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha}\right)$$

$$\leq \mathcal{C}\sup_{\alpha\in\mathbb{N}_0}\frac{\|P_{b_p/\sqrt{2}}\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha}=\mathcal{C}q_{a_p,b_p/\sqrt{2}}(\varphi).$$

Estimate (4.3) implies that there exists $\mathcal{C}$ such that for each $\varphi\in C^\infty$, $q_{a_p,b_p}(\varphi)\leq\mathcal{C}\wp_{a_p,b_p/L}(\varphi)$, $\varphi\in C^\infty$, since

$$q_{a_p,b_p}(\varphi)\leq \mathcal{C}\sup_{\alpha\in\mathbb{N}_0}\frac{\|N_{b_p}(L|x|)\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha}$$

$$\leq \mathcal{C}\sup_{\alpha\in\mathbb{N}_0}\sup_{\beta\in\mathbb{N}_0}\frac{\|\langle x\rangle^\beta\varphi^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha\left(\prod_{p=1}^{\beta}b_p/L\right)M_\beta}\leq\mathcal{C}\wp_{a_p,b_p/L}(\varphi).$$

Let us now prove the equivalence of the families $\{p_{a_p,b_p};(a_p),(b_p)\in\mathcal{R}\}$ and $\{q_{a_p,b_p};(a_p),(b_p)\in\mathcal{R}\}$. From (4.6) it follows

$$p_{a_p,b_p}(\varphi)=\sup_{\alpha\in\mathbb{N}_0}\frac{\|\left(P_{b_p}\varphi\right)^{(\alpha)}\|_\infty}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha}$$

$$=\sup_{\alpha\in\mathbb{N}_0}\frac{1}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\|P_{b_p}^{(\gamma)}\varphi^{(\alpha-\gamma)}\|_\infty$$

$$\leq\sup_{\alpha\in\mathbb{N}_0}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{\gamma!}{\left(\prod_{p=1}^{\alpha-\gamma}a_p\right)M_{\alpha-\gamma}\left(\prod_{p=1}^{\gamma}a_p\right)M_\gamma r^\gamma}\|P_{b_p/2}\varphi^{(\alpha-\gamma)}\|_\infty$$

$$\leq\sup_{\gamma\in\mathbb{N}_0}\frac{\gamma!}{\left(\prod_{p=1}^{\gamma}a_p/2\right)M_\gamma}\sup_{\alpha\in\mathbb{N}_0}\frac{1}{2^\alpha}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\sup_{\alpha-\gamma\in\mathbb{N}_0}\frac{\|P_{b_p/2}\varphi^{(\alpha-\gamma)}\|_\infty}{\left(\prod_{p=1}^{\alpha-\gamma}a_p/2\right)M_{\alpha-\gamma}}$$

$$\leq\mathcal{C}q_{a_p/2,b_p/2}(\varphi).$$

Let (a_p), $(c_p)\in\mathcal{R}$ and let $(b_p)\in\mathcal{R}$ be such that $2b_n>c_n/L$, $n\in\mathbb{N}$, where L is the constant from (4.3). This implies

$$\exp[N_{c_p}(L|x|)-N_{2b_p}(|x|)]\leq 1,\quad x\in\mathbb{R}.$$

The above inequality, (4.7) and (M.1) imply that there exist $\mathcal{C}$ and $r>0$ such that for each $\varphi\in C^\infty$,

$$q_{a_p,c_p}(\varphi)\leq\sup_{\alpha\in\mathbb{N}_0}\frac{1}{\left(\prod_{p=1}^{\alpha}a_p\right)M_\alpha}\|P_{c_p}\varphi^{(\alpha)}\|_\infty$$

$$\leq \sup_{\alpha\in N_0} \frac{1}{\left(\prod_{p=1}^{\alpha} a_p\right) M_\alpha} \| P_{c_p} \left(\frac{P_{b_p}}{P_{b_p}}\varphi\right)^{(\alpha)} \|_\infty$$

$$\leq \sup_{\alpha\in N_0} \frac{1}{\left(\prod_{p=1}^{\alpha} a_p\right) M_\alpha} \| P_{c_p} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \left(\frac{1}{P_{b_p}}\right)^{(\alpha-\gamma)} (P_{b_p}\varphi)^{(\gamma)} \|_\infty$$

$$\leq C \sup_{\alpha\in N_0} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \frac{1}{\left(\prod_{p=1}^{\alpha} a_p\right) M_\alpha} \cdot$$

$$\cdot \| \exp[N_{c_p}(L \mid x \mid)] \frac{(\alpha-\gamma)!}{r^{\alpha-\gamma}} \exp[-N_{2b_p}(\mid x \mid)] (P_{b_p}\varphi)^{(\gamma)} \|_\infty$$

$$\leq C \sup_{\alpha\in N_0} \frac{1}{2^\alpha} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \frac{1}{(\prod_{p=1}^{\gamma} a_p/2) M_\gamma} \frac{(\alpha-\gamma)!}{(\prod_{p=1}^{\alpha-\gamma} a_p/2) M_{\alpha-\gamma} r^{\alpha-\gamma}} \cdot$$

$$\cdot \| \exp[N_{c_p}(L \mid x \mid) - N_{2b_p}(\mid x \mid)] (P_{b_p}\varphi)^{(\gamma)} \|_\infty$$

$$\leq C \sup_{\alpha\in N_0} \frac{1}{\left(\prod_{p=1}^{\alpha} a_p/2\right) M_\alpha} \| (P_{b_p}\varphi)^{(\alpha)} \|_\infty = p_{a_p/2, b_p}(\varphi). \square$$

Remark From the preceding proof it follows that for given $a > 0$ (resp. $(a_p) \in \mathcal{R}$) there exists $b > 0$ (resp. $(b_p) \in \mathcal{R}$) such that $a < b$ (resp. $(a_p) \preceq (b_p)$), such that $\mathcal{D}_b^{M_p} \subset \mathcal{D}_a^{M_p}$ (resp. $\mathcal{D}_{b_p}^{M_p} \subset \mathcal{D}_{a_p}^{M_p}$) and the inclusion mapping is continuous.

Definition 4.6 *The Hilbert transform $\mathcal{H}_a$ (resp. $\mathcal{H}_{a_p}$), on the space $\mathcal{D}_a^{M_p}$ (resp. $\mathcal{D}_{a_p}^{M_p}$) is defined by*

$$(\mathcal{H}_a\varphi)(x) = \frac{1}{P_a(x)}\, PV \int_{-\infty}^{\infty} \frac{P_a(x-t)\varphi(x-t)}{t} dt, \qquad \varphi \in \mathcal{D}_a^{M_p}$$

$$\left(resp. \;\; (\mathcal{H}_{a_p}\varphi)(x) = \frac{1}{P_{a_p}(x)}\, PV \int_{-\infty}^{\infty} \frac{P_{a_p}(x-t)\varphi(x-t)}{t} dt, \quad \varphi \in \mathcal{D}_{a_p}^{M_p} \right). \tag{4.8}$$

Proposition 4.7 *1. $\mathcal{H}_a$ (resp. $\mathcal{H}_{a_p}$) is a linear continuous mapping from $\mathcal{D}_a^{M_p}$ (resp. $\mathcal{D}_{a_p}^{M_p}$) onto itself.*

2. $\mathcal{H}_a\mathcal{H}_a\varphi = -\varphi, \quad \varphi \in \mathcal{D}_a^{M_p} \quad \left(resp. \;\; \mathcal{H}_{a_p}\mathcal{H}_{a_p}\varphi = -\varphi, \quad \varphi \in \mathcal{D}_{a_p}^{M_p}\right)$.

Proof: The proof is given only in the case $* = \{M_p\}$, since the case $* = (M_p)$ is analogous. The linearity and continuity of $\mathcal{H}_{a_p}$ follows immediately from the fact that it is defined as the composition of the following linear and continuous mappings

$$T_{a_p} : \mathcal{D}_{a_p}^{M_p} \to \mathcal{D}_{L^2}^{\{M_p\}}, \quad \varphi \mapsto P_{a_p}\varphi,$$

$$\mathcal{H} : \mathcal{D}_{L^2}^{\{M_p\}} \to \mathcal{D}_{L^2}^{(M_p)}, \quad \varphi \mapsto (\mathcal{H}\varphi)(\cdot) = PV \int_{-\infty}^{\infty} \frac{\varphi(t)}{t - \cdot},$$

$$T_{a_p}^{-1} : \mathcal{D}_{L^2}^{\{M_p\}} \to \mathcal{D}_{a_p}^{M_p}, \quad \varphi \mapsto \varphi / P_{a_p},$$

where $\mathcal{H}$ denotes the Hilbert transform defined on $\mathcal{D}_{L^2}^{\{M_p\}}$. Note that in [46] the Hilbert transform is considered only on $\mathcal{D}_{L^2}^{(M_p)}$ i. e. in the Beurling case but in can be examined in a similar way in the Roumieu case.

From the definition $\mathcal{H}_{a_p}$ and the properties of the Hilbert transform on $\mathcal{D}_{L^2}^{\{M_p\}}$ ($\mathcal{H}\mathcal{H}\phi = -\phi$, $\phi \in \mathcal{D}_{L^2}^{\{M_p\}}$) it follows that for each $a_p \in \mathcal{R}$ and each $\varphi \in \mathcal{D}_{a_p}^{M_p}$

$$\begin{aligned} \mathcal{H}_{a_p}(\mathcal{H}_{a_p}\varphi) = T_{a_p}^{-1}(\mathcal{H}T_{a_p}(T_{a_p}^{-1}(\mathcal{H}(T_{a_p}\varphi)))) &= T_{a_p}^{-1}(\mathcal{H}(\mathcal{H}(T_{a_p}\varphi))) \\ &= T_a^{-1}(-T_a\varphi) = -\varphi. \end{aligned}$$

This completes the proof. □

The generalized Hilbert transform H_a (resp. H_{a_p}) on the dual space $\mathcal{D}_a'^{M_p}$ (resp. $\mathcal{D}_{a_p}'^{M_p}$) is defined by

$$\langle \mathrm{H}_a f, \varphi \rangle = -\langle f, \mathcal{H}_a \varphi \rangle, \qquad \varphi \in \mathcal{D}_a^{M_p}$$

$$\Big(\text{resp. } \langle \mathrm{H}_{a_p} f, \varphi \rangle = -\langle f, \mathcal{H}_{a_p} \varphi \rangle, \qquad \varphi \in \mathcal{D}_{a_p}^{M_p}\Big).$$

Theorem 4.8 *1. H_a (resp. H_{a_p}) is a linear continuous mapping of $\mathcal{D}_a'^{M_p}$ (resp. $\mathcal{D}_{a_p}'^{M_p}$) onto itself.*

2. $\mathrm{H}_a(\mathrm{H}_a f) = -f$ (resp. $\mathrm{H}_{a_p}(\mathrm{H}_{a_p} f) = -f$), for each $f \in \mathcal{D}_a'^{M_p}$ (resp. $f \in \mathcal{D}_{a_p}'^{M_p}$).

3. Let $f \in \mathcal{D}_a'^{M_p}$ *(resp.* $f \in \mathcal{D}_a'^{M_p}$*). Then*

$$\langle \mathcal{F}(\mathrm{H}_a f), \varphi \rangle = \begin{cases} -i\langle \mathcal{F}f, \varphi\rangle, & \varphi \in \mathcal{D}^{(M_p)}, \quad supp\ \varphi \subset (0,\infty); \\ i\langle \mathcal{F}f, \varphi\rangle, & \varphi \in \mathcal{D}^{(M_p)}, \quad supp\ \varphi \subset (-\infty,0). \end{cases}$$

$$\left(resp.\ \langle \mathcal{F}(\mathrm{H}_{a_p} f), \varphi \rangle = \begin{cases} -i\langle \mathcal{F}f, \varphi\rangle, & \varphi \in \mathcal{D}^{\{M_p\}}, \quad supp\ \varphi \subset (0,\infty); \\ i\langle \mathcal{F}f, \varphi\rangle, & \varphi \in \mathcal{D}^{\{M_p\}}, \quad supp\ \varphi \subset (-\infty,0). \end{cases} \right)$$

4. Let (M.2) and (M.3) be fulfilled. Assume that $f \in \mathcal{D}_a'^{M_p}$ *, (resp.* $f \in \mathcal{D}_{a_p}'^{M_p}$*),* $0 < a < b$ *(resp.* $(a_p) \preceq (b_p)$*) and that* b *(resp.* b_p*) is chosen so that* $f|_{\mathcal{D}_b^{M_p}} \in \mathcal{D}_b'^{M_p}$ *(resp.* $f|_{\mathcal{D}_{b_p}^{M_p}} \in \mathcal{D}_{b_p}'^{M_p}$*) (see the remark after Theorem 4.5). The difference* $(\mathrm{H}_a f - \mathrm{H}_b f)|_{\mathcal{D}_b^{M_p}}$ *(resp.* $(\mathrm{H}_{a_p} f - \mathrm{H}_{b_p} f)|_{\mathcal{D}_{b_p}^{M_p}}$*) is an ultrapolynomial of class* (M_p) *(resp.* $\{M_p\}$*).*

5. If $f, g \in \mathcal{D}_a'^{M_p}$ *and* $f|_{\mathcal{D}^{(M_p)}} = g|_{\mathcal{D}^{(M_p)}}$ *(resp.* $f, g \in \mathcal{D}_{a_p}'^{M_p}$ *and* $f|_{\mathcal{D}^{\{M_p\}}} = g|_{\mathcal{D}^{\{M_p\}}}$*) then the difference* $\mathrm{H}_a f - \mathrm{H}_a g$ *(resp.* $\mathrm{H}_{a_p} f - \mathrm{H}_{a_p} g$*) is an ultrapolynomial of class* (M_p) *(resp.* $\{M_p\}$*).*

Proof: We will prove the assertion only in the case $* = \{M_p\}$. Parts 1. and 2. follows immediately from the previous theorem. Let us prove part 3. Let $\varphi \in \mathcal{D}^{\{M_p\}}$ be such that $supp\varphi \subset (0,\infty)$. From the property of the Fourier and Hilbert transforms of an $f \in L^2$

$$\mathcal{F}(Hf)(x) = -i\ sgn\ (x)(\mathcal{F}f)(x), \qquad x \in \mathrm{R},$$

it follows

$$\langle \mathcal{F}(\mathrm{H}_{a_p} f), \varphi\rangle = \langle \mathrm{H}_{a_p} f, \mathcal{F}\varphi\rangle = \langle f, T_{a_p}^{-1}\mathcal{H}T_{a_p}\mathcal{F}\varphi\rangle$$
$$= \langle f, T_{a_p}^{-1}\mathcal{H}\mathcal{F}(P_{a_p}(D)\varphi)\rangle = \langle f, T_{a_p}^{-1}\mathcal{F}(-i(P_{a_p}(D)\varphi)\rangle$$
$$= -i\langle f, T_{a_p}^{-1}T_{a_p}\mathcal{F}\varphi\rangle = -i\langle f, \mathcal{F}\varphi\rangle = -i\langle \mathcal{F}f, \varphi\rangle.$$

In a similar way we can prove part 3. in the case $\varphi \in \mathcal{D}^{\{M_p\}}$ and $supp\varphi \subset (-\infty, 0)$.

Let us prove part 4.. For any $\varphi \in \mathcal{D}^{(M_p)}$ with $supp\varphi \subset (0,\infty)$,

$$\langle \mathcal{F}(\mathrm{H}_{a_p} f - \mathrm{H}_{b_p} f), \varphi\rangle = \langle \mathcal{F}(\mathrm{H}_{a_p} f), \varphi\rangle - \langle \mathcal{F}(\mathrm{H}_{b_p} f), \varphi\rangle$$

$$= -i\langle \mathcal{F}f, \varphi\rangle - (-i)\langle \mathcal{F}f, \varphi\rangle = 0.$$

Analogously, we have

$$\langle \mathcal{F}(\mathrm{H}_{a_p}f - \mathrm{H}_{b_p}f), \varphi\rangle = 0, \quad \varphi \in \mathcal{D}^{\{M_p\}}, \quad supp\varphi \subset (-\infty, 0).$$

Therefore, $supp\mathcal{F}(\mathrm{H}_{a_p}f - \mathrm{H}_{b_p}f) \subset \{0\}$. [34, Theorem 3.1] implies the existence of an ultradifferential operator $P(D)$, such that

$$\mathcal{F}(\mathrm{H}_{a_p}f - \mathrm{H}_{b_p}f) = P(D)\delta. \tag{4.9}$$

Applying the inverse Fourier transform on (4.9) we get

$$(\mathrm{H}_{a_p}f - \mathrm{H}_{b_p}f) = P(x), \quad x \in \mathrm{R},$$

i.e. 4. holds.

Assertion 5. follows from the fact that

$$supp\mathcal{F}(\mathrm{H}_{a_p}f - \mathrm{H}_{a_p}g) \subset \{0\},$$

which can be proved analogously as part 4.. □

Using the fact that for each $f \in \mathcal{S}'^{(M_p)}$ (resp. $\mathcal{S}'^{\{M_p\}}$) there is $a > 0$ (resp. $a_p \in \mathcal{R}$) such that f has a linear and continuous extension F on $\mathcal{D}_a^{M_p}$ (resp. $\mathcal{D}_{a_p}^{M_p}$), we define the Hilbert transform $\mathrm{H}^{(M_a)}f$ (resp. $\mathrm{H}^{\{M_a\}}f$) of f by

$$\mathrm{H}^{(M_p)}f = \mathrm{H}_a F, \left(\text{resp. } \mathrm{H}^{\{M_p\}}f = \mathrm{H}_{a_p}F\right).$$

It is determined uniquely up to entire function of ultrapolynomial growth.

Chapter 5

Convolution of Ultradistributions

There are several definitions of convolutions in the space of Schwartz's distributions and in its proper subspaces. They are analyzed in many books and papers ([56], [62], [12], [59], [1], [16], [23], [24], [70], [63], [30], [65], [61]). In the theory of ultradistributions mainly the convolution of two ultradistributions one of which has a compact support was considered. On the base of such consideration, Braun, Meise, Taylor, Voigt and their collaborators (see [42] and references there) deeply studied convolution equations in ultradistribution spaces. A convolution of two arbitrary Beurling type ultradistributions was investigated in [49] where it was proved the equivalence of so-called Schwartz's and Vladimirov's definitions of convolution for ultradistributions. In the chapter, which results are obtained in cooperation with prof. Pilipović and prof. Kamiński, we investigate in details the equivalence of several definitions of the convolution of Beurling type ultradistributions. Also, we introduce several definitions of ultratempered convolutions of Beurling type ultradistributions and prove their equivalence. The fact that ultradistributions are infinite sums of derivatives of appropriate continuous functions on bounded open sets makes the problem of equivalence of various definitions of convolutions non-trivial. As in the distribution theory the space of integrable ultradistributions is crucial for definitions of convolutions. Since in the Roumieu case the structure of such space is difficult and not-known enough, in the Roumieu case we have only some partial results concerning the convolution. Therefore, in the chapter we will consider only the Beurling case.

Let us first introduce the notations which will be used only in this chapter. All the spaces and functions which are mentioned in the chapter are defined on $\mathbf{R}^d$, with the exception of those with specially denoted domain The letter d denotes a fixed element of $\mathbf{N}$. The constant function equal to 1 on $\mathbf{R}^d$ (resp. $\mathbf{R}^{2d}$) is denoted by 1_x (resp. $1_{x,y}$). If ϑ is a function, $\vartheta^{\triangle}$ denotes $\vartheta(x+y)$, Let

$$P_r(\zeta) = (1+\zeta_1^2+\ldots+\zeta_d^2)\prod_{p\in\mathbf{N}}\left(1+\frac{\zeta_1^2+\ldots+\zeta_d^2}{r^2m_p^2}\right),\quad \zeta=(\zeta_1,\ldots,\zeta_d)\in\mathbf{C}^d, \tag{5.1}$$

where r is a positive constant. Conditions (M.1), (M.2) and (M.3) imply that P_r is an ultradifferential operator of the class (M_p) (see [33]).

We introduce the classes of sequences (see [16], [30] and [49]) as follows. A sequence (η_j) of elements of $\mathcal{D}^{(M_p)}$ is an *approximate unit* if it converges to 1_x in $\mathcal{E}^{(M_p)}$ and if there exist m and C, such that for each $j\in\mathbf{N}$

$$\sup_{\alpha\in\mathbf{N}^d}\left(\frac{m^{|\alpha|}}{M_{|\alpha|}}\parallel\eta_j^{(\alpha)}\parallel_\infty\right)<C,$$

where $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_d$ for $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_d)\in\mathbf{N}^d$. If moreover, for every compact set $K\subset\mathbf{R}^d$ there exists $j_0\in\mathbf{N}$, such that

$$\eta_j(x,y)=1,\ \ (x,y)\in K,\ \ j\geq j_0.$$

(η_j) will be called a *strong approximate unit A sequence η_j of elements of* $\mathcal{D}^{(M_p)}(\mathbf{R}^d)$ *is a* special unit sequence *if it is of the form* $\eta_j(x_1,\ldots,x_d)=\sum_{i=j}^{p}\eta(x_1/j,\ldots x_d/j)$, $j\in\mathbf{N}_0$, *where* $\eta\in\mathcal{D}^{(M_p)}$ *and* $\eta=1$ *in some neighborhood of zero in* $\mathbf{R}^d$.

Following [23] we say that the space of ultradistributions of class (M_p) *is* permitted *if for any* $f\in F$, $(\rho_k\cdot f)*\delta_k$ *and* $\rho_k\cdot(f*\delta_k)$ *converge to* f *as* $k\to\infty$, *where* ρ_k *is an approximate unit and* δ_k *belongs to* $\mathcal{D}^{(M_p)}$, $\delta_k\geq 0$, $\int_{\mathbf{R}}\delta_k=1$ *and supp* $\delta_k\subset[-\alpha_k,\alpha_k]$, *where* $\alpha_k\to 0$ *as* $k\to\infty$. *A space*

F of ultradistributions has the property (C_{M_p}) (see [49]) if and only if for each barrelled space E and each linear mapping $L : E \to F$ holds that L is continuous mapping if it is continuous as a mapping $E \to \mathcal{D}'^{(M_p)}$. Examples of the spaces which posses the property (C_{M_p}) are L^1 and $\mathcal{D}'^{(M_p)}_{L^1}$ ([49]).

5.1 On the Definition of Convolution

In this section we show the equivalence of various definitions of convolutions of elements of $\mathcal{D}'^{(M_p)}$. The form of the main theorem is similar to the corresponding one given by Shiraishi ([59]) for distributions. But in the proof of it some nontrivial problems appear, for example the Leibniz formula could not be used since ultradistributions are infinite sums of derivatives (i. e. ultraderivatives) of corresponding continuous functions on a bounded open set.

Following the approach of Schwartz ([57]), Vladimirov ([62]) and Chevalley ([12]) we have the next definitions of convolutions of $S, T \in \mathcal{D}'^{(M_p)}$.

Definition 5.1 ([49]) *The convolution $S *^{5.1} T \in \mathcal{D}'^{(M_p)}$ is defined by*

$$\langle S *^{5.1} T, \vartheta\rangle = \langle (S_x \otimes T_y)\vartheta(x+y), 1_{x,y}\rangle, \quad \vartheta \in \mathcal{D}^{(M_p)}, \quad \text{if}$$

$$\text{for each } \vartheta \in \mathcal{D}^{(M_p)}, \quad (S_x \otimes T_y)\vartheta^{\triangle} \in \mathcal{D}'^{(M_p)}_{L^1}(\mathrm{R}^{2d}). \tag{5.2}$$

Definition 5.2 ([49]) *The convolution $S *^{5.2} T$ is defined by*

$$\langle S *^{5.2} T, \vartheta\rangle = \lim_{j\to\infty} \langle S_x \otimes T_y, \eta_j(x,y)\vartheta(x+y)\rangle, \quad \vartheta \in \mathcal{D}^{(M_p)}, \quad \text{if}$$

$$\left.\begin{array}{l}\text{for every } \vartheta \in \mathcal{D}^{(M_p)} \text{ and every strong approximate unit } (\eta_j) \\ \text{the sequence } \left(\langle S_x \otimes T_y, \eta_j(x,y)\vartheta(x+y)\rangle\right)_j \\ \text{converges to a constant.}\end{array}\right\} \tag{5.3}$$

Definition 5.3 *The convolution $S *^{5.3} T$ is defined by*

$$\langle S *^{5.3} T, \vartheta\rangle = \lim_{j\to\infty} \langle S_x \otimes T_y, \eta_j(x,y)\vartheta(x+y)\rangle, \quad \vartheta \in \mathcal{D}^{(M_p)}, \quad \text{if}$$

$$\left.\begin{array}{l}\text{for every } \vartheta \in \mathcal{D}^{(M_p)} \text{ and every special unit sequence } (\eta_j),\\ \text{the sequence } \left(\langle S_x \otimes T_y, \eta_j(x,y)\vartheta(x+y)\rangle\right)_j\\ \text{converges to a constant.}\end{array}\right\} \quad (5.4)$$

Definition 5.4 *The convolution* $S *^{5.4} T$ *is defined by*

$$\langle S *^{5.4} T, \vartheta\rangle = \langle S(\check{T} * \vartheta), 1_x\rangle, \quad \vartheta \in \mathcal{D}^{(M_p)}, \quad \text{if}$$

$$\text{for each } \vartheta \in \mathcal{D}^{(M_p)}, \quad S(\check{T} * \vartheta) \in \mathcal{D}'^{(M_p)}_{L^1}. \quad (5.5)$$

Definition 5.5 *The convolution* $S *^{5.5} T$ *is defined by*

$$\langle S *^{5.5} T, \vartheta\rangle = \langle (\check{S} * \vartheta)T, 1_x\rangle, \quad \vartheta \in \mathcal{D}^{(M_p)}, \quad \text{if}$$

$$\text{for each } \vartheta \in \mathcal{D}^{(M_p)}, \quad (\check{S} * \vartheta)T \in \mathcal{D}'^{*}_{L^1}. \quad (5.6)$$

Definition 5.6 *The convolution* $S *^{5.6} T$ *is defined by*

$$\langle (S *^{5.6} T) * \vartheta, \psi\rangle = \langle (S * \vartheta)(\check{T} * \psi), 1_x\rangle, \quad \text{if}$$

$$\text{for each } \vartheta, \psi \in \mathcal{D}^{(M_p)} \quad (S * \vartheta)(\check{T} * \psi) \in L^1. \quad (5.7)$$

Note, similarly as in the distribution theory one can prove that the mapping $\mathcal{D}^{(M_p)} \to \mathcal{D}'^{(M_p)}$, $\vartheta \mapsto A_\vartheta$, *where*

$$\langle A_\vartheta, \psi\rangle = \int_{\mathbb{R}^d} (\check{S} * \vartheta)(x)(\check{T} * \psi)(x)dx, \quad \psi \in \mathcal{D}^{(M_p)},$$

is a continuous linear and translation invariant mapping from $\mathcal{D}^{(M_p)}$ *into* $\mathcal{E}^{(M_p)}$. *This implies that there exist a unique ultradistribution* G *such that* $G * \vartheta = A_\vartheta$, $\vartheta \in \mathcal{D}^{(M_p)}$. *So* $S *^{5.6} T = G$.

Definition 5.7 *The convolutions* $S *^{5.7} T$, $S *^{5.7'} T$, $S *^{5.7''} T$ *are defined by*

$$\lim_{j\to\infty} (\eta_j S) * T, \quad (5.8)$$

$$\lim_{j\to\infty} S(\tilde{\eta}_j * T), \quad (5.9)$$

$$\lim_{j\to\infty} (\eta_j S)(\tilde{\eta}_j * T), \quad (5.10)$$

respectively, if the limits (5.8), (5.9) and (5.10) respectively exist in $\mathcal{D}'^{(M_p)}$ *for any strong approximate units* (η_j), $(\tilde{\eta}_j)$.

Definition 5.8 *The convolutions $S *^{5.8} T$, $S *^{5.8'} T$, $S *^{5.8''} T$ are defined by*

$$\lim_{j\to\infty} (\eta_j S) * T, \tag{5.11}$$

$$\lim_{j\to\infty} S(\tilde{\eta}_j * T), \tag{5.12}$$

$$\lim_{j\to\infty} (\eta_j S)(\tilde{\eta}_j * T), \tag{5.13}$$

respectively, if the limits (5.11), (5.12) and (5.13) exist in $\mathcal{D}'^{(M_p)}$, respectively, for any special unit sequences (η_j), $(\tilde{\eta}_j)$.

In [49] is proved that the definitions 5.1 and 5.2 are equivalent and that they imply definitions 5.4 and 5.5, and question whether all these definitions are equivalent was left as an open problem. In the distribution theory the equivalence of definitions which are analogous to definitions 5.1, 5.4, 5.5 and 5.6 was proved by Shiraishi ([59]), Dierolf and Voigt ([16]) proved that the definitions given by Schwartz and Vladimirov are equivalent, and Kaminski [31] made an analysis of all the definitions of convolution in relation to various sequences which approximate the unit (see also [65]).

We will need the following assertions

Lemma 5.9 ([33], [13]) *Let K be a compact neighborhood of zero, and $r > 0$. There is $u \in \mathcal{D}^{M_p}_{K,r/2}$ and $\xi \in \mathcal{D}^{(M_p)}_K$ such that*

$$P_r(D)u = \delta + \xi. \tag{5.14}$$

Lemma 5.10 ([49]) *If $f, g \in \mathcal{D}'^{(M_p)}$ are convolvable in the sense of one of the equivalent definitions 5.1 and 5.2 and $P(D)$ is an ultradifferential operator of class (M_p) then*

$$P(D)(g * f) = g * P(D)f.$$

The main assertion of the section the following.

Theorem 5.11 *All above definitions of convolutions of ultradistributions are equivalent, i. e. they define the same ultradistribution.*

Proof: *We will first prove that conditions (5.2), (5.5), (5.6) and (5.7) are equivalent.*

(5.7) ⇒ (5.2). Let $\vartheta \in \mathcal{D}^{(M_p)}$ *be fixed. The mapping* $\mathcal{D}^{(M_p)} \to \mathcal{D}'^{(M_p)}$ *defined by,*

$$\psi \mapsto (S * \vartheta)(\check{T} * \psi), \tag{5.15}$$

is continuous. Since L^1 *has property* (C_{M_p}) *and* $(S*\vartheta)(\check{T}*\psi)$ *belongs to* L^1, *it follows that the mapping* $\mathcal{D}^{(M_p)} \to L^1$ *defined by (5.15) is continuous. This implies the continuity of the mapping*

$$\mathrm{R}^d \to L^1, \qquad y \mapsto (S * \vartheta)(T * \psi(\cdot - y)),$$

where $\psi, \vartheta \in \mathcal{D}^{(M_p)}$ *are fixed. Because of that, for fixed* $\phi \in \mathcal{D}^{(M_p)}$,

$$y \mapsto \int_{\mathrm{R}^d} |\phi(y)(S * \vartheta)(\cdot)(\check{T} * \psi)(\cdot - y)| dy, \quad y \in \mathrm{R}^d,$$

belongs to L^1. *The Fubini theorem implies that for each* $\vartheta, \psi, \phi \in \mathcal{D}^{(M_p)}$, *the function*

$$(x, y) \mapsto (S * \vartheta)(x)(\check{T} * \psi)(x - y), \; (x, y) \in \mathrm{R}^{2d},$$

is from $L^1(\mathrm{R}^{2d})$. *By the change of variables it follows that for each* $\vartheta, \psi, \phi \in \mathcal{D}^{(M_p)}$, $(S * \vartheta)_x(T * \check{\psi})_y \psi^{\triangle}$ *is from* $L^1(\mathrm{R}^{2d})$. *Assume that* Q *and* K *are compact neighborhoods of zero in* R^d *and that* Q *is a subset of the interior of* K. *The mapping*

$$\mathcal{D}_K^{(M_p)} \times \mathcal{D}_K^{(M_p)} \times \mathcal{D}_K^{(M_p)} \to L^1(\mathrm{R}^{2d}), \; (\vartheta, \psi, \phi) \mapsto ((S * \vartheta)_x \otimes (T * \check{\psi})_y)\phi^{\triangle},$$

is separately continuous. Since $\mathcal{D}_K^{(M_p)}$ *is a Fréchet space, the above mapping is continuous. Thus, for some* $r > 0$ *and* C,

$$\| (S * \vartheta)_x \otimes (T * \check{\psi})_y \phi^{\triangle} \|_{L^1(\mathrm{R}^{2d})} \tag{5.16}$$

$$\leq C \left(\| \vartheta \|_{\mathcal{D}^{(M_p)}_{K,r_p}} + \| \psi \|_{\mathcal{D}^{(M_p)}_{K,r_p}} + \| \phi \|_{\mathcal{D}^{(M_p)}_{K,r_p}} \right), \quad \vartheta, \phi, \psi \in \mathcal{D}_K^{(M_p)}.$$

Let $\vartheta, \phi, \psi \in \mathcal{D}_{Q,r}^{M_p}$ *and let* (ϑ_n), (ψ_n) *and* (ϕ_n) *be sequences of elements of* $\mathcal{D}_K^{(M_p)}$ *such that* $(\vartheta_n) \to \vartheta$, $(\psi_n) \to \psi$, $(\phi_n) \to \phi$ *in* $\mathcal{D}_{Q,r}^{M_p}$; *since* $\mathcal{D}^{(M_p)}$ *is permitted such sequences exist. (5.16) implies*

$$\| ((S * \vartheta_n)_x \otimes (T * \check{\psi}_n)_y)\phi_n^{\triangle} \|_{L^1(\mathrm{R}^{2d})}$$

$$\leq C\left(\| \vartheta_n \|_{\mathcal{D}^{M_p}_{K,r}} + \| \psi_n \|_{\mathcal{D}^{M_p}_{K,r}} + \| \phi_n \|_{\mathcal{D}^{M_p}_{K,r}}\right).$$

The sequence $(((S * \vartheta_n)_x \otimes (T * \check{\psi}_n)_y)\phi_n^{\triangle})_n$, *converges in* $L^1(\mathbb{R}^{2d})$. *Also it converges to* $((S * \vartheta)_x \otimes (T * \check{\psi})_y)\phi^{\triangle}$ *in* $\mathcal{D}'^{(M_p)}$ *([33]). Therefore*

$$\| ((S * \vartheta)_x (T * \check{\psi})_y)\phi^{\triangle} \|_{L^1(\mathbb{R}^{2d})} \tag{5.17}$$

$$\leq C\left(\| \vartheta \|_{\mathcal{D}^{M_p}_{K,r}} + \| \psi \|_{\mathcal{D}^{M_p}_{K,r}} + \| \phi \|_{\mathcal{D}^{M_p}_{K,r}}\right)$$

$$= C\left(\| \vartheta \|_{\mathcal{D}^{M_p}_{Q,r}} + \| \phi \|_{\mathcal{D}^{M_p}_{Q,r}} + \| \psi \|_{\mathcal{D}^{M_p}_{Q,r}}\right) < \infty.$$

Lemma 5.9 implies that there exist $u \in \mathcal{D}^{M_p}_{Q,r}$ *and* $\xi \in \mathcal{D}^{(M_p)}_Q$ *such that*

$$\delta = P_{2r}(D)u + \xi\,. \tag{5.18}$$

Thus, for each $\phi \in \mathcal{D}^{(M_p)}_K$,

$$(S_x \otimes T_y)\phi^{\triangle} = ((S * P_{2r}(D)u + S * \xi)_x \otimes (T * P_{2r}(D)u + T * \xi)_y)\,\phi^{\triangle}$$

$$= ((S * P_{2r}(D)u)_x \otimes (T * P_{2r}(D)u)_y)\,\phi^{\triangle} + \tag{5.19}$$

$$+ ((S * \xi)_x \otimes (T * P_{2r}(D)u)_y)\,\phi^{\triangle} +$$

$$+ ((S * P_{2r}(D)u)_x \otimes (T * \xi)_y)\,\phi^{\triangle} + ((S * \xi)_x \otimes (T * \xi)_y)\,\phi^{\triangle}.$$

From (5.17) it follows that $((S * u)_x \otimes (T * u)_y)\phi^{\triangle}$, $((S * \xi)_x \otimes (T * u)_y)\phi^{\triangle}$, $((S * u)_x \otimes (T * \xi)_y)\phi^{\triangle}$ *and* $((S * \xi)_x \otimes (T * \xi)_y\phi^{\triangle})$ *belong to* L^1. *Let us prove that* $((S * P_{2r}(D)u)_x \otimes (T * P_{2r}(D)u)_y)\phi^{\triangle}$ *belongs to* $\mathcal{D}'^{(M_p)}_{L^1}$, *which imply that all the terms in (5.19) are from* $\mathcal{D}'^{(M_p)}_{L^1}$ *This imply (5.7)* $\Rightarrow$ *(5.2). Applying Lemma 5.10 we obtain*

$$((S * P_{2r}(D)u)_x \otimes (T * P_{2r}(D)u)_y)\,\phi^{\triangle} \tag{5.20}$$

$$= (P_{2r}(D_x)P_{2r}(D_y)(S * u)_x \otimes (T * u)_y)\,\phi^{\triangle}$$

$$= \sum_{\alpha}\sum_{\beta} a_\alpha a_\beta \sum_{i \leq \alpha}\sum_{j \leq \beta} (-1)^{i+j}\binom{\alpha}{i}\binom{\beta}{j}\cdot$$

$$\cdot\partial_x^{\alpha-i}\partial_y^{\beta-j}\left(((S * u)_x \otimes (T * u)_y)(\phi^{(i+j)})^{\triangle}\right).$$

Therefore, for each $w \in \mathcal{D}^{(M_p)}(\mathbb{R}^{2d})$,

$$\langle ((S * P_{2r}(D)u)_x \otimes (T * P_{2r}(D)u)_y)\,\phi^{\triangle},\, w(x,y)\rangle$$

$$= \sum_{\alpha}\sum_{\beta} a_\alpha a_\beta \sum_{i\le\alpha}\sum_{j\le\beta}(-1)^{i+j}\binom{\alpha}{i}\binom{\beta}{j}\cdot$$
$$\cdot\langle(((S*u)_x \otimes (T*u)_y)(\phi^{(i+j)})^{\triangle},\ w^{(\alpha-i,\beta-j)}\rangle.$$

Since the mapping

$$\mathcal{D}^{(M_p)} \longrightarrow L^1(\mathbf{R}^{2d}) \quad \phi \mapsto (S*\vartheta)_x \otimes (T*\check{\psi})_y \phi^{\triangle},$$

where $\psi, \vartheta \in \mathcal{D}^{(M_p)}$ *are fixed, is continuous and* $\{\phi^{(\gamma)}/M_{|\gamma|}; \gamma \in \mathbf{N}^d\}$ *is a bounded set in* $\mathcal{D}^{(M_p)}$*, the set*

$$\{((S*\vartheta)_x \otimes (T*\check{\psi})_y)(\phi^{(\gamma)})^{\triangle}/M_{|\gamma|};\ \gamma \in \mathbf{N}^d\}$$

is bounded in $L^1(\mathbf{R}^{2d})$*. The inequality* $M_p M_q \le M_{p+q}$*,* $p,q \in \mathbf{N}_0$*, which follows from (M.1), implies that there exist* C *and* $h > 0$ *such that for each* $i,j \in \mathbf{N}^d$*,*

$$|\langle((S*u)_x \otimes (T*u)_y)\frac{(\phi^{(i+j)})^{\triangle}}{M_{|i|}M_{|j|}},\ \nu(x,y)\rangle| \le C \sum_{\omega\in\mathbf{N}^{2d}} \frac{h^{|\omega|}}{M_{|\omega|}} \parallel \nu^{(\omega)} \parallel_{L^\infty(\mathbf{R}^{2d})}.$$

Put $\omega = (\omega_1,\omega_2)$*,* $\omega_1,\omega_2 \in \mathbf{N}_0^d$*. From (M.1), (M.2) and (5.20) it follows*

$$|\langle((S*P_{2r}(D)u)_x \otimes (T*P_{2r}(D)u)_y)\,\phi^{\triangle},\ \nu)\rangle|$$
$$\le C \sum_{\alpha\in\mathbf{N}^d}\sum_{\beta\in\mathbf{N}^d}\sum_{i\le\alpha}\sum_{j\le\beta}\binom{\alpha}{i}\binom{\beta}{j} a_\alpha a_\beta M_{|i|}M_{|j|}\cdot$$
$$\cdot\sum_{\substack{\omega_1\in\mathbf{N}^d\\ \omega_2\in\mathbf{N}^d}} \frac{h^{|\omega_1|+|\omega_2|}}{M_{|\omega_1|+|\omega_2|}} \parallel \nu^{(\alpha-i+\omega_1,\beta-j+\omega_2)} \parallel_{L^\infty(\mathbf{R}^{2d})}$$
$$\le C \sum_{\alpha\in\mathbf{N}^d}\sum_{\beta\in\mathbf{N}^d}\sum_{i\le\alpha}\sum_{j\le\beta}\sum_{\substack{\omega_1\in\mathbf{N}^d\\ \omega_2\in\mathbf{N}^d}}\binom{\alpha}{i}\binom{\beta}{j}\frac{L^{|\alpha|+|\beta|}}{M_{|\alpha|}M_{|\beta|}}\cdot$$
$$\cdot M_{|i|}M_{|j|}\frac{h^{|\omega_1|+|\omega_2|}}{M_{|\omega_1|+|\omega_2|}} \parallel \nu^{(\alpha-i+\omega_1,\beta-j+\omega_2)} \parallel_{L^\infty(\mathbf{R}^{2d})}$$
$$\le C \sum_{\alpha\in\mathbf{N}^d}\sum_{\beta\in\mathbf{N}^d}\sum_{i\le\alpha}\sum_{j\le\beta}\sum_{\substack{\omega_1\in\mathbf{N}^d\\ \omega_2\in\mathbf{N}^d}}\binom{\alpha}{i}\binom{\beta}{j} L^{|\alpha|+|\beta|}h^{|\omega_1|+|\omega_2|}\cdot$$
$$\cdot\frac{H^{|\alpha|+|\omega_1|+|\beta|+|\omega_2|}}{M_{|\alpha|-|i|+|\omega_1|+|\beta|-|j|+|\omega_2|}} \parallel \nu^{(\alpha-i+\omega_1,\beta-j+\omega_2)} \parallel_{L^\infty(\mathbf{R}^{2d})}$$

$$\leq C \sum_{\alpha\in\mathbb{N}^d}\sum_{\beta\in\mathbb{N}^d}\sum_{i\leq\alpha}\sum_{j\leq\beta}\sum_{\substack{\omega_1\in\mathbb{N}^d\\ \omega_2\in\mathbb{N}^d}}\binom{\alpha}{i}\binom{\beta}{j}\frac{1}{4^{|\alpha|+|\beta|+|\omega_1|+|\omega_2|}}\cdot$$

$$\cdot\sum_{\gamma\in\mathbb{N}^{2d}}\frac{((1+L)(1+h)(1+4h))^{|\gamma|}}{M_{|\gamma|}}\parallel \nu^{(\gamma)}\parallel_{L^\infty(\mathbb{R}^{2d})}\cdot$$

This implies that $((S*P_r(D)u)_x\otimes(T*P_r(D)u)_y)\phi^{\triangle}$ *belongs to* $\mathcal{D}'^{(M_p)}_{L^1}$.

Implications (5.2) $\Rightarrow$ *(5.5) and (5.2)* $\Rightarrow$ *(5.6) were proved in [49, Proposition 6.].*

Let us prove (5.5) $\Rightarrow$ *(5.7). The implication (5.6)* $\Rightarrow$ *(5.7) can be proved analogously. The mappings*

$$\mathcal{D}^{(M_p)}\longrightarrow\mathcal{D}'^{(M_p)}_{L^1},\ \ \psi\mapsto S(\check{T}*\psi),$$

$$\mathbb{R}^d\times\mathcal{D}'^{(M_p)}_{L^1}\longrightarrow\mathcal{D}'^{(M_p)}_{L^1},\ \ (y,U)\mapsto U(\cdot-y),$$

are continuous. This implies that for every $\psi\in\mathcal{D}^{(M_p)}$ *the mapping*

$$\mathbb{R}^d\longrightarrow\mathcal{D}'^{(M_p)}_{L^1},\ \ y\mapsto(S(\cdot-y))(\check{T}*\psi)$$

is continuous. Therefore (see [8]),

$$\int\vartheta(y)(S(\cdot-y))(\check{T}*\psi)(\cdot)dy\in\mathcal{D}'^{(M_p)}_{L^1},\ \text{for each}\ \vartheta\in\mathcal{D}^{(M_p)},$$

where the above integral is defined by

$$\langle\int\vartheta(y)(S(\cdot-y))(\check{T}*\psi)(\cdot)dy,\omega\rangle$$

$$=\int\langle\vartheta(y)(S(\cdot-y))(\check{T}*\psi)(\cdot),\omega(y)\rangle dy,\ \ \omega\in\mathcal{D}^{(M_p)}.$$

For any $\phi\in\mathcal{D}^{(M_p)}$ *there holds*

$$\langle\int\vartheta(y)(S(x-y))(\check{T}*\psi)(x)dy,\ \phi(x)\rangle$$

$$=\int\vartheta(y)\langle(S(x-y))(\check{T}*\psi)(x),\ \phi(x)\rangle dy$$

$$=\langle\vartheta(y)\langle S(x-y),\ (\check{T}*\psi)\phi(x)\rangle dy=\langle\int\vartheta(y)S(x-y)dy,\ (\check{T}*\psi)(x)\phi(x)\rangle$$

$$=\langle(S*\vartheta),\ (\check{T}*\psi)\phi\rangle=\langle(S*\vartheta)(\check{T}*\psi),\ \phi\rangle.$$

This yields that $(S * \vartheta)(\check{T} * \psi) \in \mathcal{D}'^{(M_p)}_{L^1}$*, for each* $\vartheta, \psi \in \mathcal{D}^{(M_p)}$ *and [50, Theorem 3.] implies that* $((S * \vartheta)(\check{T} * \psi)) * \phi \in L^1$*, for each* $\phi, \vartheta, \psi \in \mathcal{D}^{(M_p)}$*. According to Lemma 5.9,*

$$(S * \vartheta)(\check{T} * \psi) = ((S * \vartheta)(\check{T} * \psi)) * P_r(D)u + ((S * \vartheta)(\check{T} * \psi)) * \xi.$$

The similar arguments as in the proof of the implication (5.7) $\Rightarrow$ *(5.2) imply* $(S * \vartheta)(\check{T} * \psi) \in L^1$*, for each* $\vartheta, \psi \in \mathcal{D}^{(M_p)}$*.*

(5.3)$\Rightarrow$*(5.10) & (5.4)*$\Rightarrow$ *(5.13). Since for an arbitrary* $\vartheta \in \mathcal{D}^{(M_p)}$*,*

$$\begin{aligned} \langle (\eta_j S) * (\tilde{\eta}_j T), \vartheta \rangle &= \langle (\eta_j S)_x \otimes (\tilde{\eta}_j T)_y, \vartheta^{\Delta} \rangle \\ &= \langle (S_x \otimes T_y)\vartheta(x+y), \eta_j(x)\tilde{\eta}_j(y) \rangle, \end{aligned}$$

it follows from (5.3) that the limit (5.10) exists for all strong approximate units (η_j), $(\tilde{\eta}_j) \subset \mathcal{D}^{(M_p)}$*, i.e. condition (5.10) is fulfilled and* $S *^{5.10} T = S *^{5.2} T$*. Similarly (5.4) implies (5.13) and* $S *^{5.8''} T = S *^{5.3} T$*.*

(5.10)$\Rightarrow$*(5.8) & (5.13)* $\Rightarrow$*(5.11). Note that (5.10) implies that for any strong approximate units* (η_j), $(\tilde{\eta}_j) \subset \mathcal{D}^{(M_p)}$ *and* $\vartheta \in \mathcal{D}^{(M_p)}$*,*

$$\lim_{i,j\to\infty} \langle (\eta_i S) * (\tilde{\eta}_j T), \vartheta \rangle = \langle S *^{5.7''} T, \vartheta \rangle. \tag{5.21}$$

In fact, if (5.8) were not true, there would exist $\vartheta \in \mathcal{D}^{(M_p)}$, $\epsilon > 0$ *and increasing sequences* (i_k) *and* (j_k) *of positive integers such that*

$$|\langle (\eta_{i_k}) * (\tilde{\eta}_{j_k} T), \vartheta \rangle - \langle S *^{5.7''} T, \vartheta \rangle| > \epsilon.$$

But since (η_{i_k}) *and* $(\tilde{\eta}_{j_k})$ *are again strong approximate units, the above inequality would contradict (5.10). Now (5.21) yields*

$$\langle S *^{5.7''} T, \vartheta \rangle = \lim_{i\to\infty} \lim_{j\to\infty} \langle (\eta_j S) * (\tilde{\eta}_j T), \vartheta \rangle = \lim_{i\to\infty} \langle (\eta_i) * T, \vartheta \rangle,$$

which implies (5.8) and the identity $S *^{5.7''} T = S *^{5.7} T$*. In the same way one proves that (5.13) implies (5.11) and* $S *^{5.8''} T = S *^{5.8} T$*.*

The implications (5.10)$\Rightarrow$*(5.9) and (5.13)*$\Rightarrow$*(5.12) follows from the preceding ones by symmetry.*

(5.8)$\Rightarrow$*(5.5) & (5.11)*$\Rightarrow$*(5.5). Since*

$$\langle (\eta_j S) * T, \vartheta \rangle = \langle \eta_j S, \vartheta * \check{T} \rangle = \langle S(\check{T} * \vartheta), \eta_j \rangle,$$

we infer from (5.8) (resp. (5.11)) that $S(\check{T} * \vartheta) \in \mathcal{D}'^{(M_p)}_{L^1}$ *for* $\vartheta \in \mathcal{D}^{(M_p)}$, *so (5.5) holds and* $S *^{5.7} T = S *^{5.4} T$ *(resp.* $S *^{5.8} T = S *^{5.4} T$*).*

(5.9)⇒(5.6) and (5.12)⇒(5.6) follows from the preceding ones by symmetry.

One can prove by standard arguments that

$$S *^{5.1} T = S *^{5.2} T = S *^{5.3} T = S *^{5.4} T = S *^{5.5} T = S *^{5.6} T. \square$$

5.2 Ultratempered Convolution

In this section we introduce the notion of ultratempered convolution of ultradistribution of Beurling type, $\mathcal{S}'^{(M_p)}$*-convolution, by giving several equivalent definitions of it. In the theory of distributions the equivalence of various definitions of* $\mathcal{S}'$*-convolution and of* $\mathcal{K}'\{M_p\}$*-convolution was proved by Shiraishi ([59]), Dierolf, Vogt ([16]) and Uryga ([61]).*

If $T \in \mathcal{S}'^{(M_p)}$ *and* $S \in \mathcal{E}'^{(M_p)}$*, then the convolution* $T * S$ *(in the sense of previous section) belongs to* $\mathcal{S}'^{(M_p)}$ *and*

$$\langle T * S, \vartheta\rangle = \langle T, \check{S} * \vartheta\rangle, \qquad \vartheta \in \mathcal{S}^{(M_p)}.$$

This implies that for every ultradifferential operator $P(D)$ *of class* (M_p)

$$\langle P(D)(T * S), \vartheta\rangle = \langle (P(D)T) * S, \vartheta\rangle = \langle T * (P(D)S), \vartheta\rangle, \quad \vartheta \in \mathcal{S}^{(M_p)}. \tag{5.22}$$

In the proof of the main theorem of this section we will use that $\mathcal{O}_M^{(M_p)}$ *is nuclear. which is proved in [51].*

Theorem 5.12 *Let* $S, T \in \mathcal{D}'^{(M_p)}$*. The following conditions are equivalent*

1. $\vartheta^{\triangle}(S_x \otimes T_y) \in \mathcal{D}'^{(M_p)}_{L^1}(\mathbf{R}^{2d})$ *for every* $\vartheta \in \mathcal{S}^{(M_p)}$*;*
2. *For every* $\vartheta \in \mathcal{S}^{(M_p)}$ *and a strong approximate unit* (η_j) *defined on* $\mathbf{R}^{2d}$*, the sequence* $(\langle (S_x \otimes T_y), \eta_j \vartheta^{\triangle}\rangle)_j$ *converges to a constant.*
3. $(\check{S} * \vartheta)T \in \mathcal{D}'^{(M_p)}_{L^1}$ *for every* $\vartheta \in \mathcal{S}^{(M_p)}$*;*
4. $(\check{T} * \vartheta)S \in \mathcal{D}'^{(M_p)}_{L^1}$ *for every* $\vartheta \in \mathcal{S}^{(M_p)}$*;*

5. $(S*\vartheta)(\check{T}*\psi)\in L^1$ *for every* $\vartheta\in\mathcal{D}^{(M_p)}$ *and* $\psi\in\mathcal{S}^{(M_p)}$;

6. $(S*\vartheta)(\check{T}*\psi)\in L^1$ *for every* $\vartheta\in\mathcal{S}^{(M_p)}$ *and* $\psi\in\mathcal{D}^{(M_p)}$;

7. $(S*\vartheta)(\check{T}*\psi)\in L^1$ *for every* $\vartheta,\psi\in\mathcal{S}^{(M_p)}$.

Proof: *The proof of 1$\Longleftrightarrow$2 follows directly from [49, Proposition 4].*

1 $\Rightarrow$ 3 It is easy to check that $(\zeta_x\otimes 1_y)\vartheta^{\triangle}\in\mathcal{S}^{(M_p)}(\mathrm{R}^{2d})$, *for every* $\zeta\in\mathcal{D}^{(M_p)}$ *and* $\vartheta\in\mathcal{S}^{(M_p)}$. *Thus, for an arbitrary strong approximate unit* (η_n) *and* $\phi\in\mathcal{D}^{(M_p)}$,

$$
\begin{aligned}
&\langle(\check{S}*\vartheta)T,\eta_n\phi\rangle\\
&=\lim_{m\to\infty}\langle T_x,\langle S_y,(\eta_n\phi)_x\otimes(\eta_m)_y\vartheta^{\triangle}\rangle\rangle\\
&=\lim_{m\to\infty}\langle T_x\otimes S_y,((\eta_n\phi)_x\otimes(\eta_m)_y)\vartheta^{\triangle}\rangle\\
&=\lim_{m\to\infty}\langle\vartheta^{\triangle}(T_x\otimes S_y),(\eta_n\phi)_x\otimes(\eta_m)_y\rangle\\
&=\langle\vartheta^{\triangle}(T_x\otimes S_y),(\eta_n\phi)_x\otimes 1_y\rangle.
\end{aligned}
$$

Since the limit

$$\lim_{n\to\infty}\langle\vartheta^{\triangle}(T_x\otimes S_y),(\eta_n\phi)\otimes 1\rangle$$

exists, it follows that $(\check{S}*\vartheta)T\in\mathcal{D}'^{(M_p)}_{L^1}$.

4$\Rightarrow$5 The proof is carried out in a similar manner as in the proof of (5.5)$\Rightarrow$(5.7) in Theorem 5.11, so it will be omitted.

5$\Rightarrow$1 For each $\alpha,\beta\in\mathcal{D}^{(M_p)}$,

$$
\begin{aligned}
\langle((S*\vartheta)_x\otimes\psi_y)T^{\triangle},\alpha_x\otimes\beta_y\rangle &= \langle(S*\vartheta)(x)\psi(y)T(x+y)\alpha(x)\beta(y),1_{x,y}\rangle \qquad (5.23)\\
&=\langle(S*\vartheta)_x(\check{T}*(\beta\psi))_x\alpha_x,1\rangle.
\end{aligned}
$$

The mapping $\psi\mapsto(S*\vartheta)(\check{T}*\psi)$ *from* $\mathcal{S}^{(M_p)}$ *into* $\mathcal{D}'^{(M_p)}$ *is continuous. Since* $(S*\vartheta)(\check{T}*\psi)$ *belongs to* L^1, *and* L^1 *has the property* (C_{M_p}) *it is continuous as the mapping from* $\mathcal{S}^{(M_p)}$ *into* L^1. *This implies that the composition mapping*

$$\mathcal{O}_M^{(M_p)}\to\mathcal{S}^{(M_p)}\to L^1\quad a\mapsto\psi a\mapsto(S*\vartheta)(\check{T}*\psi a)$$

is continuous as well. Therefore $((S*\vartheta)\otimes\psi)T^{\triangle}$ *can be extended on* $\dot{\mathcal{B}}^{(M_p)}\hat{\otimes}_\pi\mathcal{O}_M^{(M_p)}$ *as a continuous mapping. Since the space* $\mathcal{O}_M^{(M_p)}$ *is nuclear we have*

$$\dot{\mathcal{B}}^{(M_p)}\hat{\otimes}_\pi\mathcal{O}_M^{(M_p)}=\dot{\mathcal{B}}^{(M_p)}\hat{\otimes}_\epsilon\mathcal{O}_M^{(M_p)}\supset\dot{\mathcal{B}}^{(M_p)}(\mathrm{R}^d)\hat{\otimes}_\epsilon\dot{\mathcal{B}}^{(M_p)}(\mathrm{R}^d)\supset\dot{\mathcal{B}}^{(M_p)}(\mathrm{R}^{2d}),$$

where the corresponding inclusion mappings are continuous.

Let $\phi \in \dot{\mathcal{B}}^{(M_p)}(\mathrm{R}^{2d})$, *and let*

$$\phi_n = \sum_{i=1}^{m_n} (\phi_{i,n})_x \otimes (\psi_{i,n})_y \to \phi \quad \text{as in } \dot{\mathcal{B}}^{(M_p)}(\mathrm{R}^{2d}), \quad n \to \infty.$$

Clearly $(\phi_n) \to \phi$ *in the sense of topology of* $\dot{\mathcal{B}}^{(M_p)}(\mathrm{R}^d)\hat{\otimes}_\epsilon \dot{\mathcal{B}}^{(M_p)}(\mathrm{R}^d)$ *and thus*

$$\langle ((S * \vartheta)_x \otimes \psi_y) T^{\Delta}, \phi_n \rangle \to \langle ((S * \vartheta)_x \otimes \psi_y) T^{\Delta}, \phi \rangle, \quad n \to \infty.$$

This implies that $((S * \vartheta) \otimes \psi) T^{\Delta} \in \mathcal{D}'^{(M_p)}_{L^1}$.

The implications 1⇒3⇒6 ⇒1 can be proved in a similar way as 5⇒1.

The implication 7⇒5 is clear.

(5)⇒(7). Let (5) holds. In the same way as in [59] we get that $(S*\vartheta)(\check{T}*\psi) \in \mathcal{D}'^{(M_p)}_{L^1}$, *for each* $\vartheta, \psi \in \mathcal{S}^{(M_p)}$. *Analogously as in the proof of (5.5)⇒(5.7) in Theorem 5.11, using continuity of the mapping*

$$\mathcal{D}^{(M_p)} \to L^1, \quad \phi \mapsto (S * \vartheta)(\check{T} * \psi)\phi$$

one can prove that $(S * \vartheta)(\check{T} * \psi) \in L^1$, *for each* $\vartheta, \psi \in \mathcal{S}^{(M_p)}$. □

Chapter 6

Hypoellipticity in $\mathcal{D}'^{*}_{L^q}$

The results of Malgrange, Ehrenpreis and Hörmander on the solvability and hypoellipticity of convolution equations in Schwartz's spaces stimulated many mathematicians to study such problems in various subspaces of distributions. We cite here only results of Zielezny ([68], [69]) and Pahk ([46]), since they are connected with our results. In the spaces of ultradistributions convolution equations were studied by Braun, Meise, Taylor, Voigt and their cooperators (see [5], [43] and references there) and by Pilipović ([51]), who started investigations of hypoelliptic convolution equations in ultradistribution spaces similarly as it was done for distribution spaces by Zielezny. In the chapter we study hypoelliptic convolution equations in the Beurling and Roumieu ultradistribution spaces $\mathcal{D}'^{(M_p)}_{L^q}$ *and* $\mathcal{D}'^{\{M_p\}}_{L^q}$, $q \in [1,\infty]$. *The spaces were investigated by Pilipović, Corănesku, Charamichael, Pathak (see [48], [15], [9], [10]). They are generalizations of the space* $\mathcal{D}'_{L^q}$. *An analogous problem but in the distribution spaces* $\mathcal{D}'_{L^q}$, $q \in [1,\infty]$, *was investigated by Phak ([46]). Some Phak's considerations are easily transferred to the problem which we consider, but many problems appeared to be specific for ultradistributions and they have been solved.*

In this chapter we suppose that the conditions (M.1), (M.2) and (M.3) hold, and that the Fourier transform of $\varphi \in L^1$ *is given by*

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{R}} \varphi(x) e^{-i\xi x} dx.$$

By $\mathcal{O}'^{*}_{c}$ we denote the space of convolution operators-convolutors of $\mathcal{S}'^{*}$, which explicit characterization is given in [50, Proposition 9]. In this chapter we will use only one of the properties of this space, namely, the fact that the Fourier transform is isomorphism of $\mathcal{O}^{*}_{M}$ onto $\mathcal{O}'^{*}_{c}$,

We define hypoelliptic convolution operators in $\mathcal{D}'^{*}_{L^\infty}$ as follows: An ultradistribution $S \in \mathcal{D}'^{*}_{L^1}$ is hypoelliptic in $\mathcal{D}'^{*}_{L^\infty}$ if every solution U in $\mathcal{D}'^{*}_{L^\infty}$ of the convolution equation

$$S * U = V \tag{6.1}$$

belongs to $\mathcal{D}^{*}_{L^\infty}$, when V is in $\mathcal{D}^{*}_{L^\infty}$. In that case equation (6.1) is also called hypoelliptic in $\mathcal{D}'^{*}_{L^\infty}$.

The space of convolution operators in $\mathcal{D}'^{*}_{L^\infty}$ is $\mathcal{D}'^{*}_{L^1}$, therefore hypoelliptic convolution operators in $\mathcal{D}'^{*}_{L^\infty}$ has to be characterized as a subspace of $\mathcal{D}'^{*}_{L^1}$. Because of lack of differentiability of their Fourier transforms, in this paper we consider only the subclasses of $\mathcal{D}'^{*}_{L^1}$ containing $\mathcal{O}'^{*}_{c}$, whose Fourier transforms are C^∞-functions of ultrapolynomial growth.

In this classes we characterize hypoelliptic convolution operators in $\mathcal{D}'^{*}_{L^\infty}$. But we have an example of hypoelliptic convolution operator in $\mathcal{D}'^{*}_{L^\infty}$ which is not in this class.

We will now establish a necessary and sufficient condition for a convolution operator to be hypoelliptic in $\mathcal{D}'^{*}_{L^\infty}$. The result is proved only for subclasses of convolution operators $\mathcal{D}'^{*}_{L^\infty}$ and the proof is based on an idea similar to the one for distribution spaces used in [46], [68] and [69], however some problems appeared to be specific for ultradistributions and they have been solved.

Definition 6.1 *An ultradistribution $S \in \mathcal{D}'^{(M_p)}_{L^1}$ (resp. $\mathcal{D}'^{\{M_p\}}_{L^1}$) is said to be of class H_a, $a > 0$, (resp. H_{a_p}, $(a_p) \in \mathcal{R}$) if the Fourier transform $\hat{S}$ is C^∞-function such that there exists $\ell > 0$ (resp. $(\ell_p) \in \mathcal{R}$) such that*

$$\sum_{\alpha \in \mathbb{N}} \frac{1}{a^\alpha M_\alpha} \hat{S}^{(\alpha)}(x) = \mathcal{O}(\exp[M(\ell|x|)]),$$
$$\left(resp. \; \sum_{\alpha \in \mathbb{N}} \frac{1}{\left(\prod_{1 \le \beta \le \alpha} a_\beta\right) M_\alpha} \hat{S}^{(\alpha)} = \mathcal{O}(\exp[M(\ell|x|)]) \right), \quad |x| \to \infty. \tag{6.2}$$

The above defined class of ultradistributions will be used for our study of hypoellipticity in $\mathcal{D}'^{(M_p)}_{L^\infty}$ *(resp.* $\mathcal{D}'^{\{M_p\}}_{L^\infty}$*).*

Lemma 6.2 *Let* S *be an ultradistribution whose Fourier transform is of the form*

$$\hat{S} = \sum_{j\in\mathbb{N}} a_j \delta_{\xi_j}, \tag{6.3}$$

where ξ_j *is a sequence of real numbers, such that*

$$2^j < 2|\xi_{j-1}| < |\xi_j|, \quad j \in \mathbb{N}, \tag{6.4}$$

and a_j *are complex numbers such that for some (resp. each)* $m > 0$

$$|a_j| = \mathcal{O}(\exp[M(m|\xi_j|)]). \tag{6.5}$$

1. S *belongs to* $\mathcal{D}'^{(M_p)}_{L^\infty}$ *(resp.* $\mathcal{D}'^{\{M_p\}}_{L^\infty}$*);*
2. S *belongs to* $\mathcal{D}^{(M_p)}_{L^\infty}$ *(resp.* $\mathcal{D}^{\{M_p\}}_{L^\infty}$*) if and only if for each (resp. some)* $k > 0$,
$$|a_j| = o(\exp[-M(k|\xi_j|)]). \tag{6.6}$$

Note, according to [35, Lemma 3.4] in the Roumieu case condition (6.5) (i. e. "for each $m > 0$ *(6.5) holds") is equivalent to*

$$|a_j| = \mathcal{O}(\exp[N_{m_p}(|\xi_j|)]), \tag{6.7}$$

for some $(m_p) \in \mathcal{R}$*, and condition (6.6) is equivalent to*

$$|a_j| = o(\exp[-N_{k_p}(|\xi_j|)]) \tag{6.8}$$

for each $(k_p) \in \mathcal{R}$*.*

Proof: *1. Let us first prove that the sum* $S = \sum_{j\in\mathbb{N}} a_j e^{ix\xi_j}$ *converges in* $\mathcal{D}'^{(M_p)}_{L^\infty}$ *(resp.* $\mathcal{D}'^{\{M_p\}}_{L^\infty}$*). Suppose that* $\varphi \in \mathcal{D}^{(M_p)}_{L^1}$ *(resp.* $\mathcal{D}^{\{M_p\}}_{L^1}$*). Using the fact that for each ultrapolynomial* P *of class* (M_p) *(resp.* $\{M_p\}$*)*

$$|P(\xi)\mathcal{F}^{-1}\varphi(\xi)| \le \|P(D)\varphi\|_1,$$

we conclude that for each $b > 0$, some $\ell > 0$ and C (resp. for each $(b_p) \in \mathcal{R}$, some $(\ell_p) \in \mathcal{R}$ and C),

$$|\mathcal{F}^{-1}\varphi(\xi)| \leq \frac{C}{P_b(\xi)} \sum_{\alpha\in\mathbf{N}_0} \frac{1}{\ell^\alpha M_\alpha}\|\varphi^{(\alpha)}\|_1$$

$$\left(\text{resp. } |\mathcal{F}^{-1}\varphi(\xi)| \leq \frac{C}{P_{b_p}(\xi)} \sum_{\alpha\in\mathbf{N}_0} \frac{1}{(\prod\limits_{1\leq\beta\leq\alpha} \ell_\beta)M_\alpha}\|\varphi^{(\alpha)}\|_1\right), \quad \xi\in\mathbf{R}. \tag{6.9}$$

Hence for $b > 0$, such that $b < 2m$ and some $\ell > 0$ (resp. $(b_p) \in \mathcal{R}$, such that $b_p < 2m_p$, $p \in \mathbf{N}$, and some $(\ell_p) \in \mathcal{R}$)

$$|\langle S,\varphi\rangle| \leq \sum_{j\in\mathbf{N}} |a_j\langle e^{ix\xi_j},\varphi\rangle| \leq \sum_{j\in\mathbf{N}} |a_j||\mathcal{F}^{-1}\varphi(\xi_j)|$$

$$\leq C\sum_{j\in\mathbf{N}} \exp[M(m|\xi_j|)-M(2b|\xi_j|)] \left(\sum_{\alpha\in\mathbf{N}} \frac{1}{\ell^\alpha M_\alpha}\|\varphi^{(\alpha)}\|_1\right) \leq C\left(\sum_{\alpha\in\mathbf{N}_0} \frac{\ell^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\|_1\right)$$

$$\left(\text{resp. } |\langle S,\varphi\rangle| \leq C\left(\sum_{\alpha\in\mathbf{N}_0} \frac{1}{(\prod\limits_{1\leq\beta\leq\alpha} \ell_\beta)M_\alpha}\|\varphi^{(\alpha)}\|_1\right)\right),$$

which imply that S converge in $\mathcal{D}'^{(M_p)}_{L^\infty}$ (resp. $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^\infty}$). In the Roumieu case the convergence in $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^\infty}$ imply that S converge in $\mathcal{D}'^{\{M_p\}}_{L^\infty}$.

2. Suppose that S belongs to $\mathcal{D}^{(M_p)}_{L^\infty}$ (resp. $\mathcal{D}^{\{M_p\}}_{L^\infty}$). For every ultraderivative $P(D)$ of class (M_p) (resp. $\{M_p\}$) and every $\varphi \in \mathcal{D}^{(M_p)}_{L^1}$ (resp. $\mathcal{D}^{\{M_p\}}_{L^1}$),

$$\langle e^{ixu}(P(D)S(x)),\varphi(x)\rangle \to 0 \quad \text{as } |u|\to\infty,\ u\in\mathbf{R}.$$

This follows from the next calculation,

$$|\langle e^{ixu}(P(D)S(x)),\varphi(x)\rangle| = \left|\frac{1}{u}\int_{\mathbf{R}}(P(D)S(x))\varphi(x)De^{ixu}dx\right|$$

$$\leq \frac{1}{|u|}\int_{\mathbf{R}}|D(P(D)S(x))\varphi(x)|dx \leq \frac{C}{|u|}.$$

Note, $\mathcal{D}^*_{L^1} \subset \mathcal{D}^*_{L^2}$, $\varphi\in L^2$. Passing to the Fourier transform we get

$$\langle e^{ixu}(P(D)S(x)),\varphi(x)\rangle = \langle \mathcal{F}(e^{ixu}(P(D)S(x)))(\xi),\mathcal{F}(\varphi)(\xi)\rangle$$

$$= \langle P(\xi+u)\hat{S}(\xi+u), \hat{\varphi}(\xi)\rangle = \langle \hat{S}(\xi), P(\xi)\hat{\varphi}(\xi-u)\rangle.$$

Therefore, for each ultraderivative $P(D)$ of class (M_p) (resp. $\{M_p\}$)

$$\sum_{j\in\mathbf{N}} a_j P(\xi_j)\hat{\varphi}(\xi_j - u) \to 0 \qquad as\ |u| \to \infty,\ u \in \mathbf{R} \tag{6.10}$$

Let us fix $\varphi \in \mathcal{D}'^{(M_p)}_{L^1}$ (resp. $\mathcal{D}'^{\{M_p\}}_{L^1}$) so that

$$|\hat{\varphi}(0)| \geq 1 \tag{6.11}$$

and

$$\hat{\varphi}(\xi) = 0 \quad for \quad |\xi| \geq 1. \tag{6.12}$$

Suppose that condition (6.6) is not satisfied. There is $c \in \mathbf{N}$ (resp. $(c_p) \in \mathcal{R}$) and $A > 0$, such that

$$\exp[M(c|\xi_j|)]|a_j| \geq A \quad (resp.\ \exp[N_{c_p}(|\xi_j|)]|a_j| \geq A\ ) \tag{6.13}$$

for a subsequence of (a_j), which we may take as the whole sequence without loss of generality. Let $u_j = \xi_j$, $j \in \mathbf{N}$. Making use of (6.4) and (6.12) we obtain

$$\sum_{\substack{j\in\mathbf{N}\\ j\neq k}} a_j P(\xi_j)\hat{\varphi}(\xi_j - u_k) = 0.$$

On the other hand conditions (6.11) and (6.13) imply that if $P = P_c$ (resp. $P = P_{c_p}$)

$$|a_k|P_c(\xi_k)\hat{\varphi}(0) \geq A \quad (resp.\ |a_k|P_{c_p}(\xi_k)\hat{\varphi}(0) \geq A\ ).$$

This contradicts the convergence (6.10).

Conversely, if (6.6) hold then

$$\sup_{\alpha\in\mathbf{N}_0} \frac{h^\alpha}{M_\alpha}\|S^{(\alpha)}\|_\infty \leq \sup_{\alpha\in\mathbf{N}_0} \sum_{j\in\mathbf{N}} \|a_j(i\xi_j)^\alpha e^{i\xi_j}\|_\infty < \infty,$$

which imply $S \in \mathcal{D}^{(M_p)}_{L^\infty}$ (resp. $S \in \mathcal{D}^{\{M_p\}}_{L^\infty}$). □

Theorem 6.3 *Let S be an ultradistribution in $\mathcal{D}'^{(M_p)}_{L^1}$ (resp. $\mathcal{D}'^{\{M_p\}}_{L^1}$) which is of class H_a (resp. H_{a_p}) then S is hypoelliptic in $\mathcal{D}'^{(M_p)}_{L^\infty}$ (resp. $\mathcal{D}'^{\{M_p\}}_{L^\infty}$) if and only if there exist $k > 0$ and $\xi_0 > 0$ (resp. for every $k > 0$ there exists $\xi_0 > 0$), such that*

$$|\hat{S}(\xi)| \geq \exp[M(k|\xi|)], \quad \xi \in \mathbf{R}, \quad |\xi| \geq \xi_0. \tag{6.14}$$

Proof: *1. Suppose that condition (6.14) is not fulfilled then there exists a sequence* ξ_j *defined as in Lemma 6.2 and such that*

$$|\hat{S}(\xi_j)| < \exp[M(j|\xi_j|)], \quad j \in \mathbf{N}.$$

The series $U = \sum_{j\in\mathbf{N}} e^{ix\xi_j}$ *converges in* $\mathcal{D}'^{*}_{L^\infty}$ *but it does not in* $\mathcal{D}^{*}_{L^\infty}$. *On the other hand,*

$$S * U = \sum_{j\in\mathbf{N}} \hat{S}(\xi_j) e^{ix\xi_j}.$$

Applying the Lemma 6.2 we conclude that $S * U$ *is in* $\mathcal{D}^{*}_{L^\infty}$. *Thus* S *is not hypoelliptic in* $\mathcal{D}'^{*}_{L^\infty}$.

2. Let ψ *be an element of* $\mathcal{D}^{(M_p)}$ *(resp.* $\mathcal{D}^{\{M_p\}}$*), such that*

$$\psi(\xi) = \begin{cases} 1, & |\xi| < |\xi_0|, \\ 0, & |\xi| \geq |\xi_0| + 1. \end{cases}$$

We define the Fourier transform $\hat{R}$ *of* R *by the formula*

$$\hat{R}(\xi) = \begin{cases} 0, & |\xi| < |\xi_0|, \\ \dfrac{1-\psi(\xi)}{\hat{S}(\xi)}, & |\xi| \geq |\xi_0|. \end{cases}$$

The above definition of R *has sense since* S *is of class* H_a *(resp.* H_{a_p}*).*

There exists $b > 0$ *(resp.* $(b_p) \in \mathcal{R}$*) such that*

$$\hat{Q}(\xi) = \frac{\hat{R}(\xi)}{P_b(\xi)} \quad (resp.\ \hat{Q}(\xi) = \frac{\hat{R}(\xi)}{P_{b_p}(\xi)})$$

and all its derivatives belong to L^1. *We will prove it only in the Roumieu case, since the proof in the Beurling case is analogous. By the iterated "chain rule"*

$$\partial^\alpha\left(\frac{1}{\hat{S}}\right) = \sum_{1\leq\gamma\leq\alpha} \sum_{\alpha_1+\ldots+\alpha_\gamma=\alpha} C_{\alpha_1\alpha_2\ldots\alpha_\gamma} \frac{(\partial^{\alpha_1}\hat{S})(\partial^{\alpha_2}\hat{S})\ldots(\partial^{\alpha_\gamma}\hat{S})}{\hat{S}^{\gamma+1}}, \quad \alpha \in \mathbf{N}$$

Applying the estimates of derivatives of $1/P_b$ *(resp.* $1/P_{b_p}$*), which is mentioned in the fourth chapter, we obtain that for each fixed* $\alpha \in \mathbf{N}_0$ *and* $(b_p) \in \mathcal{R}$ *such that* $2b_p > H^{\alpha-1}\ell_p$, $p \in \mathbf{N}$, *where* $(\ell_p) \in \mathcal{R}$ *is the same as in (6.2) there exist* $r > 0$, C *and* $C_\alpha = C(\alpha)$ *such that*

$$|\partial^\alpha \hat{Q}(\xi)| \leq \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \left|\left(\frac{1}{P_{b_p}(\xi)}\right)^{(\alpha-\beta)}\right| \left|\left(\frac{1}{\hat{S}}(\xi)\right)^{(\beta)}\right|$$

$$\leq \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \frac{(\alpha-\beta)!}{r^{\alpha-\beta}} \exp[-N_{b_p}(|\xi|)]\cdot$$

$$\cdot \sum_{1\leq\gamma\leq\beta} \sum_{\beta_1+\ldots+\beta_\gamma=\beta} C^{\gamma} C_{\beta_1\beta_2\ldots\beta_\gamma} \frac{m^{\beta} M_{\beta_1} M_{\beta_2}\ldots M_{\beta_\gamma} \exp[\gamma N_{\ell_p}(|\xi|)]}{\exp[(\gamma+1)M(k|\xi|)]}$$

$$\leq C_\alpha \exp[-M(k|\xi|)], \qquad \xi \in \mathbf{R}.$$

Therefore the following integration by parts

$$|Q(x)| = \frac{1}{2\pi}\left|\int_{\mathbf{R}} e^{ix\xi}\hat{Q}(\xi)d\xi\right|$$

$$\leq \frac{1}{2\pi}\frac{1}{(1+|x|^2)^{\alpha/2}}\left|\int_{\mathbf{R}} e^{ix\xi}(1-\triangle)^{\alpha/2}\hat{Q}(\xi)d\xi\right| < C\frac{1}{(1+|x|^2)^{\alpha/2}},$$

where C depends on the choice of α, has sense. It follows that Q is an L^1 function and so the ultradistribution $R = P_b(D)Q$ (resp. $R = P_{b_p}(D)Q$) is in $\mathcal{D}'^{(M_p)}_{L^1}$ (resp. $\mathcal{D}'^{\{M_p\}}_{L^1}$). Furthermore,

$$\hat{S}(\xi)\hat{R}(\xi) = 1 - \psi(\xi).$$

By the inverse Fourier transform, we see that R a parametrix for S, that is:

$$S * R = \delta - W,$$

where $\hat{W} = \psi$.

Now assume that $S * U = V$, where $V \in \mathcal{D}^{*}_{L^\infty}$, $S \in \mathcal{D}'^{*}_{L^\infty}$ and $U \in \mathcal{D}'^{*}_{L^\infty}$. We have

$$U = U*\delta = U*(S*R) + U*W = (U*S)*R + U*W = V*R + U*W.$$

It is easy to check that $V * R$ and $U * W$ belongs to $\mathcal{D}^{*}_{L^\infty}$ and so U is in $\mathcal{D}^{*}_{L^\infty}$. □

The fact that the Fourier transform is topological isomorphism from $\mathcal{O}'^{*}_c$ onto $\mathcal{O}^{*}_M$ implies that every ultradistribution in $\mathcal{O}^{(M_p)}_c$ (resp. $\mathcal{O}'^{\{M_p\}}_c$) is of class H_a (resp. H_{a_p}). Therefore:

Corollary 6.4 *Let S be an ultradistribution in $\mathcal{O}'^{(M_p)}_c$ (resp. $\mathcal{O}'^{\{M_p\}}_c$) then S is hypoelliptic in $\mathcal{D}'^{(M_p)}_{L^\infty}$ (resp. $\mathcal{D}'^{\{M_p\}}_{L^\infty}$) if and only if there exist $k > 0$ and $\xi_0 > 0$ (resp. for every $k > 0$ there exists $\xi_0 > 0$) such that (6.14) holds.*

Corollary 6.5 *The same assumptions as in the Theorem 1 imply that every solution U in $\mathcal{D}'^{*}_{L^q}$, $q \in [1,\infty]$, of the equation (6.1) is in $\mathcal{D}^{*}_{L^q}$ whenever V is in $\mathcal{D}^{*}_{L^q}$.*

Proof: *Analogously as in the proof of the sufficiency of the theorem $R \in \mathcal{D}'^{*}_{L^1}$ and $U = V * R + U * W$. Since $\mathcal{D}^{*}_{L^q} * \mathcal{D}'^{*}_{L^1} \subset \mathcal{D}^{*}_{L^q}$ we have that U is in $\mathcal{D}^{*}_{L^q}$.* □

If the give convolution operator S is in $\mathcal{D}'^{*}_{L^1}$ then we have the following weak version of the regularity theorem.

Theorem 6.6 *If an ultradistribution $S \in \mathcal{D}'^{*}_{L^1}$ satisfies condition (6.14) every solution U in $\mathcal{D}'^{*}_{L^\infty}$ of the equation (6.1) with $V \in \mathcal{D}^{*}_{L^2}$ is in $\mathcal{D}^{*}_{L^\infty}$.*

Proof: *Applying the same argument as in Theorem 6.3, we construct the continuous function $\hat{R}(\xi)$ and $b > 0$ (resp. $(b_p) \in \mathcal{R}$) so that*

$$\hat{Q}(\xi) = \frac{\hat{R}(\xi)}{P_b(\xi)} \quad (resp.\ \hat{Q}(\xi) = \frac{\hat{R}(\xi)}{P_{b_p}(\xi)}), \qquad \xi \in \mathbf{R},$$

is in L^2. By Plancherel's theorem Q is in L^2 and $R = P_b(D)Q$ (resp. $R = P_{b_p}(D)Q$) is in $\mathcal{D}'^{*}_{L^2}$, also

$$U = U * \delta = V * R + U * V.$$

Since V is in $\mathcal{D}^{*}_{L^2}$, $V * R$ and $U * W$ belong to $\mathcal{D}^{*}_{L^\infty}$, U is in $\mathcal{D}^{*}_{L^\infty}$. □

We give now two examples of hypoelliptic operators, one of which is not of class H_a (resp. H_{a_p}).

Example 6.7 *Let $S = e^{-|t|}$. Since $\hat{S}(\xi) = 1/(1+\xi^2)$ is in $\mathcal{O}'^{*}_{c}$ and satisfies the condition of the theorem S is hypoelliptic in $\mathcal{D}'^{*}_{L^\infty}$.*

Example 6.8 *Let $S = 1/(1+x^2)+\delta$. Its Fourier transform $\hat{S}(\xi) = e^{-|\xi|}+1$ is not a C^1-function but it satisfies condition (6.14). From the fact that $1/(1+x^2) \in \mathcal{D}^{*}_{L^1}$, which follows from the estimation*

$$\left|\left(1/(1+\xi^2)\right)^{(\beta)}\right| \leq 3^{\beta}(\beta+1)!, \qquad \beta \in \mathbf{N},\ \xi \in \mathbf{R},$$

and the fact that $\mathcal{D}^{*}_{L^1} * \mathcal{D}'^{*}_{L^\infty} \subset \mathcal{D}^{*}_{L^\infty}$ it follows that S is hypoelliptic convolution operator in $\mathcal{D}^{*}_{L^\infty}$.

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