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On some classes of multipliers and semigroups in the spaces of ultradistributions and hyperfunctions

-doctoral dissertation-

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О неким класама мултипликатора и семигрупа на просторима ултрадистрибуција и хиперфункција

-докторска дисертација-

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Dedicated to my family

Contents

Abstrakt

U disertaciji se proučavaju prostor konvolutora i multiplikatora na prostorima temperiranih ultradistributcija. Dokazane su teoreme koji karakterišu elemente prostora konvolutora i multiplikatora. Date su strukturne teoreme za ultradistribucione polugrupe i eksponenecijalne polugrupe. Furijeve huperfunkciske polugrupe i hiperfunkciske polugrupe sa generatorima koji su negusto definisani su analizirani, takodje su date strukturne teoreme i spektralne karakterizacije kao i dovoljni uslovi za postojenje na takvih polugrupa za operator A koji ne mora biti gust. Apstraktni Koshijev problem je proučavan za težinske Banahove prostore kao i za odgovarajuće prostora ultradistribucija. Takodje su date i primene za odredjene klase jednačina.

Abstract

We are study the spaces of convolutors and multipliers in the spaces of tempered ultradistributions. There given theorems which gives us the characterization of all the elements which belongs to spaces of convolutors and multipliers. Structural theorem for ultradistribution semigroups and exponential ultradistribution semigroups is given. Fourier hyperfunction semigroups and hyperfunction semigroups with non-densely defined generators are analyzed and also structural theorems and spectral characterizations give necessary and sufficient conditions for the existence of such semigroups generated by a closed not necessarily densely defined operator A. The abstract Cauchy problem is considered in the Banach valued weighted Beurling ultradistribution setting and given some applications on particular equations.

Preface

The first part of this work is devoted to the spaces of convolutors and multipliers in the space of tempered ultradistributions. We give theorems describing their structure in the Beurling and Roumieu case. All the proofs we give only for the Roumieu case, using the definition of Roumieu ultradistributions given in [13]. The space of multipliers and the space of convolutors are topologically isomorphic.

The second part of this work is on the Generalized semigroups and abstract Cauchy problem in weighted ultradistribution spaces in Beurling case. Generalized semigroups firstly were introduced as distribution semigroups with densely defined generators by J.L. Lions, and after that many authors showed scientific interest in that area. After that, distribution semigroups with non-densely defined generators, ultradistribution semigroups and hyperfunction semigroups were considered. The theory of the Generalized semigroups, makes more wide concept that can be applied directly in many differential and integral equations, which can be modeled as an abstract Cauchy problem on some Banach space.

Using tempered ultradistributions we define exponential ultradistribution semigroups. Furthermore, we give structural characterizations for ultradistribution semigroups and exponential ultradistribution semigroups. Some of these results are already known, but we give them for completeness.

In the definition of infinitesimal generators for distribution and ultradistribution semigroups in the non-quasi-analytic case, all authors use test functions supported by $[0, \infty)$. That approach cannot be used in the case of Fourier hyperfunction semigroups since in the quasi-analytic case only the zero function has this property. Because of that, we define such semigroups on test spaces $\mathcal{P}_*$ and $\mathcal{P}_{*,a}$ $(a > 0)$ and we give the axioms for such semigroups as well as the definition of infinitesimal generator on subspaces of quoted spaces consisting of functions ϕ with the property $\phi(0) = 0$ and $\phi'(0) = 0$. Following the ultradistribution case, we give structural theorems for hyperfunction semigroups.

Following G.Da Prato and E.Sinestrari, [25], we consider the abstract Cauchy problem in the space of Banach-valued ultradistributions $\mathcal{D}'^{(s)}_{L^p}(0, T; E)$ and different type of solutions of the abstract Cauchy problem. The closed operator A in the abstract Cauchy problem satisfy Hille-Yosida condition. In order to investigate the abstract Cauchy problem, previously we give the definitions and basic properties on $\mathcal{D}'^{(s)}_{L^p}(0, T; E)$. There are several cases on A and E which we consider in the ultradistributional setting.

Novi Sad, 2014 — Daniel Velinov

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At the end I want to express eternal gratitude to my parents, my brother and Ana, for their everlasting love, moral support and encouragement.

Chapter 0

Introduction

Main goal in discourse is to study generalized semigroups, especially ultradistribution semigroups and Fourier hyperfunction semigroups. Also the spaces of convolutors and spaces of multipliers in the space of tempered ultradistributions and their connection is given. This material is divided into five chapters.

The first chapter is devoted to the notation and some results which are already known and used here to obtain new results. Here we recollect some basic facts on the spaces of ultradistributions and tempered ultradistributions. In the next section we consider more general spaces, the space of Fourier hyperfunctions and the space of hyperfunctions. Some known results on distribution semigroups and ultradistribution semigroups are mentioned, which ideas and proofs are used in the next chapters. Results of G. Da Prato and E. Sinestrari, [25] are placed in section devoted on the Cauchy problem. We extend these results in the last chapter.

In Chapter 2, the spaces of convolutors and multipliers in the space of tempered ultradistributions are studied. We give structure theorems for the space of convolutors in the Roumieu case, as well as the completeness of $O_C'^{(M_p)}$, resp. $O_C'^{\{M_p\}}$. Also the space of multipliers $O_M^{(M_p)}$, resp. $O_M^{\{M_p\}}$ is considered. Characterization theorem for the space of multipliers in Roumieu case is given. The Fourier transform gives a topological isomorphism between the space of multipliers and the space of convolutors in Roumieu case.

In the next two chapters, we consider ultradistribution semigroups, exponential ultradistribution semigroups and Fourier semigroups. The approach of J. L. Lions to distribution semigroups [67] with the densely defined generators was the inspiration for many mathematicians to investigate various classes of semigroups which generalize C_0-semigroups, see [3], [6], [15], [35], [44], [54], [61], [62], [76] and [107]. Ultradistribution semigroups with densely defined generators were considered by J. Chazarain in [15] (see also [10], [20], [24], [29], [43], [107] and references therein) while H. Komatsu [53] considered ultradistribution semigroups with non-densely defined generators as well as Laplace hyperfunction semigroups. We also refer to R. Beals [9]-[10] for the theory of ω-ultradistribution semigroups with the densely defined generators, to P. C. Kunstmann [63] and to the monograph of I. Melnikova and A. Filinkov [70] for ultradistribution semigroups with the non-densely defined generators. In [58], ultradistribution semigroups are analyzed, following the approaches of P. C. Kunstmann [62] and S. Wang [104], where distributions semigroups are considered.

On the other hand, S. Ōuchi [76] was the first who introduced the class of hyperfunction semigroups, more general than distribution and ultradistribution semigroups. Furthermore, generators of hyperfunction semigroups in the sense of [76] are not necessarily densely defined. An analysis of R. Beals [9, Theorem 2'] gives an example of a densely defined operator A in the Hardy space $H^2(\mathbb{C}_+)$ which generates a hyperfunction semigroup of [76] but this operator is not a generator of any ultradistribution semigroup, and any (local) integrated C-semigroups, $C \in L(H^2(\mathbb{C}_+))$.

Moreover, we analyze Fourier-hyperfunction semigroups with non-densely defined generators continuing over the investigations of Roumieu type ultradistribution semigroups and construct examples of tempered ultradistribution semigroups (see [58]) and Fourier-hyperfunction semigroups with non-densely defined generators. Here one cannot use the same approach as in the distribution and ultradistribution semigroups with non-densely defined generators, since the test space of Fourier hyperfunctions are rapidly decreasing real analytic functions. Also structural theorem for Fourier hyperfunction semigroups is given. The main interest is the existence of fundamental solutions for the Cauchy problems with initial data being hyperfunctions.

Da Prato and Sinestrari [25] have studied the Cauchy problem

$$u'(t) = Au(t) + f(t), u(0) = u_0, \tag{1}$$

where A is a closed operator in a Banach space E with not necessarily dense domain in E but satisfying the Hille-Yosida condition. Here $u_0 \in E$, f is the E-valued continuous or L^p- function on $[0,T]$. They have considered various classes of equations and types of solutions illustrating their theory. Regularity properties of solutions are extended much later in [93].

In the last chapter, the results of [25] for (1), mentioned above, are extended to weighted Schwartz spaces of distributions and Beurling space of ultradistributions [48]-[50]. Since the weighted Schwartz space $\mathcal{D}'_{L^p}$ ([97]) can be involved in this theory similarly as Beurling type spaces, and the second ones are more delicate, the investigations are focused to the Beurling case, more precisely to the space of ultradistributions $\mathcal{D}'^{(s)}_{L^p}((0,T)\times U))$, U is a bounded domain in $\mathbb{R}^n$, related to the Gevrey sequence $p!^s, s>1$ (see [84] for $U=\mathbb{R}^n$). In order to apply results of [25] in this abstract setting, we study the topological structure of $\mathcal{D}^s_{L^p,h}(U), p \in [1,\infty]$ (with special analysis for $p=\infty$) as well as the closure of $\mathcal{D}^{(s)}(U)$ in these spaces, corresponding projective limits, tensor products, their duals as well as vector valued versions of these spaces. As a special result, we obtain that $\mathcal{D}^{(s)}_{L^p}(U)$ is nuclear for bounded U. Also we have that all spaces $\mathcal{D}^{(s)}_{L^p}(U)$ are isomorphic to $\dot{\mathcal{B}}^{(s)}(U)$ for bounded U. Both assertions do not hold for $U=\mathbb{R}^n$. The main results in this section are related to the structure of quoted spaces. Such preparatory results are needed for the formulation of the Cauchy problem in this abstract setting and for the application of results in [25].

I want to stress that already known results have citation next to them, in order to distinguish them from the new results.

Chapter 1

Preliminaries

1.1 Spaces of Ultradistributions

Let (M_p) be a sequence of positive numbers. In sequel, some of the following conditions will be assumed on (M_p):
$(M.1)$ (Logarithmic convexity) $M_p^2 \leq M_{p-1}M_{p+1}$ for $p \in \mathbb{N}$;
$(M.2)$ (Stability under ultradifferential operators) For some $A, H \geq 0$

$$M_p \leq AH^p \min_{0\leq q\leq p} M_{p-q}M_q, \quad p, q \in \mathbb{N};$$

$(M.3)$ (Strong non-quasi-analyticity)

$$\sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq Aq\frac{M_q}{M_{q+1}}, \quad q \in \mathbb{N},$$

and weaker conditions on (M_p):
$(M.2)'$ (Stability under differential operators) For some $A, H > 1$

$$M_{p+1} \leq AH^{p+1}M_p, \quad p \in \mathbb{N};$$

$(M.3)'$ (Non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

The Gevrey sequence $M_p = p!^s$, $s > 1$ satisfies all the above conditions. Here we always assume that $M_0 = 1$. For sequence (M_p), the associate function $M(\rho)$ on $(0, +\infty)$ is defined by

$$M(\rho) = \sup_{p\in\mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

In the sequel, whenever compactly supported ultradifferentiable functions are considered, we assume that (M_p) satisfies $(M.1)$, $(M.2)$ and $(M.3)'$. Next, we give the definition and several important properties of the spaces $\mathcal{D}_K^{M_p,r}, \mathcal{D}_K^{(M_p)}, \mathcal{D}_K^{\{M_p\}}$,

$\mathcal{D}^{(M_p)}(\Omega), \mathcal{D}^{\{M_p\}}(\Omega), \mathcal{E}^{(M_p)}(\Omega), \mathcal{E}^{\{M_p\}}(\Omega)$, (see [48], [50], [13]). Let K be a regular compact set in $\mathbb{R}^n$ and Ω an open set in $\mathbb{R}^n$. Denote:

$$\mathcal{E}^{\{M_p\},r}(K) = \{\varphi \in C^\infty(K) : \|D^\alpha \varphi\|_{C(K)} \leq C r^{|\alpha|} M_{|\alpha|}\},$$

$|\alpha| = 0, 1, 2, \dots$ and for some constant $C \geq 0$,

$$\mathcal{D}_K^{\{M_p\},r} = \{\varphi \in C^\infty(\mathbb{R}^n) \text{with compact support} : \|D^\alpha \varphi\|_{C(K)} \leq C r^{|\alpha|} M_{|\alpha|}\},$$

$|\alpha| = 0, 1, 2, \dots$ and for some constant $C \geq 0$. Both spaces are Banach spaces with norm to be the infimum of the constant C in the upper estimate, i.e. $\|\varphi\| = \sup\limits_{x \in K, \alpha} \frac{|D^\alpha \varphi(x)|}{r^{|\alpha|} M_{|\alpha|}}$. Also define as a locally convex spaces,

$$\mathcal{E}^{(M_p)}(K) = \varprojlim_{r \to 0} \mathcal{E}^{\{M_p\},r}(K);$$

$$\mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}(K);$$

$$\mathcal{E}^{\{M_p\}}(K) = \varinjlim_{r \to \infty} \mathcal{E}^{\{M_p\},r}(K);$$

$$\mathcal{E}^{\{M_p\}}(\Omega) = \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}(K);$$

$$\mathcal{D}_K^{(M_p)} = \varprojlim_{r \to 0} \mathcal{D}_K^{\{M_p\},r};$$

$$\mathcal{D}^{(M_p)}(\Omega) = \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{(M_p)};$$

$$\mathcal{D}_K^{\{M_p\}} = \varinjlim_{r \to \infty} \mathcal{D}_K^{\{M_p\},r};$$

$$\mathcal{D}^{\{M_p\}}(\Omega) = \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{\{M_p\}}.$$

Theorem 1.1.1. *[48] $\mathcal{E}^{(M_p)}(K)$, $\mathcal{E}^{(M_p)}(\Omega)$ and $\mathcal{D}_K^{(M_p)}$ are (FS)-spaces, $\mathcal{E}^{\{M_p\}}(K)$, $\mathcal{D}_K^{\{M_p\}}$ and $\mathcal{D}^{\{M_p\}}(\Omega)$ are (DFS)-spaces and $\mathcal{D}^{(M_p)}(\Omega)$ is an (LFS)-space. In particular these spaces are separable complete bornologic Montel and Schwartz spaces. Every bounded set in $\mathcal{D}_K^{\{M_p\}}$ or $\mathcal{D}^{(M_p)}(\Omega)$ ($\mathcal{E}^{\{M_p\}}(K)$) is a bounded set in some $\mathcal{D}_K^{\{M_p\},r}$ ($\mathcal{E}^{\{M_p\},r}(K)$).*
$\mathcal{E}^{\{M_p\}}(\Omega)$ is a complete Schwartz spaces. In particular, it is semi-reflexive.
If (M_p) satisfies $(M.2)'$, then all the spaces defined above are nuclear.

Theorem 1.1.2. *[48] A sequence of a positive numbers (M_p), satisfies condition $(M.1)$ if and only if*

$$M_p = M_0 \sup_\rho \frac{\rho^p}{e^{M(\rho)}}.$$

Theorem 1.1.3. *[48] The sequence (M_p) satisfies $(M.2)$ if and only if for some $A, H > 0$,*

$$2M(\rho) \leq M(H\rho) + \log(AM_0).$$

If $f \in L^1$ then its Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx, \;\; \xi \in \mathbb{R}^n.$$

For $f \in L^1$ its Laplace transform is defined by

$$(\mathcal{L}f)(\zeta) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx, \;\; \zeta \in \mathbb{C}^n.$$

By $\Re$ we denote the set of positive sequences which monotonically increases to infinity. For $r_p \in \Re$ and K a compact set in $\mathbb{R}^n$, we denote by $\mathcal{D}^{\{M_p\}}_{K,r_p}$ the space of smooth functions φ on $\mathbb{R}^n$ supported by K such that

$$\|\varphi\|_{K,r_p} = \sup\{\frac{| D^p\varphi(x) |}{N_p}; \;\; p \in \mathbb{N}^n, \;\; x \in K\} < \infty,$$

where $N_p = M_p \prod_{i=1}^{|p|} r_i, \quad p \in \mathbb{N}^n$. Clearly, this is a Banach space. It is proved in [50] that

$$\mathcal{D}^{\{M_p\}}_K = \text{proj} \lim_{r_p \in \Re} \mathcal{D}^{\{M_p\}}_{K,r_p}.$$

If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $r > 0$, resp. $r_p \in \Re$, we put

$$\mathcal{D}^{(M_p)}_{\Omega,r} = \text{ind} \lim_{K \subset\subset \Omega} \mathcal{D}^{\{M_p\}}_{K,r}, \quad \mathcal{D}^{\{M_p\}}_{\Omega,r_p} = \text{ind} \lim_{K \subset\subset \Omega} \mathcal{D}^{\{M_p\}}_{K,r_p}.$$

The associated function for the sequence N_p is

$$N_{r_p}(\rho) = \sup\{\log_+ \frac{\rho^p}{N_p}; \;\; p \in \mathbb{N}\}, \;\; \rho > 0.$$

Note, for given r_p and every $k > 0$ there is $\rho_0 > 0$ such that

$$N_{r_p}(\rho) \le M(k\rho), \;\; \rho > \rho_0. \tag{1.1}$$

When (M_p) satisfies conditions $(M.1)$, $(M.2)$ and $(M.3)$ one defines ultradifferential operators as follows:
It is said that $P(\xi) = \sum_{\alpha \in \mathbf{N_0^n}} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n$, is an ultrapolynomial of the class (M_p) resp. $\{M_p\}$, whenever the coefficients a_α satisfy the estimate

$$| a_\alpha | \le \frac{CL^\alpha}{M_\alpha}, \;\; \alpha \in \mathbb{N}^n, \tag{1.2}$$

for some $L > 0$ and $C > 0$ resp. for every $L > 0$ and some $C_L > 0$. The corresponding operator $P(D) = \sum_\alpha a_\alpha D^\alpha$ is an ultradifferential operator of the class (M_p), resp. $\{M_p\}$.

Assume now $(M.1)$, $(M.2)$ and $(M.3)$ and put

$$\begin{aligned} P_r(\zeta) &= (1+\zeta^2) \prod_{p \in \mathbb{N}^n} \Big(1 + \frac{\zeta^2}{r^2 m_p^2}\Big), \quad \textit{resp.} \\ P_{r_p}(\zeta) &= (1+\zeta^2) \prod_{p \in \mathbb{N}^n} \Big(1 + \frac{\zeta^2}{r_p^2 m_p^2}\Big), \; \zeta \in \mathbb{C}^n, \end{aligned} \tag{1.3}$$

where $m_p = M_p/M_{p-1}$ and $r > 0$ resp. $r_p \in \Re$. Conditions $(M.1)$, $(M.2)$ and $(M.3)$ imply that P_r resp. P_{r_p} is an ultradifferential operator of the class (M_p) resp. of the class $\{M_p\}$ (see [48]). The following estimates

$$\begin{aligned} \mid P_r(\xi) \mid &\geq e^{M(r|\xi|)}, \quad \xi \in \mathbb{R}^n, \\ \mid P_{r_p}(\xi) \mid &\geq e^{N_{r_p}(|\xi|)}, \quad \xi \in \mathbb{R}^n, \end{aligned} \tag{1.4}$$

hold.
Assume $(M.1)$, $(M.2)$ and $(M.3)$. Denote by $\mathcal{S}_2^{M_p,m}(\mathbb{R}^n)$, $m > 0$, the space of smooth functions φ which satisfy

$$\sigma_{m,2}(\varphi) := \Big(\sum_{p,q \in \mathbb{N}^n} \int_{\mathbb{R}^n} \Big| \frac{m^{p+q} \langle x \rangle^p \varphi^{(q)}(x)}{M_p M_q} \Big|^2 dx \Big)^{1/2} < \infty, \tag{1.5}$$

supplied with the topology induced by the norm $\sigma_{m,2}$. If instead of 2 put $p \in [1, \infty]$ in (1.5) one obtains the equivalent sequence of norms $\sigma_{m,p}$, $m > 0$.

The spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ of tempered ultradistributions of Beurling and Roumieu type respectively, are defined as the strong duals of the spaces

$$\mathcal{S}^{(M_p)} = \lim \operatorname{proj}_{m \to \infty} \mathcal{S}_2^{M_p,m}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{S}^{\{M_p\}} = \lim \operatorname{ind}_{m \to 0} \mathcal{S}_2^{M_p,m}(\mathbb{R}^n),$$

respectively. The common notation for symbols (M_p) and $\{M_p\}$ will be $*$.

All the good properties of $\mathcal{S}^*$ and its strong dual follow from the equivalence of the sequence of norms $\sigma_{m,p}$, $m > 0$,, $p \in [1, \infty]$ with the each of the following sequences of norms [60], [13] :

(a) $\sigma_{m,p}$, $m > 0$, $p \in [1, \infty]$ is fixed ;

(b) $s_{m,p}$, $m > 0$, $p \in [1, \infty]$ is fixed, where

$$s_{m,p}(\varphi) := \sum_{\alpha,\beta \in \mathbb{N}^n} \frac{m^{\alpha+\beta} \|x^\beta \varphi^{(\alpha)}\|_p}{M_\alpha M_\beta};$$

(c) s_m, $m > 0$, where $s_m(\varphi) := \sup_{\alpha \in \mathbb{N}^n} \dfrac{m^\alpha \|\varphi^{(\alpha)} e^{M(m\cdot)}\|_{L_\infty}}{M_\alpha}$;

In [13] it is proved that

$$\mathcal{S}^{\{M_p\}} = \operatorname{proj} \lim_{r_i, s_j \in \Re} S^{M_p}_{r_i,s_j},$$

$$\text{where} \quad S^{M_p}_{r_i,s_j} = \{\varphi \in C^\infty(\mathbb{R}^n);\ \gamma_{r_i,s_j}(\varphi) < \infty\},$$

$$\text{and} \quad \gamma_{r_i,s_j}(\varphi) := \sum_{p,q \in \mathbb{N}^n} \{ \frac{\| \langle x \rangle^p \varphi^{(q)} \|_{L^\infty}}{(\prod_{i=1}^p r_i) M_p (\prod_{j=1}^q s_j) M_q} \}.$$

Due to [82, Theorem 2], we have the following representation theorems for tempered ultradistributions in the case when (M.1), (M.2) and (M.3) are valid.

Let $T \in \mathcal{S}'^{*}_{+}(\mathbb{R}, E)$. Then there exist an ultradifferential operator of (M_p)-class, $P_L(d/dt)$ and $L > 0$, formally of the form

$$P_L(d/dt) = \prod_{p=1}^{\infty} (1 + \frac{L^2}{m_p^2} d^2/dt^2) = \sum_{p=0}^{\infty} a_p d^p/dt^p,$$

resp., of $\{M_p\}$-class, $P_{L_p}(d/dt)$, $(L_p)_p$ is a sequence tending to zero, formally of the form

$$P_{L_p}(d/dt) = \prod_{p=1}^{\infty}(1 + \frac{L_p^2}{m_p^2} d^2/dt^2) = \sum_{p=0}^{\infty} a_p d^p/dt^p$$

and a continuous function $f : \mathbb{R} \to E$ with the properties supp$f \subset (-a, \infty)$, for some $a > 0$, $||f(t)|| \leq Ae^{M(k|t|)}$, $t \in \mathbb{R}$, for some $k > 0$ and $A > 0$, resp., for every $k > 0$ and a corresponding $A > 0$, and that $T = P_L(-id/dt)f$ in (M_p)-case on $\mathcal{S}^{(M_p)}(\mathbb{R})$, resp., $T = P_{L_p}(-id/dt)f$ in $\{M_p\}$-case on $\mathcal{S}^{\{M_p\}}(\mathbb{R})$.

Note that $\mathcal{F} : \mathcal{S}^* \to \mathcal{S}^*$ is a topological isomorphism and that the Fourier transformation on $\mathcal{S}'^*$ is defined as usual.

1.2 Spaces of Hyperfunctions

The basic facts about hyperfunctions and Fourier hyperfunctions of M. Sato can be found on an elementary level in the monograph of A. Kaneko [40] (see also [72], [32], [41]-[42]).

Let E be a Banach space, Ω be an open set in $\mathbb{C}$ containing an open set $I \subset \mathbb{R}$ as a closed subset and let $\mathcal{O}(\Omega)$ be the space of $E-$valued holomorphic functions on Ω endowed with the topology of uniform convergence on compact sets of Ω. The space of $E-$valued hyperfunctions on I is defined as $\mathcal{B}(I, E) := \mathcal{O}(\Omega \setminus I, E)/\mathcal{O}(\Omega, E)$. A representative of $f = [f(z)] \in \mathcal{B}(I, E)$, $f \in \mathcal{O}(\Omega \setminus I, E)$ is called a defining function of f. The space of hyperfunctions supported by a compact set $K \subset I$ with values in E is denoted by $\Gamma_K(I, \mathcal{B}(E)) = \mathcal{B}(K, E)$. It is the space of continuous linear mapping from $\mathcal{A}(K)$ into E, where $\mathcal{A}(K)$ is the inductive limit type space of analytic functions in neighborhoods of K endowed with the appropriate topology [47]. Denote by $\mathcal{A}(\mathbb{R})$ the space of real analytic functions on $\mathbb{R}$: $\mathcal{A}(\mathbb{R})$ =proj $\lim_{K \subset\subset \mathbb{R}} \mathcal{A}(K)$. The space of continuous linear mappings from $\mathcal{A}(\mathbb{R})$ into E, denoted by $\mathcal{B}_c(\mathbb{R}, E)$, is consisted of compactly supported elements of $\mathcal{B}(K, E)$, where K varies through the family of all compact sets in $\mathbb{R}$. We denote by $\mathcal{B}_+(\mathbb{R}, E)$ the space of $E-$valued hyperfunctions whose supports are contained in $[0, \infty)$. As in the scalar case ($E = \mathbb{C}$), if $f \in \mathcal{B}_c(\mathbb{R}, E)$ and supp$f \subset \{a\}$, then $f = \sum_{n=0}^{\infty} \delta^{(n)}(\cdot - a)x_n$, $x_n \in E$, where $\lim\limits_{n\to\infty} (n!||x_n||)^{1/n} = 0$. Let $\mathbb{D} = \{-\infty, +\infty\} \cup \mathbb{R}$ be the radial compactification of $\mathbb{R}$. Put $I_\nu = (-1/\nu, 1/\nu)$, $\nu > 0$. For $\delta > 0$, the space $\tilde{\mathcal{O}}^{-\delta}(\mathbb{D} + iI_\nu)$ is defined as a subspace of $\mathcal{O}(\mathbb{R} + iI_\nu)$ with the property that for every $K \subset\subset I_\nu$ and $\varepsilon > 0$ there exists a suitable $C > 0$ such that $|F(z)| \leq Ce^{-(\delta-\varepsilon)|Rez|}$, $z \in \mathbb{R} + iK$. Then $\mathcal{P}_*(\mathbb{D})$:=indlim$_{n\to\infty} \tilde{\mathcal{O}}^{-1/n}(\mathbb{D} + iI_n)$ is the space of all rapidly decreasing, real analytic functions (cf. [40, Definition 8.2.1]) and the space of Fourier hyperfunctions $\mathcal{Q}(\mathbb{D}, E)$ is the space of continuous linear mappings from $\mathcal{P}_*(\mathbb{D})$ into E endowed with the strong topology. We point out that Fourier hyperfunctions were firstly introduced by M. Sato in [88] who called them slowly increasing hyperfunctions. Let us note that the sub-index $*$ in $\mathcal{P}_*(\mathbb{D})$ does not have the meaning as in the case of ultradistributions. This is often used notation in the literature (cf. [40]). Recall, the restriction mapping $\mathcal{Q}(\mathbb{D}, E) \to \mathcal{B}(\mathbb{R}, E)$ is surjective.

Theorem 1.2.1. *[40] The space $\mathcal{B}_*(\mathbb{R}^n)$ of hyperfunctions with compact support, the space $\mathcal{Q}^{-\delta}$ of exponentially decreasing hyperfunctions of type $-\delta$, and the space $\mathcal{Q}$ of Fourier hyperfunctions are in the following subspace relationships via natural embeddings.*

$$\mathcal{B}_*(\mathbb{R}^n) \hookrightarrow \mathcal{Q}^{-\delta} \hookrightarrow \mathcal{Q}.$$

In addition, the definition of the Fourier transforms for these spaces is consistent with embeddings.

Recall [40], an operator of the form $P(d/dt) = \sum_{k=0}^{\infty} b_k(d/dt)^k$ is called a local operator if $\lim\limits_{k\to\infty}(|b_k|k!)^{1/k} = 0$. Note that the composition and the sum of two local operators is again a local operator.

The main structural property of $\mathcal{Q}(\mathbb{D})$ says that every element $f \in \mathcal{Q}(\mathbb{D})$ is of the form $f = P(d/dt)F$, where P is a local operator and F is a continuous slowly increasing function (for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|F(t)| \leq C_\varepsilon e^{\varepsilon|t|}$, $t \in \mathbb{R}$). More precisely, we have the following global structural theorem (cf. [40, Proposition 8.1.6, Lemma 8.1.7, Theorem 8.4.9]), reformulated here with a sequence $(L_p)_p$:

Let, formally,

$$P_{L_p}(d/dt) = \prod_{p=1}^{\infty}(1 + \frac{L_p^2}{p^2}d^2/dt^2) = \sum_{p=0}^{\infty} a_p d^p/dt^p, \tag{1.6}$$

where $(L_p)_p$ is a sequence decreasing to 0. This is a local operator and we call it hyperfunction operator.

Let $T \in \mathcal{Q}(\mathbb{D}, E)$. There is a local operator $P_{L_p}(-id/dt)$ (with a corresponding sequence $(L_p)_p$) and a continuous slowly increasing function $f : \mathbb{R} \to E$ (for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $||f(x)|| \leq C_\varepsilon e^{\varepsilon|x|}$, $x \in \mathbb{R}$) such that $T = P_{L_p}(-id/dt)f$.

If a hyperfunction is compactly supported, $\operatorname{supp} f \subset K$, $f \in \mathcal{B}(K, E)$, then we have the above representation with a corresponding local operator $P_{L_p}(-id/dt)$ and a continuous $E-$valued function in a neighborhood of K.

1.3 Some results from the theory of Generalized semigroups

In this section we briefly give some definitions and theorems of the theory of generalized semigroups. We will extensively use these results in the subsequent chapters. The theory of generalized semigroups has started to develop after the paper of J.L.Lions [67] on distribution semigroups. From now on, unless otherwise stated, by E we denote a Banach space. In [67], the distribution semigroups are defined by:

Definition 1.3.1. [67] A distribution G is a distribution semigroup if following condition are satisfied:

(i) $G \in \mathcal{D}'_+(L(E;E))$, $\quad G = 0, \quad$ for $\quad t < 0$;

(ii) $G(\varphi * \psi) = G(\varphi)G(\psi)$ for all $\varphi, \psi \in \mathcal{D}_0$, where $\mathcal{D}_0$ is the space of all $\varphi \in \mathcal{D}$ such that $\varphi(t) = 0$ for $t < 0$;

(iii) Let $\varphi \in \mathcal{D}_0$, $x \in E$ and $y = G(\varphi)x$. The distribution Gy is equal to a continuous function u on $[0, +\infty)$ and $u(t) = 0$ for $t < 0$ with range in E and $u(0) = y$;

(iv) The range of the all elements $G(\varphi)x$, where $\varphi \in \mathcal{D}_0$, $x \in E$ is dense in E;

(v) If for all $\varphi \in \mathcal{D}_0$, $G(\varphi)x = 0$, $x \in E$ then $x = 0$.

The sequence $\rho_n \in \mathcal{D}_0$ is regularizing if $\rho_n \longrightarrow \delta$ in the space of measures with compact support, equipped with the weak topology. Let $T \in \mathcal{E}'$ and $T = 0$ for $t < 0$. Note that for $\rho \in \mathcal{D}_0$, $T * \rho \in \mathcal{D}_0$. For such regularizing sequence ρ_n the following two properties can be proven:

a) $G(\rho_n)x \longrightarrow x$, when $\rho_n \longrightarrow \delta$,

b) $G(T * \rho_n)x$ converge to some $y \in E$, when $T \in \mathcal{E}'$.

We say that x is in the domain of $G(T)$, $(x \in D(G(T)))$, if there exists regularizing sequence ρ_n such that $G(\rho_n)x \longrightarrow x$ and $G(T * \rho_n)x$ converge in E. The limit of $G(T * \rho_n)x$ is denoted by $G(T)x$. Then,

$$G(\rho_n)x \longrightarrow x$$

$$G(T * \rho_n)x \longrightarrow G(T)x.$$

We deonte by $\overline{G}(T)$ the closure of the operator $G(T)$. The following theorem holds:

Theorem 1.3.1. *[67] Let $T \in \mathcal{E}'$ is zero for $t < 0$. Define the closed operator with dense domain $\overline{G}(T)$ with*

$$\overline{G}(T)G(\varphi)x = G(T)G(\varphi)x = G(T * \varphi)x, \quad \varphi \in \mathcal{D}_0, x \in E.$$

The operator $A = \overline{G}(-\delta')$ is an infinitesimal generator for the distribution semigroup G. Note that $\overline{G}(\delta) = I$.

The space $D(A)$ is the domain of the infinitesimal generator A, with the norm $\|x\| + \|Ax\|$. For $U \in \mathcal{D}'_+(L(E; D(A)))$ and $V \in \mathcal{D}'_+(L(D(A)); E)$ the compositions $U*V$ and $V*U$ are defined on $\mathcal{D}'_+(L(D(A); D(A)))$ and $\mathcal{D}'_+(L(E;E))$, respectively. The distribution $-\delta \otimes A + \delta' \otimes I$ is an element of $\mathcal{D}'_+(L(D(A); E))$.

Theorem 1.3.2. *[67] Let G be a distribution semigroup with infinitesimal generator A. Then:*

i) $G \in \mathcal{D}'_+(L(E; D(A)))$;

ii) $\Big(- A + \frac{\partial}{\partial t}\Big) * G = \delta \otimes I_E$;

iii) $G * \Big(-A + \frac{\partial}{\partial t}\Big) = \delta \otimes I_{D(A)}$,

where I_A *and* $I_{D(A)}$ *are identity mapping on* A *and* $D(A)$*, respectively.*

This corollary is closely related to the solutions of abstract Cauchy problem, having distribution as an initial value.

Corollary 1.3.1. *[67] If* G *is a distribution semigroup with infinitesimal generator* A*, then the equation*

$$-Au + \frac{\partial}{\partial t} u = T, \qquad u \in \mathcal{D}'_+(D(A)),$$

where $T \in \mathcal{D}'_+(E)$ *has an unique solution* $u = G * T$*. If* $suppT \subset [\alpha, \infty)$ *then* $suppu \subset [\alpha, \infty)$*.*

The opposite direction also holds:

Theorem 1.3.3. *[67] Let* A *be a closed operator with dense domain in* E*. If the solution*

$$-Au + \frac{\partial}{\partial t} u = T,$$

where $u \in \mathcal{D}'_+(D(A))$ *and* $T \in \mathcal{D}'_+(E)$*, has an unique solution depending of continuity of* T *and suppose that* T *is zero for* $t < \alpha$ *then* u *is zero for* $t < \alpha$*, then* A *is infinitesimal generator of an unique distribution semigroup.*

Following the idea on L. Schwartz on considering the space of all rapidly decreasing functions $\mathcal{S}$ and its dual, J.L. Lions, [67] introduced and studied the exponential distribution semigroups.

A distribution G is an exponential distribution semigroup if G is a distribution semigroup and the following additional condition is satisfied:

there exists ξ_0, such that $e^{-\xi t} G \in \mathcal{S}'(L(E;E))$, for $\xi > \xi_0$.

From the above definition it is clear that all the results on distribution semigroups hold for exponential distribution semigroups. Note that now, for the definition of $\overline{G}(T)$ there is no need T to be with compact support, but it is only enough for T to have a suitable growth at infinity.

Theorem 1.3.4. *[67] Necessary and sufficient condition for the closed linear operator* A *with dense domain in* E *to be infinitesimal generator of exponential distribution semigroup are:*

i) there exists ξ_0*, such that* $-A + p$*, for* $p = \xi + i\eta$*,* $\xi > \xi_0$ *is an isomorphism from* $D(A)$ *to* E*;*

ii) If $G(p) = (-A + p)^{-1}$*,* $\xi > \xi_0$ *then* $\|G(p)\| \leq pol(|p|)$*.*

P.C. Kunstmann [62], gave new definition of the distribution semigroups extending the previous theory of distribution semigroups, in direction that the generators of the distribution semigroups are not densely defined. Similarly like Lions, he treated the abstract Cauchy problem as a convolution equation which showed to be very useful.

By $*_0$ we denote the mapping $*_0 : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$,

$$f * g(t) = \int_0^t f(s)g(t-s)\,ds\,,$$

which is bilinear, separately continuous, associative and coincides on $\mathcal{D}_0 \times \mathcal{D}_0$ with the usual convolution. The definition of P.C. Kunstmann of distribution semigroups is:

Definition 1.3.2. [62] An element $G \in \mathcal{D}_0'(L(E))$ is a pre-distribution semigroup (shorty pre-DSG) if for all $\varphi, \psi \in \mathcal{D}$

$$G(\varphi)G(\psi) = G(\varphi *_0 \psi). \tag{1.7}$$

A pre-DSG is non-degenerate or just distribution semigroup (shorty DSG) if

$$\mathcal{N}(G) = \bigcap \{\mathrm{Ker} G(\varphi) : \varphi \in \mathcal{D}_0\} = \{0\}. \tag{1.8}$$

A pre-DSG it is called dense if

$$\mathcal{R}(G) = \bigcup \{\mathrm{Im} G(\varphi) : \varphi \in \mathcal{D}_0\} \tag{1.9}$$

is dense in E.

Let $T \in \mathcal{E}_0'$. Then

$$\tilde{G} = \{(x, y) \in E \times E : \forall \varphi \in \mathcal{D}_0 \ G(T * \varphi)x = G(\varphi)y\}.$$

$\tilde{G}(T)$ is a closed linear operator in E for all T in $\mathcal{E}_0'$. The generator A of G is the closed linear operator $A := G(-\delta')$. If the DSG G is a dense, then its generator is a dense operator.

In the definition of J.L.Lions, G is an element of $\mathcal{D}_0'(L(E; E))$ satisfying for all

$$\varphi, \psi \in \mathcal{D}_0, \qquad G(\varphi)G(\psi) = G(\varphi * \psi) \tag{1.10}$$

and in addition for all $\varphi \in \mathcal{D}_0$, $\psi\mathcal{D}_0$, $x \in E$, there exists continuous function u on $[0, \infty)$ such that for all $\psi \in \mathcal{D}$ $G(\psi)G(\varphi)x = \int_0^\infty \psi(t)u(t)\,dt$. The definition of Lions requires $u(0) = G(\varphi)x$, but by (1.10), $u(t) = G(\tau_t\varphi)x$, where $\tau_t\varphi(x) = \varphi(x-t)$, for $t > 0$, this condition is automatically fulfilled by the continuity of u. In the sequel by DSG-L, we mean the distribution semigroup in the sense of definition of Lions, and when DSG is used it means that the previous definition (definition of P.C. Kunstmann) is in consideration. Note that being DSG-L is equivalent to being a dense DSG. In the case of dense DSG the definitions of the generators by J.L. Lions and P.C. Kunstmann coincide. When the DSG is non-densely defined that is not a case. In fact if A is a generator of a non-densely defined DSG, then the generator in sense of J.L. Lions coincides with with the closure of the restriction of A to $D_\infty = \cap_{n=1}^\infty D(A^n)$. Also in [62] is given that the non-densely defined DSG cannot be obtain simply by dropping condition of denseness of DSG-L.

Example 1.3.1. [62] Let G be a non-dense DSG with generator A. Then $D(A)$ is not dense in E, so there are $x \in E$ and $x^* \in (D(A))^\circ$, such that $\langle x^*, x\rangle \neq 0$. Then $\tilde{G} := G + \delta \otimes \langle x^*, \cdot\rangle$ doesn't satisfy (1.7).

Let E and D be Banach spaces and $P \in \mathcal{D}'_0(L(D,E))$. A fundamental solution for P is an element G of $\mathcal{D}'_0(L(E,D))$ which satisfies $P*G = \delta\otimes I_E$ and $G*P = \delta\otimes I_D$, where I_E and I_D are identical mappings on E and D respectively. Furthermore, a few important results from [62] are listed, which are generalized in subsequent chapters to the case of ultradistributions and Fourier hyperfunctions.

Theorem 1.3.5. *[62] Let A be a closed operator in E. Then A generates a DSG G if and only if there is a fundamental solution for $P = \delta' \otimes I - \delta \otimes A \in \mathcal{D}'_0(L(D(A),E))$ where $D(A)$ is supplied with the graph norm and I denotes the inclusion $D(A) \longrightarrow E$.*

Theorem 1.3.6. *[62] Let G be a DSG with generator A. Then the following statements are equivalent:*

a) G is a dense DSG;

b) $G(\cdot)^*$ is a DSG in E^*;

c) A is densely defined.

Corollary 1.3.2. *[62] Let A be a closed operator in E. A is the generator of a DSG if and only if there are constants $\alpha, \beta, C > 0$ and $n \in \mathbb{N}$ such that*

$$\Lambda := \{\lambda \in \mathbb{C} : Re\lambda \geq \alpha \log(1 + |\lambda|) + \beta\} \subset \rho(A)$$

and

$$\forall \lambda \in \Lambda : \|R(\lambda, A)\| \leq C(1 + |\lambda|)^n.$$

Theorem 1.3.7. *[62] Let G be a pre-DSG in E, $a > 0$ and $n \in \mathbb{N}$. Suppose G has a strongly continuous representation F of order n on $[0, a]$. Then*

a) For all $s, t \in [0, a]$ with $s + t \in [0, a]$,

$$F(t)F(s) = \int_t^{t+s} \frac{(s+t-r)^{n-1}}{(n-1)!} F(r)\, dr - \int_0^s \frac{(s+t-r)^{n-1}}{(n-1)!} F(r)\, dr\,.$$

Furthermore

$$KerT = \bigcup_{t\in(0,a]} KerF(t) \qquad \text{and} \qquad KerS = \Big(\overline{\bigcup_{t\in(0,a]} ImF(t)}\Big)^\circ,$$

where $T \in L(\mathcal{N}(G))$ denotes the kernel operator of G and S denotes the kernel operator of $G(\cdot)^*$.

b) If G is a DSG with generator A, then

$$(x,y)\in A \Leftrightarrow \forall t\in[0,a]: F(t)x-\frac{t^n}{n!}x=\int_0^t F(s)y\,ds\,.$$

For the needs of Chapter 3 and Chapter 4, we give the definitions of (local) K-convoluted C-semigroup and α times integrated C-semigroup:

Let A be a closed operator, K be a locally integrable function on $[0,\tau)$, $0<\tau\le\infty$, and let $\Theta(t):=\int_0^t K(s)ds$, $0\le t\le\tau$. If there exists a strongly continuous operator family $(S_K(t))_{t\in[0,\tau)}$ such that, for $t\in[0,\tau)$, $S_K(t)C=CS_K(t)$, $S_K(t)A\subset AS_K(t)$, $\int_0^t S_K(s)xds\in D(A),\ x\in E$ and

$$A\int_0^t S_K(s)xds=S_K(t)x-\Theta(t)Cx,\ x\in E,$$

then $(S_K(t))_{t\in[0,\tau)}$ is called a (local) K-convoluted C-semigroup having A as a subgenerator.

For $\tau=\infty$, we say that $(S_K(t))_{t\ge0}$ is an exponentially bounded K-convoluted $C-$semigroup generated by A if, additionally, there exist $M>0$ and $\omega\in\mathbb{R}$ such that $||S_K(t)||\le Me^{\omega t},\ t\ge0$. $(S_K(t))_{t\in[0,\tau)}$ is called non-degenerate, if the assumption $S_K(t)x=0$, for all $t\in[0,\tau)$, implies $x=0$.

If $K(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha>0$, then we also say that $(S_K(t))_{t\in[0,\tau)}$ is an α-times integrated C-semigroup having A as a subgenerator. Then it is straightforward to see that A is a subgenerator of an n-times integrated C-semigroup, for any $n\in\mathbb{N}$ with $n\ge\alpha$. Usually, we will have $C=I$ in the definitions of (local)K-convoluted semigroups (see [57]).

In Chapter 3 it will be given a structural theorem through convoluted semigroups using [70]. Using the results of P.C. Kunstmann, M. Kostić and S. Pilipović [58] considered ultradistribution semigroups with non-densely defined generators. They mentioned also ultradistribution semigroups in sense of J.L. Lions, shortly noted as UDSG-L. We recall from [58] the definitions of L-ultradistribution semigroups and ultradistribution semigroups (following [67] ,[62] and [104]).

Definition 1.3.3. [58] Let $G\in\mathcal{D}'^{*}_{+}(\mathbb{R},L(E))$. It is an L-ultradistribution semigroup of $*$-class if:
$(U.1)$ $G(\phi*\psi)=G(\phi)G(\psi),\ \phi,\ \psi\in\mathcal{D}^*_0(\mathbb{R})$;
$(U.2)$ $\mathcal{N}(G):=\bigcap_{\phi\in\mathcal{D}^*_0(\mathbb{R})}N(G(\phi))=\{0\}$;
$(U.3)$ $\mathcal{R}(G):=\bigcup_{\phi\in\mathcal{D}^*_0(\mathbb{R})}R(G(\phi))$ is dense in E;
$(U.4)$ For every $x\in\mathcal{R}(G)$ there exists a function $u\in C([0,\infty),E)$ satisfying

$$u(0)=x \text{ and } G(\phi)x=\int_0^\infty\phi(t)u(t)dt,\ \phi\in\mathcal{D}^*(\mathbb{R}).$$

If $G \in \mathcal{D}'^*_+(\mathbb{R}, L(E))$ satisfies

$(U.5)$ $G(\phi *_0 \psi) = G(\phi)G(\psi),\ \phi,\ \psi \in \mathcal{D}^*(\mathbb{R})$ $\left(f *_0 g(t) := \int\limits_0^t f(t-s)g(s)ds,\ t \in \mathbb{R}\right)$,

then it is a pre-(UDSG) of $*$-class. If $(U.5)$ and $(U.2)$ are fulfilled for G, then G is an ultradistribution semigroup of $*$-class, in short, (UDSG). A pre-(UDSG) G is dense if it additionally satisfies $(U.3)$.

If $G \in \mathcal{D}'^*_+(\mathbb{R}, L(E))$, then the condition:
$(U.2)'$ $\operatorname{supp} G(\cdot)x \not\subseteq \{0\}$, for every $x \in E \setminus \{0\}$, is equivalent to $(U.2)$.

The next example shows the differences between L-ultradistribution semigroups and ultradistribution semigroups. In fact, the example 1.3.1 in context of ultradistribution semigroups is paraphrased in next example.

Example 1.3.2. Let A and E be as in Example 3.1.1. Choose an element $x \in E$ and afterwards a functional $x^* \in (D(A))^\circ$ with $\langle x^*, x\rangle = 1$. Let G be as in Example 3.1.1 and $\tilde{G} := G + \delta \otimes \langle x^*, \cdot\rangle x$. Then $\tilde{G}$ satisfies $(U.1)$, $(U.2)$, $(U.4)$, but not $(U.5)$.

In context of Lemma 2.2, Lemma 2.4, Lemma 2.6-2.7 and Lemma 3.6 in [62] in sense of ultradistribution semigroups, in [58] the following two theorems are given:

Theorem 1.3.8. *[58] Let G be a pre-(UDSG) of $*$-class, $F := E/\mathcal{N}(G)$ and q be the corresponding canonical mapping $q : E \longrightarrow F$. Then:*

- *a) Define $H \in \mathcal{D}_0^{*'}(L(F))$ by $qG(\varphi) := H(\varphi)q$ for all $\varphi \in \mathcal{D}^*$. Then H is a (UDSG) of $*$-class in F.*
- *b) $\overline{\langle \mathcal{R}(G)\rangle} = \overline{\mathcal{R}(G)}$.*
- *c) Assume that G is not dense. Put $R := \overline{\mathcal{R}(G)}$ and $H := G_{|R}$. Then H is a dense pre-(UDSG) of $*$-class in R.*
- *d) The adjoint G^* of G satisfies $\mathcal{N}(G^*) = \overline{\mathcal{R}(G)}^\circ$. (Herein $\overline{\mathcal{R}(G)}^\circ$ denotes the polar of $\overline{\mathcal{R}(G)}$.)*
- *e) If E is reflexive, then $\mathcal{N}(G) = \overline{\mathcal{R}(G^*)}^\circ$.*
- *f) G^* is a (UDSG) of $*$-class on E^* if and only if G is a dense pre-(UDSG) of $*$-class. If E is reflexive, then G^* is a dense pre-(UDSG) of $*$-class on E^* if and only if G is a (UDSG) of $*$-class.*
- *g) G is a (UDSG) of $*$-class if and only if $(U.1)$, $(U.2)$ and $G(\varphi_+) = G(\varphi)$, $\varphi \in \mathcal{D}^*$ hold.*
- *h) $\mathcal{N}(G) \cap \langle \mathcal{R}(G)\rangle = \{0\}$.*
- *i) Assume that $(U.3)$ holds. Then we have the following equivalence relation:*

$$[(U.1) \wedge (U.2) \wedge (U.4)] \Longleftrightarrow [(U.5) \wedge (U.2)].$$

Theorem 1.3.9. *[58] Let G be a (UDSG) of $*$-class and let $S, T \in \mathcal{E}_0^{*'}$, $\varphi \in \mathcal{D}_0^*$, $\psi \in \mathcal{D}^*$ and $x \in E$. Then the following holds:*

a) $(G(\varphi)x, G(\underbrace{T * T * \ldots * T}_{m} * \varphi) \in G(T)^m,\ m \in \mathbb{N}.$

b) $G(S)G(T) \subseteq G(S * T)$ *with* $G(S)G(T) = D(G(S * T)) \cap D(G(T))$ *and* $G(S) + G(T) \subseteq G(S + T)$.

c) $(G(\psi)x, G(-\psi')x - \psi(0)x) \in G(-\delta')$.

d) If G *is dense, its generator is densely defined.*

In the same way as in the case of distributions, the definition of ultradistribution fundamental solution is given by the following definition:

Definition 1.3.4. [58] Let D and E are Banach spaces and let $P \in \mathcal{D}_0^{*'}(L(D, E))$. Then $G \in \mathcal{D}_0^{*'}(L(E, D))$ is said to be an ultradistribution fundamental solution for P when $P * G = \delta \otimes I_E$ and $G * P = \delta \otimes I_D$.

Every (UDSG) is uniquely determined by its generator.

Theorem 1.3.10. *[58] Let* A *be a closed operator. If* A *generates a (UDSG)* G *of* $*$*-class, then* G *is an ultradistribution fundamental solution for*

$$P := \delta' \otimes I_{D(A)} - \delta \otimes A \in \mathcal{D}_0^{*'}(L([D(A)], E)).$$

In particular, if $T \in \mathcal{D}_0^{*'}(E)$*, then* $u = G * T$ *is the unique solution of*

$$-Au + \frac{\partial}{\partial t}u = T, \quad u \in \mathcal{D}_0^{*'}([D(A)]),$$

and the supposition $suppT \subseteq [\alpha, \infty)$ *implies* $supp \subseteq [\alpha, \infty)$*. Conversely, if* $G \in \mathcal{D}_0^{*'}(L(E, [D(A)]))$ *is an ultradistribution fundamental solution for* P*, then* G *is a pre-(UDSG) of* $*$*-class in* E *generated by the closure of the operator* $\mathcal{A} \equiv \{(q(x), q(y)) : (x, y) \in A\}$.

Theorem 1.3.11. *[58] Assume that* $(M.3)$ *holds. Let* $T \in \mathcal{D}_0^{*'}(E)$ *and let* A *be a closed, densely defined operator. Assume that the equation*

$$-Au + \frac{\partial}{\partial t}u = T, \quad u \in \mathcal{D}_0^{*'}([D(A)])$$

has a unique solution depending continuously on T*, and that the assumption* $suppT \subseteq [\alpha, \infty)$ *implies* $suppu \subseteq [\alpha, \infty)$*. Furthermore, assume that for* $T = \delta$ *the corresponding solution* u *fulfills* $suppu(\cdot, x) \nsubseteq \{0\}$, $x \in E \backslash \{0\}$*. Then* A *is the generator of an L-ultradistribution semigroup of* $*$*-class.*

Theorem 1.3.12. *[58] There exists an ultradistribution fundamental solution of* $*$*-class for a closed linear operator* A *if and only if there exist* $l > 0$ *and* $\beta > 0$*, in the Beurling case, resp., for every* $l > 0$ *there exists* $\beta_l > 0$*, in the Roumieu case, such that:*

$$\Lambda_{l,\beta}^{(M_p)} \subseteq \rho(A), \quad \textit{resp.,} \quad \Lambda_{l,\beta_l}^{\{M_p\}} = \{\lambda \in \mathbb{C} : Re\lambda \geq M(l|\lambda|) + \beta_l\} \subseteq \rho(A)$$

and

$$\|R(\lambda : A)\| \leq \beta e^{M(l|\lambda|)}, \lambda \in \Lambda_{l,\beta}^{(M_p)}, \quad \textit{resp.,} \quad \|R(\lambda : A)\| \leq \beta_l e^{M(l|\lambda|)}, \lambda \in \Lambda_{l,\beta_l}^{\{M_p\}}.$$

In the same paper M. Kostić and S. Pilipović [58], considered exponential ultradistribution fundamental solutions.

Definition 1.3.5. [58] Suppose G is an ultradistribution fundamental solution of $*$-class for a closed linear operator A, resp., G is a (UDSG) of $*$-class generated by A. Then it is said that G is an exponential ultradistribution fundamental solution of $*$-class for A, resp., an exponential (UDSG) of $*$-class, (EUDSG) in short, if $(U.7)$ there exists $\omega \geq 0$ such that $e^{-\omega\cdot}G \in \mathcal{S}^{*'}(L(E))$ holds. Conditions $(U.5)$ and $(U.7)$ define an exponential pre-(UDSG).

Theorem 1.3.13. *[55] Suppose A is a closed linear operator. Then there exists an exponential ultradistribution fundamental solution of $*$-class for A if and only if there exist $a \geq 0$, $k > 0$ and $L > 0$, in the Beurling case, resp., there exists $a \geq 0$ such that, for every $k > 0$ there exists $L_k > 0$, in the Roumieu case, such that:*

$$\{\lambda \in \mathbb{C} : Re\lambda > a\} \subseteq \rho(A) and$$

$$\|R(\lambda : A)\| \leq Le^{M(k|\lambda|)}, \lambda \in \mathbb{C}, Re\lambda > a, resp.,$$

$$\|R(\lambda : A)\| \leq L_k e^{M(k|\lambda|)} \quad \textit{for all} \quad k > 0 \quad \textit{and} \quad \lambda \in \mathbb{C}, Re\lambda > a.$$

In the case of hyperfunction semigroups, S.Ōuchi is the first who studied more deeply, while using the approach of J.L. Lions, Y. Ito considered Fourier hyperfunction semigroups with densely defined generators. But, that kind of approach can not be used because of the analyticity of the test functions. In Chapter 4 Fourier hyperfunction semigroups will be defined using substantially new approach.

1.4 Some results connected to the Cauchy problem

The Cauchy problem has been extensively studied in past three decades, [5], [25], [70]. We point out some references, for another approaches to the abstract Cauchy problem with non-densely defined A through the theory of integrated, convoluted, distribution or ultradistribution semigroups, [2]-[7], [34], [62]-[66], [74], [70].

Consider the homogeneous Cauchy problem

$$u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x$$

where A is a linear closed densely defined operator on a Banach space E, $x \in E$ By $[D(A)]$ is denoted the Banach space with the norm $\|x\|_{A^n} = \|x\| + \|Ax\| + \ldots + \|A^n x\|$.

Theorem 1.4.1. *[70] Let A be a generator of n-times integrated semigroup $\{V(t) : t \geq 0\}$, $n \in \mathbb{N}$. Then* *a) for $x \in D(A)$, $t \geq 0$*

$$V(t)x \in D(A), \quad AV(t)x = V(t)Ax,$$

and

$$V(t)x = \frac{t^n}{n!}x + \int_0^t V(s)Ax\,ds;$$

b) for $x \in \overline{D(A)}$, $t \geq 0$

$$\int_0^t V(s)x\,ds \in D(A)$$

and

$$A\int_0^t V(s)x\,ds = V(t)x - \frac{t^n}{n!}x;$$

c) for $x \in D(A^n)$, $n \in \mathbb{N}$

$$V^{(n)}(t)x = V(t)A^n x + \sum_{k=0}^{n-1} \frac{t^k}{k!}A^k x;$$

d) for $x \in D(A^{n+1})$

$$\frac{d}{dt}V^{(n)}(t)x = AV^{(n)}(t)x = V^{(n)}(t)Ax;$$

e) the homogeneous Cauchy problem is (n,ω)-well possed.

We recall from G. Da Prato and E. Sinestrari, [25] some results which in the last chapter will be used to give more general theory on the spaces of weighted ultradistributions. At the beginning, some definition and notation will be given which will be used in the sequel. They considered different type of solutions of the Cauchy problem,

$$u'(t) = Au(t) + f(t), \quad u(0) = u_0, \quad t \in [0,T]$$

where A is a closed operator $A : D(A) \subseteq E \longrightarrow E$ and $f : [0,T] \longrightarrow E$, $u_0 \in E$ are given. Here A is not need to be densely defined. Some of the spaces which are considered here are spaces with E-valued functions:

$$C(0,T;E) = \{u : [0,T] \longrightarrow E, u \text{ is continuous}\}$$

with norm $\|u\|_{C(0,T;E)} = \sup\limits_{0\leq t\leq T} \|u(t)\|$ and

$$C^n = \{u : [0,T] \longrightarrow E, \ u^{(k)} \in C(0,T;E), k = 0,1,\ldots,n\}$$

where $n \in \mathbb{N}$ and $u^{(k)}$ denotes the Fréchet derivative. Further, the definitions of L^p and Sobolev E valued spaces are given as follows:

$$L^p(0,T;E) = \{u : [0,T] \longrightarrow E; u \text{ is strongly integrable and} \|u(\cdot)\|^p \text{is integrable}\}$$

for $1 \leq p < \infty$ with norm $\|u\|_{L^p(0,T;E)} = (\int_0^T \|u(t)\|^p \, dt)^{1/p}$.
The E-valued Sobolev spaces:

$$W^{1,p}(0,T;E) = \{u : [0,T] \longrightarrow E; u(u) = u_0 + \int\limits_0^t u'(s) \, ds, \;\; t \in [0,T]$$

for some $u_0 \in E$ and $u' \in L^p(0,T;E)\}$ for $1 \leq p < \infty$
and norm $\|u\|_{W^{1,p}(0,T;E)} = \|u\|_{L^p(0,T;E)} + \|u'\|_{L^p(0,T;E)}$.

Let $A : E \longrightarrow E$ be a closed linear operator in the Banach space E and $f \in L^p(0,T;E)$ for $1 \leq p < \infty$, $u_0 \in E$. A strict solution in L^p of

$$u'(t) = Au(t) + f(t), u(0) = u_0, t \in [0,T] a.e. \tag{1.11}$$

is a function $u \in W^{1,p}(0,T;E) \cap L^p(0,T;D(A))$ satisfying (1.11).

Let $f \in C(0,T;E)$ and $u_0 \in E$. A strict solution in C of

$$u'(t) = Au(t) + f(t), u(0) = u_0, t \in [0,T] \tag{1.12}$$

is a function $u \in C^1(0,T : E) \cap C(0,T;D(A))$ verifying (1.12).
Note that a strict solution in C of (1.12) is a strict solution in L^p of (1.11). The opposite direction is not true in general.

A function $u \in L^p(0,T;E)$ is called an F-solution in L^p of (1.11) if for each $k \in \mathbb{N}$, there is $u_k \in W^{1,p}(0,T;E) \cap L^p(0,T;D(A))$ such that by setting

$$u_k'(t) - Au_k(t) = f_k(t), u_k(0) = u_{0k}, t \in [0,T] a.e. \tag{1.13}$$

follows

$$\lim_{k \to \infty} (\|u_k - u\|_{L^p(0,T;E)} + \|f_k - f\|_{L^p(0,T;E)} + \|u_{0k} - u_0\|_E) = 0. \tag{1.14}$$

Note that if u is a strict solution in L^p then u is an F-solution in L^p. The opposite direction is not true in the general case.

Let $f \in L^1(0,T;E)$ and $u_0 \in E$. Then $u : [0,T] \longrightarrow E$ is an integral solution of (1.11) if $u \in C(0,T;E)$, $\int_0^t u(s) \, ds \in D(A)$ for $t \in [0,T]$ and

$$u(t) = u_0 + A \int\limits_0^t u(s) \, ds + \int\limits_0^t f(s) \, ds \,, \quad t \in [0,T]. \tag{1.15}$$

From the above definitions it is clear that $u \in C(0,T;E)$ is an integral solution of (1.11) if and only if $v(t) = \int\limits_0^t u(s) \, ds$, $t \in [0,T]$ is a strict solution in C of

$$v'(t) = Av(t) + u_0 + \int\limits_0^t f(s),, \quad v(0) = 0, \;\; t \in [0,T].$$

Rewriting $u(t)$ as $u(t) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} u(s)\, ds \in \overline{D(A)}$ and by the definition of integral solution, an integral solution has values in $\overline{D(A)}$. The integral solution of (1.11) is unique.

Theorem 1.4.2. *[25] Let $f \in L^p(0,T;E)$ and $u_0 \in E$. If u is an integral solution of (1.11) belonging to $W^{1,p}(0,T;E)$ or to $L^p(0,T;D(A))$ then u is a strict solution in L^p of (1.11).*

Now, let B be the operator defined by $Bu = -u'$ with domain $D(B) = \{u \in W^{1,p}(0,T;E) : u(0) = 0\}$ and let $B_n = n^2R(n:B) - n = nBR(n:B)$, $n \in \mathbb{N}$ be the Yoshida approximations of B.

Theorem 1.4.3. *[25] Given $f \in L^p(0,T;E)$ and $u_0 \in E$ there exists for each $n \in \mathbb{N}$ a unique $v_n \in L^p(0,T;D(A))$ verifying*

$$B_n(v_n - u_0) + Av_n + f = 0$$

and the following estimates hold

$$\|v_n(t)\| \leq M(\|u_0\| + \frac{\|f(t)\|}{n} + \int_0^t \|f(s)\|\, ds), \quad t \in [0,T] a.e.$$

$$\|v_n\|_{L^p(0,T;E)} \leq M(1+T)(\|u_0\| + \|f\|_{L^p(0,T;E)}).$$

The solutions in the previous theorem approximate in $L^p(0,T;E)$ each possible F-solution in L^p.

Theorem 1.4.4. *[25] Given $f \in L^p(0,T;E)$ and $u_0 \in E$ let v_n be the solution of the approximating problem in the previous theorem. If u is an F-solution in L^p of (1.11) then*

$$\lim_{n\to\infty} \|u - v_n\|_{L^p(0,T;E)} = 0.$$

Theorem 1.4.5. *[25] If u is an F-solution in L^p of (1.11) then $u \in C(0,T;E)$, $u(t) \in \overline{D(A)}$ for each $t \in [0,T]$, $u(0) = u_0$ and*

$$\|u(t)\| \leq M(\|u_0\| + \int_0^t \|f(s)\|\, ds), \quad t \in [0,T];$$

so the F-solution in L^p is unique. In addition, if u_k verify (1.13) and (1.14) then

$$\lim_{k\to\infty} \|u_k - u\|_{C(0,T;E)} = 0\,.$$

Lemma 1.4.1. *[25] If $f \in C^3(0,T;E)$, $f(0) = f'(0) = f''(0) = 0$ and $u_0 = 0$ then the problem (1.12) has a strict solution in C.*

Theorem 1.4.6. *[25] Problem (1.11) has a unique F-solution in L^p for each $f \in L^p(0,T;E)$ and $u_0 \in \overline{D(A)}$.*

Theorem 1.4.7. *[25] Let $f \in W^{1,p}(0,T;E)$, $u_0 \in D(A)$ and*

$$Au_0 + f(0) \in \overline{D(A)}. \tag{1.16}$$

Then there exists a unique $u \in C^1(0,T;E) \cap C(0,T;D(A))$ verifying

$$u'(t) = Au(t) + f(t), \ \ u(0) = u_0, \ t \in [0,T].$$

Moreover $v = u'$ is an F-solution in L^p of the problem

$$v'(t) = Av(t) + f'(t), \ \ v(0) = Au_0 + f(0), \ t \in [0,T] a.e.$$

Here the condition (1.16) is a compatibility condition between f and u_0: if there exists a strict solution in C of (1.12) then $Au_0 + f(0) \in \overline{D(A)}$ since

$$Au_0 + f(0) = u'(0) = \lim_{t \to 0} \frac{u(t) - u(0)}{t} \in \overline{D(A)}.$$

Theorem 1.4.8. *[25] Let $f \in L^p(0,T;D(A))$, $u_0 \in D(A)$, $Au_0 \in \overline{D(A)}$. Then there exists a unique $u \in W^{1,p}(0,T;E) \cap C(0,T;D(A))$ such that*

$$u'(t) = Au(t) + f(t), \ \ u(0) = u_0, \ t \in [0,T] a.e.$$

Moreover, $v = Au$ is an F-solution in L^p of the problem

$$v'(t) = Av(t) + Af(t), \ \ v(0) = Au_0, \ t \in [0,T] a.e.$$

Chapter 2

Some Classes of Multipliers and Convolutors in the Spaces of Tempered Ultradistributions

In [97] and [33] convolution operators and multipliers of the space $\mathcal{S}$ were studied by L. Schwartz and J. Horvath. Later, G. Sampson, Z. Zielezny, [87], [108] characterized convolution operators of the spaces $\mathcal{K}'_p$, $p \geq 1$. D. H. Pahk, [78] considered convolution operators in $\mathcal{K}'_e$. Topological structure of the spaces of multipliers and convolutors in $\mathcal{K}'_M$ was studied by S. Abdulah, [1]. The convolution in ultradistributon spaces were considered in [38] by S. Pilipović, A. Kaminski, D. Kovačević, while convolutors in the spaces of ultradistributions were investigated in [13], [38], [39], [59], [82], [85], [86].

The main interest in this chapter are convolutors and multipliers in the space of tempered ultradistributions of Beurling and Roumieu type and their characterization. To motivate the research on convolutors, consider the following example:

Let $P(D) = \sum\limits_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ (with suitable assumptions on coefficients), then the equation $P(D)u = v$ can be rewritten in the form $P(\delta) * u = v$. Hence, considering equations of the type $S * u = v$ one generalizes the concept of ultradifferential operators with constant coefficients. In order to consider such equations, S must be an ultradistribution that has well-defined convolution with elements of $\mathcal{S}^{(M_p)}$ resp. $\mathcal{S}^{\{M_p\}}$.

2.1 The space of Convolutors

Assume that $(M.1)$, $(M.2)$ and $(M.3)$ holds.

Definition 2.1.1. The space of the convolutors O'^*_C, of $\mathcal{S}'^*$ is the space of all $S \in \mathcal{S}'^*$ such that the convolution $S * \varphi$ is in $\mathcal{S}^*$, for every $\varphi \in \mathcal{S}^*$, and the mapping

$$\varphi \longrightarrow S * \varphi\,, \quad \mathcal{S}^* \longrightarrow \mathcal{S}^* \quad \text{is continuous.}$$

Recall from [85] several results.

Proposition 2.1.1. *[85] If $\varphi \in \mathcal{S}^*$ and $S \in \mathcal{S}'^*$ then,*

$$(S * \varphi)(x) = \langle S(t), \varphi(x-t) \rangle, \ \ x \in \mathbb{R}^n,$$

is a smooth function which satisfies the following condition:

There is $k > 0$, resp. there is $k_p \in \Re$, such that for every operator P of class $*$ and $\varphi \in \mathcal{S}^*$

$$\begin{aligned} P(D)(S*\varphi)(x) &= O(e^{M(k|x|)}), \mid x \mid \to \infty, \quad \textit{resp.} \\ P(D)(S*\varphi)(x) &= O(e^{N_{k_p}(|x|)}), \ \ \mid x \mid \to \infty. \end{aligned} \tag{2.1}$$

From the definition, for $S \in O_C'^*$ the mapping

$$T \to S * T, \quad \mathcal{S}'^* \to \mathcal{S}'^*, \quad \text{is continuous.}$$

Proposition 2.1.2. *Let $S \in \mathcal{S}'^*$. The following statements are equivalent.*

a) S is a convolutor.

b) For every $\varphi \in \mathcal{D}^*$, $\ S * \varphi \in \mathcal{S}^*$.

c) For every $r > 0$, resp. there exist $k > 0$

$$\{e^{M(r|x|)} S(\cdot - x); \ \ x \in \mathbb{R}\} \ \textit{resp.} \ \{e^{M(k|x|)} S(\cdot - x); \ \ x \in \mathbb{R}\},$$

is bounded in $\mathcal{D}'^*$.

d) For every $r > 0$, resp. there exist $k > 0$, there is $l > 0$, resp. there is $k_p \in \Re$, and L^∞ functions F_1 and F_2 such that

$$S = P_l(D) F_1 + F_2, \quad \textit{resp.} \quad S = P_{k_p}(D) F_1 + F_2,$$

and

$$\|e^{M(r|x|)} (\mid F_1(x) \mid + \mid F_2(x) \mid)\|_{L^\infty} < \infty$$

resp.

$$\|e^{M(k|x|)} (\mid F_1(x) \mid + \mid F_2(x) \mid)\|_{L^\infty} < \infty.$$

Proof. It will be proven only Roumieu case. Beurling case is similar.
a) $\Rightarrow$ b) It's obvious.
b) $\Rightarrow$ c) Let $\varphi \in \mathcal{D}^*$.

$$\langle e^{M(k|x|)} \tau_x S_t, \varphi(t) \rangle = \langle e^{M(k|x|)} S_t, \varphi(t+x) \rangle =$$

$$= e^{M(k|x|)} (S * \check{\varphi})(-x)\,.$$

$$\mid e^{M(k|x|))} (S * \check{\varphi})(-x) \mid \leq C s_k (S * \check{\varphi})\,.$$

c) $\Rightarrow$ d) For this part the following lemma of H. Komatsu [51] is needed.

Lemma 2.1.1. *Let K be a compact neighborhood of zero, $r > 0$, and $r_p \in \Re$.*

i) There are $u \in \mathcal{D}^{(M_p)}_{K,r/2}$ and $\psi \in \mathcal{D}^{(M_p)}_K$ such that

$$P_r(D)u = \delta + \psi\,, \tag{2.2}$$

where P_r is of form (1.3).

ii) There are $u \in C^\infty$ and $\psi \in \mathcal{D}^{\{M_p\}}_K$ such that

$$P_{r_p}(D)u = \delta + \psi\,, \tag{2.3}$$

$$\operatorname{supp} u \subset K, \quad \sup_{x\in K}\left\{\frac{|\partial^\alpha u(x)|}{\prod_{j=1}^{|\alpha|} r_j\, M_\alpha}\right\} \longrightarrow 0, \quad |\alpha| \to \infty\,, \tag{2.4}$$

where P_{r_p} is of form (1.3).

Let Ω be a bounded open set in $\mathbb{R}^n$ which contains zero and $K = \overline{\Omega}$. Let B be a bounded set in $\mathcal{D}^{\{M_p\}}_K$. For $\varphi \in B$

$$\mid \langle e^{M(k|x|)}\tau_x S_t, \varphi(t)\rangle \mid = e^{M(k|x|)} \mid (S * \check{\varphi})(-x) \mid \leq C\,, \tag{2.5}$$

for all $x \in \mathbb{R}^n$ where $C > 0$ does not depend on $\varphi \in B$. Denote by $L^1_{exp(-M(k|\cdot|))}$ the space of locally integrable functions f on $\mathbb{R}^n$ such that $f(\cdot)e^{-M(k|\cdot|)} \in L^1(\mathbb{R}^n)$ supplied with the norm

$$\|f\|_{L^1,exp(-M(k|\cdot|))} = \|f(\cdot)e^{-M(k|\cdot|)}\|_{L^1}\,.$$

Let B_1 be the closed unit ball in the space $L^1_{exp(-M(k|\cdot|))}$, $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}$ and $\varphi \in B$. Then,

$$\mid \langle S * \psi, \varphi\rangle \mid = \mid \langle (S * \check{\varphi})(-x), \psi\rangle \mid \leq \tag{2.6}$$

$$\leq \|S * \check{\varphi}(-x) \cdot e^{M(k|x|)}\|_{L^\infty} \cdot \|\psi\|_{L^1,exp(-M(k|\cdot|))} \leq C\|\psi\|_{L^1,exp(-M(k|\cdot|))} \leq C\,.$$

Hence

$$\mid \langle S * \psi, \varphi\rangle \mid \leq C\|\psi\|_{L^1,exp(-M(k|\cdot|))} \tag{2.7}$$

for all $\varphi \in B$ and $\psi \in \mathcal{D}^{\{M_p\}}$. From (2.6) it follows that

$$\{S * \psi \mid\, \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}\}$$

is bounded set in $\mathcal{D}'^{\{M_p\}}_K$, and because $\mathcal{D}'^{\{M_p\}}_K$ is barrelled, the set is equicontinuous. There exist $k_p \in \Re$ and $\varepsilon > 0$ such that

$$\mid \langle S * \theta, \check{\psi}\rangle \mid \leq 1\,, \psi \in B_1 \cap \mathcal{D}^{\{M_p\}},\ \theta \in V_{k_p}(\varepsilon),$$

where

$$V_{k_p}(\varepsilon) = \{\chi \in \mathcal{D}^{\{M_p\}}_K \mid \|\chi\|_{K,k_p} \leq \varepsilon\}\,. \tag{2.8}$$

The same inequality holds for the closure $\overline{V_{k_p}(\varepsilon)}$ of $V_{k_p}(\varepsilon)$ in $\mathcal{D}^{\{M_p\}}_{K,k_p}$. If $\theta \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, then for some $L_\theta > 0$, $\|\theta/L_\theta\|_{K,k_p} < \varepsilon$. Hence $\theta/L_\theta \in \overline{V_{k_p}(\varepsilon)}$ and $| \langle S*\theta, \check{\psi}\rangle | \leq L_\theta$, for $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}$. It follows that for $\psi \in \mathcal{D}^{\{M_p\}}$

$$| \langle S*\theta, \check{\psi}\rangle | \leq L_\theta \|\psi\|_{L^1, exp(-M(k|\cdot|))} . \tag{2.9}$$

Because $\mathcal{D}^{\{M_p\}}$ is dense in $L^1_{exp(-M(k|\cdot|))}$ it follows that for every θ in $\mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, $S*\theta$ is a continuous functional on $L^1_{exp(-M(k|\cdot|)}$. Thus $S*\theta$ belongs to $L^\infty_{exp(M(k|\cdot|))} = \{f \in L_{1,loc} \,|\, \|f(\cdot)e^{M(k|\cdot|)}\|_{L_\infty} < \infty\}$, since the space $L^\infty_{exp(M(k|\cdot|))}$ is the dual of the space $L^1_{exp(-M(k|\cdot|)}$. Hence,

$$\|S*\theta(x)\|_{L^\infty, exp(M(k|\cdot|)} \leq L_\theta ,$$

where $L_\theta > 0$ is a constant which depends of θ. From Lemma 2.1.1 for the chosen $k_p \in \Re$ and Ω there exist $\tilde{k_p}$ and $u \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$ and $\psi \in \mathcal{D}^{\{M_p\}}_{\Omega}$ such that

$$S = P_{\tilde{k_p}}(D)(u*S) + (\psi * S) .$$

Now it's obvious that $F_1 = u*S$ and $F_2 = \psi * S$ satisfy the conditions in d).
d) $\Rightarrow$ a) Assume that $F_2 = 0$. The general case can be proved analogously. It is enough to prove that $\varphi \longrightarrow F*\varphi$ is a continuous mapping from $\mathcal{S}^{\{M_p\}}$ to $\mathcal{S}^{\{M_p\}}$. Then, a) will hold because of the continuity of the operator $P_{k_p}(D)$ and the fact that $P_{k_p}(D)(S*\varphi) = P_{k_p}(D)S*\varphi$. Observe that the continuity of the mapping $\varphi \longrightarrow F*\varphi$ will follow if it is proved that for every r which is bigger than some fixed r_0, there exist l such that $\varphi \longrightarrow F*\varphi$ is a continuous mapping from $\mathcal{S}^{M_p,r}_\infty$ to $\mathcal{S}^{M_p,l}_\infty$ (because $\mathcal{S}^{\{M_p\}}$ is a inductive limit of $\mathcal{S}^{M_p,r}_\infty$). For the k in the condition d) we choose r_0, small enough such that for all $r \leq r_0$ the integral

$$\int_{\mathbb{R}^n} e^{-M(k|t|)} e^{M(r|t|)} dt$$

converge. Fix r such that $r \leq r_0$. Note that

$$\frac{r^p \mid x \mid^p}{2^p M_p} \leq \frac{r^p \mid x-t \mid^p}{M_p} + \frac{r^p \mid t \mid^p}{M_p} \leq \tag{2.10}$$

$$\leq e^{M(r|x-t|)} + e^{M(r|t|)} \leq 2e^{M(r|x-t|)} e^{M(r|t|)}$$

and the last inequality holds since the function $M(\rho)$ is nonnegative. For the associated function there exist $\rho_0 > 0$ such that for $\rho \leq \rho_0$, $M(\rho) = 0$ and for $\rho > \rho_0$, $M(\rho) > 0$ (for the properties of the associated function we refer to [48]). If $\mid x \mid > \dfrac{2\rho_0}{r}$ then from the inequality (2.10) it follows that

$$e^{M(\frac{r}{2}|x|)} \leq 2e^{M(r|x-t|)} e^{M(r|t|)} .$$

If $\mid x \mid \leq \dfrac{2\rho_0}{r}$, there exist $c > 0$ such that $e^{M(\frac{r}{2}|x|)} \leq c$. Hence, it follows that for all $x \in \mathbb{R}^n$, the following inequality holds

$$e^{M(\frac{r}{2}|x|)} \leq 2(c+1)\, e^{M(r|x-t|)} e^{M(r|t|)}$$

and obtain that

$$e^{-M(r|x-t|)} \le Ce^{M(r|t|)}e^{-M(\frac{r}{2}|x|)} ,$$

where $C = 2(c+1)$. Let $l < r/4$. Then,

$$\frac{l^\alpha \mid F * D^\alpha \varphi(x) \mid e^{M(l|x|)}}{M_\alpha} \le \frac{l^\alpha}{M_\alpha} \int\limits_{\mathbb{R}^n} \mid F(t) \mid\mid D^\alpha\varphi(x-t) \mid dt\, e^{M(l|x|)} =$$

$$= \Big(\frac{l}{r}\Big)^\alpha \frac{1}{M_\alpha} \int\limits_{\mathbb{R}^n} \mid F(t) \mid \frac{e^{M(k|t|)}}{e^{M(k|t|)}} \mid D^\alpha\varphi(x-t) \mid r^\alpha \frac{e^{M(r|x-t|)}}{e^{M(r|x-t|)}} dt\, e^{M(l|x|)} \le$$

$$\le C'\Big(\frac{l}{r}\Big)^\alpha s_r(\varphi) \int\limits_{\mathbb{R}^n} e^{-M(k|t|)}e^{M(r|t|)}dt\, e^{-M(\frac{r}{2}|x|)}\, e^{M(l|x|)}.$$

Because of the way that l is chosen, it follows that

$$s_l(F*\varphi) = \sup_\alpha \frac{l^\alpha \| \mid F * D^\alpha\varphi(x) \mid e^{M(l|x|)}\|_{L^\infty}}{M_\alpha} \le C'' s_r(\varphi) ,$$

where C'' is a constant which does not depend on φ. It is shown that $\varphi \longrightarrow F*\varphi$, is continuous mapping from $\mathcal{S}_\infty^{M_p,r}$ to $\mathcal{S}_\infty^{M_p,l}$. Hence, $\varphi \longrightarrow F*\varphi$ is continuous mapping from $\mathcal{S}^{\{M_p\}}$ to $\mathcal{S}^{\{M_p\}}$. □

It is clear that the ultratempered convolution of $S_1, S_2 \in O_C'^*$ is in $O_C'^*$ (see [38]). As well for any $T \in \mathcal{S}'^*$, and $\psi \in \mathcal{S}^*$,

$$\langle (S_1 * S_2) * T, \psi\rangle = \langle S_1 * T, \check{S}_2 * \psi\rangle = \tag{2.11}$$

$$= \langle T * S_2, \check{S}_1 * \psi\rangle = \langle T, (S_1 * S_2) * \psi\rangle\,.$$

Supply $O_C'^*$ with the topology from $L_{\mathrm{s}}(\mathcal{S}^*, \mathcal{S}^*)$ and denote it by $O_{C,\mathrm{s}}'^*$. The same topology on this space is induced by $L_{\mathrm{s}}(\mathcal{S}'^*, \mathcal{S}'^*)$.

Proposition 2.1.3. *The strong topology on $L(\mathcal{S}'^*, \mathcal{S}'^*)$ induces the same topology on $O_C'^*$.*

Proof. Let U be a neighborhood of zero in $\mathcal{S}'^*$. Without loss of generality it can be assumed that

$$U = U(V'; B') = \{S \in O_C'^*(\mathcal{S}'^*; \mathcal{S}'^*) \,|\, S * T \in V' \,, for\, all\ T \in B'\} ,$$

where B' is bounded subset in $\mathcal{S}'^*$ and V' is a neighborhood of zero in $\mathcal{S}'^*$. Assume that

$$V' = V'(B, \varepsilon) = \{T \in \mathcal{S}'^* \mid \mid \langle T, \varphi\rangle \mid < \varepsilon\, for\, all\, \varphi \in B\} ,$$

where B is bounded in $\mathcal{S}^*$, and $\varepsilon > 0$. Let

$$V = \{\varphi \in \mathcal{S}^* \mid \mid \langle T, \varphi\rangle \mid < \varepsilon\, for\, all\, T \in B'\} .$$

Since $\mathcal{S}^*$ is barreled is follows that V is a neighborhood of zero in $\mathcal{S}^*$. Without loss of generality it can be assumed that $B = \check{B} = \{\check{\varphi} \mid \varphi \in B\}$ and $B' = \check{B}' = \{\check{T} \mid T \in B'\}$. Let

$$W = W(V, B) = \{S \in O'^*_C(\mathcal{S}'^*; \mathcal{S}'^*) \mid S * \varphi \in V \, for \, all \, \varphi \in B\}.$$

We will show that $W(V, B) \subset U(V', B')$. Let $S \in W(V, B)$, $T \in B'$ and $\varphi \in B$. Then

$$| \langle S * T, \varphi \rangle | = | \langle T, \check{S} * \varphi \rangle | < \varepsilon .$$

Hence $S * T \in V'$ for all $T \in B'$. So it is shown that the topology induced by $L_s(\mathcal{S}'^*, \mathcal{S}'^*)$ is stronger than the topology induced by $L_s(\mathcal{S}^*, \mathcal{S}^*)$. The other direction is similar and it is omitted. □

Proposition 2.1.4. *$O'^*_{C,s}$ is complete.*

Proof. Let $\{S_\mu\}$ be a Cauchy net in $O'^*_{C,s}$. Then $\{\tilde{S}_\mu\}$ is a Cauchy net in $L_s(\mathcal{S}^*, \mathcal{S}^*)$, where $\tilde{S}_\mu : \mathcal{S}^* \longrightarrow \mathcal{S}^*$ are induced continuous linear operators by S_μ, $\tilde{S}_\mu(\varphi) = S_\mu * \varphi$. Since $\mathcal{S}^*$ is complete and bornological [102], Corollary 1 of Theorem 32.2, $L_s(\mathcal{S}^*, \mathcal{S}^*)$ is complete, there exists $R \in L_s(\mathcal{S}^*, \mathcal{S}^*)$, such that $\tilde{S}_\mu \longrightarrow R$. Define $T \in \mathcal{S}'^*$, by $\langle T, \varphi \rangle = R(\varphi)(0)$. For $\varphi \in \mathcal{S}^*$, $R(\varphi) = T * \varphi$, since for $x \in \mathbb{R}^n$

$$R(\varphi)(x) = \lim_\mu (S_\mu * \varphi)(x) = \lim_\mu (S_\mu * (\tau_x \varphi))(0) =$$

$$= R(\tau_x \varphi)(0) = \langle T, \tau_x \varphi \rangle = T * \varphi(x) .$$

Thus for $\varphi \in \mathcal{S}^*$, $T * \varphi \in \mathcal{S}^*$ and the map $\varphi \longrightarrow T * \varphi$ is continuous. It follows that $T \in O'^*_C$, and moreover $S_\mu \longrightarrow T$ in O'^*_C since $\tilde{T} = R$. □

Proposition 2.1.5. *A sequence S_n from $O'^*_{C,s}$ converges to zero in $O'^*_{C,s}$ if and only if for every $k > 0$ resp. there exist $k > 0$, there exists $r > 0$, resp. there exists $k_p \in \Re$ and sequences of L^∞ functions F_{1n} and F_{2n}, such that*

$$S_n = P_r(D) F_{1n} + F_{2n}, \quad resp. \; S_n = P_{k_p}(D) F_{1n} + F_{2n}, \tag{2.12}$$

$F_{1n}, F_{2n} \in O'^*_C$,

$$\| e^{M(k|x|)} (| F_{1n} | + | F_{2n} |) \|_{L^\infty} < \infty$$

and

$$F_{1n} \longrightarrow 0, \quad F_{2n} \longrightarrow 0 \qquad in \; O'^*_{C,s} , \tag{2.13}$$

Proof. The proof of the proposition is similar with the proof of the Proposition 2.1.2, but it will be given for the sake of completeness. Let S_n be a sequence in $O'^{\{M_p\}}_C$ which converges to zero in $O'^{\{M_p\}}_{C,s}$. Let Ω be a bounded open set in $\mathbb{R}^n$ which contains zero and $K = \overline{\Omega}$. Let $\varphi \in \mathcal{D}^{\{M_p\}}_K$ be fixed. Then $S_n * \varphi \longrightarrow 0$ in $\mathcal{S}^{\{M_p\}}$. Because $\mathcal{S}^{\{M_p\}}$ is a (DFS) space, it follows that there exist $k > 0$ such that $S_n * \varphi \in \mathcal{S}^{M_p,k}_\infty$, and is bounded there, i.e.

$$\sup_\alpha \frac{k^\alpha \| e^{M(k|x|)} D^\alpha (S_n * \varphi)(x) \|_{L^\infty}}{M_\alpha} \le C_\varphi, \forall n \in \mathbb{N} ,$$

where C_φ is a constant which depends only on φ. So,

$$\|e^{M(k|x|)}(S_n * \varphi)(x)\|_{L^\infty} \leq C_\varphi \, , \forall n \in \mathbb{N} \, .$$

Let $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}$, then

$$| \langle S_n * \psi, \check{\varphi} \rangle | = | \langle S_n * \varphi, \check{\psi} \rangle | \leq \|S_n * \varphi\|_{L^\infty_{exp(M(k|\cdot|))}} \leq C_\varphi \, , \tag{2.14}$$

for all $n \in \mathbb{N}$, where B_1 is the closed unit ball in $L^1_{exp(-M(k|\cdot|))}$.

From (2.14) it follows that

$$\{S_n * \psi | \ \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}, n \in \mathbb{N}\}$$

is weakly bounded set in $\mathcal{D}'^{\{M_p\}}_K$, and because $\mathcal{D}^{\{M_p\}}_K$ is barreled, the set is equicontinuous (see [91], Theorem 5.2) . There exist $k_p \in \Re$ and $\delta > 0$ such that

$$| \langle S_n * \theta, \check{\psi} \rangle | \leq 1 \, , \, \theta \in V_{k_p}(\delta) \, , \psi \in B_1 \cap \mathcal{D}^{\{M_p\}} \, , \, n \in \mathbb{N},$$

where $V_{k_p}(\delta) = \{\chi \in \mathcal{D}^{\{M_p\}}_K | \, \|\chi\|_{K,k_p} \leq \delta\}$. The same inequality holds for the closure $\overline{V_{k_p}(\delta)}$ of $V_{k_p}(\delta)$ in $\mathcal{D}^{\{M_p\}}_{K,k_p}$. If $\theta \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, then for some $L_\theta > 0$, $\|\theta/L_\theta\|_{K,k_p} < \delta$, hence $\theta/L_\theta \in \overline{V_{k_p}(\delta)}$ and

$$| \langle S_n * \theta, \check{\psi} \rangle | \leq L_\theta \, , \, \psi \in B_1 \cap \mathcal{D}^{\{M_p\}} \, , \, n \in \mathbb{N} \, .$$

It follows that for $\psi \in \mathcal{D}^{\{M_p\}}$

$$| \langle S_n * \theta, \check{\psi} \rangle | \leq L_\theta \|\psi\|_{L^1, exp(-M(k|\cdot|))} \, . \tag{2.15}$$

Because $\mathcal{D}^{\{M_p\}}$ is dense in $L^1_{exp(-M(k|\cdot|))}$ it follows that for every θ in $\mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, $S_n * \theta$ are continuous functionals on $L^1_{exp(-M(k|\cdot|))}$ and uniformly bounded. Thus $S_n * \theta$ belong to $L^\infty_{exp(M(k|\cdot|))}$. Hence,

$$\|S_n * \theta(x)\|_{L^\infty, exp(M(k|\cdot|))} \leq L_\theta \, , \forall n \in \mathbb{N} \, ,$$

where $L_\theta > 0$ is a constant which depends on θ. From Lemma 2.1.1, for the chosen $k_p \in \Re$ and Ω, there exist $\tilde{k_p}$ and $u \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$ and $\psi \in \mathcal{D}^{\{M_p\}}_\Omega$ such that

$$S_n = P_{\tilde{k_p}}(D)(S_n * u) + (S_n * \psi) \, .$$

Let $F_{1n} = S_n * u$ and $F_{2n} = S_n * \psi$. It's obvious that $u \in O'^{\{M_p\}}_C$, hence $F_{1n}, F_{2n} \in O'^{\{M_p\}}_C$. $F_{1n} = S_n * u \longrightarrow 0$ and $F_{2n} = S_n * \psi \longrightarrow 0$ in $O'^{\{M_p\}}_C$.

Conversely, let $F_n \longrightarrow 0$ and $F_{1n} \longrightarrow 0$ in $O'^{\{M_p\}}_C$, $S_n = P_{k_p}(D)F_n + F_{1n}$, for some $k_p \in \Re$. Assume that $F_{1n} = 0$ for all $n \in \mathbb{N}$. The general case is proved similarly. Let $M(B,V)$ is a neighborhood of zero in $O'^{\{M_p\}}_C$, where B is a bounded set in $\mathcal{S}^{\{M_p\}}$, and V is a open neighborhood of zero in $\mathcal{S}^{\{M_p\}}$. Since, $P_{k_p}(D) : \mathcal{S}^{\{M_p\}} \longrightarrow \mathcal{S}^{\{M_p\}}$ is continuous , there exist open neighborhood V_0 such that $P_{k_p}(D)(V_0) \subset V$. Since $F_n \longrightarrow 0$ in $O'^{\{M_p\}}_C$, and $M(B,V_0)$ is a neighborhood of zero, there exists n_0, such that for all $n \geq n_0$, $F_n \in M(B,V_0)$. Thus, $F_n * \varphi \in V_0$, for all $\varphi \in B$ and $n \geq n_0$, and it follows that

$$P_{k_p}(D)(F_n * \varphi) \subset P_{k_p}(D)(V_0) \subset V \, .$$

□

Remark 2.1.1. The inclusion $O'^{*}_{C,s} \hookrightarrow \mathcal{S}'^{*}$ is continuous. Let V be a open neighborhood in $\mathcal{S}'^{*}$. Let us consider this neighborhood of O'^{*}_{C}:

$$W = \{S \in O'^{*}_{C} \,|\, S * \delta \in V\} .$$

Then it is obvious that from $S \in W$, it follows that $S \in V$.

From the convergence of F_{1n}, F_{2n} to zero in O'^{*}_{C}, in the above proposition, it follows convergence in $\mathcal{S}'^{*}$.

Denote by $\mathcal{E}\mathcal{S}'^{*}$ the space of elements f from $\mathcal{S}'^{*}$ such that for every $S \in O'^{*}_{C}$, $S * f \in \mathcal{E}^{*}$ and the mapping

$$S \to S * f, \;\; O'^{*}_{C,\mathrm{s}} \to \mathcal{E}^{*} \text{ is continuous.}$$

Proposition 2.1.6. *(i)* $\mathcal{E}\mathcal{S}'^{*} \subset \mathcal{E}^{*} \cap \mathcal{S}'^{*}$.

(ii) If $f \in \mathcal{E}\mathcal{S}'^{*}$ *and* $S \in O'^{*}_{C}$ *then* $S * f \in \mathcal{E}\mathcal{S}'^{*}$.

Proof. (i) It is clear from the definition of $\mathcal{E}\mathcal{S}'^{*}$ that if $f \in \mathcal{E}\mathcal{S}'^{*}$, $f \in \mathcal{S}'^{*}$ and because δ is in $O'^{\{M_p\}}_{C}$, $\delta * f = f$ is an element in $\mathcal{E}^{*}$.

(ii) From (i) it follows that $S * f \in \mathcal{S}'^{*}$. Let $T \in O'^{*}_{C}$. So,

$$T * (S * f) = (T * S) * f$$

is in $\mathcal{E}^{*}$. It is obvious that the mapping $T \longrightarrow T * (S * f)$ is continuous, since the mappings $T \longrightarrow T * S \longrightarrow (T * S) * f = T * (S * f)$ are continuous. Hence, $S * f \in \mathcal{E}\mathcal{S}'^{*}$. □

Note that $\mathcal{S}^{*}$ is subset of $\mathcal{E}\mathcal{S}'^{*}$.

2.2 The Space of Multipliers

Again we assume $(M.1)$, $(M.2)$ and $(M.3)$ hold.

As in [59] and [82] , the definition of the space of multipliers is given in the sequel.

Definition 2.2.1. Define O^{*}_{M} as the space of functions φ from $\mathcal{E}^{*}$ such that $\varphi \in O^{*}_{M}$ if and only if

$$\begin{aligned} &\text{for every } \psi \in \mathcal{S}^{*},\; \varphi\psi \in \mathcal{S}^{*} \quad \text{and the mapping} \\ &\psi \to \varphi\psi, \quad \mathcal{S}^{*} \longrightarrow \mathcal{S}^{*} \quad \text{is continuous .} \end{aligned} \tag{2.16}$$

From the definition, for $\varphi \in O^{*}_{M}$ the mapping

$$T \to \varphi T, \quad \mathcal{S}'^{*} \to \mathcal{S}'^{*}, \quad \text{is continuous.}$$

In the proof of the next proposition the following function will be needed:

$$\psi(x) = \sum_{j=1}^{\infty} \frac{\rho(x - x_j)}{e^{M(k|x_j|)}} , \tag{2.17}$$

where the function $\rho \in \mathcal{D}^{\{M_p\}}$, has values in $[0,1]$, and $\operatorname{supp}\rho \subset \{x : | x | \leq 1, x \in \mathbb{R}^n\}$, $\rho(x) = 1$, for $x \in \{x | \ | x | \leq 1/2\}$. $\{x_j\}$ is a sequence of real numbers such that $| x_j | > 2$ and $| x_{j+1} | \geq | x_j | + 2$, $j \in \mathbb{N}$.

Since $\rho \in \mathcal{D}^{\{M_p\}}$, there exist h and C such that $\sup_{x,\alpha} | D^\alpha \rho | < C h^\alpha M_\alpha$. It will be shown that $\psi \in \mathcal{S}^{\{M_p\}}$. Choose r such that $rh < \frac{1}{2}$ and $r < \frac{k}{H4\sqrt{2}}$. Using that $\frac{| x |}{| x_j |} \leq 2$, one obtains,

$$\sum_{\alpha,\beta} \int_{\mathbb{R}^n} \frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta} | D^\alpha \psi(x) |^2}{M_\alpha^2 M_\beta^2} dx \leq$$

$$\leq \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \int_{|x-x_j|\leq 1} \frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta} C^2 h^{2\alpha} M_\alpha^2}{M_\alpha^2 M_\beta^2 e^{2M(k|x_j|)}} dx \leq$$

$$\leq \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \int_{|x-x_j|\leq 1} \frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta} C^2 h^{2\alpha}}{M_\beta^2 e^{2M(k|x_j|)}} dx \leq$$

$$\leq \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \int_{|x-x_j|\leq 1} \frac{r^{2\alpha} r^{2\beta} 2^\beta | x |^{2\beta} C^2 h^{2\alpha}}{M_\beta^2 e^{2M(k|x_j|)}} dx \leq$$

$$\leq C_1 \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \frac{(rh)^{2\alpha}(r\sqrt{2})^{2\beta}}{M_\beta^2 e^{2M(k|x_j|)}} \int_{|x-x_j|\leq 1} | x |^{2\beta} dx \leq$$

$$\leq C_2 \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \frac{(rh)^{2\alpha}(r\sqrt{2})^{2\beta} | x_j |^{2\beta}}{M_\beta^2 e^{2M(k|x_j|)}} \leq$$

$$\leq C_2 \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \frac{(rh)^{2\alpha}(r2\sqrt{2})^{2\beta} | x_j |^{2\beta} M_{\beta+1}^2}{M_\beta^2 k^{2\beta+2} | x_j |^{2\beta+2}} \leq$$

$$\leq \frac{C_2 A}{k^2} \sum_{\alpha,\beta} \sum_{j=1}^{\infty} (rh)^{2\alpha} \Big(\frac{2r\sqrt{2}H}{k}\Big)^{2\beta} \frac{1}{| x_j |^2} \leq C' .$$

The proof of the next proposition in (M_p)-case is given in [59] and [82].

Proposition 2.2.1. *Let $\varphi \in C^\infty$. The following statements are equivalent:*

(i) $\varphi \in O_M^*$.

(ii) For every $h > 0$, resp. for every $k > 0$, there exist $k > 0$, resp. there exist $h > 0$,

$$\sup_{\alpha \in \mathbb{N}_0^n} \{\frac{h^\alpha \| e^{-M(k|\cdot|)} \varphi^{(\alpha)} \|_{L^\infty}}{M_\alpha}\} < \infty.$$

(iii) For every $\psi \in \mathcal{S}^*$ and every $r > 0$, resp. for some $r > 0$.

$$\sigma_{m,\psi}(\varphi) := \sigma_{m,\infty}(\psi\varphi) < \infty.$$

(iv) In Roumieu case, for every $\psi \in \mathcal{S}^{\{M_p\}}$ *and for every* $r_i, s_j \in \Re$

$$\gamma_{r_i,s_j,\psi}(\varphi) := \gamma_{r_i,s_j}(\psi\varphi) < \infty.$$

Proof. Only the proof for the Roumieu case will be given.

(iii)$\Leftrightarrow$ (iv) It is obvious. It will be proved (iii)$\Rightarrow$ (ii)$\Rightarrow$(i)$\Rightarrow$(iii).

(iii)$\Rightarrow$(ii) First φ is in $\mathcal{E}^{\{M_p\}}$ it will be proven. Let K be a fixed compact set in $\mathbb{R}^n$ and take $\chi \in \mathcal{D}^{\{M_p\}}$, with values in $[0,1]$ and $\chi(x) = 1$ on a neighborhood of K. Then there exist r such that

$$\sup_\alpha \frac{r^\alpha \|D^\alpha(\varphi(x)\chi(x))\|_{L^\infty(K)}}{M_\alpha} \leq$$

$$\leq \sup_\alpha \frac{r^\alpha \|e^{M(r|x|)} D^\alpha(\varphi(x)\chi(x))\|_{L^\infty(\mathbb{R}^n)}}{M_\alpha} = C s_r(\varphi\chi) < \infty\,.$$

Then, $D^\alpha(\varphi(x)\chi(x)) = D^\alpha\varphi(x)$ for $x \in K$. Thus $\varphi \in \mathcal{E}^{\{M_p\}}$.

Let (ii) does not hold. Then there exist k such that for all $n \in \mathbb{N}$,

$$\sup_\alpha \frac{\|e^{-M(k|x|)} D^\alpha\varphi(x)\|_{L^\infty}}{n^\alpha M_\alpha} = \infty\,.$$

Since $\varphi \in \mathcal{E}^{\{M_p\}}$ for every compact set K, there exist C and $n_K \in \mathbb{N}$ such that for $n \geq n_K$

$$\sup_\alpha \frac{\|e^{-M(k|x|)} D^\alpha(\varphi(x))\|_{L^\infty(K)}}{n^\alpha M_\alpha} < C\, n \geq n_K\,.$$

Hence, it can be chosen α_j and x_j, where $\mid x_{j+1} \mid > \mid x_j \mid +2$, such that

$$\frac{e^{-M(k|x_j|)} \mid D^{\alpha_j}\varphi(x_j) \mid}{j^{\alpha_j} M_{\alpha_j}} \geq 1\,.$$

Now take ψ as in (2.17), where k and the sequence $\{x_j\}$ are taken to be the ones chosen here. Then $\varphi\psi \in \mathcal{S}^{\{M_p\}}$, i.e. there exist l such that

$$\sup_\alpha \frac{l^\alpha \|e^{M(k|x|)} D^\alpha(\varphi(x)\psi(x))\|_{L^\infty}}{M_\alpha} < \infty\,.$$

Then there exist j_0 such that for all $j \geq j_0$, $l > 1/j$.

$$\sup_\alpha \frac{l^\alpha \|e^{M(l|x|)} D^\alpha(\varphi(x)\psi(x))\|_{L^\infty}}{M_\alpha} \geq$$

$$\geq \frac{l^{\alpha_j} e^{M(l|x_j|)} \mid D^{\alpha_j}(\varphi(x_j)\psi(x_j)) \mid}{M_{\alpha_j}} \geq$$

$$\geq \frac{1}{j^{\alpha_j}} \frac{e^{M(l|x_j|)} \mid D^{\alpha_j}\varphi(x_j) \mid}{e^{M(k|x_j|)} M_{\alpha_j}} \geq e^{M(l|x_j|)}\,.$$

This implies that $\varphi\psi$ is not in $\mathcal{S}_\infty^{M_p,l}$, which is a contradiction with the above assumption.

(ii)⇒(i) From the condition (ii) it is obvious that $\varphi \in \mathcal{E}^{\{M_p\}}$. It is enough to prove that for every $r > 0$ there is $l > 0$ such that the mapping $\psi \longrightarrow \varphi\psi$ for $\mathcal{S}_\infty^{M_p,r}$ to $\mathcal{S}_\infty^{M_p,l}$ is continuous. Let $r > 0$ be fixed. Put $k = r/4$. By (ii), there exist h such that

$$\sup_\alpha \frac{h^\alpha \| e^{-M(k|x|)} D^\alpha \varphi(x) \|_{L^\infty}}{M_\alpha} < \infty .$$

If $l < h/4$ and $l < r/4$, then

$$\frac{l^\alpha \| e^{M(l|x|)} D^\alpha (\varphi(x)\psi(x)) \|_{L^\infty}}{M_\alpha} \leq$$

$$\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{l^\alpha \| e^{M(l|x|)} D^\beta \varphi(x) D^{\alpha-\beta} \psi(x) \|_{L^\infty}}{M_{\alpha-\beta} M_\beta} =$$

$$= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{(2l)^\alpha \| e^{M(l|x|)} D^\beta \varphi(x) e^{-M(k|x|)} e^{M(k|x|)} h^\beta}{2^\alpha h^\beta} \cdot$$

$$\cdot \frac{D^{\alpha-\beta}\psi(x) e^{M(r|x|)} e^{-M(r|x|)} r^{\alpha-\beta} \|_{L^\infty}}{r^{\alpha-\beta} M_{\alpha-\beta} M_\beta} \leq$$

$$\leq C s_r(\psi) \| e^{-M(r|x|)} e^{M(k|x|)} e^{M(l|x|)} \|_{L^\infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{1}{2^\alpha} \left(\frac{2l}{h}\right)^\beta \left(\frac{2l}{r}\right)^{\alpha-\beta} \leq C' s_r(\psi) ,$$

where the last inequality holds because of the way that l is chosen.
(i)⇒(iii) It is obvious. □

Remark 2.2.1. It is obvious that if $\varphi \in O_M^*$, then $\varphi \in \mathcal{S}'^*$.

Denote by $L(\mathcal{S}^*, \mathcal{S}^*)$ the space of continuous linear mappings from $\mathcal{S}^*$ into $\mathcal{S}^*$; O_M^* is its subspace. With $L_s(\mathcal{S}^*, \mathcal{S}^*)$, denote the space $L(\mathcal{S}^*, \mathcal{S}^*)$ with the strong topology. Also O_M^* can be equipped with the topology induced by $L_s(\mathcal{S}'^*, \mathcal{S}'^*)$. Similarly as in proposition 2.1.3 it can be proven that the topologies induced by $L_s(\mathcal{S}^*, \mathcal{S}^*)$ and $L_s(\mathcal{S}'^*, \mathcal{S}'^*)$ are the same. The space O_M^* equipped with this topology is denoted by $O_{M,s}^*$.

Proposition 2.2.2. *The Fourier transformation is a topological isomorphism of* $O_{M,s}^*$ *onto* $O_{C,s}'^*$.

Proof. Only the Roumieu case will be shown. Using Proposition 2.1.2 d), there exist $k > 0$ and there exist $k_p \in \Re$ such that $S = P_{k_p}(D)F + F_1$, where F and F_1 satisfy the growth condition given in proposition 2.1.2. Without loss of generality it may be assumed that $F_1 = 0$. By $(M.2)$, are the following estimates for the derivatives of the Fourier transform of F can be obtained:

$$| D^\alpha \mathcal{F}(F) | = | \mathcal{F}(x^\alpha F) | = \left| \int_{\mathbb{R}^n} e^{-ix\xi} x^\alpha F(x) dx \right| \leq \tag{2.18}$$

$$\leq \int_{\mathbb{R}^n} | x |^\alpha | F(x) | dx \leq C_1 \int_{\mathbb{R}^n} | x |^\alpha e^{-M(k|x|)} dx \leq$$

$$\leq C_2 \int_{\mathbb{R}^n} \frac{|x|^\alpha}{\langle x\rangle^{\alpha+n+1}} M_{\alpha+n+1}\Big(\frac{c}{k}\Big)^{|\alpha|+n+1} dx \leq C M_\alpha M_{n+1}\Big(\frac{Hc}{k}\Big)^{|\alpha|+n+1} .$$

In ([48]), page 88, the following estimate of the analytic function $P_{k_p}(\zeta)$ is given: For every L, there is C such that

$$| P_{k_p}(\zeta) |\leq ACe^{M(\sqrt{n}LH|\zeta|)} , \zeta \in \mathbb{C}^n .$$

Using this and the Cauchy integral formula, it is obtained that for every $L > 0$ there exist $C > 0$ such that

$$| D^\alpha P_{k_p}(\xi) |\leq C\frac{\alpha!}{d^\alpha} \cdot e^{M(Lc'|\xi|)} , \tag{2.19}$$

where $c' > 0$ is a constant that does not depend on L. It is also known that, for every $m > 0$,

$$\frac{m^k k!}{M_k} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \tag{2.20}$$

Let $m > 0$ be arbitrary. Let L be a constant such that

$$\|e^{-M(m|\xi|)} e^{M(Lc'|\xi|)}\|_{L^\infty} < \infty ,$$

and h is chosen such that $2h < 1$ and $2hHc < k$. From (2.18), (2.19), (2.20) and $(M.1)$ one obtains,

$$\sup_\alpha \frac{h^\alpha \|e^{-M(m|\xi|)} D^\alpha (P_{r_p}(\xi)\hat{F}(\xi))\|_{L^\infty}}{M_\alpha} \leq$$

$$\leq \sup_\alpha \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{(2h)^\alpha \|e^{-M(m|\xi|)} D^{\alpha-\beta} P_{r_p}(\xi) D^\beta \hat{F}(\xi)\|_{L^\infty}}{2^\alpha M_{\alpha-\beta} M_\beta} \leq$$

$$\leq C \sup_\alpha \frac{1}{2^\alpha} \|e^{-M(m|\xi|)} e^{M(Lc'|\xi|)}\|_{L^\infty} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{(\alpha-\beta)!}{M_{\alpha-\beta} d^{\alpha-\beta}} M_{n+1} (2h)^\beta \Big(\frac{Hc}{k}\Big)^{|\beta|+n+1} \leq$$

$$\leq C_1 \sup_\alpha \|e^{-M(m|\xi|)} e^{M(Lc'|\xi|)}\|_{L^\infty} \frac{1}{2^\alpha} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \Big(\frac{2hHc}{k}\Big)^{|\beta|} \leq$$

$$\leq C_2 \|e^{-M(m|\xi|)} e^{M(Lc'|\xi|)}\|_{L^\infty} \leq C_3 .$$

By proposition 2.2.1 (ii), it follows that $\hat{S} \in O_M^{\{M_p\}}$ and it is obvious that the mapping $S \longrightarrow \hat{S}$ is injective.

Now, it will be proven that the Fourier transform from $O_M^{\{M_p\}}$ to $O_C'^{\{M_p\}}$ is an injective mapping. Let $\varphi \in O_M^{\{M_p\}}$ and $\psi \in \mathcal{S}^{\{M_p\}}$. The mappings

$$\hat{\psi} \longrightarrow \psi \longrightarrow \varphi\psi \longrightarrow \mathcal{F}(\varphi\psi) = \Big(\frac{1}{2\pi}\Big)^n \hat{\varphi} * \hat{\psi}$$

are continuous from $\mathcal{S}^{\{M_p\}}$ to $\mathcal{S}^{\{M_p\}}$. Hence, $\hat{\varphi} \in O_C'^{\{M_p\}}$ and the mapping $\varphi \longrightarrow \hat{\varphi}$ is injective from $O_M^{\{M_p\}}$ into $O_C'^{\{M_p\}}$. Now it is enough to see that the same things hold for the $\bar{\mathcal{F}} = (2\pi)^n \mathcal{F}^{-1}$ and the fact that $\mathcal{F}$ is isomorphism on $\mathcal{S}^{\{M_p\}}$ and $\mathcal{S}'^{\{M_p\}}$ with an inverse $\mathcal{F}^{-1}$. Because $\mathcal{F} : \mathcal{S}^* \longrightarrow \mathcal{S}^*$ is a topological isomorphism it is obvious that it is also a topological isomorphism from $O_{M,\mathrm{s}}^*$ to $O_{C,\mathrm{s}}'^*$. □

Proposition 2.2.3. *The bilinear mappings*

$$O^*_{M,s} \times \mathcal{S}^* \to \mathcal{S}^*, \;\; (\alpha, \psi) \to \alpha\psi,$$

$$O^*_{M,s} \times \mathcal{S}'^* \to \mathcal{S}'^*, \;\; (\alpha, f) \to \alpha f,$$

are hypocontinuous.

Proof. It is obvious that the bilinear mappings are separately continuous. It will be proved that only the mapping $T : O^*_{M,s} \times \mathcal{S}^* \longrightarrow \mathcal{S}^*$, defined by $T(\varphi, \psi) = \varphi\psi$ is hypocontinuous. Since $\mathcal{S}^*$ is barreled space, from [91] Theorem 5.2., it follows that for every open set V in $\mathcal{S}^*$, and every bounded set B in $O^*_{M,s}$, then there is an open set W in $\mathcal{S}^*$ such that $T(B \times W) \subset V$. Now, let V_1 is arbitrary open set in $\mathcal{S}^*$ and let B_1 be a bounded set in $\mathcal{S}^*$. Then, for the open set W_1 in $O^*_{M,s}$, where $W_1 = \{\psi \in O^*_{M,s} \,|\, \varphi\psi \in V, \; for\, all \;\psi \in B\}$, follows $T(W_1 \times B_1) \subset V_1$. □

Proposition 2.2.4. *The space $O^*_{M,\mathrm{s}}$ is nuclear.*

Proof. Since the space $\mathcal{S}^*$ is reflexive and the space S'^* is nuclear, from [91] it follows that $L_s(\mathcal{S}^*, \mathcal{S}^*)$ is nuclear space. Thus, the space $O^*_{M,\mathrm{s}}$ is nuclear as a subspace of a nuclear space. □

Chapter 3

Structural Theorems for Ultradistribution Semigroups

Researches on Generalized semigroups started after the paper of J.L. Lions, [67] on distribution semigroups. The literature related to generalizations of C_0-semigroups is very reach especially the literature related to various classes of integrated semigroups of W. Arendt [2] and its generalizations and extensions, [5], [7], [70], [26] and [106] (see also [3], [22], [34], [43], [64], [65], [68], [74], [100], [101]). In first part of this chapter some preparatory results and results from [58] related to ultradistribution semigroups will be given in order to continue the investigation on exponential ultradistribution semigroups. Ultradifferentiable operators are used in order to clarify relations between exponentially bounded and tempered ultradistribution semigroups and convoluted semigroups.

Then, in the second part of this chapter, a structural characterizations for ultradistribution semigroups and exponential ultradistribution semigroups are given. Five conditions for ultradistribution semigroups and the corresponding five conditions for exponential ultradistribution semigroups and relations between them are given in Theorem 3.2.1.

3.1 Ultradistribution semigroups and Exponential ultradistribution semigroups

Now, ultradistribution semigroups in the framework of exponential distributions will be considered, which will defined through tempered ultradistributions.

Definition 3.1.1. Let $a \geq 0$. Then

$$\mathcal{SE}^*_a(\mathbb{R}) := \{\phi \in C^\infty(\mathbb{R}) : e^{at}\phi \in \mathcal{S}^*(\mathbb{R})\}.$$

Define the convergence in this space by

$$\phi_n \to 0 \text{ in } \mathcal{SE}^*_a(\mathbb{R}) \text{ iff } e^{a\cdot}\phi_n \to 0 \text{ in } \mathcal{S}^*(\mathbb{R}).$$

Denote by $\mathcal{SE}'^*_a(\mathbb{R}, E)$ the space $L(\mathcal{SE}^*_a(\mathbb{R}), E)$ which is formed from all continuous linear mappings from $\mathcal{SE}^*_a(\mathbb{R})$ into E equipped with the strong topology.

It holds,

$$F \in \mathcal{SE}'^{*}_{a}(\mathbb{R}, E) \text{ if and only if } \quad e^{-a\cdot}F \in \mathcal{S}'^{*}(\mathbb{R}, E). \tag{3.1}$$

In the sequel the fact that the composition and the sum of ultradifferential operators of the Beurling, resp., the Roumieu class, are ultradifferential operators of the Beurling, resp., the Roumieu class will be used. The next proposition is a structural type theorem for the space $\mathcal{SE}'^{*}_{a}(\mathbb{R}, E),\ a \geq 0$ and can be viewed of independent interest.

Proposition 3.1.1. *Let $G \in \mathcal{SE}'^{*}_{a}(\mathbb{R}, E)$. Then there exists an ultrapolynomial P of $*$-class and a function $g \in C(\mathbb{R}, E)$ with the property that there exist $k > 0$ and $C > 0$, resp., for every $k > 0$ there exists an appropriate $C_k > 0$ such that*

$$e^{-ax}||g(x)|| \leq C_k e^{M(k|x|)},\ x \in \mathbb{R} \ \textit{ and } \ G = P(d/dt)g.$$

Proof. Assertion only in the Beurling case will be proved. Since $e^{-a\cdot}G \in \mathcal{S}'^{(M_p)}(\mathbb{R}, E)$, one can use the same arguments as in [82] in order to see that there exist an ultrapolynomial P of (M_p)-class and a function $g_1 \in C(\mathbb{R}, E)$ with the property that there exist $k > 0$ and $C_k > 0$ such that

$$||g_1(x)|| \leq C_k e^{M(k|x|)} \text{ and that } G = e^{ax}P(d/dt)g_1(x).$$

Put $g(x) = e^{ax}g_1(x),\ x \in \mathbb{R}$. By Leibnitz formula, we have

$$e^{ax}P(d/dt)g_1(x) = \sum_{j=0}^{\infty}(\sum_{k=0}^{\infty}\binom{j+k}{j}(-1)^k a^k a_{k+j})(e^{ax}g_1(x))^{(j)}$$

and we will prove the assertion if we show that $b_j \leq C\frac{L^j}{M_j},\ j \in \mathbb{N}_0$, for some $C,\ L > 0$, where $b_j = \sum_{k=0}^{\infty}\binom{k+j}{j}a^k a_{k+j},\ j \in \mathbb{N}_0$.

We will use the following inequality,

$$\binom{j+k}{j} \leq 2^{k+1}k^k e^j,\ j,k \in \mathbb{N}. \tag{3.2}$$

This follows from

$$\binom{j+k}{j} \leq (j+k)^k \leq 2^k j^k + 2^k k^k \leq 2^k(k^k e^j + k^k) \leq 2^k k^k(e^j + 1),\ j,k \in \mathbb{N},$$

where we use $j^k \leq k^k e^j,\ j,k \in \mathbb{N}$. This is clear for $k \geq j$. Let us prove this for $k < j$. Put $k = \varepsilon j$ and note, if $\varepsilon \in (0,1)$, then $\varepsilon \ln \varepsilon \in (-1, 0)$ and

$$\varepsilon j \ln j \leq \varepsilon j \ln j + \varepsilon j \ln \varepsilon + j.$$

This implies $j^k \leq k^k e^j, k < j$.

Now we will estimate b_j using the estimate

$$|a_{k+j}| \leq C\frac{h^{k+j}}{M_{k+j}} \text{ for some } \ h > 0,\ C > 0$$

and that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$M_j k^k \leq C_\varepsilon \varepsilon^{k+j} M_{k+j}.$$

With this we have

$$M_j|b_j| \leq 2\sum_{k=0}^{\infty} \frac{h^{k+j} M_j 2^k k^k e^j a^k}{M_{k+j}} \leq 2C(he)^j \sum_{k=0}^{\infty} \frac{(2ha)^k M_j k^k}{M_{k+j}}, j \in \mathbb{N}$$

and choosing ε enough small, we obtain the convergence of the last series. This implies that there exist $L > 0$ and $C > 0$ such that $|b_j| \leq CL^j/M_j,\ j \in \mathbb{N}$.

This ends the proof of the proposition. □

In the sequel, we will use the phrase "G is an ultradistribution fundamental solution for A" if G is an ultradistribution fundamental solution for $P := \delta' \otimes I_{D(A)} - \delta \otimes A \in \mathcal{D}'^*_+(\mathbb{R}, L([D(A)], E))$.

Definition 3.1.2. An L-ultradistribution semigroup G of $*$-class is exponential, EL-ultradistribution semigroup of $*$-class, if in addition to $(U.1)-(U.4)$, G fulfills:

$$(U.7)\ \ (\exists a \geq 0)(e^{-a\cdot}G \in \mathcal{S}'^*(\mathbb{R}, L(E))).$$

Conditions $(U.7)$, $(U.5)$, resp., $(U.7)$, $(U.5)$ and $(U.2)$, define an exponential pre-(UDSG) resp., exponential (UDSG) and they are denoted by pre-(EUDSG), resp., (EUDSG).

Remark 3.1.1. Let G be an (EUDSG). Then we have:

1. G has an extension on $\mathcal{SE}^*_a(\mathbb{R}) : G(\phi) = \langle e^{-a\cdot}G, e^{a\cdot}\phi\rangle$.

2. Let $\psi \in \mathcal{SE}^*_a(\mathbb{R})$ and $\psi_+ := \psi \mathbf{1}_{[0,\infty)}$. Then $\psi_+ * \phi \in \mathcal{SE}^*_a(\mathbb{R})$ for every $\phi \in \mathcal{D}^*_0(\mathbb{R})$ and it it can be checked that $G(\psi) = G(\psi_+)$.

3. Let $w \in \mathcal{E}^*(\mathbb{R})$ satisfy $w(t) = 0,\ t \in (-\infty, -r)$, for some $r > 0$, and $w(t) = 1,\ t \geq 0$. Then, $we^{-\lambda\cdot} \in \mathcal{SE}^*_a(\mathbb{R}),\ Re\lambda > a$. Then the Laplace transform of G, defined by

$$\mathcal{L}(G)(\lambda) = \hat{G}(\lambda) = G(e^{-\lambda t}) := G(w(t)e^{-\lambda t}),\ Re\lambda > a,$$

   does not depend on w. Since $||G(w(t)e^{(a-\lambda)t})|| \leq C||w(t)e^{-\lambda t}||_{M_p,k}$, for some k, resp., for every k, by the usual procedure one obtains $||\hat{G}(\lambda)|| \leq C|P(\lambda)|,\ Re\lambda > a$, where P is an appropriate ultrapolynomial of $*$-class.

4. Put $E^+_\lambda(t) = e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t),\ t \in \mathbb{R},\ Re\lambda > a$. We define $G(E^+_\lambda)(x) = y$ iff $G(E^+_\lambda *_0 \phi)(x) = G(\phi)(y)$, for every $\phi \in \mathcal{SE}^*_a(\mathbb{R})$. Then $\hat{G}(\lambda) = G(E^+_\lambda)$, for every $Re\lambda > a$.

In the next assertions, we will use ultrapolynomials with the following properties:

The Beurling case: There exist constant $L > 0$ and an ultrapolynomial $\tilde{P}_L(\lambda),\ Re\lambda > 0$ of $(M_p)-$class such that for some constants C and $L_1 > 0$,

$$e^{M(L|\lambda|)} \leq |\tilde{P}_L(\lambda)| \leq Ce^{M(L_1|\lambda|)},\ Re\lambda > 0. \tag{3.3}$$

The Roumieu case: There exist a strictly decreasing sequence (L_p) tending to zero and an ultrapolynomial $\tilde{P}_{L_p}(\lambda),\ Re\lambda > 0$, of $\{M_p\}-$class such that, for every $L > 0$, there exists $C > 0$ such that

$$|\tilde{P}_{L_p}(\lambda)| \leq Ce^{M(L|\lambda|)},\ Re\lambda > 0 \ \text{ and} \tag{3.4}$$

$$e^{M(\varepsilon(|\lambda|))} \leq |\tilde{P}_{L_p}(\lambda)|,\ Re\lambda > 0,$$

for some subordinate function $\varepsilon(\rho),\ \rho \geq 0$. Let us recall that $\varepsilon(\cdot)$ is a subordinate function (cf. [48]) if it is an increasing continuous function which fulfills $\varepsilon(0) = 0$ and $\lim\limits_{\rho\to\infty} \frac{\varepsilon(\rho)}{\rho} = 0$.

Theorem 3.1.1. *Suppose that* $a \geq 0$ *and that* $f : \{\lambda \in \mathbb{C} :\ Re\lambda > a\} \to E$ *is an analytic function satisfying* $||f(\lambda)|| \leq C|P(\lambda)|,\ Re\lambda > a$, *where* $C > 0$ *and* P *is an ultradifferential operator of* $*$*-class with the property* $|P(\lambda)| > 0,\ Re\lambda > a$. *Suppose that* $\tilde{P}$ *is an ultradifferential operator of* $*$*-class with the property (3.3), resp., (3.4)* ($\tilde{P} = \tilde{P}_L$ *or* $\tilde{P} = \tilde{P}_{L_p}$). *Then*

$$(\exists M > 0)(\exists h \in C^\infty([0,\infty), E))(\forall n \in \mathbb{N}_0)(h^{(n)}(0) = 0)$$

such that $||h(t)|| \leq Me^{at},\ t \geq 0,$ *and*

$$f(\lambda) = P(\lambda)\tilde{P}(\lambda)\int_0^\infty e^{-\lambda t}h(t)dt,\ Re\lambda > a. \tag{3.5}$$

Proof. Let $\bar{a} > a$ and

$$h(t) = \frac{1}{2\pi i}\int_{\bar{a}-i\infty}^{\bar{a}+i\infty} \frac{e^{\mu t}}{P(\mu)\tilde{P}(\mu)} f(\mu)d\mu,\ t \geq 0. \tag{3.6}$$

Cauchy's theorem implies that the definition of h is independent of $\bar{a} > a$. Applying the standard arguments we conclude that $h \in C([0,\infty), E)$ fulfills $h(0) = 0$, (3.5) and that there exists $M > 0$ such that $||h(t)|| \leq Me^{at},\ t \geq 0$ and that

$$||h(t)|| \leq Me^{\bar{a}t}\int_{\bar{a}-i\infty}^{\bar{a}+i\infty} \frac{d\mu}{\tilde{P}(\mu)},\ t \geq 0.$$

Polynomial $\tilde{P}$ is introduced in order to make possible the differentiation of (3.6) under the integral sign and so we conclude that $h \in C^\infty([0,\infty), E)$. It remains to be shown that $h^{(n)}(0) = 0,\ n \in \mathbb{N}$. Denote $\gamma(r) = \{\bar{a} + re^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\}$ and fix an $n \in \mathbb{N}$. By the Cauchy formula

$$h^{(n)}(0) = \frac{1}{2\pi i}\int_{\bar{a}-i\infty}^{\bar{a}+i\infty} \frac{\mu^n f(\mu)d\mu}{P(\mu)\tilde{P}(\mu)} = \frac{1}{2\pi i}\lim_{r\to\infty}\int_{\gamma(r)} \frac{\mu^n f(\mu)}{P(\mu)\tilde{P}(\mu)}d\mu = 0.$$

This completes the proof of the theorem. □

Theorem 3.1.2. *Let A be closed and densely defined. Then A generates an EL-ultradistribution semigroup of $*$-class iff the following conditions are true:*

(i) There exists $a \geq 0$ such that $\{\lambda \in \mathbb{C} : Re\lambda > a\} \subset \rho(A)$.

(ii) There exist an ultrapolynomial P of $*$-class with the property $|P(\lambda)| > 0,\ Re\lambda > a$ and a positive constant $C > 0$ such that

$$||R(\lambda : A)|| \leq C|P(\lambda)|,\ Re\lambda > a.$$

(iii) $R(\lambda : A)$ is the Laplace transform of some G which satisfies $(U.2)'$.

Proof. ($\Leftarrow$) Let $\tilde{P}(\lambda)$ be an ultrapolynomial with the same properties as in Theorem 3.1.1. According to Theorem 3.1.1, we know that there exist a constant $M > 0$ and a function $S \in C^\infty([0,\infty), E)$ satisfying $S^{(n)}(0) = 0,\ n \in \mathbb{N}_0$, $||S(t)|| \leq Me^{at},\ t \geq 0$ and

$$R(\lambda : A) = P(\lambda)\tilde{P}(\lambda)\int_0^\infty e^{-\lambda t}S(t)dt,\ Re\lambda > a,$$

Note that $P(\lambda)\tilde{P}(\lambda) = \mathcal{L}(P(-d/dt)\tilde{P}(-d/dt))$. This implies
$R(\lambda : A) = \mathcal{L}(G)(\lambda),\ Re\lambda > a$, where

$$G = P(-d/dt)\tilde{P}(-d/dt)S \text{ and } e^{-a\cdot}G \in \mathcal{S}'^*_+(\mathbb{R}, L(E)).$$

Furthermore, it is straightforward to see that

$$(\delta' \otimes I_{D(A)} - \delta \otimes A) * G = \delta \otimes I_E,$$

$$G * (\delta' \otimes I_{D(A)} - \delta \otimes A) = \delta \otimes I_{D(A)}$$

and, due to (iii), G is an EL-ultradistribution semigroup of $*$-class.

($\Rightarrow$) Suppose $e^{-\tilde{a}\cdot}G \in \mathcal{S}'^*_+(\mathbb{R}, L(E, [D(A)]))$, for some $\tilde{a} \geq 0$. Let $a \in (\tilde{a}, \infty)$ and $\lambda \in \{z \in \mathbb{C} : Rez > a\}$ be fixed. Then we have $(\delta' + \lambda\delta) * E^+_\lambda = \delta$. Suppose $\phi \in \mathcal{S}^*(\mathbb{R})$ and $x \in E$. Then

$$G((\delta' + \lambda\delta) *_0 E^+_\lambda *_0 \phi)x = G(\phi)x,$$

$$G(\delta' *_0 E^+_\lambda *_0 \phi)x + \lambda G(E^+_\lambda *_0 \phi)x = G(\delta')G(E^+_\lambda *_0 \phi)x + \lambda\hat{G}(\lambda)G(\phi)x.$$

It follows

$$-A(\hat{G}(\lambda)G(\phi)x) + \lambda\hat{G}(\lambda)G(\phi)x = G(\phi)x.$$

Since $(U.3)$ is assumed, one can prove by the usual procedure that $(-A+\lambda)\hat{G}(\lambda) = I$. As we have already noted, one obtains $||\hat{G}(\lambda)|| \leq C|P(\lambda)|,\ Re\lambda > a$, where P is an appropriate ultrapolynomial of $*$-class. This finishes the proof. □

Remark 3.1.2. Since $(M.2)$ holds, condition (ii) in the formulation of Theorem 3.1.2 is equivalent to say that there exist $k > 0$ and $C > 0$ in the Beurling case, resp., for every $k > 0$ there exists a corresponding $C_k > 0$ in the Roumieu case, so that $||R(\lambda : A)|| \leq Ce^{M(k|\lambda|)},\ Re\lambda > a$, resp., $||R(\lambda : A)|| \leq C_k e^{M(k|\lambda|)},\ Re\lambda > a$.

Concerning the proof of Theorem 3.1.2, we have the next corollary.

Corollary 3.1.1. *Suppose A is a closed linear operator. If A fulfills (i) and (ii) of Theorem 3.1.2, then there exists an exponential ultradistribution fundamental solution for A. If, additionally, (iii) of Theorem 3.1.2 holds, then A generates an (EUDSG).*

In the sequel (M_p) satisfies $(M.1)$, $(M.2)$ and $(M.3)$. The purpose of $(M.3)$ is again the use of [50, Theorem 4.8].

The following assertion is well known in the theory of ultradistributions (cf. [30], [48] and [53, Theorem 4.7]). Note, we use d/dx instead of $-id/dx$ which is used in [48] and [53].

Let $T \in \mathcal{D}'^{*}_{+}(\mathbb{R}, E)$. *Then for every* $a > 0$ *there exist an ultradifferential operator of* (M_p)*-class, formally of the form*

$$P_L(d/dt) = \prod_{p=1}^{\infty}(1 + \frac{L^2}{m_p^2} d^2/dt^2) = \sum_{p=0}^{\infty} a_p d^p/dt^p, \tag{3.7}$$

where $L > 0$ *is some constant, resp., of* $\{M_p\}$*-class, formally of the form*

$$P_{L_p}(d/dt) = \prod_{p=1}^{\infty}(1 + \frac{L_p^2}{m_p^2} d^2/dt^2) = \sum_{p=0}^{\infty} a_p d^p/dt^p, \tag{3.8}$$

where $(L_p)_p$ *is a sequence decreasing to* 0, *and a continuous function* $f : (-a, a) \to E$ *such that* $T = P_L(-id/dt)f$, *on* $\mathcal{D}^{(M_p)}((-a, a))$, *in* (M_p)*-case, resp.,* $T = P_{L_p}(-id/dt)f$, *on* $\mathcal{D}^{\{M_p\}}((-a, a))$, *in* $\{M_p\}$*-case.*

Note that in the above representation theorem, we do not have that f is supported by $[0, a)$.

Remark 3.1.3. The previous assertions remain true for ultradifferential operators

$$\tilde{P}_L(d/dt) = \prod_{p=1}^{\infty}(1 + \frac{L}{m_p} d/dt) = \sum_{p=0}^{\infty} a_p d^p/dt^p, \ L > 0, \tag{3.9}$$

$$\tilde{P}_{L_p}(d/dt) = \prod_{p=1}^{\infty}(1 + \frac{L_p}{m_p} d/dt) = \sum_{p=0}^{\infty} a_p d^p/dt^p, \tag{3.10}$$

where (L_p) is a sequence decreasing to 0 (see [48, Theorem 10.3]). Define also $\tilde{P}_L(\zeta) = \prod_{p=1}^{\infty}(1 + \frac{L\zeta}{m_p}), \ Re\zeta > 0$ and, analogously, $\tilde{P}_{L_p}(\zeta), \ Re\zeta > 0$.

Example 3.1.1. [58]

Examples which are to follow are based on the estimates of ultrapolynomials given in [48, Propositions 4.5, 4.6-Theorem 10.1, 10.2]. Let $E = E_{M_p} := \{f \in \mathcal{C}^{\infty}[0, 1] : \|f\|_{M_p} := \sup\limits_{p \geq 0} \frac{\|f^{(p)}\|_{\infty}}{M_p} < \infty\}$ and $A = A_{M_p} := -d/ds, \ D(A_{M_p}) := \{f \in E_{M_p} : f' \in E_{M_p}, f(0) = 0\}$. Then

$$||R(\lambda : A)|| \leq e^{M(r|\lambda|)}, \ Re\lambda > 0, \ \text{for some } r > 0.$$

Note that A generates an exponentially bounded K-convoluted semigroup $(S(t))_{t\geq 0}$ (cf. [57]).

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If (M_p) fulfills $(M.1)$, $(M.2)$ and $(M.3)$, one obtains the following: there exist $C,\ C_1>0,\ L_1>0$ such that

$$e^{M(L|\zeta|)} \leq |\tilde{P}_L(\zeta)| \leq Ce^{M(L_1|\zeta|)},\ Re\zeta>0, \quad |a_p| \leq C_1 L_1^p/M_p,\ p\in\mathbb{N}_0.$$

In the case of $M_p=p!^s$, with suitable constants,

$$e^{cL|\zeta|^{1/s}} \leq |\tilde{P}_{L,s}(\zeta)| \leq Ce^{L_1|\zeta|^{1/s}},\ Re\zeta>0,$$

holds and if $1<s'<s$, with another constants,

$$e^{\tilde{c}L|\zeta|^{1/s'}} \leq |\tilde{P}_{L,s'}(\zeta)| \leq \tilde{C}e^{\tilde{L}_1|\zeta|^{1/s'}},\ Re\zeta>0, \quad \text{holds.}$$

Let $s=1/\gamma>s'=1/\gamma',\ \gamma<1$ and $u\in E_{p!^{1/\gamma}}$. Note that

$$\mathcal{L}^{-1}(\frac{e^{-t\zeta}\mathcal{L}(u\mathbf{1}_{[0,1]})(\zeta)}{\tilde{P}_{L,s'}(\zeta)})(x)=\mathcal{L}^{-1}(\frac{\mathcal{L}(u\mathbf{1}_{[0,1]})(\zeta)}{\tilde{P}_{L,s'}(\zeta)})(x-t),\ x,\ t\geq 0.$$

This and the estimates of $P_{L,s'}$ imply that

$$[0,\infty)\ni t\mapsto \mathcal{L}^{-1}(\frac{e^{-t\zeta}\mathcal{L}(u\mathbf{1}_{[0,1]})(\zeta)}{\tilde{P}_{L,s'}(\zeta)})_{|[0,1]}\in E_{p!^{1/\gamma}}$$

is a continuous mapping.

Put $\tilde{K}(\zeta)=\tilde{P}^{-1}_{L,s'}(\zeta),\ Re\zeta>0$. We obtain the structural theorem for the K-convoluted semigroup generated by A on $E_{p!^{1/\gamma}}$ as follows. Define

$$S(t)u(x)=\mathbf{1}_{[0,1]}(x)\int_0^t \mathcal{L}^{-1}(\frac{e^{\tau\zeta}\mathcal{L}(u\mathbf{1}_{[0,1]})(\zeta)}{\tilde{P}_{L,s'}(\zeta)})(x)d\tau,$$

$$x\in[0,1],\ u\in E_{p!^{1/\gamma}},\ t\geq 0\ .$$

One can prove that $S(t)\in L(E_{p!^{1/\gamma}}),\ t\geq 0$ and that $S(0)=0$. Then, with

$$\Big(G(\phi)u\Big)(x)=\mathbf{1}_{[0,1]}(x)\langle \tilde{P}_{L,s'}(-\frac{d}{dt})\frac{d}{dt}S(t)u(x),\phi(t)\rangle$$

$$=\langle (u\mathbf{1}_{[0,1]})(x-t),\phi(t)\rangle_{|[0,1]},$$

$$x\in[0,1],\ u\in E_{p!^{1/\gamma}},\ \phi\in\mathcal{D}^{(p!^{s'})}(\mathbb{R}),$$

is given an exponentially bounded, non-dense ultradistribution semigroup of Beurling class on $\mathcal{D}^{(p!^{s'})}(\mathbb{R})$ generated by A; furthermore, G verifies $(U.2)$ and $H=G_{|\mathcal{R}(G)}$ is a regular ultradistribution semigroup of (M_p)-class.

Note, the polynomial boundedness of $||R(\cdot:A)||$ is crucial in [9]-[10] and [20, Proposition 2.6]. In this example, the polynomial boundedness of $||R(\cdot:A)||$ is not satisfied (cf. [61]).

Roumieu case.

Let $(L_p)_p$ be a sequence which strictly decrease to zero. Then for every $L > 0$ there exists $C > 0$ such that

$$|\tilde{P}_{L_p}(\lambda)| \leq Ce^{M(L|\lambda|)},\ Re\lambda > 0,$$

and that $|a_p| \leq CL^p/M_p,\ p \in \mathbb{N}_0$. Moreover, there exists a subordinate function $\varepsilon(\rho),\ \rho \geq 0$ such that

$$e^{M(\varepsilon(|\lambda|))} \leq |\tilde{P}_{L_p}(\lambda)|,\ Re\lambda > 0.$$

Let E and A be as above; then A generates an exponentially bounded, K-convoluted semigroup $(S(t))_{t\geq 0}$, where $\tilde{K}(\lambda) = \tilde{P}^{-1}_{L_p,s'}(\lambda),\ Re\lambda > 0$, for $s > s'$.

Then with

$$G(\phi)u(x) = \mathbf{1}_{[0,1]}(x)\langle \tilde{P}_{L_p,s'}(-\frac{d}{dt})\frac{d}{dt}S(t)u(x), \phi(t)\rangle$$

$$= \langle (u\mathbf{1}_{[0,1]})(x-t), \phi(t)\rangle_{|[0,1]},$$

$$x \in [0,1],\ u \in E_{p!^{1/\gamma}},\ \phi \in \mathcal{D}^{\{p!^{s'}\}}(\mathbb{R}),$$

is defined a non-dense ultradistribution semigroup of exponential growth, (EUDSG), of Roumieu class on $\mathcal{D}'^{\{p!^{s'}\}}(\mathbb{R})$ generated by A. As before, we obtain

$$S(t)u(x) = \mathbf{1}_{[0,1]}(x)\int_0^t \mathcal{L}^{-1}(\frac{e^{-\tau\xi}\mathcal{L}(u\mathbf{1}_{[0,1]})(\xi)}{\tilde{P}_{L_p,s'}(\xi)})(x)d\tau,$$

$$x \in [0,1],\ u \in E_{p!^{1/\gamma}},\ t \geq 0.$$

3.2 Structural theorems for ultradistribution semigroups and exponential ultradistribution semigroups

The use of ultradistribution semigroups is the main tool in the analysis of some classes of pseudo-differential evolution systems with constant coefficients given in [6]. We also refer to [75] for more examples of differential operators generating ultradistribution semigroups.

Following the investigation of H. Komatsu [53], in the framework of Denjoy-Carleman-Komatsu theory of ultradistributions and P. C. Kunstmann [63], in the theory of ω-ultradistributions, we define the next regions:

$\Omega^{(M_p)} := \{\lambda \in \mathbb{C} : Re\lambda \geq M(k|\lambda|) + C\}$, for some $k > 0,\ C > 0$, resp.,

$\Omega^{\{M_p\}} := \{\lambda \in \mathbb{C} : Re\lambda \geq M(k|\lambda|)+C_k\}$, for every $k > 0$ and a corresponding $C_k > 0$. By Ω^* is denoted either $\Omega^{(M_p)}$ or $\Omega^{\{M_p\}}$.

We need the following estimations of ultrapolynomials:

Lemma 3.2.1. *(a) Let P_L be of the form* (3.7). *Then there exist $C,\ C_1 > 0,\ L_1,\ L_2 > 0$ such that*

$$e^{2M(L|\zeta|)} \leq |P_L(\zeta)| \leq Ce^{M(L_1|\zeta|)}, \quad \textit{if} \quad |Im\zeta| < \frac{|Re\zeta|}{2} + \frac{1}{L}$$

and $|a_p| \leq C_1 L_2^p / M_p,\ p \in \mathbb{N}_0$.

(b) Let $(L_p)_p$ be a sequence which strictly decreases to zero and P_{L_p} be defined by (3.8). *Then there exists $C > 0$ such that, for every $k > 0$, there exists $C_k > 0$, such that*

$$|P_{L_p}(\zeta)| \leq C_k e^{M(k|\zeta|)}, \quad \textit{if} \quad |Im\zeta| < \frac{|Re\zeta|}{2} + \frac{1}{L_1},$$

and (with another C_k, for given $k > 0$) $|a_p| \leq C_k k^p / M_p,\ p \in \mathbb{N}_0$. Moreover, there exists a subordinate function $\varepsilon(\rho),\ \rho \geq 0$, such that

$$e^{2M(\varepsilon(|\zeta|))} \leq |P_{L_p}(\zeta)|, \quad \textit{if} \quad |Im\zeta| < \frac{|Re\zeta|}{2} + \frac{1}{L}.$$

Proof. We will prove only the part in Beurling case

$$e^{2M(L|\zeta|)} \leq |P_L(\zeta)|, \quad \text{if} \quad |Im\zeta| < \frac{|Re\zeta|}{2} + \frac{1}{L}.$$

Note that for any $c > 0$, the inequality $x^2 - y^2 \geq 0$ ($\zeta = x + iy$) implies $|1 + c\zeta^2| \geq c|\zeta|^2$. Also, $|1 + c\zeta^2| > c|\zeta|^2$, for all sufficiently small $|\zeta|$. Thus, by the simple calculation we have that

$$|1 + \frac{L^2\zeta^2}{m_p^2}| \geq \frac{L^2}{m_p^2}|\zeta|^2, \quad \text{if} \quad |Im\zeta| < \frac{|Re\zeta|}{2} + \frac{1}{L}.$$

This implies

$$|P_L(\zeta)| = |\prod_{p=1}^{\infty}(1 + \frac{L^2}{m_p^2}\zeta^2)| \geq \prod_{p=1}^{\infty}(\frac{L^2}{m_p^2}|\zeta|^2) \geq e^{2M(|\zeta|)}, \quad \text{if} \quad |Im\zeta| < \frac{|Re\zeta|}{2} + \frac{1}{L}.$$

□

Remark 3.2.1. In Theorem 3.2.1 which is to follow, in the case of tempered ultradistribution semigroups (and similarly in the case of exponentially bounded ultradistribution semigroups), we use Theorems 3.1.1, 3.1.2 and Corollary 3.1.1, where the inverse Laplace transform is performed on the straight line connecting points $\bar{a} - i\infty$ and $\bar{a} + i\infty$, where $\bar{a} > 0$. With a suitable choice of L, resp., $(L_p)_p$, we have that this line lies in the domain $|Im(i\zeta)| < \frac{|Re(i\zeta)|}{2} + \frac{1}{L}$, resp., $|Im(i\zeta)| < \frac{|Re(i\zeta)|}{2} + \frac{1}{L_1}$, where we have the quoted estimates for $P_L(-i\lambda)$, resp., $P_{L_p}(-i\lambda)$. Let us explain this in the Beurling case with more details. Choose any $L \in (0, \frac{1}{\bar{a}})$ and put $K(t) = \frac{1}{2\pi i} \int\limits_{\bar{a}-i\infty}^{\bar{a}+i\infty} \frac{e^{\lambda t}}{P_L(-i\lambda)} d\lambda,\ t \geq 0$. Then K is an exponentially bounded, continuous function defined on $[0, \infty)$ and we shall simply write $K = \mathcal{L}^{-1}(\frac{1}{P_L(-i\lambda)})$.

Now, we give a structural characterizations for ultradistribution semigroups and exponential ultradistribution semigroups. Some of these characterizations are proved in [53], [55], [57], [63], and [70] in another contexts. We will indicate this in Theorem 3.2.1. For the need of the proof of Theorem 3.2.1 we give following Lemma:

Lemma 3.2.2. *Let $P_L(d/dt)$ and $P_{L_p}(d/dt)$ be of the form* (3.7) *and* (3.8)*, respectively. The mappings*

$$P_L(id/dt) : \mathcal{S}^{(M_p)}(\mathbb{R}) \to \mathcal{S}^{(M_p)}(\mathbb{R}), \ \ \phi \mapsto P_L(id/dt)\phi,$$

$$P_{L_p}(id/dt) : \mathcal{S}^{\{M_p\}}(\mathbb{R}) \to \mathcal{S}^{\{M_p\}}(\mathbb{R}), \ \ \phi \mapsto P_{L_p}(id/dt)\phi,$$

are continuous linear bijections.

Proof. We will prove the lemma in the Beurling case. Let $\phi \in \mathcal{S}^{(M_p)}(\mathbb{R})$. Then

$$\mathcal{F}(P_L(id/dt)\phi)(\xi) = P_L(-\xi)\hat{\phi}(\xi) = P_L(\xi)\hat{\phi}(\xi), \ \xi \in \mathbb{R}.$$

One can prove by standard arguments that $P_L(\xi)\hat{\phi} \in \mathcal{S}^{(M_p)}(\mathbb{R})$. We have to prove that $\hat{\phi}/P_L(\xi) \in \mathcal{S}^{(M_p)}(\mathbb{R})$.

Notice that there exists $r > 0$ such that, for every $\xi \in \mathbb{R}$, the circle $k_\xi(r)$, with the center ξ and the radius r, is contained in the domain $|Im\zeta| < 1/C$ where the estimates of Lemma 3.2.1 are satisfied. By Cauchy's formula, with suitable constants, it follows

$$|(P_L^{-1})^{(n)}(\xi)| \leq C\frac{n!}{r^n}\sup\{|P_L^{-1}(\xi + re^{i\theta})| : \theta \in [0, 2\pi]\} \leq$$

$$\leq C\frac{n!}{r^n}e^{M(L(|\xi|+r))} \leq C_1\frac{n!}{r^n}e^{M((L+1)|\xi|)}, \ \xi \in \mathbb{R}, \ n \in \mathbb{N}_0.$$

Now it is easy to prove that for every $h > 0$,

$$\sup\{\frac{h^n|(\hat{\phi}/P_L)^{(n)}(\xi)|e^{M(h|\xi|)}}{M_n} : \xi \in \mathbb{R}, \ n \in \mathbb{N}_0\} < \infty$$

which is equivalent with $\hat{\phi}/P_L \in \mathcal{S}^{(M_p)}(\mathbb{R})$. □

For the needs of the next theorem, we list the following statements:

(a) A generates a (UDSG) of $*$-class G.

(a)' A generates a (EUDSG) of $*$-class G.

(b) A generates a (UDSG) of $*$-class G such that, for every $a > 0$, G is of the form $G = P_L^a(-id/dt)S_K^a$ on $\mathcal{D}^{(M_p)}((-\infty, a))$ in (M_p)-case, (resp., $G = P_{L_p}^a(-id/dt)S_K^a$ on $\mathcal{D}^{\{M_p\}}((-\infty, a))$ in $\{M_p\}$-case), where $S_K^a : (-\infty, a) \to L(E, [D(A)])$ is continuous, $S_K^a(t) = 0, \ t \leq 0$.

(b)' A generates a (EUDSG) of $*$-class G so that G is of the form $G = P_L(-id/dt)S_K$ on $\mathcal{SE}_a^{(M_p)}(\mathbb{R})$ in (M_p)-case, (resp., $G = P_{L_p}(-id/dt)S_K$ in $\{M_p\}$-case), where $S_K : \mathbb{R} \to L(E, [D(A)])$ is continuous, $S_K(t) = 0,\ t \leq 0$ and $e^{-at}||S_k(t)|| \leq Ae^{M(k|t|)}$, for some $k > 0$ and $A > 0$, resp., for every $k > 0$ and corresponding $A > 0$, $t \in \mathbb{R}$.

(c) For every $a > 0$, A is the generator of a local non-degenerate K_a-convoluted semigroup $(S^a_{K_a}(t))_{t\in[0,a)}$, where $K_a = \mathcal{L}^{-1}(\frac{1}{P^a_L(-i\lambda)})$ in (M_p)-case, resp., $K_a = \mathcal{L}^{-1}(\frac{1}{P^a_{L_p}(-i\lambda)})$ in $\{M_p\}$-case and P^a_L, resp., $P^a_{L_p}$, is an ultradifferential operator of $*$-class such that for $0 < a < b$ the restriction of $P^b_L S^b_K$, resp., $P^b_{L_p} S^b_K$, on $\mathcal{D}^*((-\infty, a))$ is equal to $P^a_L S^a_K$, resp., $P^a_{L_p} S^a_K$.

(c)' A is the generator of a global, exponentially bounded non-degenerate K-convoluted semigroup $(S_K(t))_{t\geq 0}$, where $K = \mathcal{L}^{-1}(\frac{1}{P_L(-i\lambda)})$ in (M_p)-case, resp., $K = \mathcal{L}^{-1}(\frac{1}{P_{L_p}(-i\lambda)})$ in $\{M_p\}$-case.

(d) There exists an ultradistribution fundamental solution of $*$-class for A, denoted by G, with the property $\mathcal{N}(G) = \{0\}$.

(d)' There exists an exponential ultradistribution fundamental solution of $*$-class G for A, with the property $\mathcal{N}(G) = \{0\}$.

(e) $\rho(A) \supset \Omega^*$ and

$$||R(\lambda : A)|| \leq Ce^{M(k|\lambda|)},\ \lambda \in \Omega^{(M_p)},$$

for some $k > 0$ and $C > 0$ in (M_p)-case, resp.,

$$||R(\lambda : A)|| \leq C_k e^{M(k|\lambda|)},\ \lambda \in \Omega^{\{M_p\}},$$

for every $k > 0$ and a corresponding $C_k > 0$ in $\{M_p\}$-case.

(e)' $\rho(A) \supset \{\lambda \in \mathbb{C} : Re\lambda > a\}$ and

$$||R(\lambda : A)|| \leq Ce^{M(k|\lambda|)},\ Re\lambda > a,$$

for some $a, k > 0$ and $C > 0$ in (M_p)-case, resp.,

$$||R(\lambda : A)|| \leq C_k e^{M(k|\lambda|)},\ Re\lambda > a,$$

for every $k > 0$ and a corresponding $a, C_k > 0$ in $\{M_p\}$-case.

Note that in statements (a)', (b)', (c)', (d)' and (e)' we consider tempered ultradistribution semigroups, i.e., exponential ultradistribution semigroups (EUDSG) with $a = 0$. In this way, we simplify the exposition.

Theorem 3.2.1. *(a) $\Leftrightarrow$ (d); (a)' $\Leftrightarrow$ (d)'; (c) $\Rightarrow$ (d); (c)' $\Rightarrow$ (d)'; (d) $\Rightarrow$ (e); (d)' $\Rightarrow$ (e)'; if (M_p) additionally satisfies $(M.3)$, then $(a)' \Rightarrow$ (c)'.*

Proof. (a) $\Leftrightarrow$ (d): This equivalence is proved in [57], when $\mathcal{N}(G) \neq \{0\}$. The statement (a) $\Rightarrow$ (d) is direct consequence of [58, Theorem 2 (c)]. We give here the sketch of the proof of the opposite direction. Let $G \in \mathcal{D}_+^{*'}(\mathbb{R}, L(E, [D(A)]))$ be an ultradistributional fundamental solution of $*$-class for A. By the direct calculation we have that A is closable operator.

Let $\tilde{A}$ generates G. If (x, y) belongs to the closure of A, then there exists a sequence $(x_n, y_n)_n$ in A such that $(x_n, y_n) \to (x, y)$, when $n \to \infty$, in $E \times E$. Let $\phi \in \mathcal{D}_0^*(\mathbb{R})$ be fixed. For $\varphi \in \mathcal{D}_0^*(\mathbb{R})$ we have

$$\|G(\varphi)(G(-\phi')x - G(\phi)y)\| =$$

$$= \|G(\varphi)[G(\phi')(x_n - x) - G(\phi')x_n + G(\phi)(y_n - y) - G(\phi)y_n]\| =$$

$$= \|G(\varphi)[G(\phi')(x_n - x) + G(\phi)(y_n - y)]\| \leq (\|G(\varphi *_0 \phi')\| + \|G(\varphi *_0 \phi\|)/k\,,$$

for $k \in \mathbb{N}$. So it follows $G(-\phi')x = G(\phi)y$ for all $\phi \in \mathcal{D}_0^*(\mathbb{R})$. Since G is a ultradistribution fundamental solution of $*$-class for A we have $A \subset \tilde{A}$. It implies that $\mathcal{D}_+'^*(\mathbb{R}, [\overline{D(A)}])$ is an isomorphic to a subspace of $\mathcal{D}_+'^*(\mathbb{R}, [D(\tilde{A})])$. From the first part of the theorem we have that G is a fundamental ultradistribution solution for $P := \delta' \otimes Id_{D[\tilde{A}]} - \delta \otimes \tilde{A}$. So G^* is an isomorphism from $\mathcal{D}_+'^*(\mathbb{R}, E)$ onto $\mathcal{D}_+'^*(\mathbb{R}, [D(A)])$ and onto $\mathcal{D}_+'^*(\mathbb{R}, [D(\tilde{A})])$ which implies that $\mathcal{D}_+'^*(\mathbb{R}, [D(A)]) = \mathcal{D}_+'^*(\mathbb{R}, [D(\tilde{A})])$, so $[D(A)] = [D(\tilde{A})]$.

The statement (a)' $\Leftrightarrow$ (d)' can be proved similarly using that G can be extended continuously on $\mathcal{ES}^*(\mathbb{R})$, [58].

(d) $\Rightarrow$ (e)[70]: We will consider only Beurling case. Proof in Roumieu case is similar. Let G be a ultradistribution solution of class (M_p) for A. For $0 < r < r'$ we construct a function $k(t) \in \mathcal{D}^{(M_p)}(\mathbb{R})$ such that $k(t) = 0$ for $t < -1$, $k(t) = 1$ for $0 \leq t \leq r$ and $k(t) = 0$ for $t > r'$. For $\lambda \in \mathbb{C}$ we define $k_\lambda := e^{-\lambda t}k(t)$ and $\tilde{G}(\lambda) := G(k_\lambda)$. Since G is an ultradistribution fundamental solution of class (M_p) for A it holds that $\tilde{G}'(\lambda) - A\tilde{G}(\lambda) = I_E k_\lambda(0) = I_E$. From linearity of $G(\varphi)$ we have

$$\tilde{G}'(\lambda) = -G([k_\lambda]') = -G(-\lambda e^{-\lambda t}k(t) + e^{-\lambda t}k'(t)) = \lambda\tilde{G}(\lambda) - G(e^{-\lambda t}k'(t))\,.$$

So we get that

$$(\lambda I - A)\tilde{G}(\lambda) = I_E + G(e^{-\lambda t}k'(t))\,.$$

In order to show existence of a resolvent, we estimate function $e^{-\lambda t}\varphi(t)$ for any $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R})$ and then we estimate $G(e^{-\lambda t}k(t))$ and $G(e^{-\lambda t}k'(t))$. Let $\mathrm{supp}\varphi \cap \mathbb{R}_+ \subset K = [a, b]$. Hence,

$$[e^{-\lambda t}\varphi(t))]^{(n)} = e^{-\lambda t}\sum_{j=0}^{n}\binom{n}{j}(-\lambda)^{n-j}\varphi^{(j)}(t)\,.$$

Using $(M.1)$ we obtain:

$$\sup_{t\in K}|[e^{-\lambda t}\varphi(t))]^{(n)}| \leq e^{-aRe\lambda}\sum_{j=0}^{n}\sup_{t\in K}\binom{n}{j}\frac{|\varphi^{(j)}(t)|}{M_j L^j}\cdot\frac{|\lambda|^{n-j}}{L^{n-j}}M_j L^n \leq$$

$$\leq \|\varphi\|_{M_p,L,K} e^{-aRe\lambda} \sum_{j=0}^{n} \sup_{n-j} \frac{|\lambda|^{n-j}}{L^{n-j} M_{n-j}} L^n \binom{n}{j} M_j M_{n-j} \leq$$

$$\leq \|\varphi\|_{M_p,L,K} e^{M(\frac{\lambda}{L})-aRe\lambda} L^n \sum_{j=0}^{n} \binom{n}{j} M_j M_{n-j} \leq$$

$$\leq \|\varphi\|_{M_p,L,K} e^{M(\frac{\lambda}{L})-aRe\lambda} L^n 2^n M_n ,$$

for every $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R})$. Now, we give estimate for $G(e^{-\lambda t}k(t))$. Using the structure theorem [48, Theorem 8.1] for $G \in {\mathcal{D}_+}^{(M_p)'}(\mathbb{R}, L(E, [D(A)]))$ and a relatively compact set $K_1 \subset [0, \infty)$ there exist $f_n \in \mathcal{C}'\{\bar{K}_1, L(E, [D(A)])\}$ and constants $s, C' > 0$ such that:

$$G \mid_{K_1} = \sum_{n=0}^{\infty} f_n^{(n)} ,$$

where

$$\|f_n\|_{\{\mathcal{C}', L(E,[D(A)])\}} \leq C' \frac{s^p}{M_p}, \quad \text{and}\, K = \bar{K}_1 .$$

Now,

$$\|G(e^{-\lambda t}k(t))\|_{L(E,[D(A)])} = \| \sum_{n=0}^{\infty} \langle f_n^{(n)}, e^{-\lambda t}k(t) \rangle \|_{L(E,[D(A)])} \leq$$

$$\leq \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{C}'\{,L(E,[D(A)])\}} \|[e^{-\lambda t}k(t)]^{(n)}\|_{\mathcal{C}(K)} \leq C' \sum_{n=0}^{\infty} \frac{s^n \|[e^{-\lambda t}k(t)]^{(n)}\|_{\mathcal{C}(K)} L^n}{M_n L^n} \leq$$

$$\leq 2C' \|e^{-\lambda t}k(t)\|_{M_p,L,K} \sum_{n=0}^{\infty} L^n s^n .$$

Because $K = [0, r]$ for $k(t)$ and $K = [r, r']$ for $k'(t)$, we have following estimates for $L = \frac{1}{2s}$:

$$\|G(e^{-\lambda t}k'(t))\| \leq 2C' e^{M(2s\lambda)-rRe\lambda} \|k'\|_{M_p,\frac{1}{4s},K} ,$$

$$\|G(e^{-\lambda t}k(t))\| \leq 2C e^{M(2s\lambda)} \|k\|_{M_p,\frac{1}{4s},K} .$$

We put $2C' e^{M(2s\lambda)-rRe\lambda} = \delta < 1$ and we obtain that the operator $\lambda I - A$ has inverse on

$$\Omega^{(M_p)} = \{\lambda \in \mathbb{C} \,:\, Re\lambda \geq \frac{1}{c}[M(2s\lambda) - \ln(\frac{\delta}{2C'})] := M(k(\lambda)) + C\} .$$

The estimate of the resolvent in $\Omega^{(M_p)}$ is

$$\|R(\lambda : A)\| \leq \frac{2C'}{1-\delta} e^{M(k|\lambda|)} \|k\|_{M_p,\frac{1}{2s},K} .$$

Finally, for some constants k and C holds

$$\|R(\lambda : A)\| \leq C e^{M(k|\lambda|)} \quad \lambda \in \Omega^{(M_p)} .$$

(d)' $\Rightarrow$ (e)'[55] : We will give a proof for Beurling case. The Roumeiu case is quite similar. Let G be a exponential fundamental ultradistribution solution of (M_p)-class for A, i.e G is a fundamental ultradistribution solution and $G \in \mathcal{SE}'^{(M_p)}_{\omega}(\mathbb{R}, L(E))$ for $\omega \geq 0$. Let $s > 0$. We define a function $g \in \mathcal{E}^{(M_p)}(\mathbb{R})$ such that $g(t) = 0$ for $t < -s$ and $g(t) = 1$ for $t \geq 0$. The definition of $\tilde{G}(\lambda) := G(g(t)e^{-\lambda t}) := G(e^{-\omega t}(g(t)e^{(\omega-\lambda)t}))$ have meaning since the function $t \mapsto g(t)e^{(\omega-\lambda)t}$, when $t \in \mathbb{R}$ and for all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \omega$, is in $\mathcal{S}^{(M_p)}(\mathbb{R})$. Because G is a fundamental ultradistribution solution for $A - \omega I$, for $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R})$, $x \in E$ we have that,

$$(A - \omega I)G(e^{-\omega t}\varphi)x = G(-e^{-\omega t}\varphi')x - \varphi(0)x\,.$$

Using that $\mathcal{D}^{(M_p)}(\mathbb{R})$ is dense in $\mathcal{S}^{(M_p)}(\mathbb{R})$, we get that the previous equation holds for all $\mathcal{S}^{(M_p)}(\mathbb{R})$. Let we put $\varphi(t) = g(t)e^{(\omega-\lambda)t} \in \mathcal{S}^{(M_p)}(\mathbb{R})$. Then $\mathrm{supp}G \subseteq [0, \infty)$ and we obtain:

$$A\tilde{G}(\lambda)x = AG(e^{-\lambda t}\varphi)x = \lambda\tilde{G}(\lambda)x - \varphi(0)x, \quad \mathrm{Re}\lambda > \omega\,.$$

From this equation, $(\lambda I - A)\tilde{G}(\lambda)x = x$, $x \in E$, $\mathrm{Re}\lambda > \omega$. $\tilde{G}(\lambda)A \subseteq A\tilde{G}(\lambda)$ holds for $\mathrm{Re}\lambda > \omega$ we have $\tilde{G}(\lambda)(\lambda I - A)x = x$, for $x \in D(A)$ and $\mathrm{Re}\lambda > \omega$. We put $\omega = a$ so we have proved the first part of the statement. From the discussion above, it is clear that $R(\lambda : A)x = \tilde{G}(\lambda)x$, for $x \in E$, $\mathrm{Re}\lambda > a$. Using $(M.1)$ we obtain that

$$\|R(\lambda : A)\| = \|\tilde{G}(\lambda)\| = \|G(e^{-\omega t}(g(t)e^{(\omega-\lambda)t}))\| \leq$$

$$\leq C'' \sup_{t\in K} \frac{(g(t)e^{(\omega-\lambda)t})^{(p)}}{M_p h^p} \leq C'' \sup_{t\in K} \sum_{j=0}^{p} \binom{p}{j} \frac{g^{(p-j)}(t) \cdot (e^{(\omega-\lambda)t})^{(j)}}{M_p h^p} \leq$$

$$\leq C'' \sup_{t\in K} \sum_{j=0}^{p} \binom{p}{j} \frac{g^{(p-j)}(t)}{M_{p-j}h^{p-j}} \cdot \frac{(\omega-\lambda)^j e^{(\omega-\lambda)t}}{M_j h^j} \leq$$

$$\leq C' \sup_{t\in K} \sum_{j=0}^{p} \binom{p}{j} \frac{(\omega-\lambda)^j e^{(\omega-\lambda)t}}{M_j h^j} \leq Ce^{M(k|\lambda|)}\,.$$

(a)' $\Rightarrow$ (c)':

We will prove this assertion in the Beurling case by the use of already mentioned structural theorem for elements of $\mathcal{SE}'^{(M_p)}_{a}(L(E))$:

$$G(\phi) = \langle \phi, P_L(-id/dt)S(t)\rangle, \ \phi \in \mathcal{S}^{(M_p)}(\mathbb{R}),$$

where, for an appropriate $k > 0$,

$$e^{-at}||S(t)|| \leq e^{M(k|t|)}, \ t \in \mathbb{R}.$$

Fix an $x \in E$. By Theorem [58, Theorem 2 (c)],

$$AG(\phi)x = -\langle \phi', P_L(-id/dt)S(t)x\rangle - \phi(0)x, \ \text{for all } \phi \in \mathcal{S}^{(M_p)}(\mathbb{R}).$$

Since $1 = P_L(-id/dt)\mathcal{L}^{-1}(1/P_L(-i\cdot))$ in the sense of ultradistributions, we have, for every $\phi \in \mathcal{S}^{(M_p)}(\mathbb{R})$,

$$0 = \langle \phi'(t), (P_L(-id/dt)A\int\limits_0^t S(s)x\,ds - P_L(-id/dt)S(t)x$$

$$+P_L(-id/dt)\int\limits_0^t \mathcal{L}^{-1}(1/P_L(-i\cdot))(s)x\,ds)\rangle$$

$$= \langle P_L(id/dt)\phi'(t), (A\int\limits_0^t S(s)x\,ds - S(t)x + \int\limits_0^t \mathcal{L}^{-1}(1/P_L(-i\cdot))(s)x\,ds)\rangle.$$

Assume that $\psi \in \mathcal{D}(\mathbb{R})$ and $\phi \in \mathcal{S}^{(M_p)}(\mathbb{R})$ so that $\psi = P_L(id/dt)\phi$ (cf. Lemma 3.2.2). This implies

$$A\int\limits_0^t S(s)x\,ds - S(t)x + \int\limits_0^t \mathcal{L}^{-1}(1/P_L(-i\cdot))(s)x\,ds = const, \tag{3.11}$$

in the sense of Beurling ultradistributions on $(0,\infty)$. We obtain that $const = 0$ by putting $x = 0$ in (3.11). Since the left side of (3.11) is continuous on $\mathbb{R}$, we have

$$A\int_0^t S(s)x\,ds = S(t)x - \Theta(t)x = 0, \quad \text{where} \quad \Theta(t) = \int_0^t \mathcal{L}^{-1}(1/P_L(-i\cdot))(s)\,ds,$$

for all $t \geq 0$. This completes the proof of (a)' $\Rightarrow$ (c)'.

Let us show (c) $\Rightarrow$ (d) in the Beurling case. The proof of (c)' $\Rightarrow$ (d)' is similar. Define G on $\mathcal{D}^{(M_p)}((-\infty,a))$, for all $a > 0$, by

$$G := P_L^a(-id/dt)S^a_{K_a}, \quad \text{where} \quad P_L^a = \sum_{p=0}^{\infty} a_p(d/dt)^p.$$

Then G is a continuous linear mapping from $\mathcal{D}^{(M_p)}(\mathbb{R})$ into $L(E)$ which commutes with A. Moreover, supp$G \subset [0,\infty)$. Let $\phi \in \mathcal{D}^{(M_p)}((-\infty,a))$ and $x \in E$. We have,

$$G(-\phi')x - AG(\phi)x = -\sum_{p\geq 0} a_p(-i)^p \int\limits_0^a \phi^{(p+1)}(s)S^a_{K_a}(s)x\,ds$$

$$-\sum_{p\geq 0} a_p(-i)^p \int\limits_0^a \phi^{(p)}(s)AS^a_{K_a}(s)x\,ds = -\sum_{p\geq 0} a_p(-i)^p \int\limits_0^a \phi^{(p+1)}(s)S^a_{K_a}(s)x\,ds$$

$$+\sum_{p\geq 0} a_p(-i)^p \int\limits_0^a \phi^{(p+1)}(s)(S^a_{K_a}(s)x - \Theta_a(s)x)\,ds =$$

$$= \sum_{p \geq 0} a_p (-i)^p \int_0^a \phi^{(p)}(s) K_a(s) x \, ds = \phi(0)x.$$

Hence, $G \in \mathcal{D}'^{(M_p)}(\mathbb{R}, L(E, [D(A)]))$ is an ultradistribution fundamental solution for A. Clearly, $\mathcal{N}(G) = \{0\}$. □

Chapter 4

Hyperfunction Semigroups

S. Ōuchi [76] was the first who introduced the class of hyperfunction semigroups, more general than that of distribution and ultradistribution semigroups and in [77] he considered the abstract Cauchy problem in the space of hyperfunctions. Furthermore, generators of hyperfunction semigroups in the sense of [76] are not necessarily densely defined. A.N. Kochubei, [46] considered hyperfunction solutions on abstract differential equations of higher order.
The definition of Fourier hyperfunction semigroups is intrinsically different than that of ultradistribution semigroups because test functions with the support bounded on the left cannot be used. Fourier hyperfunction semigroups with densely defined infinitesimal generators were introduced by Y. Ito [36] related to the corresponding Cauchy problem [35]. There are given structural and spectral characterizations of Fourier hyperfunction semigroups and exponentially bounded Fourier hyperfunction semigroups with non-dense infinitesimal generators, their relations with the convoluted semigroups and to the corresponding Cauchy problems. Spectral properties of hyperfunction semigroups give a new insight to S. Ōuchi's results.

4.1 Hyperfunction and Fourier hyperfunction Type Spaces

The spaces of Fourier hyperfunctions were also analyzed by J. Chung, S.-Y. Chung and D. Kim in [16]-[17]. Following this approach, we have that $\mathcal{P}_*(\mathbb{D})$ is (topologically) equal to the space of $C^\infty-$functions ϕ defined on $\mathbb{R}$ with the property: $(\exists h > 0)(||\phi||_{n!,h} < \infty)$, where the norms $||\cdot||_{n!,h},\ h > 0$, are defined by $||\phi||_{n!,h} := \sup\{||\phi^{(n)}(x)||e^{|x|/h}/(h^n n!) : n \in \mathbb{N},\ x \in \mathbb{R}\}$, equipped with the corresponding inductive limit topology when $h \to +\infty$. The next lemma can be proved by the standard arguments using the norms $||\phi||_{n!,h}$.

Lemma 4.1.1. *If $\phi,\ \psi \in \mathcal{P}_*(\mathbb{D})$, then $\phi *_0 \psi = \int_0^t \phi(\tau)\psi(t-\tau)\,d\tau\,, t > 0$ is in $\mathcal{P}_*(\mathbb{D})$ and the mapping $*_0 : \mathcal{P}_*(\mathbb{D}) \times \mathcal{P}_*(\mathbb{D}) \to \mathcal{P}_*(\mathbb{D})$ is continuous.*

Proof. Suppose $x \in \mathbb{R},\ n \in \mathbb{N}$ and $h_1 > 0$ fulfill $||\phi||_{h_1} < \infty$. Suppose that $h > 2h_1$ satisfies $||\psi||_{\frac{h}{2}} < \infty$ and put $h_2 = \frac{hh_1}{h-h_1}$. We will use the next inequality which holds for every t,

$$\frac{|x|}{h} \leq \frac{|x-t|}{h} + \frac{|t|}{h} \leq \frac{|x-t|}{h} + \frac{|t|}{h_1} - \frac{|t|}{h_2}.$$

We have

$$\sup_{n\in\mathbb{N},\ x\in\mathbb{R}} \frac{e^{|x|/h}\Big|\Big(\int_0^x \phi(t)\psi(x-t)\,dt\Big)^{(n)}\Big|}{h^n n!} \leq$$

$$\leq \sup_{n\in\mathbb{N},\ x\in\mathbb{R}} \frac{e^{|x|/h}\int_0^x |\phi(t)\psi^{(n)}(x-t)|\,dt}{h^n n!} + \sum_{j=0}^{n-1} \sup_{n\in\mathbb{N},\ x\in\mathbb{R}} \frac{e^{|x|/h}|\phi^{(j)}(x)||\psi^{(n-1-j)}(0)|}{h^n n!} =$$

$$= I + II.$$

We will estimate separately I and II.

$$I \leq \sup_{t\in\mathbb{R}}\Big(|\phi(t)|e^{\frac{|t|}{h_1}}\Big)\Big(\int_0^x e^{-\frac{|t|}{h_2}}\,dt\Big) \sup_{n\in\mathbb{N},\ x,t\in\mathbb{R}} \frac{|\psi^{(n)}(x-t)|e^{|x-t|/h}}{h^n n!},$$

$$II \leq \frac{1}{2^n}\sum_{j=0}^{n-1} \sup_{j\in\mathbb{N},\ x\in\mathbb{R}} \frac{e^{|x|/h}|\phi^{(j)}(x)|}{(h/2)^j j!} \sup_{n-j\in\mathbb{N}} \frac{|\psi^{(n-1-j)}(0)|}{(h/2)^{n-j}(n-j)!}$$

This gives $\phi *_0 \psi \in \mathcal{P}_*(\mathbb{D})$ while the continuity of the mapping $*_0 : \mathcal{P}_*(\mathbb{D})\times\mathcal{P}_*(\mathbb{D}) \to \mathcal{P}_*(\mathbb{D})$ follows similarly. This completes the proof of the lemma. $\square$

Now we will transfer the definitions and assertions for Roumieu tempered ultradistributions to Fourier hyperfunctions.

Definition 4.1.1. Let $a \geq 0$. Then

$$\mathcal{P}_{*,a}(\mathbb{D}) := \{\phi \in C^\infty(\mathbb{R}) : e^{a\cdot}\phi \in \mathcal{P}_*(\mathbb{D})\}.$$

Define the convergence in this space by

$$\phi_n \to 0 \text{ in } \mathcal{P}_{*,a}(\mathbb{D}) \text{ iff } e^{a\cdot}\phi_n \to 0 \text{ in } \mathcal{P}_*(\mathbb{D}).$$

We denote by $\mathcal{Q}_a(\mathbb{D}, E)$ the space of continuous linear mappings from $\mathcal{P}_{*,a}(\mathbb{D})$ into E endowed with the strong topology.

We have:

$$F \in \mathcal{Q}_a(\mathbb{D}, E) \text{ iff } e^{-a\cdot}F \in \mathcal{Q}(\mathbb{D}, E). \tag{4.1}$$

Proposition 4.1.1. *Let $G \in \mathcal{Q}_a(\mathbb{D}, L(E))$. Then there exists a local operator P and a function $g \in C(\mathbb{R}, L(E))$ with the property that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$e^{-ax}||g(x)|| \leq C_\varepsilon e^{\varepsilon|x|},\ x \in \mathbb{R} \ \text{ and } \ G = P(d/dt)g.$$

Proof. From the structure theorem for the space $\mathcal{Q}(\mathbb{D}, L(E))$ and since $e^{-a\cdot}G \in \mathcal{Q}(\mathbb{D}, L(E))$, there exists a local operator P and a function g_1 with the property that for every $\varepsilon > 0$ there is corresponding $C_\varepsilon > 0$ such that

$$\|g_1(x)\| \leq C_\varepsilon e^{\varepsilon|x|},\ x \in \mathbb{R} \quad \text{and} \quad G = e^{ax}P(d/dt)g_1\,.$$

We put $g(x) = e^{ax}g_1(x)$, $x \in \mathbb{R}$. Using Leibnitz formula , we have

$$e^{ax}P(d/dt)g_1(x) = \sum_{t=0}^{\infty}(\sum_{k=0}^{\infty}\binom{t+k}{t}(-1)^k a^k b_{k+t})(e^{ax}g_1(x))^{(t)}\,.$$

The assertion will be proved if we show that $\lim\limits_{|t|\to\infty}(|c_t|t!)^{\frac{1}{t}} = 0$, where $c_t = \sum\limits_{k=0}^{\infty}\binom{k+t}{k}a^k b_{k+t}$. To prove this, we use

$$\binom{t+k}{k} \leq (t+k)^k \leq 2^k k^k + 2^k t^k \leq 2^k(k^k + k^k e^t) = 2^k k^k(1+e^t)\,,$$

where we used $t^k \leq k^k e^t$. The last inequality is clear for $k \geq t$. For $k < t$, we put $k = \nu t$. First let we note that $\nu \ln \nu \in (-1, 0)$. Then $\nu t \ln t \leq \nu t \ln t + \nu t \ln \nu + t$. Hence $t^k \leq k^k e^t$. Now,

$$c_t = \sum_{k=0}^{\infty} 2^k k^k(1+e^t)a^k b_{k+t} = \sum_{k=0}^{\infty}(2a)^k k^k(1+e^t)b_{k+t}\,.$$

The coefficients b_{k+t} are coefficients of a local operator, so for every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $t + k > M$, $|b_{k+t}|(t+k)! < \varepsilon^{t+k}$. With this we have

$$t!|c_t| \leq (1+e^t)\sum_{k=0}^{\infty}\frac{(2a)^k k^k(t+k)!t!|b_{k+t}|}{(t+k)!} \leq \sum_{k=0}^{\infty}\frac{(2a)^k(1+e)^t e^k k!t!(t+k)!|b_{t+k}|}{(t+k)!} \leq$$

$$\leq \sum_{k=0}^{\infty}\frac{(2a)^k e^k(1+e^t)k!t!(t+k)!|b_{t+k}|}{t!k!} \leq \sum_{k=0}^{\infty}(2ae)^k(1+e^t)\varepsilon^{t+k} =$$

$$= (1+e^t)\varepsilon^t\sum_{k=0}^{\infty}(2ae\varepsilon)^k$$

and the assertion follows since we can choose ε arbitrary small. □

Remark 4.1.1. By Lemma 4.1.1, one can easily prove that, if $\phi,\ \psi \in \mathcal{P}_{*,a}(\mathbb{D})$, then $\phi *_0 \psi \in \mathcal{P}_{*,a}(\mathbb{D})$ and the mapping $*_0 : \mathcal{P}_{*,a}(\mathbb{D}) \times \mathcal{P}_{*,a}(\mathbb{D}) \to \mathcal{P}_{*,a}(\mathbb{D})$ is continuous.

For the needs of the Laplace transform we define the space $\mathcal{P}_*([-r,\infty]), r > 0$. Note that $[-r,\infty]$ is compact in $\mathbb{D}$.

$\mathcal{P}_*([-r,\infty], h)$ is defined as the space of smooth functions ϕ on $(-r,\infty)$ with the property $||\phi||_{*,-r,h} < \infty$, where

$$||\phi||_{*,-r,h} := \sup\Big\{\frac{||\phi^{(\alpha)}(x)||e^{|x|/h}}{h^\alpha \alpha!} : \alpha \in \mathbb{N}_0,\ x \in (-r,\infty)\Big\}.$$

Then

$$\mathcal{P}_*([-r,\infty]) := \operatorname{ind}\lim_{h\to+\infty}\mathcal{P}_*([-r,\infty], h).$$

Lemma 4.1.2. *$\mathcal{P}_*(\mathbb{D})$ is dense in $\mathcal{P}_*([-r,\infty])$.*

Proof. This is a consequence of Lemma 8.6.4 in [40]. □

For $a \geq 0$, we define the space

$$\mathcal{P}_{*,a}([-r,\infty]) := \{\phi : e^{a\cdot}\phi \in \mathcal{P}_*([-r,\infty])\}.$$

The topology of $\mathcal{P}_{*,a}([-r,\infty])$ is defined by:

$$\lim_{n\to\infty} \phi_n = 0 \text{ in } \mathcal{P}_{*,a}([-r,\infty]) \text{ iff } \lim_{n\to\infty} e^{a\cdot}\phi_n = 0 \text{ in } \mathcal{P}_*([-r,\infty]).$$

If $a \geq 0$ and $e^{-a\cdot}G \in \mathcal{Q}_+(\mathbb{D}, L(E))$, then G can be extended to an element of the space of continuous linear mappings from $\mathcal{P}_{*,a}([-r,\infty])$ into $L(E)$ equipped with the strong topology. This extension is unique because of Lemma 4.1.2. We will use this for the definition of the Laplace transform of G.

4.2 Fourier hyperfunction semigroups

The definition of (exponential) Fourier hyperfunction semigroup with densely defined infinitesimal generators of Y. Ito (see [36, Definition 2.1]) is given on the basis of the space $\mathcal{P}_0$ whose structure is not clear to authors. Our definition is different and related to non-densely defined infinitesimal generators.

In the sequel, we use the notation $\mathcal{Q}_+(\mathbb{D}, L(E))$ for the space of vector-valued Fourier hyperfunctions supported by $[0,\infty]$. More precisely, if $f \in \mathcal{Q}_+(\mathbb{D}, L(E))$ is represented by $f(t,\cdot) = F_+(t+i0,\cdot) - F_-(t-i0,\cdot)$, where F_+ and F_- are defining functions for f (see [40, Definition 1.3.6, Definition 8.3.1]) and γ_+ and γ_- are piecewise smooth paths connecting points $-a$ ($a > 0$) and ∞ such that γ_+ and γ_- lie respectively in the upper and the lower half planes as well as in a strip around $\mathbb{R}$ depending on f, then for any $\psi \in \mathcal{P}_*(\mathbb{D})$,

$$\int_{\mathbb{R}} f(t)\psi(t)\,dt = \int_0^\infty f(t)\psi(t)\,dt := \int_{\gamma_+} F_+(z)\psi(z)\,dz - \int_{\gamma_-} F_-(z)\psi(z)\,dz.$$

Since we will use the duality approach of Chong and Kim, we use the notation $\langle f, \psi\rangle$ for the above expression.

Let $\varphi \in \mathcal{P}_*$ and $f(t,\cdot) = F_+(t+i0,\cdot) - F_-(t-i0,\cdot)$ be an element in $\mathcal{Q}_+(\mathbb{D}, L(E))$. Then

$$\varphi(t)f(t,\cdot) := \varphi(t)F_+(t+i0,\cdot) - \varphi(t)F_-(t-i0,\cdot).$$

We denote by $\mathcal{P}_*^0$ the subspace of $\mathcal{P}_*$ consisting of functions ϕ with the property $\phi(0) = 0$. Also, we consider $\mathcal{P}_*^{00}$, a subspace of $\mathcal{P}_*$ consisting of functions ψ with the properties $\psi(0) = 0$ and $\psi'(0) = 0$. Note, any $\psi \in \mathcal{P}_*$ can be written in the form

$$\psi(t) = \psi(0)\phi_0(t) + \theta(t),\ t \in \mathbb{R}, \text{ respectively}, \tag{4.2}$$

$$\psi(t) = \psi(0)\phi_0(t) + \psi'(0)\phi_1(t) + \tilde{\theta}(t),\ t \in \mathbb{R}, \tag{4.3}$$

where ϕ_0 and ϕ_1 are fixed elements of $\mathcal{P}_*$ with the properties $\phi_0(0) = 1$, $\phi_0'(0) = 0$, $\phi_1(0) = 0$, $\phi_1'(0) = 1$ and θ varies over $\mathcal{P}_*^0$ respectively $\tilde{\theta}$ varies over $\mathcal{P}_*^{00}$. We define $\mathcal{P}_{*a}^0$ as a space of functions $\phi \in \mathcal{P}_{*,a}$ with the property $\phi(0) = 0$ and $\mathcal{P}_{*,a}^{00}$, as a space of functions $\phi \in \mathcal{P}_{*,a}$ with the property $\phi(0) = 0, \phi'(0) = 0$ and note that the similar decompositions as (4.2) and (4.3) hold for elements of $\mathcal{P}_{*,a}^0$ and $\mathcal{P}_{*,a}^{00}$, respectively.

Definition 4.2.1. An element $G \in \mathcal{Q}_+(\mathbb{D}, L(E))$ is called a pre-Fourier hyperfunction semigroup, if the next condition is valid

(H.1) $G(\phi *_0 \psi) = G(\phi)G(\psi),\ \phi,\ \psi \in \mathcal{P}_*(\mathbb{D})$.

Further on, a pre-Fourier hyperfunction semigroup G is called a Fourier hyperfunction semigroup, (FHSG) in short, if in addition, the following holds

(H.2) $\mathcal{N}(G) := \bigcap_{\phi\in\mathcal{P}_*^{00}(\mathbb{D})} N(G(\phi)) = \{0\}$.

If the next condition also holds:

(H.3) $\mathcal{R}(G) := \bigcup_{\phi\in\mathcal{P}_*^{00}(\mathbb{D})} R(G(\phi))$ is dense in E, then G is called a dense (FHSG).

If $e^{-a\cdot}G \in \mathcal{Q}_+(\mathbb{D}, L(E))$, for some $a > 0$, and (H.1) holds with $\phi,\ \psi \in \mathcal{P}_{*,a}(\mathbb{D})$ then G is called exponentially bounded pre-Fourier hyperfunction semigroup. If (H.2) and (H.3) hold with $\phi \in \mathcal{P}_{*,a}^{00}(\mathbb{D})$, then G is called a dense exponential Fourier hyperfunction semigroup, dense (EFHSG), in short.

Let A be a closed operator. We denote by $[D(A)]$ the Banach space $D(A)$ endowed with the graph norm $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$, $x \in D(A)$. Like in [35, Definition 2.1, Definition 3.1], we give the following definitions:

Definition 4.2.2. Let A be a closed operator. Then $G \in \mathcal{Q}_+(\mathbb{D}, L(E, [D(A)]))$ is a Fourier hyperfunction solution for A if $P * G = \delta \otimes I_E$ and $G * P = \delta \otimes I_{[D(A)]}$, where $P := \delta' \otimes I_{D(A)} - \delta \otimes A \in \mathcal{Q}_+(\mathbb{D}, L([D(A)], E))$; G is called an exponential Fourier hyperfunction solution for A if, additionally,

$$e^{-a\cdot}G \in \mathcal{Q}_+(\mathbb{D}, L(E, [D(A)])), \text{ for some } a > 0.$$

Similarly, if G is an exponential Fourier hyperfunction solution for A which fulfills (H.3), then G is called a dense, exponential Fourier hyperfunction solution for A.

Let $a \geq 0$ and $\alpha \in \mathcal{P}_{*a}$, be an even function such that $\int \alpha(t)\,dt = 1$. Let sgn $(x) := 1, x > 0$, sgn $(x) := -1, x < 0$ and sgn $(0) := 0$. A net of the form $\delta_\varepsilon = \alpha(\cdot/\varepsilon)/\varepsilon, \varepsilon \in (0,1)$, is called delta net in $\mathcal{P}_{*a}$. Changing α with the above properties, one obtains a set of delta nets in $\mathcal{P}_{*a}$. Clearly, every delta net converges to δ as $\varepsilon \to 0$ in $\mathcal{Q}(\mathbb{D})$. We define, for $x \in \mathbb{R}$,

$$\delta *_0 \phi(x) := 2\text{sgn }(x) \lim_{\varepsilon\to 0} \delta_\varepsilon *_0 \phi(x) = \phi(x),\ \phi \in \mathcal{P}_{*,a}^0,$$

$$\delta' *_0 \phi(x) := 2\text{sgn }(x) \lim_{\varepsilon\to 0} \delta_\varepsilon' *_0 \phi(x) = \phi'(x),\ \phi \in \mathcal{P}_{*,a}^{00}.$$

Definition 4.2.3. Let $a \geq 0$ and G be an (EFHSG). Then

1. $G(\delta)x := y$ iff $G(\delta *_0 \phi)x = G(\phi)y$ for every $\phi \in \mathcal{P}_{*,a}^0(\mathbb{D})$.
2. $G(-\delta')x := y$ if $G(-\delta' *_0 \phi)x = G(\phi)y$ for every $\phi \in \mathcal{P}_{*,a}^{00}(\mathbb{D})$.

$A = G(-\delta')$ is called the infinitesimal generator of G.

Thus $G(\delta)$ is the identity operator. In order to prove that $G(-\delta')$ is a single-valued function, we have to prove that for every $x \in E$, $G(-\delta')x = y_1$ and $G(-\delta')x = y_2$ imply $y_1 = y_2$. This means that we have to prove that

$$G(\phi')x = G(\phi)y_1,\ G(\phi')x = G(\phi)y_2,\ \phi \in \mathcal{P}_*^{00} \Longrightarrow y_1 = y_2.$$

Proposition 4.2.1. *If* $G(\phi')x = 0$ *for every* $\phi \in \mathcal{P}_{*,a}^{00}$, *then* $x = 0$.

Proof. We shall prove that the assumption $G(\phi)y = 0$ for every $\phi \in \mathcal{P}_{*,a}^{0}$ implies that $y = 0$. By (4.2), we have that for any $\phi_0 \in \mathcal{P}_{*,a}$ such that $\phi_0(0) = c \neq 0$

$$G(\psi)y = \frac{\psi(0)}{c}G(\phi_0)y, \psi \in \mathcal{P}_{*,a}.$$

Now let ϕ, ψ be arbitrary elements of $\mathcal{P}_{*,a}$. Since $G(\phi *_0 \psi)y = G(\phi)G(\psi)y$ and $\phi *_0 \psi(0) = 0$, it follows, with $z = G(\psi)y$,

$$G(\phi *_0 \psi)y = G(\phi)z = 0,\ \phi \in \mathcal{P}_{*,a} \Longrightarrow z = 0.$$

Thus, for any $\psi \in \mathcal{P}_{*,a}$, we have $G(\psi)y = 0$ which finally implies $y = 0$.

Now, we will prove the assertion. By (4.3) we have that for every $\psi \in \mathcal{P}_{*,a}$

$$G(\psi')x = \psi(0)G(\phi_0')x + \psi'(0)G(\phi_1')x = 0.$$

Denote by P_{10} the set of all $\phi_0 \in \mathcal{P}_*$ with the properties $\phi_1(0) = c \neq 0, \phi_1'(0) = 0$ and by P_{01} the set of all $\phi_1 \in \mathcal{P}_*$ with the properties $\phi_0(0) = 0, \phi_0'(0) = c \neq 0$.

We have the following cases:

$$(\forall\phi_0 \in P_{10})(\forall\phi_1 \in P_{01})(G(\phi_0)x = 0,\ G(\phi_1)x = 0);$$

$$(\forall\phi_0 \in P_{10})(\exists\phi_1 \in P_{01})(G(\phi_0)x = 0,\ G(\phi_1)x \neq 0);$$

$$(\exists\phi_0 \in P_{10})(\forall\phi_1 \in P_{01})(G(\phi_0)x \neq 0,\ G(\phi_1)x = 0);$$

$$(\exists\phi_0 \in P_{10})(\exists\phi_1 \in P_{01})(G(\phi_0)x \neq 0,\ G(\phi_1)x \neq 0).$$

In the first case we have, by (4.3), $G(-\psi')x = 0, \psi \in \mathcal{P}_{*,a}$. This implies, by the standard arguments, that $G(\psi)x = C\int_{\mathbb{R}} \psi(t)\,dt\,x = 0, \psi \in \mathcal{P}_{*,a}$ and this holds for $C = 0$. Consider the fourth case. In this case we have that

$$G(\psi')x = C_1\langle\delta,\psi\rangle x + C_2\langle\delta',\psi\rangle x$$

and thus,

$$G(\psi')x = C_1\langle\delta,\psi\rangle x + C_2\langle\delta',\psi\rangle x + C_3\langle\mathbf{1},\psi\rangle x,$$

where $\langle\mathbf{1},\psi\rangle x = \int_{\mathbb{R}} \psi(t)\,dt\,x$. Now, by the semigroup property it follows $C_1 = C_2 = C_3 = 0$ and with this we conclude as above that $x = 0$. We can handle out the second and the third case in a similar way. This completes the proof of the assertion. □

4.2.1 Laplace transform and the characterizations of Fourier hyperfunction semigroups

The assertions of this section related to the Laplace transform are new but some of them are quite simple. They are based on the technics developed by Komatsu [48]-[53].

Note, for every $r > 0$, $E_\lambda = e^{-\lambda\cdot} \in \mathcal{P}_*((-r, \infty])$, for every $\lambda \in \mathbb{C}$ with $Re\lambda > 0$. So, we can define the Laplace transform of $G \in \mathcal{Q}_+(\mathbb{D}, L(E))$ by

$$\mathcal{L}G(\lambda) = \hat{G}(\lambda) := G(E_\lambda),\ Re\lambda > 0.$$

Proposition 4.2.2. *There exists a suitable local operator P such that*

$$|\hat{G}(\lambda)| \leq |P(\lambda)|,\ Re\lambda > 0.$$

The proof of this assertion it is even simpler than the proof of the corresponding assertion in the case of Roumieu ultradistributions.

If $e^{-a\cdot}G \in \mathcal{Q}_+(\mathbb{D}, L(E))$, we define the Laplace transform of G by

$$\mathcal{L}(G)(\lambda) = \hat{G}(\lambda) := G(E_\lambda),\ Re\lambda > a.$$

It is an analytic function defined on $\{\lambda \in \mathbb{C} : Re\lambda > a\}$ and there exists a local operator P such that $|\hat{G}(\lambda)| \leq |P(\lambda)|, Re\lambda > a$.

Remark 4.2.1. Similarly to the corresponding Roumieu case, one can prove the next statement:
If $G \in \mathcal{Q}_+(\mathbb{D}, L(E, [D(A)]))$ is a Fourier hyperfunction solution for A, then G is a pre-Fourier hyperfunction semigroup. It can be seen, as in the case of ultradistributions, that we do not have that G must be an (FHSG).

Structural properties of the Fourier hyperfunction semigroups are similar to that of ultradistribution semigroups of Roumieu class. For the essentially different proofs of corresponding results we need the next lemma where we again use the Fourier transform instead of Laplace transform.

Lemma 4.2.1. *Let P_{L_p} be of the form (1.6). The mapping*

$$P_{L_p}(id/dt) : \mathcal{P}_*(\mathbb{D}) \to \mathcal{P}_*(\mathbb{D}),\ \ \phi \mapsto P_{L_p}(id/dt)\phi$$

is a continuous linear bijection.

Proof. Due to [40, Proposition 8.2.2], $\phi \in \mathcal{P}_*(\mathbb{D})$ implies $\mathcal{F}(\phi) \in \mathcal{P}_*(\mathbb{D})$. Thus, for some $n \in \mathbb{N}$, every $\varepsilon > 0$ and a corresponding $C_\varepsilon > 0$, $|\mathcal{F}(\phi)(z)| \leq C_\varepsilon e^{(-1/n-\varepsilon)|Rez|}$, $z \in \mathbb{R} + I_n$. By [40, Proposition 8.1.6, Lemma 8.1.7, Theorem 8.4.9], with some simple modifications, we have

$$Ce^{\frac{A|\zeta|}{r(|\zeta|+1)}} \leq |P_{L_p}(\zeta)|,\ |\eta| \leq \frac{|\xi|}{2} + \frac{1}{L_1},\ \zeta = \xi + i\eta, \tag{4.4}$$

for some $C,\ A > 0$ and some monotone increasing function r with the properties $r(0) = 1,\ r(\infty) = \infty$. This implies that there exists an integer $n_0 \in \mathbb{N}$ such that

$$\mathcal{F}(\phi)/P_{L_p} \in \tilde{\mathcal{O}}^{-1/n_0}(\mathbb{R} + iI_{n_0}).$$

Thus, its inverse Fourier transform $\mathcal{F}^{-1}(\mathcal{F}(\phi)/P_{L_p})$ is an element of $\mathcal{P}_*(\mathbb{D})$. □

Using the properties of local operators as well as norms $||\cdot||_{h,p!}$, as in the case of Roumieu tempered ultradistributions, one obtains the following assertions.

Theorem 4.2.1. *Suppose that* $f:\{\lambda\in\mathbb{C}:\ Re\lambda>a\}\to E$ *is an analytic function satisfying*

$$||f(\lambda)||\le C|P(\lambda)|,\ Re\lambda>a,$$

for some $C>0$, *some local operator* P *with the property* $|P(\lambda)|>0,\ Re\lambda>a$. *Suppose, further, that a local operator* $\tilde{P}$ *satisfies (4.4). Then*

$$(\exists M>0)(\exists h\in C^{\infty}([0,\infty);E))(\forall j\in\mathbb{N})(h^{(j)}(0)=0)$$

such that $||h(t)||\le Me^{at},\ t\ge 0,$ *and*

$$f(\lambda)=P(\lambda)\tilde{P}(\lambda)\int_0^{\infty}e^{-\lambda t}h(t)\,dt,\ Re\lambda>a.$$

Theorem 4.2.2. *Let* A *be closed and densely defined. Then* A *generates a dense* $(EFHSG)$ *if and only if the following conditions hold:*

(i) $\{\lambda\in\mathbb{C}:Re\lambda>a\}\subset\rho(A)$.

(ii) There exists a local operator P *with the property* $|P(\lambda)|>0,\ Re\lambda>a$, *a local operator* $\tilde{P}$ *with the properties as in the previous theorem and* $C>0$ *such that*

$$||R(\lambda:A)||\le C|P(\lambda)\tilde{P}(\lambda)|,\ Re\lambda>a.$$

(iii) $R(\lambda:A)$ *is the Laplace transform of some* G *which satisfies* $(H.2)$.

Proof. We will prove the theorem for $a=0$.

($\Leftarrow$): Theorem 4.2.1 implies that $R(\lambda:A)$ is of the form

$$R(\lambda:A)=P(\lambda)\tilde{P}(\lambda)\int_0^{\infty}e^{-\lambda t}S(t)\,dt,\ Re\lambda>0,$$

where $S\in C^{\infty}([0,\infty))$, $S^{(j)}(0)=0,\ j\in\mathbb{N}_0$ and for every $\varepsilon>0$ there exists $M>0$ such that $||S(t)||\le M,\ t\ge 0$ This implies $R(\lambda:A)=\mathcal{L}(G)(\lambda),\ Re\lambda>0$, where $G=P(-d/dt)\tilde{P}(-d/dt)S,$ and $G\in\mathcal{Q}_+(\mathbb{D},E)$. Since

$$(\delta'\otimes I_{D(A)}-\delta\otimes A)*G=\delta\otimes I_E,$$

$$G*(\delta'\otimes I_{D(A)}-\delta\otimes A)=\delta\otimes I_{D(A)},$$

and (iii) holds, we have that G is a Fourier hyperfunction semigroup.

($\Rightarrow$): Put $E_\lambda^+=E_\lambda H, R_\lambda^+=R_\lambda H$, where H is Heaviside's function. Let $G\in\mathcal{Q}_+(\mathbb{D},L(E,D(A)))$ and $\lambda\in\{z\in\mathbb{C}:Re\lambda>a\}\subset\rho(A)$ be fixed. Then $(\delta'+\lambda\delta)*E_\lambda^+=\delta$. Now let $\phi\in\mathcal{P}_*(\mathbb{D})$ and $x\in E$. Then

$$G((\delta'+\lambda\delta)*_0E_\lambda^+*_0\phi)=G(\phi)x,$$

and

$$G(\delta'*_0R_\lambda^+*_0\phi)x+\lambda G(\delta*_0E_\lambda^+*_0\phi)x=G(\delta')G(E_\lambda^+*_0\phi)x+\lambda\hat{G}(\lambda)G(\phi)x\,.$$

Hence,

$$-A(\hat{G}(\lambda)G(\phi)x) + \lambda\hat{G}(\lambda)G(\phi)x = G(\phi)x\,.$$

Since $(H.3)$ is assumed $(-A+\lambda)\hat{G}(\lambda) = I$, so $\|\hat{G}(\lambda)\| \leq C|P(\lambda)|$, $Re\lambda > a$, where P is an appropriate local operator. □

Corollary 4.2.1. *Suppose A is a closed linear operator. If A generates an (EFHSG), (i), (ii) and (iii) of Theorem 4.2.2 hold.*

If (i) and (ii) of Theorem 4.2.2 hold, then G, defined in the same way as above, is a Fourier hyperfunction fundamental solution for A. If (iii) is satisfied, then G is an (EFHSG) generated by A.

We note that in Corollary 4.2.1 the operator A is non–densely defined.

Now we will prove a theorem related to Fourier hyperfunction semigroups. As in the case of ultradistributions, the theorem can be proved for (EFHSG) but for the sake of simplicity, we will assume that $a = 0$.

We need one more theorem.

Theorem 4.2.3. *Let A be a closed operator in E. If A generates a (FHSG) G, then G is an Fourier hyperfunction fundamental solution for*

$$P := \delta' \otimes I_{D(A)} - \delta \otimes A \in \mathcal{Q}_+(\mathbb{D}, L([D(A)], E)).$$

In particular, if $T \in \mathcal{Q}_+(\mathbb{D}, E)$, then $u = G * T$ is the unique solution of

$$-Au + \frac{\partial}{\partial t}u = T,\ u \in \mathcal{Q}_+(\mathbb{D}, [D(A)]). \tag{4.5}$$

If supp$T \subset [\alpha, \infty)$, then supp$u \subset [\alpha, \infty)$.

Conversely, if $G \in \mathcal{Q}_+(\mathbb{D}, L(E, [D(A)]))$ is a Fourier hyperfunction fundamental solution for P and $\mathcal{N}(G) = \{0\}$, then G is an (FHSG) in E.

Proof. ($\Rightarrow$) One can simply check that $(G(\psi)x, G(-\psi')x - \psi(0)x) \in G(-\delta')$ and G is a fundamental solution for P. The uniqueness of the solution $u = G * T$ of (4.5) is clear as well as the support property for the solution u if supp$T \subset [\alpha, \infty)$.

The part ($\Leftarrow$) can be proved in the same way as in the Theorem 3.2.1, part (d)$\Rightarrow$ (a). □

We give following statements in order to simplify the exposition of the next theorem:

(1) A generates an (FHSG) G.

(2) A generates an (FHSG) of the form $G = P_{L_p}(-id/dt)S_{a,K}$, where $S_K : \mathbb{R} \to L(E)$ is exponentially slowly increasing continuous function and $S_K(t) = 0,\ t \leq 0$.

(3) A is the generator of a global K-convoluted semigroup $(S_K(t))_{t\geq 0}$, where $K = \mathcal{L}^{-1}(\frac{1}{P_{L_p}(-i\lambda)})$.

(4) The problem

$$(\delta\otimes(-A)+\delta'\otimes I_E)*G=\delta\otimes I_E,\ \ G*(\delta\otimes(-A)+\delta'\otimes I_{D(A)})=\delta\otimes I_{D(A)}$$

has a unique solution $G\in\mathcal{Q}_+(\mathbb{D},L(E,[D(A)]))$ with $\mathcal{N}(G)=\{0\}$.

(5) For every $\varepsilon>0$ there exists $K_\varepsilon>0$ such that

$$\rho(A)\supset\{\lambda\in\mathbb{C}:\ Re\lambda>0\}$$

and

$$||R(\lambda:A)||\leq K_\varepsilon e^{\varepsilon|\lambda|},\ Re\lambda>0.$$

Theorem 4.2.4. *(1) ⇔ (4); (1) ⇒ (3); (3) ⇒ (4); (4) ⇒ (5);*

Proof. The equivalence of (1) and (4) can be proved in the same way as in the case of ultradistribution semigroups, Theorem 3.2.1. For the proof of (1) ⇒ (3) we have to use Lemma 4.2.1 (see Theorem 3.2.1 (a)' ⇒ (c)'). The implication (4) ⇒ (5) is a consequence of Theorem 4.2.2 and Corollary 4.2.1. In the case when the infinitesimal generator is densely defined Y. Ito [35] proved the equivalence of a slightly different assertion (4), without the assumption $\mathcal{N}(G)=\{0\}$, and (5). Our assertion is the stronger one since it is based on the strong structural result of Theorem 4.2.2. □

Operators which satisfy (5) may be given using the analysis of P.C. Kunstmann [61, Example 1.6] with suitable chosen sequence $(M_p)_{p\in\mathbb{N}}$.

The definition of a hyperfunction fundamental solution G for a closed linear operator A can be found in the paper [76] of S. Ōuchi. For the sake of simplicity, we shall also say, in that case, that A generates a hyperfunction semigroup G. The next assertion is proved in [76]:

A closed linear operator A generates a hyperfunction semigroup if and only if for every $\varepsilon>0$ there exist suitable $C_\varepsilon,\ K_\varepsilon>0$ so that

$$\rho(A)\supset\Omega_\varepsilon:=\{\lambda\in\mathbb{C}:\ Re\lambda\geq\varepsilon|\lambda|+C_\varepsilon\}$$

and

$$||R(\lambda:A)||\leq K_\varepsilon e^{\varepsilon|\lambda|},\ \lambda\in\Omega_\varepsilon.$$

We will give some results related to hyperfunction and convoluted semigroups in terms of spectral conditions and the asymptotic behavior of $\tilde{K}$. We refer to [3] for the similar results related to n-times integrated semigroups, $n\in\mathbb{N}$, to [34] for α-times integrated semigroups, $\alpha>0$ and to [70, Theorem 1.3.1] for convoluted semigroups. Since we focus our attention on connections of convoluted semigroups with hyperfunction semigroups, we use the next conditions for K :

(P1) K is exponentially bounded, i.e., there exist $\beta\in\mathbb{R}$ and $M>0$ so that $|K(t)|\leq Me^{\beta t}$, for a.e. $t\geq0$.

(P2) $\tilde{K}(\lambda)\neq0,\ Re\lambda>\beta.$

In general, the second condition does not hold for exponentially bounded functions, cf. [5, Theorem 1.11.1] and [57]. Following analysis in [22] and [55, Theorem 2.7.1, Theorem 2.7.2], in our context, we can give the following statements:

Theorem 4.2.5. *Let K satisfy $(P1)$ and $(P2)$ and let $(S_K(t))_{t\in[0,\tau)}$, $0<\tau\leq\infty$, be a K-convoluted semigroup generated by A. Suppose that for every $\varepsilon>0$ there exist $\varepsilon_0\in(0,\tau\varepsilon)$ and $T_\varepsilon>0$ such that*

$$\frac{1}{|\tilde{K}(\lambda)|}\leq T_\varepsilon e^{\varepsilon_0|\lambda|},\ \ \lambda\in\Omega_\varepsilon\cap\{\lambda\in\mathbb{C}:Re\lambda>\beta\}.$$

Then for every $\varepsilon>0$ there exist $\overline{C}_\varepsilon>0$ and $\overline{K}_\varepsilon>0$ such that

$$\Omega^1_\varepsilon=\{\lambda\in\mathbb{C}:Re\lambda\geq\varepsilon|\lambda|+\overline{C}_\varepsilon\}\subset\rho(A)\ \text{and}\ ||R(\lambda:A)||\leq\overline{K}_\varepsilon e^{\varepsilon_0|\lambda|},\ \ \lambda\in\Omega^1_\varepsilon.$$

Proof. Let M and β be as in (P1) and let $\varepsilon>0$ be fixed. Define

$$R(\lambda:t):=\frac{1}{\tilde{K}(\lambda)}\int\limits_0^t e^{-\lambda s}S_K(s)ds,\ \ Re\lambda>\beta,\ \ t\in[0,\tau).$$

Fix an $x\in E$. Proceeding as in [70, Theorem 1.3.1], it follows

$$\begin{aligned}(\lambda I-A)R(\lambda:t)x&=\frac{1}{\tilde{K}(\lambda)}\Big(\lambda\int_0^t e^{-\lambda s}S_K(s)xds-A\int_0^t e^{-\lambda s}S_K(s)xds\Big)\\&=\frac{1}{\tilde{K}(\lambda)}\Big(\lambda\int_0^t e^{-\lambda s}\Theta(s)xds-e^{-\lambda t}A\int_0^t S_K(s)xds\Big)\\&=\frac{1}{\tilde{K}(\lambda)}\Big(\int_0^t e^{-\lambda s}K(s)xds-e^{-\lambda t}S_K(t)x\Big)\end{aligned}$$

$$=I-\frac{1}{\tilde{K}(\lambda)}\Big(e^{-\lambda t}S_K(t)x+\int\limits_t^\infty e^{-\lambda s}K(s)xds\Big):=I-B_t(\lambda)x,\ \ Re\lambda>\beta,\ \ t\in[0,\tau).$$

Our goal is to find the domain Ω^1_ε such that for $\lambda\in\Omega^1_\varepsilon$, $Re\lambda>\beta$, we can estimate $B_t(\lambda)$:

$$\begin{aligned}||B_t(\lambda)||&\leq\frac{1}{|\tilde{K}(\lambda)|}\Big(e^{-Re\lambda t}||S_K(t)||+M\int\limits_t^\infty e^{(\beta-Re\lambda)s}ds\Big)\\&\leq\frac{1}{|\tilde{K}(\lambda)|}\Big(e^{-Re\lambda t}||S_K(t)||+M\frac{e^{(\beta-Re\lambda)t}}{Re\lambda-\beta}\Big)\\&\leq T_\varepsilon e^{\varepsilon_0|\lambda|}e^{(\beta-Re\lambda)t}(e^{-\beta t}||S_K(t)||+\frac{M}{Re\lambda-\beta}),\ \ t\in[0,\tau).\end{aligned}$$

Fix $t\in(0,\tau)$ and put $||S_K(t)||=C_0$. Let $\beta_1\in(\beta,\infty)$. Assume that $Re\lambda>\beta_1$ and let us find an additional condition on $Re\lambda$ such that

$$||B_t(\lambda)||\leq T_\varepsilon(e^{-\beta t}||S_K(t)||+\frac{M}{Re\lambda-\beta})e^{\varepsilon_0|\lambda|+(\beta-Re\lambda)t}$$

$$\leq T_\varepsilon(e^{-\beta t}C_0 + \frac{M}{\beta_1-\beta})e^{\varepsilon_0|\lambda|+(\beta_1-Re\lambda)t} \leq \delta < 1.$$

Now we see that with

$$\overline{C}_\varepsilon := \beta_1 + C_\varepsilon + |\beta - \ln\frac{\delta}{T_\varepsilon(e^{-\beta t}C_0 + \frac{M}{\beta_1-\beta})}|,$$

$$\overline{K}_\varepsilon := \frac{tT_\varepsilon}{1-\delta} \geq |\int_0^t e^{-\lambda s}S_K(s)ds|\frac{T_\varepsilon}{1-\delta}$$

and Ω^1_ε as in the theorem, we have

$$||B_t(\lambda)|| \leq \delta,\ \lambda \in \Omega^1_\varepsilon.$$

Since $R(\lambda : t)$ and $B_t(\lambda)$ commute with A, it follows $\Omega^1_\varepsilon \subset \rho(A)$ and

$$||R(\lambda : A)|| = ||R(\lambda : t)(I - B_t(\lambda))^{-1}|| \leq \frac{1}{|\tilde{K}(\lambda)|}|\int_0^t e^{-\lambda s}S_K(s)ds|\frac{1}{1-\delta}$$

$$\leq \overline{K}_\varepsilon e^{\varepsilon_0|\lambda|} \leq \overline{K}_\varepsilon e^{\varepsilon\tau|\lambda|},\ \lambda \in \Omega^1_\varepsilon.$$

□

This consideration also gives the following:

Proposition 4.2.3. *Let $K \in L^1_{loc}([0,\tau))$ for some $0 < \tau \leq 1$ and let A generate a K-convoluted semigroup $(S_K(t))_{t\in[0,\tau)}$. If K can be extended to a function K_1 in $L^1_{loc}([0,\infty))$ which satisfies (P1) so that its Laplace transform has the same estimates as in Theorem* 4.2.5*, then A generates S. Ōuchi's hyperfunction semigroup.*

We state now the assertion which naturally corresponds to Theorem 4.2.5. The asymptotic properties of $\tilde{K}$ are slightly different now.

Theorem 4.2.6. *Assume that for every $\varepsilon > 0$ there exist $C_\varepsilon > 0$ and $M_\varepsilon > 0$ so that $\Omega_\varepsilon \subset \rho(A)$ and that $||R(\lambda : A)|| \leq M_\varepsilon e^{\varepsilon|\lambda|},\ \lambda \in \Omega_\varepsilon$.*

(a) Assume that K is an exponentially bounded function with the following property for its Laplace transform: There exists $\varepsilon_0 > 0$ such that for every $\varepsilon > 0$ exists $T_\varepsilon > 0$ with

$$|\tilde{K}(\lambda)| \leq T_\varepsilon e^{-\varepsilon_0|\lambda|},\ \lambda \in \Omega_\varepsilon. \tag{4.6}$$

If $\tau > 0$ and $K_{|[0,\tau)} \neq 0$ ($K_{|[0,\tau)}$ is the restriction of K on $[0,\tau)$), then A generates a local K-semigroup on $[0,\tau)$.

(b) Assume that K is an exponentially bounded function, $\tau > 0$ and $K_{|[0,\tau)} \neq 0$. Assume that for every $\varepsilon > 0$ there exist $T_\varepsilon > 0$ and $\varepsilon_0 \in (\varepsilon(1+\tau),\ \infty)$ such that (4.6) *holds. Then A generates a local K-semigroup on $[0,\tau)$.*

Proof. (a) Suppose that $\tau > 0$ and $K_{|[0,\tau)} \neq 0$. Choose a $\varepsilon \in (0,1)$ with $\frac{\varepsilon_0}{\varepsilon} - 1 > \tau$. Let $\Gamma_\varepsilon = \partial(\Omega_\varepsilon)$ be upwards oriented. Define, for $t \in [0, \frac{\varepsilon_0}{\varepsilon} - 1)$,

$$S_K(t) := \frac{1}{2\pi i} \int\limits_{\Gamma_\varepsilon} e^{\lambda t} \tilde{K}(\lambda) R(\lambda : A) d\lambda.$$

Then prescribed assumptions imply that $(S_K(t))_{t\in[0,\frac{\varepsilon_0}{\varepsilon}-1)}$ is a norm continuous operator family which commutes with A. The proof of (a) (for $t \in [0, \frac{\varepsilon_0}{\varepsilon} - 1)$) is almost completely contained in the proof of [70, Theorem 1.3.2]; note only that Cauchy formula (applied to a contour $\Gamma^R = \partial(\{\lambda \in \mathbb{C} : \; Re(\lambda) < R\} \cap \Omega_\varepsilon)$ and $f(\lambda) = \tilde{K}(\lambda)R(\lambda : A)$) implies

$$\int\limits_{\Gamma_\varepsilon} \tilde{K}(\lambda) R(\lambda : A) d\lambda = 0.$$

The assumption on K implies that $(S_K(t))_{t\in[0,\tau)}$ is a non-degenerate K-convoluted semigroup with the generator A, which ends the proof of (a). The same arguments work for (b). □

Connections of hyperfunction and ultradistribution semigroups with (local integrated) regularized semigroups seems to be more complicated. In this context, there is a example (essentially due to R. Beals [9]) which shows that there exists a densely defined operator A on the Hardy space $H^2(\mathbb{C}_+)$ which has the following properties:

1. A is the generator of S. Ōuchi's hyperfunction semigroup.

2. A is not a subgenerator of a local α-times integrated C-semigroup, for any injective $C \in L(H^2(\mathbb{C}_+))$ and $\alpha > 0$.

It is essentially due to R. Beals [9].

Example 4.2.1. [55] Let $\psi(t) = \frac{t}{\ln(t+1)}, \; t > 0, \; \psi(0) = 1$. Then ψ is nonnegative, continuous, concave function on $[0,\infty)$ with $\lim\limits_{t\to\infty} \psi(t) = \infty, \; \lim\limits_{t\to\infty} \frac{\psi(t)}{t} = 0$ and

$$\int\limits_1^\infty \frac{\psi(t)}{t^2} dt = \infty.$$

Note that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $\varepsilon t + C_\varepsilon \geq \psi(t), \; t \geq 0$. Let A be a closed, densely defined linear operator acting on $E := H^2(\mathbb{C}_+)$ such that:

$$\Omega(\psi) := \{\lambda \in \mathbb{C} : Re\lambda \geq \psi(|Im\lambda|)\} \subset \rho(A),$$

$$||R(\lambda : A)|| \leq \frac{M}{1 + Re\lambda}, \; \lambda \in \Omega(\psi),$$

and that, for every $\tau \in (0,\infty)$, there does not exist a solution of

$$\begin{cases} u \in C([0,\tau), [D(A)]) \cap C^1([0,\tau), E), \\ u'(t) = Au(t), \; t \in (0,\tau), \\ u(0) = x, \end{cases}$$

unless $x = 0$. The existence of such an operator is proved in [9, Theorem 2']. Since $\rho(A) \neq \emptyset$, it follows $\overline{\bigcap_{n\in\mathbb{N}} D(A^n)} = E$; see for instance [61]. Suppose that A is a subgenerator of a local k-times integrated C-semigroup on $[0,\tau)$, for some injective $C \in L(E)$ and $k \in \mathbb{N}$. Then the problem

$$\begin{cases} u \in C([0,\tau),[D(A)]) \cap C^1([0,\tau),E), \\ u'(t) = Au(t),\ t \in [0,\tau), \\ u(0) = x, \end{cases}$$

has a unique solution for all $x \in C(D(A^{k+1}))$ (cf. [65]). It follows $C(D(A^{k+1})) = \{0\}$ and this is a contradiction. Hence, A is not a subgenerator of any α-times integrated C-semigroup, for any injective $C \in L(H^2(\mathbb{C}_+))$ and $\alpha > 0$. Moreover, A does not generate a C-distribution semigroup ([54]). On the other hand, it is easy to see that $\Omega_\varepsilon \subset \Omega_\psi \subset \rho(A)$ and the growth rate of resolvent shows that A generates S. Ōuchi's hyperfunction semigroup. It can be easily proved that A is not the generator of any ultradistribution semigroup of $*$-class in the sense of [53] and we refer to J. Kisyński [45] for similar results within the theory of (degenerate) distribution semigroups.

It is clear that there exists an operator A which generates an entire C-regularized group but not a hyperfunction semigroup.

Chapter 5

On the Solution of the Cauchy Problem in the Weighted Spaces of Beurling Ultradistributions

In this chapter first we solve (1) in the space of Banach valued ultradistributions $\mathcal{D}'^{(s)}_{L^p}(0,T;E)$, i.e.

$$\langle \mathbf{u}'(t), \varphi(t)\rangle = A\langle \mathbf{u}(t), \varphi(t)\rangle + \langle \mathbf{f}(t), \varphi(t)\rangle,\ \forall \varphi \in \mathcal{D}^{(s)}_{L^q}(0,T),$$

where $A : D(A) \subseteq E \to E$ is closed operator which satisfies the Hille-Yosida condition

$$\|(\lambda-\omega)^k R(\lambda : A)^k\| \leq C,\ \text{for } \lambda > \omega,\ k \in \mathbb{Z}_+.$$

Then, after extensive preparations we give the two most important results of this chapter, Theorem 5.4.2 and Proposition 5.5.1.

This chapter is divided into five sections.
The Banach space $\mathcal{D}^{(s)}_{L^q,h}(U)$ and its dual $\mathcal{D}'^{(s)}_{L^p,h}(U)$ are explained in Section 5.1. Section 5.2 is devoted to the Beurling type test spaces $\mathcal{D}^{(s)}_{L^p}(U)$ and their correspon-ding duals. In Section 5.3 we consider the vector valued ultradistribution spaces $\mathcal{D}'^{(s)}_{L^p}(U;E)$ and $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$, where U is a bounded open subset of $\mathbb{R}^n$. The bound-ness of U is important since it implies nuclearity of $\mathcal{D}^{(s)}_{L^p}(U)$ and $\mathcal{D}'^{(s)}_{L^p}(U)$ which in turn will imply a very important kernel theorem when E is equal to $\mathcal{D}'^{(s)}_{L^p}(U)$. In the end of this section we are particulary interested in the spaces $\mathcal{D}'^{(s)}_{L^p}(U;E)$ when E is a Banach space. We start Section 5.4 by defining the Banach space $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ consisting of sequences of Bochner L^p functions with certain growth condition. In this abstract setting we define the Cauchy problem (1) and recall from [25] two types of solutions of (1). Then, using the proof in [25] we prove the existence of such solutions in $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ and use this to prove existence of solution of (1) in the space of Banach-valued ultradistributions $\mathcal{D}'^{(s)}_{L^p}(0,T;E)$. We

apply in Section 5.5 results of Section 5.4 for several important instances of A and E considered by Da Prato and Sinestrari in [25], but in our ultradistributional setting.

5.1 Banach spaces of Weighted Ultradistributions

5.1.1 Basic Banach Spaces

Let U be an open subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. Let $\mathcal{D}^s_{L^p,h}(U)$ be the space of all $\varphi \in \mathcal{C}^\infty(U)$ such that the norm $\left(\sum_{\alpha\in\mathbb{N}^n} \frac{h^{p|\alpha|}\,\|D^\alpha\varphi\|^p_{L^p(U)}}{\alpha!^{ps}}\right)^{1/p}$ is finite (with the obvious meaning when $p=\infty$). One can simply prove:

Lemma 5.1.1. *$\mathcal{D}^s_{L^p,h}(U)$ is a Banach space, when $1 \leq p \leq \infty$.*

Let$1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Let $\mathcal{D}^{(s)}_{L^p,h}(U)$ denotes the closure of $\mathcal{D}^{(s)}(U)$ in $\mathcal{D}^s_{L^p,h}(U)$. Denote by $\mathcal{D}'^{(s)}_{L^p,h}(U)$ the strong dual of $\mathcal{D}^{(s)}_{L^q,h}(U)$. Then, $\mathcal{D}'^{(s)}_{L^p,h}(U)$ is continuously injected in $\mathcal{D}'^{(s)}(U)$, for $1 \leq p \leq \infty$. Denote by $\mathcal{C}_0(U)$ the space of all continuous functions f on U such that for every $\varepsilon > 0$ there exists $K \subset\subset U$ such that $|f(x)| < \varepsilon$ when $x \in U\backslash K$.

Lemma 5.1.2. *Let $\varphi \in \mathcal{D}^{(s)}_{L^\infty,h}(U)$. Then for every $\varepsilon > 0$ there exist $K \subset\subset U$ and $k \in \mathbb{Z}_+$ such that*

$$\sup_{\alpha\in\mathbb{N}^d}\sup_{x\in U\backslash K} \frac{h^{|\alpha|}\,|D^\alpha\varphi(x)|}{\alpha!^s} \leq \varepsilon \ \textit{and}\ \sup_{|\alpha|\geq k} \frac{h^{|\alpha|}\,\|D^\alpha\varphi\|_{L^\infty(U)}}{\alpha!^s} \leq \varepsilon.$$

5.1.2 Duals of Banach spaces

The main goal in this subsection is to give a representation of the elements of $\mathcal{D}'^{(s)}_{L^p,h}(U)$, $1 \leq p \leq \infty$. In order to do that, first we will construct a Banach space which will contain $\mathcal{D}^{(s)}_{L^p,h}(U)$ as a closed subspace. It is worth to note that the main idea of this constructions is due to Komatsu [48].

For $1 \leq p < \infty$ define

$$Y_{h,L^p} = \left\{(\psi_\alpha)_{\alpha\in\mathbb{N}^n} \,\middle|\, \psi_\alpha \in L^p(U),\ \|(\psi_\alpha)_\alpha\|_{Y_{h,L^p}} = \right.$$

$$\left. = \left(\sum_{\alpha\in\mathbb{N}^n} \frac{h^{p|\alpha|}\,\|\psi_\alpha\|^p_{L^p(U)}}{\alpha!^{ps}}\right)^{1/p} < \infty\right\}.$$

Then one easily verifies that Y_{h,L^p} is a Banach space, with the norm $\|\cdot\|_{Y_{h,L^p}}$, for $1 \leq p < \infty$. Let $p = \infty$. Define

$$Y_{h,L^\infty} = \left\{ (\psi_\alpha)_{\alpha\in\mathbb{N}^n} \middle| \psi_\alpha \in \mathcal{C}_0(U),\ \lim_{|\alpha|\to\infty} \frac{h^{|\alpha|}\,\|\psi_\alpha\|_{L^\infty(U)}}{\alpha!^s} = 0 \right\},$$

with the norm $\|(\psi_\alpha)_\alpha\|_{Y_{h,L^\infty}} = \sup_{\alpha\in\mathbb{N}^n} \frac{h^{|\alpha|}}{\alpha!^s}\|\psi_\alpha\|_{L^\infty(U)}$. One easily verifies that it is a Banach space.

Let $\tilde{U}$ be the disjoint union of countable number of copies of U, one for each $\alpha \in \mathbb{N}^n$, i.e. $\tilde{U} = \bigsqcup_{\alpha\in\mathbb{N}^n} U_\alpha$, where $U_\alpha = U$. Equip $\tilde{U}$ with the disjoint union topology. Then $\tilde{U}$ is Hausdorff locally compact space. Moreover every open set in $\tilde{U}$ is σ-compact. For each $1 \leq p < \infty$, one can define a Borel measure μ_p on $\tilde{U}$ by $\mu_p(E) = \sum_\alpha \frac{h^{|\alpha|p}}{\alpha!^{ps}}|E \cap U_\alpha|$, for E a Borel subset of $\tilde{U}$, where $|E\cap U_\alpha|$ is the Lebesgue measure of $E \cap U_\alpha$. It is obviously locally finite, σ-finite and $\mu(K) < \infty$ for every compact subset K of $\tilde{U}$. By the properties of $\tilde{U}$ described above, μ_p is regular (both inner and outer regular). We obtained that μ_p is a Radon measure. It follows that Y_{h,L^p} is exactly $L^p(\tilde{U}, \mu_p)$, for $1 \leq p < \infty$. In particular, Y_{h,L^p} is a reflexive (B)-space for $1 < p < \infty$. For $p = \infty$, we will prove that Y_{h,L^∞} is isomorphic to $\mathcal{C}_0(\tilde{U})$. For $\psi \in \mathcal{C}_0(\tilde{U})$ denote by ψ_α the restriction of ψ to U_α. By the definition of $\tilde{U}$, K is compact subset of $\tilde{U}$ if and only if $K \cap U_\alpha \neq \emptyset$ for only finitely many $\alpha \in \mathbb{N}^n$ and for those α, $K \cap U_\alpha$ is compact subset of U_α. Now, one easily verifies that $\psi_\alpha \in \mathcal{C}_0(U)$ and $\lim_{|\alpha|\to\infty} \|\psi_\alpha\|_{L^\infty(U)} = 0$. Moreover, if $\psi_\alpha \in \mathcal{C}_0(U)$, $\alpha \in \mathbb{N}^n$, are such that $\lim_{|\alpha|\to\infty} \|\psi_\alpha\|_{L^\infty(U)} = 0$ then the function ψ on $\tilde{U}$, defined by $\psi(x) = \psi_\alpha(x)$, when $x \in U_\alpha$ is an element of $\mathcal{C}_0(\tilde{U})$. We obtain that

$$\mathcal{C}_0(\tilde{U}) = \left\{ (\psi_\alpha)_{\alpha\in\mathbb{N}^n} \middle| \psi_\alpha \in \mathcal{C}_0(U),\ \forall\alpha\in\mathbb{N}^n,\ \lim_{|\alpha|\to\infty} \|\psi_\alpha\|_{L^\infty(U)} = 0 \right\}.$$

Observe that the mapping $(\psi_\alpha)_{\alpha\in\mathbb{N}^n} \mapsto (\tilde{\psi}_\alpha)_{\alpha\in\mathbb{N}^n}$, where $\tilde{\psi}_\alpha = \frac{h^{|\alpha|}}{\alpha!^s}\psi_\alpha$, is an isometry from Y_{h,L^∞} onto $\mathcal{C}_0(\tilde{U})$. For the purpose of the next proposition we will denote by ι the inverse mapping of this isometry, i.e. $\iota : \mathcal{C}_0(\tilde{U}) \to Y_{h,L^\infty}$.

Note that $\mathcal{D}^{(s)}_{L^p,h}(U)$ can be identified with a closed subspace of Y_{h,L^p} by the mapping $\varphi \mapsto ((-D)^\alpha\varphi)_{\alpha\in\mathbb{N}^n}$. This is obvious for $1 \leq p < \infty$ and for $p = \infty$ it follows from Lemma 5.1.2. Since Y_{h,L^p} is reflexive for $1 < p < \infty$ so is $\mathcal{D}^{(s)}_{L^p,h}(U)$ as a closed subspace of a reflexive Banach space.

Observe that spaces $L^p(U)$, for $1 \leq p \leq \infty$, resp, $(\mathcal{C}_0(U))'$, are continuously injected into $\mathcal{D}'^{(s)}_{L^p,h}(U)$, resp. $\mathcal{D}'^{(s)}_{L^1,h}(U)$. For $\alpha \in \mathbb{N}^n$ and $F \in L^p(U)$, resp. $F \in (\mathcal{C}_0(U))'$, we define $D^\alpha F \in \mathcal{D}'^{(s)}_{L^p,h}(U)$, resp. $D^\alpha F \in \mathcal{D}'^{(s)}_{L^1,h}(U)$, by

$$\langle D^\alpha F, \varphi\rangle \ = \ \int_U F(x)(-D)^\alpha\varphi(x)dx,\ \varphi \in \mathcal{D}^{(s)}_{L^q,h}(U),\ \text{resp.}$$

$$\langle D^\alpha F, \varphi\rangle = \int_U (-D)^\alpha \varphi(x) dF,\ \varphi \in \mathcal{D}^{(s)}_{L^\infty,h}(U).$$

It is easy to verify that $D^\alpha F$ is well defined element of $\mathcal{D}'^{(s)}_{L^p,h}(U)$, resp. $\mathcal{D}'^{(s)}_{L^1,h}(U)$, and in fact it is equal to its ultradistributional derivative when we regard F as an element of $\mathcal{D}'^{(s)}(U)$.

Proposition 5.1.1. *Let $1 < p \le \infty$. For every $T \in \mathcal{D}'^{(s)}_{L^p,h}(U)$, there exist $C > 0$ and $F_\alpha \in L^p(U)$, $\alpha \in \mathbb{N}^n$, such that*

$$\left(\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|F_\alpha\|^p_{L^p(U)}\right)^{1/p} \le C \text{ and } T = \sum_{|\alpha|=0}^{\infty} D^\alpha F_\alpha. \tag{5.1}$$

When $p = 1$, for every $T \in \mathcal{D}'^{(s)}_{L^1,h}(U)$, there exist $C > 0$ and Radon measures $F_\alpha \in (\mathcal{C}_0(U))'$, $\alpha \in \mathbb{N}^n$, such that

$$\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{s}}{h^{|\alpha|}} \|F_\alpha\|_{(\mathcal{C}_0(U))'} \le C \text{ and } T = \sum_{|\alpha|=0}^{\infty} D^\alpha F_\alpha. \tag{5.2}$$

Moreover, if B is a bounded subset of $\mathcal{D}'^{(s)}_{L^p,h}(U)$, then there exists $C > 0$ independent of $T \in B$ and for each $T \in B$ there exist $F_\alpha \in L^p(U)$, $\alpha \in \mathbb{N}^n$, for $1 < p \le \infty$, resp. $F_\alpha \in (\mathcal{C}_0(U))'$, $\alpha \in \mathbb{N}^n$, for $p = 1$, such that (5.1), resp. (5.2), holds.

If $F_\alpha \in L^p(U)$, $\alpha \in \mathbb{N}^n$, for $1 < p \le \infty$, resp. $F_\alpha \in (\mathcal{C}_0(U))'$, $\alpha \in \mathbb{N}^n$, for $p = 1$, are such that $\left(\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|F_\alpha\|^p_{L^p(U)}\right)^{1/p} < \infty$, for $1 < p \le \infty$, resp. $\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{s}}{h^{|\alpha|}} \|F_\alpha\|_{(\mathcal{C}_0(U))'} < \infty$, for $p = 1$, then the series $\sum_{|\alpha|=0}^{\infty} D^\alpha F_\alpha$ converges absolutely in $\mathcal{D}'^{(s)}_{L^p,h}(U)$, resp. $\mathcal{D}'^{(s)}_{L^1,h}(U)$.

Proof. Let Y_{h,L^q} be as in the above discussion. Extend T by the Hahn-Banach theorem to a continuous functional on Y_{h,L^q} and denote it again by T, for $1 \le q \le \infty$. For $q = \infty$, $\tilde{T} = T \circ \iota$ is a functional on $\mathcal{C}_0(\tilde{U})$. Then, for $1 < p \le \infty$, there exists $g \in L^p(\tilde{U}, \mu_q)$ such that $T\left((\psi_\alpha)_{\alpha\in\mathbb{N}^n}\right) = \int_{\tilde{U}} (\psi_\alpha)_{\alpha\in\mathbb{N}^n} g d\mu_q$, $(\psi_\alpha)_{\alpha\in\mathbb{N}^n} \in Y_{h,L^q}$. For $p = 1$, there exists $g \in \left(\mathcal{C}_0(\tilde{U})\right)'$ such that $\tilde{T}(\psi) = \int_{\tilde{U}} \psi dg$, for $\psi \in \mathcal{C}_0(\tilde{U})$. Hence, for $(\psi_\alpha)_{\alpha\in\mathbb{N}^n} \in Y_{h,L^\infty}$, we have

$$T\left((\psi_\alpha)_{\alpha\in\mathbb{N}^n}\right) = \tilde{T}\left((\tilde{\psi}_\alpha)_{\alpha\in\mathbb{N}^n}\right) = \int_{\tilde{U}} (\tilde{\psi}_\alpha)_{\alpha\in\mathbb{N}^n} dg,$$

where $(\tilde{\psi}_\alpha)_\alpha = \iota^{-1}\left((\psi_\alpha)_\alpha\right) = \left(\frac{h^{|\alpha|}}{\alpha!^s}\psi_\alpha\right)_\alpha$. Put $F_\alpha = \frac{h^{|\alpha|q}}{\alpha!^{qs}} g_{|U_\alpha}$, for $1 \le q < \infty$. For $q = \infty$, put $F_\alpha = \frac{h^{|\alpha|}}{\alpha!^s} g_{|U_\alpha}$. Then $F_\alpha \in L^p(U)$, for $1 \le q < \infty$, respectively

$F_\alpha \in (\mathcal{C}_0(U))'$ for $q = \infty$. Moreover, for $1 < q < \infty$,

$$\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|F_\alpha\|^p_{L^p(U)} = \sum_{\alpha\in\mathbb{N}^n} \frac{h^{|\alpha|q}}{\alpha!^{qs}} \left\|g_{|U_\alpha}\right\|^p_{L^p(U)} = \|g\|^p_{L^p(\tilde{U},\mu_q)} < \infty.$$

Also, it is easy to verify that, for $q = 1$, $\sup_\alpha \dfrac{\alpha!^s}{h^{|\alpha|}} \|F_\alpha\|_{L^\infty(U)} = \|g\|_{L^\infty(\tilde{U},\mu_1)} < \infty$. For $q = \infty$ we have

$$\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^s}{h^{|\alpha|}} \|F_\alpha\|_{(\mathcal{C}_0(U))'} = \sum_{\alpha\in\mathbb{N}^n} \left\|g_{|U_\alpha}\right\|_{(\mathcal{C}_0(U))'} = \|g\|_{\left(\mathcal{C}_0(\tilde{U})\right)'} < \infty,$$

where in the second equality we used that $\left\|g_{|U_\alpha}\right\|_{(\mathcal{C}_0(U))'} = \left|g_{|U_\alpha}\right|(U_\alpha) = |g|(U_\alpha)$ (we denote by $|g|$ the total variation of the measure g and similarly for $g_{|U_\alpha}$). Moreover

$$T\left((\psi)_{\alpha\in\mathbb{N}^n}\right) = \sum_{\alpha\in\mathbb{N}^n} \int_U \psi_\alpha(x) F_\alpha(x)dx,$$

for $1 \leq q < \infty$. For $q = \infty$ we have

$$T\left((\psi_\alpha)_{\alpha\in\mathbb{N}^n}\right) = \int_{\tilde{U}} (\tilde{\psi}_\alpha)_{\alpha\in\mathbb{N}^n} dg = \sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^s}{h^{|\alpha|}} \int_U \tilde{\psi}_\alpha dF_\alpha = \sum_{\alpha\in\mathbb{N}^n} \int_U \psi_\alpha dF_\alpha.$$

So, for $1 \leq q < \infty$, if $\varphi \in \mathcal{D}^{(s)}_{L^q,h}(U)$, we obtain

$$\langle T, \varphi\rangle = \sum_{\alpha\in\mathbb{N}^n} \int_U (-D)^\alpha \varphi(x) F_\alpha(x)dx = \sum_{\alpha\in\mathbb{N}^n} \langle D^\alpha F_\alpha, \varphi\rangle.$$

Similarly, $\langle T, \varphi\rangle = \sum_\alpha \langle D^\alpha F_\alpha, \varphi\rangle$ when $q = \infty$. Moreover, by these calculations, it follows that for $1 \leq q < \infty$

$$\sum_{\alpha\in\mathbb{N}^n} |\langle D^\alpha F_\alpha, \varphi\rangle| \leq \left(\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|F_\alpha\|^p_{L^p(U)}\right)^{1/p} \left(\sum_{\alpha\in\mathbb{N}^n} \frac{h^{|\alpha|q} \|D^\alpha\varphi\|^q_{L^q(U)}}{\alpha!^{qs}}\right)^{1/q}.$$

Hence the partial sums of $\sum_\alpha D^\alpha F_\alpha$ converge absolutely in $\mathcal{D}'^{(s)}_{L^p,h}(U)$, when $1 < p \leq \infty$. When $p = 1$, the proof that the partial sums of $\sum_\alpha D^\alpha F_\alpha$ converge absolutely in $\mathcal{D}'^{(s)}_{L^1,h}(U)$ is similar and we omit it. If B is a bounded subset of $\mathcal{D}'^{(s)}_{L^p,h}(U)$, by the Hahn-Banach theorem it can be extended to a bounded set B_1 in Y'_{h,L^q}, for $1 \leq q < \infty$, resp. to a bounded set B_1 in $\mathcal{C}_0(\tilde{U})$ for $q = \infty$ (ι is an isometry). Hence, there exists $C > 0$ independent of $T \in B_1$ and for each $T \in B_1$ there exists $g \in L^p(\tilde{U}, \mu_q)$, for $1 < p \leq \infty$, resp. $g \in \left(\mathcal{C}_0(\tilde{U})\right)'$, for $p = 1$, such that $\|g\|_{L^p(\tilde{U})} \leq C$, resp. $\|g\|_{\left(\mathcal{C}_0(\tilde{U})\right)'} \leq C$. Defining F_α as above, one obtains (5.1), resp. (5.2), with the desired uniform estimate independent of $T \in B$.

The last part of the proposition is easy and it is omitted. □

5.2 Ultradistribution Spaces

5.2.1 Beurling type test spaces

For $1 \leq p \leq \infty$, define locally convex spaces $\mathcal{B}^{(s)}_{L^p}(U) = \lim\limits_{\substack{\longleftarrow \\ h\to\infty}} \mathcal{D}^{s}_{L^p,h}(U)$. Then $\mathcal{B}^{(s)}_{L^p}(U)$ is a Frechét space. Denote by $\mathcal{D}^{(s)}_{L^p}(U)$ the closure of $\mathcal{D}^{(s)}(U)$ in $\mathcal{B}^{(s)}_{L^p}(U)$ for $1 \leq p < \infty$ and $\dot{\mathcal{B}}^{(s)}(U)$ the closure of $\mathcal{D}^{(s)}(U)$ in $\mathcal{B}^{(s)}_{L^\infty}(U)$. Hence, when $U = \mathbb{R}^n$, these spaces coincide with the spaces $\mathcal{D}^{(s)}_{L^p}(\mathbb{R}^n)$, for $1 \leq p < \infty$, resp. $\dot{\mathcal{B}}^{(s)}$ defined in [84]. All of these spaces are Frechét spaces as well as $X_{L^p} = \lim\limits_{\substack{\longleftarrow \\ h\to\infty}} \mathcal{D}^{(s)}_{L^p,h}(U)$ $1 \leq p \leq \infty$.

Lemma 5.2.1. *Let X_{L^p} be as above and $1 \leq p \leq \infty$.*

i) $\mathcal{D}^{(s)}(U)$ is dense in X_{L^p}.

ii) X_{L^p} is a closed subspace of $\mathcal{B}^{(s)}_{L^p}(U)$ and the topology of X_{L^p} is the same as the induced one from $\mathcal{B}^{(s)}_{L^p}(U)$. Hence X_{L^p} and $\mathcal{D}^{(s)}_{L^p}(U)$, for $1 \leq p < \infty$, resp. X_{L^∞} and $\dot{\mathcal{B}}^{(s)}(U)$ when $p = \infty$, are isomorphic locally convex spaces.

Proof. Since $\mathcal{D}^{(s)}(U)$ is dense in each $\mathcal{D}^{(s)}_{L^p,h}(U)$ it follows that $\mathcal{D}^{(s)}(U) \subseteq X_{L^p}$ and it is dense in X_{L^p}. The proof of $i)$ is complete. To prove $ii)$ note that $X_{L^p} \subseteq \mathcal{B}^{(s)}_{L^p}(U)$. Let φ_j, $j \in \mathbb{N}$, be a sequence in X_{L^p} which converges to $\varphi \in \mathcal{B}^{(s)}_{L^p}(U)$ in the topology of $\mathcal{B}^{(s)}_{L^p}(U)$. Then φ_j converges to φ in $\mathcal{D}^{s}_{L^p,h}(U)$ for each h. But $\varphi_j \in \mathcal{D}^{(s)}_{L^p,h}(U)$, $j \in \mathbb{N}$ and $\mathcal{D}^{(s)}_{L^p,h}(U)$ is a closed subspace of $\mathcal{D}^{s}_{L^p,h}(U)$ with the same topology. It follows that $\varphi \in \mathcal{D}^{(s)}_{L^p,h}(U)$ and φ_j converges to φ in $\mathcal{D}^{(s)}_{L^p,h}(U)$ for each h. Hence $\varphi \in X_{L^p}$. Moreover, since the inclusion $X_{L^p} \to \mathcal{B}^{(s)}_{L^p}(U)$ is obviously continuous and X_{L^p} and $\mathcal{B}^{(s)}_{L^p}(U)$ are Frechét spaces and the image of X_{L^p} under the inclusion is closed subspace of $\mathcal{B}^{(s)}_{L^p}(U)$ by the open mapping theorem it follows that X_{L^p} is isomorphic with its image under this inclusion (isomorphic as l.c.s.). $\square$

By the above lemma we obtain that $\mathcal{D}^{(s)}_{L^p}(U) = \lim\limits_{\substack{\longleftarrow \\ h\to\infty}} \mathcal{D}^{(s)}_{L^p,h}(U)$, for $1 \leq p < \infty$ and $\dot{\mathcal{B}}^{(s)}(U) = \lim\limits_{\substack{\longleftarrow \\ h\to\infty}} \mathcal{D}^{(s)}_{L^\infty,h}(U)$, for $p = \infty$ and the projective limits are reduced. For $1 < p \leq \infty$, denote by $\mathcal{D}'^{(s)}_{L^p}(U)$ the strong dual of $\mathcal{D}^{(s)}_{L^q}(U)$. Denote by $\mathcal{D}'^{(s)}_{L^1}(U)$ the strong dual of $\dot{\mathcal{B}}^{(s)}(U)$. Since $\mathcal{D}^{(s)}(U)$ is continuously and densely injected into $\mathcal{D}^{(s)}_{L^q}(U)$, for $1 \leq q < \infty$ and into $\dot{\mathcal{B}}^{(s)}(U)$, $\mathcal{D}'^{(s)}_{L^p}(U)$ are continuously injected into $\mathcal{D}'^{(s)}(U)$, for $1 \leq p \leq \infty$. One easily verifies that ultradifferential operators of class (s) act continuously on $\mathcal{D}^{(s)}_{L^p}(U)$, for $1 \leq p < \infty$ and on $\dot{\mathcal{B}}^{(s)}(U)$. Hence they act continuously on $\mathcal{D}'^{(s)}_{L^p}(U)$, for $1 \leq p \leq \infty$. For $1 < p < \infty$, since all $\mathcal{D}^{(s)}_{L^p,h}(U)$ are reflexive (B)-spaces, the inclusion $\mathcal{D}^{(s)}_{L^p,h_2}(U) \to \mathcal{D}^{(s)}_{L^p,h_1}(U)$, for $h_2 > h_1$ is weakly compact mapping, hence $\mathcal{D}^{(s)}_{L^p}(U)$ is a (FS^*)-space, in particular it is reflexive.

From now on we suppose that U is bounded open set in $\mathbb{R}^n$.

Proposition 5.2.1. *Let $1 \leq p < \infty$ and $h_1 > h$. We have the continuous inclusions $\mathcal{D}^{(s)}_{L^\infty,h_1}(U) \to \mathcal{D}^{(s)}_{L^p,h}(U)$ and $\mathcal{D}^{(s)}_{L^p,2^s h}(U) \to \mathcal{D}^{(s)}_{L^\infty,h}(U)$. In particular, the spaces $\mathcal{D}^{(s)}_{L^p}(U)$, $1 \leq p < \infty$ and $\dot{\mathcal{B}}^{(s)}(U)$ are isomorphic among each other.*

Proof. Let $1 \leq p < \infty$ and $\varphi \in \mathcal{D}^{(s)}_{L^p,h}(U)$. It is obvious that for each $\alpha \in \mathbb{N}^n$, $D^\alpha \varphi \in W^{m,p}_0(U)$, for any $m \in \mathbb{Z}_+$. Hence, by the Sobolev imbedding theorem it follows that for each $\alpha \in \mathbb{N}^n$, $D^\alpha\varphi$ extends to a uniformly continuous function on $\overline{U}$. Now, let $\varphi \in \mathcal{D}^s_{L^\infty,h_1}(U)$. Then

$$\begin{aligned}
\left(\sum_{\alpha\in\mathbb{N}^n} \frac{h^{p|\alpha|}\,\|D^\alpha\varphi\|^p_{L^p(U)}}{\alpha!^{ps}}\right)^{1/p} &\leq |U|^{1/p}\left(\sum_{\alpha\in\mathbb{N}^n} \frac{h^{p|\alpha|}h_1^{p|\alpha|}\,\|D^\alpha\varphi\|^p_{L^\infty(U)}}{h_1^{p|\alpha|}\alpha!^{ps}}\right)^{1/p}\\
&\leq C|U|^{1/p}\sup_{\alpha\in\mathbb{N}^n}\frac{h_1^{|\alpha|}\,\|D^\alpha\varphi\|_{L^\infty(U)}}{\alpha!^s}.
\end{aligned}$$

Hence, the inclusion $\mathcal{D}^s_{L^\infty,h_1}(U) \to \mathcal{D}^s_{L^p,h}(U)$ is continuous. Moreover, if $\varphi \in \mathcal{D}^{(s)}_{L^\infty,h_1}(U)$, then there exist $\varphi_j \in \mathcal{D}^{(s)}(U)$, $j \in \mathbb{Z}_+$, such that $\varphi_j \to \varphi$, as $j \to \infty$, in $\mathcal{D}^s_{L^\infty,h_1}(U)$. But then $\varphi_j \to \varphi$, as $j \to \infty$, in $\mathcal{D}^s_{L^p,h}(U)$. Hence, $\mathcal{D}^{(s)}_{L^\infty,h_1}(U)$ is continuously injected into $\mathcal{D}^{(s)}_{L^p,h}(U)$. It follows that for each $\varphi \in \mathcal{D}^{(s)}_{L^\infty,h_1}(U)$, $\alpha \in \mathbb{N}^n$, $D^\alpha\varphi$ can be extended to a uniformly continuous function on $\overline{U}$. Let $\varphi \in \mathcal{D}^{(s)}_{L^p,2^s h}(U)$. Fix $m \in \mathbb{Z}_+$, such that $mp > n$. Denote by $C_1 = \max_{|\alpha|\leq m} \alpha!^s/h^{|\alpha|}$. By the Sobolev imbedding theorem we have

$$\begin{aligned}
\frac{h^{|\beta|}\|D^\beta\varphi\|_{L^\infty(U)}}{\beta!^s} &\leq C'\frac{h^{|\beta|}}{\beta!^s}\left(\sum_{|\alpha|\leq m}\|D^{\alpha+\beta}\varphi\|^p_{L^p(U)}\right)^{1/p}\\
&\leq C'\left(\sum_{|\alpha|\leq m}\frac{h^{(|\alpha|+|\beta|)p}\alpha!^{ps}}{\beta!^{ps}\alpha!^{ps}h^{|\alpha|p}}\|D^{\alpha+\beta}\varphi\|^p_{L^p(U)}\right)^{1/p}\\
&\leq C'C_1\left(\sum_{|\alpha|\leq m}\frac{(2^s h)^{(|\alpha|+|\beta|)p}}{(\alpha+\beta)!^{ps}}\|D^{\alpha+\beta}\varphi\|^p_{L^p(U)}\right)^{1/p}\\
&\leq C'C_1\left(\sum_{\gamma\in\mathbb{N}^n}\frac{(2^s h)^{|\gamma|p}}{\gamma!^{ps}}\|D^{\gamma}\varphi\|^p_{L^p(U)}\right)^{1/p}.
\end{aligned}$$

We obtain that $\mathcal{D}^{(s)}_{L^p,2^s h}(U)$ is continuously injected in $\mathcal{D}^s_{L^\infty,h}(U)$. Moreover, if $\varphi_j \in \mathcal{D}^{(s)}(U)$, $j \in \mathbb{Z}_+$, are such that $\varphi_j \to \varphi$, when $j \to \infty$, in $\mathcal{D}^{(s)}_{L^p,2^s h}(U)$, then $\varphi_j \to \varphi$, when $j \to \infty$, in $\mathcal{D}^s_{L^\infty,h}(U)$. Hence, $\mathcal{D}^{(s)}_{L^p,2^s h}(U)$ is continuously injected into $\mathcal{D}^{(s)}_{L^\infty,h}(U)$. □

Proposition 5.2.1 implies that, we no longer need to distinguish the spaces $\mathcal{D}^{(s)}_{L^p}(U)$ since they are all isomorphic to $\dot{\mathcal{B}}^{(s)}(U)$. Hence their duals are all isomorphic to $\mathcal{D}'^{(s)}_{L^1}(U)$.

Proposition 5.2.2. *Let U be bounded open subset of $\mathbb{R}^n$.*

i) Let $h>0$ be fixed. Every element φ of $\mathcal{D}^{(s)}_{L^p,h}(U)$ for $1\leq p\leq\infty$, can be extended to $\mathcal{C}^\infty$ function on $\mathbb{R}^n$ with support in $\overline{U}$. Moreover $\mathcal{D}^{(s)}_{L^\infty,h}(U)$ can be identified with a closed subspace of $\mathcal{D}^{s,h}_{\overline{U}}$;

ii) $\dot{\mathcal{B}}^{(s)}(U)$ can be identified with a closed subspace of $\mathcal{D}^{(s)}_{\overline{U}}$;

iii) $\dot{\mathcal{B}}^{(s)}(U)$ is a nuclear (FS)-spaces. Moreover, in the representation $\dot{\mathcal{B}}^{(s)}(U)=\varprojlim_{h\to\infty}\mathcal{D}^{(s)}_{L^\infty,h}(U)$, the linking inclusions in the projective limit $\mathcal{D}^{(s)}_{L^\infty,h_1}(U)\to\mathcal{D}^{(s)}_{L^\infty,h}(U)$ are compact for $h_1>h$.

Proof. To prove the first part of *i*), note that by Proposition 5.2.1, $\mathcal{D}^{(s)}_{L^p,h}(U)$ is continuously injected into $\mathcal{D}^{(s)}_{L^\infty,h/2^s}(U)$. Hence it is enough to prove it for $\mathcal{D}^{(s)}_{L^\infty,h}(U)$. Let $\varphi\in\mathcal{D}^{(s)}_{L^\infty,h}(U)$. Then there exist $\varphi_j\in\mathcal{D}^{(s)}(U)$, $j\in\mathbb{Z}_+$, such that $\varphi_j\to\varphi$, as $j\to\infty$ in $\mathcal{D}^{(s)}_{L^\infty,h}(U)$. So for $\varepsilon>0$ there exists $j_0\in\mathbb{Z}_+$ such that for $j,k\geq j_0$, $j,k\in\mathbb{Z}_+$, we have $\sup_{\alpha\in\mathbb{N}^n}\dfrac{h^{|\alpha|}\left\|D^\alpha\varphi_k-D^\alpha\varphi_j\right\|_{L^\infty(U)}}{\alpha!^s}\leq\varepsilon$. Since all φ_j, $j\in\mathbb{Z}_+$, have compact support in U and $\mathcal{D}^{(s)}(U)\subseteq\mathcal{D}^{s,h}_{\overline{U}}$ we obtain that

$$\sup_{\alpha\in\mathbb{N}^n}\frac{h^{|\alpha|}\left\|D^\alpha\varphi_k-D^\alpha\varphi_j\right\|_{L^\infty(\mathbb{R}^n)}}{\alpha!^s}\leq\varepsilon$$

for all $j,k\geq j_0$, $j,k\in\mathbb{Z}_+$. Hence, φ_j is a Cauchy sequence in the Banach space $\mathcal{D}^{s,h}_{\overline{U}}$ so it must converge to an element $\psi\in\mathcal{D}^{s,h}_{\overline{U}}$. Hence $\psi(x)=\varphi(x)$, when $x\in U$ and obviously $\psi(x)=0$ when $x\in\mathbb{R}^n\backslash U$ (since all φ_j, $j\in\mathbb{Z}_+$, have compact support in U). This proofs the first part of *i*). To prove the second part, consider the mapping $\varphi\mapsto\widetilde{\varphi}$, $\mathcal{D}^{(s)}_{L^\infty,h}(U)\to\mathcal{D}^{s,h}_{\overline{U}}$, where $\widetilde{\varphi}(x)=\varphi(x)$, when $x\in U$ and $\widetilde{\varphi}(x)=0$, when $x\in\mathbb{R}^n\backslash U$. By the above discussion, this is well defined mapping. Moreover, one easily sees that it is an isometry, which completes the proof of *i*). Observe that *ii*) follows from *i*) since $\dot{\mathcal{B}}^{(s)}(U)=\varprojlim_{h\to\infty}\mathcal{D}^{(s)}_{L^\infty,h}(U)$ and $\mathcal{D}^{(s)}_{\overline{U}}=\varprojlim_{h\to\infty}\mathcal{D}^{s,h}_{\overline{U}}$. The first part of *iii*) follows from *ii*) since $\dot{\mathcal{B}}^{(s)}(U)$ is a closed subspace of the nuclear (FS)-space $\mathcal{D}^{(s)}_{\overline{U}}$ (Komatsu in [48] proves the nuclearity of $\mathcal{D}^{(s)}_{\overline{U}}$ when $\overline{U}$ is regular compact set, but the proof is valid for general $\overline{U}$; the regularity of $\overline{U}$ is used by Komatsu [48] for the definition and nuclearity of $\mathcal{E}^{(s)}(\overline{U})$). For the second part, by Proposition 2.2 of [48] the inclusion $\mathcal{D}^{s,h_1}_{\overline{U}}\to\mathcal{D}^{s,h}_{\overline{U}}$ is compact. Since $\mathcal{D}^{(s)}_{L^\infty,h_1}(U)$, resp. $\mathcal{D}^{(s)}_{L^\infty,h}(U)$, is closed subspace of $\mathcal{D}^{s,h_1}_{\overline{U}}$, resp. $\mathcal{D}^{s,h}_{\overline{U}}$, one obtains the compactness of the inclusion under consideration. □

5.2.2 Weighted Beurling spaces of ultradistributions

Proposition 5.2.3. *Let $T \in \mathcal{D}'^{(s)}_{L^1}(U)$. For every $1 \leq p \leq \infty$ there exist $h, C > 0$ and $F_\alpha \in \mathcal{C}(\overline{U})$, $\alpha \in \mathbb{N}^n$, such that*

$$\left(\sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|F_\alpha\|^p_{L^\infty(U)}\right)^{1/p} \leq C \text{ and } T = \sum_{\alpha \in \mathbb{N}^n} D^\alpha F_\alpha, \tag{5.3}$$

where the last series converges absolutely in $\mathcal{D}'^{(s)}_{L^1}(U)$. Moreover, if B is a bounded subset of $\mathcal{D}'^{(s)}_{L^1}(U)$ and $1 \leq p \leq \infty$, then there exist $h, C > 0$ independent of $T \in B$ and for each $T \in B$ there exist $F_\alpha \in \mathcal{C}(\overline{U})$, $\alpha \in \mathbb{N}^n$, such that (5.3) holds.

Conversely, for $1 \leq p \leq \infty$, if $F_\alpha \in L^p(U)$, $\alpha \in \mathbb{N}^n$, are such that

$$\left(\sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|F_\alpha\|^p_{L^p}\right)^{1/p} < \infty$$

for some $h > 0$ then the series $\sum_{|\alpha|=0}^{\infty} D^\alpha F_\alpha$ converges absolutely in $\mathcal{D}'^{(s)}_{L^p,h}(U)$ and hence also in $\mathcal{D}'^{(s)}_{L^1}(U)$.

Proof. First the second part of the proposition will be proved. If $F_\alpha \in L^p(U)$, $\alpha \in \mathbb{N}^n$, are as above, the absolute convergence of $\sum_{|\alpha|=0}^{\infty} D^\alpha F_\alpha$ in $\mathcal{D}'^{(s)}_{L^p,h}(U)$ follows by Proposition 5.1.1 for $1 < p \leq \infty$ and can be easily verified for $p = 1$. By Proposition 5.2.1, $\dot{\mathcal{B}}^{(s)}(U)$ is continuously and densely injected into $\mathcal{D}^{(s)}_{L^q,h}(U)$, where q is the conjugate of p, i.e. $p^{-1} + q^{-1} = 1$ (the part about the denseness follows from the fact that $\mathcal{D}^{(s)}(U) \subseteq \dot{\mathcal{B}}^{(s)}(U)$ is dense in $\mathcal{D}^{(s)}_{L^q,h}(U)$). Hence $\mathcal{D}'^{(s)}_{L^p,h}(U)$ is continuously injected into $\mathcal{D}'^{(s)}_{L^1}(U)$ and one obtains that $\sum_{|\alpha|=0}^{\infty} D^\alpha F_\alpha$ converges absolutely in $\mathcal{D}'^{(s)}_{L^1}(U)$.

To prove the first part, fix $1 < p \leq \infty$ and let q to be the conjugate of p. Since $\dot{\mathcal{B}}^{(s)}(U) = \varprojlim_{h \to \infty} \mathcal{D}^{(s)}_{L^\infty,h}(U)$ and the projective limit is reduced with compact linking mappings (cf. Proposition 5.2.2), $\mathcal{D}'^{(s)}_{L^1}(U) = \varinjlim_{h \to \infty} \mathcal{D}'^{(s)}_{L^1,h}(U)$ as locally convex space, where the inductive limit is injective with compact linking mappings. If B is bounded subset of $\mathcal{D}'^{(s)}_{L^1}(U)$ there exists $h_1 > 0$ such that $B \subseteq \mathcal{D}'^{(s)}_{L^1,h}(U)$ and is bounded there. By Proposition 5.2.1, if take $h = 2^s h_1$, $\mathcal{D}^{(s)}_{L^q,h}(U)$ is continuously injected into $\mathcal{D}^{(s)}_{L^\infty,h_1}(U)$. Obviously, $\mathcal{D}^{(s)}_{L^q,h}(U)$ is dense in $\mathcal{D}^{(s)}_{L^\infty,h_1}(U)$ (since $\mathcal{D}^{(s)}(U)$ is). Hence, $\mathcal{D}'^{(s)}_{L^1,h_1}(U)$ is continuously injected into $\mathcal{D}'^{(s)}_{L^p,h}(U)$, so B is a bounded subset of $\mathcal{D}'^{(s)}_{L^p,h}(U)$. Now, by Proposition 5.1.1, for each $T \in B$ there exist $\tilde{F}_\alpha \in L^p(U)$, $\alpha \in \mathbb{N}^n$, such that

$$\left(\sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!^{ps}}{h^{p|\alpha|}} \|\tilde{F}_\alpha\|^p_{L^p(U)}\right)^{1/p} \leq C' \text{ and } T = \sum_{\alpha \in \mathbb{N}^n} D^\alpha \tilde{F}_\alpha$$

and the constant C' is the same for all $T \in B$. Let $L(x) \in \mathcal{C}(\mathbb{R}^n)$ be a fundamental solution of $\Delta^n L = \delta$ (Δ is the Laplacian). Define $G_\alpha(x) = \int_U L(x-y)\tilde{F}_\alpha(y)dy$, $\alpha \in \mathbb{N}^n$. Obviously $G_\alpha \in \mathcal{C}(\overline{U})$, $\alpha \in \mathbb{N}^n$ and $\|G_\alpha\|_{L^\infty(U)} \leq C_1\|\tilde{F}_\alpha\|_{L^p(U)}$, for all $\alpha \in \mathbb{N}^n$. Hence $\left(\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{ps}}{h^{p|\alpha|}}\|G_\alpha\|^p_{L^\infty(U)}\right)^{1/p} \leq C_2$ and C_2 is independent of $T \in B$. Let $\Delta^n = \sum_\beta c_\beta D^\beta$ and define $F_\alpha = \sum_{\beta\leq\alpha} c_\beta G_{\alpha-\beta}$, $\alpha \in \mathbb{N}^n$. The obviously $F_\alpha \in \mathcal{C}(\overline{U})$ for all $\alpha \in \mathbb{N}^n$. Note that $c_\beta \neq 0$ only for finitely many $\beta \in \mathbb{N}^n$. Put $C_3 = \sum_\beta \frac{\beta!^s}{h^{|\beta|}}|c_\beta|$. Then

$$\begin{aligned}
&\left(\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{ps}}{(2^{s+1}h)^{|\alpha|p}}\|F_\alpha\|^p_{L^\infty(U)}\right)^{1/p}\\
&\quad\leq \left(\sum_{\alpha\in\mathbb{N}^n} \frac{1}{2^{|\alpha|p}}\left(\sum_{\beta\leq\alpha} \frac{(\alpha-\beta)!^s\beta!^s}{h^{|\alpha|-|\beta|}h^{|\beta|}}|c_\beta|\,\|G_{\alpha-\beta}\|_{L^\infty(U)}\right)^p\right)^{1/p}\\
&\quad\leq C_2C_3\left(\sum_{\alpha\in\mathbb{N}^n}\frac{1}{2^{|\alpha|p}}\right)^{1/p}
\end{aligned}$$

and the last is independent of $T \in B$. Now one easily obtains that $T = \sum_\alpha D^\alpha F_\alpha$ which completes the first part of the proposition when $1 < p \leq \infty$. Note that the case $p = 1$ follows from this for any $\tilde{h} > h$. □

5.3 Vector-valued Spaces of Ultradistributions

Let now E be a complete locally convex space. As we saw above, $\mathcal{D}'^{(s)}_{L^1}(U)$ and $\mathcal{D}'^{(s)}_{L^p,h}(U)$, $1 \leq p \leq \infty$, are continuously injected in $\mathcal{D}'^{(s)}(U)$. Following Komatsu [50], (see also [63]) we define the spaces $\mathcal{D}'^{(s)}_{L^1}(U;E)$ and $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$, $1 \leq p \leq \infty$, of E-valued ultradistributions of type $\mathcal{D}'^{(s)}_{L^1}(U)$ and $\mathcal{D}'^{(s)}_{L^p,h}(U)$ respectively, as

$$\mathcal{D}'^{(s)}_{L^1}(U;E) = \mathcal{D}'^{(s)}_{L^1}(U)\varepsilon E = L_\epsilon\left(\left(\mathcal{D}'^{(s)}_{L^1}(U)\right)'_c, E\right), \text{ resp.} \tag{5.4}$$

$$\mathcal{D}'^{(s)}_{L^p,h}(U;E) = \mathcal{D}'^{(s)}_{L^p,h}(U)\varepsilon E = L_\epsilon\left(\left(\mathcal{D}'^{(s)}_{L^p,h}(U)\right)'_c, E\right). \tag{5.5}$$

The subindex c stands for the topology of compact convex circled convergence on the dual of $\mathcal{D}'^{(s)}_{L^1}(U)$, resp. $\mathcal{D}'^{(s)}_{L^p,h}(U)$, from the duality

$$\left\langle \mathcal{D}'^{(s)}_{L^1}(U), \left(\mathcal{D}'^{(s)}_{L^1}(U)\right)'\right\rangle, \text{ resp. } \left\langle \mathcal{D}'^{(s)}_{L^p,h}(U), \left(\mathcal{D}'^{(s)}_{L^p,h}(U)\right)'\right\rangle.$$

If denote by ι, resp. ι_p, the inclusion $\mathcal{D}'^{(s)}_{L^1}(U) \to \mathcal{D}'^{(s)}(U)$, resp. $\mathcal{D}'^{(s)}_{L^p,h}(U) \to \mathcal{D}'^{(s)}(U)$, then $\mathcal{D}'^{(s)}_{L^1}(U;E)$, resp. $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$, is continuously injected into $\mathcal{D}'^{(s)}(U;E) = \mathcal{D}'^{(s)}(U)\varepsilon E = L_b\left(\mathcal{D}^{(s)}(U), E\right)$ by the mapping $\iota\,\varepsilon\,\mathrm{Id}$, resp. $\iota_p\,\varepsilon\,\mathrm{Id}$ (cf. [50]). In [98] is proved that when both spaces are complete. The same holds for their ε tensor product. Hence, $\mathcal{D}'^{(s)}_{L^1}(U;E)$ and $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$ are complete. Since $\mathcal{D}'^{(s)}_{L^1}(U)$ and $\mathcal{D}'^{(s)}_{L^p,h}(U)$ are barreled (the former is a (DFS)-space as the strong dual of a (FS)-space, hence barreled), every bounded subset of $\left(\mathcal{D}'^{(s)}_{L^1}(U)\right)'_c$ or $\left(\mathcal{D}'^{(s)}_{L^p,h}(U)\right)'_c$ is equicontinuous (and vice versa). Hence, the ϵ topology on the right hand sides of (5.4) and (5.5) is the same as the topology of bounded convergence. Moreover, since $\dot{\mathcal{B}}^{(s)}(U)$ is a (FS)-space and $\mathcal{D}'^{(s)}_{L^1}(U)$ is a (DFS)-space they are both Montel spaces. Hence $\mathcal{D}'^{(s)}_{L^1}(U;E) = L_b\left(\dot{\mathcal{B}}^{(s)}(U), E\right)$. For $1 < p < \infty$, $\mathcal{D}'^{(s)}_{L^p,h}(U;E) = L_b\left(\mathcal{D}^{(s)}_{L^q,h}(U)_c, E\right)$, since $\mathcal{D}^{(s)}_{L^q,h}(U)$ are reflexive, where $\mathcal{D}^{(s)}_{L^q,h}(U)_c$ is the space $\mathcal{D}^{(s)}_{L^q,h}(U)$ equipped with topology of compact convex circled convergence from the duality $\left\langle \mathcal{D}^{(s)}_{L^q,h}(U), \mathcal{D}'^{(s)}_{L^p,h}(U)\right\rangle$. Since $\dot{\mathcal{B}}^{(s)}(U)$ is a nuclear (FS)-space (by Proposition 5.2.2) $\mathcal{D}'^{(s)}_{L^1}(U)$ is a nuclear (DFS)-space and hence it satisfies the weak approximation property by Corollary 2 pg.110 of [91] (for the definition of the weak approximation property see [98]). Hence Proposition 1.4 of [50] implies $\mathcal{D}'^{(s)}_{L^1}(U;E) = \mathcal{D}'^{(s)}_{L^1}(U)\varepsilon E \cong \mathcal{D}'^{(s)}_{L^1}(U)\hat{\otimes} E$ where the π and the ϵ topologies coincide on $\mathcal{D}'^{(s)}_{L^1}(U)\hat{\otimes} E$ since $\mathcal{D}'^{(s)}_{L^1}(U)$ is nuclear. Later we will need the following kernel theorem.

Theorem 5.3.1. *Let U_1 and U_2 be bounded open sets in $\mathbb{R}^{n_1}_x$ and $\mathbb{R}^{n_2}_y$ respectively. Then we have the following canonical isomorphisms of locally convex spaces*

i) $\dot{\mathcal{B}}^{(s)}(U_1)\hat{\otimes}\dot{\mathcal{B}}^{(s)}(U_2) \cong \dot{\mathcal{B}}^{(s)}(U_1 \times U_2)$.

ii) $\mathcal{D}'^{(s)}_{L^1}(U_1)\hat{\otimes}\mathcal{D}'^{(s)}_{L^1}(U_2) \cong \mathcal{D}'^{(s)}_{L^1}(U_1 \times U_2) \cong \mathcal{D}'^{(s)}_{L^1}(U_1)\varepsilon\mathcal{D}'^{(s)}_{L^1}(U_2)$
$\cong L_b\left(\dot{\mathcal{B}}^{(s)}(U_1), \mathcal{D}'^{(s)}_{L^1}(U_2)\right) \cong \mathcal{D}'^{(s)}_{L^1}\left(U_1; \mathcal{D}'^{(s)}_{L^1}(U_2)\right) \cong \mathcal{D}'^{(s)}_{L^1}\left(U_2; \mathcal{D}'^{(s)}_{L^1}(U_1)\right)$.

Proof. First we prove *i*). Since $\dot{\mathcal{B}}^{(s)}(U_1)$ and $\dot{\mathcal{B}}^{(s)}(U_2)$ are nuclear (Proposition 5.2.2) the π and the ϵ topologies coincide on $\dot{\mathcal{B}}^{(s)}(U_1) \otimes \dot{\mathcal{B}}^{(s)}(U_2)$. Moreover, one easily verifies that $\dot{\mathcal{B}}^{(s)}(U_1)\otimes\dot{\mathcal{B}}^{(s)}(U_2)$ can be regarded as a subspace of $\dot{\mathcal{B}}^{(s)}(U_1\times U_2)$ by identifying $\varphi \otimes \psi$ with $\varphi(x)\psi(y)$. Since $\mathcal{D}^{(s)}(U_1 \times U_2)$ is continuously and densely injected in $\dot{\mathcal{B}}^{(s)}(U_1 \times U_2)$ and $\mathcal{D}^{(s)}(U_1) \otimes \mathcal{D}^{(s)}(U_2)$ is a dense subspace of $\mathcal{D}^{(s)}(U_1 \times U_2)$ (see Theorem 2.1 of [49]) we obtain that $\mathcal{D}^{(s)}(U_1) \otimes \mathcal{D}^{(s)}(U_2)$ and hence $\dot{\mathcal{B}}^{(s)}(U_1) \otimes \dot{\mathcal{B}}^{(s)}(U_2)$ is a dense subspace of $\dot{\mathcal{B}}^{(s)}(U_1 \times U_2)$. Observe that the bilinear mapping $(\varphi, \psi) \mapsto \varphi(x)\psi(y)$, $\dot{\mathcal{B}}^{(s)}(U_1) \times \dot{\mathcal{B}}^{(s)}(U_2) \to \dot{\mathcal{B}}^{(s)}(U_1 \times U_2)$ is continuous (it is separately continuous and hence continuous since all spaces under consideration are Frechét spaces). One obtains that the π topology on $\dot{\mathcal{B}}^{(s)}(U_1) \otimes \dot{\mathcal{B}}^{(s)}(U_2)$ is stronger than the induced one by $\dot{\mathcal{B}}^{(s)}(U_1 \times U_2)$. Hence, to obtain $\dot{\mathcal{B}}^{(s)}(U_1)\hat{\otimes}\dot{\mathcal{B}}^{(s)}(U_2) \cong \dot{\mathcal{B}}^{(s)}(U_1\times U_2)$, it is enough to prove that the ϵ topology on $\dot{\mathcal{B}}^{(s)}(U_1) \otimes \dot{\mathcal{B}}^{(s)}(U_2)$ is weaker than the induced one by $\dot{\mathcal{B}}^{(s)}(U_1 \times U_2)$. Let A'

and B' be equicontinuous subsets of $\mathcal{D}'^{(s)}_{L^1}(U_1)$ and $\mathcal{D}'^{(s)}_{L^1}(U_2)$ respectively. Hence, there exist $h, C > 0$ such that

$$\sup_{T\in A'} |\langle T,\varphi\rangle| \leq C \sup_{x,\alpha} \frac{h^{|\alpha|}\,|D^\alpha\varphi(x)|}{\alpha!^s} \text{ and } \sup_{S\in B'} |\langle S,\psi\rangle| \leq C \sup_{y,\beta} \frac{h^{|\beta|}\,\left|D^\beta\psi(y)\right|}{\beta!^s}$$

Then for $\chi \in \dot{\mathcal{B}}^{(s)}(U_1) \otimes \dot{\mathcal{B}}^{(s)}(U_2)$, $T \in A'$ and $S \in B'$,

$$\begin{aligned}
&|\langle T(x) \otimes S(y), \chi(x,y)\rangle| \\
&= |\langle T(x), \langle S(y), \chi(x,y)\rangle\rangle| \leq C \sup_{x,\alpha} \frac{h^{|\alpha|}\,|\langle S(y), D^\alpha_x \chi(x,y)\rangle|}{\alpha!^s} \\
&\leq C^2 \sup_{x,y,\alpha,\beta} \frac{h^{|\alpha|+|\beta|}\left|D^\alpha_x D^\beta_y \chi(x,y)\right|}{\alpha!^s\beta!^s} \leq C^2 \sup_{x,y,\alpha,\beta} \frac{(2^s h)^{|\alpha|+|\beta|}\left|D^\alpha_x D^\beta_y \chi(x,y)\right|}{(\alpha+\beta)!^s}.
\end{aligned}$$

Hence, we obtain that the ϵ topology is weaker than the topology induced by $\dot{\mathcal{B}}^{(s)}(U_1 \times U_2)$.

ii) Since $\dot{\mathcal{B}}^{(s)}(U_1)$ and $\dot{\mathcal{B}}^{(s)}(U_2)$ are nuclear (FS)-spaces (by Proposition 5.2.2), $\mathcal{D}'^{(s)}_{L^1}(U_1)$ and $\mathcal{D}'^{(s)}_{L^1}(U_2)$ are nuclear (DFS)-spaces. Hence the π and the ϵ topologies on the tensor product $\mathcal{D}'^{(s)}_{L^1}(U_1) \otimes \mathcal{D}'^{(s)}_{L^1}(U_2)$ coincide and by *i*) (using the fact that $\mathcal{D}'^{(s)}_{L^1}(U_1)$ and $\mathcal{D}'^{(s)}_{L^1}(U_2)$ are nuclear (DFS)-spaces) we have $\mathcal{D}'^{(s)}_{L^1}(U_1 \times U_2) \cong \left(\dot{\mathcal{B}}^{(s)}(U_1)\hat{\otimes}\dot{\mathcal{B}}^{(s)}(U_2)\right)' \cong \mathcal{D}'^{(s)}_{L^1}(U_1)\hat{\otimes}\mathcal{D}'^{(s)}_{L^1}(U_2)$. Other isomorphisms in the assertion on U follow by the discussion before the theorem. □

5.3.1 Banach-valued ultradistributions

Let now E be a Banach space and denote by $L^p(U;E)$, $1 \leq p \leq \infty$, the Bochner L^p space. If $\varphi \in \mathcal{C}_{L^\infty}(U)$ (the space of bounded continuous functions on U) and $\mathbf{F} \in L^1(U;E)$ then one easily verifies that $\varphi\mathbf{F} \in L^1(U;E)$. We will need the following lemma.

Lemma 5.3.1. *(variant of du Bois-Reymond lemma for Bochner integrable functions) Let* $\mathbf{F} \in L^1(U;E)$ *is such that* $\int_U \mathbf{F}(x)\varphi(x)dx = 0$ *for all* $\varphi \in \mathcal{D}^{(s)}(U)$. *Then* $\mathbf{F}(x) = 0$ *a.e.*

Proof. Observe first that for each $e' \in E'$ and $\varphi \in \mathcal{D}^{(s)}(U)$, we have

$$\int_U e' \circ \mathbf{F}(x)\varphi(x)dx = e'\left(\int_U \mathbf{F}(x)\varphi(x)dx\right) = 0.$$

Since $\mathcal{D}^{(s)}(U)$ is dense in $\mathcal{D}(U)$, by the du Bois-Reymond lemma it follows that $e' \circ \mathbf{F} = 0$ a.e. for each $e' \in E'$. Since $\mathbf{F}$ is strongly measurable $\mathbf{F}(U)$ is separable subset of E. Let D be a countable dense subset of $\mathbf{F}(U)$. Denote by L the set of all finite linear combinations of the elements of D with scalars from $\mathbb{Q}+i\mathbb{Q}$. Then L is countable. Denote by $\tilde{E}$ the closure of L in E. Then $\tilde{E}$ is a separable Banach space and $\mathbf{F}(U) \subseteq \tilde{E}$. Thus $\tilde{E}'_\sigma$ is separable (by Theorem 1.7 of Chapter 4 of [91]; σ

stands for the weak* topology). Let $\tilde{V} = \{\tilde{e}'_1, \tilde{e}'_2, \tilde{e}'_3, ...\}$ be a countable dense subset of $\tilde{E}'_\sigma$. Extend each $\tilde{e}'_j$, $j \in \mathbb{Z}_+$, by the Hahn-Banach theorem to a continuous functional of E and denote this extension by e'_j, $j \in \mathbb{Z}_+$. Arguments given above imply that $e'_j \circ \mathbf{F} = 0$ a.e. for each $j \in \mathbb{Z}_+$ and in fact $\tilde{e}'_j \circ \mathbf{F} = 0$ a.e., $j \in \mathbb{Z}_+$, since e'_j is extension of $\tilde{e}'_j$ and $\mathbf{F}(U) \subseteq \tilde{E}$. Hence $P_j = \{x \in U | \, \tilde{e}'_j \circ \mathbf{F}(x) \neq 0\}$ is a set of measure 0, for each $j \in \mathbb{Z}_+$ and so is $P = \bigcup_j P_j$. We will prove that $\mathbf{F}(x) = 0$ for every $x \in U \backslash P$. Assume that there exists $x_0 \in U \backslash P$ such that $\mathbf{F}(x_0) \neq 0$. Then there exists $\tilde{e}' \in \tilde{E}'$ such that $\tilde{e}' \circ \mathbf{F}(x_0) \neq 0$ i.e. $|\tilde{e}' \circ \mathbf{F}(x_0)| = c > 0$. Then there exists $\tilde{e}'_k \in \tilde{V}$ such that $|\tilde{e}' \circ \mathbf{F}(x_0) - \tilde{e}'_k \circ \mathbf{F}(x_0)| \leq c/2$. Since $\tilde{e}'_k \circ \mathbf{F}(x_0) = 0$, by the definition of P, we have

$$c = |\tilde{e}' \circ \mathbf{F}(x_0)| \leq |\tilde{e}' \circ \mathbf{F}(x_0) - \tilde{e}'_k \circ \mathbf{F}(x_0)| + |\tilde{e}'_k \circ \mathbf{F}(x_0)| \leq c/2,$$

which is a contradiction. Hence $\mathbf{F}(x) = 0$ for all $x \in U \backslash P$ and the proof is complete. □

Denote by δ_x the delta ultradistribution concentrated at x. For $\alpha \in \mathbb{N}^n$ and $x \in U$ one easily verifies that $D^\alpha \delta_x \in \mathcal{D}'^{(s)}_{L^1,h}(U)$ for any $h > 0$ and hence, by Proposition 5.2.1, $D^\alpha \delta_x \in \mathcal{D}'^{(s)}_{L^p,h}(U)$ for any $h > 0$ and $1 \leq p \leq \infty$. For the next proposition we need the following result.

Lemma 5.3.2. *Let $h > 0$, $\alpha \in \mathbb{N}^n$ and $1 \leq p \leq \infty$. The set $G_\alpha = \{D^\alpha \delta_x | \, x \in U\} \subseteq \mathcal{D}'^{(s)}_{L^p,h}(U)$ is precompact in $\mathcal{D}'^{(s)}_{L^p,h}(U)$.*

Proof. Let $0 < h_1 < h/2^s$. By Proposition 5.2.1 we have the continuous inclusion $\mathcal{D}^{(s)}_{L^q,h}(U) \to \mathcal{D}^{(s)}_{L^\infty,h/2^s}(U)$. Proposition 5.2.2 implies that the inclusion $\mathcal{D}^{(s)}_{L^\infty,h/2^s}(U) \to \mathcal{D}^{(s)}_{L^\infty,h_1}(U)$ is compact. Hence we have the compact dense inclusion $\mathcal{D}^{(s)}_{L^q,h}(U) \to \mathcal{D}^{(s)}_{L^\infty,h_1}(U)$ (the denseness follows from the fact that $\mathcal{D}^{(s)}(U) \subseteq \mathcal{D}^{(s)}_{L^q,h}(U)$ is dense in $\mathcal{D}^{(s)}_{L^\infty,h_1}(U)$). So, the dual mapping $\mathcal{D}'^{(s)}_{L^1,h_1}(U) \to \mathcal{D}'^{(s)}_{L^p,h}(U)$ is compact inclusion. Observe that, for $\varphi \in \mathcal{D}^{(s)}_{L^\infty,h_1}(U)$,
$|\langle D^\alpha \delta_x, \varphi \rangle| \leq \dfrac{\alpha!^s}{h_1^{|\alpha|}} \|D^\alpha \varphi\|_{\mathcal{D}^{(s)}_{L^\infty,h_1}(U)}$, $\forall x \in U$. Hence G_α is bounded in the Banach space $\mathcal{D}'^{(s)}_{L^1,h_1}(U)$, thus precompact in $\mathcal{D}'^{(s)}_{L^p,h}(U)$. □

Proposition 5.3.1. *Each $\mathbf{F} \in L^p(U; E)$ can be regarded as an E-valued ultradistribution by $\overline{\mathbf{F}}(\varphi) = \int_U \mathbf{F}(x)\varphi(x)dx$. In this way $L^p(U; E)$ is continuously injected into $\mathcal{D}'^{(s)}_{L^1}(U; E)$ for $1 \leq p \leq \infty$ and in $\mathcal{D}'^{(s)}_{L^p,h}(U; E)$ for $1 < p < \infty$.*

Proof. Let $\mathbf{F} \in L^p(U; E)$. First we will prove that $L^p(U; E)$ is continuously injected into $\mathcal{D}'^{(s)}_{L^1}(U; E)$. If $\varphi \in \dot{\mathcal{B}}^{(s)}(U)$ then

$$\left\| \int_U \mathbf{F}(x)\varphi(x)dx \right\|_E \leq \int_U \|\mathbf{F}(x)\|_E |\varphi(x)| dx \leq \|\mathbf{F}\|_{L^p(U;E)} \|\varphi\|_{L^q(U)}. \tag{5.6}$$

Since U is bounded, $\|\varphi\|_{L^q(U)} \leq |U|^{1/q} \|\varphi\|_{L^\infty(U)}$. Hence $\overline{\mathbf{F}} \in L_b\left(\dot{\mathcal{B}}^{(s)}(U), E\right) = \mathcal{D}'^{(s)}_{L^1}(U; E)$ and the mapping $\mathbf{F} \mapsto \overline{\mathbf{F}}$ is continuous from $L^p(U; E)$ into $\mathcal{D}'^{(s)}_{L^1}(U; E)$.

To prove that it is injective let $\overline{\mathbf{F}} = 0$ i.e. $\displaystyle\int_U \mathbf{F}(x)\varphi(x)dx = 0$ for all $\varphi \in \dot{\mathcal{B}}^{(s)}(U)$. Since U is bounded $L^p(U;E) \subseteq L^1(U;E)$. Now, Lemma 5.3.1 implies that $\mathbf{F} = 0$.

Next, we prove that $L^p(U;E)$ is continuously injected into $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$ for $1 < p < \infty$. Consider the set $G = \{\delta_x |\, x \in U\} \subseteq \mathcal{D}'^{(s)}_{L^p,h}(U)$. It is precompact in $\mathcal{D}'^{(s)}_{L^p,h}(U)$ by Lemma 5.3.2. Fix $\mathbf{F} \in L^p(U;E)$ and note that (5.6) still holds when $\varphi \in \mathcal{D}^{(s)}_{L^q,h}(U)$. Let $V = \{e \in E |\, \|e\|_E \le \varepsilon\}$ be a neighborhood of zero in E and $\tilde{G} = \dfrac{\|\mathbf{F}\|_{L^p(U;E)}|U|^{1/q}}{\varepsilon} G$. Since G is precompact so is $\tilde{G}$. But then, for $\varphi \in \tilde{G}^\circ$,

$$\|\mathbf{F}\|_{L^p(U;E)}\|\varphi\|_{L^q(U)} \le |U|^{1/q}\|\mathbf{F}\|_{L^p(U;E)} \sup_{x\in U} |\langle \delta_x, \varphi\rangle| \le \varepsilon.$$

Hence $\overline{\mathbf{F}}(\varphi) \in V$ for all $\varphi \in \tilde{G}^\circ$. We obtain that $\overline{\mathbf{F}} \in L\left(\mathcal{D}^{(s)}_{L^q,h}(U)_c, E\right)$ since the topology of precompact convergence on $\mathcal{D}^{(s)}_{L^q,h}(U)$ coincides with the topology of compact convex circled convergence ($\mathcal{D}'^{(s)}_{L^p,h}(U)$ is a Banach space). The continuity of the mapping $\mathbf{F} \mapsto \overline{\mathbf{F}}$ follows from (5.6) since the bounded sets of $\mathcal{D}^{(s)}_{L^q,h}(U)$ are the same for the initial topology and the topology of compact convex circled convergence. The proof of the injectivity is the same as above. □

By Proposition 5.3.1, from now on we will use the same notation for $\mathbf{F} \in L^p(U;E)$ and its image in $\mathcal{D}'^{(s)}_{L^1}(U;E)$, resp. $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$ for $1 < p < \infty$.

For $\alpha \in \mathbb{N}^n$ and $\mathbf{F} \in L^p(U;E)$, $1 < p < \infty$, define $D^\alpha \mathbf{F} \in \mathcal{D}'^{(s)}_{L^p,h}(U;E)$ by

$$D^\alpha \mathbf{F}(\varphi) = \int_U \mathbf{F}(x)(-D)^\alpha \varphi(x)dx,\ \varphi \in \mathcal{D}^{(s)}_{L^q,h}(U).$$

As in Proposition 5.3.1, one can prove that this is well defined element of $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$. One only has to use the set G_α from Lemma 5.3.2 instead $G = \{\delta_x |\, x \in U\}$. Observe that $D^\alpha \mathbf{F}$ coincides with the ultradistributional derivative of $\mathbf{F}$ when we regard $\mathbf{F}$ as an element of $\mathcal{D}'^{(s)}_{L^1}(U;E)$ or $\mathcal{D}'^{(s)}(U;E)$.

Theorem 5.3.2. *Let $1 < p < \infty$ and $\mathbf{F}_\alpha \in L^p(U;E)$, $\alpha \in \mathbb{N}^n$, are such that, for some fixed $h > 0$, $\left(\sum_\alpha \dfrac{\alpha!^{ps}}{h^{|\alpha|p}}\|\mathbf{F}_\alpha\|^p_{L^p}\right)^{1/p} < \infty$. Then the partial sums $\sum_{|\alpha|=0}^{n} D^\alpha \mathbf{F}_\alpha$ converge absolutely in $\mathcal{D}'^{(s)}_{L^p}(U;E)$ and $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$.*

The partial sums converge absolutely in $\mathcal{D}'^{(s)}_{L^1}(U;E)$ also in the cases $p = 1$ and $p = \infty$.

Proof. Let $1 < p < \infty$. To prove that the partial sums converge absolutely in $\mathcal{D}'^{(s)}_{L^p,h}(U;E) = L_b\left(\mathcal{D}^{(s)}_{L^q,h}(U)_c, E\right)$ let B be a bounded subset of $\mathcal{D}^{(s)}_{L^q,h}(U)_c$. Since the bounded sets of $\mathcal{D}^{(s)}_{L^q,h}(U)$ are the same for the initial topology and the topology of compact convex circled convergence we may assume that B is the closed unit ball in $\mathcal{D}^{(s)}_{L^q,h}(U)$. We obtain

$$\sum_{|\alpha|=0}^{n} \sup_{\varphi\in B} \left\| \int_U \mathbf{F}_\alpha(x)(-D)^\alpha \varphi(x)dx \right\|_E$$

$$\leq \sup_{\varphi\in B} \sum_{|\alpha|=0}^{\infty} \int_U \|\mathbf{F}_\alpha(x)\|_E |D^\alpha\varphi(x)|dx \leq \sup_{\varphi\in B} \sum_{|\alpha|=0}^{\infty} \|\mathbf{F}_\alpha\|_{L^p(U;E)} \|D^\alpha\varphi\|_{L^q(U)}$$

$$\leq \left(\sum_{|\alpha|=0}^{\infty} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|\mathbf{F}_\alpha\|^p_{L^p(U;E)} \right)^{1/p} \cdot \sup_{\varphi\in B} \left(\sum_{|\alpha|=0}^{\infty} \frac{h^{|\alpha|q}}{\alpha!^{qs}} \|D^\alpha\varphi\|^q_{L^q(U)} \right)^{1/q},$$

for any $n \in \mathbb{Z}_+$. Since $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$ is complete it follows that the partial sums converge absolutely in $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$ to an element of $\mathcal{D}'^{(s)}_{L^p,h}(U;E)$. The proof for $\mathcal{D}'^{(s)}_{L^1}(U;E)$ is similar. □

Observe that each $\mathbf{F} \in \mathcal{C}(\overline{U};E)$ is in $L^p(U;E)$ for any $1 \leq p \leq \infty$. To see this, note that $\mathbf{F}$ is separately valued since it is continuous and $\overline{U}$ is a subset of $\mathbb{R}^n$. Moreover it is easy to see that it is weakly measurable. Hence Pettis' theorem implies that $\mathbf{F}$ is strongly measurable. Now the claim follows since U is bounded $\|\mathbf{F}(\cdot)\|_E$ is in $L^p(U)$, for any $1 \leq p \leq \infty$.

Theorem 5.3.3. *Let $\mathbf{f} \in \mathcal{D}'^{(s)}_{L^1}(U;E)$ and $1 \leq p \leq \infty$. Then there exists $h > 0$ and $\mathbf{F}_\alpha \in \mathcal{C}(\overline{U};E)$, $\alpha \in \mathbb{N}^n$, such that*

$$\left(\sum_{\alpha} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|\mathbf{F}_\alpha\|^p_{L^p(U;E)} \right)^{1/p} < \infty \tag{5.7}$$

and $\mathbf{f} = \sum_{|\alpha|=0}^{\infty} D^\alpha \mathbf{F}_\alpha$, where the series converges absolutely in $\mathcal{D}'^{(s)}_{L^1}(U;E)$.

Conversely, let $\mathbf{F}_\alpha \in L^p(U;E)$, $\alpha \in \mathbb{N}^n$, be such that (5.7) holds. Then there exists $\mathbf{f} \in \mathcal{D}'^{(s)}_{L^1}(U;E)$ such that $\mathbf{f} = \sum_{|\alpha|=0}^{\infty} D^\alpha \mathbf{F}_\alpha$ and the series converges absolutely in $\mathcal{D}'^{(s)}_{L^1}(U;E)$.

Proof. First, note that the second part of the theorem follows by Theorem 5.3.2. To prove the first part, let $\mathbf{f} \in \mathcal{D}'^{(s)}_{L^1}(U;E) = L_b\left(\dot{\mathcal{B}}^{(s)}(U), E\right)$. Since $\dot{\mathcal{B}}^{(s)}(U)$ is nuclear (by Proposition 5.2.2) and E is a Banach space $\mathbf{f}$ is nuclear. Hence there exists a sequence e_j, $j \in \mathbb{N}$, in the closed unit ball of E, an equicontinuous sequence f_j, $j \in \mathbb{N}$, of $\mathcal{D}'^{(s)}_{L^1}(U)$ and a complex sequence λ_j, $j \in \mathbb{N}$, such that $\sum_j |\lambda_j| < \infty$, such that

$$\mathbf{f}(\varphi) = \sum_{j=0}^{\infty} \lambda_j \langle f_j, \varphi \rangle e_j.$$

Since $\{f_j | j \in \mathbb{N}\}$ is equicontinuous subset of $\mathcal{D}'^{(s)}_{L^1}(U)$, it is bounded and by Proposition 5.2.3, there exist $h, C > 0$ and $F_{j,\alpha} \in \mathcal{C}(\overline{U})$ such that

$$f_j = \sum_{|\alpha|=0}^{\infty} D^\alpha F_{j,\alpha} \text{ and } \sup_j \left(\sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!^{ps}}{h^{|\alpha|p}} \|F_{j,\alpha}\|^p_{L^\infty(U)} \right)^{1/p} \leq C.$$

Define $\mathbf{F}_\alpha(x) = \sum_j \lambda_j F_{j,\alpha}(x) e_j$. To prove that $\mathbf{F}_\alpha \in \mathcal{C}(\overline{U}; E)$, observe that for each $j \in \mathbb{N}$, $\lambda_j F_{j,\alpha}(x) e_j \in \mathcal{C}(\overline{U}; E)$ and the series $\sum_j \lambda_j F_{j,\alpha}(x) e_j$ converges absolutely in the Banach space $\mathcal{C}(\overline{U}; E)$. Hence $\mathbf{F}_\alpha \in \mathcal{C}(\overline{U}; E)$. Moreover

$$\frac{\alpha!^s}{h^{|\alpha|}} \|\mathbf{F}_\alpha(x)\|_E \leq \sum_{j=0}^{\infty} |\lambda_j| \frac{\alpha!^s}{h^{|\alpha|}} \|F_{j,\alpha}\|_{L^\infty(U)} \leq C \sum_{j=0}^{\infty} |\lambda_j|, \quad \text{for all } x \in \overline{U}.$$

We obtain $\sup_\alpha \frac{\alpha!^s}{h^{|\alpha|}} \|\mathbf{F}_\alpha\|_{\mathcal{C}(\overline{U};E)} < \infty$. Since U is bounded, (5.7) holds for any $h_1 > h$. One easily verifies that the series $\sum_{j,\alpha} \lambda_j \langle D^\alpha F_{j,\alpha}, \varphi \rangle e_j$ converges absolutely in E for each fixed $\varphi \in \dot{\mathcal{B}}^{(s)}(U)$. Hence $\mathbf{f}(\varphi) = \sum_{|\alpha|=0}^{\infty} D^\alpha \mathbf{F}_\alpha(\varphi)$, for each fixed $\varphi \in \dot{\mathcal{B}}^{(s)}(U)$. By Theorem 5.3.2, $\sum_{|\alpha|=0}^{\infty} D^\alpha \mathbf{F}_\alpha$ converges absolutely in $\mathcal{D}'^{(s)}_{L^1}(U; E)$, hence $\mathbf{f} = \sum_{|\alpha|=0}^{\infty} D^\alpha \mathbf{F}_\alpha$. □

5.4 On the Cauchy Problem in $\tilde{\mathcal{D}}'^s_{L^p,h}(0, T; E)$

In this section E is the Banach space with the norm $\|\cdot\|$, and $D(A)$ is the domain of a closed linear operator A, endowed with the graph norm $\|u\|_{D(A)} = \|u\| + \|Au\|$. We use standard notation for the symbols $R(\lambda : A)$, $\rho(A)$. The results obtained in previous sections will often be applied in the one dimensional case (i.e. $n = 1$) when a bounded open set U is equal to the interval $(0, T)$. In this case we will use the more descriptive notations $L^p(0, T; E)$, $\mathcal{D}^s_{L^p,h}(0, T)$, $\mathcal{D}^{(s)}_{L^p,h}(0, T)$, $\dot{\mathcal{B}}^{(s)}(0, T)$, $\mathcal{D}'^{(s)}_{L^p,h}(0, T)$, $\mathcal{D}'^{(s)}_{L^1}(0, T)$, $\mathcal{D}'^{(s)}_{L^p,h}(0, T; E)$ and $\mathcal{D}'^{(s)}_{L^1}(0, T; E)$ for the spaces $L^p(U; E)$, $\mathcal{D}^s_{L^p,h}(U)$, $\mathcal{D}^{(s)}_{L^p,h}(U)$, $\dot{\mathcal{B}}^{(s)}(U)$, $\mathcal{D}'^{(s)}_{L^p,h}(U)$, $\mathcal{D}'^{(s)}_{L^1}(U)$, $\mathcal{D}'^{(s)}_{L^p,h}(U; E)$ and $\mathcal{D}'^{(s)}_{L^1}(U; E)$, respectively. Note that by Sobolev imbedding theorem, every derivative of $\varphi \in \mathcal{D}^s_{L^p,h}(0, T)$ can be extended to uniformly continuous function on $[0, T]$. As in [25], we define the E-valued Sobolev space $W^{1,p}(0, T; E)$ as the space of all $\mathbf{F} : [0, T] \to E$, such that $\mathbf{F}(t) = F_0 + \int_0^t \mathbf{F}'(s) ds$, $t \in [0, T]$, for some $F_0 \in E$ and $\mathbf{F}'(t) \in L^p(0, T; E)$, with the norm $\|\mathbf{F}\|_{W^{1,p}(0,T:E)} = \|\mathbf{F}\|_{L^p(0,T;E)} + \|\mathbf{F}'\|_{L^p(0,T;E)}$, $1 \leq p < \infty$. Observe that if $\mathbf{F} \in W^{1,p}(0, T; E)$ then $\mathbf{F}$ is continuous function with values in E which is a.e. differentiable and its derivative is equal to $\mathbf{F}'$ a.e.

Let $1 \le p < \infty$. Define $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ as a space of all sequences $\mathbf{f} = (\mathbf{F}_\alpha)_\alpha$, $\mathbf{F}_\alpha \in L^p(0,T;E)$, $\alpha \in \mathbb{N}$, such that

$$\|\mathbf{f}\|_{\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)} = \left(\sum_{\alpha\in\mathbb{N}} \frac{\alpha!^{ps}}{h^{p\alpha}} \|\mathbf{F}_\alpha\|^p_{L^p(0,T;E)} \right)^{1/p} < \infty. \tag{5.8}$$

One easily verifies that it is a Banach space with the norm (5.8). Each $\mathbf{f} \in \tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ generates an element of $L\left(\mathcal{D}^{s}_{L^q,h}(0,T),E\right)$ by

$$\langle \mathbf{f},\varphi\rangle = \mathbf{f}(\varphi) = \sum_{\alpha\in\mathbb{N}} (-1)^\alpha \int_0^T \mathbf{F}_\alpha(t)\varphi^{(\alpha)}(t)\,dt \in E.$$

Moreover, one easily verifies that the mapping $\mathbf{f} \mapsto \langle \mathbf{f},\cdot\rangle$,
$\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E) \to L_b\left(\mathcal{D}^{s}_{L^q,h}(0,T),E\right)$ is continuous.

Remark 5.4.1. It is worth to note that this mapping is not injective. To see this let $\psi \in \mathcal{D}^{(s)}(0,T)$, $\psi \ne 0$. Take nonzero element e of E and define $\mathbf{F}(x) = \psi'(x)e$ and $\mathbf{G}(x) = \psi(x)e$, $x \in (0,T)$. Obviously $\mathbf{F},\mathbf{G} \in L^p(0,T;E)$, for any $1 \le p \le \infty$. Define $\mathbf{f},\mathbf{g} \in \tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ by $\mathbf{f} = (\mathbf{F},0,0,...)$ and $\mathbf{g} = (0,\mathbf{G},0,...)$. Observe that, for $\varphi \in \mathcal{D}^{s}_{L^q,h}(0,T)$,

$$\langle \mathbf{f},\varphi\rangle = e\int_0^T \psi'(x)\varphi(x)dx = -e\int_0^T \psi(x)\varphi'(x)dx = \langle \mathbf{g},\varphi\rangle.$$

Hence $\langle \mathbf{f},\cdot\rangle$ and $\langle \mathbf{g},\cdot\rangle$ are the same element of $L_b\left(\mathcal{D}^{s}_{L^q,h}(0,T),E\right)$.

Note that $L^p(0,T;E)$ can be continuously imbedded in $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ by $\mathbf{F} \mapsto (\mathbf{F},0,0,\ldots)$.

Let $1 \le p < \infty$. Define $\tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)$ as the space of all sequences $\mathbf{f} = (\mathbf{F}_\alpha)_\alpha$, where $\mathbf{F}_\alpha \in W^{1,p}(0,T;E)$ and

$$\|\mathbf{f}\|_{\tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)} = \left(\sum_{\alpha\in\mathbb{N}} \frac{\alpha!^{ps}}{h^{p\alpha}} \left(\|\mathbf{F}_\alpha\|^p_{L^p(0,T;E)} + \|\mathbf{F}'_\alpha\|^p_{L^p(0,T;E)} \right) \right)^{1/p} < \infty.$$

Equipped with the norm $\|\cdot\|_{\tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)}$, it becomes a Banach space.
$\tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)$ is continuously injected into $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$. For $\mathbf{f} = (\mathbf{F}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)$, $\mathbf{f}' = \tilde{\mathbf{f}} = (\tilde{\mathbf{F}}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$, where $\tilde{\mathbf{F}}_\alpha = \mathbf{F}'_\alpha$ is the classical derivative a.e. in $(0,T)$.
Moreover, the mapping $\mathbf{f} \mapsto \mathbf{f}'$, $\tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E) \to \tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$, is continuous.

Our main assumption is that the Hille-Yosida condition holds for the resolvent of the operator A:

$$\|(\lambda-\omega)^k R(\lambda:A)^k\| \le C, \text{ for } \lambda > \omega,\ k \in \mathbb{Z}_+. \tag{5.9}$$

From now on we will always denote these constants by ω and C.

5.4.1 Various types of solutions

We need the following technical lemma.

Lemma 5.4.1. *Let* $1 \le p < \infty$ *and* $\mathbf{g} = (\mathbf{G}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;D(A))$. *Then for every* $\varphi \in \mathcal{D}^s_{L^q}(0,T)$, $\langle \mathbf{g}, \varphi\rangle \in D(A)$ *and*

$$A\sum_{\alpha=0}^{\infty}(-1)^\alpha \int_0^T \mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt = \sum_{\alpha=0}^{\infty}(-1)^\alpha \int_0^T A\mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt.$$

Proof. First observe that for each $\alpha \in \mathbb{N}$, $\mathbf{G}_\alpha\varphi^{(\alpha)} \in L^1(0,T;D(A))$ and $A\mathbf{G}_\alpha\varphi^{(\alpha)} \in L^1(0,T;E)$ since $\mathbf{G}_\alpha(t) \in L^p(0,T;D(A))$ and $\varphi \in \mathcal{D}^s_{L^q}(0,T)$. Then

$$A\int_0^T \mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt = \int_0^T A\mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt. \tag{5.10}$$

Moreover, observe that

$$\sum_{\alpha=0}^{\infty}\left\|\int_0^T \mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt\right\|_{D(A)} \le \|(\mathbf{G}_\alpha)_\alpha\|_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;D(A))}\|\varphi\|_{\mathcal{D}^s_{L^q}(0,T)}.$$

We obtain that $\sum_{\alpha=0}^{\infty}(-1)^\alpha \int_0^T \mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt$ converges absolutely in $D(A)$, i.e. $\langle \mathbf{g}, \varphi\rangle \in D(A)$. Hence

$$A\sum_{\alpha=0}^{\infty}(-1)^\alpha \int_0^T \mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt = \sum_{\alpha=0}^{\infty}(-1)^\alpha A\int_0^T \mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt,$$

which, together with (5.10), completes the proof of the lemma. □

Let $u_{0,\alpha} \in E$, $\alpha \in \mathbb{N}$, be such that

$$\left(\sum_{\alpha=0}^{\infty}\frac{\alpha!^{ps}}{h^{p\alpha}}\|u_{0,\alpha}\|_E^p\right)^{1/p} < \infty. \tag{5.11}$$

Then the constant functions $\tilde{\mathbf{U}}_\alpha(t) = u_{0,\alpha}$, $t \in [0,T]$, are such that $\tilde{\mathbf{U}}_\alpha \in L^p(0,T;E)$ and (5.8) holds. Hence $(\tilde{\mathbf{U}}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$. In the sequel, if $u_{0,\alpha}$, $\alpha \in \mathbb{N}$, are such elements we will denote the corresponding constant functions simply by $u_{0,\alpha}$ and the element $(u_{0,\alpha})_\alpha$ of $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ that they generate by u_0. We also use the notation $\|u_0\|_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)}$ for the norm of this element of $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$.

We recall from [25] the definition of two types of solutions of the Cauchy problem (1) (here they are restated to fit in our setting). We also define weak version of them. Let $A : D(A) \subseteq E \to E$ be a closed linear operator in the Banach space E, $\mathbf{f} \in \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ and $u_{0,\alpha} \in E$, $\alpha \in \mathbb{N}$.

1. We say that $\mathbf{u} = (\mathbf{U}_\alpha)_\alpha$ is a strict solution, respectively, strict weak solution, in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ of (1) if $\mathbf{u} \in \tilde{\mathcal{D}}'^s_{W^{1,p},h}(0,T;E) \cap \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;D(A))$ and

$$\mathbf{U}'_\alpha(t) = A\mathbf{U}_\alpha(t) + \mathbf{F}_\alpha(t),\ t \in [0,T]\ a.e. \text{ and } \mathbf{U}_\alpha(0) = u_{0,\alpha},\ \forall\alpha \in \mathbb{N},$$

respectively, for each $\varphi \in \mathcal{D}^s_{L^q,h}(0,T)$ it satisfies

$$\langle \mathbf{u}'(t), \varphi(t)\rangle = A\langle \mathbf{u}(t), \varphi(t)\rangle + \langle \mathbf{f}(t), \varphi(t)\rangle \text{ and } \mathbf{U}_\alpha(0) = u_{0,\alpha},\ \forall \alpha \in \mathbb{N}. \quad (5.12)$$

We know by Lemma 5.4.1 that $\langle \mathbf{u}(t), \varphi(t)\rangle \in D(A)$ for each $\varphi \in \mathcal{D}^s_{L^q,h}(0,T)$. Also, note that in both cases (of strict or of strict weak solution of (1)) we have

$$\|u_{0,\alpha}\|^p_E \le 2^p T^{-1} \|\mathbf{U}_\alpha\|_{L^p(0,T;E)} + 2^p T^{p/q} \|\mathbf{U}'_\alpha\|_{L^p(0,T;E)}.$$

Hence $u_0 = (u_{0,\alpha})_\alpha$ satisfies (5.11).

2. We say that $\mathbf{u} \in \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ is an F-solution, respectively, F-weak solution in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ of (1), if for every $k \in \mathbb{N}$ there is $\mathbf{u}_k = (\mathbf{U}_{k,\alpha})_\alpha \in \tilde{\mathcal{D}}'^s_{W^{1,p},h}(0,T;E) \cap \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;D(A))$ such that from

$$\mathbf{U}'_{k,\alpha}(t) = A\mathbf{U}_{k,\alpha}(t) + \mathbf{F}_{k,\alpha}(t),\ t \in [0,T]\, a.e. \text{ and } \mathbf{U}_{k,\alpha}(0) = u_{0,k,\alpha}$$

we have

$$\lim_{k\to\infty} \Big(\|\mathbf{u}_k - \mathbf{u}\|_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)} + \|\mathbf{f}_k - \mathbf{f}\|_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)} +$$

$$+ \|u_{0,k} - u_0\|_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)} \Big) = 0,$$

respectively, from

$$\begin{aligned} \langle \mathbf{u}'_k(t), \varphi(t)\rangle &= A\langle \mathbf{u}_k(t), \varphi(t)\rangle + \langle \mathbf{f}_k(t), \varphi(t)\rangle,\ \forall \varphi \in \mathcal{D}^s_{L^q,h}(0,T) \\ &\quad \text{and } \mathbf{U}_{k,\alpha}(0) = u_{0,k,\alpha},\ \forall k, \alpha \in \mathbb{N} \end{aligned}$$

we have that for every $\varphi \in \mathcal{D}^s_{L^q,h}(0,T)$,

$$\lim_{k\to\infty} (\|\langle \mathbf{u}_k - \mathbf{u}, \varphi\rangle\|_E + \|\langle \mathbf{f}_k - \mathbf{f}, \varphi\rangle\|_E + \|\langle u_{0,k} - u_0, \varphi\rangle\|_E) = 0. \quad (5.13)$$

From the above definitions it is clear that a strict, resp. a strict weak solution, in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ is an F-solution, resp. F-weak solution in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$.

Remark 5.4.2. If a strict weak solution of (1) in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ exists then it is not unique. To see this let $\psi \in \mathcal{D}^{(s)}(0,T)$ and $e \in D(A)$ such that $\psi \ne 0$ and $e \ne 0$. Define $\mathbf{v} = (\mathbf{V}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ by $\mathbf{V}_0(t) = \psi'(t)e$, $\mathbf{V}_1(t) = -\psi(t)e$ and $\mathbf{V}_\alpha(t) = 0$, for $\alpha \ge 2$, $\alpha \in \mathbb{N}$. Obviously $\mathbf{v} \in \tilde{\mathcal{D}}'^s_{W^{1,p},h}(0,T;E) \cap \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;D(A))$ and $\mathbf{V}_\alpha(0) = 0$, $\forall \alpha \in \mathbb{N}$. Moreover, it is easy to verify that the operators $\langle \mathbf{v}, \cdot\rangle, \langle \mathbf{v}', \cdot\rangle \in L\left(\mathcal{D}^s_{L^q,h}(0,T), E\right)$ are in fact the zero operator. Hence, if $\mathbf{u}$ is a strict weak solution of (1) in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ then so is $\mathbf{u} + \mathbf{v}$.

One can use the same construction to prove that the F-weak solution in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ of (1) is also not unique.

Now we consider the existence of such solutions of the Cauchy problem (1).

Proposition 5.4.1. *If* $\mathbf{u}$ *is a strict, resp. a* F*-solution, of the Cauchy problem (1), then it is also strict weak, resp.* F*-weak solution, of (1).*

Proof. The proof follows from Lemma 5.4.1 and the fact that the mapping $\mathbf{g} \mapsto \langle \mathbf{g}, \cdot \rangle$, $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E) \to L_b\left(\mathcal{D}^s_{L^q,h}(0,T), E\right)$ is continuous. □

The proof of the next theorem heavily relies on the results obtained in [25]. Parts in brackets are consequences of Proposition 5.4.1.

Theorem 5.4.1. *i) The Cauchy problem (1) has an F-solution (resp. an F-weak solution) in $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ for every $\mathbf{f} = (\mathbf{F}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ and $u_0 = (u_{0,\alpha})_\alpha$ such that $(u_{0,\alpha})_\alpha$ satisfies (5.11) and $u_{0,\alpha} \in \overline{D(A)}$, $\forall \alpha \in \mathbb{N}$. In the case of F-solution, it is unique.*

ii) The Cauchy problem (1) has a strict solution (resp. strict weak solution) in $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$ for every $\mathbf{f} = (\mathbf{F}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)$ and $u_0 = (u_{0,\alpha})_\alpha$ such that $u_{0,\alpha} \in D(A)$ and $Au_{0,\alpha} + \mathbf{F}_\alpha(0) \in \overline{D(A)}$, $\forall \alpha \in \mathbb{N}$ and $(u_{0,\alpha})_\alpha$ and $(Au_{0,\alpha})_\alpha$ satisfies (5.11). In the case of strict solution, it is unique.

Proof. First we will prove *i*). By Theorem 1.4.6 (see also the Appendix of [25]) for each fixed $\alpha \in \mathbb{N}$, the problem $\mathbf{U}'_\alpha = A\mathbf{U}_\alpha + \mathbf{F}_\alpha$, $\mathbf{U}_\alpha(0) = u_{0,\alpha}$ has a F-solution in $L^p(0,T;E)$. In other words, there exist $\mathbf{U}_{k,\alpha} \in W^{1,p}(0,T;E) \cap L^p(0,T;D(A))$, $\mathbf{F}_{k,\alpha} \in L^p(0,T;E)$, $u_{0,k,\alpha} \in E$, $k \in \mathbb{Z}_+$, such that $\mathbf{U}'_{k,\alpha} = A\mathbf{U}_{k,\alpha} + \mathbf{F}_{k,\alpha}$, $\mathbf{U}_{k,\alpha}(0) = u_{0,k,\alpha}$ and

$$\lim_{k\to\infty} \Big(\|\mathbf{U}_{k,\alpha} - \mathbf{U}_\alpha\|_{L^p(0,T;E)} + \|\mathbf{F}_{k,\alpha} - \mathbf{F}_\alpha\|_{L^p(0,T;E)} + \tag{5.14}$$

$$\|u_{0,k,\alpha} - u_{0,\alpha}\|_E \Big) = 0.$$

Moreover, by Theorem 1.4.5 (see also Theorem A.1 of the Appendix of [25]), each $\mathbf{U}_\alpha$ is in fact in $\mathcal{C}(0,T;E)$, $\mathbf{U}_\alpha(t) \in \overline{D(A)}$, $\forall t \in [0,T]$, $\mathbf{U}_\alpha(0) = u_{0,\alpha}$ and

$$\|\mathbf{U}_\alpha(t)\| \le Ce^{\omega t}\left(\|\mathbf{U}_\alpha(0)\| + \int_0^t e^{-\omega s}\|\mathbf{F}_\alpha(s)\|ds\right),\ t \in [0,T]. \tag{5.15}$$

Using this estimate one easily verifies that $\mathbf{u} = (\mathbf{U}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$. We will prove that this is an F-solution of (1).

Let $k \in \mathbb{Z}_+$. Take $n_k \in \mathbb{Z}_+$ such that

$$\sum_{\alpha=n_k}^{\infty} \frac{\alpha!^{ps}}{h^{p\alpha}}\|\mathbf{F}_\alpha\|^p_{L^p(0,T;E)} \le \frac{1}{(2k)^p},\ \sum_{\alpha=n_k}^{\infty} \frac{\alpha!^{ps}}{h^{p\alpha}}\|\mathbf{U}_\alpha\|^p_{L^p(0,T;E)} \le \frac{1}{(2k)^p}$$

$$\text{and } \sum_{\alpha=n_k}^{\infty} \frac{\alpha!^{ps}}{h^{p\alpha}}\|u_{0,\alpha}\|^p_E \le \frac{1}{(2k)^p}.$$

For each $0 \le \alpha \le n_k - 1$, by (5.14) we can take $\mathbf{F}_{k_\alpha,\alpha}$, $\mathbf{U}_{k_\alpha,\alpha}$ and $u_{0,k_\alpha,\alpha}$ such that

$$\sum_{\alpha=0}^{n_k-1} \frac{\alpha!^{ps}}{h^{p\alpha}}\Big(\|\mathbf{U}_{k_\alpha,\alpha} - \mathbf{U}_\alpha\|^p_{L^p(0,T;E)} + \|\mathbf{F}_{k_\alpha,\alpha} - \mathbf{F}_\alpha\|^p_{L^p(0,T;E)}$$

$$+ \quad \|u_{0,k_\alpha,\alpha} - u_{0,\alpha}\|_E^p\Big) \le \frac{1}{(2k)^p}$$

and $\mathbf{U}'_{k_\alpha,\alpha} = A\mathbf{U}_{k_\alpha,\alpha} + \mathbf{F}_{k_\alpha,\alpha}$, $\mathbf{U}_{k_\alpha,\alpha}(0) = u_{0,k_\alpha,\alpha}$. For $0 \le \alpha \le n_k - 1$ define $\mathbf{V}_{k,\alpha} = \mathbf{U}_{k_\alpha,\alpha}$, $v_{0,k,\alpha} = u_{0,k_\alpha,\alpha}$ and $\mathbf{G}_{k,\alpha} = \mathbf{F}_{k_\alpha,\alpha}$. For $\alpha \ge n_k$ put $\mathbf{V}_{k,\alpha} = 0$, $v_{0,k,\alpha} = 0$ and $\mathbf{G}_{k,\alpha} = 0$. Then $\mathbf{v}_k = (\mathbf{V}_{k,\alpha})_\alpha \in \tilde{\mathcal{D}}'^s_{W^{1,p},h}(0,T;E) \cap \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;D(A))$, $\mathbf{g}_k = (\mathbf{G}_{k,\alpha})_\alpha \in \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ and $v_{0,k} = (v_{0,k,\alpha})_\alpha$ is such that $\displaystyle\sum_{\alpha=0}^{\infty} \frac{(\alpha!)^{ps}}{h^{p\alpha}} \|v_{0,k,\alpha}\|_E^p < \infty$. Also $\mathbf{v}_k(0) = v_{0,k}$. By definition, we have $\mathbf{V}'_{k,\alpha} = A\mathbf{V}_{k,\alpha} + \mathbf{G}_{k,\alpha}$ for all $\alpha \in \mathbb{N}$. We will prove that $\mathbf{v}_k \to \mathbf{u}$, $\mathbf{g}_k \to \mathbf{f}$ and $v_{0,k} \to u_0$ in $\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$, hence $\mathbf{u}$ is F-solution of (1). Let $\varepsilon > 0$. Take $k_0 \in \mathbb{Z}_+$ such that $1/k_0 \le \varepsilon$. For $k \ge k_0$, $k \in \mathbb{Z}_+$, we have

$$\begin{aligned}
&\|\mathbf{v}_k - \mathbf{u}\|^p_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)} \\
&\qquad = \sum_{\alpha=0}^{n_k-1} \frac{\alpha!^{ps}}{h^{p\alpha}} \|\mathbf{V}_{k,\alpha} - \mathbf{U}_\alpha\|^p_{L^p(0,T;E)} + \sum_{\alpha=n_k}^{\infty} \frac{\alpha!^{ps}}{h^{p\alpha}} \|\mathbf{U}_\alpha\|^p_{L^p(0,T;E)} \\
&\qquad \le \sum_{\alpha=0}^{n_k-1} \frac{\alpha!^{ps}}{h^{p\alpha}} \|\mathbf{U}_{k_\alpha,\alpha} - \mathbf{U}_\alpha\|^p_{L^p(0,T;E)} + \frac{\varepsilon^p}{2^p} \le \frac{2\varepsilon^p}{2^p}.
\end{aligned}$$

Hence $\|\mathbf{v}_k - \mathbf{u}\|_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)} \le \varepsilon$. Similarly, $\|\mathbf{g}_k - \mathbf{f}\|_{\tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)} \le \varepsilon$ and $\left(\displaystyle\sum_{\alpha=0}^{\infty} \frac{\alpha!^{ps}}{h^{p\alpha}} \|v_{0,k,\alpha} - u_{0,\alpha}\|^p_{L^p(0,T;E)}\right)^{1/p} \le \varepsilon$, for $k \ge k_0$. It remains to prove the uniqueness. If $\tilde{\mathbf{u}} = (\tilde{\mathbf{U}}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;E)$ is another F-solution of (1) then $\tilde{\mathbf{U}}_\alpha$ is a F-solution to the problem $\tilde{\mathbf{U}}'_\alpha(t) = A\tilde{\mathbf{U}}_\alpha(t) + \mathbf{F}_\alpha(t)$, $\tilde{\mathbf{U}}_\alpha(0) = u_{0,\alpha}$, for each $\alpha \in \mathbb{N}$. But, theorem 1.4.5 (see also Theorem A.1 of the Appendix of [25]) implies that the F-solution to this problem must be unique, hence $\tilde{\mathbf{U}}_\alpha = \mathbf{U}_\alpha$ which proofs the desired uniqueness.

To prove ii), observe that Theorem 1.4.7 (see also Theorem A.2 of the Appendix of [25]) implies that for each $\alpha \in \mathbb{N}$ there exists $\mathbf{U}_\alpha \in \mathcal{C}^1(0,T;E) \cap \mathcal{C}(0,T;D(A))$ such that

$$\mathbf{U}'_\alpha(t) = A\mathbf{U}_\alpha(t) + \mathbf{F}_\alpha(t),\ \forall t \in [0,T] \text{ and } \mathbf{U}_\alpha(0) = u_{0,\alpha} \tag{5.16}$$

and it satisfy (5.15) and

$$\|\mathbf{U}'_\alpha(t)\| \le Ce^{\omega t}\left(\|Au_{0,\alpha} + \mathbf{F}_\alpha(0)\| + \int_0^t e^{-\omega s}\|\mathbf{F}'_\alpha(s)\|ds\right),\ t \in [0,T]. \tag{5.17}$$

Moreover, by (5.16) and (5.17), we have

$$\|A\mathbf{U}_\alpha(t)\| \le Ce^{2|\omega|T}\left(\|Au_{0,\alpha}\| + \|\mathbf{F}_\alpha(0)\| + T^{1/q}\|\mathbf{F}'_\alpha\|_{L^p(0,T;E)}\right) + \|\mathbf{F}_\alpha(t)\|,\ t \in [0,T].$$

Since $\mathbf{f} \in \tilde{\mathcal{D}}'^s_{W^{1,p},h}(0,T;E)$ and $(u_{0,\alpha})_\alpha$ and $(Au_{0,\alpha})_\alpha$ satisfy (5.11), by the above estimate and (5.15) and (5.17) we can conclude $\mathbf{u} = (\mathbf{U}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^s_{W^{1,p},h}(0,T;E) \cap \tilde{\mathcal{D}}'^s_{L^p,h}(0,T;D(A))$. Hence $\mathbf{u}$ is a strict solution. The uniqueness follows from Theorem 1.4.7 (see also Theorem A.2 of the Appendix of [25]) by similar arguments as in i). $\square$

5.4.2 Solutions in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)$

Let $\mathbf{g}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$. By Theorem 5.3.3 for $1<p<\infty$, there exists $h_1>0$ and $\mathbf{G}_\alpha\in L^p(0,T;E)$, $\alpha\in\mathbb{N}$, such that

$$\sum_{\alpha=0}^{\infty}\frac{\alpha!^{ps}}{h_1^{p\alpha}}\|\mathbf{G}_\alpha\|^p_{L^p(0,T;E)}<\infty \text{ and } \mathbf{g}=\sum_{\alpha=0}^{\infty}\mathbf{G}_\alpha^{(\alpha)}. \tag{5.18}$$

For the moment, for $\mathbf{g}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)=L_b\left(\dot{\mathcal{B}}^{(s)}(0,T),E\right)$, denote by $\mathbf{g}(\varphi)$ the action of $\mathbf{g}$ on $\varphi\in\dot{\mathcal{B}}^{(s)}(0,T)$. On the other hand, put $\tilde{\mathbf{g}}=(\mathbf{G}_\alpha)_\alpha\in\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$. By the way we define the operator $\langle\tilde{\mathbf{g}},\cdot\rangle\in L_b\left(\mathcal{D}^{(s)}_{L^q,h}(0,T),E\right)$, one easily verifies that $\mathbf{g}(\varphi)=\langle\tilde{\mathbf{g}},\varphi\rangle$ for all $\varphi\in\dot{\mathcal{B}}^{(s)}(0,T)\subseteq\mathcal{D}^{(s)}_{L^q,h}(0,T)$. Hence, if $\mathbf{g}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ has the representation (5.18) we will denote by $\langle\mathbf{g},\cdot\rangle$ the action $\mathbf{g}(\cdot)$.

Let $\mathbf{g}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ has the representation (5.18). Define $\tilde{\mathbf{G}}_0=0$ and $\tilde{\mathbf{G}}_\alpha(t)=\displaystyle\int_0^t\mathbf{G}_{\alpha-1}(s)ds$, $t\in[0,T]$ for $\alpha\in\mathbb{Z}_+$. Then, obviously, $\tilde{\mathbf{G}}_\alpha\in W^{1,p}(0,T;E)$, $\tilde{\mathbf{G}}_\alpha(0)=0$, $\tilde{\mathbf{G}}'_\alpha=\mathbf{G}_{\alpha-1}$ a.e. for all $\alpha\in\mathbb{Z}_+$, and if we put $h>h_1$ we have

$$\sum_{\alpha=0}^{\infty}\frac{\alpha!^{ps}}{h^{p\alpha}}\left(\|\tilde{\mathbf{G}}_\alpha\|^p_{L^p(0,T;E)}+\|\tilde{\mathbf{G}}'_\alpha\|^p_{L^p(0,T;E)}\right)<\infty. \tag{5.19}$$

By Theorem 5.3.3, $\sum_{\alpha=1}^{\infty}\tilde{\mathbf{G}}_\alpha^{(\alpha)}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$. Also, for $\varphi\in\dot{\mathcal{B}}^{(s)}(0,T)$,

$$\begin{aligned}\sum_{\alpha=1}^{\infty}(-1)^\alpha\int_0^T\tilde{\mathbf{G}}_\alpha(t)\varphi^{(\alpha)}(t)dt &= \sum_{\alpha=0}^{\infty}(-1)^\alpha\int_0^T\tilde{\mathbf{G}}'_{\alpha+1}(t)\varphi^{(\alpha)}(t)dt\\ &= \sum_{\alpha=0}^{\infty}(-1)^\alpha\int_0^T\mathbf{G}_\alpha(t)\varphi^{(\alpha)}(t)dt=\langle\mathbf{g},\varphi\rangle,\end{aligned}$$

i.e. $\mathbf{g}=\sum_{\alpha=1}^{\infty}\tilde{\mathbf{G}}_\alpha^{(\alpha)}$. In other words, for $\mathbf{g}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ and $1<p<\infty$ we can always find $h>0$ such that $\mathbf{g}=\sum_\alpha\tilde{\mathbf{G}}_\alpha^{(\alpha)}$, where $\tilde{\mathbf{G}}_\alpha\in W^{1,p}(0,T;E)$, $\tilde{\mathbf{G}}_\alpha(0)=0$, $\alpha\in\mathbb{N}$, such that (5.19) holds. Moreover, in this notation, if we put $\tilde{\mathbf{f}}=(\tilde{\mathbf{G}}'_\alpha)_\alpha\in\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$, then $\langle\tilde{\mathbf{f}},\cdot\rangle$ and the E-valued ultradistribution $\mathbf{g}'\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ (where $\mathbf{g}'$ is the ultradistributional derivative of $\mathbf{g}$) generate the same element in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)\cong L_b\left(\dot{\mathcal{B}}^{(s)}(0,T),E\right)$. To see this, for $\varphi\in\dot{\mathcal{B}}^{(s)}(0,T)$ we calculate as follows

$$\langle\tilde{\mathbf{f}},\varphi\rangle=\sum_{\alpha=0}^{\infty}(-1)^\alpha\int_0^T\tilde{\mathbf{G}}'_\alpha(t)\varphi^{(\alpha)}(t)dt=-\sum_{\alpha=0}^{\infty}(-1)^\alpha\int_0^T\tilde{\mathbf{G}}_\alpha(t)\varphi^{(\alpha+1)}(t)dt$$

which is exactly the value at φ of the ultradistributional derivative of $\mathbf{g}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$.

We consider the equation $\mathbf{u}'=A\mathbf{u}+\mathbf{f}$ in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)$. In other words, $\mathbf{f}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ is given, we search for $\mathbf{u}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ such that, for every $\varphi\in$

$\dot{\mathcal{B}}^{(s)}(0,T)$, $\langle \mathbf{u}, \varphi\rangle \in D(A)$ and $\langle \mathbf{u}', \varphi\rangle = A\langle \mathbf{u}, \varphi\rangle + \langle \mathbf{f}, \varphi\rangle$. By the above discussion, for $1 < p < \infty$, there exists $h > 0$ and $\mathbf{F}_\alpha \in W^{1,p}(0,T;E)$, $\mathbf{F}_\alpha(0) = 0$, $\alpha \in \mathbb{N}$, such that (5.19) holds (with $\mathbf{F}_\alpha$ and $\mathbf{F}'_\alpha$ in place of $\tilde{\mathbf{G}}_\alpha$ and $\tilde{\mathbf{G}}'_\alpha$) and $\mathbf{f} = \sum_{\alpha=0}^{\infty} \mathbf{F}_\alpha^{(\alpha)}$. If we put $\tilde{\mathbf{f}} = (\mathbf{F}_\alpha)_\alpha$, then $\tilde{\mathbf{f}} \in \tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)$. For $u_{0,\alpha} = 0 \in D(A)$ put $u_0 = (u_{0,\alpha})_\alpha$. Then the conditions of Theorem 5.4.1 *ii)* are satisfied, hence there exists $\tilde{\mathbf{u}} = (\mathbf{U}_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E) \cap \tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;D(A))$ which is a strict weak solution of $\tilde{\mathbf{u}}' = A\tilde{\mathbf{u}} + \tilde{\mathbf{f}}$ in $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$. If we put $\mathbf{u} = \sum_{\alpha=0}^{\infty} \mathbf{U}_\alpha^{(\alpha)} \in \mathcal{D}'^{(s)}_{L^1}(0,T;E)$, by the above discussion, $\langle \mathbf{u}, \varphi\rangle \in D(A)$, $\forall \varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$ (since this holds for $\tilde{\mathbf{u}}$) and $\mathbf{u}$ is a solution of $\mathbf{u}' = A\mathbf{u} + \mathbf{f}$ in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)$. Moreover, by Theorem 5.3.2 this $\mathbf{u}$ as well as $\mathbf{f}$ are in fact elements of $\mathcal{D}'^{(s)}_{L^p,h}(0,T;E)$. Thus, we proved the following theorem.

Theorem 5.4.2. *Let $A : D(A) \subseteq E \to E$ be a closed operator which satisfies the Hille-Yosida condition and $\mathbf{f} \in \mathcal{D}'^{(s)}_{L^1}(0,T;E)$. Then the equation $\mathbf{u}' = A\mathbf{u} + \mathbf{f}$ always has a solution $\mathbf{u} \in \mathcal{D}'^{(s)}_{L^1}(0,T;E)$. Moreover, $\mathbf{u} \in \mathcal{D}'^{(s)}_{L^p,h}(0,T;E)$ where $1 < p < \infty$ and $h > 0$ are such that*

$$\sum_{\alpha=0}^{\infty} \frac{\alpha!^{ps}}{h^{p\alpha}} \left(\|\mathbf{F}_\alpha\|^p_{L^p(0,T;E)} + \|\mathbf{F}'_\alpha\|^p_{L^p(0,T;E)} \right) < \infty,$$

with $\mathbf{f} = \sum_\alpha \mathbf{F}_\alpha^{(\alpha)}$, where $\mathbf{F}_\alpha \in W^{1,p}(0,T;E)$, $\mathbf{F}_\alpha(0) = 0$, $\alpha \in \mathbb{N}$.

5.5 Applications

Theorem 5.4.2 is applicable in variety of different situations. We collect some of them in the next proposition. First we need the following definition given in [79].

Definition 5.5.1. Let Ω be bounded open domain with smooth boundary in $\mathbb{R}^n$ and $m \in \mathbb{Z}_+$. We say that $A(x, \partial_x) = \sum_{|\alpha| \leq 2m} a_\alpha(x)\partial_x^\alpha$ where $a_\alpha \in \mathcal{C}^{2m}(\overline{\Omega})$, is strongly elliptic if there exists $c > 0$ such that

$$\mathrm{Re}(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geq c|\xi|^{2m}, \ \forall x \in \overline{U}, \ \forall \xi \in \mathbb{R}^n.$$

Proposition 5.5.1. *The operator $A : D(A) \subseteq E \to E$ is closed operator which satisfies the Hille-Yosida condition in each of the following situations:*

i) *([25]) $E = \mathcal{C}([0,1])$, $Av = -v'$, $D(A) = \{v \in \mathcal{C}^1([0,1]) |\, v(0) = 0\}$;*

ii) *([25]) for $\kappa \in (0,1)$, $E = \mathcal{C}_0^\kappa([0,1]) = \{v \in \mathcal{C}^\kappa([0,1]) |\, v(0) = 0\}$, $Av = -v'$, $D(A) = \{v \in \mathcal{C}^{1+\kappa}([0,1]) |\, v(0) = v'(0) = 0\}$;*

iii) *([25]) $E = \mathcal{C}([0,1])$, $Av = v''$, $D(A) = \{v \in \mathcal{C}^2([0,1]) |\, v(0) = v(1) = 0\}$;*

iv) *([25]) for Ω bounded open set with regular boundary in $\mathbb{R}^n$, $E = \mathcal{C}(\overline{\Omega})$, $Av = \Delta v$, $D(A) = \{v \in \mathcal{C}(\overline{\Omega}) |\, v_{|\partial\Omega} = 0,\ \Delta v \in \mathcal{C}(\overline{\Omega})\}$ (here Δ is the Laplacian in the sense of distributions in Ω);*

v) ([79]) let Ω be bounded open domain with smooth boundary in $\mathbb{R}^n$ and $m \in \mathbb{Z}_+$. Let $A(x,\partial_x)$ be strongly elliptic. Define $E = L^p(\Omega)$, $Av = -A(x,\partial_x)v$, $D(A) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$, for $1 < p < \infty$ and for $p = 1$ define $E = L^1(\Omega)$, $Av = -A(x,\partial_x)v$, $D(A) = \{v \in W^{2m-1,1}(\Omega) \cap W_0^{m,1}(\Omega) | \, A(x,\partial_x)v \in L^1(\Omega)\}$.

In particular, for $\mathbf{f} \in \mathcal{D}'^{(s)}_{L^1}(0,T;E)$, the equation $\mathbf{u}'_t = A\mathbf{u} + \mathbf{f}$ always has solution in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)$.

Proof. The facts that $A : D(A) \subseteq E \to E$ is closed operator which satisfies the Hille-Yosida condition when A and E are defined as in $i)-iv)$ are proven in Section 14 of [25]. When A and E are defined as in $v)$ Theorem 7.3.5, pg. 214, of [79] for the case $1 < p < \infty$, resp. Theorem 7.3.10, pg. 218, of [79] for the case $p = 1$, implies that A is closed operator which satisfies the Hille-Yosida condition (in fact these theorems state that A is the infinitesimal generator of analytic semigroup on $L^p(\Omega)$, $1 \le p < \infty$). Now, the fact that the equation $\mathbf{u}'_t = A\mathbf{u} + \mathbf{f}$ has solution in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ follows from Theorem 5.4.2. □

5.5.1 Parabolic equation in $\mathcal{D}'^{(s)}_{L^1}(U)$

In this subsection U is a bounded domain in $\mathbb{R}^n$ with smooth boundary. For the brevity in notation, let $\tilde{\mathcal{D}}'^{s}_{L^p,h}(U)$, resp. $\tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(U)$, be the space $\tilde{\mathcal{D}}'^{s}_{L^p,h}(0,T;E)$, resp. $\tilde{\mathcal{D}}'^{s}_{W^{1,p},h}(0,T;E)$, when $E = \mathbb{C}$. Also, for $k \in \mathbb{Z}_+$, by $\tilde{\mathcal{D}}'^{s}_{W^{k,p},h}(U)$ we denote the space of all sequences $(F_\alpha)_\alpha$, $F_\alpha \in W^{k,p}(U)$, $\forall \alpha \in \mathbb{N}^n$, for which

$$\|(F_\alpha)_\alpha\|_{\tilde{\mathcal{D}}'^{s}_{W^{k,p},h}(U)} = \left(\sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!^{ps}}{h^{p\alpha}} \|F_\alpha\|^p_{W^{k,p}(U)}\right)^{1/p} < \infty.$$

It is easy to verify that it becomes a Banach space with the norm $\|\cdot\|_{\tilde{\mathcal{D}}'^{s}_{W^{k,p},h}(U)}$.

Let $m \in \mathbb{Z}_+$, $A(x,\partial_x) = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial_x^\alpha$, where $a_\alpha \in \mathcal{E}^{(s)}(V)$ for some open set $V \subseteq \mathbb{R}^n$ and $U \subset\subset V$. We assume that $A(x,\partial_x)$ is a strongly elliptic operator. Obviously, $A(x,\partial_x)$ is continuous operator on $\dot{\mathcal{B}}^{(s)}(U)$ and on $\mathcal{D}'^{(s)}_{L^1}(U)$. Denote by $\tilde{A} : D(\tilde{A}) \subseteq L^2(U) \to L^2(U)$ the following unbounded operator

$$D(\tilde{A}) = W^{2m,2}(U) \cap W_0^{m,2}(U), \quad \tilde{A}(\varphi) = A(x,\partial_x)\varphi, \, \varphi \in D(\tilde{A}).$$

For such $A(x,\partial_x)$ the following a priori estimate holds (see Theorem 7.3.1, pg. 212, of [79]).

Proposition 5.5.2. *[79] Let $A(x,\partial_x)$ be strongly elliptic operator of order $2m$ on a bounded domain U with smooth boundary ∂U in $\mathbb{R}^n$ and let $1 < p < \infty$. Then, there exists a constant $\tilde{C} > 0$ such that*

$$\|\varphi\|_{W^{2m,p}(U)} \le \tilde{C}\left(\|A(x,\partial_x)\varphi\|_{L^p(U)} + \|\varphi\|_{L^p(U)}\right), \, \forall \varphi \in W^{2m,p}(U) \cap W_0^{m,p}(U).$$

Moreover, Theorem 7.3.5, pg. 214, of [79], yields that $-\tilde{A}$ is the infinitesimal generator of an analytic semigroup of operators on $L^2(U)$. In particular $-\tilde{A}$ is closed and it satisfies the Hille-Yosida condition (5.9) for some $\omega, C > 0$.
Now we can prove the theorem announced in the introductions. Note that we need to prove the theorem for $\mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$, since $\mathcal{D}'^{(s)}_{L^p}((0,T)\times U)$ and $\mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$ are isomorphic locally convex spaces.

Theorem 5.5.1. *Let U be a bounded domain in $\mathbb{R}^n$ with smooth boundary and $A(x,\partial_x)$ strongly elliptic operator of order $2m$ on U. Then for each $f \in \mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$ there exists $u \in \mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$ such that $u'_t + A(x,\partial_x)u = f$ in $\mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$.*

Proof. Denote by A the following unbounded operator:

$$\begin{aligned} A\tilde{f} &= (-A(x,\partial_x)F_\alpha)_\alpha \left(= (-\tilde{A}F_\alpha)_\alpha\right), \\ D(A) &= \left\{\tilde{f} = (F_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{W^{2m,2},h}(U) \mid F_\alpha \in W_0^{m,2}(U),\ \forall \alpha \in \mathbb{N}^n\right\}. \end{aligned}$$

Then, obviously, $A : D(A) \subseteq \tilde{\mathcal{D}}'^{s}_{L^2,h}(U) \to \tilde{\mathcal{D}}'^{s}_{L^2,h}(U)$ is a linear operator. Since $\tilde{A}$ is closed, by Proposition 5.5.2, it is easy to verify that A is closed. For $\lambda > \omega$, define $B_\lambda : \tilde{\mathcal{D}}'^{s}_{L^2,h}(U) \to \tilde{\mathcal{D}}'^{s}_{L^2,h}(U)$, by $B_\lambda(\tilde{f}) = (R(\lambda : -\tilde{A})F_\alpha)_\alpha$. For $\tilde{f} = (F_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{L^2,h}(U)$,

$$\|B_\lambda \tilde{f}\|_{\tilde{\mathcal{D}}'^{s}_{L^2,h}(U)} = \left(\sum_{|\alpha|=0}^{\infty} \frac{\alpha!^{2s}}{h^{2|\alpha|}} \|R(\lambda : -\tilde{A})F_\alpha\|^2_{L^2(U)}\right)^{1/2} \leq \frac{C}{\lambda-\omega}\|\tilde{f}\|_{\tilde{\mathcal{D}}'^{s}_{L^2,h}(U)}.$$

Hence B_λ is well defined continuous operator. For $(F_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{L^2,h}(U)$, by the Hille-Yosida condition for $-\tilde{A}$, Proposition 5.5.2 and the fact that $\tilde{A}R(\lambda : -\tilde{A}) = \mathrm{Id} - \lambda R(\lambda : -\tilde{A})$, we obtain

$$\left\|R(\lambda : -\tilde{A})F_\alpha\right\|_{W^{2m,2}(U)} \leq \tilde{C}\left(1 + \frac{C(\lambda+1)}{\lambda-\omega}\right)\|F_\alpha\|_{L^2(U)}.$$

This implies that $B_\lambda(F_\alpha)_\alpha = (R(\lambda : -\tilde{A})F_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{W^{2m,2},h}(U)$. Obviously $R(\lambda : -\tilde{A})F_\alpha \in W_0^{m,2}(U)$, for each $\alpha \in \mathbb{N}^n$. Hence, the image of B_λ is contained in $D(A)$. Conversely, for $(F_\alpha)_\alpha \in D(A)$, let $G_\alpha = (\lambda + \tilde{A})F_\alpha$, for each $\alpha \in \mathbb{N}^n$. Then $(G_\alpha)_\alpha \in \tilde{\mathcal{D}}'^{s}_{L^2,h}(U)$ and $B_\lambda(G_\alpha)_\alpha = (F_\alpha)_\alpha$. Hence, the image of B_λ is $D(A)$. Also, $(\lambda - A)B_\lambda = \mathrm{Id}$ and $B_\lambda(\lambda - A) = \mathrm{Id}$. We obtain that $\lambda > \omega$ is in the resolvent of A, $R(\lambda : A) = B_\lambda$, and similarly as above one can prove that $\|(\lambda-\omega)^k R(\lambda : A)^k\|_{\mathcal{L}\left(\tilde{\mathcal{D}}'^{s}_{L^2,h}(U)\right)} \leq C$, i.e. A satisfies the Hille-Yosida condition.

We want to solve the equation $u'_t(t,x) + A(x,\partial_x)u(t,x) = f(t,x)$ in $\mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$. For the simplicity of notation put $U_1 = (0,T)\times U$. By Proposition 5.2.3, there exist $h > 0$ and $F_{\alpha,\beta}(t,x) \in \mathcal{C}\left(\overline{U_1}\right)$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$ such that

$$f = \sum_{\alpha,\beta} \partial_t^\alpha \partial_x^\beta F_{\alpha,\beta} \quad \text{and} \quad \sum_{\alpha,\beta} \frac{(\alpha!\beta!)^{2s}}{h^{2(\alpha+|\beta|)}}\|F_{\alpha,\beta}\|^2_{L^\infty(\overline{U_1})} < \infty. \tag{5.20}$$

Let $E=\tilde{\mathcal{D}}'^{s}_{L^2,h}(U)$. Let $C_1'=1+\sup\limits_{\beta\in\mathbb{N}^n} h^{|\beta|}/\beta!^s$ and put $C_1=(1+T+|U|)C_1'$. Let L_f be the mapping $\varphi\mapsto L_f(\varphi)$, $\dot{\mathcal{B}}^{(s)}(0,T)\to E$ defined by $L_f(\varphi)=(\tilde{F}_{\varphi,\beta})_\beta$, where $\tilde{F}_{\varphi,\beta}(x)=\sum\limits_\alpha(-1)^\alpha\int_0^T F_{\alpha,\beta}(t,x)\varphi^{(\alpha)}(t)dt$. We prove that it is well defined and continuous mapping. First we prove that $\tilde{F}_{\varphi,\beta}$ is continuous function on $\overline{U}$ for each $\beta\in\mathbb{N}^n$ and $\varphi\in\dot{\mathcal{B}}^{(s)}(0,T)$. For $\varepsilon>0$, by (5.20), we can find $k_0\in\mathbb{Z}_+$ such that $\sum\limits_{\alpha+|\beta|\geq k_0}\frac{(\alpha!\beta!)^{2s}}{h^{2(\alpha+|\beta|)}}\|F_{\alpha,\beta}\|^2_{L^\infty(\overline{U_1})}<\frac{\varepsilon^2}{(4C_1)^2}$. For each $\alpha\in\mathbb{N}$, $\beta\in\mathbb{N}^n$, $F_{\alpha,\beta}$ is uniformly continuous (since $\overline{U_1}$ is compact in $\mathbb{R}^{n+1}$), hence there exists $\delta>0$ such that for every $t,t'\in[0,T]$, $x,x'\in\overline{U}$ such that $|t-t'|\leq\delta$ and $|x-x'|\leq\delta$,

$$\sum_{\alpha+|\beta|=0}^{k_0-1}\frac{(\alpha!\beta!)^{2s}}{h^{2(\alpha+|\beta|)}}\left|F_{\alpha,\beta}(t,x)-F_{\alpha,\beta}(t',x')\right|^2<\frac{\varepsilon^2}{(2C_1)^2}.$$

Hence

$$\left|\tilde{F}_{\varphi,\beta}(x)-\tilde{F}_{\varphi,\beta}(x')\right|$$

$$\leq\|\varphi\|_{\mathcal{D}^{(s)}_{L^2,h}(0,T)}\left(\sum_{\alpha=0}^{\infty}\frac{(\alpha!)^{2s}}{h^{2\alpha}}\int_0^T\left|F_{\alpha,\beta}(t,x)-F_{\alpha,\beta}(t,x')\right|^2dt\right)^{1/2}\leq$$

$$\leq\varepsilon\|\varphi\|_{\mathcal{D}^{(s)}_{L^2,h}(0,T)}$$

and the continuity of $\tilde{F}_{\varphi,\beta}$ follows. Also, one easily verifies that

$$\left(\sum_\beta\frac{\beta!^{2s}}{h^{2|\beta|}}\left\|\tilde{F}_{\varphi,\beta}\right\|^2_{L^\infty(U)}\right)^{1/2}\leq$$

$$\leq T^{1/2}\|\varphi\|_{\mathcal{D}^{(s)}_{L^2,h}(U)}\left(\sum_{\alpha,\beta}\frac{(\alpha!\beta!)^{2s}}{h^{2(\alpha+|\beta|)}}\|F_{\alpha,\beta}\|^2_{L^\infty(\overline{U_1})}\right)^{1/2}.$$

Since $\left\|\tilde{F}_{\varphi,\beta}\right\|_{L^2(U)}\leq|U|^{1/2}\left\|\tilde{F}_{\varphi,\beta}\right\|_{L^\infty(U)}$, we obtain that L_f is well defined and $L_f\in L\left(\dot{\mathcal{B}}^{(s)}(0,T),E\right)$. Now, as $L_b\left(\dot{\mathcal{B}}^{(s)}(0,T),E\right)\cong\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ denote by $\mathbf{f}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ the mapping L_f.

Now, Theorem 5.4.2 implies that there exists $\mathbf{u}\in\mathcal{D}'^{(s)}_{L^1}(0,T;E)$ such that $\mathbf{u}'=A\mathbf{u}+\mathbf{f}$ in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)$. Each element $\mathbf{g}=(G_\alpha)_\alpha\in E=\tilde{\mathcal{D}}'^{s}_{L^p,h}(U)$ generates an element of $L_b\left(\dot{\mathcal{B}}^{(s)}(U),\mathbb{C}\right)=\mathcal{D}'^{(s)}_{L^1}(U)$ (see Section 5.4) by $\langle S(\mathbf{g}),\psi\rangle=\sum\limits_\beta(-1)^{|\beta|}\int_U G_\beta(x)\partial^\beta_x\psi(x)dx$ and one easily verifies that the mapping $S:E\to$

$\mathcal{D}'^{(s)}_{L^1}(U)$, $\mathbf{g} \mapsto S(\mathbf{g})$, is continuous. Hence, we have the continuous mapping $\varphi \mapsto S(\langle \mathbf{u}, \varphi\rangle)$, given by

$$\dot{\mathcal{B}}^{(s)}(0,T) \xrightarrow{\langle \mathbf{u},\cdot\rangle} E \xrightarrow{S} \mathcal{D}'^{(s)}_{L^1}(U).$$

Since $\varphi \mapsto S(\langle \mathbf{u}, \varphi\rangle) \in L_b\left(\dot{\mathcal{B}}^{(s)}(0,T), \mathcal{D}'^{(s)}_{L^1}(U)\right) \cong \mathcal{D}'^{(s)}_{L^1}(U_1)$ (where the isomorphism follows from Theorem 5.3.1), denote by $u \in \mathcal{D}'^{(s)}_{L^1}(U_1)$ this ultradistribution. Then, for $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$, $\psi \in \dot{\mathcal{B}}^{(s)}(U)$, $\langle u(t,x), \varphi(t)\psi(x)\rangle = \langle S(\langle \mathbf{u}, \varphi\rangle), \psi\rangle$. Since $\langle \mathbf{u}', \varphi\rangle = -\langle \mathbf{u}, \varphi'\rangle$, for all $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$ we have
$\langle u'_t(t,x), \varphi(t)\psi(x)\rangle = -\langle u(t,x), \varphi'(t)\psi(x)\rangle = \langle S(\langle \mathbf{u}', \varphi\rangle), \psi\rangle$, for all $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$, $\psi \in \dot{\mathcal{B}}^{(s)}(U)$. Also, for $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$, since $\langle \mathbf{u}, \varphi\rangle \in D(A)$, $\langle \mathbf{u}, \varphi\rangle = (G_{\varphi,\beta})_\beta \in D(A)$. Then, by the definition of A, $A\langle \mathbf{u}, \varphi\rangle = (-\tilde{A}G_{\varphi,\beta})_\beta \in E$. Now, for $\psi \in \dot{\mathcal{B}}^{(s)}(U)$,

$$\begin{aligned}
&\left\langle S\left((-\tilde{A}G_{\varphi,\beta})_\beta\right), \psi\right\rangle \\
&\quad= -\sum_\beta (-1)^{|\beta|}\int_U \tilde{A}G_{\varphi,\beta}(x)\partial_x^\beta\psi(x)dx \\
&\quad= -\sum_\beta (-1)^{|\beta|}\int_U G_{\varphi,\beta}(x)\,{}^tA(x,\partial_x)\partial_x^\beta\psi(x)dx = -\langle S(\langle \mathbf{u},\varphi\rangle), {}^tA(x,\partial_x)\psi\rangle \\
&\quad= -\langle u(t,x), \varphi(t)\,{}^tA(x,\partial_x)\psi(x)\rangle = -\langle A(x,\partial_x)u(t,x), \varphi(t)\psi(x)\rangle,
\end{aligned}$$

i.e. $\langle S(A\langle \mathbf{u},\varphi\rangle), \psi\rangle = -\langle A(x,\partial_x)u(t,x), \varphi(t)\psi(x)\rangle$ for all $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$, $\psi \in \dot{\mathcal{B}}^{(s)}(U)$. Moreover, observe that for $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$, $\psi \in \dot{\mathcal{B}}^{(s)}(U)$, we have

$$\begin{aligned}
\langle S(\langle \mathbf{f},\varphi\rangle), \psi\rangle &= \sum_\beta (-1)^{|\beta|}\int_U \tilde{F}_{\varphi,\beta}(x)\partial_x^\beta\psi(x)dx \\
&= \sum_{\alpha,\beta} (-1)^{\alpha+|\beta|}\int_{U_1} F_{\alpha,\beta}(t,x)\varphi^{(\alpha)}(t)\partial_x^\beta\psi(x)dtdx \\
&= \langle f(t,x), \varphi(t)\psi(x)\rangle,
\end{aligned}$$

where, in the second equality, we used the definition of $\tilde{F}_{\varphi,\beta}$ and Fubini's theorem since $\sum_{\alpha,\beta}\int_{U_1} |F_{\alpha,\beta}(t,x)|\left|\varphi^{(\alpha)}(t)\right|\left|\psi^{(\beta)}(x)\right|dtdx < \infty$ by (5.20). Now, since $\mathbf{u}' = A\mathbf{u} + \mathbf{f}$ in $\mathcal{D}'^{(s)}_{L^1}(0,T;E)$, for every $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$, $\langle \mathbf{u}'(t), \varphi(t)\rangle = A\langle \mathbf{u}(t), \varphi(t)\rangle + \langle \mathbf{f}(t), \varphi(t)\rangle$ in E. Then $S(\langle \mathbf{u}', \varphi\rangle) = S(A\langle \mathbf{u}, \varphi\rangle) + S(\langle \mathbf{f}, \varphi\rangle)$ in $\mathcal{D}'^{(s)}_{L^1}(U)$. Hence, for $\varphi \in \dot{\mathcal{B}}^{(s)}(0,T)$, $\psi \in \dot{\mathcal{B}}^{(s)}(U)$, we have

$$\begin{aligned}
\langle u'_t(t,x), \varphi(t)\psi(x)\rangle &= \langle S(\langle \mathbf{u}',\varphi\rangle), \psi\rangle = \langle S(A\langle \mathbf{u},\varphi\rangle), \psi\rangle + \langle S(\langle \mathbf{f},\varphi\rangle), \psi\rangle \\
&= -\langle A(x,\partial_x)u(t,x), \varphi(t)\psi(x)\rangle + \langle f(t,x), \varphi(t)\psi(x)\rangle.
\end{aligned}$$

Since $\dot{\mathcal{B}}^{(s)}(0,T)\hat{\otimes}\dot{\mathcal{B}}^{(s)}(U) \cong \dot{\mathcal{B}}^{(s)}(U_1)$ by Theorem 5.3.1, we obtain the claim in the theorem. □

Example 5.5.1. An interesting application of this theorem is obtained by taking $A(x,\partial_x)$ to be $-\Delta_x$ (Δ_x is the Laplacian $\partial^2_{x_1}+...+\partial^2_{x_n}$) and U to be arbitrary bounded domain with smooth boundary in $\mathbb{R}^n$. Then $-\Delta_x$ is strongly elliptic operator of order 2 on U. The above theorem then asserts that for $f \in \mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$ the equation $u'_t - \Delta_x u = f$ always has solution in $\mathcal{D}'^{(s)}_{L^1}((0,T)\times U)$.

Example 5.5.2. If $U = (0,T_1) \subseteq \mathbb{R}$ and A is differentiation in x, arguing as above, one can prove the following assertion: Let $f \in \mathcal{D}'^{(s)}_{L^1}((0,T)\times(0,T_1))$. The equation $u'_t + u'_x = f$ always has a solution in $\mathcal{D}'^{(s)}_{L^1}((0,T)\times(0,T_1))$.

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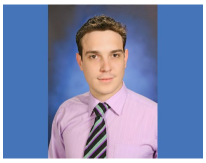
Short Biography



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M. Kostić, S. Pilipović and D. Velinov, *Hyperfunction semigroups*, preprint; `arXiv: 1306.1098 [math.FA]`

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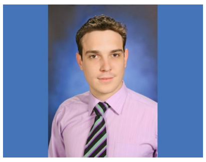

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Abstract: We are study the spaces of convolutors and multipliers in the spaces of tempered ultradistributions. There given theorems which gives us the characterization of all the elements which belongs to spaces of convolutors and multipliers. Structural theorem for ultradistribution semigroups and exponential ultradistribution semigroups is given. Fourier hyperfunction semigroups and hyperfunction semigroups with non-densely defined generators are analyzed and also structural theorems and spectral characterizations give necessary and sufficient conditions for the existence of such semigroups generated by a closed not necessarily densely defined operator A. The abstract Cauchy problem is considered in the Banach valued weighted Beurling ultradistribution setting and given some applications on particular equations.

AB

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Важна напомена:
ВН

Извод: У дисертацији се проучавају простор конволутора и мултипликатора на просторима темперираних ултрадистрибутција. Доказане су теореме који карактеришу елементе простора конволутора и мултипликатора. Дате су структурне теореме за ултрадистрибуционе полугрупе и експоненецијалне полугрупе. Фуријеве хуперфункциске полугрупе и хиперфункциске полугрупе са генераторима који су негусто дефинисани су анализирани, такође су дате структурне теореме и спектралне карактеризације као и довољни услови за постојење на таквих полугрупа за оператор A који не мора бити густ. Апстрактни Кошијев проблем је проучаван за тежинске Банахове просторе као и за одговарујуће простора ултрадистрибуција. Такође су дате и примене за одређене класе једначина.

ИЗ

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