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Split delta shocks and applications to conservation law systems

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Contents

Introduction

Starting from the last decade in the previous century non-standard solutions to conservation laws become more popular. Usually, these solutions are unbounded and correspond to some kind of a measure instead of functions with bounded variation.

There are a lot of approaches in solving systems with above type of singular solutions. Let us mention some of them:

- The numerical evidence for existing of unbounded weak solutions, [19].
- Using measure valued solutions, but for BV solutions only, [10]
- Using Vol'pert product ([48]) to obtain measure solutions, [21].
- Vanishing viscosity, [46].
- Methods of geometrical optics and generalized solutions, [36].
- Deep analysis, smooth, box approximations, singular shocks, weighted measure spaces, [18].
- Using theoretical measure theory, [1].
- Generalized variational principle, [11].
- Sticky particle method, [2].
- Split delta shocks, [30], [33].
- Vanishing pressure, [5], [6], [28].
- Smooth approximations, sided delta functions, [31].
- Weak asymptotic methods, [8], [9].
- Shadow waves, [32].

This thesis is devoted to applications of split delta shocks to conservation law systems. The main result is a definition of an inverse of split delta shock and its applications to a fairly general class of conservation laws. The main example is well known chromatography model fully described in [27] and [37]. Some results are given in [45], [14], [43], and [47]. The main result is briefly described in [29].

The thesis is organized as follows.

In the first chapter, we give definitions and properties of different spaces used in solving both linear and nonlinear PDE's, especially hyperbolic PDE's and systems.

In the second chapter, we present some classical ideas how to solve one dimensional hyperbolic semilinear and quasilinear quasilinear system. The main tool is the method of characteristics and all solutions are strong and local in general.

The third chapter is devoted to basic properties and BV solutions to Riemann and initial data problems for one dimensional conservation law systems. We introduce elementary solutions, shocks, rarefaction waves and contact discontinuities.

A bit original look at split delta shock solutions is given in the fourth chapter. Delta shocks are added to other classical elementary waves defined in the previous chapter in order to solve a wider class of problems arising in the science and technology.

The original part is the fifth chapter. It contains the definition of a split delta shock inverse and its application to different problems. A complete analysis is done for the singular chromatography model.

Chapter 1

Basic spaces

1.1 Classical function spaces

1.1.1 Space of differentiable functions

Denote by $\Omega \subset \mathbb{R}^n$ an open set, its closure by $\bar{\Omega}$ and a boundary by $\partial\Omega$.

$C^k(\Omega)$ is the set of all functions $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$, but all functions in conservation laws are real-valued) with continuous derivatives of order k, $0 \le k \le \infty$.

$C^k(\bar{\Omega})$ is the set of all functions $u \in C^k(\Omega)$ such that there exists a function $\phi \in C^k(\Omega')$, $u \equiv \phi$ on $\bar{\Omega} \subset \Omega'$, where Ω' is an open set.

$C^k_b(\Omega)$ consists of functions from $C^k(\Omega)$ bounded together with all their derivatives. It satisfying

$$C^k(\mathbb{R}^n)\Big|_{\Omega} \subset C^k(\bar{\Omega}) \subset C^k(\Omega).$$

If Ω is bounded, then $C^k(\bar{\Omega}) \subset C^k_b(\Omega)$.

Denote by $\operatorname{supp} u$, $u : \Omega \to \mathbb{R}$, the complement of the largest open set Ω' such that $u\big|_{\Omega'} = 0$. The set $\operatorname{supp} u$ is called support of the function u since $\Omega \in \mathbb{R}^n$,

$$\operatorname{supp} u = \overline{\{x \in \Omega : \ u(x) \neq 0\}}.$$

Note that $A \Subset B$ means that there exists a compact K such that $A \subset K \subset B$.

$$C^k_0(\Omega) = \{u \in C^k(\Omega) : \operatorname{supp} u \Subset \Omega\}.$$

Elements of C^∞_0 are called test functions.

1.1.2 L^p−spaces

A set $A \subset \Omega \subset \mathbb{R}^n$ is of Lebesgue measure zero, $\mathcal{L}(A) = 0$, if for each $\varepsilon > 0$ there exists a numerable union $\bigcup_{i\in\mathbb{N}} C_i$ of parallelepipeds $C_i \subset \mathbb{R}^n$ such that $\operatorname{mes} \bigcup_{i=1}^{\infty} C_i < \varepsilon$ (the measure of parallelepipeds is the product of its edges lengths). For a definition of Lebesgue measure for sets in $\mathbb{R}^n$ one can look in [40], for example.

In the set of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ (all elementary functions and their compositions are Lebesgue measurable, for example). We define the equivalence relation "equal almost everywhere in Ω", $f \sim g$, if $\mathcal{L}(\{x : \; f(x) \neq g(x)\}) = 0$.

Let $1 \leq p \leq \infty$. From now on Ω will be an open connected set. "Measurable" stands for Lebesgue measurable.

$$L^p(\Omega) = \{f/_\sim : \Omega \to \mathbb{R} : \; f \text{ is measurable, } \int_\Omega |f(x)|^p \, dx < \infty\}$$

is Banach space with the norm

$$\|f\|_{L^p(\Omega)} = \Big(|f(x)|^p \, dx\Big)^{1/p}.$$

$L^2(\Omega)$ is Hilbert space with the product $(f|g)$ defined by

$$(f|g) = \int_\Omega f(x)\overline{g(x)} \, dx,$$

where $\overline{g(x)}$ stands for complex conjugate of $g(x)$. If we are in the space of real-valued functions (which will usually be the case), then

$$(f|g) = \int_\Omega f(x)g(x) \, dx.$$

For $p = \infty$ we have a different definition:

We define

$$\begin{aligned} L^\infty(\Omega) = \{f/_\sim : \Omega \to \mathbb{R} : \; & f \text{ is measurable and there exists real} \\ & M \text{ such that } |f(x)| \leq M, \text{ for every } x \in \Omega\}. \end{aligned} \tag{1.1}$$

$L^\infty(\Omega)$ is also Banach space with the norm $\|f\|_{l^\infty} = \inf M$, where the constant M is from (1.1).

The most important spaces are L^2−spaces and L^1_{loc}−spaces which are defined by

$$\begin{aligned} L^1_{loc}(\Omega) = \{f/_\sim : \Omega \to \mathbb{R} : \; & f \text{ is measurable and for every} \\ & K \Subset \Omega \text{ we have } \int_k |f(x)|^p \, dx < \infty\}. \end{aligned} \tag{1.2}$$

Functions from L^1_{loc} are called locally integrable ones.

Hölder inequality

$$\int_\Omega |u(x)v(x)|dx \leq \|u\|_{L^p}\|v\|_{L^q}, \quad u \in L^p(\Omega),\, v \in L^q(\Omega),\, \frac{1}{p}+\frac{1}{q}=1 \quad (1.3)$$

will be often used. The special case $p = q = 2$ is called Schwartz inequality.

Corollaries of Hoölder inequality are:

1. $\mathrm{mes}(\Omega)^{-1/p}\|u\|_{L^p} \leq \mathrm{mes}(\Omega)^{-1/q}\|u\|_{L^q}$, $u \in L^q(\Omega)$, $p \leq q$,
2. $\|u\|_{L^q} \leq \|u\|^\lambda_{L^p}\|u\|^\lambda_{L^r}$, $u \in L^r(\Omega)$, $p \leq q \leq r$, $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$,
3. $\int_\Omega u_1 \cdot \ldots \cdot u_m\, dx \leq \|u\|_{L^{p_1}} \cdot \ldots \cdot \|u\|_{L^{p_m}}$, $u_i \in L^{p_i}(\Omega)$, $i = 1, \ldots, m$, $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1$.

1.2 Weak solutions

1.2.1 Weak derivative

Denote by $|\alpha| = \alpha_1 + \ldots + \alpha_n$ multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$ and

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \cdot \ldots \cdot \partial^{\alpha_n}x_n}$$

(is $\alpha_i = 0$ for some i, there is no derivative with respect to the variable x_i.)

Definition 1.1. A function $f \in L^1_{loc}(\Omega)$ has $\alpha-$th weak derivative, $|\alpha| \leq m$, denoted again by $\partial^\alpha f$, if there exist a function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f(x)\partial^\alpha\phi(x)\, dx = (-1)^{|\alpha|}\int_\Omega g(x)\phi(x)\, dx,$$

for every $\phi \in C^\infty_0$. The function g will be called $\alpha-$th weak derivative for f.

Theorem 1.1. *If there exists a weak derivative for a locally integrable function u, then u is almost everywhere differentiable and the weak derivative equals to a strong at the points where it exists.*

1.2.2 Weak solution of partial differential equations

Notion of a weak solution is not defined in a unique manner. It should be defined to fit a physical problem as much as it can.

First, we shall give the definition for first order systems. Later on, the definition will be easily adopted to an equation of higher order.

Definition 1.2. A system of first order partial differential equation is in divergence form if it can be written as

$$\partial_t a_0(t,x,u) + \partial_{x_1} a_1(t,x,u) + \ldots + \partial_{x_n} a_n(t,x,u) = b(t,x,u), \tag{1.4}$$

where $u = u(t, x_1, x_2, \ldots, x_n)$ is a vector-valued function. Let that u satisfies initial condition $u(x,0) = u_0(x)$. It is said that $u \in \Big(L^1_{loc}([0,T) \times \Omega)\Big)^n$ is weak solution to the system (1.4) with the above given initial data if

$$\begin{aligned}
&\int_0^t \int_\Omega \partial_t \phi(t,x) a_0(t,x,u) + \partial_{x_1} \phi(t,x) a_1(t,x,u) + \ldots \\
&+ \partial_{x_n} \phi(t,x) a_n(t,x,u)\, dxdt + \int_\Omega u_0(x) \phi(x,0)\, dx \\
= &\int_0^t \int_\Omega b(t,x,u) \phi(t,x)\, dxdt,
\end{aligned} \tag{1.5}$$

for every $\phi \in C_0^\infty\big((-\infty,\infty) \times \Omega\big)$.

As one can see, vector-valued function u is not necessary differentiable and the name “weak solution” comes from that fact. Also, it is easy to check, using integration by parts, that every C^1-solution of (1.4) also satisfied (1.5), i.e. it is weak solution, too. For practical reasons we shall use the following simpler (and weaker) condition instead of (1.5):

$$\begin{aligned}
&\int_0^t \int_\Omega \partial_t \phi(t,x) a_0(t,x,u) + \partial_{x_1} \phi(t,x) a_1(t,x,u) + \ldots \\
&+ \partial_{x_n} \phi(t,x) a_n(t,x,u)\, dxdt \\
= &\int_0^t \int_\Omega b(t,x,u) \phi(t,x)\, dxdt, \\
&\lim_{t \to 0} u(t,x) = u_0 \text{ almost everywhere in } \Omega,
\end{aligned} \tag{1.6}$$

for every $\phi \in C_0^\infty\big((0,\infty) \times \Omega\big)$. Note that now ϕ is defined on a smaller domain, i.e. it equals zero on the $x-$axes ($t = 0$).

Remark 1.1. If a system is not given in the divergence form, then a definition of a weak solution is much more difficult to give and more specific.

Systems where t is distinguished variable are called evolution systems (or systems “written in evolution form”).

1.3 Distribution spaces

In this section we shall present a simplified version of distribution theory by using a convergence in vector spaces instead of a topology.

Mapping from a vector space over some field into that field (usually $\mathbb{R}$ or $\mathbb{C}$) is called functional. Let us introduce a convergence in the set $C_0^\infty(\Omega)$.

Definition 1.3. A sequence $\{\phi_j\} \subset C_0^\infty(\Omega)$ converge to zero as $j \to \infty$ if

- There exists a compact $K \Subset \Omega$ such that $\operatorname{supp} \phi_j \subset K$ for every $j \in \mathbb{N}$.
- For each $\alpha \in \mathbb{N}_0^n$, $\|\partial^\alpha \phi\|_{L^\infty(\Omega)} \to 0$ as $j \to \infty$. This convergence is denoted by $\xrightarrow{\mathcal{D}}$.

The set $C_0^\infty(\Omega)$ with the convergence defined in this way will be denoted by $\mathcal{D}(\Omega)$. Elements of this space will be called test functions.

Definition 1.4. Linear continuous functional S with the domain $\mathcal{D}(\Omega)$ is called distribution. Its acting on the test function ϕ is denoted by $\langle S, \phi \rangle$.

Continuity is understood in the means of convergence: S is continuous if for each sequence of test functions $\{\phi_j\}_j$ converging to zero as $j \to \infty$ it holds $\langle S, \phi_j \rangle \to 0$ as $j \to \infty$. Vector space of distributions is denoted by $\mathcal{D}'(\Omega)$.

Now, we shall give some important examples of distributions. The first one show how locally integrable function can be treated as distributions and the second one is an example of distributions which cannot be treated as a usual function.

Example 1.1. Let $f \in L^1_{loc}(\Omega)$ and ϕ be a test function. Then mapping from $\mathcal{D}$ into $\mathbb{R}$ defined by

$$S_f : \langle S_f, \phi \rangle = \int_\Omega f(x)\phi(x)\, dx$$

define a distribution. Functional S_f is obviously linear and

$$|\langle S_f, \phi \rangle| \leq \|\phi\|_{L^\infty(\Omega)} \int_{\operatorname{supp} \phi} |f(x)|\, dx.$$

That means that if a sequence $\{\phi_j\}$ converges to zero in $\mathcal{D}$, then $\langle S_f, \phi \rangle \to 0$ as $j \to \infty$, i.e. S_f is a distribution.

Example 1.2. Let $a \in \Omega$. Relation $\langle \delta_a, \phi \rangle = \phi(a)$ defines Dirac delta distribution at the point a. If $a = 0$, then we write just δ instead of δ_0.

1.3.1 Properties and operations with distributions

1. For a sequence of distributions $\{S_j\} \subset \mathcal{D}'(\Omega)$ is said to converge to zero if

   $$\langle S_j, \phi \rangle \to 0 \quad \text{as } j \to \infty$$

   for every $\phi \in \mathcal{D}(\Omega)$. Convergence in the distribution space is denoted by $\xrightarrow{\mathcal{D}'}$. (In distribution theory this convergence is called "weak".) Convergence to zero is enough since the distribution space is a vector one: $S_j \to T$, $T \in \mathcal{D}'(\Omega)$ if and only if $\langle S_j - T, \phi \rangle \to 0$ as $j \to \infty$ for every test function ϕ.

2. $S \in \mathcal{D}'(\Omega)$ is zero on $\omega \subset \Omega$ is $\langle S, \phi \rangle = 0$ for every test function ϕ with a support in ω.

Definition 1.5. Support of a distribution $S \in \mathcal{D}'(\Omega)$, $\operatorname{supp} S$, is a complement of the maximum open set where $S = 0$ (i.e. set of points in Ω which do not have a neighborhoods ω where $S = 0$.)

Definition 1.6. $\mathcal{E}'(\Omega)$ in the space of distributions with compact support.

Example 1.3. $\operatorname{supp} \delta = \{0\}$, because for each $x \in \Omega$, $x \neq 0$, there exists its neighborhoods ω not containing zero and there exist a test function ϕ with a support in Ω. In the same way we can define

$$\langle \delta_a, \phi \rangle = \phi(a) = 0.$$

Definition 1.7. Distributional derivative S of order $\alpha \in \mathbb{N}_0^n$ is defined by

$$\langle \partial^\alpha S, \phi \rangle := (-1)^{|\alpha|} \langle S, \partial^\alpha \phi \rangle$$

for every $\phi \in \mathcal{D}(\Omega)$. Since $\partial^\alpha \phi$ is also in $\mathcal{D}(\Omega)$, one can see that the definition makes sense, i.e. each distribution has a derivative of every order. That fact is the main reason why distributions are so important.

Lemma 1.1. *Differentiation is a continuous operation in the distribution space.*

Example 1.4. We can easily calculate derivative of the delta distribution

$$\langle \partial^\alpha \delta, \phi \rangle = (-1)^{|\alpha|} \langle \delta, \partial^\alpha \phi \rangle = (-1)^{|\alpha|} \partial^\alpha \phi(x).$$

One can easily verify the following. If $g \in L^1_{loc}(\Omega)$ is $\alpha-$th weak derivative of $f \in L^1_{loc}(\Omega)$, then $S_g = \partial^\alpha S_f$, where S_f (or S_g) is the distributional image of f (or g).

Example 1.5. Define Heaviside function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

Since H is locally integrable function we can identify it with a distribution defined on $\mathbb{R}$, we will show that its derivative is the delta distribution. Let ϕ be an arbitrary test function on $\mathbb{R}$. Then

$$\langle H', \phi\rangle = -\langle H, \phi'\rangle = \int_0^\infty \phi'(x)\, dx = \phi(0) = \langle \delta, \phi\rangle.$$

If $W^k(\Omega)$ stands for the space of locally integrable functions on Ω having all derivatives of order less or equal to k, then

$$C^k(\Omega) \subset W^k(\Omega) \subset \mathcal{D}'(\Omega).$$

(Here, function is identified with its image in the space of distributions.)

If $f \in C^\infty(\Omega)$, then we can define its product with a distribution S, $T = Sf$, in the following way

$$\langle T, \phi\rangle := \langle S, f\phi\rangle, \quad \phi \in \mathcal{D}(\Omega).$$

But there is no general definition of the product if f is not. This is the main disadvantage of distributions.

Definition 1.8. We say that $T \in \mathcal{D}'(\Omega)$ is zero at an open set $U \subset \Omega$, $T|_U = 0$ if $\langle T, \varphi\rangle = 0$ for every $\varphi \in \mathcal{D}(\Omega)$, $\varphi|_U = 0$. Let us now define support of a distribution:

$$\operatorname{supp} T = \operatorname{compl}\left(\bigcup_{U \text{ is open }, T|_U = 0} U \right).$$

Denote by $\mathcal{E}'(\Omega)$ the subspace of distributions having a compact support. It is isomorphic with a dual space of smooth functions with the uniform convergence of all its derivatives.

1.3.2 Fourier transform

In this subsection, we will assume that $\Omega = \mathbb{R}^n$.

In order to extend the Fourier transform for distributions, we have to restrict the space $\mathcal{D}'(\mathbb{R}^n)$ to a smaller one.

Definition 1.9. Define

$$\mathcal{S} = \{\phi \in \mathcal{C}^\infty(\mathbb{R}^n) : \lim_{|x|\to\infty} |x^\alpha \partial^\beta \phi(x)| = 0 \text{ for every } \alpha, \beta \in \mathbb{N}_0^n\}$$

with the convergence: We say that a sequence $\{\phi_j\}_j \subset \mathcal{S}(\mathbb{R}^n)$ converge to 0 in $\mathcal{S}$, denoted by $S_j \stackrel{\mathcal{S}}{\to} 0$, as $j \to \infty$, if

$$\lim_{j\to\infty} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi_j(x)| = 0, \text{ for every } \alpha, \beta \in \mathbb{N}_0^n.$$

Definition 1.10. The dual of $\mathcal{S}(\mathbb{R}^n)$ is called the space of tempered distributions (Schwartz distributions) denoted by $\mathcal{S}'(\mathbb{R}^n)$.

The classical Fourier transform is defined on $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$ in the following way. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. Then

$$\mathcal{F}(f)(x) = \hat{f}(x) := (2\pi)^{-n/2} \int f(y) e^{ix\cdot y} \mathrm{d}y \in L^q(\mathbb{R}^n),$$

where q satisfies $1/p + 1/q = 1$

We have the following theorem.

Theorem 1.2. *(a) If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C^0(\mathbb{R}^n)$.*

(b) If $y_j f(y) \in L^q(\mathbb{R}^n)$, $q \geq 1$, then there exists $\partial_{x_j}\hat{f}$, $\hat{f} \in L^p(\mathbb{R}^n)$, $1/p + 1/q = 1$, and

$$\partial_{x_j}\hat{f}(x) = -i(\widehat{y_j f(y)})(x), \ x \in \mathbb{R}^n.$$

(c) If $\partial_{y_j} f(y) \in L^p(\mathbb{R}^n) \cup \mathcal{C}(\mathbb{R}^n)$, $1 \leq p \leq 2$, then

$$(\widehat{\partial_{y_j} f(y)})(x) = i x_j \hat{f}(x), \ x \in \mathbb{R}^n.$$

If $f \in L^2$, then $\mathcal{F}(\mathcal{F}(f(x))) = (2\pi)^{-n} f(-x)$. As $\mathcal{S} \subset L^2$, the assertions above hold for $f \in \mathcal{S}$. Also, we have the following theorem.

Theorem 1.3. *Let $\phi, \psi \in \mathcal{S}$. Then*

- *$\phi \mapsto \hat{\phi}$ is injective and continuous iz $\mathcal{S}$ na $\mathcal{S}$.*
- *$(\widehat{\hat{\phi}})(x) = (2\pi)^{-n}\phi(-x)$.*
- *$(\widehat{-i\partial_{x_j}\phi(x)})(y) = y_j \hat{\phi(x)}(y)$.*
- *$(\widehat{x_j \phi(x)})(y) = i\partial_{y_j}\hat{\phi(x)}(y)$.*

- $\widehat{(\phi * \psi)} = (2\pi)^{n/2}\hat{\phi}\hat{\psi}$.
- $\widehat{(\phi\psi)} = (2\pi)^{n/2}\hat{\phi} * \hat{\psi}$.

Definition 1.11. For $S \in \mathcal{S}$, let us define its Fourier transform $\hat{S}$ by

$$\langle \hat{S}, \phi \rangle := \langle S, \hat{\phi} \rangle, \text{ za svako } \psi \in \mathcal{S}.$$

It is well defined, since $\phi_j \xrightarrow{\mathcal{S}} 0$ implies $\langle \hat{S}, \hat{\phi}_j \rangle = \langle S, \hat{\phi}_j \rangle \to 0$, due to (i) from the above theorem.

Example 1.6. If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = (2\pi)^{-n/2} \int e^{-i0\cdot x}\phi(x)\mathrm{d}x = (2\pi)^{-n/2}\langle \mathbf{1}, \phi \rangle,$$

or $\hat{\delta} = (2\pi)^{-n/2}\mathbf{1}$.

One can show in the same way that

$$\begin{aligned}
&\langle \widehat{e^{ia\cdot x}}(y), \phi(y) \rangle = \langle e^{ia\cdot x}, \hat{\phi}(x) \rangle = \int e^{ia\cdot x}\hat{\phi}(x)\mathrm{d}x \\
=&(2\pi)^{-n/2} \int\int e^{-ix\cdot(y-a)}\phi(y)\mathrm{d}y\mathrm{d}x \\
&\text{(after a change of variables)} \\
=&(2\pi)^{-n/2} \int\int e^{-ix\cdot y}\phi(y+a)\mathrm{d}y\mathrm{d}x \\
=&(2\pi)^{-n/2}\langle \mathbf{1}, \widehat{\phi(y+a)} \rangle \\
&\text{(as above)} \\
=&\langle \delta, \phi(y+a) \rangle = \langle \delta_a, \phi \rangle.
\end{aligned}$$

Thus, $\widehat{e^{ia\cdot x}}(y) = \delta_a(y)$.

Theorem 1.4. *Mapping $S \mapsto \hat{S}$ iz $\mathcal{S}'$ on $\mathcal{S}'$ is injective and continuous. Also*

(i) If $S \in \mathcal{E}'$, $T \in \mathcal{S}'$, then $\hat{S} \in \mathcal{C}^\infty$ i $T * S \in \mathcal{S}'$.

(ii) If $S \in \mathcal{E}'$, $T \in \mathcal{S}'$, then $\widehat{(S * T)} = \hat{S}\hat{T}$.

(iii) If P is a polynomial, then

$$(\widehat{P(\partial)u(x)})(\xi) = P(i\xi)\hat{u}(\xi)$$

for every $u \in \mathcal{S}'$.

(iv) $(\widehat{y_j S(y)})(x) = i\partial_{x_j}\hat{S}(x)$.

The inverse Fourier transform is defined by

$$\mathcal{F}^{-1}(\phi)(x) = (2\pi)^{-n/2}\int e^{ix\cdot y}\phi(y)\mathrm{d}y.$$

It has the same properties as the Fourier one and

$$\mathcal{F}(\mathcal{F}^{-1}(\phi)) = \mathcal{F}^{-1}(\mathcal{F}(\phi)) = (2\pi)^{-n}\phi.$$

1.4 Sobolev spaces

1.4.1 Definitions

Let $m \in \mathbb{N}_0$, $p \geq 1$ and Ω be an open subset of $\mathbb{R}^n$. Denote by $W^k(\Omega)$ the vector space of all locally integrable functions on Ω which has all weak derivatives of order less or equal to k. We are in position to define its subspaces which will have the advantage to be normed (space $W^k(\Omega)$ is only locally convex, with topology defined by a sequence of seminorms).

Definition 1.12. Sobolev space $H^{m,p}(\Omega)$ is the set of functions $u \in W^m(\Omega)$, such that $\partial^\alpha u \in L^p(\Omega)$ for every $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$. Norm is given by

$$\|u\|_{H^{m,p}(\Omega)} = \|u\|_{m,p,\Omega} := \Big(\int_\Omega \sum_{|\alpha|\leq m} |\partial^\alpha u(x)|^p \, dx\Big)^{1/p}.$$

One can use the rquivalent norm given by

$$\|u\|'_{H^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^p(\Omega)}.$$

In the rest of the text, we will not distinguish them by symbols, that is, for any of these norms we use the symbol $\|u\|_{H^{m,p}(\Omega)}$.

If $p = 2$, we omit that number in the upper index for Sobolev's spaces or norms.

It is easy to see that the spaces of Sobolev can be viewed as subspaces of the space of tempered distributions.

We will now dedicate our attention to the very important one because it occurs naturally in many physical phenomena described by partial differential equations, the space $H^1(\Omega)$.

Lemma 1.2. *The space $H^1(\Omega)$ has an inner product compatible with the above defined norm. The product is given by*

$$\begin{aligned}(u|v) &= \int_\Omega u(x)v(x)\mathrm{dx} + \int_\Omega \nabla u(x)\nabla v(x)\mathrm{dx} \\ &= \int_\Omega u(x)v(x)\mathrm{dx} + \int_\Omega \Big(\sum_{j=1}^{n} \partial_{x_j} u(x)\partial_{x_j} v(x)\mathrm{dx}\Big).\end{aligned} \tag{1.7}$$

Proof. The norm induced by (1.7) is

$$\sqrt{(u|u)} = \sqrt{\int_\Omega |u(x)|^2\mathrm{dx} + \int_\Omega |\nabla u(x)|^2\mathrm{dx}},$$

and it is exactly the same as one of the norms for $H^1(\Omega)$ given above. □

Lemma 1.3. *$H^1(\Omega)$ is Hilbert space.*

Proof. We have to show that that space is complete. We will use the elementary fact that $L^2(\Omega)$ is complete without a proof.

Define

$$\begin{aligned} F : H^1(\Omega) &\to \big(L^2(\Omega)\big)^{n+1} \\ v &\mapsto (v, \partial_{x_1}, ..., \partial_{x_n}) = w := (v := w_0, w_1, ..., w_n).\end{aligned}$$

Let us define a norm in $\big(L^2(\Omega)\big)^{n+1}$ by

$$\|w\| = \sqrt{\sum_{j=0}^{n} \int_\Omega |w_j(x)|^2\mathrm{dx}},$$

taht is consistent with the product topology of $\big(L^2(\Omega)\big)^{n+1}$.

F is an isometry and linear bijection of the space $H^1(\Omega)$ onto $F(H^1(\Omega)) \subset \big(L^2(\Omega)\big)^{n+1}$ in addition. One can easily check that.

Since $L^2(\Omega)$ is complete, then $\big(L^2(\Omega)\big)^{n+1}$ is also complete, so for each Cauchy sequence $\{v^{(j)}\}_j \subset H^1(\Omega)$ there exists $w \in \big(L^2(\Omega)\big)^{n+1}$ such that $F(v^{(j)}) \to w$, u $\big(L^2(\Omega)\big)^{n+1}$, as $j \to \infty$. That is

$$\begin{aligned}&v^{(j)} \to w_0, \\ &\partial_{x_k} v^{(j)} \to w_k,\ k = 1, ..., n \ \text{ u } L^2(\Omega),\ j \to \infty.\end{aligned}$$

Since $\phi \in \mathcal{C}_0^\infty(\Omega)$, the Hölder inequality implies

$$\begin{aligned}&|\int_\Omega (v^{(j)} - w_0)\phi \mathrm{dx}| \\ \leq &\|v^{(j)} - w_0\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \to 0,\ j \to \infty,\end{aligned}$$

because $v^{(j)} \xrightarrow{L^2} w_0$, i.e.

$$v^{(j)} \xrightarrow{\mathcal{D}'} w_0$$

too.

Using the same arguments, one can see that

$$\partial_{x_k} v^{(j)} \xrightarrow{\mathcal{D}'} w_k.$$

Due to the fact that differentiation is continuous in the distribution space, we have

$$\partial_{x_k} v^{(j)} \xrightarrow{\mathcal{D}'} \partial_{x_k} w_0.$$

That is

$$w_k = \partial_{x_k} w_0,\ k = 1, ..., n.$$

So, $w = F(w_0)$, ant the assertion follows from that fact that F is an isometry. □

Similarly, one can prove the following theorem.

Theorem 1.5. *$H^m(\Omega)$ is Hilbert space for each $m \in \mathbb{N}_0$.*

If $p \geq 1$, $H^{m,p}(\Omega)$ is only Banach space.

Now, we will use the fact that the Fourier transform maps L^2 onto L^2 as well as other its properties.

Denote $\langle\xi\rangle = \sqrt{1+\xi^2}$. Then $H^m(\mathbb{R}^n)$ norm of u is equivalent to

$$\sup_{\xi\in\mathbb{R}^n} \sum_{j=0}^m \|\langle\xi\rangle^j \hat{u}\|_{L^2(\Omega)}.$$

In order to define a value of a Sobolev function on the boundary $\partial\Omega$.

Definition 1.13. The space $H_0^{m,p}(\Omega)$ is defined to be a closure of $\mathcal{C}_0^\infty(\Omega)$ in the $H^{m,p}(\Omega)$–norm: $v \in H_0^{m,p}(\Omega)$ means that there is a sequence $\{\phi_j\}_j \subset \mathcal{C}_0^\infty(\Omega)$ such that

$$\phi_j \xrightarrow{H^{m,p}} v,\ j \to \infty.$$

We will use the following interpretation. $u|_{\partial\Omega} = 0$, $u \in H^{m,p}(\Omega)$ in the weak sense if $u \in H_0^{m,p}(\Omega)$. We say that $u = v$ na $\partial\Omega$, $v \in H^{m,p}(\Omega)$ if and only if $u - v \in H_0^{m,p}(\Omega)$.

Embedding theorems

We will give only a few among a lot of variations theorems about embedding Sobolev spaces into some other ones.

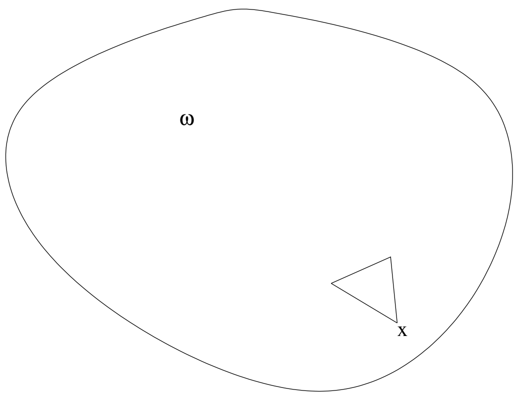
Definition 1.14. We say that a Banach space B_1 is embedded into a Banach space B_2, $B_1 \to B_2$ if there is a bounded injective and linear mapping from B_1 in B_2.

Theorem 1.6. *For an open $\Omega \subset \mathbb{R}^n$ there holds:*

$$H^{m,p}(\Omega) \to L^q(\Omega),\ mp > n,\ p \leq q \leq \infty$$
$$H^{m,p}(\Omega) \to L^q(\Omega),\ mp = n,\ p \leq q < \infty$$
$$H^{m,p}(\Omega) \to \mathcal{C}_b^0(\Omega),\ mp > n$$

Theorem 1.7. *Let Ω be bounded and posses the conic property: For each $x \in \Omega$ there is a cone with height h and center in the point x which is subset of Ω. Then*

$$H^{m,p}(\Omega) \to L^q(\Omega),\ p \leq q \leq np/(n - mp)$$
$$H^{m+j,1}(\Omega) \to \mathcal{C}_b^j(\Omega),\ mp > n$$



The conic property

Chapter 2

Formulation of the Cauchy problem for a system of quasilinear equations of hyperbolic type

2.1 Formulation of the problem

A system of quasilinear equations of hyperbolic type

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = b, \tag{2.1}$$

where $u = u(x,t) \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t > 0$, and $A = A(x,t,u(x,t)) \in \mathbb{R}^n \times \mathbb{R}^n$, we can write in characteristic form

$$l^k\Big(\frac{\partial u}{\partial t} + \xi_k\frac{\partial u}{\partial x}\Big) = f_k, \quad k = 1,\dots,n,$$

where $\{l^k\}$ are left eigenvectors and $\{\xi^k\}$ are eigenvalues of the matrix A, $k = 1,\dots,n$.

Hyperbolicity means that all n eigenvectors $\{l^k\}$ are left eigenvectors and $k = 1,\dots,n$ are linearly independent. The system is strictly hyperbolic if all eigenvalues $\{\xi^k\}$, $k = 1,\dots,n$, are real and different.

2.2 Solving the Cauchy problem by method of characteristics

Assume a solution $u(x,t)$ of the system (2.1) is known having the initial condition

$$u(x,0) = u^0(x), \quad a \leq x \leq b.$$

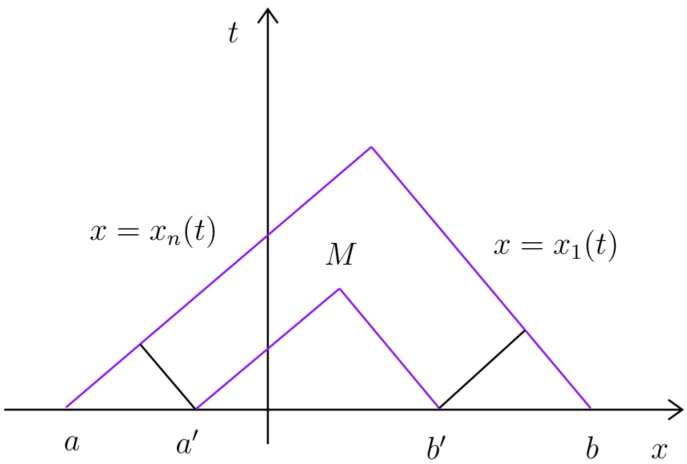


Figure 2.1: Domain of determinacy

Suppose that we can draw the characteristics $x = x_k(x^0, t^0, \tau)$ defined by the equations

$$\frac{dx_k}{d\tau} = \xi_k(x_k, \tau, u(x_k, \tau)), \quad k = 1, 2, \dots, n$$

through a point $M = (x^0, t^0)$ of the half plane $t > 0$ until their intersection with the axis $t = 0$. Let that they intersect this axis at certain point and we call the two extreme intersection points a' and b' $(a' < b')$ (see Figure 2.1). The segment $a' \leq x \leq b'$ of the initial axis $t = 0$ is called the domain of dependence for the solution $u(x,t)$ at the point M.

In the case of semilinear systems the domain of determinacy G can be calculated independently of u and it is given by

$$G = \{(x,t) | \, t \geq 0, \, X_n(t) \leq x \leq X_1(t)\},$$

where $X_1(t)$ and $X_n(t)$ denote the solutions of the differential equations

$$\frac{dX_n(t)}{dt} = \max_{k=1,\dots,n} \xi_k(X_n(t), t), \quad \frac{dX_1(t)}{dt} = \min_{k=1,\dots,n} \xi_k(X_1(t), t),$$

which assume for $t = 0$ the values $X_n(0) = a$ and $X_1(0) = b$. In the other words, X_1 and X_n are extremal characteristics. As usual, we assume $\xi_1 \leq \cdots \leq \xi_n$ without losing in generality.

Similarly, we define the domain of influence for the interval $[a, b]$,

$$D = \{(x, t) | \, t \geq 0, \, X_1(t) \leq x \leq X_n(t)\}.$$

It contain all the points influenced by the initial values defined on interval $[a, b]$.

The fact that there is a nonzero time interval that influence reach a point $x > b$ or $x < a$ is called the finite speed of propagation property.

Multiplication by a matrix Q containing the right eigenvalues as columns, we write the semilinear system

$$\frac{\partial u}{\partial t} + A(x, t)\frac{\partial u}{\partial x} = b(x, t, u) \tag{2.2}$$

in the diagonalized ()canonical) form (or written in invariants),

$$\frac{\partial v_j}{\partial t} + \xi_j(x, t)\frac{\partial v_j}{\partial x} = g_j(x, t, v). \tag{2.3}$$

Here, the function g is obtained by collecting all the terms that do not contain derivative of u.

Suppose that on some segments $[a, b]$ of the axis $t = 0$ for the system (2.2) there are given the initial conditions

$$u(x, 0) = u^0(x). \tag{2.4}$$

Setting $v_j^0(x) = l^j(x, 0)u^0(x)$, we obtain the initial conditions

$$v(x, 0) = v^0(x) \tag{2.5}$$

for the system (2.3). Functions $v(x, t)$ continuous in G are called a solution of the Cauchy problem (2.3, 2.5) in the broad sense if $r(x, 0) = r_0(x)$ and each of the functions $v_j(x, t)$ is continuously differentiable in t along the corresponding characteristic $x = x_j(\xi, \tau, t)$ and

$$\frac{d}{dt}v_j(x_j(\xi, \tau, t), t) = g_j(x_j, t, v(x_j, t)). \tag{2.6}$$

The vector $u(t, x)$ obtained from a vector $v(x, t)$ according to

$$u_j = \lambda_a^j v_a = \lambda^j r, \quad v = (v_1, \ldots, v_n), \, \Lambda = [l_i^j(x, t)]_{i,j}, \, \Lambda^{-1} = [\lambda_i^j(x, t)]_{i,j},$$

is called a solution in the broad sense of the Cauchy problem (2.2, 2.4).

Let us present some results for hyperbolic semilinear systems from the book [35].

Integrating (2.6) from 0 to t, we get

$$v_j(x,t) = u_j^0(x_j(x,t,0)) + \int_0^t g_j(x_j(x,t,\tau),\tau,u(x_j(x,t,\tau),\tau))\mathrm{d}\tau \quad (2.7)$$

This system is equivalent for differentiable solution to (2.6) provided that $g(x,t,u(x,t))$ is continuously differentiable. But system (2.7) may have locally integrable solutions in the cases when the original system does not have smooth solutions or if the function g is not smooth enough. We call such solutions the mild solutions. One can prove that these kind of solutions are week ones, but the contrary is maybe false.

Theorem 2.1. *Let K be an interval in $\mathbb{R}$. Denote by K_T the area bounded by K, the extremal characteristics form end ponts of the interval K and the line $t = T$.*
(a) Let $u^0 \in \mathcal{C}(K_0)$. Thet there exists $T > 0$ such that (2.7) has a solution $v \in \mathcal{C}(K_T)$.
(b) For every $T_0 > 0$ there exists at most one solution $u \in \mathcal{C}(K_T)$.

One can prove the theorem by Fixed Point Method for a space of continuous functions.

Sometimes, we can obtain global solution, i.e. solution for every T. Also, a solution obtained in such a way has additional regularity provided a regularity of given data. The following theorems are also proved in [35].

Theorem 2.2. *Suppose that $\nabla_u g(x,t,u)$ is uniformnly bounded as (x,t) lies in a compact set. Then the initial data problem (2.3), (2.5) has a unique solution $v \in \mathcal{C}(\mathbb{R}^2)$ if $v^0 \in \mathcal{C}(\mathbb{R})$.*

Theorem 2.3. *Let $v^0 \in \mathcal{C}^k(\mathbb{R})$, $1 \le k \le \infty$ and suppose that there exists a solution $v \in \mathcal{C}(K_T)$ od (2.3), (2.5) for $T_0 > 0$. Then $v \in \mathcal{C}^k(\mathbb{R} \times (0,T))$.*

The quasilinear case when A depends on u also is more complicated. Now, a system could be hyperbolic or not for different solutions since a set of eigenvalues and eigenvectors depends on u in general. In one space dimension, one can find a fairly general method for a construction of local solution in [12]. The proof is based on the method of characteristics and successive approximations.

Let us note that one can rarely expect that a global classical solution for quasilinear initial data problem exists. For example, let us take the simplest example, so called Burgers' inviscid equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

with the initial data $u(x, 0) = u^0(x)$. Using the method of characteristics one can find that a classical solution exists until

$$T = -\frac{1}{\sup_{x\in\mathbb{R}} {u^0}'(x)}.$$

Then, the solution blow up (its gradient tends to infinity, and we have so called gradient catastrophe) and we have to deal with discontinuous initial data and weak solutions after that point. That we will do in the rest of this thesis.

Chapter 3

System of conservation laws

A very important class of homogeneous hyperbolic equations called conservation laws.

The simplest case of conservation law in one space dimension is the partial differential equation (PDE) of the form

$$\partial_t u + \partial_x\big(f(u)\big) = 0, \qquad u(x,0) = u_0(x).$$

Let us to start to one dimensional case is much better understood than more dimensional cases.

And we will discuss a lot of examples, and how to use the Rankine-Hugoniot (RH) condition in examples, from literature given in the introduction.

3.1 Single 1-D equation

3.1.1 Rankine-Hugoniot condition

Let $u \in C^1\big(\mathbb{R} \times [0,\infty)\big)$ be solution to the following partial differential equation

$$\begin{aligned} u_t + \big(f(u)\big)_x &= 0 \\ u(x,0) &= u_0(x). \end{aligned} \tag{3.1}$$

Take $\varphi \in C_0^1\big(\mathbb{R} \times [0,\infty)\big)$, i.e. smooth function such that its support intersected by $\mathbb{R} \times [0,\infty)$ is compact. Then

$$\begin{aligned}
0 &= \int_0^\infty \int_{-\infty}^\infty \big(u_t(x,t) + f(u)_x\big)\varphi(x,t)\,dtdx \\
&= -\int_0^\infty \int_{-\infty}^\infty f(u)\varphi_x\,dtdx + \int_{-\infty}^\infty u(x,t)\varphi(x,t)\,dx\Big|_{t=0}^{t=\infty} - \int_0^\infty \int_{-\infty}^\infty u\varphi_t\,dxdt \\
&= -\int_0^\infty \int_{-\infty}^\infty \big(u\varphi_t + f(u)\varphi_x\big)\,dxdt - \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,dx.
\end{aligned}$$

The above calculation inspired the following definition of weak solution for (3.1).

Definition 3.1. $u \in L^\infty\big(\mathbb{R} \times (0,\infty)\big)$ (u is bounded function up to a set of Lebesgue measure zero) is called weak solution of (3.1) if

$$\int_0^\infty \int_{-\infty}^\infty \big(u\varphi_t + f(u)\varphi_x\big)\,dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,dx = 0,$$

for every $\varphi \in C_0^1\big(\mathbb{R} \times [0,\infty)\big)$.

Remark 3.1. 1. All classical solutions are also weak.

2. If u is a weak solution, then u is also a distributive solution.

3. If $u \in C^1\big(\mathbb{R} \times [0,\infty)\big)$ is a weak solution, then it is a classical, too.

In the next step, we will prove important theorem about necessary and sufficient conditions for existence of piecewise differentiable weak solution to some conservation law.

Theorem 3.1. *Necessary and sufficient condition that*

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t),\ t \geq 0 \\ u_d(x,t), & x > \gamma(t),\ t \geq 0, \end{cases}$$

where u_l and u_d are C^1 solutions on their domains, be a weak solution to (3.1) is

$$\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}. \tag{3.2}$$

Proof. The proof will be given in few steps

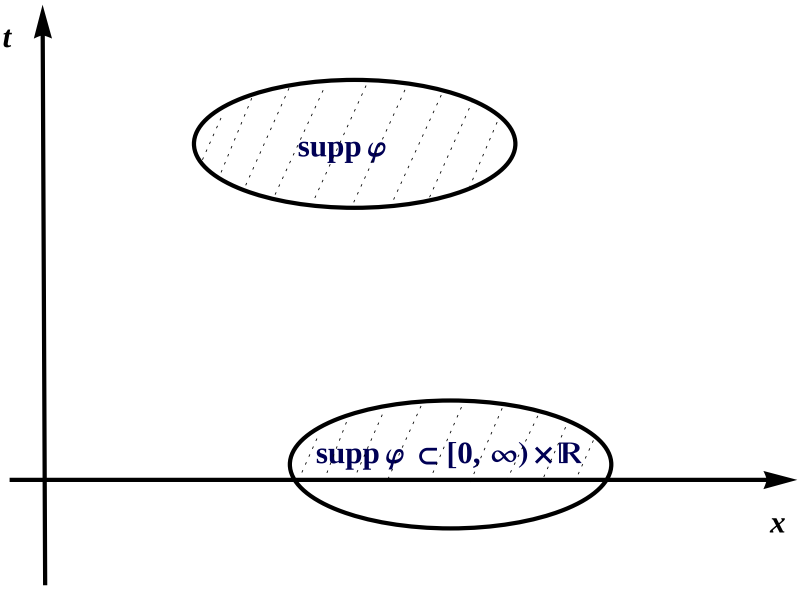


Figure 3.1: Supports of test functions in half plane

1. Let

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t),\ t \geq 0 \\ u_d(x,t), & x > \gamma(t),\ t \geq 0 \end{cases},$$

where u_l and u_d are defined above, be a weak solution to (3.1). Then

$$\int_0^\infty \int_{-\infty}^\infty \big(u\varphi_t + f(u)\varphi_x\big)\, dxdt + \int_{-\infty}^\infty u(x,0)\varphi(x,0)\, dx = 0,$$

for every $\varphi \in C_0^1\big(\mathbb{R} \times [0,\infty)\big)$.

Also, $(u_l)_t + f(u_l)_x = 0$ for $x < \gamma(t)$ and $t > 0$ as well as $(u_d)_t + f(u_d)_x = 0$ for $x > \gamma(t)$ and $t > 0$. That is consequence of the fact

$$0 = \iint u_l\varphi_t + f(u_l)\varphi_x\, dxdt - \iint (u_l)_t\varphi + \big(f(u_l)\big)_x \varphi\, dxdt,$$

for every φ, supp$\varphi \subset \{(x,t) :\ x < \gamma(t),\ t > 0\}$ and C^1 function u_l. And since φ is arbitrary, we have

$$(u_l)_t + \big(f(u_l)\big)_x = 0.$$

The same argument hold for u_d, too.

2. We have

$$\int_0^\infty \int_{-\infty}^\infty \big(u\varphi_t + f(u)\varphi_x\big)\,dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,dx$$
$$= \int_0^\infty \int_{-\infty}^{\gamma(t)} \big(u_l\varphi_t + f(u_l)\varphi_x\big)\,dxdt + \int_0^\infty \int_{\gamma(t)}^\infty \big(u_d\varphi_t + f(u_d)\varphi_x\big)\,dxdt$$
$$+ \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,dx.$$

3. Let us calculate the first integral from above. It holds

$$\frac{d}{dt}\int_{-\infty}^{\gamma(t)} u_l\varphi\,dx = \dot\gamma(t)u_l\big(\gamma(t),t\big)\varphi\big(\gamma(t),t\big) + \int_{-\infty}^{\gamma(t)} \big(u_l\varphi_t + f(u_l)\varphi_x\big)\,dx.$$

That implies

$$\int_0^\infty \int_{-\infty}^{\gamma(t)} u_l\varphi_t\,dxdt = -\int_0^\infty \int_{-\infty}^{\gamma(t)} (u_l)_t\varphi\,dxdt$$
$$- \int_0^\infty \dot\gamma(t)u_l\big(\gamma(t),t\big)\varphi\big(\gamma(t),t\big)\,dt + \int_0^\infty \frac{d}{dt}\int_{-\infty}^{\gamma(t)} u_l\varphi\,dxdt,$$

on the other hand

$$\int_0^\infty \int_{-\infty}^{\gamma(t)} f(u_l)\varphi_x\,dxdt = -\int_0^\infty \int_{-\infty}^{\gamma(t)} f(u_l)_x\varphi\,dxdt$$
$$+ \int_0^\infty f\big(u_l(\gamma(t),t)\big)\varphi\big(\gamma(t),t\big)\,dt.$$

Adding these terms and using the fact that u_l is a solution of PDE on the left-hand side of the curve $(\gamma(t),t)$, one gets the following

$$\int_0^\infty \big(f(u_l) - \dot\gamma u_l\big)\varphi\,dt + \int_0^\infty \frac{d}{dt}\int_{-\infty}^{\gamma(t)} u_l\varphi\,dxdt$$

as a value of that integral.

4. Analogously, concerning the right-hand side, one can see that the second integral equals

$$-\int_0^\infty \big(f(u_d) - \dot\gamma u_d\big)\varphi\,dt + \int_0^\infty \frac{d}{dt}\int_{-\infty}^{\gamma(t)} u_d\varphi\,dxdt.$$

5. After adding all the above integrals one gets

$$0 = \int_0^\infty \big(f(u_l) - f(u_d) - (u_l - u_d)\dot{\gamma}\big)\varphi \, dt + \int_0^\infty \frac{d}{dt} \int_{-\infty}^\infty u\varphi \, dx dt \\ + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\, dx,$$

and

$$\int_{-\infty}^\infty u(x,t)\varphi(x,t)\, dx \Big|_{t=0}^{t=\infty} = -\int_{-\infty}^\infty u_0(x)\varphi(x,0)\, dx.$$

That is true if

$$\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}.$$

Obviously, the above condition is sufficient. Condition (3.2) is called Rankine-Hugoniot (RH) condition.

□

3.1.2 Rarefaction waves

Solution of equation (3.1) of the form $u(x,t) = \tilde{u}\big(\frac{x}{t}\big)$ is called self-similar solution.

Lets try to find such a solution of (3.1) in a simple way, just by substituting a function of this form into the equation. After differentiation we have

$$-\frac{x}{t^2}\tilde{u}'\Big(\frac{x}{t}\Big) + f'\Big(\tilde{u}\Big(\frac{x}{t}\Big)\Big)\frac{1}{t}\tilde{u}'\Big(\frac{x}{t}\Big) = 0.$$

After multiplication of the equation with t and the substitution $\frac{x}{t} \mapsto y$ one gets the ODE

$$\tilde{u}'(y)\big(f'\big(\tilde{u}(y)\big) - y\big) = 0.$$

After neglecting constant, so called trivial solutions ($\tilde{u}' \neq 0$), one can see that solution is given by the implicit relation

$$f'(u) = y, \quad \text{i.e. } \tilde{u}(y) = f'^{-1}(y),$$

if f' is bijection (locally). One can interpret the initial data in the following way:

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0 \end{cases} \Rightarrow \tilde{u}(+\infty) = u_d, \ \tilde{u}(-\infty) = u_l. \tag{3.3}$$

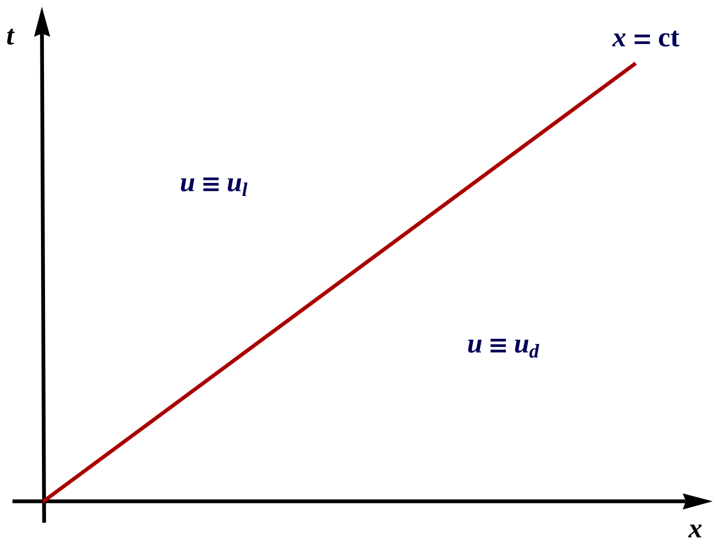


Figure 3.2: Shock wave

If $f'' > 0$ (f is convex), then f' is an increasing function and solution $\tilde{u}$ to the equation satisfying (3.3) exists if $u_l < u_d$. Such solution is called centered rarefaction wave (the initial data has singularity at zero).

Let use some examples, to know how to use the condition (3.2) to find the solution of the Riemann problem.

Example 3.1. Consider the following Riemann problem

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u_0 = \begin{cases} u_l \in \mathbb{R}, & x < 0 \\ u_d \in \mathbb{R}, & x > 0 \end{cases}. \end{aligned} \tag{3.4}$$

Since u_l and u_d are constants, there exist two trivial solutions of (3.4) out of the discontinuity curve, and RH-condition gives

$$\dot{\gamma} = \frac{u_d^2 - u_l^2}{2(u_d - u_l)} = \frac{u_d + u_l}{2},$$

i.e. $\dot{\gamma}(t) = ct$, $c = \frac{u_l + u_d}{2}$ (see Figure 3.2).

$$u(x,t) = \begin{cases} u_l, & x < ct \\ u_d, & x > ct \end{cases} \tag{3.5}$$

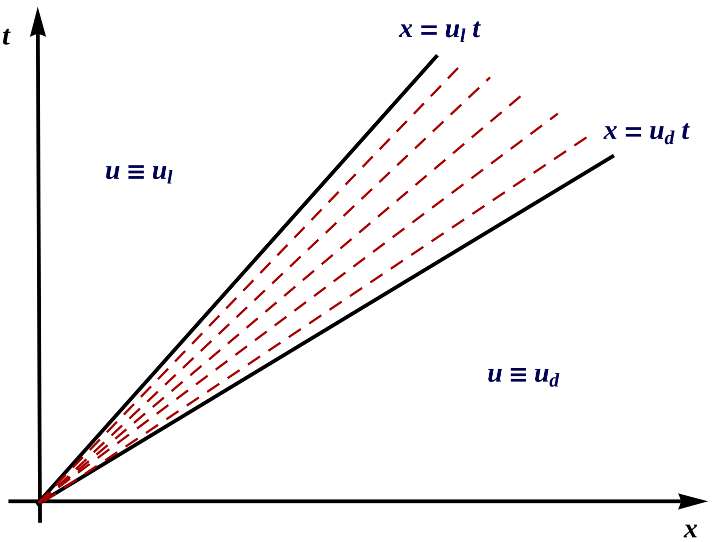


Figure 3.3: Rarefaction wave

if $u_l < u_d$, then except the above solution there exist also following solution (Figure 3.3)

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{t}, & u_l t \le x \le u_d t \\ u_d, & x > u_d t, \end{cases} \tag{3.6}$$

or (Figure 3.4)

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{t}, & u_l t \le x \le at \\ a, & at \le x \le \frac{a+u_d}{2} t \\ u_d, & x > \frac{a+u_d}{2} t, \end{cases} \tag{3.7}$$

for some $a \in (u_l, u_d)$.

Example 3.2. Let us multiply partial differential equation (3.4) by u and transfer it into divergence form.

$$\begin{aligned} u_t + u u_x &= 0 \quad / \cdot u \\ u u_t + u^2 u_x &= 0 \\ \left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x &= 0. \end{aligned}$$

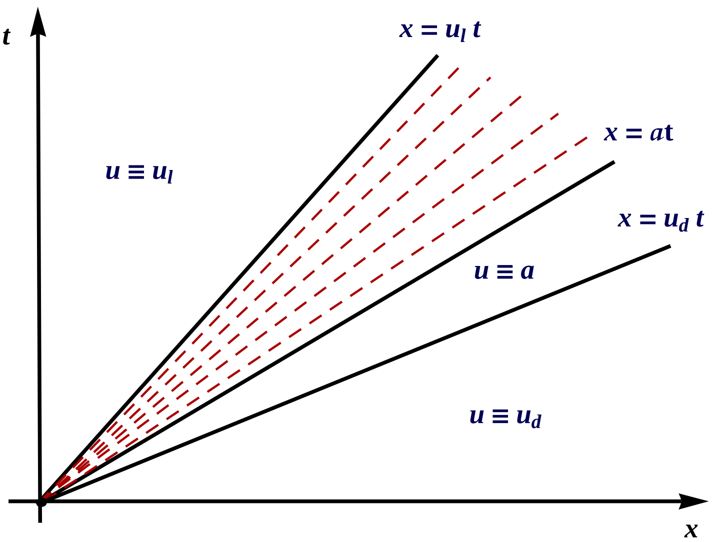


Figure 3.4: Non-entropic weak solution

After nonlinear change of variables $\frac{1}{2}u^2 \mapsto v$, one gets the following conservation law

$$v_t + \left(\frac{2\sqrt{2}}{3}v^{\frac{3}{2}}\right)_x = 0$$

$$v\Big|_{t=0} = \begin{cases} v_l = \frac{1}{2}u_l^2 & x < 0 \\ v_d = \frac{1}{2}u_d^2, & x > 0. \end{cases}$$

RH-condition gives the following speed of shock wave c and the discontinuity line is $\gamma = ct$:

$$\begin{aligned}\dot{\gamma}(t) = \frac{\left[\frac{3}{2}v^{\frac{3}{2}}\right]}{[v]} &= \frac{\frac{2\sqrt{2}}{3}\frac{1}{2}\left(u_d^2\right)^{\frac{3}{2}} - \frac{2\sqrt{2}}{3}\frac{1}{2}\left(u_l^2\right)^{\frac{3}{2}}}{\frac{1}{2}\left(u_d^2 - u_l^2\right)} \\ &= \frac{\frac{1}{3}\left(u_d^3 - u_l^3\right)}{\frac{1}{2}\left(u_d^2 - u_l^2\right)} \neq \frac{u_l + u_d}{2},\end{aligned}$$

in general. (For example, for $u_l = 1, u_d = 0$ one has $\frac{1/3}{1/2} \neq \frac{1}{2}$.)

3.1.3 Linear hyperbolic systems

We will look at linear systems before we start with systems of conservation laws.

Homogeneous linear scalar Cauchy problem with constant coefficients

$$\begin{aligned} u_t + \lambda u_x &= 0 \\ u(x,0) &= \bar{u}(x),\ \lambda \in C(\mathbb{R}),\ \bar{u} \in C^1\big([0,\infty) \times \mathbb{R}\big) \end{aligned} \tag{3.8}$$

has a simple solution in a traveling wave form

$$u(x,t) = \bar{u}(x - \lambda t). \tag{3.9}$$

If $\bar{u} \in L^1_{loc}$, then the above function (3.9) is a weak solution to (3.8), what one can show easily.

Let a homogeneous system with constant coefficients

$$\begin{aligned} u_t + Au_x &= 0 \\ u(x,0) &= \bar{u}(x) \end{aligned} \tag{3.10}$$

be given, where A is $n \times n$ hyperbolic matrix with real characteristic values $\lambda_1 < \ldots < \lambda_n$ and left-hand side l_i (resp. right-hand sided r_i) $i = 1, \ldots, n$ eigenvectors.

They are chosen in a way that $l_i R_i = \delta_{ij},\ i,j = 1, \ldots, n$. Denote by $u_i := l_i u$ coordinates of the vector $u \in \mathbb{R}^n$ with respect to the $\{r_1, \ldots, r_n\}$. Multiplying (3.10) from the left-hand side with l_i one gets

$$(u_i)_t + \lambda_i (u_i)_x = (\lambda_i u)_t + \lambda_i (l_i u)_x = l_i u_t + l_i A u_x = 0.$$

So, (3.10) decouples into n scalar Cauchy problems, which can be solved same (3.8), one. Using (3.9) one can see that

$$u(x,t) = \sum_{i=1}^{n} \bar{u}_i(x - \lambda_i t) r_i(u) \tag{3.11}$$

is solution to (3.10) because

$$u_t(x,t) = \sum_{i=1}^{n} -\lambda_i \big(l_i \bar{u}_x(x - \lambda_i t)\big) r_i = -Au_x(x,t).$$

Thus, initial profile $\bar{u}$ decouples into sum of n waves with speeds $\lambda_1, \ldots, \lambda_n$. As a special case, take Riemann problem

$$\bar{u}(x) = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0. \end{cases}$$

We use write down a solution to (3.11) using

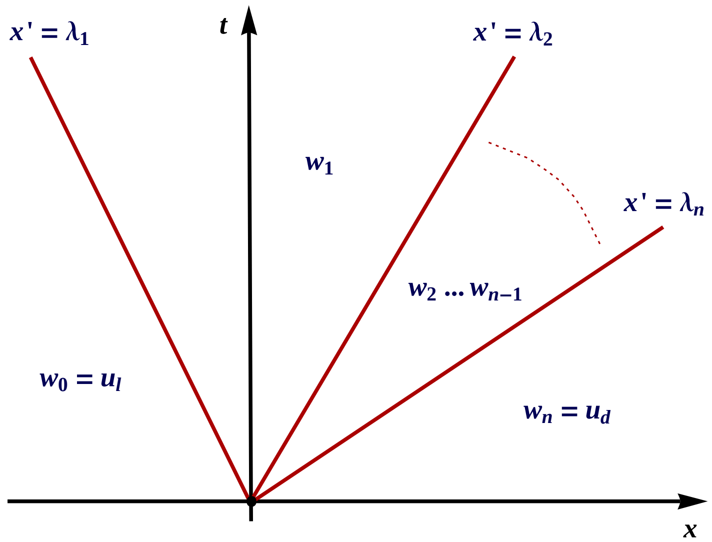


Figure 3.5: Waves and linear system

$$u_d - u_l = \sum_{j=1}^{n} c_j r_j$$

and we define the intermediate states by

$$w_i := u^l + \sum_{j \leq i} c_j r_j, \qquad i = 0, \dots, n$$

such that $w_i - w_{i-1}$ is $(i-n)-$th characteristic vector of A. Solution is of the form (Figure 3.5)

$$u(x,t) = \begin{cases} w_0 = u_l, & \frac{x}{t} < \lambda_i \\ \dots, & \\ w_i, & \lambda_i < \frac{x}{t} < \lambda_{i+1} \\ \dots, & \\ w_n = u_d, & \frac{x}{t} > \lambda_n. \end{cases} \tag{3.12}$$

.

3.2 Quasilinear hyperbolic system of balance laws in n dimensions

We shall briefly give some notions for conservation and balance laws in general case when $x \in \mathbb{R}^n$. That is given just for the sake of completeness since we are not dealing with singular solutions in more than one space dimension.

Consider the following system of balance laws

$$\partial_t H\big(U(x,t),x,t\big) + \operatorname{div} G\big(U(x,t),x,t\big) = \Pi\big(U(x,t),x,t\big), \tag{3.13}$$

where $x \in \mathbb{R}^m$ and $t \geq 0$. Here, matrix functions F, G and Π are at least continuous (for our purposes, but one can permit lower regularity like in porous flow equations).

Also $\dim(U) = m \times 1$, $U = [U^1, \dots, U^m]$, $\dim(H) = m \times 1$, $\Pi = m \times 1$, $\dim(\Pi) = m \times 1$, $\dim(G) = m \times n$, $G = (G_1, \dots, G_n)$ and G_α is a row matrix. Here and bellow, all operators acting on $(x,t)-$space are capitalized (Div, for example) while the ones action on $x-$spaces are not (div, for example), D denotes the differential regarded as a row operation,

$$D = \Big[\frac{\partial}{\partial U^1}, \dots, \frac{\partial}{\partial U^n}\Big].$$

The system (3.13) is called a canonical (evolutionary) form if $H(U,x,t) \equiv U$.

Definition 3.2. The system (3.13) is called hyperbolic in the $t-$direction if the following holds.

For a fixed $U \in \Omega$ (physical domain) and $U \in S^{m-1}$ (the unit sphere), the matrix $DH(U,x,t)$ (with dimension $n \times n$) is nonsingular, while the eigenvalue problem

$$\Big(\sum_{\alpha=1}^{m} \nu_\alpha DG_\alpha(U,x,t) - \lambda DH(U,x,t)\Big)R = 0$$

has real eigenvalues

$$\lambda_1(\nu; U,x,t), \dots, \lambda_n(\nu; U,x,t),$$

called characteristic speeds, and n linearly independent eigenvectors

$$R_1(\nu; U,x,t), \dots, R_n(\nu; U,x,t).$$

Note that there are no strictly hyperbolic systems in more than one space dimension, see [7].

3.3 Elementary waves for conservation laws in one space dimension

Systems for $n > 1$ dimensions are still not well understood, there are a lot of open questions. One of te most important question is to find an appropriate space for a weak solution to system of conservation laws.

One can find the class of functions with finite total variation being very useful for one dimensional systems of conservation laws.

Definition 3.3. Total variation of a function v is defined by

$$\mathrm{TV}(v) = \sup \sum_{j=1}^{N} |v(\xi_j) - v(\xi_{j-1})|, \tag{3.14}$$

where the supremum is taken by all partitions of the real line

$$-\infty = \xi_0 < \xi_1 < \ldots < \xi_N = \infty.$$

We can write (3.14) in the form

$$\mathrm{TV}(v) = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |v(x) - v(x-\varepsilon)|\, dx.$$

Let

$$\begin{aligned} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \ldots, u_n) &= 0 \\ &\vdots \\ \frac{\partial}{\partial t} u_n + \frac{\partial}{\partial x} f_n(u_1, \ldots, u_n) &= 0 \end{aligned} \tag{3.15}$$

be $n \times n$ one-dimensional conservation laws system, where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$.

Denote by $A(u) := Df(u)$ Jacobi matrix of f at a point u. The above system using vector notation

$$u_t + f(u)_x = 0. \tag{3.16}$$

If a solution is smooth enough (C^1), the qusilinear form

$$u_t + A(u)u_x = 0 \tag{3.17}$$

defines the equivalent system. That system is called strictly hyperbolic if all characteristic values of $A(u)$ are real and distinct. They are ordered in the following way

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u).$$

If there exist n linearly independent characteristic vector, the system is called hyperbolic.

Left-hand sided $l_1(u), \ldots, l_n(u)$ and right-hand sided $r_1(u), \ldots, r_n(u)$ characteristic vectors are determined in a way that it holds

$$l_i(u)r_i(u) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

To avoid technical complications we consider 1-D system

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0, \tag{3.18}$$

with F be a C^3 map from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

3.3.1 Riemann invariants

Definition 3.4. We can say that an Riemann invarian of (3.18) is a smooth scalar-valued function such that

$$D\omega(U)R_i(U) = 0, \quad U \in \Omega.$$

We say that the system (3.18) has a coordinate system of Riemann invariants if there exist n scalar-valued functions $(\omega_1, \ldots, \omega_n)$ on Ω such that ω_j is an $i-$Riemann invariant of the system for $i, j = 1, \ldots, n$, $i \neq j$.

We have the following theorem.

Theorem 3.2. *The functions $(\omega_1, \ldots, \omega_n)$ form a coordinate system of Riemann invariants for (3.18) if and only if*

$$D\omega_i R_j(U) \begin{cases} = 0, & \text{if } i \neq j \\ \neq 0, & \text{if } i = j. \end{cases}$$

In other words, $D\omega_i$ is left $i-th$ eigenvector of the matrix DF.

It is convenient to normalize eigenvectors $R_1, \ldots, R_n$ if the Riemann coordinate system exists such that

$$D\omega_i R_j(U) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

Multiplying $i-$th equation of the system (3.18) by $D\omega_i$, $i = 1, \ldots, n$ we get

$$\partial_t \omega_i + \lambda_i \partial_x \omega_i = 0, \quad i = 1, \ldots, n.$$

3.3.2 Shock waves

Like in the case of $n = 1$, we will suppose that $x = \gamma(t)$ defines a discontinuity curve of piecewise smooth solutions $u_l(x,t)$ and $u_d(x,t)$, i.e.

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t) \\ u_d(x,t), & x > \gamma(t). \end{cases}$$

In order that u defines a weak solution one has to find γ from Rankine-Hugoniot conditions for system

$$\dot\gamma(u_d - u_l) = f(u_d) - f(u_l). \tag{3.19}$$

Now u_d, u_l, $f(u_d)$ and $f(u_l)$ are $n-$dim vectors. That means that discontinuity curve $x = \gamma(t)$ can not be found in direct way like in the case of a single equation. That is, it is not true for each pair of constant initial vectors u_l, u_d there exists a shock wave solution. Denote by

$$A(u,v) := \int A(\theta u + (1-\theta)v)\, d\theta$$

averaged matrix, where $\lambda_i(u,v)$, $i = 1,\dots,n$ are its characteristic values. Then (3.19) can be written as

$$\dot\gamma(u_d - u_l) = f(u_d) - f(u_l) = A(u_d,u_l)(u_d - u_l). \tag{3.20}$$

In the other word, RH conditions hold if (u_d, u_l) is a characteristic speed $\dot\gamma$ equals its characteristic value.

3.3.3 Rarefaction waves

Let us find solution of the form $u = u\big(\frac{x}{t}\big)$ (self-similar solutions) for system (3.17)

$$u_t + A(u)u_x = -\frac{x}{t^2}u'(y) + \frac{1}{t}A\big(u(y)\big)u'(y) = 0,$$

where $y = \frac{x}{t}$. From the last equation it follows

$$A(u)u' = yu',$$

u' is equal to the right-hand sided characteristic vector r_i and $y = \lambda_i$, for $i = 1,\dots,n$.

3.3.4 Entropy conditions

As one could see, even for the case $n = 1$ there is problem of uniqueness for weak solutions.

In order to choose physically relevant solution we can use so called entropy conditions. The solution will satisfy it is called admissible.

Entropy condition 1 - vanishing viscosity. A weak solution u to (3.15) is admissible if there exists a sequence of smooth solutions u_ε to

$$u_{\varepsilon t} + A(u_\varepsilon)u_{\varepsilon x} = \varepsilon u_{\varepsilon xx}$$

which converges to u in L^1 as $\varepsilon \to 0$.
Entropy condition 2 - entropy inequality. C^1 function $\eta : \mathbb{R}^n \to \mathbb{R}$ is called entropy for system (3.15) with appropriate entropy flux $q : \mathbb{R}^n \to \mathbb{R}$ if

$$D\eta(u)Df(u) = Dq(u), \qquad u : \mathbb{R}^n \to \mathbb{R}^n. \tag{3.21}$$

Note that (3.21) implies

$$\big(\eta(u)\big)_t + \big(q(u)\big)_x = 0,$$

for $u \in C^1$ as a solution to (3.15). When one substitutes $u_t = -Df(u)u_x$ into the above equation,

$$D\eta(u)u_t + Dq(u)u_x = D\eta(u)\big(-Df(u)u_x\big) + Dq(u)u_x = 0.$$

A weak solution u to (3.15) is admissible if

$$\big(\eta(u)\big)_t + \big(q(u)\big)_x \leq 0$$

in a distributional sense, i.e.

$$-\int \eta(u)\varphi_t + q(u)\varphi_x \geq 0,$$

for every $\varphi \geq 0$, $\varphi \in C_0^\infty\big(\mathbb{R} \times [0,\infty)\big)$. Thus, $D\eta(u)u_t + Dq(u)u_x = 0$ outside discontinuity, and

$$\dot{x}_\alpha\Big(\eta\big(u(x_{\alpha^+})\big) - \eta\big(u(x_{\alpha^-})\big)\Big) \geq q\big(u(x_{\alpha^+})\big) - \eta\big(u(x_{\alpha^-})\big)$$

on the discontinuity curve $x = \dot{x}_\alpha(t)$.
Entropy condition 3 - Lax condition. Shock wave connecting states u_l and u_d and has a speed $\dot{\gamma} = \lambda_i(u_l, u_d)$ is admissible if

$$\lambda_i(u_l) \geq \lambda_i(u_l, u_d) = \dot{\gamma} \geq \lambda_i(u_d). \tag{3.22}$$

Because of the ordering of characteristic values

$$\lambda_j(u_l) > \dot{\gamma}, \quad j > i$$
$$\lambda_j(u_d) < \dot{\gamma}, \quad j < i$$

such a wave is called $i-$th shock wave.

3.3.5 Rarefaction (RW) and shock wave (SW) curves

Fix $u_0 \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$. Integral curve for vector field r_i through u_0 is called $i-$th rarefaction curve (RW_i). One can get it explicitly solving the Cauchy problem

$$\frac{du}{d\sigma} = r_i(u), \quad u(0) = u_0. \tag{3.23}$$

That curve will be denoted by

$$\sigma \mapsto R_i(\sigma)(u_0).$$

Next to the above definition, u_0 can be joined with $u \in \mathrm{RW}_i(u_0)$ by a single rarefaction wave.

We note that a curve parametrization depends on a choice of r_i. If $|r_i| = 1$ then the curve is parametrized by its length.

We fix $u_0 \in \mathbb{R}^n$ again. Let u be a right-hand state which can be joined to u_0 with $i-$th shock wave. (We use RH conditions and also Lax condition (3.22).)

Values of u lies on a curve $W_i(s, u_l)$ for some s. A shock speed is then $c = c_i(s, u_l)$. So, the vector $u - u_l$ is a right-hand side $i-$th eigenvector of the averaged matrix $A(u, u_l)$. We can use the some theorem of linear algebra that is true if and only if $u - u_l$ is an orthogonal to all left eigenvectors l_j for every $j \neq i$. That means

$$l_j(u_l)(u - u_l) = 0, \ \forall j \neq i, \quad \dot{\gamma} = \lambda_i(u, u_l). \tag{3.24}$$

Also we can see that (3.24) is the system of $n - 1$ scalar equation with n variables $(u_1, \dots, u_n)$.

Linearizing (3.24) in a neighborhood of u_0 we get the linear system

$$l_j(u_l)(u - u_l) = 0, \quad j \neq i$$

which it has a solution $w = u_l + Cr_i(u_l)$, $c \in \mathbb{R}$.

By Implicit function theorem, a set of solutions forms a regular curve (C^1-curve) which can be connected to u_l with a tangent vector r_i in the point u_l. That curve is called the curve of $i-$th shock wave and denoted by S_i.

And also both of the above curves exist in neighborhood of u_l (if f is smooth enough).

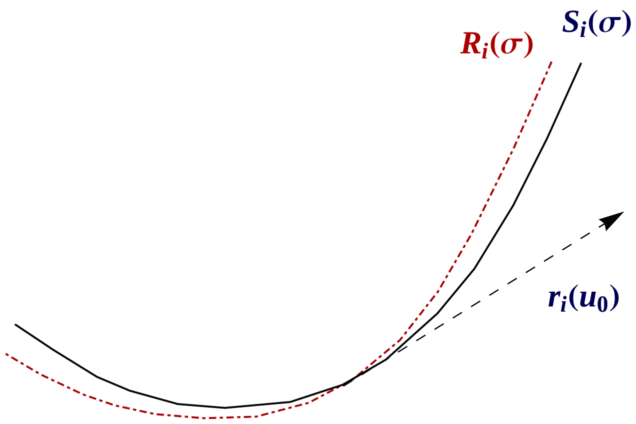


Figure 3.6: Shock wave and rarefaction curve

3.3.6 Riemann problem

In the rest of this thesis we are dealing with piecewise constant initial data

$$u|_{t=0} = \begin{cases} u_0, & x < 0 \\ u_1, & x > 0. \end{cases}$$

The most of this section is from [4].

Definition 3.5. We say that the $i-$th characteristic field is genuinely non-linear if

$$D\lambda_i(u)r_i(u) \neq 0.$$

If

$$D\lambda_i(u)r_i(u) = 0$$

then $i-$th field is said to be linearly degenerate.

We note that in this case when $i-$th field is genuinely nonlinear one can chose the orientation of r_i (by choosing its sign, eventually) such that

$$D\lambda_i(u)r_i(u) > 0.$$

The system (3.15) is called strictly hyperbolic with smooth coefficients. For each $i \in \{1, \dots, n\}$, $i-$th characteristic field is either genuinely nonlinear or linearly degenerate.

Assumption 1. Assume that $i-$th field be genuinely nonlinear and suppose that u_d lies on a positive part of rarefaction curve starting from u_d, i.e. $u_d = R(\sigma)(u_l)$, for some $\sigma > 0$.

Theorem 3.3. *Assume us define*

$$\lambda_i(s) = \lambda_i\big(R_i(s)(u_l)\big),$$

for every $s \in [0, \sigma]$, *by genuine nonlinearity. The mapping* $s \mapsto \lambda_i(s)$ *is strictly increasing. Let* $t \geq 0$, *the function*

$$u(x,t) = \begin{cases} u_l, & x < t\lambda_i(u_l) \\ R_i(s)(u_l), & x = t\lambda_i(s) \\ u_d = R_i(\sigma)(u_l), & x > t\lambda_i(u_d) \end{cases} \tag{3.25}$$

such that $\frac{x}{t} = y = \lambda_i(s)$, $s \in [0, \sigma]$, *is a piecewise smooth solution to Riemann problem*

$$u_t + f(u)_x = 0$$

$$u\Big|_{t=0} := u_0 = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0. \end{cases}$$

Proof. It is easily to see that

$$\lim_{t \to 0} \|u(x,t) - u_0\|_{L^1} = 0.$$

So, the initial data are satisfied. Also, the equation (3.15) trivially holds true for $x < t\lambda_i(u_l)$ and $x > t\lambda_i(u_d)$, because $\partial_t u = \partial_x u = 0$. Suppose that $x = t\lambda_i(s)$, for some $s \in (0, \sigma)$. Since u is constant along each halfline $\{(x,t) : x = t\lambda_i(s)\}$, then there holds

$$\partial_t u(x,t) + \lambda_i(s)\partial_x u(x,t) = 0. \tag{3.26}$$

Since

$$\begin{aligned} \partial_x u = \frac{\partial u}{\partial x} &= \frac{dR_i(s)(u_l)}{ds}\Big(\frac{d\lambda_i(s)}{ds}\Big)^{-1}\frac{d\lambda_i}{dx} \\ &= r_i(u)\Big(\frac{d\lambda_i(s)}{ds}\Big)^{-1}\frac{1}{t}, \end{aligned}$$

such that $\partial_x u$ is eigenvector for the Jacobian matrix $A(u)$, when $\lambda_i(s) = \lambda_i(u(t,x))$, i.e.

$$A(u)\partial_x u = \lambda_i \partial_x u.$$

We note that the assumption $\sigma > 0$ is crusial for the above construction of a solution. If $\sigma < 0$, (3.25) would define a triple function in the area $\frac{x}{t} \in [\lambda_i(u_d), \lambda_i(u_l)]$. □

Assumption 2 (Shock waves)**.** Let $i-$th characteristic field be genuinely nonlinear and let u_d be connected with u_l by $i-$shock wave, $u_d = S_i(\sigma)(u_l)$.

Then $\lambda := \lambda_i(u_d, u_l)$ is the speed of that wave and

$$u(t,x) = \begin{cases} u_l, & x < \lambda t \\ u_d, & x > \lambda t \end{cases} \tag{3.27}$$

is piecewise constant solution to the above Riemann problem.

We note that in the case $\sigma < 0$ this solution is admissible in the Lax-sense, because

$$\lambda_i(u_d) < \lambda_(u_l, u_d) < \lambda_i(u_l),$$

while $\sigma > 0$, we would have

$$\lambda_i(u_l) < \lambda_i(u_d)$$

and Lax condition could not be satisfied.

Assumption 3. Let the $i-$th characteristic field is linearly degenerate and $u_d = R_i(\sigma)(u_l)$ for some σ.

By the assumption, λ_i is constant along that curve. i.e. $D\lambda_i r_i = 0$.

Let put $\lambda = \lambda_i(u_l)$, we can see that piecewise constant function given by (3.27) solves the above Cauchy problem, because the Rankine-Hugoniot conditions is satisfied at discontinuity curve.

$$\begin{aligned} f(u_d) - f(u_l) &= \int_0^\sigma Df\big(R_i(s)(u_l)\big) r_i\big(R_i(s)(u_l)\big)\, ds \\ &= \int_0^\sigma \lambda_i\big(R_i(s)(u_l)\big) r_i\big(R_i(s)(u_l)\big)\, ds \\ &= \lambda_i(u_l) \int_0^\sigma \frac{dR_i(s)(u_l)}{ds}\, ds = \lambda_i(u_l)\big(R_i(\sigma)(u_l) - u_l\big) \end{aligned}$$

We will use here that

$$\begin{aligned} \frac{d}{ds}\lambda_i\big(R_i(s)(u_l)\big) &= D\lambda_i\big(R_i(s)(u_l)\big)\frac{dR_i(s)(u_l)}{ds} \\ &= \big(D\lambda_i R_i\big)\big(R_i(s)(u_l)\big) = 0, \end{aligned}$$

as well as the definition of linear degeneracy.

In that case Lax conditions hold, thus regardless to the sign of σ, because

$$\lambda_i(u_d) = \lambda_i(u_l, u_d) = \lambda_i(u_l),$$

because for the above calculations, we can deduce that

$$R_i(\sigma)(u_0) = S_i(\sigma)(u_0), \quad \text{for every } \sigma.$$

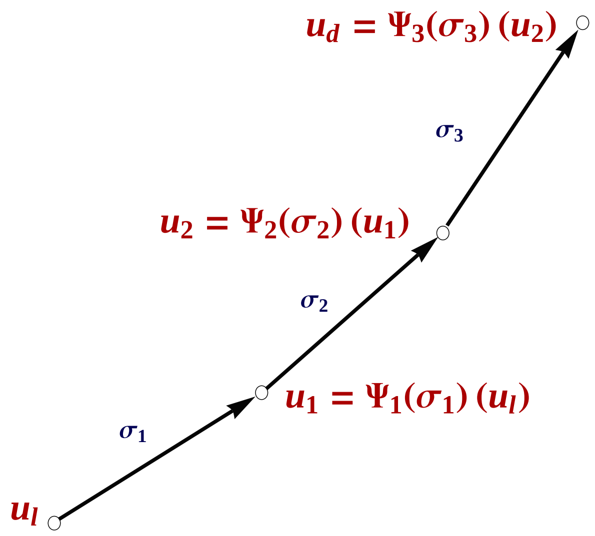


Figure 3.7: Sketch of a solution to Riemann problem

3.3.7 General solutions

As we have seen before, the set of points $\{u_d : u \in \mathbb{R}^n\}$ which could be connected with a left-hand side state of Riemann problem produces a curve in $\mathbb{R}^2$.

In order to connect two arbitrary points $u_l, u_d \in \mathbb{R}^n$ with an entropic solution of Riemann problem, we can insert at most $n-1$ vectors

$$u_l =: u_0, u_1, u_2, \dots, u_{n-1}, u_n := u_d.$$

Note that between each pair (u_l, u_1), $(u_1, u_2), \dots, (u_{n-1}, u_d)$ there exist the previously described elementary waves: rarefaction, shock waves or contact discontinuity. That is possible for sure if the total variation of the initial data is small enough. We can see that in illustration in Figure 3.7. Here Ψ denotes any kind of elementary wave.

Remark 3.2. For bounded initial date, we can approximate it by piecewise constant functions. So, there are Riemann problems which have to be simultaneously solved. And also we can by one solution in the form of elementary waves can be easily find, but the important problem is how to deal with a huge number of mutual wave interaction.

There are two famous methods to do it.

- Glimm scheme ([13]). Before the first interaction of the initial elementary waves, we can approximates a solution with new piecewise constant function by choosing finite number of points in a random way.

That because a new initial data and procedure is repeated as many times as needed. Rarefaction wave is approximated by a fan of non-admissible shock waves in this procedure.

The procedure will converge for small enough variation of initial states, i.e. when the total variation of the initial data is small enough. We can also be sure that each approximation is independent of the previous ones.

There are a lot of technical problems concerning the above scheme, so a lot of effort was given to find a new procedure. Later on, randomness was excluded the assumptions above, see [26].

- Front - tracking method ([3, 38]). The following scheme is the good choice both for proving solution existence and numerical approximation of a solution.

  Again, rarefaction is approximated with a fan of non-entropic shock waves. But now waves are permitted to interact. In a point of interaction there is a new Riemann problem. Also we can solve it accurately or approximately. In the later case, one constructs non physical shock wave with small amplitude, but with the larger speed of all possible waves in order to prevent blow - up effect.

  After that we can again use the same method for later interactions. Again, this procedure will converge when total variation of the initial data is small enough.

Chapter 4

Split delta shocks

In this part we are introducing the concept of split delta shocks. For an overview one can see in [34].

4.1 The definition of split delta shocks

Let us now briefly characterize what we mean by a solution in the form of a split delta shock wave. We suppose $\overline{\mathbb{R}^2_+}$ is divided into finitely disjoint open sets $\Omega_i \neq \emptyset$, $i = 1, ..., n$ with piecewise smooth boundary curves Γ_i, $i = 1, ..., m$, that is $\Omega_i \cap \Omega_j = \emptyset$, $\cup_{i=1}^{n} = \overline{\mathbb{R}^2_+}$ such that $\overline{\Omega_i}$ is the closure of Ω_i.

Let $C(\overline{\Omega_i})$ be the space of bounded and continuous real - valued functions on $\overline{\Omega_i}$, equipped with the L^∞ - norm. Suppose $M(\overline{\Omega_i})$ be the space of measures on $\overline{\Omega_i}$. We will consider the spaces

$$C_\Gamma = \prod_{i=1}^{n} C(\overline{\Omega_i}), \quad M_\Gamma = \prod_{i=1}^{n} M(\overline{\Omega_i}).$$

The product of the element $G = (G_1, ..., G_n) \in C_\Gamma$ and $D = (D_1, ..., D_n) \in M_\Gamma$ is defined as an element $D \cdot G = (D_1 G_1, ..., D_n G_n) \in M_\Gamma$, where each component is defined as the usual product of continuous function and a measure. Every measure on $\overline{\Omega_i}$ can be seen as a measure on $\overline{\mathbb{R}^2_+}$ with support in $\overline{\Omega_i}$. This way we can get a mapping

$$m : M_\Gamma \to M(\overline{\mathbb{R}^2_+}),$$
$$m(D) = D_1 + D_2 + ... + D_n.$$

A typical example is obtained when $\overline{\mathbb{R}^2_+}$ is divided into regions Ω_1, Ω_2 by a piecewise smooth curve $x = \gamma(t)$. The delta function $\delta(x - \gamma(t)) \in M(\overline{\mathbb{R}^2_+})$

along the line $x = \gamma(t)$ can be split in a non unique way into a left - hand side $D^- \in M(\overline{\Omega_1})$ and the right - hand component $D^+ \in M(\overline{\Omega_2})$ such that

$$\begin{aligned}\delta(x-\gamma(t)) &= \alpha_0(t)D^- + \alpha_1(t)D^+ \\ &= m(\alpha_0(t)D^- + \alpha_1(t)D^+)\end{aligned}$$

with $\alpha_0(t) + \alpha_1(t) = 1$. The solution concept which allows to incorporate such two sided delta functions as well as shock waves is modeled along the lines of the classical weak solution concept and proceeds as follows:

Step 1: perform all nonlinear operations of functions in the space C_Γ.

Step 2: perform multiplications with measures in the space M_Γ.

Step 3: Map the space M_Γ into $M(\overline{\mathbb{R}^2_+})$ by means of the map m and embed it into the space of distributions.

Step 4: perform the differentiation in the sense of distributions and require that the equation is satisfied in this sense.

We note that in the case of absence of a measure part (Step 2), this is precisely the concept of a weak solution to equations in divergence form.

Example 4.1. Consider the following combination of shocks in u and v along the curve $\Gamma : x = ct$ and a two sided delta function along Γ:

$$\begin{aligned}u(x,t) &= \begin{cases} u_0, & x < ct \\ u_1, & x > ct \end{cases} \\ v(x,t) &= \left\{\begin{array}{ll} v_0, & x < ct \\ v_1, & x > ct \end{array}\right\} + \alpha_0(t)D^-_\Gamma + \alpha_1(t)D^+_\Gamma\end{aligned}$$

Observing that the derivative of the Heaviside function along Γ is the delta function on Γ, we will get the following weak derivatives:

$$v_t(x,t) = (-c[v] + \alpha_0'(t) + \alpha_1'(t))\delta(x-ct) - c(\alpha_0(t) + \alpha_1(t))\delta'(x-ct),$$

where $[v]$ denotes the jump in v along Γ, and

$$((u-1)v)_x(x,t) = [(u-1)v]\delta(x-ct) + ((u_0-1)v_0\alpha_0(t) + (u_1-1)v_1\alpha_1(t))\delta'(x-ct).$$

Thus equation $v_t + ((u-1)v)_x = 0$ is satisfied if and only if

$$\begin{aligned}-c[v] + \alpha_0'(t) + \alpha_1'(t) + [(u-1)v] &= 0, \\ -c(\alpha_0(t) + \alpha_1(t)) + (u_0-1)v_0\alpha_0(t) + (u_1-1)v_1\alpha_1(t) &= 0.\end{aligned}$$

4.2 Simplified magnetohydrodynamics model

We start the study of different models from the literature that contain solutions with SDSs.

The first model equation solved in the paper [33]

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &= 0 \\ v_t + \big((u-1)v\big)_x &= 0 \end{aligned} \tag{4.1}$$

both the Riemann problem and all possible interactions of two waves (given by three constant states as the initial data). The system is introduced in [15]. More precisely, it is derived from a simplified model magnetohydrodynamics. We get the eigenvalues of the above system $\lambda_1(u,v) = u-1,\ \lambda_2(u,v) = u$, and the right - hand side eigenvectors are$r_1(u,v) = (0,1)^T,\ r_2(z,v) = (1,v)^T$. The first characteristic field is linearly degenerate and the second is genuinely nonlinear. Thus, there are three types of solution for the Riemann data

$$(u,v)(x,0) = \begin{cases} (u_0, v_0), & x < 0, \\ (u_1, v_1), & x > 0. \end{cases}$$

(i) When $u_1 > u_0$ the solution is a contact discontinuity followed by a rarefaction wave

$$u(x,t) = \begin{cases} u_0, & x \le u_0 t, \\ \dfrac{x}{t}, & u_0 t < x < u_1 t, \\ u_1, & x \ge u_1 t \end{cases}$$

$$v(x,t) = \begin{cases} v_0, & x \le (u_0 - 1)t, \\ v_1 \exp(u_0 - u_1), & (u_0 - 1)t < x < u_0 t, \\ v_1 \exp\left(\dfrac{x}{t}, -u_1\right), & u_0 t < x < u_1 t, \\ v_1, & x > u_1 t \end{cases}$$

(ii) If $u_1 < u_0 < u_1 + 2$, the solution is given in the form of contact discontinuity followed by a shock wave,

$$u(x,t) = \begin{cases} u_0, & x \le ct, \\ u_1, & x > ct, \end{cases}$$

$$v(x,t) = \begin{cases} v_0, & x \le (u_0 - 1)t, \\ v_*, & (u_0 - 1)t < x < ct, \\ v_1, & x \ge ct, \end{cases}$$

where $v_* = v_1 \frac{2-u_0-u_1}{2+u_1-u_0}$.

(iii) If $u_0 \geq u_1 + 2$ the solution is given in the form of delta shock wave,

$$u(x,t) = \begin{cases} u_0, & x \leq ct \\ u_1, & x > ct \end{cases}$$
$$v(x,t) = \left\{ \begin{array}{ll} v_0, & x \leq ct \\ v_1, & x > ct \end{array} \right\} + \alpha_0(t) D^- + \alpha_1(t) D^+,$$

such that D^- and D^+ are the left and right hand side delta functions with the support on the line $x = ct$,

$$c = (u_0 + u_1)/2,$$

$$\alpha_0(t) = \frac{\delta t(c - (u_1 - 1))}{u_0 - u_1}, \quad \alpha_1(t) = \frac{\delta t(c - (u_0 - 1))}{u_0 - u_1}.$$

$\alpha(t) = \alpha_0(t) + \alpha_1(t)$ is called the strength of the delta shock wave, and

$$s := c(v_1 - v_0) - \big((u_1 - 1)v_1 - (u_0 - 1)v_0\big)$$

is called the Rankine - Hugoniot deficit (see [18]).

4.3 Simplified chromatography equations

The full chromatograpy system is given by the following equations

$$\begin{aligned} &\Big((1 + \frac{A}{1 - u + v})u\Big)_t + u_x = 0, \\ &\Big((1 + \frac{B}{1 - u + v})v\Big)_t + v_x = 0. \end{aligned} \tag{4.2}$$

The physical condition is given by $A < B$, and the physical domain for solutions is defined by $1 - u + v > 1$ or $v - u > -1$.

In [27] and [37] one can find all relevant things about that system. Let us note that the real model has determined values for $(x, 0)$, $x > 0$ and $(0, t)$, $t > 0$ instead of the standard initial data, as we have assumed above.

The Riemman problem for the simplified model in nonlinear chromatography given gy the system

$$\begin{aligned} u_t + \Big((1 + \frac{1}{1 - u + v})u\Big)_x &= 0, \\ v_t + \Big((1 + \frac{1}{1 - u + v})v\Big)_x &= 0. \end{aligned} \tag{4.3}$$

is solved in [45]. The system has

$$\lambda_i = \frac{1}{1-u+v}, \quad \lambda_j = \frac{1}{(1-u+v)^2}$$

as eigenvalues. Note that $i=1$, $j=2$ if $-1 < v-u < 1$, and $i=2$, $j=1$ if $v-u \geq 1$.Note that the system is strictly hyperbolic if $u \neq v$ with i - th field being linearly degenerate and j-th field being genuinely nonlinear.

The Riemann problem

$$(u,v)(x,0) = \begin{cases} (u_0, v_0), & x < 0, \\ (u_1, v_1), & x > 0. \end{cases}$$

We can study here only the case $v_0 - u_0 < 0,\ v_1 - u_1 > 0$ when there are no elementary wave solutions. In that case, we will try with SDS solution

$$\begin{aligned} u(x,t) &= \left\{ \begin{array}{ll} u_0, & x \leq ct \\ u_1, & x > ct \end{array} \right. + \alpha_0(t)D^- + \alpha_1(t)D^+ \\ v(x,t) &= \left\{ \begin{array}{ll} v_0, & x \leq ct \\ v_1, & x > ct \end{array} \right. + \beta_0(t)D^- + \beta_1(t)D^+, \end{aligned} \tag{4.4}$$

A split delta function can be multiplied only with continuous function on the domains $\{(x,t) : x \leq ct\}$ and $\{(x,t) : x \geq ct\}$. That is, $\frac{1}{1-u+v}$ has to be continuous on these sets. That is possible if delta parts in $1-u+v$ cancel each other, i.e. when

$$\alpha_0 + \alpha_1 = \beta_0 + \beta_1. \tag{4.5}$$

Exchange of (4.4) into the system (4.3) gives the following two equations. The first equation is satisfied if the following relation is true

$$\begin{aligned} &\Big(-c[u] + \alpha_0 + \alpha_1 + \Big[\big(1 + \frac{1}{1-u+v}\big)u\Big]\Big)\delta(x-ct) \\ &+ \Big(-c(\alpha_0+\alpha_1)t + \Big(\big(1 + \frac{1}{1-u_0+v_0}\big)u_0\alpha_0 \\ &+ \big(1 + \frac{1}{1-u_1+v_1}\big)u_1\alpha_1\Big)t\Big)\delta'(x-ct) = 0, \end{aligned}$$

while the second one holds if

$$\begin{aligned} &\Big(-c[v] + \beta_0 + \beta_1 + \Big[\big(1 + \frac{1}{1-u+v}\big)v\Big]\Big)\delta(x-ct) \\ &+ \Big(-c(\beta_0+\beta_1)t + \Big(\big(1 + \frac{1}{1-u_0+v_0}\big)v_0\beta_0 \\ &+ \big(1 + \frac{1}{1-u_1+v_1}\big)v_1\beta_1\Big)t\Big)\delta'(x-ct) = 0. \end{aligned}$$

Note that these relations produce four equations

$$\begin{aligned}
&\alpha_0 + \alpha_1 = k_1 := c[u] - \Big[(1 + \frac{1}{1-u+v})u\Big],\\
&\Big((1 + \frac{1}{1-u_0+v_0})u_0 - c\Big)\alpha_0 + \Big((1 + \frac{1}{1-u_1+v_1})u_1 - c\Big)\alpha_1 = 0,\\
&\beta_0 + \beta_1 = k_2 := c[v] - \Big[(1 + \frac{1}{1-u+v})v\Big],\\
&\Big((1 + \frac{1}{1-u_0+v_0})v_0 - c\Big)\beta_0 + \Big((1 + \frac{1}{1-u_1+v_1})v_1 - c\Big)\beta_1 = 0.
\end{aligned}$$

The condition $\alpha_0+\alpha_1 = \beta_0+\beta_1$ implies $k_1 = k_2$ and that condition determines the speed by

$$c[u] - \Big[\Big(1 + \frac{1}{1-u+v}\Big)u\Big] = c[v] - \Big[\Big(1 + \frac{1}{1-u+v}\Big)v\Big],$$

i.e.

$$c = 1 + \frac{1}{(1-u_0+v_0)(1-u_1+v_1)}.$$

Chapter 5

Inverse of a split delta shock with applications

5.1 The definition of an inverse of split delta shock

The definition of the split delta shocks is well adopted to the case when a given system is linear in one of dependent variables. Then we can easily perform all necessary multiplication of split delta shocks with piecewise continuous functions as described above. But we have seen that it is possible to solve some other systems (like (4.3) above). It was possible to solve it because two split delta functions annihilate in the denominators there. Now, we will extend the split delta solutions to some other systems where there is a division with delta functions and there is no their annihilation in denominators.

So, let us define an inverse of a split delta function in the following way. Suppose that

$$u(x,t) = \begin{cases} u_0, & x \leq ct \\ u_1, & x > ct \end{cases} + \alpha_0 D^-(x-ct) + D^+(x-ct).$$

We define $\frac{1}{u} \in C_\Gamma,\ \Gamma = \{(x,t) : x = ct\}$, to be a function satisfying $\frac{1}{u} \cdot u = 1$ in the M_Γ sense. Using the above definition that means

$$\frac{1}{u} \cdot \left(\begin{cases} u_0, & x \leq ct \\ u_1, & x > ct \end{cases} + \alpha_0 D^-(x-ct) + D^+(x-ct) \right)$$

$$1 + \frac{\alpha_0(t)}{u_0} D^-(x-ct) + \frac{\alpha_1}{u_1} D^+(x-ct) \xrightarrow{m} 1 + \left(\frac{\alpha_0}{u_0} + \frac{\alpha_1}{u_1}\right) \cdot \delta(x-ct).$$

Thus,

$$\frac{\alpha_0(t)}{u_0} + \frac{\alpha_1(t)}{u_1} = 0$$

should hold in order to justify the above calculation.

5.2 System given in a general form

Let us consider Riemann problem

$$\begin{aligned} u_t + \left(\frac{a_0 + a_1 u}{v} + \frac{b_0 + b_1 v}{u} \right)_x &= 0, \quad u(x,0) = \begin{cases} u_0, & x < 0, \\ u_1, & x > 0, \end{cases} \\ v_t + \left(\frac{\overline{a_0} + \overline{a_1} u}{v} + \frac{\overline{b_0} + \overline{b_1} v}{u} \right)_x &= 0, \quad v(x,0) = \begin{cases} v_0, & x < 0, \\ v_1, & x > 0, \end{cases} \end{aligned} \tag{5.1}$$

We suppose that $(u, v) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is a physical domain, i.e. a set of all possible values for (u, v). Let us look for a solution in the form of two component split delta shock

$$\begin{aligned} u(x,t) &= \begin{cases} u_0, & x \le ct \\ u_1, & x > ct \end{cases} + \alpha_0 t D^- + \alpha_1 t D^+ =: \hat{u} + \alpha_0 t D^- + \alpha_1 t D^+ \\ v(x,t) &= \begin{cases} v_0, & x \le ct \\ v_1, & x > ct \end{cases} + \beta_0 t D^- + \beta_1 t D^+ =: \hat{v} + \beta_0 t D^- + \beta_1 t D^+, \end{aligned} \tag{5.2}$$

where the support of all split delta function components is $x = ct$.

In the sequel, notation $[u]$ is used for a jump in $\hat{u}$. For a given point (u_0, v_0) in a physical domain Ω for (5.1), a set of all (u_1, v_1) in the domain such that there exists a split delta shock connecting these states is called split delta locus denoted by $L((u_0, v_0))$.

Theorem 5.1. *There is a split delta shock solution to (5.1) if there exists c such that u_i, v_i, $i = 0, 1$ satisfy*

$$\begin{aligned} a_1 \left[\frac{u}{v}\right] \frac{k_1}{[u]} + b_1 \left[\frac{v}{u}\right] \frac{k_2}{[v]} &= ck_1 \\ \bar{a}_1 \left[\frac{u}{v}\right] \frac{k_1}{[u]} + \bar{b}_1 \left[\frac{v}{u}\right] \frac{k_2}{[v]} &= ck_2, \ v_1 \neq v_0, \ u_1 \neq u_0. \end{aligned} \tag{5.3}$$

Here

$$k_1 := c[u] - \left[\frac{a_0 + a_1 u}{v} + \frac{b_0 + b_1 v}{u}\right], \ k_2 := c[v] - \left[\frac{\bar{a}_0 + \bar{a}_1 u}{v} + \frac{\bar{b}_0 + \bar{b}_1 v}{u}\right],$$

are so called Rankine–Hugoniot deficits for the first and second equation.

Proof. The definition of the inverses of u i v gives the following equations

$$\begin{aligned} &\frac{\alpha_0}{u_0} + \frac{\alpha_1}{u_1} = 0, \\ &\frac{\beta_0}{v_0} + \frac{\beta_1}{v_1} = 0. \end{aligned} \tag{5.4}$$

Using the procedure for split delta shock calculations, from the first equation in (5.1) one gets

$$\begin{aligned} &- c[u]\delta + \left[\frac{a_0 + a_1 u}{v} + \frac{b_0 + b_1 v}{u}\right]\delta + (\alpha_0 + \alpha_1)\delta \\ &- c + (\alpha_0 + \alpha_1)t\delta' + \left(\frac{a_1}{v_0}\alpha_0 + \frac{a_1}{v_1}\alpha_1 + \frac{b_1}{u_0}\beta_0 + \frac{b_1}{u_1}\beta_1\right)t\delta' = 0 \end{aligned}$$

where the support of δ and δ' is the line $x = ct$. The above equality is true if and only if

$$\alpha_0 + \alpha_1 = c[u] - \left[\frac{a_0 + a_1 u}{v} + \frac{b_0 + b_1 v}{u}\right] =: k_1 \tag{5.5}$$

$$c(\alpha_0 + \alpha_1) = \frac{a_1}{v_0}\alpha_0 + \frac{a_1}{v_1}\alpha_1 + \frac{b_1}{u_0}\beta_0 + \frac{b_1}{u_1}\beta_1. \tag{5.6}$$

With the same arguments, one gets

$$\beta_0 + \beta_1 = c[v] - \left[\frac{\overline{a_0} + \overline{a_1} u}{v} + \frac{\overline{b_0} + \overline{b_1} v}{u}\right] =: k_2 \tag{5.7}$$

$$c(\beta_0 + \beta_1) = \frac{\overline{a_1}}{v_0}\alpha_0 + \frac{\overline{a_1}}{v_1}\alpha_1 + \frac{\overline{b_1}}{u_0}\beta_0 + \frac{\overline{b_1}}{u_1}\beta_1 \tag{5.8}$$

from the second equation in (5.1).

A general algorithm. If $u_0 \neq u_1$ and $v_0 \neq v_1$ then the variables $\alpha_0, \alpha_1, \beta_0, \beta_1$ are uniquely determined by the following systems

$$\begin{aligned} &\alpha_0 + \alpha_1 = k_1, \quad \beta_0 + \beta_1 = k_2, \\ &\frac{\alpha_0}{u_0} + \frac{\alpha_1}{u_1} = 0, \quad \frac{\beta_0}{v_0} + \frac{\beta_1}{v_1} = 0. \end{aligned} \tag{5.9}$$

We have used (5.2), (5.5) and (5.7). All possible values for c and a relation between left and right - hand initial data are determined combining (5.5) and (5.6) and solving the following system of equations (quadratic in c)

$$\begin{aligned} &a_1\left(\frac{\alpha_0}{v_0} + \frac{\alpha_1}{v_1}\right) + b_1\left(\frac{\beta_0}{u_0} + \frac{\beta_1}{u_1}\right) = ck_1, \\ &\overline{a_1}\left(\frac{\alpha_0}{v_0} + \frac{\alpha_1}{v_1}\right) + \overline{b_1}\left(\frac{\beta_0}{u_0} + \frac{\beta_1}{u_1}\right) = ck_2. \end{aligned}$$

After solving (5.9) and inserting a solution in the above system one gets

$$\begin{aligned} a_1 \left[\frac{u}{v}\right] \frac{k_1}{[u]} + b_1 \left[\frac{v}{u}\right] \frac{k_2}{[v]} &= ck_1 \\ \overline{a_1} \left[\frac{u}{v}\right] \frac{k_1}{[u]} + \overline{b_1} \left[\frac{v}{u}\right] \frac{k_2}{[v]} &= ck_2. \end{aligned} \tag{5.10}$$

□

In general, we expect that one could get a value(s) for c and a curve with possible right - hand states that could be connected the left - hand ones by a split delta shock. Of course, there are a lot of specific situations. We will look at some of them in this thesis. For a real model one has to check whether $(u_1, v_1) \in \Omega$ and an admissibility condition for split delta shocks, too. The most usual admissibility condition is that split delta shock are required to be overcompressive, i.e. all characteristics should run into the shock curve. Another admissible solution is delta shock that propagates along a characteristic. It is called a delta contact discontinuity (see [33]). That is possible for systems having a linearly degenerate field.

5.2.1 Some special cases

1. $b_0 = b_1 = \overline{a_0} = \overline{a_1} = 0$. In this case, (5.10) reduces to $a_1 \left[\frac{u}{v}\right] = c[u]$, $\overline{b_1} \left[\frac{v}{u}\right] = c[v]$. That is, a speed c is uniquely determined with split delta locus given by the relation

$$\begin{aligned} L((u_0, v_0)) = \Big\{ (u_1, v_1) \in \Omega : a_1 \left(\frac{u_1}{v_1} - \frac{u_0}{v_0}\right) (v_1 - v_0) \\ = \overline{b_1} \left(\frac{v_1}{u_1} - \frac{v_0}{u_0}\right) (u_1 - u_0) \Big\}. \end{aligned}$$

   We note that this relation can be easily solved now (quadratic equation for v_1 or u_1), contrary to the general case given in (5.10).

2. $b_0 = b_1 = \overline{b_0} = \overline{b_1} = 0$. Now, there is only one condition for an inverse

$1/v$, relation (5.2). Equations (5.5) – (5.8) are reduced to

$$\alpha_0 + \alpha_1 = c[u] - \left[\frac{a_0 + a_1 u}{v}\right] =: k_1, \tag{5.11}$$

$$c(\alpha_0 + \alpha_1) = \frac{a_1}{v_0}\alpha_0 + \frac{a_1}{v_1}\alpha_1, \tag{5.12}$$

$$\beta_0 + \beta_1 = c[v] - \left[\frac{\overline{a_0} + \overline{a_1} u}{v}\right] =: k_2, \tag{5.13}$$

$$c(\beta_0 + \beta_1) = \frac{\overline{a_1}}{v_0}\alpha_0 + \frac{\overline{a_1}}{v_1}\alpha_1. \tag{5.14}$$

One could easily see that the above equations imply $k_1 = \frac{a_1}{\overline{a_1}} k_2$ and that relation uniquely determined a speed c of a split delta shock and (5.14) is satisfied. Provided $u_0 \neq u_1$ and $v_0 \neq v_1$, one could also see that β_0 and β_1 are determined from (5.2) and (5.13) while α_0 and α_1 are determined from (5.11) and (5.12). That means there are no restriction on (u_1, v_1) and $L((u_0, v_0)) = \Omega$. Of course, one can exclude all non - physical and non - admissible points, but that depends on a concrete model.

5.2.2 Chromatography system – singular case

Here, we will present results from the paper [29] for the following chromatography system

$$\begin{aligned} &\partial_t u + \left(\frac{u}{1-u+v}\right)_x = 0 \\ &\partial_t v + \left(\frac{v}{1-u+v}\right)_x = 0 \end{aligned} \tag{5.15}$$

taking a singular choice $A = B$ made by some simplifications and change of dependent variables in the full one, system (4.2). We assume the Riemann initial data

$$\begin{aligned} u(x,0) &= \begin{cases} u_0, & x < 0, \\ u_1, & x > 0, \end{cases} \\ v(x,0) &= \begin{cases} v_0, & x < 0, \\ v_1, & x > 0, \end{cases} \end{aligned} \tag{5.16}$$

Theorem 5.2. *There exists a unique solution to Riemann problem for (5.15) in the region where u, v and $1-u+v$ are non-negative. The solution consists of elementary waves, vacuum states and split delta shocks. Uniqueness holds in the sense of distributions.*

Proof. System (4.2) has the eigenvalues $\lambda_a = \frac{1}{1-u+v}$ and $\lambda_b = \frac{1}{(1-u+v)^2}$ with the appropriate eigenvectors $r_a = (1,1)$ and $r_b = (1, v/u)$. The a–field is linearly degenerate, while b–field is genuinely nonlinear for $v \neq u$.

The constant discontinuity curve for a–field starting in the point (u_0, v_0) is given by

$$\frac{dv}{du} = 1, \quad v(u_0) = v_0,$$

i.e.

$$Cd_a: \ v = u - u_0 + v_0.$$

The rarefaction curve for b–field starting in the point (u_0, v_0) is given by

$$\frac{dv}{du} = \frac{v}{u}, \quad v(u_0) = v_0,$$

i.e.

$$R_b: \ v = \frac{v_0}{u_0}u, \ u > u_0.$$

The shock curve for b–field starting in the point (u_0, v_0) is given by

$$\frac{\left(\frac{u}{1-u+v}\right) - \left(\frac{u_0}{1-u_0+v_0}\right)}{\left(\frac{v}{1-u+v}\right) - \left(\frac{v_0}{1-u_0+v_0}\right)}$$

i.e.

$$S_b: \ v = \frac{v_0}{u_0}u, \ u < u_0$$

and the appropriate shock speed is

$$c = \frac{u0 - u}{u0 - v0 - 1}.$$

The analysis depends on position of (u_0, v_0). Let us denote by I the region where $v \geq u$ and by II the one where $u > v > u - 1$.

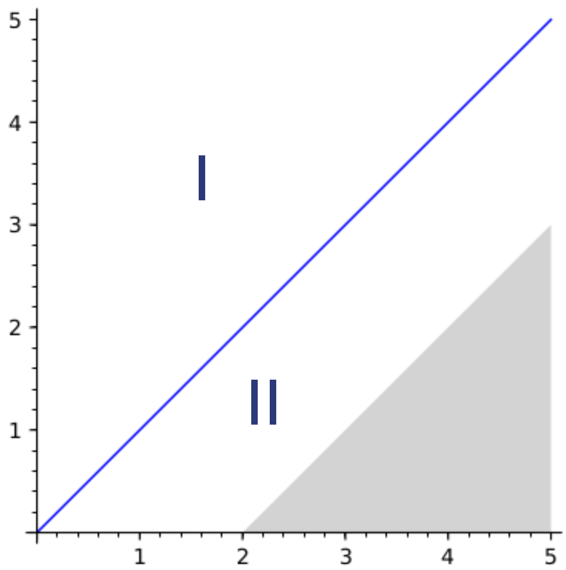


Figure 5.1: Regions in (u, v)–plane

In I, $\lambda_1 = \lambda_b$, $r_1 = r_b$ and $\lambda_2 = \lambda_a$, $r_2 = r_a$. The opposite holds in II.

Region I Assume the initial data given in (5.16). If $(u_i, v_i) \in I$, $i = 0, 1$. a solution is the following combination $S_1 + Cd_2$ or $R_1 + Cd_2$. See the illustration at Figure 5.2

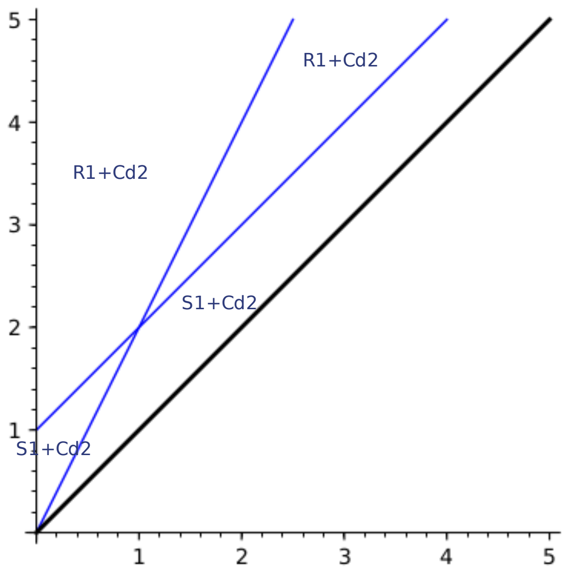


Figure 5.2: Solution in region I

Region II If $(u_i, v_i) \in II$, $i = 0, 1$. a solution is the following combination $Cd_1 + S_2$ or $Cd_1 + R_2$. See the illustration at Figure 5.3

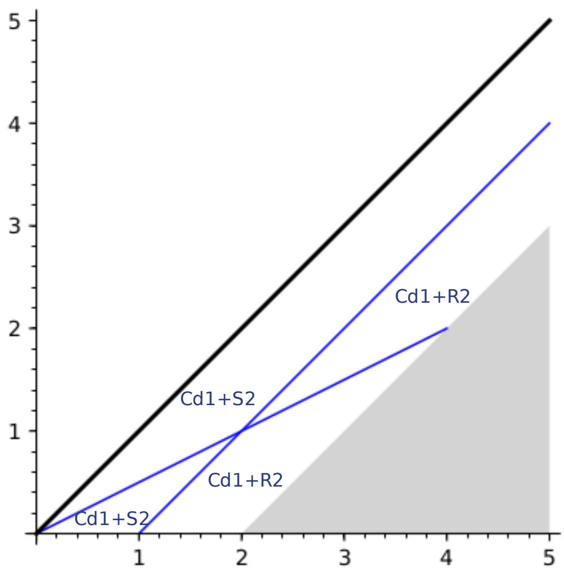


Figure 5.3: Solution in region II

If initial data values lie in different regions, there is no direct solution. The curves change a type the situation is a bit complicated. We could try with connections with states were u or v equals zero ("vacuum in u or v" variable). In one situation it can be done by using only elementary waves, but the other one requires the split delta shock solution.

Case 1. Suppose that $(u_0, v_0) \in I$ and $(u_1, v_1) \in II$. Then one could connect (u_0, v_0) with $(0, 0)$ by S_1 with speed $c_0 = \frac{1}{(1-u_0+v_0)^2} \in (0, 1)$. Then, one can connect the point $(0, 0)$ with some $(u_s, 0)$ by a rarefaction wave in u while $v = 0$: u is a solution to the scalar equation $u_t + \left(\frac{1}{1-u}\right)_x = 0$, $\lambda(0, 0) = 1 > c_0$ and $\lambda(u_s, 0) = \frac{1}{1-u_s} > 1$. The value u_s is chosen such that $(u_s, 0)$ could be connected by a contact discontinuity following the vacuum rarefaction wave which speed equals $c_1 = \frac{1}{1-u_s} = \lambda(u_s, 0)$. See the illustration in Figure 5.4

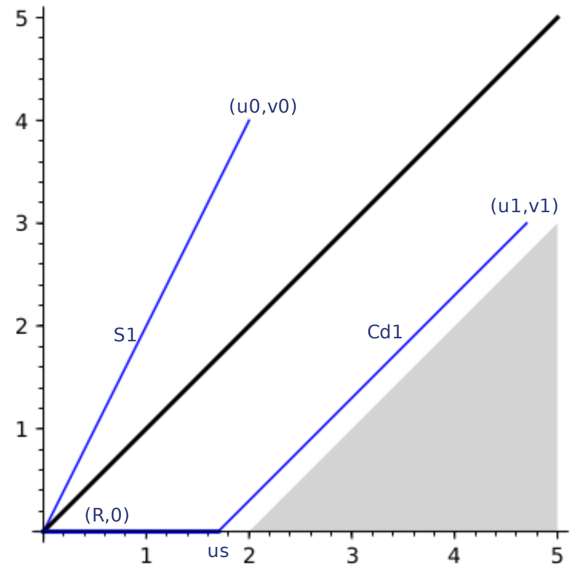


Figure 5.4: Connecting I on the left and II on the right

Case 2. Let $(u_0, v_0) \in II$ and $(u_1, v_1) \in I$. Then, there is no classical solution to the problem. One can try to connect (u_0, v_0) to $(0, 0)$ by an S_2 with speed $c_0 = \frac{1}{1-u_0+v_0} > 1$. If we want to connect $(0, 0)$ to some $(u_s, v_s) \in I$ (or $u_s = 0$), a speed would be $c_s = \frac{1}{1-u_s+v_s} < 1 < c_0$ that is impossible.

Let us try with a split delta shock solution of the form (5.2). One can use the definition for inverse since $1 - u + v$ is again split delta shock function.

The inverse condition (5.4) is now

$$\frac{\beta_0 - \alpha_0}{1 - u_0 + v_0} + \frac{\beta_1 - \alpha_1}{1 - u_1 + v_1} = 0, \tag{5.17}$$

with $\alpha_0 + \alpha_1 \geq 0$, $\beta_0 + \beta_1 \geq 0$. Using a similar calculations as in the proof of Theorem 5.1 one could see that the following equations should be satisfied.

$$\begin{aligned} \alpha_0 + \alpha_1 &= \kappa_1 := c[u] - \left[\frac{u}{1 - u + v}\right] \\ c(\alpha_0 + \alpha_1) &= \frac{\alpha_0}{1 - u_0 + v_0} + \frac{\alpha_1}{1 - u_1 + v_1} \\ \beta_0 + \beta_1 &= \kappa_2 := c[v] - \left[\frac{v}{1 - u + v}\right] \\ c(\beta_0 + \beta_1) &= \frac{\beta_0}{1 - u_0 + v_0} + \frac{\beta_1}{1 - u_1 + v_1}. \end{aligned}$$

One can find α_0 and α_1 from the first two, and β_0 and β_1 from the last two equations since $v_1 - v_0 - (u_1 - u_0) > 0$. Substitution of these values into (5.17) gives the condition $\kappa_1 = \kappa_2$. From that condition one can calculate a speed c,

$$c = \frac{1}{(1 - u_0 + v_0)(1 - u_1 + v_1)}.$$

The overcompressiblity condition

$$\underbrace{\frac{1}{1 - u_0 + v_0}}_{=\lambda_1(u_0, v_0)} \geq \underbrace{\frac{1}{(1 - u_0 + v_0)(1 - u_1 + v_1)}}_{=c} \geq \underbrace{\frac{1}{(1 - u_1 + v_1)^2}}_{=\lambda_2(u_1, v_1)}$$

is satisfied since $u_0 > v_0$ and $u_1 < v_1$. That completes the proof. □

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