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Generalized Stochastic Processes with Applications in Equation Solving

–doctoral dissertation–

Uopšteni stohastički procesi sa primenama u rešavanju jednačina

–doktorska disertacija–

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Predgovor

Predmet istraživanja ove doktorske disertacije su uopšteni stohastički procesi. Uopšteni stohastički procesi se javljaju kao rešenja široke klase stohastičkih parcijalnih diferencijalnih jednačina koje nemaju rešenje u klasičnom smislu. Stoga, uopšteni stohastički procesi predstavljaju dobar teorijski okvir za proučavanje stohastičkih parcijalnih diferencijalnih jednačina.

U disertaciji su proučavani uopšteni stohastički procesi Kolomboovog tipa, ili kraće Kolomboovi stohastički procesi. Kolomboovi stohastički procesi su bili predmet proučavanja u mnogim radovima navedenim u literaturi, ali ni u jednom od tih radova nisu proučavane njihove probabilističke osobine. Glavni cilj istraživanja ove doktorske disertacije su probabilističke osobine Kolomboovih stohastičkih procesa. Takođe, cilj nam je da dobijene teorijske rezultate primenimo na rešavanje jedne klase stohastičkih parcijalnih diferencijalnih jednačina.

U prvom delu dat je kratak pregled osnovnih pojmova Kolomboove teorije uopštenih funkcija. Posebna pažnja je posvećena osobinama translatorno invarijantnih Kolomboovih uopštenih funkcija. Potom su definisani Kolomboovi stohastički procesi i uvoden je pojam vrednosti Kolomboovog stohastičkog procesa u tačkama sa kompaktnim nosačem. To nam je omogućilo da pokažemo merljivost odgovarajuće slučajne promenljive sa vrednostima u Kolomboovoj algebri uopštenih konstanti sa kompaktim nosačem, snabdevenoj toplogijom generisanom oštrim otvorenim loptama. Dat je pregled osnovnih osobina gausovskih Kolomboovih stohastičkih procesa.

U drugom delu disertacije proučavano je uopšteno očekivanje, uopštena korelacijska funkcija, uopštena kovarijansa i uopštena karakteristična funkcija Kolomboovog stohastičkog procesa. Nakon toga, definisani su Kolomboovi stohastički procesi sa nezavisnim vrednostima i data je karakterizacija takvih procesa preko njihove uopštene korelacijske funkcije u Kolomboovoj algebri uopštenih konstanti. U disertaciji su proučavane i osobine stacionarnih Kolomboovih stohastičkih procesa, pri čemu je napravljena razlika između stroge stacionarnosti i slabe stacionarnosti. Kolomboovi stohastički procesi sa stacionarnim priraštajima su definisani preko stacionarnosti gradijenta procesa.

U završnom delu disertacije dat je metod za rešavanje jedne klase stohastičkih parcijalnih diferencijalnih jednačina u okvirima stacionarnih gausovskih temperiranih Kolomboovih stohastičkih procesa. Predstavljena metoda za rešavanje stohastičkih parcijalnih diferencijalnih jednačina koristi tehnike Furijeove transformacije. Kao ilustracija, predstavljeno je rešenje stacionarne Klajn–Gordonove jednačine.

Ova doktorska disertacija predstavlja rezultat višegodišnjeg naučno–istraživačkog rada pod mentorstvom Akademika dr Stevana Pilipovića i profesora dr Michael-a Oberguggenberger-a, kao i saradnje sa profesoricom dr Dorom Seleši. Doktorska disertacija nastala je kao rezultat rada na sledećim projektima:

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Sombor, 16. maj 2018. Snežana Gordić

Preface

The subject of this doctoral dissertation are generalized stochastic processes. Generalized stochastic processes occur as solutions of a wide class of stochastic partial differential equations that do not have a solution in the classical sense. Therefore, generalized stochastic processes represent a good theoretical framework for studying stochastic partial differential equations.

Generalized stochastic processes of Colombeau-type, or shortly Colombeau stochastic processes, are considered in the dissertation. Colombeau stochastic processes were considered in many of the papers listed in bibliography, but none of these papers addressed the question of probabilistic properties of Colombeau stochastic processes. The main aim of the research in the dissertation are the probabilistic properties of Colombeau stochastic processes. Also, our aim is to apply the obtained theoretical results for solving a class of stochastic partial differential equations.

In the first part of the dissertation, a brief overview of basic notions of Colombeau theory of generalized functions is given. Special attention is devoted to the property of translational invariance of Colombeau generalized functions. Thereafter, Colombeau stochastic processes are defined. The notion of point values of Colombeau stochastic processes in compactly supported generalized points was introduced. This allowed us to prove measurability of the corresponding random variable with values in a Colombeau algebra of compactly supported generalized constants, endowed with the topology generated by sharp open balls. An overview of the basic characteristics of Gaussian Colombeau stochastic processes is given.

In the second part of the dissertation generalized expectation, generalized correlation functions, generalized covariances and generalized characteristic functions of Colombeau stochastic processes were studied. After that, Colombeau stochastic processes with independent values are defined and the characterization of such processes is given. The properties of stationary Colombeau stochastic processes are studied, distinguishing between strict stationarity and weak stationarity. Colombeau stochastic processes with stationary increments are defined via stationarity of the gradient of the process.

In the closing part of the dissertation a method for solving a class of stochastic partial differential equations in the framework of stationary Gaussian tempered Colombeau stochastic processes is presented. The presented method uses the technique of the Fourier transform. As an illustration, the stationary Klein–Gordon equation is considered.

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> *Great things in business are never done by one person. They are done by a team of people.*
>
> — **Steve Jobs**
> (1955 - 2011)

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Contents

1 Introduction

> *The scientific man does not aim at an immediate result. He does not expect that his advanced ideas will be readily taken up. His work is like that of the planter - for the future. His duty is to lay the foundation for those who are to come and point the way.*
>
> — **Nikola Tesla**
> (1856 - 1943)

The theory of generalized stochastic (random) processes, or shortly GSPs, was introduced independently in the mid 1950s by K. Itô and I. M. Gel'fand (see [Itô54] and [Gel55]). Ever since, GSPs have been investigated by many authors. The theory of GSPs continues to be an active topic of research. This theory has proved to have many useful applications.

The Colombeau approach into the settings of GSPs was introduced by M. Oberguggenberger and F. Russo (see [Obe95] and [Rus94]) in the mid 1990s. GSPs of Colombeau-type are used in solving some nonlinear stochastic problems.

The subject of the research within this doctoral dissertation are Colombeau generalized stochastic processes or shortly Colombeau stochastic processes (CSPs). Aims of the research of this dissertation are probabilistic properties of CSPs and applications in equation solving.

1.1 From Ordinary Stochastic Processes to Generalized Stochastic Processes

An ordinary (classical) stochastic process (OSP) is a mapping from an open subset O of $\mathbb{R}^d$ into a topological vector space of random variables over a probability space $(\Omega, \mathcal{F}, P)$. On the contrary, a GSP is a linear and continuous functional from a space of test functions over O into a space of random variables.

In [GV64] was given a physical justification for the concept of GSP. The concept of an OSP is based upon the assumption that it is possible to measure the value of the process at the particular time t. But, the concept of the value $u(t)$ of the OSP u at the time t is a mathematical idealization. In practice, one must use some device during a certain measurement. The reading on the device is not the value of random variable u at time t, but rather a certain averaged value

$$\langle u, \phi \rangle = \int u(t)\phi(t)\, dt,$$

where $\phi(t)$ is a function characterizing the device. These considerations suggest introducing a new definition of a stochastic process. Which properties should have a new process? A new process u should be linear in ϕ. Also, different devices should give approximately the same readings, i.e. if ϕ is close enough to ϕ_1 then $\langle u, \phi \rangle$ is close to $\langle u, \phi_1 \rangle$. Therefore, a new process should be a continuous linear random functional.

Note that a GSP extends the notion of an OSP in a similar way as the Schwartz generalized function is used to extend the notion of an ordinary (classical) function. This similarity is not surprising since we can regard the GSP as a vector-valued Schwartz generalized function.

The theory of GSPs has several advantages. The main advantage of GSPs is the fact that the family of GSPs is closed under differentiation, i.e. the derivative of a GSP always exists and is itself a GSP. In contrast, the family of OSPs is not closed under differentiation. For example, the derivative of Brownian motion is not an OSP, it is GSP. Some processes, for example white noise, can not be treated as OSPs, but can be treated as a GSPs.

There are three equivalent ways to look on an OSP:

I An OSP $u(\omega, x)$, $\omega \in \Omega$, $x \in O$, can be regarded as a family of random variables $u(\cdot, x)$, $x \in O$.

II An OSP $u(\omega, x)$, $\omega \in \Omega$, $x \in O$, can be regarded as a family of trajectories $u(\omega, \cdot)$, $\omega \in \Omega$.

III An OSP $u(\omega, x)$, $\omega \in \Omega$, $x \in O$, can be regarded as a family of functions

$$u : \Omega \times O \to \mathbb{R}^d$$

such that

- for each fixed $\omega \in \Omega$, $u(\omega, \cdot)$ is $\mathbb{R}^d-$valued random variable, and
- for each fixed $x \in O$, $u(x, \cdot)$ is an $\mathbb{R}^d-$valued random variable.

If one replaces the space of random variables with a space of generalized random variables, or if one replaces the space of trajectories with the space of deterministic generalized functions, then different types of GSPs are obtained. In such a way, we can obtain

- processes generalized with respect to the x argument,
- processes generalized with respect to the ω argument, and
- processes generalized with respect to both the x and ω arguments.

Each of these approaches have been investigated by many authors. We will give an overview of various definitions of the GSPs in the following section.

1.2 Generalized Stochastic Processes from the Mid 1950s to the Present

As mentioned in the previous section, GSPs can be introduced in several ways depending on the type of generalization. Let us give a historical overview of definitions of the concept of GSPs.

According to the definition given by K. Itô in [Itô54], a second-order GSP is a linear and continuous operator from the Schwartz space $\mathcal{D}(O)$ of test functions into a space $L^2(\Omega)$ of random variables with finite second moment. In contrast, a second-order OSP is a mapping from O into a space $L^2(\Omega)$. K. Itô investigated conditions for second-order GSP to be inducible by second-order OSPs and gave a response in the stationary case. The general case is solved in [Mei80].

A GSP, in the terminology of I. M. Gel'fand and N. Ya. Vilenkin (see [GV64]), is a continuous linear random functional u on the space of infinitely differentiable functions having bounded supports. Recall that u is a random functional if with every test function ϕ there is associated random variable $u(\phi)$. Note that this concept of GSP originates from I. M. Gel'fand (see [Gel55]).

H. Inaba and B. T. Tapley used in [IT75] the definition of a GSP as a continuous mapping from a certain space of test functions to the space $L^2(\Omega)$. Also, they defined the white Gaussian process as a GSP and gave its characterization.

GSPs as linear continuous mappings from a certain space of test functions to some space of classical or generalized random variables are considered in [PS07b; Sel07b; Urb61]. S. Pilipović and D. Seleši in [PS07b] referred to GSPs defined in this sense as to GSPs of type (I). The Wick product for GSPs of type (I) is developed in [PS07a; Sel07b].

Another approach is to define a GSP as a mapping $u : O \times V \to \mathbb{C}$ such that

(i) for every $\phi \in V$, $u(\cdot, \phi)$ is a random variable on $(\Omega, \mathcal{F}, P)$, and

(ii) for every $\omega \in \Omega$, $u(\omega, \cdot)$ is an element in V', where V denotes a topological vector space and V' its dual space.

In particular, if V is taken to be the space $\mathcal{D}(O)$ then u is a random Schwartz distribution as defined by M. Ullrich in [Ull57]. It has been shown that the GSP defined by M. Ullrich is a special case of the GSP in the sense of I. M. Gel'fand. Also, it has been shown that Ullrich's definition is not equivalent to the definition given by K. Itô. In [Ull59], a stochastic integral representation for a random Schwartz distribution is given.

In [SM71], C. Swartz and D. E. Myers considered the case where V is a space $K\{M_p\}$ and on inductive limits of $K\{M_p\}$ spaces. The reader is referred to [GS86] for the definition and basic properties of space $K\{M_p\}$. Z. Lozanov–Crvenković and S. Pilipović gave representation theorems for Gaussian GSPs on the space $K\{M_p\}$ (see [LCP94b]). Also, they studied and gave representation theorems for the GSPs on $\Omega \times \mathcal{A}$, where $\mathcal{A}$ is the Zemanian space (see [LCP89]). GSPs defined as mappings on $\Omega \times V$ were studied in [Han59; Crv89; Kit72; LCP88; LC92; LCP92; PS07b; Sel07a]. S. Pilipović and D. Seleši referred to GSPs defined in this sense as to GSPs of type (II) (see [PS07b; Sel07a; Sel07c]).

In [PS07b], it is shown that the classes of GSPs of type (I) and (II) are not contained in each other.

J. B. Walsh defined in [Wal86] a GSP as a measurable mapping $u : \Omega \to \mathcal{D}'(O)$. For each fixed test function $\phi \in \mathcal{D}(O)$, the mapping $\Omega \to \mathbb{R}$, $\omega \mapsto \langle u(\omega), \phi \rangle$ is a random variable.

On the other hand, stochastic processes with paths in the Colombeau algebra of generalized functions $\mathcal{G}(O)$ are considered in [MPS09; Obe95; OR98a; OR98b; OR01; RĆS12; NR02a; LCP97]. In this sense, GSP is defined as a mapping $u : \Omega \to \mathcal{G}(O)$ with the property that there exists a sequence of functions $u_n : \Omega \times O \to \mathbb{C}$, $n \in \mathbb{N}$, such that $u_n(\omega, \cdot)$ represents $u(\omega)$ for almost all $\omega \in \Omega$, and for every $n \in \mathbb{N}$, $(\omega, x) \to u_n(\omega, x)$ is an OSP. GSPs of this type are called Colombeau stochastic processes (CSPs). Note that this definition of a GSP is a generalization of the Walsh definition, which is due to the fact that Colombeau algebra contains the space of Schwartz distributions $\mathcal{D}'(O)$ as subspace. In the next section we will give an overview of existing results in theory of CSPs.

Another possibility is to generalize stochastic process with respect to the ω argument. T. Hida, Y. Kondratiev, B. Øksendal, H.-H. Kuo, their coauthors and many others (see [HKPS93; HØUZ96; Kuo96]) have developed a theory of GSPs via chaos expansions. In [HØUZ96] a GSP is defined as a measurable mapping $\mathbb{R}^d \mapsto (\mathcal{S})_{-1}$, where $(\mathcal{S})_{-1}$ denotes the Kondratiev space for the Gaussian measure. The Kondratiev spaces are infinite-dimensional analogues of the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. Note that one can consider other spaces of generalized stochastic functions instead of Kondratiev space $(\mathcal{S})_{-1}$. S. Pilipović and D. Seleši in [PS07a] referred to GSPs defined in this sense as to GSPs of type (O). Malliavin calculus for GSPs is studied in [LPS11; LS17; Lev11].

Processes generalized with respect to both arguments were introduced in [PS07a]. The authors generalized and unified the notion of GSP in Inaba's sense and the notion of GSP via chaos expansions. Thus, they considered GSPs as linear continuous mappings defined on the space of Zemanian test functions $\mathcal{A}$ and taking values in the Kondratiev space $(\mathcal{S})_{-1}$.

1.3 Generalized Stochastic Processes of Colombeau-type

In this section we give an overview of the existing results on the theory of generalized stochastic processes of Colombeau-type. M. Oberguggenberger and F. Russo were pioneers in theory of CSPs.

In [Obe95], M. Oberguggenberger introduced algebras of Colombeau-type generalized stochastic processes. The purpose of the Colombeau approach to the stochastic setting was to obtain an existence theory for the stochastic ordinary differential equation (SODE)

$$\begin{cases} \dot{u}(t) = f(u(t), v(t)), \quad t \in \mathbb{R}, \\ u(0) = a, \end{cases}$$

where v is any generalized stochastic process and f can be a nonlinear, discontinuous or generalized function.

Also, the applications to stochastic analysis of Colombeau theory were developed in [Rus94]. F. Russo investigated the relation between Colombeau generalized functions and stochastic integrals and equations of Stratonovich type. In this paper, the following SODE was considered in Colombeau sence

$$\begin{cases} \dot{u}(t) = f(u(t))w(t) + g(u(t)), \quad t \in \mathbb{R}_+, \\ u(0) = X, \end{cases}$$

where f and g are smooth functions, w is a white noise and X is a random variable. In the same paper, the author studied non-linear SPDE. Namely, a wave equation which is perturbed by a white noise was considered.

In [AHR96; OR98a; OR98b; RO99; OR01; AHR01; ORĆ05; NR02a] SPDEs are solved by regularization methods in the Colombeau framework. In [AHR96], the authors considered linear and nonlinear stochastic wave equations perturbed by a space-time Gaussian white noise in space dimension $d \geq 2$. In the nonlinear case the solution is a Colombeau random generalized function. A semilinear wave equation is considered in [OR98a; OR98b; OR01]. In [RO99] a semilinear stochastic heat equation driven by a space-time Gaussian white noise is considered. In [AHR01] the object of study was a two-space dimensional heat equation perturbed by white noise in a bounded volume. The paper [NR02a] is devoted to a one-dimensional nonlinear stochastic wave equation with additive white noise. Linear SODEs with generalized positive noise processes are considered in [ORĆ05].

In order to study a nonlinear stochastic wave and Klein–Gordon equations, the authors in [NR02b] have chosen to make use of the Colombeau approach. In this paper one-dimensional and three-dimensional nonlinear stochastic wave equation were considered. The existence and uniqueness of solution to the Klein–Gordon equation with Lipschitz and cubic nonlinearities have been proved.

Different types of CSPs were considered in [LCP97]. Stationary CSPs determined by Schwartz GSPs are studied. The authors defined CSPs with independent values, and a characterization of such processes is given if they are determined by Schwartz GSPs.

Gaussian Colombeau stochastic processes (GCSPs) were introduced and analyzed in [LCP97; LCP94a; MPS09]. Necessary and sufficient conditions for existence of a GCSP in terms of its generalized correlation function are given in [MPS09].

Paper [RĆS12] is devoted to some classes of nonlinear SODE containing generalized delta processes in the framework of the Colombeau theory and chaos expansions of GSPs. The nonlinear SODE can be replaced with an infinite system of ordinary differential equations (ODEs) by chaos expansion method. The system of ODEs can be solved by deterministic Colombeau methods. Lastly, it remains to sum up solutions and to prove convergence of the chaos expansions series. Paper [CO11] is devoted to an algebra of generalized functions of Colombeau-type in the context of the Hida stochastic distributions.

A theory of the Caputo fractional derivatives in the algebra of CSPs is introduced in [RĆS11]. It has been shown that a Caputo fractional derivative of a one-dimensional CSP u defined on $[0, \infty)$ is a CSP itself only if u satisfies certain conditions. The fractional derivatives of a Colombeau generalized process defined on entire $\mathbb{R}^d$ were introduced in [RĆS13]. It has been shown that a Caputo partial fractional derivative of a compactly supported multidimensional CSP is a CSP itself, but not with compact support. Colombeau fractional derivatives stochastic processes are introduced and analyzed in [RĆ10]. In [RĆ11], it has been shown that the Caputo fractional derivative of a CSP defined on $(0, \infty)$ is a CSP itself. Also, the Riemann-Liouville fractional derivative of a CSP defined on $[0, \infty)$ is introduced. As an illustration of the theory, a Cauchy problem with fractional derivatives is considered. The Cauchy problem for some classes of fractional differential equations in the Colombeau framework is considered in [JRĆ15; JRĆ17; JRĆ18; Jap16].

1.4 Motivation and Problem Statement

In the previous section we have seen that CSPs are widely studied. Many authors used the Colombeau theory to solve various classes of nonlinear SODEs and SPDEs. On the other hand, probabilistic properties of CSPs are not studied.

In this dissertation, the object of study are the CSPs with values in spaces of random variables with finite moments up to order p, the CSPs with values in spaces of random variables with all finite moments, and the CSP with values in the space of real valued measurable functions endowed with almost sure convergence. Two main aims of research are formulated:

1. Studying the probabilistic properties of CSPs, and
2. Application of the obtained results to solving some classes of SPDEs.

In order to achieve the first aim of research, we will investigate measurability of CSPs, expectation and correlation functions of CSPs, characteristic functions of CSPs, CSPs with independent values, and stationary CSPs.

In order to illustrate the application of the theory, at the end of the dissertation we give the method for solving a class of SPDEs in the framework of stationary Gaussian tempered CSPs. The stationary Klein–Gordon equation driven by higher order derivatives of white noise is considered.

1.5 Results

Main aims of research in the disseration have been achieved. All the results have been obtaind in joint work with Michael Oberguggenberger, Stevan Pilipović and Dora Seleši. The results are published in [GOPS18b] and [GOPS18a]. Some results have been presented at conferences:

- Workshop and Conference: Wien–Innsbruck–Novi Sad–Gent,
  June 29–July 3, 2016, University of Innsbruck, Austria,
  Research talk: *Generalized Stochastic Processes in Algebras of Generalized Functions*;
- The 14th Serbian Mathematical Congress,
  May 16–19, 2018, Faculty of Sciences, University of Kragujevac, Serbia,
  Research talk: *Probabilistic Properties of Colombeau Stochastic Processes*;
- International Conference on Generalized Functions,
  Dedicated to Professor Michael Oberguggenberger's 65th birthday,
  August 27–31, 2018, Faculty of Sciences, University of Novi Sad, Serbia,
  Research talk: *Stationary Colombeau Stochastic Processes*.

1.6 Dissertation Structure

The dissertation is organized in seven chapters and three appendix chapters.

Chapter 1

The subject and aims of a research are introduced in Chapter 1. An overview of different types of GSPs is given. Motivation for the concept of GSPs is considered. The structure of the dissertation is presented.

Chapter 2

Chapter 2 is expository and it represents an overview of some basic concepts of Colombeau theory, which are necessary to understand the sequent chapters of the dissertation. We summarize definitions and the most important properties of positive and positive–definite Colombeau generalized functions. We recall results on the solvability of the equation $P(D)u = f$, where f is a given Colombeau generalized function and $P(D)$ is a differential operator of order k with generalized real constant coefficients. All theorems are stated without proof since they are well-known.

Chapter 3

Chapter 3 is titled *Translation Invariant Colombeau Generalized Functions*. Translation invariance is a very important property in the theory of stationary stochastic processes. It is known that translation invariant Colombeau generalized function over $\mathbb{R}^d$ is generalized constant. The main result of this chapter is Theorem 3.2.1, which claim that a translation invariant Colombeau generalized function over an open convex subset of $\mathbb{R}^d$ is a generalized constant.

Chapter 4

Chapter 4, titled *Colombeau Stochastic Processes*, is concerned with spaces $\mathcal{G}^k_{\mathcal{L}}(\Omega, O)$, $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ and $\mathcal{G}^k_{L^p}(\Omega, O)$, whose elements are called CSPs with values in $\mathcal{L}(\Omega)$, $\mathcal{M}^\infty(\Omega)$ and $L^p(\Omega)$, respectively. Section 4.1, Section 4.4 and Section 4.5 contain the original parts of the dissertation. Evaluation of CSPs at compactly supported generalized points gives generalized random variables in $\mathcal{G}_{\mathcal{L}}(\Omega)$, $\mathcal{G}_{\mathcal{M}^\infty}(\Omega)$ and $\mathcal{G}_{L^p}(\Omega)$. The main result (see Proposition 4.5.1) of this chapter states that the corresponding mapping from Ω to $\mathcal{R}_c$ is $(\Omega, \mathcal{B}(\mathcal{R}_c))$-measurable, where $\mathcal{B}(\mathcal{R}_c)$ denotes the σ-algebra generated by sharp open balls of $\mathcal{R}_c$. We recall the definition and basic properties of Gaussian Colombeau stochastic processes, or shortly GCSPs, from [MPS09]. Also, we recall that the space of CSPs contains the space of distributional stochastic processes as a subspace.

Chapter 5

Chapter 5, entitled *Probabilistic Properties of Colombeau Generalized Processes*, represents the original part of the dissertation. The results are published in [GOPS18b] and [GOPS18a]. First, we recall the notions of generalized expectation and generalized correlation function of CSPs from [MPS09]. We give a characterization of generalized correlation function of CSPs. Generalized characteristic functions are introduced for CSPs in $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ and $\mathcal{G}^k_{L^{kp}}(\Omega, O)$. We prove that two possibilities to embed an element of space $\mathcal{C}^\infty_{\mathcal{M}^\infty}(\Omega, O)$ into $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$, namely by convolution or as a constant sequence, are identical in the Colombeau quotient. For an element of the space $\mathcal{C}_{\mathcal{M}^\infty}(\Omega, O)$ this equality holds on the level of association. The generalized expectation and generalized correlation function can be retrieved from the generalized characteristic function. The definition of CSPs with independent values is presented. A characterization of such processes via their generalized correlation function in the set of compactly supported generalized points is given. Strictly and weakly stationary CSPs are presented. We prove that the generalized expectation of a stationary CSP is a generalized constant and we provide a special form of its generalized correlation function. Stationarity of the increments of CSPs can be defined via stationarity of the gradient of the process.

Chapter 6

Chapter 6 is titled *Applications* and it is devoted to solving a class of SPDEs in the framework of stationary GCSPs. Techniques of the Fourier transform are used in this chapter. We give a necessary condition (Theorem 6.1.1) for the existence of a stationary Gaussian solution to $P(D)u = f$, where $P(D)$ is a differential operator of order k with generalized constant coefficients and f is a weakly stationary tempered GCSP. We apply the method developed in this chapter to solve the stationary Klein–Gordon equation driven by higher order derivatives of white noise, i.e. the equation

$$(1 - \Delta)u = c + f\partial^k w,$$

where w is spatial white noise on $\mathbb{R}^d$ considered in the framework of GCSPs, a and b are constants, Δ the Laplace operator and $\partial^k w = \partial^{k_1}_{x_1}\partial^{k_2}_{x_2}\dots\partial^{k_d}_{x_d} w(\omega, x)$ the kth derivative of the white noise for $k = (k_1, k_2, \dots, k_d) \in \mathbb{N}^d_0$.

Chapter 7

Chapter 7 gives a conclusion to the dissertation and possible directions for future work.

Appendix A

Appendix A, entitled *An Overview of Background Theory*, is devoted to basic topics from real and functional analysis, distribution theory, measure and probability theory. All theorems are stated without proof since they are parts of well-developed mathematical theories.

Appendix B

List of notation and list of abberivations are given in Appendix B.

Appendix C

Appendix C is a biographical index.

2 An Overview of the Colombeau Theory

> *Pure mathematics is in its way, the property of logical ideas.*
>
> — **Albert Einstein** (1879-1955)

The aim of this chapter is to give a brief overview of key notions of Colombeau theory that will be used extensively in the subsequent chapters of the dissertation. The theorems mentioned here are known and therefore given without proofs. The reader is referred to [Col84; Col85; GKOS01; NPS98; Obe92] for proofs and further details.

Note that in this dissertation we focus our attention on the sequential approach to Colombeau-type algebras. Instead of considering nets of functions $(u_\varepsilon)_\varepsilon$, $\varepsilon \in (0,1]$, we take sequences of functions $(u_n)_n$ indexed by $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$.

2.1 Colombeau Generalized Functions

Let O be an open set in $\mathbb{R}^d$. Further on, we use the conventional notation $\mathcal{C}^k(O)$ for k times continuously differentiable functions and $\mathcal{C}^\infty(O)$ for smooth functions, $\mathcal{D}(O)$ for the smooth test functions with compact support and its dual $\mathcal{D}'(O)$ for the generalized functions. The space of Schwartz distributions with compact support is denoted by $\mathcal{E}'(O)$. Similarly, $\mathcal{S}(O)$ denotes the Schwartz space of rapidly decreasing functions and $\mathcal{S}'(O)$ the space of tempered Schwartz distributions. A general references for properties of these spaces are [SP00; Vla79; GS86]. Also, the reader is referred to Appendix A.2.

In the following $K \Subset O$ will be used to denote that K is a compact subset of O. The notation $a_n = \mathcal{O}(b_n)$ means that $a_n \leq Cb_n$, $n > n_0$, for some constant $C > 0$ and $n_0 \in \mathbb{N}$.

Elements of $\mathcal{E}(O) = (\mathcal{C}^\infty(O))^{\mathbb{N}}$ are called sequences of smooth functions and denoted by $(u_n)_n$. The space $\mathcal{E}(O)$ endowed with componentwise operations is a differential algebra.

Omitting the general construction (see [Col84; Col85; GKOS01]) we recall only the definition and basic properties of the Colombeau algebra $\mathcal{G}(O)$.

Definition 2.1.1 *Set*

$$\mathcal{E}_M(O) = \{(u_n)_n \in \mathcal{E}(O) : (\forall K \Subset O)(\forall \alpha \in \mathbb{N}_0^d)(\exists a \in \mathbb{N}) \ (\sup_{x \in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^a))\},$$

$$\mathcal{N}(O) = \{(u_n)_n \in \mathcal{E}(O) : (\forall K \Subset O)(\forall \alpha \in \mathbb{N}_0^d)(\forall b \in \mathbb{N}) \ (\sup_{x \in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^{-b}))\},$$

Elements of $\mathcal{E}_M(O)$ are called moderate sequences of functions and elements of $\mathcal{N}(O)$ are called negligible sequences of functions. The Colombeau algebra on O is defined as

$$\mathcal{G}(O) = \mathcal{E}_M(O)/\mathcal{N}(O)$$

and its elements are Colombeau generalized functions.

Notice that $\mathcal{G}(O)$ is a quotient space, thus its elements $u \in \mathcal{G}(O)$ are equivalence classes denoted by $u = [(u_n)_n]$.

The space of all moderate sequences of functions $\mathcal{E}_M(O)$ is a differential algebra with pointwise operations. It is the largest differential subalgebra of $\mathcal{E}(O)$ in which $\mathcal{N}(O)$ is a differential ideal. Hence, $\mathcal{G}(O)$ is an associative commutative differential algebra.

Addition, multiplication and differentiation are carried out componentwise in $\mathcal{G}(O)$, i.e. if $u = [(u_n)_n] \in \mathcal{G}(O)$ and $v = [(v_n)_n] \in \mathcal{G}(O)$ then these operations are given by

$$u + v = [(u_n + v_n)_n], \quad u \cdot v = [(u_n \cdot v_n)_n], \quad \partial^\alpha u = [(\partial^\alpha u_n)_n].$$

Define a sequence of mollifiers $\varphi_n \in \mathcal{S}(\mathbb{R}^d), n \in \mathbb{N}$, of the form

$$\varphi_n(x) = n^d \varphi(nx), \ x \in \mathbb{R}^d, \ n \in \mathbb{N}, \tag{2.1}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^d)$ has the following properties:

$$\int \varphi(x)\, dx = 1, \tag{2.2}$$

$$\int x^m \varphi(x)\, dx = 0, \ \ m \in \mathbb{N}. \tag{2.3}$$

The Colombeau algebra contains the space of compactly supported Schwartz generalized functions. Indeed, if $f \in \mathcal{E}'(O)$ then

$$f \hookrightarrow Cd(f) = [((f * \varphi_n)|_O)_n] = ((f * \varphi_n)|_O)_n + \mathcal{N}(O)$$

defines a linear embedding of $\mathcal{E}'(O)$ into $\mathcal{G}(O)$. Since the presheaf $O \to \mathcal{G}(O)$ is a sheaf, it follows that the above embedding can be extended to an embedding of $\mathcal{D}'(O)$ and $\mathcal{C}^\infty(O)$ into $\mathcal{G}(O)$ for any open set $O \subset \mathbb{R}^d$.

Example 2.1.1 *The Dirac delta distribution* δ *as an element of the Colombeau algebra* $\mathcal{G}(\mathbb{R}^d)$ *is given by*

$$Cd(\delta) = [(\varphi_n)_n] = (\varphi_n)_n + \mathcal{N}(\mathbb{R}^d).$$

Now the expression δ^2 *obtains a meaning in* $\mathcal{G}(\mathbb{R}^d)$*. It is defined by*

$$Cd(\delta)Cd(\delta) = [(\varphi_n^2)_n] = (\varphi_n^2)_n + \mathcal{N}(\mathbb{R}^d). \qquad \square$$

The following result shows that in order that an element $(u_n)_n \in \mathcal{E}_M(O)$ is in $\mathcal{N}(O)$ it is enough to prove negligibility of its zeroth derivative. For a proof, see [GKOS01] (Chapter 1, Theorem 1.2.3, p. 11).

Theorem 2.1.1 $(u_n)_n \in \mathcal{E}_M(O)$ *is negligible if and only if for every* $K \Subset O$ *and every* $b \in \mathbb{N}$

$$\sup_{x \in K} |u_n(x)| = \mathcal{O}(n^{-b}).$$

We have seen that all operations are carried out componentwise in $\mathcal{G}(O)$. It is the same with the definition of integrals. Let A be a Lebesgue–measurable set such that $\overline{A} \Subset O$. The integral of $u = [(u_n)_n] \in \mathcal{G}(O)$ over A is defined by

$$\int_A u(x)\, dx = \left(\int_A u_n(x) \right)_n + \mathcal{N}(O).$$

Some basic properties of the integral can be found in [GKOS01]. An important property of integration of generalized functions is given in theorem below. Its proof can be found in [GKOS01] (Chapter 1, Theorem 1.2.36, p. 46).

Theorem 2.1.2 *Let* $f \in \mathcal{D}'(O)$ *and* $\phi \in \mathcal{D}(O)$*. Then*

$$\lim_{n \to \infty} \int Cd(f)_n(x)\phi(x)\, dx = \langle f, \phi \rangle.$$

The association relation is an important notion of the Colombeau theory.

Definition 2.1.2 *Two Colombeau generalized functions* $u = [(u_n)_n] \in \mathcal{G}(O)$ *and* $v = [(v_n)_n] \in \mathcal{G}(O)$ *are called associated, denoted by* $u \approx v$*, if*

$$\lim_{n \to \infty} \int_O (u_n(x) - v_n(x))\phi(x)\, dx = 0,$$

for all $\phi \in \mathcal{D}(O)$*.*

Note that Definition 2.1.2 is independent of the chosen representatives $(u_n)_n$ and $(v_n)_n$ of u and v, respectively. Association is an equivalence relation which respects addition and differentiation. Also, the product of two continuous functions in $\mathcal{G}(O)$ is associated with their classical product.

Observe that $u = v$ implies $u \approx v$. In general, the reverse implication is false. Hence, association is weaker than equality in $\mathcal{G}(O)$. An equivalence relation between association and equality is equality in the sense of distributions; see [Col84; Col85].

Definition 2.1.3 *Two Colombeau generalized functions $u = [(u_n)_n]$ and $v = [(v_n)_n]$ of $\mathcal{G}(O)$ are called equal in the sense of distributions, if*

$$\int_O (u_n(x) - v_n(x))\phi(x)\, dx \in \mathcal{N}(O)$$

for every $\phi \in \mathcal{D}(O)$.

Definition 2.1.4 *A Colombeau generalized function $u = [(u_n)_n] \in \mathcal{G}(O)$ is associated with a Schwartz distribution $f \in \mathcal{D}'(O)$ if*

$$\lim_{n\to\infty} \int_O u_n(x)\phi(x)\, dx = \langle f, \phi \rangle,$$

for all $\phi \in \mathcal{D}(O)$. This will be denoted by $u \approx f$ and f will be called the distributional shadow of u.

Note that not all elements of $\mathcal{G}(O)$ have a distributional shadow. Genuine Colombeau generalized functions, as $\delta^2 = [(\varphi_n^2)_n]$, are not associated with any Schwartz distribution. If the distributional shadow of u exists, it is uniquely determined.

Let $u = [(u_n)_n] \in \mathcal{G}(O)$ and O' is an open subset of O. The restriction $u|_{O'} \in \mathcal{G}(O')$ is defined as $[(u_n|_{O'})_n]$. A Colombeau generalized function $u = [(u_n)_n] \in \mathcal{G}(O)$ vanishes on O' if $u|_{O'} = 0$ in $\mathcal{G}(O')$. The support of $u = [(u_n)_n] \in \mathcal{G}(O)$ is defined as

$$\operatorname{supp} u = \Big(\bigcup\{O' \subseteq O : O' \text{ open and } u \text{ vanishes on } O'\}\Big)^c.$$

Note that the support of u is the complement of the largest open set O' such that u vanishes on O'. In general, if $u = [(u_n)_n] \in \mathcal{G}(O)$ is associated with an element $f \in \mathcal{D}'(O)$, then $\operatorname{supp} u \supseteq \operatorname{supp} f$. The next example indicates that the support of u can be strictly larger than the support of f.

Example 2.1.2 *Let $u \in \mathcal{G}(\mathbb{R})$ be given by the representative*

$$u_n(x) = n\phi(nx) + \frac{1}{n}, \quad n \in \mathbb{N},$$

where $\varphi \in \mathbb{D}(\mathbb{R})$ satisfies $\phi(0) = 1$ and $\int \phi(x)\,dx = 1$. Let us show that u is associated with the Dirac delta distribution. Using a change of variable $y = nx$, the Lebesgue dominated convergence and properties of the function ϕ we obtain

$$\begin{aligned}\lim_{n\to\infty}\int u_n(x)\psi(x)\,dx &= \int \lim_{n\to\infty} n\phi(nx)\psi(x)\,dx \\ &= \lim_{n\to\infty}\int \phi(y)\psi\left(\frac{y}{n}\right)\,dy \\ &= \psi(0)\int \phi(y)\,dy \\ &= \langle \delta, \psi\rangle,\end{aligned}$$

for any $\psi \in \mathcal{D}(\mathbb{R})$. It is clear that $\operatorname{supp} u = \mathbb{R}$ and it is known that $\operatorname{supp}\delta = \{0\}$. Therefore, the support of u is strictly larger than the support of δ. □

Observe that in Definition 2.1.1 we can consider functions with continuous derivatives up to kth order instead of smooth functions and thus obtain the spaces

$$\mathcal{E}^k(O) = (\mathcal{C}^k(O))^{\mathbb{N}},$$

$$\begin{aligned}\mathcal{E}^k_M(O) &= \{(u_n)_n \in \mathcal{E}^k(O) : (\forall K \Subset O)(\forall \alpha \in \mathbb{N}_0^d, |\alpha| \le k) \\ &\qquad (\exists a \in \mathbb{N})(\sup_{x\in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^a)\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}^k(O) &= \{(u_n)_n \in \mathcal{E}^k(O) : (\forall K \Subset O)(\forall \alpha \in \mathbb{N}_0^d, |\alpha| \le k) \\ &\qquad (\forall b \in \mathbb{N})(\sup_{x\in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^{-b}))\},\end{aligned}$$

$$\mathcal{G}^k(O) = \mathcal{E}^k_M(O)/\mathcal{N}^k(O).$$

2.2 Tempered Colombeau Generalized Functions

J. F. Colombeau introduced the algebra of tempered generalized functions in [Col85]. His main aim was to develop a theory of Fourier transform in algebras of generalized functions. The concept of Colombeau tempered generalized functions is a natural generalization of Schwartz tempered distributions.

In this section, we will recall the definition and basic properties of Colombeau tempered generalized functions.

Definition 2.2.1 *Set*

$$\begin{aligned}\mathcal{E}_\tau(O) &= \{(u_n)_n \in \mathcal{E}(O) : (\forall \alpha \in \mathbb{N}_0^d)(\exists N \in \mathbb{N}) \\ &\qquad (\sup_{x\in O} |\partial^\alpha u_n(x)|(1+|x|)^{-N} = \mathcal{O}(n^N))\},\end{aligned}$$

$$\mathcal{N}_\tau(O) = \{(u_n)_n \in \mathcal{E}(O) : (\forall\alpha \in \mathbb{N}_0^d)(\exists N \in \mathbb{N})(\forall a > 0)$$
$$(\sup_{x\in O} |\partial^\alpha u_n(x)|(1+|x|)^{-N} = \mathcal{O}(n^{-a}))\}.$$

Elements of $\mathcal{E}_\tau(O)$ and $\mathcal{N}_\tau(O)$ are called moderate, respectively negligible sequences of functions. The Colombeau algebra of tempered generalized functions is defined as the quotient space

$$\mathcal{G}_\tau(O) = \mathcal{E}_\tau(O)/\mathcal{N}_\tau(O).$$

Elements of $\mathcal{G}_\tau(O)$ are called tempered Colombeau generalized functions.

Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a Schwartz tempered distribution. Then

$$f \mapsto [(f * \varphi_n)_n]$$

defines a linear embedding of $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{G}_\tau(\mathbb{R}^d)$. Observe that $\mathcal{O}_M(\mathbb{R}^d)$ is a subalgebra of $\mathcal{G}_\tau(\mathbb{R}^d)$ via the constant embedding.

It is clear that $\mathcal{E}_\tau(\mathbb{R}^d) \subset \mathcal{E}_M(\mathbb{R}^d)$ and $\mathcal{N}_\tau(\mathbb{R}^d) \subset \mathcal{N}(\mathbb{R}^d)$. It is well-known that $\mathcal{C}^\infty(\mathbb{R}^d) \subset \mathcal{E}_M(\mathbb{R}^d)$, but $\mathcal{C}^\infty(\mathbb{R}^d)$ is not contained in $\mathcal{E}_\tau(\mathbb{R}^d)$. For example, the function $e^x \in \mathcal{C}^\infty(\mathbb{R})$ is not in $\mathcal{E}_\tau(\mathbb{R})$. Notice that $\mathcal{N}_\tau(\mathbb{R}^d)$ is a differential ideal in $\mathcal{E}_\tau(\mathbb{R}^d)$. Hence, $\mathcal{G}_\tau(\mathbb{R}^d)$ is a differential algebra with componentwise operations.

Note that $\mathcal{E}_\tau(\mathbb{R}^d) \cap \mathcal{N}(\mathbb{R}^d)$ is a proper subset of $\mathcal{N}_\tau(\mathbb{R}^d)$ (see [Col85], Chapter 4, Proposition 4.1.6, p. 101). Therefore, the mapping given by

$$\iota : \mathcal{G}_\tau(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d)$$

$$(u_n)_n + \mathcal{N}_\tau(\mathbb{R}^d) \mapsto (u_n)_n + \mathcal{N}(\mathbb{R}^d)$$

is not injective and it is not an embedding. So $\mathcal{G}_\tau(\mathbb{R}^d)$ is not a subalgebra of $\mathcal{G}(\mathbb{R}^d)$. But every $(u_n)_n$ representative of $u \in \mathcal{G}_\tau(\mathbb{R}^d)$ determines a unique element of $\mathcal{G}(\mathbb{R}^d)$. Mapping ι is called a canonical mapping.

2.3 Generalized Numbers and Point Values of Colombeau Generalized Function

In this section, we recall the notion of generalized numbers, point values of Colombeau generalized functions, and compactly supported generalized points.

Classical functions are completely characterized by their point value. Within the Schwartz theory, generalized function cannot be characterized by their point values in any similar way to classical functions. For Schwartz distributions a concept of point values was given by S. Łojasiewicz in [Ło57], but an arbitrary Schwartz distribution need not have point values in this sense at arbitrary points.

Within the Colombeau theory, generalized functions do have point values. Naturally, point values of Colombeau generalized functions are defined componentwise. In contrast to classical functions Colombeau generalized functions are not completely determined by their point values. It can happen that a Colombeau generalized function is not zero, but at every (classical) point its value is zero. But, Colombeau generalized functions are completely determined by their values in generalized points.

Let $\mathbb{K}$ be a field.

Definition 2.3.1 *Let*

$$\mathcal{E}_M = \{(r_n)_n \in \mathbb{K}^{\mathbb{N}} : (\exists a \in \mathbb{N})(|r_n| = \mathcal{O}(n^a))\},$$

$$\mathcal{N} = \{(r_n)_n \in \mathcal{E}_M : (\forall b \in \mathbb{N})(|r_n| = \mathcal{O}(n^{-b}))\}.$$

The quotient space

$$\mathcal{K} = \mathcal{E}_M / \mathcal{N}$$

is called the ring of generalized numbers.

Especially, for $\mathbb{K} = \mathbb{R}$ we obtain the ring of generalized real numbers denoted by $\mathcal{R}$. In case $\mathbb{K} = \mathbb{C}$ we obtain the ring of generalized complex numbers and we denote it by $\mathcal{C}$.

It is clear that $\mathcal{N}$ is an ideal in $\mathcal{E}_M$. Hence, $\mathcal{K}$ is a ring. Observe that $\mathcal{K}$ is not a field.

Note that $\mathbb{R}$ can be embedded into $\mathcal{R}$ by the identity mapping $\mathbb{R} \hookrightarrow \mathcal{R}$, $r \mapsto (r)_n + \mathcal{N}$.

Definition 2.3.2 *Let $u = [(u_n)_n] \in \mathcal{G}(O)$ (or $u = [(u_n)_n] \in \mathcal{G}_\tau(O)$) and $x_0 \in O$. The point value of the Colombeau (tempered) generalized function u at x_0 is the equivalence class $[(u_n(x_0))_n]$ in $\mathcal{K}$.*

Example 2.3.1 *The point value of the Dirac delta distribution δ at zero is*

$$Cd(\delta)(0) = [(\varphi_n(0))_n] = [(n\varphi(0)))_n]. \qquad \square$$

Similarly as for classical functions, but now in terms of generalized numbers we have the following result. For a proof, see [GKOS01] (Chapter 1, Proposition 1.2.35, p. 33).

Theorem 2.3.1 *Let O be a connected open subset of $\mathbb{R}^d$ and $u = [(u_n)_n] \in \mathcal{G}(O)$ (or $u = [(u_n)_n] \in \mathcal{G}_\tau(O)$). Then, $Du = 0$ if and only if $u \in \mathcal{K}$.*

The next example shows that Colombeau generalized functions are not determined by prescribing all their point values.

Example 2.3.2 *All point values of the Colombeau generalized function* $Cd(x)Cd(\delta) = [(x\varphi_n(x))_n] \in \mathcal{G}(\mathbb{R})$ *are equal to zero. Indeed, for* $x_0 \neq 0$, *we have*

$$x_0\varphi_n(x_0) = nx_0\varphi(nx_0) = \mathcal{O}(n^{-b})$$

for every $b \in \mathbb{N}$, *since* $\varphi \in \mathcal{S}(\mathbb{R})$. *The case* $x_0 = 0$ *is clear. It is known that* $x \times \delta = 0$ *in* $\mathcal{D}'(O)$. *But,* $Cd(x)Cd(\delta) \neq 0$ *in* $\mathcal{G}(\mathbb{R})$. *Let us show it. Choose some* $x_0 \neq 0$ *with* $\varphi(0) \neq 0$, *and set* $x = \frac{x_0}{n}$. *Then* $x\varphi_n(x) = x_0\varphi(x_0) \neq 0$. *Hence,*

$$\sup_{x\in[-1,1]} |x\varphi_n(x)| \not\to 0 \ \text{as} \ n \to \infty,$$

and $(x\varphi_n(x))_n \notin \mathcal{N}(\mathbb{R})$. □

In [OK99] the authors introduced the notion of generalized points of O and they showed that Colombeau generalized functions are determined by their values at compactly supported generalized points.

Definition 2.3.3 *Let*

$$O_M = \{(x_n)_n \in O^{\mathbb{N}} : (\exists a \in \mathbb{N})(|x_n| = \mathcal{O}(n^a))\}$$

and on O_M *define an equivalence relation* $\sim$ *by*

$$(x_n)_n \sim (y_n)_n \Leftrightarrow (\forall b \in \mathbb{N})(|x_n - y_n| = \mathcal{O}(n^{-b})).$$

The quotient space

$$\tilde{O} = O_M / \sim$$

is called the set of generalized points. The set

$$\tilde{O}_c = \{\tilde{x} = [(x_n)_n] \in \tilde{O} : (\exists K \Subset O)(x_n \in K \text{ for all } n)\}$$

is called the set of compactly supported generalized points.

Especially, for $O = \mathbb{R}$ we obtain the set of generalized real numbers $\tilde{\mathbb{R}} = \mathbb{R}_M/\sim = \mathcal{R}$. For $\tilde{\mathbb{R}}_c$ we write $\mathcal{R}_c$.

Definition 2.3.4 *Let* $u = [(u_n)_n] \in \mathcal{G}(O)$ *and* $\tilde{x} \in \tilde{O}_c$. *The point value of* u *at the compactly supported generalized point* $\tilde{x}$ *is the equivalence class* $u(\tilde{x}) = [(u_n(x_n))_n] \in \mathcal{R}_c$.

Now, Colombeau generalized functions are completely characterized by knowing their value at compactly supported generalized point. This is the main result of paper [OK99] (see Theorem 2.4., p. 4, for a proof).

Theorem 2.3.2 *Let $u = [(u_n)_n] \in \mathcal{G}(O)$. Then $u = 0$ in $\mathcal{G}(O)$ if and only if $u(\tilde{x}) = 0$ in $\mathcal{R}_c$ for all $\tilde{x} \in \tilde{O}_c$.*

2.4 The Sharp Topology

The Colombeau theory was developed aiming to solve non-linear problems and it has become an important tool in recent years. However, structure of the Colombeau theory was purely algebraic. It was expected that it would be possible to define topology compatible with the algebraic structure, in order to have a complete set of algebraic and topological tools for studying of non-linear problems.

The first step in this direction was done by H. A. Biagioni, J. F. Colombeau and M. Oberguggenberger in the mid 1980s. H. A. Biagioni and J. F. Colombeau had introduced in [BC86] a coarser (non-Hausdorff) topology on the set of generalized complex numbers. They proved that this topology is the topology of a uniform structure. H. A. Biagioni used in [Bia88] this topology to prove the well posedness of the Cauchy problem for semilinear hyperbolic systems with generalized functions as initial conditions (the behavior of the solution changes continuously with the initial conditions).

In [Obe91] the sharp topology was used to prove continuous dependence of solutions to the Carleman system with positive measures as initial data. The sharp topology in connection with the well posedness of the Burgers' equation was introduced in [BO92]. The sharp topology is mentioned in [Bia06] (Definition 5, p. 44), though the author puts more emphasis on another topology in this book.

In the period 1992–1993 the sharp topology was independently reintroduced and analyzed by D. Scarpalézos. His results were published later in [Sca98; Sca00; Sca04]. The name "sharp topology" appeared in his papers and he was the first to investigate its properties seriosly. D. Scarpalézos was the one who developed the theory of the sharp topology. For more details on the sharp topology, the reader is referred to [AFJ09; DHPV02; DHPV04; May07].

Omitting the general construction of the sharp topology on the Colombeau algebra $\mathcal{G}(O)$, we give its construction on the set of compactly supported generalized real numbers $\mathcal{R}_c$.

Let $(r_n)_n \in \mathbb{R}^{\mathbb{N}}$ and define

$$\|(r_n)_n\| = \limsup_{n\to\infty} |r_n|^{(\log n)^{-1}}.$$

We have

$$\begin{aligned}
\|(r_n)_n\| < \infty \quad &\Leftrightarrow \quad \limsup_{n\to\infty} |r_n|^{(\log n)^{-1}} < \infty \Leftrightarrow (\exists \sigma > 0) \left(\limsup_{n\to\infty} |r_n|^{(\log n)^{-1}} = \sigma \right) \\
&\Leftrightarrow \quad (\exists C > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}) \left(n > n_0 \Rightarrow |r_n| \leq C^{\log n} \right) \\
&\Leftrightarrow \quad (\exists C > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}) \left(n > n_0 \Rightarrow |r_n| \leq n^{\log C} \right) \\
&\Leftrightarrow \quad (\exists a \in \mathbb{N}) \left(|r_n| = \mathcal{O}(n^a) \right),
\end{aligned}$$

and

$$\begin{aligned}
\|(r_n)_n\| = 0 \quad &\Leftrightarrow \quad \limsup_{n\to\infty} |r_n|^{(\log n)^{-1}} = 0 \\
&\Leftrightarrow \quad (\forall C > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}) \left(n > n_0 \Rightarrow |r_n| \leq C^{\log n} \right) \\
&\Leftrightarrow \quad (\forall C > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}) \left(n > n_0 \Rightarrow |r_n| \leq n^{\log C} \right) \\
&\Leftrightarrow \quad (\forall b \in \mathbb{N}) \left(|r_n| = \mathcal{O}(n^{-b}) \right).
\end{aligned}$$

Therefore, the ring of generalized real numbers is given by

$$\mathcal{R} = \mathcal{E}_M / \mathcal{N},$$

where

$$\mathcal{E}_M = \{(r_n)_n \in \mathbb{R}^{\mathbb{N}} : \|(r_n)_n\| < \infty\},$$

$$\mathcal{N} = \{(r_n)_n \in \mathcal{E}_M : \|(r_n)_n\| = 0\}.$$

The corresponding set of compactly supported generalized real numbers is given by

$$\mathcal{R}_c = \mathcal{E}_M^c / \mathcal{N},$$

where

$$\mathcal{E}_M^c = \{(r_n)_n \in \mathcal{E}_M : (\exists K \Subset \mathbb{R}) \, (r_n \in K)\}.$$

The mapping

$$d_c : \mathcal{E}_M^c \times \mathcal{E}_M^c \to \mathbb{R}, \quad d_c((r_n)_n, (s_n)_n) = \|(r_n - s_n)_n\|$$

is an ultra-pseudometric on $\mathcal{E}_M^c$, and the mapping

$$\tilde{d}_c : \mathcal{R}_c \times \mathcal{R}_c \to \mathbb{R}, \quad \tilde{d}_c([(r_n)_n], [(s_n)_n]) = d_c((r_n)_n, (s_n)_n)$$

is an ultrametric on $\mathcal{R}_c$. The topology defined by $\tilde{d}_c$ is called the sharp topology on $\mathcal{R}_c$. For given $[(r_n)_n] \in \mathcal{R}_c$ and $k \in \mathbb{R}_+$, we call

$$\begin{aligned}
L((r_n)_n, k)) &= \{[(s_n)_n] \in \mathcal{R}_c : \|(r_n)_n - (s_n)_n\| < k\} \\
&= \{[(s_n)_n] \in \mathcal{R}_c : \limsup_{n\to\infty} |r_n - s_n|^{(\log n)^{-1}} < k\}
\end{aligned}$$

the sharp open ball with center $[(r_n)_n]$ and radius k. In this dissertation, we shall endow the set of compactly supported generalized real numbers $\mathcal{R}_c$ with the topology generated by the sharp open balls. We will denote by $\mathcal{B}(\mathcal{R}_c)$ the σ-algebra generated by the sharp open balls of $\mathcal{R}_c$. Observe that the σ-algebra generated by the sharp open balls is smaller than the σ-algebra generated by the sharp topology in $\mathcal{R}_c$. This is a consequence of the fact that $\mathcal{R}_c$ is not separable. This fact was also observed in [May07]. We will show this in the next chapter.

2.5 Positivity and Positive Definiteness

Concepts of positivity and positive definitness are very important in the theory of GSPs. These concepts in the framework of the Colombeau algebras have been introduced by M. Oberguggenberger, S. Pilipović and D. Scarpalézos in [OPS07]. In this section we quote definitions and results given in [OPS07].

2.5.1 Positive Colombeau Generalized Functions

First, we recall the notions of positive generalized functions from $\mathcal{D}'(O)$; see [GV64; Vla79]. An $f \in \mathcal{D}'(O)$ is positive if

$$\langle f, \phi \rangle \geq 0,$$

for every $\phi \in \mathcal{D}(O)$ with $\phi \geq 0$. We write $f \geq 0$.

An element $r \in \mathcal{R}$ is positive, denoted by $r \geq 0$, if it has a representative $(r_n)_n$ such that for every $a > 0$ there exists $n_0 \in \mathbb{N}$ satisfying the inequality

$$r_n + n^{-a} \geq 0,$$

for every $n > n_0$. If the above inequality is valid for some representative of r, it is valid for every representative of r. It is not difficult to see that $r \in \mathcal{R}$ is positive if and only if there is a positive representative $(r_n)_n$ of r, i.e. there is a representative $(r_n)_n$ such that $r_n \geq 0$ for all $n \in \mathbb{N}$.

Definition 2.5.1 *A Colombeau generalized function $u \in \mathcal{G}(O)$ is positive if there exists a representative $(u_n)_n$ of u such that for every $a > 0$ and $K \Subset O$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ it holds*

$$\inf_{x \in K} u_n(x) + n^{-a} \geq 0. \tag{2.4}$$

We write $u \geq 0$.

If (2.4) holds for some representative of u, it holds for every representative of u. Note that $u \in \mathcal{G}(O)$ is positive if and only if there exists a representative $(u_n)_n$ such that $u_n(x) \geq 0$, $x \in O$, $n \in \mathbb{N}$.

The proof of the following proposition is given in [OPS07] (see Proposition 3.4, p. 1324).

Proposition 2.5.1 *A Colombeau generalized function $u = [(u_n)_n] \in \mathcal{G}(O)$ is positive if and only if for every $\tilde{x} = [(x_n))n] \in \tilde{O}_c$ the generalized number $u(\tilde{x}) = [(u_n(x_n))_n]$ is positive.*

Next we recall the notion of $\mathcal{D}'$-weakly positive Colombeau generalized function.

Definition 2.5.2 *A Colombeau generalized function $u = [(u_n)_n] \in \mathcal{G}(O)$ is $\mathcal{D}'$-weakly positive if for every positive $\phi \in \mathcal{D}(O)$*

$$z_\phi = \left[\left(\int_{\mathbb{R}^d} u_n(t)\phi(t)\,dt\right)_n\right] \geq 0.$$

Observe that positivity is a stronger property than $\mathcal{D}'$-weak positivity. If $u = [(u_n)_n] \in \mathcal{G}(O)$ satisfies $u \geq 0$ and $u \leq 0$, then $u = 0$. Also, if $u = [(u_n)_n] \in \mathcal{G}(O)$ satisfies $u \geq 0$ and $u \leq 0$ in the $\mathcal{D}'$-weak sense, then $u = 0$ in the sense of distributions, i.e. for every positive $\phi \in \mathcal{D}(O)$ it holds

$$\int_O u_n(x)\phi(x)\,dx \in \mathcal{N}(O).$$

The following proposition gives the compatibility between positivity of Colombeau generalized functions and positivity of Schwartz distribution. The proof may be found in [OPS07] (see Proposition 3.16, p. 1328).

Proposition 2.5.2 *Let $f \in \mathcal{D}'(O)$. Then $Cd(f)$ is $\mathcal{D}'$-weakly positive if and only if f is a positive Schwartz distribution.*

2.5.2 Positive-definite Colombeau Generalized Functions

Recall, see [GV64; Vla79], an $f \in \mathcal{D}'(\mathbb{R}^d)$ is positive-definite if

$$\langle f, \phi * \phi^* \rangle \geq 0,$$

where $\phi^*(x) = \overline{\phi(-x)}$, $\phi \in \mathcal{D}(\mathbb{R}^d)$.

Next we recall the definition of positive-definite Colombeau generalized function on $\mathbb{R}^d$.

Definition 2.5.3 *A Colombeau generalized function $u \in \mathcal{G}(\mathbb{R}^d)$ is positive-definite on $\mathbb{R}^d$ if it has a representative $(u_n)_n$ such that*

$$(\forall K \Subset \mathbb{R}^d)(\forall a > 0)(\exists n_0 \in \mathbb{N})(\forall \zeta_1, \dots, \zeta_m \in \mathbb{C})$$

$$\inf_{x_k, x_j \in K} \sum_{k,j=1}^{m} (u_n(x_k - x_j) + n^{-a})\zeta_k \overline{\zeta_j} \geq 0, \quad n \geq n_0.$$

The proof of the next proposition may be found in [OPS07] (see Proposition 3.6, p.1324).

Proposition 2.5.3 *The following conditions are equivalent:*

(i) $u \in \mathcal{G}(\mathbb{R}^d)$ *is positive-definite.*

(ii) There is a representative $(u_n)_n$ such that

$$(\forall K \Subset \mathbb{R}^d)(\exists n_0 \in \mathbb{N})(\forall \zeta_1, \dots, \zeta_m \in \mathbb{C})$$

$$\inf_{x_k, x_j \in K} \sum_{k,j=1}^{m} u_n(x_k - x_j)\zeta_k \overline{\zeta_j} \geq 0, \quad n \geq n_0.$$

(iii) There is a representative $(u_n)_n$ such that

$$(\forall K \Subset \mathbb{R}^d)(\forall a > 0)(\exists n_0 \in \mathbb{N})(\forall \zeta_1, \dots, \zeta_m \in \mathbb{C})$$

$$\inf_{x_k, x_j \in K} \sum_{k,j=1}^{m} u_n(x_k - x_j)\zeta_k \overline{\zeta_j} + n^{-a} \geq 0, \quad n \geq n_0.$$

Next is the definition of $\mathcal{D}'$-weakly positive-definite Colombeau generalized function on $\mathbb{R}^d$.

Definition 2.5.4 *A Colombeau generalized function $u = [(u_n)_n] \in \mathcal{G}(\mathbb{R}^d)$ is $\mathcal{D}'$-weakly positive-definite if for every $\phi \in \mathcal{D}(\mathbb{R}^d)$,*

$$z_\phi = \left[\left(\int_{\mathbb{R}^d} u_n(t)(\phi * \phi^*)(t)\, dt \right)_n \right] \geq 0.$$

Proposition 2.5.4 *Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then $Cd(f)$ is $\mathcal{D}'$-weakly positive-definite if and only if f is a positive-definite Schwartz distribution.*

Positive-definite Schwartz distributions are in connection with translation invariant positive-definite bilinear functionals; see [GV64]. Recall, a real bilinear functional $B(\phi, \psi)$ on $\mathcal{D}(\mathbb{R}^d)$ is a functional which is linear in both arguments ϕ and ψ. It is said that a real bilinear functional $B(\phi, \psi)$ on $\mathcal{D}(\mathbb{R}^d)$ is translation invariant if its value does not change under simultaneous translation of $\phi(x)$ and $\psi(x)$ by the same vector $h \in \mathbb{R}^d$, i.e.

$$B(\phi(x), \psi(x)) = B(\phi(x+h), \psi(x+h)), \quad h \in \mathbb{R}^d.$$

A real bilinear functional $B(\phi, \psi)$ on $\mathcal{D}(\mathbb{R}^d)$ is positive-definite if for every $\phi \in \mathcal{D}(\mathbb{R}^d)$

$$B(\phi, \phi) \geq 0.$$

Let $B(\phi, \psi)$ be a positive-definite real bilinear functional on $\mathcal{D}(\mathbb{R}^d)$. Recall, see [GV64], if $\phi_1, \dots, \phi_m \in \mathcal{D}(\mathbb{R}^d)$ are linearly independent functions and $\zeta_1, \dots, \zeta_m \in \mathbb{R}$, then with

$$\psi = \sum_{i=1}^{m} \zeta_i \phi_i,$$

we obtain

$$\sum_{i,j=1}^{m} B(\phi_i, \phi_j)\zeta_i\zeta_j = B(\psi, \psi) \geq 0.$$

Every real translation invariant bilinear functional on $\mathcal{D}(\mathbb{R}^d)$ can be written in the form

$$B(\phi, \psi) = \langle F(x-y), \phi(x)\psi(y) \rangle, \tag{2.5}$$

where F is a generalized function in $\mathcal{D}'(\mathbb{R}^d)$. Moreover, $B(\phi, \psi)$ is positive-definite if and only if the generalized function F, which corresponds to B via (2.5), satisfies

$$\langle F, \phi * \phi^* \rangle \geq 0,$$

i.e. F is also positive-definite.

An element $B \in \mathcal{G}(O \times O)$ is $\mathcal{D}'$-weakly positive-definite if it has a representative $(B_n)_n$ such that for every $\phi \in \mathcal{D}(O)$,

$$\left[\left(\int_{\mathbb{R}^d} B_n(x,y)\phi(x)\phi(y)\, dxdy \geq 0 \right)_n \right] \geq 0.$$

2.6 Differential Operators With Generalized Real Constant Coeficients

In this section we will briefly present results from [PS96] and we will use them in Chapter 6.

Let $P(D)$ be a differential operator of order k with coefficients in $\mathcal{R}$ of the form

$$P(D) = \sum_{|\alpha| \leq k} a_\alpha D^\alpha, \quad a_\alpha \in \mathcal{R}, \tag{2.6}$$

where $D^\alpha = i^{|\alpha|}\partial^\alpha$. Consider the equation

$$P(D)u = f, \tag{2.7}$$

in $\mathcal{G}(\mathbb{R}^d)$. In [PS96] the classical method for solving the equation (2.7) is adapted to a method of solving the family of equations

$$P_n(D)u_n = f_n, \quad n \in \mathbb{N}, \tag{2.8}$$

where

$$P_n(D) = \sum_{|\alpha|<k} a_{\alpha,n} D^\alpha, \quad n \in \mathbb{N},$$

is a family of differential operators with moderate coefficients in the Colombeau sense ($(a_{\alpha,n})_n \in \mathcal{E}_M$), and $(f_n)_n \in \mathcal{E}_M(\mathbb{R}^d)$ is a representative of a given Colombeau generalized function $f \in \mathcal{G}(\mathbb{R}^d)$.

The main result is given in the following theorem. For the proof see [PS96] (see Theorem 1, p. 311).

Theorem 2.6.1 *Let $P(D)$ be a differential operator with coefficients in $\mathcal{R}$ of the form (2.6) such that*

$$\left| \sum_{|\alpha|=k} a_{\alpha,n} c^\alpha \right| > Cn^{-r}, \;\; n \in \mathbb{N},$$

holds for some $c \in \mathbb{R}^d$, $C > 0$ and $r \in \mathbb{R}$. Then for every $(f_n)_n \in \mathcal{E}_M(\mathbb{R}^d)$ there exists a solution $(u_n)_n \in \mathcal{E}_M(\mathbb{R}^d)$ of the equation (2.8). In particular, $[(u_n)_n]$ is the solution of equation (2.7).

3 Translation Invariant Colombeau Generalized Functions

> *A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories.*
>
> — **Stefan Banach**
> (1892 - 1945)

Invariance under translations is a very important concept of the theory of stationary processes. Stationary CSPs are the subject of research in Chapter 5, Section 5.5. Therefore, this chapter is devoted to properties of translation invariant Colombeau generalized functions.

Analogies between the theory of Colombeau generalized functions and the theory of classical functions are nontrivial. It is known that if a smooth function is invariant under all translations, then it has to be a constant function. A conjecture of M. Oberguggenberger was that the same is true for a Colombeau generalized function, i.e. a translation invariant Colombeau generalized function is a constant generalized function. His conjecture was proven by S. Pilipović, D. Scarpalézos and V. Valmorin in [PSV06].

In [PSV06] (see Theorem 6, p.798), it has been shown that a Colombeau generalized function $u \in \mathcal{G}(\mathbb{R}^d)$ invariant under all translations is a generalized constant. In [GOPS18a], we show that this fact also holds for $u \in \mathcal{G}(O)$, where O is an open convex subset of $\mathbb{R}^d$.

3.1 Translation Invariant Colombeau Generalized Functions over $\mathbb{R}^d$

In this section, we study a translation invariant Colombeau generalized function over $\mathbb{R}^d$. As mentioned above, translation invariant Colombeau generalized functions over $\mathbb{R}^d$ are studied in [PSV06]. We recall the theorem on translation invariant Colombeau generalized function over $\mathbb{R}^d$ from [PSV06].

A Colombeau generalized function $u = [(u_n)_n] \in \mathcal{G}(\mathbb{R}^d)$ is said to be translation invariant if

$$u(x+h) - u(x) = 0, \quad x \in \mathbb{R}^d,$$

holds as an equality in $\mathcal{G}(\mathbb{R}^d)$ for any $h \in \mathbb{R}^d$.

Theorem 3.1.1 *Let $u = [(u_n)_n] \in \mathcal{G}(\mathbb{R}^d)$ be a translation invariant Colombeau generalized function. Then u is a constant generalized function in $\mathcal{G}(\mathbb{R}^d)$.*

In [PSV06] two proofs of above theorem are given. The Baire theorem was used in the first proof. Arguments of the parametrix were used in the second proof. We give a proof of Theorem 3.1.1 following the first proof in [PSV06]. Notice that H. Vernaeve gave two original proofs of Theorem 3.1.1 in [Ver08] as well.

PROOF. Let $R > 0$. Let us show that $u = [(u_n)_n]$ is a constant generalized function, i.e. for every $p \in \mathbb{N}$

$$\lim_{n\to\infty} \sup_{t\in B(0,R)} n^p |u_n(t) - u_n(0)| = 0.$$

By assumption, u is invariant under all translations, i.e.

$$(\forall R > 0)(\forall p \in \mathbb{N})(\forall x \in \mathbb{R}^d)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})$$

$$(n \geq n_0 \Rightarrow \sup_{t\in B(0,R+\delta)} n^p |u_n(x+t) - u_n(t)| \leq 1),$$

where $\delta > 0$ is arbitrary. Let $p, l \in \mathbb{N}$. Put

$$F_{l,p} = \{x \in \mathbb{R}^d : n > l \Rightarrow \sup_{t\in B(0,R+\delta)} n^p |u_n(x+t) - u_n(t)| \leq 1\}.$$

The sets $F_{l,p}$ are closed and

$$\bigcup_{l=1}^{\infty} F_{l,p} = \mathbb{R}^d.$$

By the Baire theorem, there exist $l_0 \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$ and $r > 0$ such that

$$B(x_0, r) \subset F_{l_0,p}.$$

Put $r_1 = \inf\{r, \delta\}$. If $h \in B(0, r_1)$, then for $n > l_0$

$$\begin{aligned}
&\sup_{t\in B(0,R)} n^p |u_n(h+t) - u_n(t)| \\
&\leq \sup_{t\in B(0,R)} n^p |u_n(h+x_0+t) - u_n(h+t)| + \sup_{t\in B(0,R)} n^p |u_n(h+x_0+t) - u_n(t)| \\
&\leq \sup_{\tau\in B(0,R+\delta)} n^p |u_n(x_0+\tau) - u_n(\tau)| + \sup_{t\in B(0,R+\delta)} n^p |u_n(h+x_0+t) - u_n(t)| \\
&\leq 2, \qquad (3.1)
\end{aligned}$$

where $\tau = h + t \in B(0, R + \delta)$. Chose $n \in \mathbb{N}$ such that $n - 1 > \frac{R}{r_1}$. Let $x \in B(0, R)$ be an arbitrary point. We can find points $x_1, x_2, \ldots, x_n \in B(0, R)$ such that $x_1 = 0$, $x_n = x$ and $d(x_i, x_{i+1}) < r_1$. We have

$$|u_n(x) - u_n(0)| \leq |u_n(x_n) - u_n(x_{n-1})| + \ldots + |u_n(x_3) - u_n(x_2)| + |u_n(x_2) - u_n(x_1)|.$$

Therefore, using estimate (3.1) we obtain

$$\sup_{t \in B(0,R)} n^p |u_n(t) - u_n(0)| \leq 2n, \quad n > l_0.$$

From this the claim of the theorem follows. ■

3.2 Translation Invariant Colombeau Generalized Functions over an Open Convex Set O

In this section, we consider properties of translation invariant Colombeau generalized functions over an open convex subset O of $\mathbb{R}^d$.

The next theorem is the generalization of the Theorem 3.1.1. Its proof was given by S. Pilipović in [GOPS18a]. The Baire theorem is again used in the proof.

Theorem 3.2.1 *Let O be an open convex set in $\mathbb{R}^d$. Assume that for every $K \Subset O$ and every $h \in \mathbb{R}$ such that $t \in K$ implies $t + h \in O$, the following holds:*

$$(\forall p \in \mathbb{N})(\exists n_p \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq n_p \Rightarrow \sup_{t \in K} n^p |u_n(t + h) - u_n(t)| \leq 1). \tag{3.2}$$

Then $[(u_n)_n]$ is a generalized constant on O, i.e. there exists $(r_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that for every $K \Subset O$ and every $p > 0$ there exists $n_p > 0$ such that

$$\sup_{x \in K} n^{p-2} |u_n(x) - r_n| \leq 1, \quad n > n_p. \tag{3.3}$$

PROOF. Assume first that O is a bounded convex set and that $K \Subset O$, where $m > 0$ be chosen so that $K \Subset O_{3m}$; O_{3m} is the set of $t \in O$ such that $d(t, \partial O) > 3m$. We know that $\overline{O_m} \supset \overline{O_{2m}} \supset \overline{O_{3m}}$ are compact in O and that they are all convex. We will show that there exists a generalized constant $[(r_n)_n]$ such that for every $p > 0$ there exists $n_p > 0$ such that (3.3) holds.

Let $p \in \mathbb{N}$. Put

$$F_{l,p} = \{h \in \overline{B(0, m)} : (n \geq l) \Rightarrow (\sup_{t \in \overline{O_m}} n^p |u_n(t + h) - u_n(t)| \leq 1)\}.$$

Then $F_{l,p}$ are closed sets and

$$\bigcup_{l\in\mathbb{N}} F_{l,p} = \overline{B(0,m)}.$$

By the Baire theorem, there exist $h_0 \in B(0,m)$ and $c \in (0,m)$ such that

$$B(h_0,c) \subset F_{l_0,p},$$

that is,

$$\sup_{t\in O_m, h\in B(h_0,c)} n^p|u_n(t+h) - u_n(t)| \leq 1, \; n \geq l_0. \tag{3.4}$$

We will show that

$$\sup_{t\in O_{2m}, h\in B(0,c)} n^p|u_n(t+h) - u_n(t)| \leq 2, \; n \geq l_0. \tag{3.5}$$

Note, if $|\omega| < m$, then $O_{2m} - \omega \in O_m$. Every $h \in B(0,c)$ can be writen as $h = h_1 - h_0, h_1 \in B(h_0,c)$. Thus, for $t \in O_m$, we write

$$\begin{aligned} |u_n(t+h) - u_n(t)| &\leq |u_n(t+h_1-h_0) - u_n(t+h_1)| + |u_n(t+h_1) - u_n(t)| \\ &= |u_n(v) - u_n(v+h_0)| + |u_n(t+h_1) - u_n(t)|, \end{aligned}$$

where $t + h_1 - h_0 = v$. Since $v \in O_m$, by (3.4), we have

$$\begin{aligned} &\sup_{t\in O_{2m}, h\in B(0,c)} n^p|u_n(t+h) - u_n(t)| \\ &\leq \sup_{v\in O_m, h\in B(h_0,c)} n^p|u_n(v) - u_n(v+h)| + \sup_{t\in O_m, h_1\in B(h_0,c)} n^p|u_n(t+h_1) - u_n(t)| \\ &\leq 2, \quad n \geq l_0. \end{aligned}$$

For the moment, consider points of $\mathbb{R}^d$ as vectors $\vec{h} \equiv h$. Any $\vec{h} \in \mathbb{R}^d$, so that $|\vec{h}| < m$, can be written as a sum of vectors $\vec{h}_i$ with the same direction as $\vec{h}$ so that $\vec{h} = \sum_{i=1}^{w} \vec{h}_i$, where $w \leq [m/c] + 1$ and $|\vec{h}_i| < c$, $i = 1, \ldots, w$. Note that the returning to the "point" notation we have that for any $t \in O_{3m}$ and $h \in B(0,m)$,

$$\begin{aligned} &|u_n(t+h) - u_n(t)| \\ &\leq |u_n(t+h) - u_n(t + \sum_{i\leq w-1} h_i)| + |u_n(t + \sum_{i\leq w-1} h_i) - u_n(t + \sum_{i\leq w-2} h_i)| \\ &\quad + \ldots + |u_n(t+h_1) - u_n(t)| \\ &\leq \left(\frac{2m}{c} + 2\right) n^{-p}, \quad n \geq l_0. \end{aligned} \tag{3.6}$$

The last estimate follows from the fact that

$$t + \sum_{i\leq j-1} h_i = s \in O_{2m},$$

for any $j = 2, \ldots, w$. So the summands become $|u_n(s+h_j) - u_n(s)|$; we have to use (3.5) and obtain (3.6). Extending l_0 and denote it by n'_p, we obtain

$$\sup_{t \in O_{3m}} n^{p-1}|u_n(t+h) - u_n(t)| \leq 1, \;\; n > n'_p. \tag{3.7}$$

Now, we will use the fact that K can be covered by a finite set of balls with the radius less than m and all laying inside O_{3m}. Denote them by $B_1, \ldots, B_j$ with centers at $t_1, \ldots, t_j$ and put

$$r_n = f_n(t_1), \quad n \in \mathbb{N},$$

where we assume that $t_1 \in K$. We note that every $t \in K$ can be connected by t_1 by a finite (at most $2j - 1$) number of segments connecting points belonging to the intersections of two balls and the centers of the balls. Points of the intersections will be denoted by

$$s_1 \in B_1 \cap B_2, \ldots, s_{j-1} \in B_{j-1} \cap B_j.$$

Let $t \in B_j$. (If $t \in B_k, k < j$, the procedure is similar.) We write

$$|u_n(t) - u_n(t_1)| \leq$$

$$|u_n(t) - u(t_j)| + |u_n(t_j) - u_n(s_{j-1})| + |u_n(s_{j-1}) - u_n(t_{j-1})| + \ldots + |u_n(s_1) - u_n(t_1)|.$$

Since

$$d(t, t_j),\ d(t_j, s_{j-1}),\ d(s_{j-1}, t_{j-1}), \ldots, d(s_1, t_1) < m$$

we apply for the each absolute value on the right hand side the same procedure as above and in this way, obtain

$$\sup_{x \in K} n^{p-1}|u_n(x) - r_n| \leq C, \quad n > n'_p.$$

Again extending n'_p to n_p, we obtain (3.3).

If O is unbounded open set and $K \Subset O$, then, since the convex hull of K is also compact, there exists an open bounded convex set O_0 such that $K \Subset O_0 \subset O$. Repeating the above given proof for K and O_0 we obtain the complete proof of the theorem. ∎

Theorem 3.1.1 and Theorem 3.2.1 will turn out to be very useful in Section 5.5. Namely, Section 5.5 is devoted to stationary CSPs. Notice that an OSP is said to be stationary when its distribution is translation invariant. An OSP is said to be weakly stationary when its expectation and its covariance are translation invariant. Therefore, the properties of translation invariant Colombeau generalized functions are important for the study of stationary CSPs.

Colombeau Stochastic Processes 4

> *The moving power of mathematical invention is not reasoning, but imagination.*
>
> — **Augustus De Morgan**
> (1806 - 1871)

The aim of this chapter is to present Colombeau stochastic processes (CSPs). Section 4.1, Section 4.4 and Section 4.5 contain original parts of the dissertation, which are published in [GOPS18b].

In Section 4.1 we define several classes of CSPs. The notion and basic properties of Gaussian Colombeau stochastic processes (GCSPs) are recalled in Section 4.2. In Section 4.3 we recall the embedding of a distributional stochastic processes into the space of Colombeau stochastic processes. This section relies on [MPS09]. Finally, in Section 4.4, we give the characterisation of CSPs via their generalized point values. The main result in this chapter is the proof of the measurability of CSPs via their point values. This result is presented in Section 4.5.

4.1 Definitions and Basic Properties

We fix a probability space $(\Omega, \mathcal{F}, P)$; $\mathcal{F}$ is a family of measurable subsets of Ω and P is a probability measure (for more details, see Appendix A.3). Let $O \subset \mathbb{R}^d$ be an open set.

In the sequel, we will use a sequence of mollifiers $(\varphi_n)_n$ defined by (2.1). Recall that a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ from the definition of a sequence of mollifiers has the properties (2.2-2.3). Additionally, we assume that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is positive-definite, i.e. $\hat{\varphi} \geq 0$, where $\hat{\varphi}$ denotes the Fourier transform of φ. For example, one can take $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^d)$, $\hat{\varphi} \geq 0$ and $\hat{\varphi} \equiv 1$ in a neighborhood of zero.

4.1.1 Colombeau Stochastic Processes with Values in $L^p(\Omega)$

Let $p \in [1, \infty]$. The space $L^p(\Omega)$ consists of random variables with finite pth moments (see Appendix A.3). Now, we introduce the Colombeau stochastic processes over O with values in $L^p(\Omega)$.

Definition 4.1.1 *Let $k \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{E}^k_{L^p}(\Omega, O) = (\mathcal{C}^k_{L^p}(\Omega, O))^{\mathbb{N}}$ be the set of sequences $(u_n(\omega, x))_n$, $\omega \in \Omega$, $x \in O$, $n \in \mathbb{N}$, such that the mapping $x \mapsto u_n(\omega, x)$ is in $\mathcal{C}^k(O)$ for a.a. $\omega \in \Omega$, and for every $x \in O$, $u_n(\cdot, x)$ is in $L^p(\Omega)$. Define:*

$$
\begin{aligned}
\mathcal{E}^k_{M,L^p}(\Omega, O) &= \Big\{(u_n)_n \in \mathcal{E}^k_{L^p}(\Omega, O) \, : \, (\forall K \Subset O)(\forall \alpha \in \mathbb{N}^d_0, \ |\alpha| \leq k) \\
&\qquad (\exists a \in \mathbb{N})\Big(\sup_{x \in K} \|\partial^\alpha u_n(\cdot, x)\|_{L^p} = \mathcal{O}(n^a)\Big)\Big\}, \\
\mathcal{N}^k_{L^p}(\Omega, O) &= \Big\{(u_n)_n \in \mathcal{E}^k_{L^p}(\Omega, O) \, : \, (\forall K \Subset O)(\forall \alpha \in \mathbb{N}^d_0, \ |\alpha| \leq k) \\
&\qquad (\forall b \in \mathbb{N})\Big(\sup_{x \in K} \|\partial^\alpha u_n(\cdot, x)\|_{L^p} = \mathcal{O}(n^{-b})\Big)\Big\}.
\end{aligned}
$$

Elements of the vector spaces $\mathcal{E}^k_{M,L^p}(\Omega, O)$ and $\mathcal{N}^k_{L^p}(\Omega, O)$ are called moderate and negligible sequences of functions with values in $L^p(\Omega)$, respectively. The elements of the quotient space

$$\mathcal{G}^k_{L^p}(\Omega, O) = \mathcal{E}^k_{M,L^p}(\Omega, O)/\mathcal{N}^k_{L^p}(\Omega, O)$$

are called Colombeau stochastic processes (CSPs) over O with values in $L^p(\Omega)$.

For the case $k = \infty$, in the above definition we will omit the superscript ∞ and use the notation $\mathcal{E}_{M,L^p}(\Omega, O)$, $\mathcal{N}_{L^p}(\Omega, O)$ and $\mathcal{G}_{L^p}(\Omega, O)$.

In the sequel, we will use the phrase CSPs u with values in $L^p(\Omega)$ if u is element of $\mathcal{G}_{L^p}(\Omega, O)$ or $\mathcal{G}^k_{L^p}(\Omega, O)$. In the cases where it is important for k to be specified, we will do that.

Observe that pathwise continuity does not imply L^p-continuity and neither does L^p-continuity imply pathwise continuity. Counterexamples are given in Appendix A.3 (see Example A.3.1 and Example A.3.2). In above definition we require pathwise continuity almost everywhere (a.e.) and pathwise differentiability k times, but in general the mappings $x \mapsto u_n(\cdot, x)$ do not have to be continuous or differentiable with respect to the L^p-norm.

Also, observe that pathwise $\mathcal{C}^k$-smoothness for a.a. $\omega \in \Omega$ can easily be modified to obtain $\mathcal{C}^k$-smoothness for every $\omega \in \Omega$. Namely, there may exist at most countably many sets $A_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \leq k$, of probability measure zero on which the mapping $x \mapsto u_n(\omega, x)$ is not of class $\mathcal{C}^k$. Then $A = \cup_{n,\alpha} A_{n,\alpha}$ is also a zero-probability set and we can modify the representatives to be of class $\mathcal{C}^k$ by letting $u_n(\omega, x) = 0$ for $\omega \in A$.

The following result is a generalization of Theorem 2.1.1.

Proposition 4.1.1 *$(u_n)_n \in \mathcal{E}_{M,L^p}(\Omega, O)$ is negligible if and only if for every $K \Subset O$ and for every $b \in \mathbb{N}$*

$$\sup_{x \in K} \|u_n(\cdot, x)\|_{L^p} = \mathcal{O}(n^{-b}).$$

PROOF. If $(u_n)_n \in \mathcal{E}_{M,L^p}(\Omega, O)$ is negligible, then it trivially satisfies negligibility of the zeroth order derivative.

Suppose that $(u_n)_n \in \mathcal{E}_{M,L^p}(\Omega, O)$ satisfies negligibility of the zeroth order derivative. By induction, it suffices to show that the same is true for $(\partial_{x_i} u_n)_n$ for any $1 \le i \le d$. Let $K \Subset O$ and set $\delta := \min(1, \text{dist}(K, \partial O))$, $K_1 := K + \overline{B}_{\delta/2}(0)$. Since $(u_n)_n \in \mathcal{E}_{M,L^p}(\Omega, O)$, there exists $a \in \mathbb{N}$ such that

$$\sup_{x \in K_1} \|\partial^2_{x_i} u_n(\cdot, x)\|_{L^p} = \mathcal{O}(n^a)$$

as $n \to \infty$. From the given condition it follows that for any $b \in \mathbb{N}$,

$$\sup_{x \in K_1} \|u_n(\cdot, x)\|_{L^p} = \mathcal{O}(n^{-(a+2b)}).$$

Using Taylor's theorem, we obtain

$$u_n(\cdot, x + n^{-(a+b)} e_i) = u_n(\cdot, x) + \partial_{x_i} u_n(\cdot, x) n^{-(a+b)} + \frac{1}{2} \partial^2_{x_i} u_n(\cdot, x_\theta) n^{-2(a+b)},$$

i.e.

$$\partial_{x_i} u_n(\cdot, x) = (u_n(\cdot, x + n^{-(a+b)} e_i) - u_n(\cdot, x)) n^{a+b} - \frac{1}{2} \partial^2_{x_i} u_n(\cdot, x_\theta) n^{-(a+b)},$$

where $x_\theta = x + \theta n^{-(a+b)} e_i \in K_1$ for some $\theta \in (0, 1)$. Therefore,

$$\begin{aligned} \sup_{x \in K} \|\partial_{x_i} u_n(\cdot, x)\|_{L^p} &\le \left(\sup_{x \in K_1} \|u_n(\cdot, x + n^{-(a+b)} e_i)\|_{L^p} + \sup_{x \in K_1} \|u_n(\cdot, x)\|_{L^p} \right) n^{a+b} \\ &\quad + \frac{1}{2} \sup_{x \in K_1} \|\partial^2_{x_i} u_n(\cdot, x)\|_{L^p} n^{-(a+b)} \\ &\le C n^{-b}. \end{aligned}$$

■

Note that the operation of multiplication is not closed in the vector space $\mathcal{E}^k_{M,L^p}(\Omega, O)$.

Proposition 4.1.2 *Let* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

(a) If $(u_n)_n \in \mathcal{N}^k_{L^p}(\Omega, O)$ *and* $(v_n)_n \in \mathcal{E}^k_{L^q}(\Omega, O)$, *then* $(u_n v_n)_n \in \mathcal{N}^k_{L^r}(\Omega, O)$.

(b) If $(u_n)_n \in \mathcal{E}^k_{M,L^p}(\Omega, O)$ *and* $(v_n)_n \in \mathcal{E}^k_{L^q}(\Omega, O)$, *then* $(u_n v_n)_n \in \mathcal{E}^k_{L^r}(\Omega, O)$.

PROOF.

(a) Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$. Let $b \in \mathbb{N}$ be arbitrary. For any $\beta \in \mathbb{N}_0^d$ with $\beta \le \alpha$, there exists $a \in \mathbb{N}$ such that

$$\sup_{x \in K} \|\partial^{\alpha-\beta} v_n\|_{L^q} = \mathcal{O}(n^a).$$

It holds

$$\sup_{x\in K} \|\partial^\beta u_n\|_{L^p} = \mathcal{O}(n^{-(a+b)}).$$

Now, we have

$$\sup_{x\in K} \|\partial^\alpha (u_n v_n)\|_{L^r} \leq C \sup_{x\in K} \|\partial^\beta u_n\|_{L^p} \|\partial^{\alpha-\beta} v_n\|_{L^q} = \mathcal{O}(n^{-b}).$$

This implies the assertions.

(b) Follow directly from the definition. ■

Another fact to note is that the operation of differentiation is not closed in the vector space $\mathcal{E}^k_{M,L^p}(\Omega, O)$, $k < \infty$. Indeed, if $(u_n)_n \in \mathcal{E}^k_{M,L^p}(\Omega, O)$ and $\alpha \in \mathbb{N}^d_0$, $|\alpha| \leq k$, then there exists $a \in \mathbb{N}$ such that

$$\sup_{x\in K} \|\partial^\beta (\partial^\alpha u_n(\omega, x))\|_{L^p} = \sup_{x\in K} \|\partial^{\alpha+\beta} u_n(\omega, x))\|_{L^p} = \mathcal{O}(n^a),$$

for every $\beta \in \mathbb{N}^d_0$, $|\beta| \leq k - |\alpha|$. Therefore, $(\partial^\alpha u_n)_n \in \mathcal{E}^{k-|\alpha|}_{M,L^p}(\Omega, O)$.

If the elements of a sequence $(u_n)_n$ do not depend on $x \in O$, then Definition 4.1.1 reduces to the following definition.

Definition 4.1.2 *Let $k \in \mathbb{N} \cup \{\infty\}$. Define:*

$$\mathcal{E}^k_{M,L^p}(\Omega) = \{(u_n)_n \in L^p(\Omega) : (\exists a \in \mathbb{N})(\|u_n(\cdot)\|_{L^p} = \mathcal{O}(n^a)\},$$

$$\mathcal{N}^k_{L^p}(\Omega) = \{(u_n)_n \in L^p(\Omega) : (\forall b \in \mathbb{N})(\|u_n(\cdot)\|_{L^p} = \mathcal{O}(n^{-b})\}.$$

Elements of the vector spaces $\mathcal{E}^k_{M,L^p}(\Omega)$ and $\mathcal{N}^k_{L^p}(\Omega)$ are called moderate and negligible sequences of random variables, respectively. Elements of the corresponding quotient space

$$\mathcal{G}_{L^p}(\Omega) = \mathcal{E}_{M,L^p}(\Omega)/\mathcal{N}_{L^p}(\Omega)$$

are called generalized random variables with values in $L^p(\Omega)$.

4.1.2 Colombeau Stochastic Processes with Values in $\mathcal{M}^\infty(\Omega)$

Denote by

$$\mathcal{M}^\infty(\Omega) = \bigcap_{1\leq p<\infty} L^p(\Omega)$$

the space of random variables with finite seminorms

$$|||\cdot|||_s = \sup_{1\leq p\leq s} \|\cdot\|_{L^p}, \quad s \in \mathbb{N}.$$

Definition 4.1.3 *Let* $\mathcal{E}_{\mathcal{M}^\infty}(\Omega, O) = (\mathcal{C}^\infty_{\mathcal{M}^\infty}(\Omega, O))^{\mathbb{N}}$ *be the set of sequences* $(u_n(\omega, x))_n$*,* $\omega \in \Omega$, $x \in O$, $n \in \mathbb{N}$*, such that the mapping* $x \mapsto u_n(\omega, x)$ *is in* $C^\infty(O)$ *for a.a.* $\omega \in \Omega$*, and for every* $x \in O$, $u_n(\cdot, x)$ *is in* $\mathcal{M}^\infty(\Omega)$*. Define:*

$$\begin{aligned}
\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O) &= \Big\{(u_n)_n \in \mathcal{E}_{\mathcal{M}^\infty}(\Omega, O) \,:\, (\forall K \Subset O)(\forall \alpha \in \mathbb{N}_0^d)(\forall s \in \mathbb{N}) \\
&\qquad (\exists a \in \mathbb{N})\Big(\sup_{x \in K} |||\partial^\alpha u_n(\cdot, x)|||_s = \mathcal{O}(n^a)\Big)\Big\}, \\
\mathcal{N}_{\mathcal{M}^\infty}(\Omega, O) &= \Big\{(u_n)_n \in \mathcal{E}_{\mathcal{M}^\infty}(\Omega, O) \,:\, (\forall K \Subset O)(\forall \alpha \in \mathbb{N}_0^d)(\forall s \in \mathbb{N}) \\
&\qquad (\forall b \in \mathbb{N})\Big(\sup_{x \in K} |||\partial^\alpha u_n(\cdot, x)|||_s = \mathcal{O}(n^{-b})\Big)\Big\}.
\end{aligned}$$

Elements of $\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)$ *and* $\mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$ *are called moderate and negligible sequences of functions with values in* $\mathcal{M}^\infty(\Omega)$*, respectively. Elements of the quotient space*

$$\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O) = \mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)/\mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$$

are called Colombeau stochastic processes (CSPs) over Ω *with values in* $\mathcal{M}^\infty(\Omega)$*.*

The spaces $\mathcal{E}_{\mathcal{M}^\infty}(\Omega, O)$ and $\mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$ are algebras. Clearly, $\mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$ is an ideal in $\mathcal{E}_{\mathcal{M}^\infty}(\Omega, O)$, and so $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ is an algebra.

The following result gives a characterization of $\mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$ as a subspace of $\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)$.

Proposition 4.1.3 $(u_n)_n \in \mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)$ *is negligible if and only if for every* $K \Subset O$*, for every* $s \in \mathbb{N}$ *and for every* $b \in \mathbb{N}$

$$\sup_{x \in K} |||u_n(\cdot, x)|||_s = \mathcal{O}(n^{-b}).$$

PROOF. The proof is analogous to the proof of Proposition 4.1.1. ■

Let $p \geq q$. Notice that

$$\mathcal{E}_{M,L^\infty}(\Omega, O) \to \mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O) \to \mathcal{E}_{M,L^p}(\Omega, O) \to \mathcal{E}_{M,L^q}(\Omega, O) \to \mathcal{E}_{M,L^1}(\Omega, O),$$

$$\mathcal{N}_{L^\infty}(\Omega, O) \to \mathcal{N}_{\mathcal{M}^\infty}(\Omega, O) \to \mathcal{N}_{L^p}(\Omega, O) \to \mathcal{N}_{L^q}(\Omega, O) \to \mathcal{N}_{L^1}(\Omega, O).$$

Therefore, for $p \geq q$, there exist canonical mappings

$$\mathcal{G}_{L^\infty}(\Omega, O) \to \mathcal{G}_{\mathcal{M}^\infty}(\Omega, O) \to \mathcal{G}_{L^p}(\Omega, O) \to \mathcal{G}_{L^q}(\Omega, O) \to \mathcal{G}_{L^1}(\Omega, O)$$

although the mapping is not injective. This means that a sequence $(u_n)_n$ determining an element of the left space determines the element of the right space.

Now, we introduce the notion of generalized random variables with values in $\mathcal{M}^\infty(\Omega)$.

Definition 4.1.4 *Define:*

$$\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega) = \{(u_n)_n \in \mathcal{M}^\infty(\Omega) : (\forall s \in \mathbb{N})(\exists a \in \mathbb{N})(|||u_n(\cdot)|||_s = \mathcal{O}(n^a)\},$$

$$\mathcal{N}_{\mathcal{M}^\infty}(\Omega) = \{(u_n)_n \in \mathcal{M}^\infty(\Omega) : (\forall s \in \mathbb{N})(\forall b \in \mathbb{N})(|||u_n(\cdot)|||_s = \mathcal{O}(n^{-b})\}.$$

Elements of the vector spaces $\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega)$ and $\mathcal{N}_{\mathcal{M}^\infty}(\Omega)$ are called moderate and negligible sequences of random variables, respectively. Elements of the corresponding quotient space

$$\mathcal{G}_{\mathcal{M}^\infty}(\Omega) = \mathcal{E}_{M,\mathcal{M}^\infty}(\Omega)/\mathcal{N}_{\mathcal{M}^\infty}(\Omega)$$

are called generalized random variables with values in $\mathcal{M}^\infty(\Omega)$.

4.1.3 Colombeau Stochastic Processes with Values in $\mathcal{L}(\Omega)$

The space $\mathcal{L}(\Omega)$ comprises the real valued random variables (measurable functions) on Ω endowed with almost sure convergence.

In this subsection, we consider a class of CSPs with values in $\mathcal{L}(\Omega)$.

Definition 4.1.5 *Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{E}^k_{\mathcal{L}}(\Omega, O)$ be the set of sequences $(u_n(\omega, x))_n$, $\omega \in \Omega$, $x \in O$, $n \in \mathbb{N}$, such that $(u_n(\omega, \cdot))_n \in (\mathcal{C}^k(\Omega))^{\mathbb{N}}$ for almost all (a.a.) $\omega \in \Omega$, and for every $x \in O$, $(u_n(\cdot, x))_n$ is a sequence of measurable functions on Ω. Define:*

$$\begin{aligned}
\mathcal{E}^k_{M,\mathcal{L}}(\Omega, O) &= \Big\{(u_n)_n \in \mathcal{E}^k_{\mathcal{L}}(\Omega, O) : \ (\text{for a.a. } \omega \in \Omega)(\forall K \Subset O) \\
&\quad (\forall \alpha \in \mathbb{N}^d_0, \ |\alpha| \le k)(\exists a \in \mathbb{N})\Big(\sup_{x \in K} |\partial^\alpha u_n(\omega, x)| = \mathcal{O}(n^a)\Big)\Big\}, \\
\mathcal{N}^k_{\mathcal{L}}(\Omega, O) &= \Big\{(u_n)_n \in \mathcal{E}^k_{\mathcal{L}}(\Omega, O) : \ (\text{for a.a. } \omega \in \Omega)(\forall K \Subset O) \\
&\quad (\forall \alpha \in \mathbb{N}^d_0, \ |\alpha| \le k)(\forall b \in \mathbb{N})\Big(\sup_{x \in K} |\partial^\alpha u_n(\omega, x)| = \mathcal{O}(n^{-b})\Big)\Big\}.
\end{aligned}$$

Elements of $\mathcal{E}^k_{M,\mathcal{L}}(\Omega, O)$ and $\mathcal{N}^k_{\mathcal{L}}(\Omega, O)$ are called moderate and negligible sequences of functions with values in $\mathcal{L}(\Omega)$, respectively. The elements of the quotient space

$$\mathcal{G}^k_{\mathcal{L}}(\Omega, O) = \mathcal{E}^k_{M,\mathcal{L}}(\Omega, O)/\mathcal{N}^k_{\mathcal{L}}(\Omega, O)$$

are called Colombeau stochastic processes (CSPs) over O with values in $\mathcal{L}(\Omega)$.

If $k = \infty$ in the above definition we will use the notation $\mathcal{E}_{M,\mathcal{L}}(\Omega, O)$, $\mathcal{N}_{\mathcal{L}}(\Omega, O)$ and $\mathcal{G}_{\mathcal{L}}(\Omega, O)$. Clearly, $\mathcal{E}^k_{M,\mathcal{L}}(\Omega, O)$ is an algebra with respect to multiplication and $\mathcal{N}^k_{\mathcal{L}}(\Omega, O)$ is an ideal in $\mathcal{E}^k_{M,\mathcal{L}}(\Omega, O)$, so we have that $\mathcal{G}^k_{\mathcal{L}}(\Omega, O)$ is an algebra.

The following result shows that in order that an element $(u_n)_n \in \mathcal{E}_{M,\mathcal{L}}(\Omega, O)$ is in $\mathcal{N}(\Omega, O)$ it is enough to prove negligibility of its zeroth derivative.

Proposition 4.1.4 *$(u_n)_n \in \mathcal{E}_{M,\mathcal{L}}(\Omega, O)$ is negligible if and only if the following condition is satisfied:*

$$(\text{for a.e. } \omega \in \Omega)(\forall K \Subset O)(\forall b \in \mathbb{N})\Big(\sup_{x\in K} |u_n(\cdot, x)| = \mathcal{O}(n^{-b})\Big)$$

PROOF. If $(u_n)_n \in \mathcal{E}_{M,\mathcal{L}}(\Omega, O)$ is negligible, then it trivially satisfies negligibility of the zeroth order derivative.

Let $(u_n)_n \in \mathcal{E}_{M,\mathcal{L}}(\Omega, O)$ satisfies negligibility of the zeroth order derivative. It suffices to prove that for any $1 \le i \le d$, $(\partial_{x_i} u_n)_n$ satisfies negligibility of zeroth order derivative. Let $K \Subset O$ and set $\delta := \min(1, \text{dist}(K, \partial O))$, $K_1 := K + \overline{B}_{\delta/2}(0)$. Let $b \in \mathbb{N}$ be given. By assumptions we can choose $a \in \mathbb{N}$ such that both of the estimates hold

$$\sup_{x\in K_1} |\partial^2_{x_i} u_n(\cdot, x)| = \mathcal{O}(n^a),$$

$$\sup_{x\in K_1} |u_n(\cdot, x)| = \mathcal{O}(n^{-(a+2b)}),$$

for a.a. $\omega \in \Omega$. Taylor expansion gives

$$\partial_{x_i} u_n(\cdot, x) = (u_n(\cdot, x + n^{-(a+b)} e_i) - u_n(\cdot, x)) n^{a+b} - \frac{1}{2}\partial^2_{x_i} u_n(\cdot, x_\theta) n^{-(a+b)},$$

where $x_\theta = x + \theta n^{-(a+b)} e_i \in K_1$ for some $\theta \in (0, 1)$. Therefore,

$$\begin{aligned}
&\sup_{x\in K} \|\partial_{x_i} u_n(\cdot, x)\|_{L^p} \\
&\le \left(\sup_{x\in K_1} \|u_n(\cdot, x + n^{-(a+b)} e_i)\|_{L^p} + \sup_{x\in K_1} \|u_n(\cdot, x)\|_{L^p}\right) n^{a+b} \\
&\quad + \frac{1}{2} \sup_{x\in K_1} \|\partial^2_{x_i} u_n(\cdot, x)\|_{L^p} n^{-(a+b)} \\
&\le Cn^{-b}. \qquad \blacksquare
\end{aligned}$$

Notice that if elements of a sequence $(u_n)_n$ do not depend on $x \in O$, then Definition 4.1.5 reduces to the definition of generalized random variables with values in $\mathcal{L}(\Omega)$.

Definition 4.1.6 *Let $\mathcal{E}_{\mathcal{L}}(\Omega)$ be the space of sequences of measurable functions on Ω. Define:*

$$\mathcal{E}_{M,\mathcal{L}}(\Omega) = \{(u_n)_n \in \mathcal{E}_{\mathcal{L}}(\Omega) : (\text{for a.a. } \omega \in \Omega)(\exists a \in \mathbb{N})\,(|u_n(\omega)| = \mathcal{O}(n^a))\},$$

$$\mathcal{N}_{\mathcal{L}}(\Omega) = \Big\{(u_n)_n \in \mathcal{E}_{\mathcal{L}}(\Omega) : (\text{for a.a. } \omega \in \Omega)(\forall b \in \mathbb{N})\,\Big(|u_n(\omega)| = \mathcal{O}(n^{-b})\Big)\Big\}.$$

Elements of $\mathcal{E}_{\mathcal{L}}(\Omega)$ and $\mathcal{N}_{\mathcal{L}}(\Omega)$ are called moderate and negligible sequences of random variables, respectively. Elements of the quotient space

$$\mathcal{G}_{\mathcal{L}}(\Omega) = \mathcal{E}_{M,\mathcal{L}}(\Omega)/\mathcal{N}_{\mathcal{L}}(\Omega)$$

are called generalized random variables with values in $\mathcal{L}(\Omega)$.

4.1.4 Tempered Colombeau Stochastic Processes with Values in $L^2(\Omega)$

In Chapter 6 we will investigate the solutions for linear SPDEs with generalized constant coefficients in the framework of CSPs. Since we will use the Fourier transform in Chapter 6 we need to switch to tempered Colombeau generalized function. Therefore, we introduce the notion of tempered Colombeau stochastic processes over $\mathbb{R}^d$ with values in $L^2(\Omega)$.

Definition 4.1.7 *Set*

$$\begin{aligned}
\mathcal{E}_{\tau,L^2}(\Omega,\mathbb{R}^d) &= \Big\{(u_n)_n \in \mathcal{E}_{L^2}(\Omega,\mathbb{R}^d) \,:\, (\forall \alpha \in \mathbb{N}_0^d)(\exists N \in \mathbb{N}) \\
&\quad \Big(\sup_{x\in\mathbb{R}^d} \|\partial^\alpha u_n(\cdot,x)\|_{L^2}(1+|x|)^{-N} = \mathcal{O}(n^N)\Big)\Big\},\\
\mathcal{N}_{\tau,L^2}(\Omega,\mathbb{R}^d) &= \Big\{(u_n)_n \in \mathcal{E}_{L^2}(\Omega,\mathbb{R}^d) \,:\, (\forall \alpha \in \mathbb{N}_0^d,\ |\alpha|\le k)(\exists N \in \mathbb{N})(\forall b \in \mathbb{N}) \\
&\quad \Big(\sup_{x\in\mathbb{R}^d} \|\partial^\alpha u_n(\cdot,x)\|_{L^2}(1+|x|)^{-N} = \mathcal{O}(n^{-b})\Big)\Big\}.
\end{aligned}$$

Elements of the vector spaces $\mathcal{E}_{\tau,L^2}(\Omega,\mathbb{R}^d)$ and $\mathcal{N}_{\tau,L^2}(\Omega,\mathbb{R}^d)$ are called moderate and negligible sequences of functions with values in $L^2(\Omega)$, respectively. The elements of the quotient space

$$\mathcal{G}_{\tau,L^2}(\Omega,\mathbb{R}^d) = \mathcal{E}_{\tau,L^2}(\Omega,\mathbb{R}^d)/\mathcal{N}_{\tau,L^2}(\Omega,\mathbb{R}^d)$$

are called tempered Colombeau stochastic processes (tempered CSPs) over $\mathbb{R}^d$ with values in $L^2(\Omega)$.

Note that the mapping

$$\mathcal{G}_{\tau,L^2}(\Omega,\mathbb{R}^d) \to \mathcal{G}_{L^2}(\Omega,\mathbb{R}^d)$$

is a canonical mapping. Thus, every representative $(u_n)_n$ of $u \in \mathcal{G}_{\tau,L^2}(\Omega,\mathbb{R}^d)$ determines a unique element of $\mathcal{G}_{L^2}(\Omega,\mathbb{R}^d)$.

Another fact to note is that the whole theory of CSPs over $\mathbb{R}^d$ with values in $L^2(\Omega)$ can be adapted word by word, with the change of negligible sets, to tempered CSPs.

4.2 Gaussian Colombeau Stochastic Processes with Values in $L^2(\Omega)$

Gaussian Colombeau stochastic processes (GCSPs) were introduced and analyzed in [LCP94a; MPS09].

In this section we recall the basic assertions related to Gaussian Colombeau stochastic processes with values in $L^2(\Omega)$. This concept originates from the corresponding one in distribution theory; see [GV64].

Definition 4.2.1 *Let $u \in \mathcal{G}_{L^2}(\Omega, O)$. It is said that u is a Gaussian Colombeau stochastic process (GCSP), if there exists a representative $(u_n)_n$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$ and arbitrary $x_1, \ldots, x_r \in O \subset \mathbb{R}^d$, the probability that $X_n = (u_n(x_1,\omega), \ldots, u_n(x_r,\omega)) \in B$, where B is a Borel set in $\mathbb{R}^r$, is*

$$P(X_n \in B) = \left(\frac{\det A_n}{(2\pi)^d}\right)^{1/2} \int_B \exp\left(-\frac{1}{2}\, s^T A_n s\right) ds, \quad n > n_0,$$

where $A_n, n \in \mathbb{N}$, is a sequence of non-degenerate positive-definite matrices, and

$$s^T A_n s = \sum_{i=1}^{r}\sum_{j=1}^{r} a_{ijn} s_i s_j, \quad n > n_0.$$

We will call $(u_n)_n$ a Gaussian representative of u. Also, instead of $n > n_0$ we will write $n \in \mathbb{N}$.

Remark 4.2.1 *As before, in above definition we can consider derivatives up to order k instead of smooth functions, i.e. we can consider processes in $\mathcal{G}^k_{L^2}(\Omega, O)$.*

The following example shows that not all representatives of GCSP are Gaussian representative.

Example 4.2.1 *Let $(c_n)_n \in \mathcal{N}$ be negligible sequence and $s(\omega, x)$ any non-Gaussian stochastic process. Then $c_n s(\omega, x)$ is a non-Gaussian negligible sequence. If $(u_n)_n$ is a Gaussian representative of a GCSP, then $(u_n + c_n s)_n$ is a non-Gaussian representative of the same GCSP.*

In the next chapter we will see some basic examples of GCSPs with values in $L^2(\Omega)$.

The proof of the following theorem can be found in [MPS09] (see Theorem 4.2, p. 267).

Theorem 4.2.1 *Partial derivatives of a GCSP in $\mathcal{G}_{L^2}(\Omega, O)$ are again GCSPs.*

4.3 Embeddings of Distributional Stochastic Processes in $\mathcal{G}_{L^p}(\Omega, O)$

In this section, we analyze the embedding of the space of distributional stochastic processes into the space of Colombeau stochastic processes. The reader is referred to [MPS09] for more details.

Recall, $\xi : \mathcal{D}(O) \times L^p(\Omega) \to \mathbb{C}$ is a distributional stochastic process (or generalized functional stochastic process) if the mapping $\phi \mapsto \xi(\cdot, \phi)$ is a strongly continuous mapping of $\mathcal{D}(O)$ into $L^p(\Omega)$.

The authors in [MPS09] have shown that distributional stochastic processes can be embedded into Colombeau-type stochastic processes. Indeed, let ξ be a distributional stochastic process on O. Denote by $(\kappa_n)_n$ a sequence of smooth functions supported by

$$O_{-1/n} = \left\{ x \in O : d(x, \mathbb{R}^d \setminus O) > \frac{1}{n} \right\}, \quad n > n_0,$$

such that $\kappa_n \equiv 1$ on $O_{-2/n}$, $n > n_0$. Then the assignment

$$\xi \hookrightarrow Cd(\xi) = [(\xi_n)_n] = [((\kappa_n \xi)(\omega, \varphi_n(x - \cdot)))_n], \ \omega \in \Omega, \ \ x \in O, \tag{4.1}$$

$n \in \mathbb{N}$, defines an embedding of the space of distributional stochastic processes into the space $\mathcal{G}_{L^p}(\Omega, O)$ of Colombeau stochastic $L^p(\Omega)$-valued processes.

Let $\xi : \mathcal{E}(O) \times L^p(\Omega) \to \mathbb{C}$ be a compactly supported distributional stochastic processes. Then we may take

$$\xi_n(\omega, x) = \xi(\omega, \varphi_n(x - \cdot)), \quad \omega \in \Omega, \, x \in O, \, n \in \mathbb{N},$$

for the representative of $Cd(\xi)$, since

$$(\kappa_n \xi)(\omega, \phi) = \xi(\omega, \phi), \quad \omega \in \Omega, \, \phi \in \mathcal{D}(O),$$

for large n.

Note that every distributional stochastic process $\xi : \mathcal{D}(O) \times L^p(\Omega) \to \mathbb{C}$ can be written in the form

$$\xi = \sum_{i=1}^{\infty} \xi \chi_i,$$

where $(\chi_i)_{i \in \mathbb{N}}$ is a partition of unity for an open cover of O consisting of bounded sets so that $K_i = \operatorname{supp} \chi_i \Subset O$, $i \in \mathbb{N}$. Clearly, $\xi \chi_i$, $i \in \mathbb{N}$, are compactly supported distributional stochastic processes. Therefore, without loss of generality we will assume that ξ is a compactly supported distributional stochastic process.

4.4 Generalized Point Values of Colombeau Stochastic Processes

In Section 2.3 we recall the notion of generalized point values of Colombeau generalized functions. Now we introduce in a similar manner the notion of generalized point values of CSP with values in $\mathcal{L}(\Omega)$, in $L^p(\Omega)$ or in $\mathcal{M}^\infty(\Omega)$, respectively.

Proposition 4.4.1 *Suppose that* $u = [(u_n)_n]$ *belongs to* $\mathcal{G}_{\mathcal{L}}(\Omega, O)$ *or* $\mathcal{G}_{L^p}(\Omega, O)$ *or* $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$*. Then, for fixed* $\tilde{x} = [(x_n)_n] \in \widetilde{O}_c$*,*

$$u(\omega, \tilde{x}) = [(u_n(\omega, x_n))_n]$$

is a generalized random variable in $\mathcal{G}_{\mathcal{L}}(\Omega)$ *or* $\mathcal{G}_{L^p}(\Omega)$ *or* $\mathcal{G}_{\mathcal{M}^\infty}(\Omega)$*.*

It is called the point value of u *(in* $\mathcal{L}(\Omega)$ *or* $L^p(\Omega)$ *or* $\mathcal{M}^\infty(\Omega)$*) at the generalized point* $\tilde{x} \in \widetilde{O}_c$*.*

PROOF. Since $\tilde{x} = [(x_n)_n] \in \widetilde{O}_c$, there exists some $K \Subset O$ such that $x_n \in K$ for all $n \in \mathbb{N}$. If $u = [(u_n)_n]$ belongs to $\mathcal{G}_{\mathcal{L}}(\Omega, O)$, then by definition $u_n(\omega) = u_n(\omega, x_n)$, $\omega \in \Omega$, is a measurable function on Ω for any $n \in \mathbb{N}$. Also, for a.a. $\omega \in \Omega$, it holds that

$$|u_n(\omega)| = |u_n(\omega, x_n)| \leq \sup_{x \in K} |u_n(\omega, x)| \leq Cn^a,$$

for some $a \in \mathbb{N}$ and therefore $(u_n)_n \in \mathcal{E}_{M,\mathcal{L}}(\Omega)$. If $u = [(u_n)_n]$ belongs to $\mathcal{G}_{L^p}(\Omega, O)$ or $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$, then in a similar way it can be shown that $(u_n)_n$ belongs to $\mathcal{E}_{M,L^p}(\Omega)$ or $\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega)$.

Let $\tilde{y} = [(y_n)_n] \in \widetilde{O}_c$ such that $\tilde{x} \sim \tilde{y}$, i.e.

$$|x_n - y_n| = \mathcal{O}(n^{-b})$$

for any $b \in \mathbb{N}$. If $u \in \mathcal{G}_{\mathcal{L}}(\Omega, O)$, then for a.a. $\omega \in \Omega$ there exists $a \in \mathbb{N}$ such that

$$\sup_{x \in K} |\nabla u_n(\omega, x)| = \mathcal{O}(n^a).$$

Let us show that $u(\omega, \tilde{x}) - u(\omega, \tilde{y}) \in \mathcal{N}_{\mathcal{L}}(\Omega)$. For arbitrary $b \in \mathbb{N}$, we have

$$\begin{aligned} |u_n(\omega, x_n) - u_n(\omega, y_n)| &\leq |x_n - y_n| \int_0^1 |\nabla u_n(\omega, x_n + \sigma(y_n - x_n))| \, d\sigma \\ &\leq C_1 n^{-(b+a)} C_2 n^a \\ &= Cn^{-b}, \end{aligned}$$

for a.a. $\omega \in \Omega$, since the point $x_n + \sigma(y_n - x_n)$ remains within some compact subset of O.

If $u \in \mathcal{G}_{L^p}(\Omega, O)$ (the proof can be conducted in a similar way if $u \in \mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$), then there exists $a \in \mathbb{N}$ such that

$$\sup_{x \in K} \|\nabla u_n(\cdot, x)\|_{L^p} = \mathcal{O}(n^a).$$

Using Minkowski's inequality in integral form (see Theorem A.3.14), we obtain

$$\begin{aligned}
&\|u_n(\cdot, x_n) - u_n(\cdot, y_n)\|_{L^p} \\
\leq\ & |x_n - y_n| \left[\int_\Omega \left| \int_0^1 |\nabla u_n(\omega, x_n + \sigma(y_n - x_n)| \, d\sigma \right|^p \, dP(\omega) \right]^{\frac{1}{p}} \\
\leq\ & |x_n - y_n| \int_0^1 \left[\int_\Omega |\nabla u_n(\omega, x_n + \sigma(y_n - x_n))|^p \, dP(\omega) \right]^{\frac{1}{p}} \, d\sigma \\
=\ & |x_n - y_n| \int_0^1 \|\nabla u_n(\cdot, x_n + \sigma(y_n - x_n)\|_{L^p} \, d\sigma \\
\leq\ & C_1 n^{-(b+a)} C_2 n^a \int_0^1 \, d\sigma \\
=\ & C n^{-b},
\end{aligned}$$

for arbitrary $b \in \mathbb{N}$. Therefore, $u(\omega, \tilde{x}) - u(\omega, \tilde{y}) \in \mathcal{N}_{L^p}(\Omega)$. ■

4.5 Measurablity of Colombeau Stochastic Processes

In this section we will prove the main result, the measurability of CSPs, which will allow us to investigate probabilistic properties of CSPs.

In the definition of CSPs we required measurability of each representative $u_n(\cdot, x)$, i.e. $u_n(\omega, x)$ is an OSP for $n \in \mathbb{N}$. We also required pathwise smoothness or that a.e. path $x \mapsto u_n(\omega, x)$ is of class $\mathcal{C}^k$. Therefore, we have the joint measurability of the mappings $(\omega, x) \mapsto u_n(\omega, x)$ in $\Omega \times O$. Recall that we can assume that the representatives $u_n(\omega, x)$ are defined and of class $\mathcal{C}^k$ for all $\omega \in \Omega$. In this section we will prove that the processes obtained by considering generalized point values of CSPs are also measurable in the appropriate sense.

Let $u = [(u_n)_n]$ be CSPs in $\mathcal{G}_{\mathcal{L}}(\Omega, O)$ or $\mathcal{G}_{L^p}(\Omega, O)$ or $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$. Similar to Proposition 4.4.1 one has that $(u_n(\omega, x_n))_n$ belongs to $\mathbb{R}_M$.

Proposition 4.5.1 *Suppose that $u = [(u_n)_n]$ belongs to $\mathcal{G}_{\mathcal{L}}(\Omega, O)$ or $\mathcal{G}_{L^p}(\Omega, O)$ or $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$. For fixed $\tilde{x} \in \widetilde{O}_c$ the mapping*

$$(\Omega, \mathcal{F}) \ni \omega \mapsto u(\omega, \tilde{x}) \in (\mathcal{R}_c, \mathcal{B}(\mathcal{R}_c))$$

is measurable.

PROOF. Let

$$\mathfrak{O} = L((y_n)_n, k) = \{[(z_n)_n] \in \mathcal{R}_c : \limsup_{n\to\infty} |y_n - z_n|^{(\log n)^{-1}} < k\}$$

be an open ball in $\mathcal{R}_c$. Then,

$$\begin{aligned} u^{-1}(\cdot, \tilde{x})(\mathfrak{O}) &= \{\omega \in \Omega : u(\omega, \tilde{x}) \in \mathfrak{O}\} \\ &= \{\omega \in \Omega : \limsup_{n\to\infty} |u_n(\omega, x_n) - y_n|^{(\log n)^{-1}} < k\} \\ &= \{\omega \in \Omega : |u_n(\omega, x_n) - y_n| < n^{\log k} \text{ for all } n > n_0 \text{ for some } n_0 \in \mathbb{N}\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\omega \in \Omega : |u_m(\omega, x_m) - y_m| < m^{\log k}\}. \end{aligned}$$

This is a measurable set. ■

As we mentioned in Chapter 2, the σ-algebra generated by the sharp open balls is smaller than the σ-algebra generated by the sharp topology in $\mathcal{R}_c$. We show this in the next example.

Example 4.5.1 *Let $\Omega = \mathbb{R}$ be endowed with the $\sigma-$algebra $\mathcal{F}$ of Lebesgue measurable sets. Consider $u : \Omega \to \mathcal{R}_c$ represented by*

$$u_n(\omega) = \omega, \quad n \in \mathbb{N},$$

i.e. the standard embedding of $\mathbb{R}$ into $\mathcal{R}_c$. Then u is not measurable with respect to the $\sigma-$algebra generated by the sharp topology. Let us show this.

Take a set $E \subseteq \mathbb{R} = \Omega$ which is not Lebesgue measurable. Denote by $L(\tilde{x}, p)$ the sharp open ball with center $\tilde{x} = [(x_n)_n] \in \mathcal{R}_c$, i.e. the equivalence class of elements $\tilde{y} = [(y_n)_n] \in \mathcal{R}_c$ such that

$$|y_n - x_n| \leq n^{-p}, \quad n \to \infty,$$

for $p > 0$. Take $\tilde{x} = u(\omega_0)$ for some $\omega_0 \in \Omega$. Then

$$u^{-1}(L(\tilde{x}, p)) = \{\omega \in \Omega : |\omega - \omega_0| \leq n^{-p}, n \to \infty\} = \{\omega_0\}$$

is Lebesgue-measurable, which is in compliance with Proposition 4.5.1.

On the other hand, the set

$$V = \bigcup_{\omega \in E} L(u(\omega), p)$$

is open in the sharp topology, but

$$u^{-1}(V) = E$$

is not Lebesgue measurable. □

5 Probabilistic Properties of Colombeau Stochastic Processes

> *It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge.*
>
> — **Pierre Simon Laplace (1749-1827)**
> "Théorie Analytique des Probabilités,"1812.

Colombeau generalized processes (CSPs) were considered in [MPS09; LCP97; NR02a; Obe95; ORĆ05; OR98a; OR98b; OR01; Sel08], but none of these papers addressed the question of their probabilistic properties.

In the previous chapter we began to study the probabilistic properties of the CSPs. We established the notion of the point value of CSPs in compactly supported generalized points and proved measurability of the corresponding random variable with values in a Colombeau algebra of compactly supported real generalized numbers $\mathcal{R}_c$, endowed with the topology generated by sharp open balls.

In this chapter, we continue to study the probabilistic properties of CSPs. The first part of this chapter is devoted to generalized expectation and generalized correlation function of CSPs with values in $L^2(\Omega)$. A structural characterization of generalized correlation functions of CSPs with values in $L^2(\Omega)$ is given. Generalized characteristic functions are introduced for CSPs in $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ and $\mathcal{G}^k_{L^{kp}}(\Omega, O)$. We close this chapter by studying CSPs with independent values and stationary CSPs.

This chapter represents an original part of dissertation. The original results are published in [GOPS18b] (Section 5.2 and Section 5.3) and [GOPS18a] (Section 5.4 and Section 5.5).

5.1 Generalized Expectation and Generalized Correlation Function

In this section, we work with CSPs with values in $L^2(\Omega)$. Namely, we consider the CSPs in $\mathcal{G}^k_{L^2}(\Omega, O)$ and $\mathcal{G}_{\mathcal{T},L^2}(\Omega, O)$.

Following [MPS09], we introduce here the definitions of generalized expectation, generalized correlation functions and generalized covariance functions of the CSPs in $\mathcal{G}^k_{L^2}(\Omega, O)$ and $\mathcal{G}_{\tau,L^2}(\Omega, O)$.

Let $u = [(u_n)_n] \in \mathcal{G}^k_{L^2}(\Omega, O)$. By the mean value theorem we have that

$$u_n(\omega, x+h) - u_n(\omega, x) = u_n'(\omega, x+\theta h)h, \quad \theta \in (0,1)$$

and

$$|E(u_n(\cdot, x+h)) - E(u_n(\cdot, x))| \leq \int_\Omega |u_n'(\omega, x+\theta h)|h \, dP(\omega) \to 0,$$

as $h \to 0$. Thus, $E(u_n(\cdot, x)) \in \mathcal{C}(O)$. Similarly, it can be shown that $(E(u_n(\cdot, x)))_n$ belongs to $\mathcal{E}^{k-1}(O)$. Now, we prove moderateness of the sequence $(E(u_n(\cdot, x)))_n$. Let $K \Subset O$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k-1$. Using the Hölder's inequality we obtain

$$\sup_{x \in K} |\partial^\alpha E(u_n(\cdot, x))| \leq \sup_{x \in K} \|\partial^\alpha u_n(\cdot, \omega)\|_{L^2} \leq C n^a$$

for some $a \in \mathbb{N}$, since $(u_n)_n \in \mathcal{E}^k_{M,L^2}(\Omega, O)$. Let $(v_n)_n \in \mathcal{N}^k_{L^2}(\Omega, O)$. Using the Hölder's inequality, it can be shown that sequence $(E((u_n + v_n)(\cdot, x)) - E(u_n(\cdot, x)))_n$ belongs to $\mathcal{N}^{k-1}(O)$. Therefore, $[(E(u_n(\cdot, x)))_n]$ is a well-defined element of $\mathcal{G}^{k-1}(O)$. In a similar way we prove that $[(E(u_n(\cdot, x)u_n(\cdot, y)))_n]$ is a well-defined element of $\mathcal{G}^{k-1}(O \times O)$. This enables us to introduce the next definition.

Definition 5.1.1 *Let $u = [(u_n)_n] \in \mathcal{G}^k_{L^2}(\Omega, O)$ (resp. $u = [(u_n)_n] \in \mathcal{G}_{\tau,L^2}(\Omega, O)$).*

- *The generalized expectation of u is an element m of $\mathcal{G}^{k-1}(O)$ (resp. $\mathcal{G}_\tau(O)$) with representative*

$$m_{u_n}(x) = E(u_n(\cdot, x)) = \int_\Omega u_n(\omega, x) \, dP(\omega), \quad x \in O, \ \ n \in \mathbb{N}.$$

- *The generalized correlation function of u is an element B of $\mathcal{G}^{k-1}(O \times O)$ (resp. $\mathcal{G}_\tau(O \times O)$) with representative*

$$B_{u_n}(x, y) = E(u_n(\cdot, x)u_n(\cdot, y)), \quad x, y \in O, \ \ n \in \mathbb{N}.$$

In addition, the generalized variance of u is an element of $\mathcal{G}^{k-1}(O \times O)$ (resp. $\mathcal{G}_\tau(O \times O)$) with representative $B_{u_n}(x, x)$, $x \in O$, $n \in \mathbb{N}$.

Let ξ be a distributional stochastic process. Suppose that $Cd(\xi) = [(\xi_n)_n]$ is the corresponding element of $\mathcal{G}_{L^2}(\Omega, O)$. Then the representatives of the generalized expectation $m = [(m_{\xi_n})_n]$ and the generalized correlation function $B = [(B_{\xi_n})_n]$, as well as the process $Cd(\xi) = [(\xi_n)_n]$ itself, depend on the choice of the mollifier function. However, they define elements of the Colombeau algebra which are equal in the sense of distributions.

Proposition 5.1.1 *Let $u \in \mathcal{G}^{k+1}_{L^2}(\Omega, O)$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$. Then*

(a) $\partial^\alpha m_{u_n}(x) = m_{\partial^\alpha u_n}(x)$, $x \in O$, $n \in \mathbb{N}$, *and*

(b) $\partial_x^\alpha \partial_y^\alpha B_{u_n}(x,y) = B_{\partial^\alpha u_n}(x,y)$, $x, y \in O$, $n \in \mathbb{N}$.

PROOF. Follows directly from Definition 5.1.1. ■

Similarly, Proposition 5.1.1 is true for elements in $\mathcal{G}_{\tau,L^2}(\Omega, O)$.

Definition 5.1.2 *Let $u \in \mathcal{G}^{k}_{L^2}(\Omega, O)$ (resp. $\mathcal{G}_{\tau,L^2}(\Omega, O)$). The generalized covariance function of u is an element of $\mathcal{G}^{k-1}(O \times O)$ (resp. $\mathcal{G}_\tau(O \times O)$) represented by*

$$C_{u_n}(x,y) = B_{u_n}(x,y) - m_{u_n}(x) m_{u_n}(y), \quad x, y \in O, \ n \in \mathbb{N}.$$

Remark 5.1.1 *In the spirit of [GV64], we use the term* generalized correlation function *for the noncentered expression $B_{u_n}(x,y) = E(u_n(\cdot,x)u_n(\cdot,y))$ and* generalized covariance function *for its centered counterpart $C_{u_n}(x,y) = B_{u_n}(x,y) - m_{u_n}(x)m_{u_n}(y)$ throughout the dissertation.*

The generalized correlation function $B = [(B_{u_n}(x,y))_n]$ is positive-definite, i.e. it has a representative with bilinear positive-definite functionals. Furthermore, its generalized covariance function $C = [(C_{u_n}(x,y))_n]$ is positive-definite.

Next, we recall some results on GCSPs from [MPS09].

Note that with A_n, $n \in \mathbb{N}$, we denote a sequence of non-degenerate positive-definite matrices from Definition 4.2.1.

Theorem 5.1.1 *Let $u \in \mathcal{G}^{k}_{L^2}(\Omega, O)$ be a GCSP with Gaussian representative $(u_n)_n$ and let $(B_{u_n})_n$ be a representative of its generalized correlation function. Then*

$$A_n = (B_{u_n}(x_i, x_j))^{-1}, \quad n \in \mathbb{N},$$

for all $x_1, \dots, x_d \in \mathbb{R}$.

PROOF. See [MPS09] (Theorem 4.1, p. 266). ■

The following theorem gives the complete characterization of GCSPs. The proof is similar to the proof of Theorem 4.3 in [MPS09].

Theorem 5.1.2 *Let $m = [(m_n(x))_n] \in \mathcal{G}^k(O)$ and $B = [(B_n(x,y))_n] \in \mathcal{G}^k(O \times O)$ be such that the generalized covariance function $C = [(C_n(x,y)))_n] \in \mathcal{G}^k(O \times O)$ is positive-definite (C_n are positive-definite). There exists a GCSPs $u \in \mathcal{G}^{k+1}_{L^p}(\Omega, O)$ with a Gaussian representative $(u_n)_n$, whose generalized expectation and generalized covariance function are m and C.*

The previous theorem implies the following result (see [MPS09] Corollary 4.1, p. 268).

Corollary 5.1.1 *Let $u = [(u_n)_n] \in \mathcal{G}^{k+1}_{L^2}(\Omega, O)$ be a CSP with generalized expectation $m = [(m_{u_n}(x))_n] \in \mathcal{G}^k(O)$ and generalized correlation function $B = [(B_{u_n}(x, y))_n] \in \mathcal{G}^k(O \times O)$. There exists a GCSP with the given generalized expectation and generalized correlation function.*

We shall now give some basic examples of GCSPs with values in $L^2(\Omega)$. First, we consider Brownian motion in the Colombeau sense, its generalized expectations and generalized correlation functions.

Example 5.1.1 *Let b be a Brownian motion*[1] *(see Appendix A.3). It is an OSP and the corresponding element of $\mathcal{G}_{L^2}(\Omega, \mathbb{R})$ is given by*

$$Cd(b) = [(b_n)_n] = [(b * \varphi_n)_n] = \left[\left(\int_{\mathbb{R}} b(s)\varphi_n(x-s)\,ds\right)_n\right].$$

The generalized expectation of $Cd(b)$ is

$$m_{Cd(b)} = \left[\left(\int_{\mathbb{R}} E(b(s))\varphi_n(x-s)\,ds\right)_n\right] = [(0)_n] = 0,$$

and the generalized correlation function is

$$\begin{aligned} B_{Cd(b)} &= \left[\left(\iint_{\mathbb{R}} E(b(s)b(t))\varphi_n(x-s)\varphi_n(y-t)\,dsdt\right)_n\right] \\ &= \left[\left(\int_0^\infty\int_0^\infty \min\{s,t\}\varphi_n(x-s)\varphi_n(y-t)\,dsdt\right)_n\right], \end{aligned}$$

where we use $E(b(s)) = 0$ and

$$E(b(s)b(t)) = \begin{cases} \min\{s,t\}, & s,t \geq 0, \\ 0, & t < 0 \text{ or } s > 0. \end{cases}$$

In the sequel, we write $b = [(b_n)_n]$ instead of $Cd(b)$, $m_b = [(m_{b_n})_n]$ instead of $m_{Cd(b)}$, and $B_b = [(B_{b_n})_n]$ instead of $B_{Cd(b)}$.

Note that after integration by parts, we obtain

$$\begin{aligned} B_{b_n}(x,y) &= \int_0^\infty\int_0^\infty \min\{s,t\}\varphi_n(x-s)\varphi_n(y-t)\,dsdt \\ &= \int_0^\infty \varphi_n(x-s)\int_0^s t\varphi_n(y-t)\,dtds + \int_0^\infty \varphi_n(y-t)\int_0^t s\varphi_n(x-s)\,dsdt \\ &= \int_0^\infty \left(\int_t^\infty \varphi_n(x-\sigma)\,d\sigma \int_t^\infty \varphi_n(y-\tau)\,d\tau\right) dt. \end{aligned}$$

[1] Brownian motion is often called the Wiener process. It is named in the honor of Norbert Wiener.

Therefore, (generalized) Brownian motion $b = [(b_n)_n] \in \mathcal{G}_{L^2}(\Omega, \mathbb{R})$ as a GCSP is defined by a zero generalized expectation and a generalized correlation function $B_b(x,y) = [(B_{b_n}(x,y))_n]$, where

$$\begin{aligned} B_{b_n}(x,y) &= \min\{s,t\} * \varphi_n(x)\varphi_n(y) \\ &= \int_0^\infty \left(\int_t^\infty \varphi_n(x-\sigma)\, d\sigma \int_t^\infty \varphi_n(y-\tau)\, d\tau \right) dt, \quad n \in \mathbb{N}. \qquad \square \end{aligned}$$

Next, we consider white noise in the Colombeau sense, its generalized expectation and its generalized correlation function.

Example 5.1.2 *White noise $w = [(w_n)_n] \in \mathcal{G}_{L^2}(\Omega, O)$ as a GCSP is defined by a zero generalized expectation and a generalized correlation function that is associated to the Dirac delta $\delta(x-y)$ supported on the diagonal. There are several ways to achieve this: one possibility is to define*

$$B_w^1(x,y) = [(\varphi_n(x-y))_n],$$

another to let

$$B_w^2(x,y) = \left[\left(\int \varphi_n(x-s)\varphi_n(y-s) ds \right)_n \right].$$

Let us show that the two processes are associated in the Colombeau sense, but not equal. First we show that $B_w^1(x,y) \approx \delta(x-y)$, i.e.

$$\lim_{n\to\infty} \iint \varphi_n(x-y)\psi(x,y)\, dxdy = \langle \delta(x,y), \psi(x,y) \rangle = \int \psi(s,s)\, ds$$

holds for every $\psi \in \mathcal{D}(\mathbb{R}^2)$. We have

$$\begin{aligned} \iint \varphi_n(x-y)\psi(x,y)\, dxdy &= \iint n\varphi(n(x-y))\psi(x,y)\, dxdy \\ &= \iint \varphi(\tau)\psi\left(\sigma + \frac{\tau}{n}, \sigma\right) d\tau d\sigma, \end{aligned}$$

where we use the change of varables $n(x-y) = \tau$, $y = \sigma$. Now, letting $n \to \infty$ we obtein by the Lebesgue dominated convergence, Fubini's theorem and properties of the mollifier function, that the latter expression converges to

$$\int \varphi(\tau)\, d\tau \int \psi(\sigma,\sigma)\, d\sigma = \int \psi(\sigma,\sigma)\, d\sigma.$$

A similar computation can be carried out to show

$$\lim_{n\to\infty} \iint \varphi_n(x-s)\varphi_n(y-s)\psi(x,y)\, dxdy = \langle \delta(x-y), \psi(x,y) = \int \psi(s,s)\, ds,$$

for every $\psi \in \mathcal{D}(\mathbb{R}^d)$, i.e. $B_w^2(x,y) \approx \delta(x-y)$. Therefore, the elements $B_w^1(x,y)$ and $B_w^2(x,y)$ are associated in $\mathcal{G}(\mathbb{R}^2)$ and they determine GCSPs which are associated as *elements of* $\mathcal{G}_{L^2}(\Omega, \mathbb{R})$*. Note that these two GCSPs are not equal. The generalized variance of white noise is given by*

$$B_w(x, x) = [(\varphi_n(0))_n]$$

in the first case and

$$B_w(x, x) = [(\|\varphi_n(\cdot)\|^2_{L^2})_n]$$

in the second case. These two processes are equal provided that the mollifier satisfies

$$\|\varphi(\cdot)\|^2_{L^2} = \varphi(0). \qquad \square$$

5.2 Characterizations of the Generalized Correlation Function

In this section, without restriction of generality, we will assume that the generalized expectation of a given process equals zero.

5.2.1 Structural Theorems

Denote the complement of the diagonal by

$$Q_O = \{(x, y) \in O \times O : \ x \neq y\}$$

and the diagonal by

$$D_O = O \times O \setminus Q_O = \{(x, y) \in O \times O : \ x = y\}.$$

In the case $O = \mathbb{R}^d$, we use notation $Q = \{(x, y) \in \mathbb{R}^{2d} : x \neq y\}$ and $D = \mathbb{R}^{2d} \setminus Q$.

First, we give a structural characterization of generalized correlation functions which correspond to embedded distributional stochastic processes.

Theorem 5.2.1 *Let* $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ *be a generalized correlation function which corresponds to a CSP* $u = [(u_n)_n]$ *over* O *with values in* $L^2(\Omega)$ *such that* $u = Cd(\xi)$*, where* ξ *is a distributional stochastic process on* O*.*

a) Let $F \in \mathcal{D}'(O \times O)$ *be the correlation functional of* ξ*. Then* B *is given by* $B = Cd(F)$*.*

b) $B(\tilde{x}, \tilde{y}) = 0$ *for all* $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$*, if and only if* F *is concentrated on the diagonal* $x = y$*, i.e. supp* $F \subseteq D_O$*.*

c) If $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_O)_c$, then B is associated to a generalized function which has a representative of the form

$$B_n^*(x,y) = \int_O \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\varphi_n^{(j)}(x-s)\varphi_n^{(k)}(y-s)ds, \quad x,y \in O, \tag{5.1}$$

where for every $n \in \mathbb{N}$ only a finite number of continuous functions $R_{j,k}$ are different from zero on any compact subset of O.

PROOF. *a)* We have

$$\begin{aligned}
&B_n(x,y) = \\
&= \int_\Omega \left(\int_O \kappa_n(s)\xi(\omega,s)\varphi_n(x-s)\,ds\right)\left(\int_O \kappa_n(s)\xi(\omega,t)\varphi_n(y-t)\,dt\right)\,dP \\
&= \iint_{O\times O} \kappa_n(s)\kappa_n(t)\varphi_n(x-s)\varphi_n(y-t)\left(\int_\Omega \xi(\omega,s)\xi(\omega,t)\,dP(\omega)\right)\,dsdt \\
&= \iint_{O\times O} \kappa_n(s)\kappa_n(t)F(s,t)\varphi_n(x-s)\varphi_n(y-t)\,dsdt \\
&= (\kappa_n(x)\kappa_n(y)F(x,y)) * \varphi_n(x)\varphi_n(y).
\end{aligned}$$

Therefore, $B = Cd(F)$.

b) We know that $\operatorname{supp} B = \operatorname{supp} F$ for embedded distributions. Now, $F \equiv 0$ in Q_O is equivalent to $B \equiv 0$ in Q_O and Theorem 2.3.2 implies the statement.

c) From *a)* and *b)* we have $B = Cd(F)$ and F is concentrated on D_O. The generalized function F has the form

$$\langle F,\theta\rangle = \int_{O\times O} \sum_{j,k\in\mathbb{N}_0} Q_{j,k}(x,y)\frac{\partial^{j+k}}{\partial x^j\partial y^k}\theta(x,y)dxdy, \quad \theta \in \mathcal{D}(O\times O),$$

where the $Q_{j,k}(x,y)$ are continuous functions, only a finite number of which are different from zero on any compact set of O. Since F is concentrated on the diagonal D_O, we obtain that

$$\langle F,\theta\rangle = \int_O \sum_{j,k\in\mathbb{N}_0} R_{j,k}(x)\left(\frac{\partial^{j+k}}{\partial x^j\partial y^k}\theta(x,y)\right)\Big|_{x=y} dx,$$

where we have put $R_{j,k}(x) = Q_{j,k}(x,x)$. The form of F over $\mathbb{R}^d$ was mentioned in [GV64], page 287. A version of this theorem is also given in Theorem 2.3.5 in [Hör03] for compactly supported distributions (see Appendix A.2). We apply the quoted result of [Hör03] but rewritten in the form of [GV64].

Put

$$B_n^*(x,y) = \int_O \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\varphi_n^{(j)}(x-s)\varphi_n^{(k)}(y-s)ds, \quad x,y \in O, \ n \in \mathbb{N}.$$

We will show that $[(B_n^*)_n] \approx F$, from which it will follow that $[(B_n^*)_n] \approx B$. For any $\phi \in \mathcal{D}(O \times O)$ we have

$$
\begin{aligned}
&\iint_{O\times O} B_n^*(x,y)\phi(x,y)dxdy \\
&\quad = \sum_{j,k\in\mathbb{N}_0} \iiint_{O\times O\times O} R_{j,k}(s)\frac{\partial^j}{\partial x^j}\varphi_n(x-s)\frac{\partial^k}{\partial y^k}\varphi_n(y-s)\phi(x,y)\,dxdyds \\
&\quad = \sum_{j,k\in\mathbb{N}_0} n^2 \iiint_{O\times O\times O} R_{j,k}(s)\frac{\partial^j}{\partial x^j}\varphi\left(n(x-s)\right)\frac{\partial^k}{\partial y^k}\varphi\left(n(y-s)\right)\phi(x,y)\,dxdyds \\
&\quad = \sum_{j,k\in\mathbb{N}_0} n^2 \iiint_{O\times O\times O} R_{j,k}(s)\varphi\left(n(x-s)\right)\varphi\left(n(y-s)\right)\frac{\partial^{j+k}}{\partial x^j\partial y^k}\phi(x,y)\,dxdyds
\end{aligned}
$$

where we applied partial integration in the last step. Now, with $t = n(x-s)$, $z = n(y-s)$, we obtain

$$
\begin{aligned}
&\iint_{O\times O} B_n^*(x,y)\phi(x,y)dxdy \\
&\quad = \sum_{j,k\in\mathbb{N}_0} \iiint_{O\times O\times O} R_{j,k}(s)\varphi(t)\varphi(z)A\left(s+\frac{t}{n}, s+\frac{z}{n}\right)\,dtdzds,
\end{aligned}
$$

where

$$
A\left(s+\frac{t}{n}, s+\frac{z}{n}\right) = \frac{\partial^{j+k}}{\partial x^j\partial y^k}\phi(x,y)\bigg|_{x=s+\frac{t}{n},y=s+\frac{z}{n}}.
$$

Letting $n \to \infty$ we obtain by the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
&\iint_{O\times O} B_n^*(x,y)\phi(x,y)dxdy \\
&\quad \to \int_O \varphi(t)dt\int_O \varphi(z)dz\int_O \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)A(s,s)ds \\
&\quad = \int_O \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\frac{\partial^{j+k}}{\partial x^j\partial y^k}\phi(x,y)\bigg|_{x=y=s} ds \\
&\quad = \langle F, \phi\rangle.
\end{aligned}
$$

Thus, $[(B_n^*)_n] \approx F$. Since $B = Cd(F) \approx F$, it follows $[(B_n^*)_n] \approx B$. ∎

Next, we give a structural characterization of a generalized correlation function which is associated to Schwartz distribution.

Proposition 5.2.1 *Let $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ be a generalized correlation function of a CSP $u = [(u_n)_n]$ over O with values in $L^2(\Omega)$. Suppose that B is associated to $F \in \mathcal{D}'(O \times O)$. If $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_O)_c$, then*

a) F is concentrated on D_O,

b) B is associated to a generalized function which has a representative of the form (5.1).

PROOF. Let $F \in \mathcal{D}'(O \times O)$ and suppose that $B \approx F$. Let $B(\tilde{x}, \tilde{y}) = 0$ for all $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$.

a) By Theorem 2.3.2, $B|_{Q_O} = 0$ in $\mathcal{G}(Q_O)$. In particular,

$$\sup_{(x,y)\in K} |B_n(x,y)| \to 0$$

as $n \to \infty$, for every compact subset K of Q_O. We prove that F is concentrated on the diagonal, i.e. $\langle F, \theta \rangle = 0$ for all $\theta \in \mathcal{D}(O \times O)$ with $\operatorname{supp} \theta \subset Q_O$. Indeed, for such θ,

$$\langle F, \theta \rangle = \lim_{n\to\infty} \iint_{O\times O} B_n(x,y)\theta(x,y)dxdy = 0.$$

b) From *a)* it follows that F is concentrated on D_O. The rest of proof is analogous to the proof of Theorem 5.2.1 *c)*. ■

5.2.2 Examples of the Generalized Correlation Function of Gaussian Colombeau Stochastic Processes.

In this section, we give two examples of the generalized correlation function of GCSPs.

The first example shows that if $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ is a generalized correlation function of some GCSP and B is associated to $F \in \mathcal{D}'(O \times O)$ which is concentrated on the diagonal D_O, then it does not necessarily follow that $\operatorname{supp} B \subsetneq (\tilde{D}_O)_c$.

Example 5.2.1 *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a positive-definite function with $\int_{\mathbb{R}} \varphi(x)\, dx = 1$. Define*

$$B_n(x,y) = n\varphi(n(x-y)) + \frac{1}{n}, \quad x, y \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Since the sum of two positive-definite functions is positive-definite, it follows that $B = [(B_n(x,y))_n]$ is positive-definite. From Theorem 5.1.2 (see also [MPS09], Theorem 4.3, p. 267) it follows that B is a generalized correlation function of some GCSP with zero generalized expectation.

Let us show that B is associated in $\mathcal{G}(\mathbb{R}^2)$ to the Dirac delta distribution $\delta(x-y)$. Using the change of variables, $t = n(x-y)$, $s = y$, we obtain

$$\begin{aligned}\iint_{\mathbb{R}^2} B_n(x,y)\phi(x,y)\, dxdy &= \iint_{\mathbb{R}^2} \left(n\varphi(n(x-y)) + \frac{1}{n}\right)\phi(x,y)\, dxdy \\ &= \iint_{\mathbb{R}^2} \left(\varphi(t) + \frac{1}{n}\right)\phi\left(\frac{t}{n} + s, s\right)\, dtds,\end{aligned}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^2)$. Now, letting $n \to \infty$ we obtain by the Lebesgue dominated convergence theorem, Fubini's theorem and properties of the mollifier function that the latter expression converges to $\int_{\mathbb{R}} \phi(s,s)\,ds$. Hence, $B \approx \delta(x-y)$.

It is known that supp $\delta(x-y) = D$. We have supp $B = \mathbb{R}^2$, so B is not concentrated on $\tilde{D}_c$. □

The second example shows that there exists a GCSP which does not have a distributional shadow and which has a generalized correlation function that does not have the form (5.1).

Example 5.2.2 *Let $B = \delta^2(x-y) \in \mathcal{G}(\mathbb{R}^2)$ be the Colombeau generalized function with the representative*

$$B_n(x,y) = \varphi_n^2(x-y) = n^2\varphi^2(n(x-y)), \;\; x,y \in \mathbb{R} \;\; n \in \mathbb{N},$$

where $\varphi \in \mathcal{S}(\mathbb{R})$ is a positive-definite function such that $\int_{\mathbb{R}} \varphi(x)\,dx = 1$. Since $\delta^2(x-y)$ is positive-definite, from Theorem 5.1.2 (see also [MPS09], Theorem 4.3, p. 267) it follows that there exists a GCSP $u = [(u_n)_n]$ whose generalized expectation is zero and generalized correlation function is B. This is an example of a GCSP which is not associated with any element of $L(\mathcal{D}(\mathbb{R}), L^2(\Omega))$, i.e. it does not have a distributional shadow. Here $L(\mathcal{D}(\mathbb{R}), L^2(\Omega))$ denotes the space of linear continuous mappings of a test space $\mathcal{D}(\mathbb{R})$ into the space $L^2(\Omega)$. Clearly, B_n is supported by the diagonal D, thus $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in \tilde{Q}_c$. We will show that B does not have the form (5.1), i.e.

$$\begin{aligned} &\langle B_n(x,y), \phi(x)\psi(y)\rangle \\ &\neq \left\langle \int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\varphi_n^{(j)}(x-s)\varphi_n^{(k)}(y-s)\,ds, \phi(x)\psi(y)\right\rangle, \end{aligned} \tag{5.2}$$

for $\phi, \psi \in \mathcal{D}(\mathbb{R})$. We have

$$\begin{aligned} \langle B_n(x,y), \phi(x)\psi(y)\rangle &= n^2 \iint_{\mathbb{R}^2} \varphi^2(n(x-y))\phi(x)\psi(y)\,dxdy \\ &= n \iint_{\mathbb{R}^2} \varphi^2(t)\phi\left(\frac{t}{n}+y\right)\psi(y)\,dtdy, \end{aligned}$$

where we used the change of variables $t = n(x-y)$, $y = y$. Now, by letting $n \to \infty$ we obtain that the latter expression converges to infinity. On the other hand, only a finite number of the functions $R_{j,k}$ are different from zero, so the sum on the right hand side of (5.2) is finite, and it converges to the finite value

$$\int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\frac{\partial^{j+k}}{\partial x^j \partial y^k}\phi(x,y)\mid_{x=y=s}\, ds$$

as $n \to \infty$ (see the proof of Theorem 5.2.1). □

5.3 Generalized Characteristic Functions of Colombeau Stochastic Processes

It is known that every classical stochastic process u is uniquely defined via its finite-dimensional distributions, while those are uniquely defined via their characteristic functions. Thus, the information about $E(e^{it(u(\cdot,x_1),u(\cdot,x_2),\ldots,u(\cdot,x_m))})$, $m \in \mathbb{N}$, provides enough to determine the process itself.

In this section, generalized characteristic functions are introduced for CSPs in $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ and $\mathcal{G}^k_{L^{kp}}(\Omega, O)$. For technical simplicity we will consider only the one-dimensional distributions $E(e^{itu(\cdot,x)})$, but one can easily carry out the proofs also for all finite-dimensional distributions.

5.3.1 Generalized Characteristic Functions of Colombeau Stochastic Processes in $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$

Let $u = [(u_n)_n]$ be a CSP with values in $\mathcal{M}^\infty(\Omega)$. Let us show that sequence $(E(e^{itu_n(\cdot,x)}))_n$ belongs to $\mathcal{E}_M(\mathbb{R} \times O)$. Using that the paths are smooth, it can be shown that sequence $(E(e^{itu_n(\cdot,x)}))_n$ belongs to $\mathcal{E}(\mathbb{R} \times O)$. We prove moderateness. Let $K_1 = [-t_0, t_0] \times K$, where $K \Subset O$ and $t_0 \in \mathbb{R}$. The kth order derivative with respect to the variable t of $E(e^{itu_n(\cdot,x)})$ is $\int_\Omega i^k u_n^k(\omega, x) e^{itu_n(\omega,x)}\, dP(\omega)$. Thus,

$$\sup_{(t,x)\in K_1} |\partial_t^k E(e^{itu_n(\cdot,x)})| \leq \sup_{x\in K} \|u_n(\cdot,x)\|^k_{L^k} \leq \sup_{x\in K} |||u_n(\cdot,x)|||^k_k \leq Cn^{ak},$$

for some $a \in \mathbb{N}$, since $(u_n)_n \in \mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)$. The mth order derivative with respect to the variable x of $E(e^{itu_n(\cdot,x)})$ is a linear combination of members of the form $E((u_n^{(i_1)}(\cdot,x))^{k_1} \cdot \ldots \cdot (u_n^{(i_s)}(\cdot,x))^{k_s} e^{itu_n(\cdot,x)})$, where $i_1k_1 + \ldots + i_sk_s = m$. So the proof of moderateness is the same as for $E((u_n'(\cdot,x))^m e^{itu_n(\cdot,x)})$:

$$\sup_{(t,x)\in K_1} \left|\int_\Omega (u_n'(\omega,x))^m e^{itu_n(\omega,x)} dP(\omega)\right| \leq \sup_{x\in K} \|u_n'\|^m_{L^m} \leq \sup_{x\in K} |||u_n'|||^m_m \leq Cn^{am},$$

for some $a \in \mathbb{N}$, since $(u_n)_n \in \mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)$. Note that here and henceforth u_n' denotes some first order derivative of u_n. In a similar way we estimate the mixed derivatives.

Let $u = [(u_n)_n]$ be a CSP with values in $\mathcal{M}^\infty(\Omega)$ and let $(v_n)_n$ be a negligible sequence of functions with values in $\mathcal{M}^\infty(\Omega)$. Let us prove negligibility of the sequence of functions $(E(e^{it(u_n+v_n)}) - E(e^{itu_n}))_n$. Let $b \in \mathbb{N}$ be arbitrary. Using the mean value theorem we obtain

$$\sup_{(t,x)\in K_1} \left|E\left(e^{it(u_n+v_n)(\cdot,x)}\right) - E\left(e^{itu_n(\cdot,x)}\right)\right| \leq$$

$$
\begin{aligned}
&\leq \sup_{(t,x)\in K_1} \left| \int_\Omega \left(e^{it(u_n+v_n)(\omega,x)} - e^{itu_n(\omega,x)} \right) dP(\omega) \right| \\
&\leq \sup_{(t,x)\in K_1} \int_\Omega |e^{itv_n(\omega,x)} - 1|\, dP(\omega) \\
&\leq \sup_{(t,x)\in K_1} |t| \int_\Omega |v_n(\omega,x)||e^{i\theta_t v_n(\omega,x)}|\, dP(\omega) \\
&\leq C \sup_{x\in K} \int_\Omega |v_n(\omega,x)|\, dP(\omega) \\
&= \mathcal{O}(n^{-b}),
\end{aligned}
$$

where θ_t lies on the segment $(0,t)$ (or $(t,0)$). We proved the negligibility of the zeroth derivative and, by Theorem 2.1.1, this is sufficient. Therefore, $[(E(e^{itu_n(\cdot,x)}))_n]$ does not depend on the representative and it is a well-defined element of $\mathcal{G}(\mathbb{R}\times O)$. This enables us to introduce the next definition.

Definition 5.3.1 *Let $u = [(u_n)_n]$ be a CSP in $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$. Then*

$$
L_u(t,x) = [(L_{u_n}(t,x))_n] = [(E(e^{itu_n(\cdot,x)}))_n] \in \mathcal{G}(\mathbb{R}\times O), \;\; t\in\mathbb{R},\, x\in O,
$$

is called the generalized characteristic function of u.

The generalized characteristic function $L_u(t,x)$ of a CSP $u \in \mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ is positive-definite in t for every $x \in O$. The proof is the same as the well known one for the classical characteristic function.

Note that if $u = [(u_n)_n]$ is a CSP with values in $\mathcal{M}^\infty(\Omega)$, then

$$
L_u(\tilde{t},\tilde{x}) = E(e^{i\tilde{t}u(\cdot,\tilde{x})}),
$$

for every $\tilde{x} \in \tilde{O}_c$ and $\tilde{t} \in \mathcal{R}_c$. Hence, we have

$$
L(\tilde{0},\tilde{x}) = \tilde{1},
$$

(where $\tilde{0} = (0,0,0,\ldots,0,\ldots)$, $\tilde{1} = (1,1,1,\ldots,1,\ldots)$) for every $\tilde{x} = [(x_n)_n] \in \tilde{O}_c$, since $L_{u_n}(0,x_n) = 1$ for every $n \in \mathbb{N}$.

5.3.2 Embedding Results

Let $u(\omega,x)$, $\omega\in\Omega$, $x \in O$, be an OSP such that $u(\omega,\cdot) \in L^1_{loc}(O)$ for a.a. $\omega \in \Omega$. Moreover, assume that $u(\cdot,x) \in \mathcal{M}^\infty(\Omega)$ for every $x \in O$. The embedding of u into the Colombeau algebra $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ is given by $u \mapsto [(u_n)_n]$, where

$$
u_n(\omega,x) = (u\kappa_n * \varphi_n)(\omega,x) = (u*\varphi_n)(\omega,x),
$$

for sufficiently large n; see Section 4.3. One can prove that the sequence of characteristic functions $(L_{u_n}(t,x))_n = (E(e^{itu_n(\cdot,x)}))_n$ is in $\mathcal{E}_M(\mathbb{R} \times O)$. Note that it would not hold without the assumption $u(\cdot,x) \in \mathcal{M}^\infty(\Omega)$.

Remark 5.3.1 *Let $u(\omega,\cdot) \in L^1_{loc}(O)$ for a.a. $\omega \in \Omega$ and assume that $u(\cdot,x)$ is a measurable function for all $x \in O$ and belongs to $\mathcal{M}^\infty(\Omega)$. Then*

$$\mathcal{C}_0^\infty(O) \ni \phi \mapsto \int u(\cdot,x)\phi(x)\,dx \in L^p(\Omega)$$

is a strongly continuous mapping from $\mathcal{C}_0^\infty(O)$ into $L^p(\Omega)$, $p \geq 1$.

It is known that we can embed an element from $\mathcal{C}^\infty_{\mathcal{M}^\infty}(\Omega, O)$ into $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ by convolution with a mollifier function or as a constant sequence. Is equality valid for the generalized characteristic functions of the corresponding elements in $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$? The next proposition gives the answer to this question.

Proposition 5.3.1 *Let $\phi \in \mathcal{C}^\infty_{\mathcal{M}^\infty}(\Omega, O)$ and assume that*

$$\sup_{x\in K} \|\phi^{(\alpha)}(\cdot,x)\|_{L^p} < \infty$$

for every $\alpha \in \mathbb{N}_0$ and every $K \Subset O$. Let

$$\phi_n(\omega,x) = (\phi(\omega,\cdot) * \varphi_n(\cdot))(x), \quad x \in O, \ \ \omega \in \Omega,$$

for sufficiently large n. Then

$$(\phi_n(\omega,x))_n - (\phi(\omega,x))_n \in \mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$$

and

$$(L_{\phi_n}(t,x))_n - (L_\phi(t,x))_n \in \mathcal{N}(\mathbb{R} \times O),$$

where $(\phi)_n$ is a constant sequence.

PROOF. One can prove easily that $(\phi_n(\omega,x))_n - (\phi(\omega,x))_n \in \mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)$. Hence, for the proof of negligibility, by Theorem 2.1.1, it is enough to prove the negligibility of the zeroth order derivative. For simplicity of exposition, we work out the case $d = 1$ only, i.e. $O \subseteq \mathbb{R}$. First,

$$\begin{aligned} &|||\phi_n(\omega,x) - \phi(\omega,x)|||_s \\ &= \sup_{1\leq p\leq s} \left(\int_\Omega |(\phi(\omega,\cdot) * \varphi_n(\cdot))(x) - \phi(\omega,x)|^p\,dP(\omega)\right)^{1/p} \\ &= \sup_{1\leq p\leq s} \left(\int_\Omega \left|\int_O \phi(\omega,x-t)\varphi(nt)\,dt - \phi(\omega,x)\right|^p\,dP(\omega)\right)^{1/p} \end{aligned}$$

$$= \sup_{1\le p\le s}\left(\int_\Omega\left|\int_O\left(\phi\left(\omega, x-\frac{z}{n}\right)-\phi(\omega,x)\right)\varphi(z)\,dz\right|^p dP(\omega)\right)^{1/p}$$

By Taylor's formula and the fact that $\int z^k\varphi(z)\,dz = 0$ for all $k\in\mathbb{N}$, the inner integral equals

$$\int_O\left(\sum_{k=1}^{b-1}\frac{\phi^{(k)}(\omega,x)}{k!}\left(-\frac{z}{n}\right)^k+\frac{\phi^{(b)}\left(\omega,x-\theta_n\frac{z}{n}\right)}{b!}\left(-\frac{z}{n}\right)^b\right)\varphi(z)\,dz$$
$$= \int_O\frac{\phi^{(b)}\left(\omega,x-\theta_n\frac{z}{n}\right)}{b!}\left(-\frac{z}{n}\right)^b\varphi(z)\,dz$$

where $0<\theta_n<1$. By Minkowski's inequality in integral form (see Appendix A.3, Theorem A.3.14),

$$\begin{aligned}&|||\phi_n(\omega,x)-\phi(\omega,x)|||_s\\ &\le \sup_{1\le p\le s}\frac{1}{b!n^b}\left(\int_\Omega\left|\int_O\phi^{(b)}\left(\omega,x-\theta_n\frac{z}{n}\right)(-z)^b\varphi(z)\,dz\right|^p dP(\omega)\right)^{1/p}\\ &\le \sup_{1\le p\le s}\frac{1}{b!n^b}\int_O\left(\int_\Omega\left|\phi^{(b)}\left(\omega,x-\theta_n\frac{z}{n}\right)\right|^p dP(\omega)\right)^{1/p}|z|^b|\varphi(z)|\,dz\,.\end{aligned}$$

The points $x-\theta_n\frac{z}{n}$, $n\in\mathbb{N}$, remain within some compact set. Since by assumption $\phi^{(b)}$ is uniformly bounded on compact sets with respect to the L^p-norm, it follows that the above expression is uniformly bounded by Cn^{-b} on every compact set for every $s\in\mathbb{N}$ and for every $b\in\mathbb{N}$ (C depends on b and s).

The fact that $(L_{\phi_n}(t,x))_n-(L_\phi(t,x))_n\in\mathcal{N}(\mathbb{R}\times O)$ is proven along the same lines. ■

In the sequel, $\mathcal{C}_{\mathcal{M}^\infty}(\Omega,O)$ denotes the space of functions ϕ such that the mapping $x\mapsto\phi(\omega,x)$ is in $\mathcal{C}(O)$ for a.a. $\omega\in\Omega$, and for every $x\in O$, $\phi(\cdot,x)$ is in $\mathcal{M}^\infty(\Omega)$.

The following proposition presents embedding results for an element of the space $\mathcal{C}_{\mathcal{M}^\infty}(\Omega,O)$. Now, equality holds on the level of association.

Proposition 5.3.2 *If $\phi\in\mathcal{C}_{\mathcal{M}^\infty}(\Omega,O)$, then $[(L_{\phi*\varphi_n}(t,x))_n]$ is associated to the Colombeau generalized function with representative $(L_\phi(t,\cdot)*\varphi_n(\cdot))(x)$, $t\in\mathbb{R}$, $x\in O$.*

PROOF. Let $\theta(t,x)\in\mathcal{D}(\mathbb{R}\times O)$. Then

$$\begin{aligned}&\int_{\mathbb{R}\times O}(L_{\phi*\varphi_n}(t,x)-(L_\phi(t,\cdot)*\varphi_n(\cdot))(x))\,\theta(t,x)\,dtdx\\ &= \int_{\mathbb{R}\times O}\left(L_{\phi*\varphi_n}(t,x)-\int_{\mathbb{R}}L_\phi(t,x-y)\varphi_n(y)\,dy\right)\theta(t,x)\,dtdx\\ &= \int_{\mathbb{R}\times O}\left(\int_\Omega e^{it(\phi*\varphi_n)(\omega,x)}\,dP(\omega)-\int_{\mathbb{R}}\int_\Omega e^{it\phi(\omega,x-y)}\varphi_n(y)dP(\omega)\,dy\right)\theta(t,x)\,dtdx\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}\times O}\int_{\Omega}\left(e^{it\int_{\mathbb{R}}\phi(\omega,x-y)\varphi_n(y)\,dy} - \int_{\mathbb{R}} e^{it\phi(\omega,x-y)}\varphi_n(y)\,dy\right)\theta(t,x)\,dP(\omega)dtdx \\
&= \int_{\mathbb{R}\times O}\int_{\Omega}\left(e^{it\int_{\mathbb{R}}\phi(\omega,x-y)n\varphi(ny)\,dy} - \int_{\mathbb{R}} e^{it\phi(\omega,x-y)}n\varphi(y)\,dy\right)\theta(t,x)\,dP(\omega)dtdx \\
&= \int_{\mathbb{R}\times O}\int_{\Omega}\left(e^{it\int_{\mathbb{R}}\phi(\omega,x-\frac{z}{n})\varphi(z)\,dz} - \int_{\mathbb{R}} e^{it\phi(\omega,x-\frac{z}{n})}\,\varphi(z)\,dz\right)\theta(t,x)\,dP(\omega)\,dt\,dx,
\end{aligned}
$$

where we used Fubini's theorem and a change of variable $ny = z$. Now, letting $n\to\infty$ we obtain by the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}\times O}(L_{\phi*\varphi_n}(t,x) - (L_\phi(t,\cdot)*\varphi_n(\cdot))(x))\,\theta(t,x)\,dtdx \\
&= \int_{\mathbb{R}\times O}\int_{\Omega}\left(e^{it\int_{\mathbb{R}}\phi(\omega,x)\varphi(z)\,dz} - \int_{\mathbb{R}} e^{it\phi(\omega,x)}\,\varphi(z)\,dz\right)\theta(t,x)\,dP(\omega)\,dt\,dx \\
&= \int_{\mathbb{R}\times O}\int_{\Omega}\left(e^{it\phi(\omega,x)\int_{\mathbb{R}}\varphi(z)\,dz} - e^{it\phi(\omega,x)}\int_{\mathbb{R}}\varphi(z)\,dz\right)\theta(t,x)\,dP(\omega)\,dt\,dx \\
&= 0.
\end{aligned}
$$

Hence, $[(L_{\phi*\varphi_n}(t,x))_n]$ is associated to the Colombeau generalized function with representative $(L_\phi(t,\cdot)*\varphi_n(\cdot))(x)$. ■

5.3.3 Generalized Characteristic Functions of Colombeau Stochastic Processes in $\mathcal{G}^k_{L^{kp}}(\Omega, O)$

In this subsection, we suppose that $k\le p$. Our goal is to define generalized characteristic functions of CSPs in $\mathcal{G}^k_{L^{kp}}(\Omega, O)$. The following proposition will enable us to achieve this goal.

Proposition 5.3.3 *If sequence $(u_n)_n$ belongs to $\mathcal{E}^k_{M,L^{kp}}(\Omega, O)$, then sequence $(e^{itu_n(\omega,x)})_n$ belongs to $\mathcal{E}^k_{M,L^p}(\Omega,\mathbb{R}\times O)$ and sequence $(E(e^{itu_n(\cdot,x)}))_n$ belongs to $\mathcal{E}^k_M(\mathbb{R}\times O)$.*

PROOF. Denote $K_1 = [-t_0,t_0]\times K$, where $K\Subset O$ and $t_0\in\mathbb{R}$. Note that the derivative with respect to variable x of order $m\le k$ of $e^{itu_n(\omega,x)}$ is a linear combination of members of the form $(u_n^{(i_1)}(\omega,x))^{k_1}\cdots(u_n^{(i_s)}(\omega,x))^{k_s}e^{itu_n(\omega,x)}$, where $i_1k_1+\ldots+i_sk_s = m$. For example, for the member $(u_n'(\omega,x))^k e^{itu_n(\omega,x)}$, we have

$$
\begin{aligned}
&\left(\int_\Omega |(u_n'(\omega,x))^k e^{itu_n(\omega,x)}|^p\,dP(\omega)\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega \underbrace{|u_n'(\omega,x)|^p\cdots|u_n'(\omega,x)|^p}_{k}\,dP(\omega)\right)^{\frac{1}{p}} \\
&\le \underbrace{\left(\int_\Omega |u_n'(\omega,x)|^{pk}\,dP(\omega)\right)^{\frac{1}{kp}}\cdots\left(\int_\Omega |u_n'(\omega,x)|^{pk}\,dP(\omega)\right)^{\frac{1}{kp}}}_{k} \\
&= \|u_n'(\cdot,x)\|^k_{L^{kp}}.
\end{aligned}
$$

Now, using $(u_n)_n \in \mathcal{E}^k_{M,L^{kp}}(\Omega, O)$, we obtain that

$$\sup_{(t,x)\in K_1} \|(u_n'(\cdot,x))^k e^{itu_n(\cdot,x)}\|_{L^p} = \mathcal{O}(n^a),$$

for some $a \in \mathbb{N}$. In a similar way we estimate the other derivatives. Hence, $(e^{itu_n(\omega,x)})_n \in \mathcal{E}^k_{M,L^p}(\Omega, \mathbb{R}\times O)$.

Using that the paths are of class $\mathcal{C}^k$, it can be shown that $(E(e^{itu_n(\cdot,x)}))_n$ belongs to $\mathcal{E}^k(\mathbb{R}\times O)$. Let $j, l \in \mathbb{N}$ such that $j + l \le k$. Using the previously proven estimates, we obtain

$$\sup_{(t,x)\in K_1} |\partial_t^j \partial_x^l E(e^{itu_n(\cdot,x)})| \le \sup_{(t,x)\in K_1} \|\partial_t^j \partial_x^l e^{itu_n(\cdot,x)}\|_{L^p} \le Cn^a,$$

for some $a \in \mathbb{N}$. Therefore, $(E(e^{itu_n(\cdot,x)}))_n \in \mathcal{E}^k_M(\mathbb{R}\times O)$. ■

Proposition 5.3.3 enables us to introduce the next definition.

Definition 5.3.2 *Let u be a CSP in $\mathcal{G}^k_{L^{kp}}(\Omega, O)$. Then*

$$L_u(t,x) = [(L_{u_n}(t,x))_n] = [(E(e^{itu_n(\cdot,x)})_n] \in \mathcal{G}^k(\mathbb{R}\times O), \quad t\in\mathbb{R},\ x\in O,$$

is called the generalized characteristic function of u.

5.3.4 Calculating the Generalized Expectation and Generalized Correlation Function

The generalized characteristic function determines the CSP. As in the classical case, both the generalized expectation and the generalized correlation function can be retrieved from the generalized characteristic function. Also, all the higher order moments (if exist) can be retrieved from the generalized characteristic function. For the purpose of the second moments (i.e. the generalized correlation function) we will denote by

$$L_u(t,s;x,y) = E(e^{i(t,s)\centerdot(u(\cdot,x),u(\cdot,y))}) = E(e^{itu(\cdot,x)}e^{isu(\cdot,y)}), \quad t,s\in\mathbb{R},\ x,y\in O,$$

the generalized characteristic function of the joint distribution of the random field $(u(\omega,x), u(\omega,y))$. Here $\centerdot$ denotes the scalar product in $\mathbb{R}^2$.

Theorem 5.3.1 *Let $u = [(u_n)_n] \in \mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$, resp. $u \in \mathcal{G}^k_{L^{kp}}(\Omega, O)$, for $k \ge 1$, $p \ge 2$, and let $L_u(t,x) \in \mathcal{G}(\mathbb{R}\times O)$, resp. $L_u(t,x) \in \mathcal{G}^k(\mathbb{R}\times O)$ be its generalized characteristic function. Furthermore, let $L_u(t,s;x,y) \in \mathcal{G}(\mathbb{R}^2\times O^2)$, resp. $L_u(t,s;x,y) \in \mathcal{G}^k(\mathbb{R}^2\times O^2)$, be the generalized characteristic function of the joint distributions.*

(a) *The generalized expectation* $m \in \mathcal{G}(O)$*, resp.* $m \in \mathcal{G}^{k-1}(O)$ *satisfies*

$$m(x) = i^{-1}\frac{d}{dt}L_u(t,x)\Big|_{t=0}.$$

(b) *The generalized correlation function* $B \in \mathcal{G}(O \times O)$*, resp.* $B \in \mathcal{G}^{k-1}(O \times O)$ *satisfies*

$$B(x,y) = -\frac{d}{dt}\frac{d}{ds}L_u(t,s;x,y)\Big|_{(t,s)=(0,0)}.$$

PROOF. Let u be given by the representative $[(u_n(\omega,x))_n]$.

(a) We have

$$\frac{d}{dt}L_{u_n}(t,x) = \frac{d}{dt}E(e^{itu_n(\cdot,x)}) = E(iu_n(\cdot,x)e^{itu_n(\cdot,x)}), \quad n \in \mathbb{N},$$

and thus

$$\frac{d}{dt}L_{u_n}(0,x) = iE(u_n(\cdot,x)) = im_n(x), \quad n \in \mathbb{N},$$

i.e.

$$m_n(x) = i^{-1}\frac{d}{dt}L_{u_n}(0,x), \quad n \in \mathbb{N}.$$

(b) Similarly,

$$\begin{aligned}\frac{d}{dt}\frac{d}{ds}L_{u_n}(t,s;x,y) &= \frac{d}{dt}\frac{d}{ds}E(e^{itu_n(\cdot,x)}e^{isu_n(\cdot,y)}) \\ &= -E(u_n(\cdot,x)u_n(\cdot,y)e^{i(tu_n(\cdot,x)+su_n(\cdot,y))}), \quad n \in \mathbb{N},\end{aligned}$$

and by smoothness of the representatives

$$\frac{d}{dt}\frac{d}{ds}L_{u_n}(t,s;x,y) = \frac{d}{ds}\frac{d}{dt}L_{u_n}(t,s;x,y).$$

Now,

$$\frac{d}{dt}\frac{d}{ds}L_{u_n}(0,0;x,y) = -E(u_n(\cdot,x)u_n(\cdot,y)) = -B_n(x,y), \quad n \in \mathbb{N},$$

so the claim follows. ■

5.3.5 Generalized Characteristic Function of Gaussian Colombeau Stochastic Process

We provide in this subsection examples of the generalized characteristic function related to GCSPs with values in $L^2(\Omega)$.

Let us compute the generalized characteristic function of a GCSP $u = [(u_n)_n] \in \mathcal{G}^1_{L^2}(\Omega,\mathbb{R})$. Let $B = [(B_{u_n})_n] \in \mathcal{G}^1(\mathbb{R}^2)$ be the generalized correlation function of u

determined by a Gaussian representative $(u_n)_n$. According to Definition 4.2.1, the distribution function of u_n is

$$P(u_n(\omega,x)\in(-\infty,b))=\frac{1}{\sqrt{2\pi B_{u_n}(x,x)}}\int_{-\infty}^{b}\exp\left(-\frac{s^2}{2B_{u_n}(x,x)}\right)ds,$$

so we obtain

$$\begin{aligned}L_{u_n}(t,x) &= \int_{\mathbb{R}} e^{itu_n(\omega,x)}\,dP(\omega)\\ &= \frac{1}{\sqrt{2\pi B_{u_n}(x,x)}}\int_{\mathbb{R}}\exp\left(it-\frac{t^2}{2B_{u_n}(x,x)}\right)dt\\ &= \exp\left(-\frac{1}{2}B_{u_n}(x,x)t^2\right).\end{aligned}$$

Therefore, the generalized characteristic function of a GCSP u with generalized correlation function B is

$$L_u(t,x)=\exp\left(-\frac{1}{2}B(x,x)t^2\right)\in\mathcal{G}^1(\mathbb{R}^2).$$

In the following two examples we consider the generalized characteristic functions of Brownian motion and white noise.

Example 5.3.1 *Recall, Brownian motion is* $b=[(b_n)_n]\in\mathcal{G}^1_{L^2}(\Omega,\mathbb{R})$ *is a GCSP with zero generalized expectation and with a representative of the generalized correlation function*

$$B_{b_n}(x,y)=\min\{\sigma,\tau\}*\varphi_n(x)\varphi_n(y),\ \ x,y\in\mathbb{R},\ n\in\mathbb{N};$$

see Example 5.1.1. Since

$$B_{b_n}(x,x)=\min\{\sigma,\tau\}*\varphi_n^2(x),\quad n\in\mathbb{N},$$

we obtain that the generalized characteristic function of a Brownian motion b *is represented by*

$$L_{b_n}(t,x)=\exp\left(-\frac{t^2}{2}\left(\min\{\sigma,\tau\}*\varphi_n^2(x)\right)\right),\ \ n\in\mathbb{N}.$$

□

Example 5.3.2 *Recall, white noise* $w=[(w_n)_n]\in\mathcal{G}^1_{L^2}(\Omega,\mathbb{R})$ *is a GCSP with zero generalized expectation and with a representative of the generalized correlation function*

$$B_{w_n}(x,y)=\varphi_n(x-y),\ \ x,y\in\mathbb{R},\ n\in\mathbb{N};$$

see Example 5.1.2. Since

$$B_{w_n}(x,x)=\varphi_n(x-x)=\varphi_n(0),\quad n\in\mathbb{N},$$

we obtain that the generalized characteristic function of a white noise w is represented by

$$L_{w_n}(t,x) = \exp\left(-\frac{1}{2}\varphi_n(0)t^2\right), \quad n \in \mathbb{N}. \qquad \square$$

5.4 Colombeau Stochastic Processes with Independent Values

The theory of GSP with independent values was developed by I. M. Gel'fand in [Gel55]. Recall, a GSP with independent values $u(\phi) = \langle u, \phi\rangle$, $\phi \in \mathcal{D}(O)$, has independent values, if the random variables $u(\phi_1)$ and $u(\phi_2)$ are mutually independent, whenever $\phi_1(x)\phi_2(x) = 0$.

In this section the main goal is to define CSPs with independent values and give a characterization of such processes via their generalized correlation function in the classical Colombeau algebra of generalized numbers.

Definition 5.4.1 *A CSP u over O with values in $L^p(\Omega)$ has independent values if it has a representative $(u_n)_n$ such that the following conditions hold:*

(1) for every $n \in \mathbb{N}$, $u_n(\omega,x)$ and $u_n(\omega,y)$ are independent random variables for $(x,y) \in K$, $K \Subset Q_O$, i.e. for every $n \in \mathbb{N}$,

$$P\{u_n(\omega,x) \in B_1 \cap u_n(\omega,y) \in B_2\} = P\{u_n(\omega,x) \in B_1\}P\{u_n(\omega,y) \in B_2\}$$

for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ and $(x,y) \in K$, $K \Subset Q_O$,

(2) for $n \neq m$, $u_n(\omega,x)$ and $u_m(\omega,y)$ are independent random variables for every $x,y \in O$, i.e. for $n \neq m$,

$$P\{u_n(\omega,x) \in B_1 \cap u_m(\omega,y) \in B_2\} = P\{u_n(\omega,x) \in B_1\}P\{u_m(\omega,y) \in B_2\}$$

for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ and $x,y \in O$.

The following example shows that not all representations of such a process are with independent values.

Example 5.4.1 *Let $(u_n)_n$ be a representative of a CSP u satisfying the conditions of Definition 5.4.1 and $(N_n)_n$ be a negligible CSP with non-independent values. Then $(u_n)_n + (N_n)_n$ is a representative of the same equivalence class (of the same CSP) which has not independent values. Therefore, not all representatives are with independent values.* $\square$

In the sequel, we will call the representatives that satisfy the conditions of Definition 5.4.1 shortly *IV-representatives*.

Remark 5.4.1 *The notion of CSPs with independent values is dependent on the existence of a special representative that satisfies the conditions i Definition 5.4.1. Note that it is not possible to give a characterization of a CSPs with independent values that would not depend on the choise of representatives.*

Note that by Proposition 4.5.1 inverse images of sharp open balls are always in $\mathcal{F}$.

Theorem 5.4.1 *Let u be a CSP over O with values in $L^p(\Omega)$ and let u have independent values. Then*

$$P\{u(\omega,\tilde{x}) \in \mathfrak{O}_1 \cap u(\omega,\tilde{y}) \in \mathfrak{O}_2\} = P\{u(\omega,\tilde{x}) \in \mathfrak{O}_1\}P\{u(\omega,\tilde{y}) \in \mathfrak{O}_2\} \tag{5.3}$$

for all open balls $\mathfrak{O}_1, \mathfrak{O}_2$ in $\mathcal{R}_c$ and $(\tilde{x},\tilde{y}) \in (\tilde{Q}_O)_c$.

PROOF. Let u have independent values and let $(u_n)_n$ be the corresponding IV-representative as stated in Definition 5.4.1. Let

$$\begin{aligned}
\mathfrak{O}_i &= L((c_{i;n})_n, k_i) = \{[(z_n)_n] \in \mathcal{R}_c : \limsup_{n\to\infty} |c_{i;n} - z_n|^{(\log n)^{-1}} < k_i\} \\
&= \{[(z_n)_n] \in \mathcal{R}_c : |c_{i;n} - z_n| < n^{\log k_i} \text{ for all } n > n_0 \text{ for some } n_0 \in \mathbb{N}\},
\end{aligned}$$

$i = 1, 2$, be two open balls in $\mathcal{R}_c$ and $(\tilde{x},\tilde{y}) = ((x_n)_n, (y_n)_n) \in (\tilde{Q}_O)_c$. There exists a compact set $K \Subset Q_O$ such that $(x_m, y_m) \in K$ for all $m \in \mathbb{N}$. Since u has independent values and $x_m \neq y_m$, it follows that $u_m(\omega, x_m)$ and $u_m(\omega, y_m)$ are independent random variables for every $m \in \mathbb{N}$, $u_n(\omega, x_n)$ and $u_m(\omega, x_m)$ are independent random variables for $n \neq m$, and $u_n(\omega, y_n)$ and $u_m(\omega, y_m)$ are independent random variables for $n \neq m$. Therefore, the events $\{\omega \in \Omega : u_m(\omega, x_m) \in B_1\}$ and $\{\omega \in \Omega : u_m(\omega, y_m) \in B_2\}$ are independent for every $m \in \mathbb{N}$, as well as the events $\{\omega \in \Omega : u_n(\omega, x_n) \in B_1\}$ and $\{\omega \in \Omega : u_m(\omega, x_m) \in B_2\}$ for $n \neq m$ and the events $\{\omega \in \Omega : u_n(\omega, y_n) \in B_1\}$ and $\{\omega \in \Omega : u_m(\omega, y_m) \in B_2\}$ for $n \neq m$.

We have

$$\begin{aligned}
A^{\tilde{x}} &= \{\omega \in \Omega : u(\omega,\tilde{x}) \in \mathfrak{O}_1\} \\
&= \{\omega \in \Omega : \limsup_{n\to\infty} |c_{1;n} - u_n(\omega, x_n)|^{(\log n)^{-1}} < k_1\} \\
&= \{\omega \in \Omega : |c_{1;n} - u_n(\omega, x_n)| < n^{\log k_1} \text{ for all } n > n_0 \text{ for some } n_0 \in \mathbb{N}\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\omega \in \Omega : |c_{1;m} - u_m(\omega, x_m)| < m^{\log k_1}\} \\
&= \bigcup_{n=1}^{\infty} A_n^{\tilde{x}},
\end{aligned}$$

where

$$A_n^{\tilde{x}} = \bigcap_{m\geq n} \{\omega \in \Omega : |c_{1;m} - u_m(\omega, x_m)| < m^{\log k_1}\} = \bigcap_{m\geq n} I_m^{x_m}.$$

It holds $A_1^{\tilde{x}} \subset A_2^{\tilde{x}} \subset A_3^{\tilde{x}} \subset \ldots$ and by continuity of the probability measure we obtain

$$P(A^{\tilde{x}}) = P\left(\bigcup_{n=1}^{\infty} A_n^{\tilde{x}}\right) = \lim_{n\to\infty} P(A_n^{\tilde{x}}) = \lim_{n\to\infty} P\left(\bigcap_{m\geq n} I_m^{x_m}\right).$$

Put

$$A_n^k = \bigcap_{m\geq n}^{k} I_m^{x_m}, \quad k \geq n.$$

Then

$$\bigcap_{m\geq n} I_m^{x_m} = \bigcap_{k=n}^{\infty} A_n^k.$$

It holds $A_n^n \supset A_n^{n+1} \supset A_n^{n+2} \supset \ldots$ and hence

$$\begin{aligned} P(A^{\tilde{x}}) &= \lim_{n\to\infty} P\left(\bigcap_{m\geq n} I_m^{x_m}\right) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_n^k\right) \\ &= \lim_{n\to\infty}\lim_{k\to\infty} P(A_n^k) = \lim_{n\to\infty}\lim_{k\to\infty} P\left(\bigcap_{m=n}^{k} I_m^{x_m}\right) \\ &= \lim_{n\to\infty}\lim_{k\to\infty} \prod_{m=n}^{k} P(I_m^{x_m}), \end{aligned} \tag{5.4}$$

where we used the independence of the events $\{\omega \in \Omega : |c_{1;m} - u_m(\omega, x_m)| < m^{\log k_1}\}$ and $\{\omega \in \Omega : |c_{1;n} - u_n(\omega, x_n)| < n^{\log k_1}\}$ for $m \neq n$. Analogously, we have

$$B^{\tilde{y}} = \{\omega \in \Omega : u(\omega, \tilde{y}) \in \mathfrak{O}_2\} = \bigcup_{n=1}^{\infty} B_n^{\tilde{y}},$$

$$B_n^{\tilde{y}} = \bigcap_{m\geq n} \{\omega \in \Omega : |c_{2;m} - u_m(\omega, y_m)| < m^{\log k_2}\} = \bigcap_{m\geq n} J_m^{y_m},$$

$$B_n^k = \bigcap_{m\geq n}^{k} J_m^{y_m}, \quad k \geq n,$$

and

$$\begin{aligned} P(B^{\tilde{y}}) &= \lim_{n\to\infty} P(B_n^{\tilde{y}}) = \lim_{n\to\infty} P\left(\bigcap_{m\geq n} J_m^{y_m}\right) \\ &= \lim_{n\to\infty}\lim_{k\to\infty} \prod_{m=n}^{k} P(J_m^{y_m}). \end{aligned} \tag{5.5}$$

We have

$$A^{\tilde{x}} \cap B^{\tilde{y}} = \left(\bigcup_{n\in\mathbb{N}} A_n^{\tilde{x}} \right) \cap \left(\bigcup_{l\in\mathbb{N}} B_l^{\tilde{y}} \right) = \bigcup_{n,l\in\mathbb{N}} (A_n^{\tilde{x}} \cap B_l^{\tilde{y}}).$$

Since $A_n^{\tilde{x}} \subset A_{n+1}^{\tilde{x}}$, $n \in \mathbb{N}$, and $B_l^{\tilde{y}} \subset B_{l+1}^{\tilde{y}}$, $l \in \mathbb{N}$, we have $A_n^{\tilde{x}} \cap B_l^{\tilde{y}} \subset A_{n+1}^{\tilde{x}} \cap B_{l+1}^{\tilde{y}}$, $n, l \in \mathbb{N}$, and therefore

$$P(A^{\tilde{x}} \cap B^{\tilde{y}}) = P\left(\bigcup_{n,l\in\mathbb{N}} (A_n^{\tilde{x}} \cap B_l^{\tilde{y}}) \right) = P\left(\bigcup_{n\in\mathbb{N}} (A_n^{\tilde{x}} \cap B_n^{\tilde{y}}) \right) = \lim_{n\to\infty} P(A_n^{\tilde{x}} \cap B_n^{\tilde{y}}).$$

Since

$$\begin{aligned} A_n^{\tilde{x}} \cap B_n^{\tilde{y}} &= \left(\bigcap_{m\geq n} I_m^{x_m} \right) \cap \left(\bigcap_{l\geq n} J_l^{y_l} \right) = \bigcap_{m,l\geq n} (I_m^{x_m} \cap J_l^{y_l}) \\ &= \bigcap_{k=n}^{\infty} \bigcap_{m,l\geq n}^{k} (I_m^{x_m} \cap J_l^{y_l}) = \bigcap_{k=n}^{\infty} C_{n;k}, \end{aligned}$$

and $C_{n;k} \supset C_{n,k+1}$, $k \geq n$, we obtain

$$\begin{aligned} P(A^{\tilde{x}} \cap B^{\tilde{y}}) &= \lim_{n\to\infty} P(A_n^{\tilde{x}} \cap B_n^{\tilde{y}}) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} C_{n;k} \right) = \lim_{n\to\infty} \lim_{k\to\infty} P(C_{n;k}) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} P\left(\bigcap_{m,l\geq n}^{k} (I_m^{x_m} \cap J_l^{y_l}) \right) = \lim_{n\to\infty} \lim_{k\to\infty} \prod_{m,l\geq n}^{k} P(I_m^{x_m} \cap J_l^{y_l}) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} \prod_{m,l\geq n}^{k} P(I_m^{x_m}) P(J_l^{y_l}), \qquad (5.6) \end{aligned}$$

where we used the independence of the events $\{\omega \in \Omega : |c_{1;m} - u_m(\omega, x_m)| < m^{\log k_1}\}$ and $\{\omega \in \Omega : |c_{2;m} - u_m(\omega, y_m)| < m^{\log k_2}\}$ in the last step.

Now, from (5.4), (5.5) and (5.6) we have

$$\begin{aligned} P(A^{\tilde{x}} \cap B^{\tilde{y}}) &= \lim_{n\to\infty} \lim_{k\to\infty} \prod_{m,l\geq n}^{k} P(I_m^{x_m}) P(J_l^{y_l}) \\ &= \left(\lim_{n\to\infty} \lim_{k\to\infty} \prod_{m,l\geq n}^{k} P(I_m^{x_m}) \right) \left(\lim_{n\to\infty} \lim_{k\to\infty} \prod_{m,l\geq n}^{k} P(J_l^{y_l}) \right) \\ &= P(A^{\tilde{x}}) P(B^{\tilde{y}}). \end{aligned}$$

■

In the next section we shall make use of the following characterization of CSPs with independent values.

Proposition 5.4.1 *Let u be a CSP over O with values in $L^2(\Omega)$ and let u have independent values. Then the generalized correlation function $B(\tilde{x},\tilde{y})$ is supported by the diagonal, i.e. $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_O)_c$.*

PROOF. Let $(u_n)_n$ be an IV-representative and suppose that the representative $(B_n)_n$ of its generalized correlation function is determined by this same $(u_n)_n$. Without restriction of generality we may assume that all generalized expectations are zero, thus we have $E(u_n(\cdot,x_n)) = N_n$, $|N_n| = \mathcal{O}(n^{-k})$ for all $k > 0$. Let $(\tilde{x},\tilde{y}) \in (\tilde{Q}_O)_c$ be arbitrary and choose its representatives such that $x_m \neq y_m$ for all pairs $(x_m, y_m) \in K$, $m \in \mathbb{N}$, $K \Subset Q_O$. Hence, by independence at different points we obtain

$$B_n(x_n,y_n) = E(u_n(\cdot,x_n)u_n(\cdot,y_n)) = E(u_n(\cdot,x_n))E(u_n(\cdot,y_n)) = N_nM_n,$$

where $|N_n| = \mathcal{O}(n^{-k})$ and $|M_n| = \mathcal{O}(n^{-k})$ for arbitrary $k > 0$. Thus, $B(\tilde{x},\tilde{y}) = 0$ in $\mathcal{R}_c$. ■

Corollary 5.4.1 *Let u and B be as in Proposition 5.4.1. If B is associated to $F \in \mathcal{D}'(O \times O)$, then B is associated to a generalized function which has a representative of the form*

$$B_n^*(x,y) = \int_O \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\varphi_n^{(j)}(x-s)\varphi_n^{(k)}(y-s)ds, \quad x,y \in O, \tag{5.7}$$

where for every $n \in \mathbb{N}$ only a finite number of continuous functions $R_{j,k}$ are different from zero on any compact subset of O.

PROOF. From Proposition 5.4.1 it follows that $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_O)_c$. Proposition 5.2.1 imply that B is associated to a generalized function which has a representative of the form (5.1). ■

5.5 Stationary Colombeau Stochastic Processes

The subject of this section are stationary CSPs. We will introduce strictly stationary CSPs and weakly stationary CSPs. In this section the main goal is to show that the generalized expectation of a stationary CSP is a generalized constant. We give a special form of generalized correlation function of a stationary CSPs.

5.5.1 Strictly Stationary Colombeau Stochastic Processes

Definition 5.5.1 *A CSP u over O with values in $L^p(\Omega)$ is called strictly stationary if it has a representative $(u_n)_n$ such that for every $n \in \mathbb{N}$, for arbitrary $x_1,\ldots,x_m \in O$ and for every $h \in \mathbb{R}^d$ such that $x_1+h,\ldots,x_m+h \in O$, the random variables*

$$(u_n(\cdot,x_1),\ldots,u_n(\cdot,x_m)) \quad \text{and} \quad (u_n(\cdot,x_1+h),\ldots,u_n(\cdot,x_m+h)) \tag{5.8}$$

are identically distributed.

In the sequel, a CSP that satisfies (5.8) is called stationary CSP.

Remark 5.5.1 *Observe that it is not possible to give a characterization of stationary CSPs independently of the representatives.*

Theorem 5.5.1 *The generalized expectation $m \in \mathcal{G}(O)$ of a stationary CSP over $O \subseteq \mathbb{R}^d$ is a generalized constant $m \in \mathcal{R}_c$.*

PROOF. If u is stationary, taking a representative $(u_n)_n$ which satisfies (5.8) and calculating

$$m_{u_n}(x+h) - m_{u_n}(x) = E(u_n(\cdot, x+h) - E(u_n(\cdot, x)) = 0,$$

we immediately obtain that

$$m_{u_n}(x) = c_n, \quad x \in O,$$

(It is known that if a smooth function is translation invariant, then it has to be a constant.) that is, $m_u = [(m_{u_n})_n]$ is a constant in $\mathcal{R}_c$. ■

Remark 5.5.2 *Not all representatives have to be stationary, but all of them have constant expectations. If $(u_n)_n$ is a representative of a CSP satisfying (5.8) and $(N_n)_n$ is a representative of a negligible non-stationary CSP, then $((u_n)_n + (N_n)_n$ is a representative of the same equivalence class which is not stationary.*

In the sequel, we will assume that O is a centrally symmetric convex open set in $\mathbb{R}^d$. This will imply that $O - O = \{z \in \mathbb{R}^2 : z = x - y,\ x, y \in O\} \cong 2O$.

Theorem 5.5.2 *Let $O \subseteq \mathbb{R}^d$ be a centrally symmetric convex open set and let u be a stationary CSP over $2O$ with values in $L^2(\Omega)$. If $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ is the generalized correlation function of u, then there exists a positive-definite generalized function $B^* = [(B_n^*)_n] \in \mathcal{G}(2O)$ such that*

$$B_n(x, y) = B_n^*(x - y), \quad x, y \in O,\ n \in \mathbb{N}.$$

PROOF. Without loss of generality we may assume that the generalized expectation of u is zero. Since u is stationary, it follows that

$$\begin{aligned} B_n(x, y) &= E(u_n(\cdot, x)u_n(\cdot, y)) \\ &= E(u_n(\cdot, x+h)u_n(\cdot, y+h)) \\ &= B_n(x+h, y+h), \quad n \in \mathbb{N}, \end{aligned} \tag{5.9}$$

for every $x, y \in O$ and every $h \in \mathbb{R}^d$ such that $x+h, y+h \in O$. Thus, the representative of its generalized correlation function is translation invariant. Putting $h = -y$ in (5.9), we obtain

$$B_n(x, y) = B_n(x - y, 0), \quad n \in \mathbb{N}.$$

Define

$$B_n^*(x - y) = B_n(x - y, 0), \quad n \in \mathbb{N}.$$

Let us show that $B^* = [(B_n^*)_n]$ is a positive-definite generalized function. Let $K \Subset O$. Let $a > 0$ and $\zeta_1, \ldots, \zeta_m \in \mathbb{R}$ be arbitrary. Then we have

$$\inf_{x_k, x_j \in K} \sum_{k,j=1}^{m} (B_n^*(x_k - x_j) + n^{-a})\zeta_k \zeta_j = \inf_{x_k, x_j \in K} \sum_{k,j=1}^{m} (B_n(x_k - x_j, 0) + n^{-a})\zeta_k \zeta_j$$

$$= \inf_{x_k, x_j \in K} \sum_{k,j=1}^{m} (B_n(x_k, x_j) + n^{-a})\zeta_k \zeta_j \geq 0,$$

for all $n \geq n_0, n_0 \in \mathbb{N}$, since B is a translation invariant positive-definite generalized function. Therefore, B^* is positive-definite. ■

This leads to:

Corollary 5.5.1 *Let $O \subseteq \mathbb{R}^d$ be a centrally symmetric convex open set and let u be a stationary CSP over $2O$ with values in $L^2(\Omega)$. If $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ is the generalized correlation function of u, then there exists a positive-definite generalized function $B^* = [(B_n^*)_n] \in \mathcal{G}(2O)$ such that*

$$B(\tilde{x}, \tilde{y}) = B^*(\tilde{x} - \tilde{y}), \quad \tilde{x}, \tilde{y} \in \tilde{O}_c.$$

Corollary 5.5.2 *Let $O \subseteq \mathbb{R}^d$ be a centrally symmetric convex open set and let u be a stationary CSP over $2O$ with independent values in $L^2(\Omega)$. Then the generalized function B^* from Corollary 5.4.1 satisfies $B^*(\tilde{z}) = 0$ for every $\tilde{z} \in \tilde{O}_c$, $\tilde{z} \neq \tilde{0}$.*

PROOF. Since u has independent values, from Proposition 5.4.1 it follows that $B(\tilde{x}, \tilde{y}) = 0$ for all $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$. Since u is stationary, from Corollary 5.5.1 it follows that $B(\tilde{x}, \tilde{y}) = B^*(\tilde{x} - \tilde{y})$. Put $\tilde{z} = \tilde{x} - \tilde{y}$, for $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$. Clearly, $\tilde{z} \neq \tilde{0}$ and $B^*(\tilde{z}) = 0$.

In the next example, we construct a stationary CSP.

Example 5.5.1 *Let m be a generalized constant and B a positive-definite generalized function. It is known by Theorem 5.1.2 (also see [LCP94a] and [MPS09]) that it is possible to construct GCSPs $u = [(u_n)_n] \in \mathcal{G}_{L^2}(\Omega, O)$ such that m is the generalized expectation of u and B is the generalized correlation function of u. Observe that process u is stationary.* □

5.5.2 Weakly Stationary Colombeau Stochastic Processes

We procees now to the study of weakly stationary CSP. As we mentioned, our goal is to prove that the generalized expectation of a weakly stationary CSP is a generalized constant.

Definition 5.5.2 *A CSP u over $2O$ with values in $L^2(\Omega)$ is called weakly stationary if its expectation $m_u \in \mathcal{G}(O)$ and correlation function $B_u \in \mathcal{G}(O \times O)$ are translation invariant, i.e.*

$$m_u(x+h) = m_u(x)$$

for all $h \in \mathbb{R}$ such that $x, x+h \in O$, and

$$B_u(x,y) = B^*(x-y), \quad x, y \in O,$$

for some positive-definite generalized function $B^* \in \mathcal{G}(2O)$.

Remark 5.5.3 *Unlike the stationary CSP, the weakly stationary CSP is defined independently of representatives.*

Recall, Theorem 3.2.1 states that a generalized function $u \in \mathcal{G}(O)$ invariant under all translation is a generalized constant. As an application of Theorem 3.2.1, we obtain the following result.

Corollary 5.5.3 *The generalized expectation $m \in \mathcal{G}(O)$ of a weakly stationary CSP over a centrally symemetric convex open set $O \subseteq \mathbb{R}^d$ is a generalized constant $m \in \mathcal{R}_c$.*

Clearly, stationarity implies weak stationarity of a process (Theorem 5.5.1 and Theorem 5.5.2). The converse is not true in general: weak stationarity is defined only via the first two moments. However, since GCSPs are completely determined via their generalized expectation and generalized correlation function (Corollary 5.1.1), it follows that every weakly stationary GCSP is also stationary. Also, derivatives of a (weakly) stationary CSP are (weakly) stationary.

5.5.3 Colombeau Stochastic Processes with Stationary Increments

I. M. Gel'fand and N. Ya. Vilenkin develop the theory of processes with stationary incerements in the framework of GSPs. In [GV64], a GSP u is called a process with stationary increments of order n, if its nth derivative is a stationary GSP, that is random variables

$$(u^{(n)}(\phi_1(x+h)), \ldots, u^{(n)}(\phi_k(x+h))) \text{ and } (u^{(n)}(\phi_1(x)), \ldots, u^{(n)}(\phi_k(x)))$$

are identically distributed for all functions $\phi_1, \ldots, \phi_k \in \mathcal{D}(O)$ and any $h \in \mathbb{R}^d$ such that $x + h \in O$.

Following [GV64], we introduce here the notion of CSP with stationary increments.

Definition 5.5.3 *A CSP u over O with values in $L^p(\Omega)$ has stationary increments if the derivative of the process ∇u is stationary.*

Next, we introduce the notion of CSP with weakly stationary increments.

Definition 5.5.4 *A CSP u over O with values in $L^p(\Omega)$ has weakly stationary increments if the derivative of the process ∇u is weakly stationary.*

Thus, the study of processes with stationary increments reduces to the study of their derivative process. This again reduces to the study of the derivatives of the generalized expectation and the generalized correlation function, i.e. to checking if

$$\nabla m(x) = \nabla E(u(\cdot, x)) = E(\nabla u(\cdot, x))$$

corresponds to the expectation of a stationary process and if

$$\nabla_x \boldsymbol{\cdot} \nabla_y B(x, y) = \nabla_x \boldsymbol{\cdot} \nabla_y E(u(\cdot, x)u(\cdot, y)) = E(\nabla_x u(\cdot, x) \boldsymbol{\cdot} \nabla_y u(\cdot, y))$$

corresponds to the generalized correlation function of a stationary process. Here $\boldsymbol{\cdot}$ denotes the scalar product in $\mathbb{R}^d$.

Example 5.5.2 *Let $d = 1$. The generalized correlation function of white noise $w = [(w_n)_n]$ is represented by*

$$B_{w_n}(x, y) = \int_{\mathbb{R}} \varphi_n(s - x)\varphi_n(s - y)\, ds, \quad n \in \mathbb{N}.$$

White noise is a stationary GCSP. Brownian motion has the generalized correlation function represented by

$$B_{b_n}(x, y) = min\{x, y\} * \varphi_n(x)\varphi(y), n \in \mathbb{N}.$$

It holds

$$\partial_x \partial_y B_{b_n}(x, y) = \int_{\mathbb{R}} \varphi_n(s - x)\varphi_n(s - y) ds, \quad n \in \mathbb{N}.$$

Thus Brownian motion has stationary increments as expected.

6 Applications

> *All happy families resemble one another, each unhappy family is unhappy in its own way.*
>
> — **Leo Tolstoy** (1828 - 1910)

> *All linear problems resemble one another, each nonlinear problem is nonlinear in its own way.*
>
> — **Akademik Teodor Atanacković**

In this chapter we present a method for solving a class of linear SPDEs in the framework of stationary GCSPs over $\mathbb{R}^d$ with values in $L^2(\Omega)$. The results presented in this chapter are an original part of the dissertation; see [GOPS18a].

In order to find the solutions to a class of SPDEs in the framework of stationary GCSPs, we use the Fourier transform. Therefore, we need to switch to tempered CSPs over $\mathbb{R}^d$ with values in $L^2(\Omega)$. Notice that the whole theory of CSPs over $\mathbb{R}^d$ with values in $L^2(\Omega)$ can be adapted word by word, with the change of negligible sets, to tempered CSPs over $\mathbb{R}^d$ with values in $L^2(\Omega)$.

6.1 Stationary Solutions to Some Class of Stochastic Partial Differential Equations

Let

$$P(D) = \sum_{|\alpha|\leq k} \tilde{a}_\alpha D_x^\alpha, \quad \tilde{a}_\alpha \in \mathcal{R}_c,$$

be a differential operator of order k with generalized constant coefficients; see also Section 2.6. Its symbol is

$$P(\xi) = \sum_{|\alpha|\leq k} \tilde{a}_\alpha \xi^\alpha, \;\; \xi \in \mathbb{R}^d.$$

Our aim is to present a method for solving the equation

$$P(D)u(\omega, x) = f(\omega, x), \;\; \omega \in \Omega, \; x \in \mathbb{R}^d, \tag{6.1}$$

where $f = [(f_n)_n]$ is a weakly stationary tempered GCSP over $\mathbb{R}^d$ with values in $L^2(\Omega)$ with generalized expectation $\tilde{m}_f = [(m_{f_n})_n] \in \mathcal{R}_c$ (it is a constant due to Corollary 5.5.3) and generalized correlation function $B_f = [(B_{f_n})_n] \in \mathcal{G}_\tau(\mathbb{R}^{2d})$. Notice that a weakly stationary GCSP is also stationary.

6.1.1 Matching the Expectation and Correlation

We interpret equation (6.1) as a family of equations

$$P_n(D)u_n(\omega, x) = f_n(\omega, x), \;\; \omega \in \Omega, \;\; x \in \mathbb{R}^d, \;\; n \in \mathbb{N}, \tag{6.2}$$

in $\mathcal{E}_{\tau,L^2}(\Omega, \mathbb{R}^d)$, where

$$P_n(D) = \sum_{|\alpha| \le k} (a_\alpha)_n D_x^\alpha, \quad n \in \mathbb{N};$$

see Section 2.6 or [PS96] for more details.

It is known that Gaussian processes are completely determined by their expectation and correlation. Therefore, we will match the expectations and correlations on the left hand side of (6.2) with the corresponding ones on the right hand side of (6.2). Note that it is the same technique as used in [MPS09]. Also, due to Corollary 5.5.3 the generalized expectation of any stationary solution u will have to be a generalized constant, while its generalized correlation function will have to be of the form $B_u^*(x - y)$ due to Theorem 5.5.2.

In the following theorem we will give a necessary condition for the existence of a stationary Gaussian solution to the equation (6.1).

Theorem 6.1.1 *Let $f = [(f_n)_n] \in \mathcal{G}_{\tau,L^2}(\Omega, \mathbb{R}^d)$ be a weakly stationary tempered GCSP with generalized expectation $\tilde{m}_f = [(m_{f_n})_n]$ and generalized correlation function $B_f = [(B_{f_n})_n]$.*

(a) The generalized expectation $\tilde{m}_u = [(m_{u_n})_n] \in \mathcal{R}_c$ of a weakly stationary solution to equation (6.1) satisfies

$$\tilde{m}_u = \begin{cases} \frac{\tilde{m}_f}{\tilde{a}_0}, & \text{if } \tilde{a}_0 \neq \tilde{0}, \\ \text{arbitrary}, & \text{if } \tilde{a}_0 = \tilde{0} \text{ and } \tilde{m}_f = \tilde{0} \\ \text{does not exist}, & \text{if } \tilde{a}_0 = \tilde{0} \text{ and } \tilde{m}_f \neq \tilde{0}. \end{cases} \tag{6.3}$$

Especially, if $\tilde{a}_0 = \tilde{0}$ and $\tilde{m}_f \neq \tilde{0}$, then equation (6.1) has no weakly stationary solutions in $\mathcal{G}_{\tau,L^2}(\Omega, \mathbb{R}^d)$.

(b) The generalized correlation function $[(B_{u_n})_n] \in \mathcal{G}_\tau(\mathbb{R}^{2d})$ of a weakly stationary solution to equation (6.1) satisfies

$$P_n(D)P_n(-D)B_{u_n}(z) = B_{f_n}(z), \;\; z = x - y \in \mathbb{R}^d, \; n \in \mathbb{N}. \tag{6.4}$$

Especially, if there exists an open set $S \subset \mathbb{R}^d$ such that

$$\hat{B}_{f_n}(\xi) > 0,$$

for $\xi \in S, n \in \mathbb{N}$, and

$$P_n(\xi)P_n(-\xi) < 0,$$

for $\xi \in S, n \in \mathbb{N}$, for all representatives of the coefficients $(a_\alpha)_n$, then B_{u_n} cannot be a positive-definite function.

(c) Let

$$|P_n(\xi)| \geq Cn^{-r}(1 + |\xi|)^k, \;\; n \in \mathbb{N}, \;\; \xi \in \mathbb{R}^d,$$

for some $C > 0$, $r > 0$, $k > 0$, for some representative of the coefficients $(a_\alpha)_n$. Then equation (6.1) has a weakly stationary solution $u = [(u_n)_n] \in \mathcal{G}_{\tau,L^2}(\Omega, \mathbb{R}^d)$ and its generalized correlation function satisfies

$$P_n(\xi)P_n(-\xi)\hat{B}_{u_n}(\xi) = \hat{B}_{f_n}(\xi), \;\; \xi \in \mathbb{R}^d, \; n \in \mathbb{N}. \tag{6.5}$$

PROOF. (a) If $u = [(u_n)_n] \in \mathcal{G}_{\tau,L^2}(\Omega, \mathbb{R}^d)$ is the solution of equation (6.1), then $(u_n)_n \in \mathcal{E}_{\tau,L^2}(\Omega, \mathbb{R}^d)$ is the solution of the family of equations (6.2). Taking expectations on both sides of equation (6.2) and using the fact that stationary processes have constant expectations, we obtain

$$P_n(D)m_{u_n} = m_{f_n}, \quad n \in \mathbb{N}. \tag{6.6}$$

Since m_{u_n} is a constant, $P_n(D)m_{u_n} = (a_0)_n m_{u_n}$ if $(a_0)_n \neq 0$, while $P_n(D)m_{u_n} = 0$ for $(a_0)_n = 0$. This means that equation (6.6) will have no solutions if $(a_0)_n = 0$ and $m_{f_n} \neq 0$. If $(a_0)_n = 0$ and $m_{f_n} = 0$, then m_{u_n} can be taken as an arbitrary constant from $\mathcal{E}_M$. Finally, if $(a_0)_n \neq 0$, then from

$$P(D)m_{u_n} = (a_0)_n m_{u_n} = m_{f_n}$$

we obtain

$$m_{u_n} = \frac{m_{f_n}}{(a_0)_n}.$$

Thus, our claim (6.3) follows.

(b) Taking expectations on both sides in the equation

$$P_n(D_x)P_n(D_y)u_n(\omega, x)u_n(\omega, y) = f_n(\omega, x)f_n(\omega, y), \;\; \omega \in \Omega, \; x, y \in \mathbb{R}^d, \; n \in \mathbb{N},$$

we obtain

$$P_n(D_x)P_n(D_y)B_{u_n}(x,y) = B_{f_n}(x,y), \quad x,y \in \mathbb{R}^d, \ n \in \mathbb{N}. \tag{6.7}$$

Since we seek for u stationary, from Theorem 5.5.2 it follows that

$$B_{u_n}(x,y) = B_{u_n}(x-y), \quad x,y \in \mathbb{R}^d, \ n \in \mathbb{N}.$$

Therefore, we rewrite equation (6.7) in the form (6.4) and $[(B_{u_n})_n] \in \mathcal{G}_\mathcal{T}(\mathbb{R}^{2d})$.

Applying the Fourier transform to (6.4), we obtain (6.5). Since $[(B_{f_n})_n]$ is a generalized correlation function of $f = [(f_n)_n]$ and thus positive definite, $\hat{B}_{f_n}$ is a positive distribution for all $n \in \mathbb{N}$. From (6.5) it follows that $P_n(\xi)P_n(-\xi)$ must be non-negative in order that $\hat{B}_{u_n}(\xi)$ can be a positive distribution. By the Bochner theorem, $B_{u_n}(x)$ will be a positive-definite function.

(c) First we fix $\omega \in \Omega$. CSPs possess smooth regular paths on the representative level, which enables one to construct pathwise solutions i.e. solutions for any fixed realization $\omega \in \Omega$.

Applying the Fourier transform to (6.1) we obtain

$$P_n(\xi)\hat{u}_n(\omega,\xi) = \hat{f}_n(\omega,\xi),$$

and since the polynomial $P_n(\xi)$ has no real zeros, we obtain

$$\hat{u}_n(\omega,\xi) = \frac{1}{P_n(\xi)}\hat{f}_n(\omega,\xi),$$

i.e.,

$$u_n(\omega,x) = S_n(x) * f_n(\omega,x), \quad \omega \in \Omega, \ x \in \mathbb{R}^d, \ n \in \mathbb{N}, \tag{6.8}$$

as the solution to equation (6.1), where

$$S_n(x) = \mathcal{F}^{-1}\left(\frac{1}{P_n(\xi)}\right)(x).$$

Clearly we have

$$|P_n(\xi)| \leq Cn^s(1+|\xi|)^k$$

for some $C > 0$ and $s > 0$. Then from the assumption

$$|P_n(\xi)| \geq Cn^{-r}(1+|\xi|)^k$$

we have that $\frac{1}{P_n(\xi)}$ is in $\mathcal{O}_M(\mathbb{R}^d)$ and $S_n = \mathcal{F}^{-1}\left(\frac{1}{P_n(\xi)}\right)$ is in $\mathcal{O}'_c(\mathbb{R}^d)$.

The sequence $(u_n)_n$ is moderate, since $S_n \in \mathcal{O}'_c(\mathbb{R}^d)$; see [Hor66].

We continue to investigate the stationarity of the solution. Since $P_n(\xi) \neq 0$ implies $\tilde{a}_0 \neq \tilde{0}$, the expectation of u_n is given by

$$m_{u_n} = \frac{m_{f_n}}{(a_0)_n}$$

as stated in (a). Indeed, from (6.8) one can also derive by Fubini's theorem that

$$\begin{aligned} m_{u_n} &= S_n * E(f_n) = m_{f_n} S_n * 1 \\ &= m_{f_n} \int_{\mathbb{R}^d} S_n(x)\, dx = m_{f_n} \hat{S}_n(0) \\ &= m_{f_n} \frac{1}{P_n(0)} = \frac{m_{f_n}}{(a_0)_n}. \end{aligned}$$

From (6.8) we obtain

$$u_n(\omega, x) u_n(\omega, y) = S_n(x) * f_n(\omega, x) \cdot S_n(y) * f_n(\omega, y)$$

and taking expectations and applying Fubini's theorem we get

$$E(u_n(\cdot, x) u_n(\cdot, y)) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_n(\xi) S_n(\eta) E(f(\cdot, x-\xi) f(\cdot, y-\eta)\, d\xi\, d\eta,$$

i.e.

$$B_{u_n}(x, y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_n(\xi) S_n(\eta) B_{f_n}(x-\xi, y-\eta) d\xi\, d\eta.$$

Since f is stationary,

$$B_{f_n}(x-\xi, y-\eta) = B_{f_n}(x-\xi-(y-\eta))$$

and we may apply the change of variables $\sigma = \xi - \eta$, $\tau = \eta$, to obtain

$$\begin{aligned} B_{u_n}(x-y) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_n(\xi) S_n(\eta) B_{f_n}(x-\xi-(y-\eta))\, d\xi\, d\eta \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_n(\sigma+\tau) S_n(\tau) B_{f_n}(x-y-\sigma)\, d\sigma\, d\tau \\ &= \int_{\mathbb{R}^d} (\check{S}_n * S_n)(\sigma) * B_{f_n}(x-y-\sigma)\, d\sigma \\ &= (\check{S}_n * S_n * B_{f_n})(x-y), \end{aligned}$$

where $\check{S}_n(\tau) = S_n(-\tau)$. Taking $z = x - y$ we obtain

$$B_{u_n}(z) = (\check{S}_n * S_n * B_{f_n})(z).$$

This is in compliance with (6.4) in (b). Taking the Fourier transform we obtain (6.5). According to the assumption in (c), $P_n(\xi)P_n(-\xi)$ is positive and bounded away from zero. It follows that

$$\hat{B}_{u_n}(\xi) = \frac{\hat{B}_{f_n}(\xi)}{P_n(\xi)P_n(-\xi)}, \quad n \in \mathbb{N},$$

is a positive distribution and hence $[(B_{u_n})_n]$ is a positive-definite generalized function. Thus, u_n given by (6.8) is weakly stationary. ■

In Section 6.2 we will apply the method developed here to solve the stationary Klein–Gordon equation driven by higher order derivatives of white noise.

6.1.2 Some Remarks on the Existence of a Solution

Let

$$P_n(\xi)P_n(-\xi) \geq 0, \ \ \xi \in \mathbb{R}^d, \ \ n \in \mathbb{N},$$

for some representative of the coefficients $(a_\alpha)_n$. Let

$$N = \{\xi \in \mathbb{R}^d : P_n(\xi) = 0\}, \quad n \in \mathbb{N},$$

and

$$V = \{\xi \in \mathbb{R}^d : P_n(\xi)P_n(-\xi) = 0\}, \quad n \in \mathbb{N}.$$

The sets N and V are assumed to be same for all $n \in \mathbb{N}$. Assume that

$$P\{\omega \in \Omega : \hat{f}_n(\omega, \xi) = 0, \xi \in N\} = 1$$

and

$$\hat{B}_{f_n}(\xi) = 0, \ \ \xi \in V, \ \ n \in \mathbb{N}.$$

Then, for the existence of a solution one needs to consider the problem of division with $P(\xi)P(-\xi)$ which is highly non-trivial. It is an old and classical result of Łojasiewicz, cf. Hörmander [Hör63]. In the case of Colombeau generalized functions, this is solved in [PS96] (see also [OPS03] for the general question of extending distributions out of a set O). This question has not been considered in the dissertation.

Let us continue with the same notation. If there exists a stationary CSP $u = [(u_n)_n] \in \mathcal{G}_{\mathcal{T},L^2}(\Omega, \mathbb{R}^d)$ as a solution to equation (6.1), then from Theorem 6.1.1 it follows that its generalized expectation and generalized correlation function are given by

$$\tilde{m}_u = \frac{\tilde{m}_f}{\tilde{a}_0} \tag{6.9}$$

and

$$P_n(\xi)P_n(-\xi)\hat{B}_{u_n}(\xi) = \hat{B}_{f_n}(\xi), \quad \xi \notin V, \ \ n \in \mathbb{N}. \tag{6.10}$$

We will illustrate in the following example that in some special cases one may still find a weakly stationary Gaussian solution. Notice that the solution does not have to be unique.

Example 6.1.1 *We illustrate the case when the sets of points ξ for which $P(\xi)P(-\xi) = 0$ and for which $\hat{B}_{f_n}(\xi) = 0$ coincide, $n \in \mathbb{N}$.*

Let $d = 1$ and consider the equation

$$\left(1 + \frac{d^2}{dx^2}\right) u = f,$$

where $f = [(f_n)_n]$ is a stationary Gaussian CSP with zero generalized expectation and generalized correlation function

$$B_{f_n}(x-y) = \varphi_n(x-y) + 2\varphi_n^{(2)}(x-y) + \varphi_n^{(4)}(x-y), \quad x, y \in \mathbb{R}, n \in \mathbb{N}.$$

Then

$$P(\xi) = 1 - \xi^2$$

and

$$P(\xi)P(-\xi) = (1-\xi^2)^2.$$

Clearly, $V = \{-1, 1\}$. Applying the Fourier transformation to the correlation function we obtain

$$\hat{B}_{f_n}(\xi) = (1 - 2\xi^2 + \xi^4)\hat{\varphi}_n(\xi) = \hat{\varphi}_n(\xi), \quad \xi \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Thus, we have

$$(1-\xi^2)^2 \hat{B}_{u_n}(\xi) = (1-\xi^2)^2 \hat{\varphi}_n(\xi), \quad \xi \in \mathbb{R} \setminus V, \quad n \in \mathbb{N}, \tag{6.11}$$

and since $\hat{\varphi}_n(\xi)$ is continuous on $\mathbb{R}$, we may extend representatives of the generalized correlation function of the solution to be

$$\hat{B}_{u_n}(\xi) = \hat{\varphi}_n(\xi), \quad \xi \in \mathbb{R}, \quad n \in \mathbb{N},$$

i.e.

$$B_{u_n}(x-y) = \varphi_n(x-y), \quad x, y \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Thus, u is a GCSP with zero generalized expectation and generalized correlation function associated to the Dirac delta function. This means that the white noise GCSP is a solution to the given equation.

This solution is not unique. Observe that

$$\hat{B}_{v_n}(\xi) = \hat{B}_{u_n}(\xi) + \delta(\xi - 1) + \delta(\xi + 1), \quad n \in \mathbb{N},$$

also satisfies equation 6.11. Therefore, a GCSP $v = [(v_n)_n]$ with zero generalized expectation and generalized correlation function given by

$$B_{v_n}(x-y) = B_{u_n}(x-y) + 2\cos(x-y), \quad n \in \mathbb{N},$$

is a solution the given equation as well. □

6.2 The stationary Klein–Gordon equation driven by higher order derivatives of white noise

Let us illustrate the method described in the previous section on the following equation

$$(\tilde{1} - \Delta_x)u(\omega, x) = \tilde{c} + \tilde{f} \cdot \partial_x^k w(\omega, x), \ \ \omega \in \Omega, \ x \in \mathbb{R}^d, \tag{6.12}$$

where $\tilde{1} = (1, 1, 1, \ldots)$, $\tilde{c}, \tilde{f} \in \mathcal{R}_c$ are generalized constants and $w = [(w_n)_n]$ is the white noise GCSP with zero generalized expectation and generalized correlation function

$$B_{w_n}(x, y) = \varphi_n(x - y), \quad x, y \in \mathbb{R}^d, \ \ n \in \mathbb{N}.$$

It is known from Theorem 4.2.1 that all derivatives of a GCSP are also GCSPs. Therefore, the process on the right hand side of equation (6.12)

$$g(\omega, x) = \tilde{c} + \tilde{f} \cdot \partial_x^k w(\omega, x)$$

is a stationary GCSP. From Proposition 5.1.1 it follows that a GCSP $g = [(g_n)_n]$ has generalized expectation

$$[(m_{g_n})_n] = [(c_n)_n] = \tilde{c} \in \mathcal{R}_c$$

and generalized correlation function

$$[(B_{g_n}(x, y))_n] = [(c_n^2 + f_n^2 \partial_x^k \partial_y^k \varphi_n(x - y))_n], \quad x, y \in \mathbb{R}^d, \ \ n \in \mathbb{N}.$$

Equation (6.6) reduces to

$$(1 - \Delta_x)m_{u_n} = c_n, \quad x \in \mathbb{R}^d, \ n \in \mathbb{N}, \tag{6.13}$$

and from this follows $m_{u_n} = c_n$. Therefore,

$$[(m_{u_n})_n] = [(c_n)_n] = \tilde{c} \in \mathcal{R}_c.$$

Equation (6.7) reduces to

$$(1 - \Delta_x)(1 - \Delta_y)B_{u_n}(x - y) = c_n^2 + f_n^2 \ \partial_x^k \partial_y^k \varphi_n(x - y), \ \ x, y \in \mathbb{R}^d, \ n \in \mathbb{N}.$$

i.e. after the change of variables $x - y = z$

$$(1 - \Delta)^2 B_{u_n}(z) = c_n^2 + f_n^2 (-1)^k \partial_z^{2k} \varphi_n(z), \ \ z \in \mathbb{R}^d, n \in \mathbb{N}. \tag{6.14}$$

Applying the Fourier transformation to (6.14) we obtain

$$(1+\|\xi\|^2)^2\hat{B}_{u_n}(\xi) = c_n^2(2\pi)^{d/2}\delta(\xi) + f_n^2\xi^{2k}\hat{\varphi}_n(\xi), \;\; \xi\in\mathbb{R}^d,\, n\in\mathbb{N}.$$

Clearly, the condition of Theorem 6.1.1 (c) holds and the right hand side is also positive. Now we have

$$\begin{aligned} B_{u_n}(z) &= c_n^2(2\pi)^{-d/2} * \mathcal{F}^{-1}\left(\frac{1}{(1+\|\xi\|^2)^2}\right)(z) \\ &\quad + f_n^2(2\pi)^{-d/2}\varphi_n * \mathcal{F}^{-1}\left(\frac{\xi^{2k}}{(1+\|\xi\|^2)^2}\right)(z), \;\; z\in\mathbb{R}^d, \end{aligned}$$

which can be expressed as

$$B_{u_n}(z) = c_n^2(2\pi)^{-d} * b^{*2}(z) + f_n^2(2\pi)^{-d}(-1)^k\left(\partial_z^k b\right)^{*2} * \varphi_n(z), \;\; z\in\mathbb{R}^d,$$

where

$$b(z) = \mathcal{F}^{-1}\left(\frac{1}{1+\|\xi\|^2}\right)(z) = 2\pi^{-d/2}\|z\|^{1-d/2}K_{d/2-1}(\|z\|), \quad z\in\mathbb{R}^d,$$

is the fundamental solution of $1-\Delta_x$ vanishing at infinity (see [Ort80], p. 128), expressed in terms of the modified Bessel function $K_{d/2}-1$.

7 Conclusion

Science is a beautiful gift to humanity, we should not distort it.

— **A. P. J. Abdul Kalam (1931-2015)**
(11th President of India)

7.1 Overview

The main aims of research were probabilistic properties of CSPs and applications in equation solving. Observe that all aims have been achieved. Therefore, this dissertation brings relevant contributions to the theory of CSPs.

The measurability of the CSPs has been proven and the transfer of probabilistic arguments into the Colombeau setting is enabled. The generalized expectation, the generalized correlation function, the generalized characteristic function were considered. Independence and stationarity of CSPs were studied.

In the natural science and engineering linear and nonlinear stochastic differential equations often appear. Therefore, there is a great need to find different methods for solving them. In this doctoral dissertation we have presented the method of solving a class of SPDEs in the framework of a stationary Gaussian tempered CSPs.

7.2 Future work

In this section we give a few directions for future research.

We gave a structural characterization of the generalized correlation function of CSPs with values in $L^2(\Omega)$. A structural characterization of the pth-order moment of CSPs with values in $L^p(\Omega)$ can be given.

If X and Y are independent random variable, then $L_{X+Y} = L_X \cdot L_Y$. In [GV64], a necessary and sufficient condition for a functional to be the characteristic functional of a GSP with independent values. Is it possible to carry out the study of CSPs with independent values with the help of their generalized characteristic functions?

In Chapter 6, the problem of division with $P(\xi)P(-\xi)$ is left for future work. Further application to SPDEs remain as possibilities for future investigation.

A An Overview of Background Theory

In this appendix chapter some basic definitions and theorems of fundamental theories, which are used in the previous chapters of the dissertation are presented. Theorems presented here are familiar and therefore given without proofs but with references for further reading.

A.1 Real and Functional Analysis

The relevant notions and theorems of real and functional analysis are briefly recalled in this section. The reader is referred to [AK95; AK99; Rud06; Kur90; MV97] for proofs of theorems.

A.1.1 Real Analysis: Basic Concepts and Theorems

In this subsection we recall three basic theorems of real analysis: Leibniz's rule, mean value theorem and Taylor's theorem.

Theorem A.1.1 (Leibniz's rule) *If f and g are n-times differentiable functions, then the product fg is n-times differentiable and*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

With the multi-index notation (see Chapter B, Section B.1), the Leibniz's rule has the following form

$$D^{\alpha}(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} f D^{\beta} g.$$

Theorem A.1.2 (Mean value theorem) *Let $O \subseteq \mathbb{R}^d$ be open, $a = (a_1, a_2, \dots, a_d) \in O$ and $h = (h_1, h_2, \dots, h_d) \in \mathbb{R}^d$ such that $[a, a+h] = \{x : x = a+th, 0 \leq t \leq 1\} \subseteq O$. If $f : O \to \mathbb{R}$ is continuous in all points in line segment $[a, a+h]$ and differentiable in all points in line segment $(a, a+h) = \{x : x = a+th, 0 < t < 1\}$, then there exist point $c = a + \theta h$, $0 < \theta < 1$ in $(a, a+h)$ such that*

$$f(a+h) - f(a) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(a_1 + \theta h_1, a_2 + \theta h_2, \dots, a_d + \theta h_d) h_i.$$

Theorem A.1.3 (Taylor's theorem) *Let O be an open subset of $\mathbb{R}^d$. Let $f : O \to \mathbb{R}$ be $(n+1)$-times continuously differentiable function and let $x \in O, h \in \mathbb{R}^d$ such that $\{x + th : 0 \le t \le 1\} \subseteq O$. Then there exist $\theta \in [0, 1]$ such that*

$$f(x+h) = \sum_{|\alpha| \le n} \frac{D^\alpha f(x)}{\alpha!} h^\alpha + \sum_{|\alpha| = n+1} \frac{D^\alpha f(x + \theta h)}{\alpha!} h^\alpha.$$

A.1.2 Functional Analysis: Basic Concepts and Theorems

A *metric* on the nonempty set M is a function $d : M \times M \to \mathbb{R}_+$ with the following properties:

(M1) $d(x, y) = 0$ if and only if $x = y$,

(M2) $d(x, y) = d(y, x)$ for all $x, y \in M$ (symmetry),

(M3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in M$ (triangle inequality).

The pair (M, d) is called a *metric space*.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (M, d) is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $m, n > n_0$. A convergent sequence is a Cauchy sequence. A Cauchy sequence converges if and only if it contains a convergent subsequence. A metric space (M, d) is said to be *complete* if every Cauchy sequence in (M, d) is convergent.

Theorem A.1.4 (Baire's theorem) *Let (M, d) be a complete metric space. If C_n, $n \in \mathbb{N}$, are closed subsets of M such that*

$$\bigcup_{n \in \mathbb{N}} C_n = M,$$

then at least one of the sets C_n contains an open ball.

An *ultrametric* is a metric which satisfies the *strong triangle inequality*

$$d(x, z) \le \max\{d(x, y), d(y, z)\}$$

for all $x, y, z \in M$. If d is an ultrametric on M, then the pair (M, d) is called an *ultrametric space*.

A pseudometric on a set M is a mapping $d : M \times M \to \mathbb{R}_+$ with the following properties:

(PM1) $(\forall x, y \in M)(x = y \Rightarrow d(x, y) = 0)$;

(PM2) $(\forall x, y \in M)(d(x, y) = d(y, x))$;

(PM3) $(\forall x, y, z \in M)(d(x,z) \leq d(x,y) + d(y,z))$.

The pair (M, d) is called a *pseudometric space*. In case axiom (PM3) is replaced by a stronger axiom

(PM3') $(\forall x, y, z \in M)(d(x,z) \leq \max\{d(x,y), d(y,z)\})$

we come to the definition of an *ulta-pseudometric*. Clearly, every ultra-pseudometric is a pseudometric, but not vice-versa.

Let M be a metric space and $x_0 \in M$. We say that $f :\to \overline{\mathbb{R}}$ is *upper semi-continuous* (resp. *lower semi-continuous*) at x_0 if

$$\limsup_{x \to x_0} f(x) \leq f(x_0) \quad (\text{resp. } \liminf_{x \to x_0} f(x) \geq f(x_0)).$$

The function f is called *upper semi-continuous* (resp. *lower semi-continuous*) if it is upper semi-continuous (resp. lower semi-continuous) at every point of its domain. Let $f_i : M \to \overline{\mathbb{R}}$ be a linear lower semi-continuous (resp. upper semi-continuous) function for every index i in a nonempty set I, and define

$$f(x) = \sup_{i \in I} f_i(x) \quad (\text{resp. } f(x) = \inf_{i \in I} f_i(x)), \quad x \in M.$$

Then f is lower semi-continuous. (resp. upper semi-continuous). A function f is lower semi-continuous if and only if $\{x \in M : f(x) > a\}$ is an open set for every $a \in \mathbb{R}$. A function f is lower semi-continuous if and only if $\{x \in M : f(x) \leq a\}$ is closed for every $a \in \mathbb{R}$.

If u is continuous function on $\mathbb{R}^d$ with compact support and v is continuous function on $\mathbb{R}^d$, then the convolution $u * v$ is the continuous function defined by

$$(u * v)(x) = \int_{\mathbb{R}^d} u(x - y) v(y)\, dy, \quad x \in \mathbb{R}^d.$$

A.2 Distribution Theory

In this section, the most important concepts and results of the Schwarz theory of generalized functions are presented. For deeper properties of Schwartz generalized functions, the reader is referred to [SP00; Hör63; Hör03; Vla79].

A.2.1 Spaces of Functions

Let us recall the definition of some function spaces needed in the dissertation.

Let $O \subseteq \mathbb{R}^d$ be an open set. The space of smooth compactly supported functions on O is denoted by $\mathcal{D}(O)$. A sequence $(\phi_n)_{n \in \mathbb{N}} \in \mathcal{D}(O)$ is said to *converge to zero in* $\mathcal{D}(O)$ if

(i) there is $K \Subset O$ such that $\operatorname{supp} \phi_n \subset K$, and

(ii) for each $\alpha \in \mathbb{N}_0^d$ the $\partial^\alpha \phi_n$ converge to zero uniformly as $n \to \infty$.

We say that $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ is *rapidly decreasing function*, if for all $\alpha, \beta \in \mathbb{N}_0^d$

$$\gamma_{\alpha,\beta}(\phi) = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta \phi(x)| < \infty.$$

The space of rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. A sequence $(\phi_n)_n \in \mathcal{S}(\mathbb{R}^d)$ is said to converge to ϕ in $\mathcal{S}(\mathbb{R}^d)$ if for every $\alpha, \beta \in \mathbb{N}_0^d$ it holds $\gamma_{\alpha,\beta}(\phi_n - \phi) \to 0$ as $n \to 0$.

By $\mathcal{O}_C(O)$ we denote the space of smooth functions with the following property: there exists $p \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^d$

$$\sup_{x \in O} (1 + |x|)^{-p} |\partial^\alpha f(x)| < \infty$$

holds.

Let $\mathcal{O}_M(O)$ be the space of functions with the following property: for every $\alpha \in \mathbb{N}_0^d$, there exists $p \in \mathbb{N}$ such that

$$\sup_{x \in O} (1 + |x|)^{-p} |\partial^\alpha f(x)| < \infty.$$

A.2.2 Schwartz Generalized Functions

Let $O \subseteq \mathbb{R}^d$ be an open set. A *Schwartz generalized function* or *Schwartz distribution* u on O is a linear functional on $\mathcal{D}(O)$, i.e. $u : \mathcal{D}(O) \to \mathbb{C}$ is linear with the following property:

$$\phi_n \to 0 \text{ in } \mathcal{D}(O) \Rightarrow u(\phi_n) \to 0 \text{ in } \mathbb{C}.$$

The space of Schwartz distributions on O is denoted by $\mathcal{D}'(O)$. We usually write $\langle u, \phi \rangle$ instead of $u(\phi)$.

Let $u : \mathcal{D}(O) \to \mathbb{C}$ be linear. Then $u \in \mathcal{D}'(O)$ if and only if for every compact set $K \Subset O$ there exist $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \phi| \tag{A.1}$$

for every $\phi \in \mathcal{D}(K)$. A Schwartz distribution $u \in \mathcal{D}'(O)$ is said to be *of finite order* if the same $k \in \mathbb{N}_0$ can be used in the estimate (A.1) for every K. The minimal $k \in \mathbb{N}_0$ satisfying the estimate (A.1) is called the *order of the Schwartz distribution*.

The space of continuous functions on O can be identified with a subspace of $\mathcal{D}'(O)$ by assigning

$$\mathcal{C}(O) \ni f \mapsto \int f(x)\phi(x)\,dx \in \mathcal{D}'(O)$$

for all $\phi \in \mathcal{D}(O)$. More generally we can embed the space $L^1_{loc}(O)$ into $\mathcal{D}'(O)$. Hence, we have $\mathcal{C}(O) \subseteq L^1_{loc}(O) \subseteq \mathcal{D}'(O)$.

If u is a *positive Schwartz distribution* on O, i.e. $\langle u, \phi\rangle \geq 0$ for all non-negative $\phi \in \mathcal{D}(O)$, then u is a positive measure.

Let $O' \subset O \subset \mathbb{R}^d$ and $u \in \mathcal{D}'(O)$. The restriction of u to O' is the Schwartz distribution $v \in \mathcal{D}'(O)$ defined by $\langle v, \phi\rangle = \langle u, \phi\rangle$, for all $\phi \in \mathcal{D}(O')$. The *support* of $u \in \mathcal{D}'(O)$ is

$$\operatorname{supp} u = \{x \in O : u = 0 \text{ in some open neighborhood of } x\}^c.$$

A *Schwartz distribution on O with compact support* is a linear functional on $\mathcal{C}^\infty(O)$, i.e. $u : \mathcal{C}^\infty(O) \to \mathbb{C}$ is linear with the following property:

$$\phi_n \to \phi \text{ in } \mathcal{C}^\infty(O) \Rightarrow \langle u, \phi_n\rangle \to \langle u, \phi\rangle \text{ in } \mathbb{C}.$$

The space of Schwartz distribution on O with compact support is denoted by $\mathcal{E}'(O)$.

Let $u : \mathcal{C}^\infty(O) \to \mathbb{C}$ be linear. Then $u \in \mathcal{E}'(O)$ if and only if there exist a compact set $K \Subset O$, $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\langle u, \phi\rangle| \leq C \sum_{|\alpha|\leq k} \|\partial^\alpha \phi\|_{L^\infty(K)}$$

for every $\phi \in \mathcal{C}^\infty(O)$.

A.2.3 Structure of Schwartz distributions

It is known [Hör03], that all Schwartz distributions are in fact of the form

$$\phi \mapsto \sum \int f_\alpha \partial^\alpha \phi\,dx, \quad \phi \in \mathcal{D}(O),$$

where the sum is locally finite, i.e. on every compact set there are only a finite number of continuous functions f_α which do not vanish identically. For a proof of the following theorem see [Hör03] (Theorem 2.3.5, p. 47).

Theorem A.2.1 *Let $x = (x', x'')$ be a splitting of the variables in $\mathbb{R}^d$ in two groups and let u be a Schwartz distribution in $\mathbb{R}^d$ of order k with compact support contained in the plane $x' = 0$. Then*

$$\langle u, \phi \rangle = \sum_{|\alpha| \le k} \langle u_\alpha, \phi_\alpha \rangle,$$

where u_α is a Schwartz distribution of compact support and order $k - |\alpha|$ in the x'' variables, $\alpha = (\alpha', 0)$ and

$$\phi_\alpha(x'') = \partial^\alpha \phi(x', x'')\Big|_{x'=0}.$$

Denote by $K(a)$ the space of all infinitely differentiable functions $\phi(x)$ defined in $\mathbb{R}^d$ and with support in the domain $G_a = \{|x_1| \le a_1, |x_2| \le x_2, \ldots, |x_d| \le a_d\}$, where $a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$. The general form of the functional in the space $K(a)$ is the following:

$$\langle u, \phi \rangle = \sum_{|\alpha| \le k} u_\alpha(x) D^\alpha \phi(x)\, dx,$$

where $u_\alpha(x)$ are bounded measurable functions in the domain $\{|x| \le a\}$. Integrating by parts, we obtain

$$\langle u, \phi \rangle = \int_{|x| \le a} U(x) D^\beta \phi(x)\, dx,$$

where $U(x)$ is a continuous function in G_a. The last expression can be written in the following form

$$\langle u, \phi \rangle = \pm \langle D^\beta U(x), \phi(x) \rangle.$$

Therefore, each Schwartz distribution in space $K(a)$ is the derivative of some continuous function. The reader is referred to [GS86] for more details.

A.2.4 Tempered Schwartz Generalized Functions

A *tempered Schwartz generalized function* or *tempered Schwartz distribution* on $\mathbb{R}^d$ is continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$, i.e. $u : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is linear with the following property:

$$\phi_n \to 0 \text{ in } \mathcal{S}(\mathbb{R}^d) \Rightarrow \langle u, \phi_n \rangle \to 0 \text{ in } \mathbb{C}.$$

The space of tempered Schwartz distributions on $\mathbb{R}^d$ is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Let $u : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ be linear. Then $u \in \mathcal{S}'(\mathbb{R}^d)$ if and only if there exist $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\langle u, \phi \rangle| \le C \sum_{|\alpha|, |\beta| \le k} \|x^\alpha D^\beta \phi\|_{L^\infty}.$$

A.2.5 The Fourier Transform

The *Fourier transform* of $u \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^d,$$

where $x \cdot \xi = \sum_{i=1}^{d} x_i \cdot \xi_i$. The *inverse Fourier transform* of $u \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$\mathcal{F}^{-1}u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} u(\xi)\, d\xi, \quad x \in \mathbb{R}^d.$$

The Fourier transform and the inverse Fourier transform are linear mappings of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$.

If $u \in \mathcal{S}(\mathbb{R}^d)$, then

$$\mathcal{F}(D_x^\alpha u) = \xi^\alpha \mathcal{F}(u).$$

Moreover, if $P(D)$ is a partial differential operator with constant coefficients, then

$$\mathcal{F}(P(D)u) = P(\xi)\mathcal{F}(u).$$

If $u \in \mathcal{S}(\mathbb{R}^d)$, then for every $h \in \mathbb{R}^d$

$$\mathcal{F}(\tau_h u) = e^{-ih\xi}\mathcal{F}(u),$$

where $(\tau_h u)(x) = u(x-h)$ is the translation by $h \in \mathbb{R}^d$. If $u, v \in \mathcal{S}(\mathbb{R}^d)$, then

$$\mathcal{F}(u * v) = (2\pi)^{d/2}\hat{u}\hat{v}.$$

Let $u, v \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} u(x)\overline{v(x)}\, dx = \int_{\mathbb{R}^d} \mathcal{F}u(\xi)\overline{\mathcal{F}v(\xi)}\, d\xi.$$

Moreover,

$$\int_{\mathbb{R}^d} |u(x)|^2\, dx = \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2\, d\xi.$$

Theorem A.2.2 (Plancherel-Parseval) *If $u, v \in L^2(\mathbb{R}^d)$, then*

1. $(u, v)_{L^2} = (\mathcal{F}u, \mathcal{F}v)_{L^2}$,
2. $\|u\|_{L^2} = \|\mathcal{F}u\|_{L^2}$.

As consequence of Plancherel-Parseval theorem, we obtain that the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ can be extended as isometry of $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

A.3 Measure and Probability Theory

This appendix section is devoted to some definitions and theorems of measure and probability theory that have been used in dissertation. The proofs of theorems can be found in [PS12], [RĆ09] and [Kal97].

A.3.1 Measure and Probability Spaces

Let Ω be a non-empty set. A *σ-algebra* in Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

(i) $\Omega \in \mathcal{F}$,

(ii) if $A \in \mathcal{F}$, then $A^c = \Omega \setminus A \in \mathcal{F}$, and

(iii) if $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called *$\mathcal{F}$-measurable space*. Elements of a σ-algebra $\mathcal{F}$ are called *$\mathcal{F}$-measurable sets*. A σ-algebra $\mathcal{G}$ is a *sub-σ-algebra* of $\mathcal{F}$ if $\mathcal{G} \subseteq \mathcal{F}$, i.e. $A \in \mathcal{G}$ implies $A \in \mathcal{F}$.

For any family $\mathcal{M}$ of subsets of Ω, the smallest σ-algebra $\mathcal{B}_{\mathcal{M}}$ containing $\mathcal{M}$, i.e.

$$\mathcal{B}_{\mathcal{M}} = \bigcap \{\mathcal{B} | \mathcal{B} \text{ is } \sigma\text{-algebra of subsets of } \Omega,\ \mathcal{M} \subseteq \mathcal{B}\},$$

is a *σ-algebra generated by $\mathcal{M}$*. The *Borel σ-algebra* of subsets of $\mathbb{R}^d$, denoted by $\mathcal{B}(\mathbb{R}^d)$ or $\mathcal{B}$, is the smallest σ-algebra of subsets of $\mathbb{R}^d$ that contains all open sets. The elements od $\mathcal{B}(\mathbb{R}^d)$ are called *Borel sets*.

Let $(\Omega, \mathcal{F})$ be a measurable space. A *measure* on $(\Omega, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to [0, \infty]$ which satisfies: If $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, are mutually disjoint sets in $\mathcal{F}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad (\sigma - \text{additivity}).$$

The triple $(\Omega, \mathcal{F}, \mu)$ is called *measure space*.

Proposition A.3.1 *Let μ be a non-trival measure on a measurable space $(\Omega, \mathcal{F})$.*

1. $\mu(\emptyset) = 0$.
2. *If $A_i \in \mathcal{F}$, $i = 1, 2, \ldots, n$, are mutually disjoint, then* $\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_n)$.
3. *If $A, B \in \mathcal{F}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*
4. *If $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, then* $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Theorem A.3.1 (Continuity of measure) *Let μ be a non-trivial measure on a measurable space $(\Omega, \mathcal{F})$, and suppose that $A_i \in \mathcal{F}$, $i \in \mathbb{N}$.*

1. *If $A_i \subseteq A_{i+1}$, $i \in \mathbb{N}$, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$.*
2. *If $A_i \supseteq A_{i+1}$, $i \in \mathbb{N}$, and $\mu(A_1) < \infty$, then $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$.*

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. All sets $N \in \mathcal{F}$ with $\mu(N) = 0$ are called the *null sets*. Note that if $M \subset N$ and N is a null set, it does not follow that M is a null set because M may not be in $\mathcal{F}$. The space $(\Omega, \mathcal{F}, \mu)$ is a *complete measure space* if and only if any subset of a null set is a null set.

Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2, \ldots, n$, be a family of measure spaces. Their product is the space $(\Omega, \mathcal{F}, \mu)$, where $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$, $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \ldots \otimes \mathcal{F}_n$ is the smallest σ-algebra containing all sets of the form $A_1 \times A_2 \times \ldots \times A_n$ for which each $A_i \in \mathcal{F}_i$, and $\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_n$ is the product measure given by

$$\mu(A_1 \times A_2 \times \ldots \times A_n) = \prod_{i=1}^{n} \mu(A_i)$$

on the sets $A_1 \times A_2 \times \ldots \times A_n$.

Any measure μ on $(\Omega, \mathcal{F})$ with $\mu(\Omega) = 1$ is called a *probability measure* and it is denoted by P. The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*. Note that Ω represents the set of outcomes of some random experiment. Elements of a σ-algebra $\mathcal{F}$ are called *events*.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A proposition $\mathcal{P}$ about the elements of Ω is said to hold *almost everywhere* (abbreviated as a.e.) with respect to μ if

$$A = \{\omega \in \Omega : \mathcal{P}(\omega) \text{ is false}\} \in \mathcal{F}$$

and $\mu(A) = 0$. In the case of a probability measure, we say *almost surely* (abbreviated as a.s.) instead of almost everywhere. We say that *almost all* (abbreviated as a.a.) elements of set A have a certain property if the subset of A for which the property fails has measure zero.

A.3.2 Measurable Functions

In this subsection, we recall the properties of measurable functions.

Let $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$, be measurable spaces. A mapping $u : \Omega_1 \to \Omega_2$ is said to be *$(\mathcal{F}_1, \mathcal{F}_2)$-measurable* or *measurable* if $u^{-1}(A) \in \mathcal{F}_1$ for all $A \in \mathcal{F}_2$. If $\Omega_1 \subseteq \mathbb{R}^{d_1}$ and $\Omega_2 \subseteq \mathbb{R}^{d_2}$ and $\mathcal{F}_i = \mathcal{B}(\Omega_i)$, $i = 1, 2$, u is said to be *Borel measurable*. Note that we have two σ-algebras on Ω_1, namely the original σ-algebra $\mathcal{F}_1$ and the σ-algebra $u^{-1}(\mathcal{F}_2) = \{u^{-1}(A) : A \in \mathcal{F}_2\}$ induced by the mapping u. Hence, u is measurable if the induced σ-algebra $u^{-1}(\mathcal{F}_2)$ is a sub-σ-algebra of $\mathcal{F}_1$.

Theorem A.3.2 *Let Ω_1 and Ω_2 be metric spaces. Every continuous mapping $u : \Omega_1 \to \Omega_2$ is $(\mathcal{B}(\Omega_1), \mathcal{B}(\Omega_2))$-measurable.*

Theorem A.3.3 *Let $(\Omega, \mathcal{F})$ be a measurable space and $u : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The following are equivalent:*

1. *u is measurable.*
2. *$u^{-1}((a, \infty)) \in \mathcal{F}$ for all $a \in \mathbb{R}$.*
3. *$u^{-1}([a, \infty)) \in \mathcal{F}$ for all $a \in \mathbb{R}$.*
4. *$u^{-1}((-\infty, a)) \in \mathcal{F}$ for all $a \in \mathbb{R}$.*
5. *$u^{-1}((-\infty, a]) \in \mathcal{F}$ for all $a \in \mathbb{R}$.*

The following result says that measurability is inherited through composition.

Theorem A.3.4 *Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and $(\Omega_3, \mathcal{F}_3)$ be measurable spaces. If $u : \Omega_1 \to \Omega_2$ is $(\mathcal{F}_1, \mathcal{F}_2)$-measurable and $v : \Omega_2 \to \Omega_3$ is $(\mathcal{F}_2, \mathcal{F}_3)$-measurable, then the composition $u \circ v : \Omega_1 \to \Omega_3$ is $(\mathcal{F}_1, \mathcal{F}_3)$-measurable.*

As a consequence, we get the following assertion.

Proposition A.3.2 *Let $(\Omega, \mathcal{F})$ be a measurable space. If $u : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable and $v : \mathbb{R} \to \mathbb{R}$ is continuous then $u \circ v$ is $\mathcal{F}$-measurable.*

Let $(\Omega, \mathcal{F})$ be a measurable space. If $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ are $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F})$, then $u+v$, $u-v$ and $u \cdot v$ are $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F})$. In particular, if u is $\mathcal{F}$-measurable, then cu is $\mathcal{F}$-measurable function for every $c \in \mathbb{R}$. If $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}_+$ are $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F})$, then $\frac{u}{v}$ is $\mathcal{F}$-measurable function.

Recall, if $(u_n)_{n \in \mathbb{N}}$ be a sequence of $\mathcal{F}$-measurable functions on a measurable space $(\Omega, \mathcal{F})$ with values in $\overline{\mathbb{R}} = [-\infty, \infty]$, then

$$\limsup_{n \to \infty} u_n(x) = \inf_{k \in \mathbb{N}} \left(\sup_{n \geq k} u_n(x) \right)$$

and

$$\liminf_{n \to \infty} u_n(x) = \sup_{k \in \mathbb{N}} \left(\inf_{n \geq k} u_n(x) \right).$$

Theorem A.3.5 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$-measurable functions on a measurable space $(\Omega, \mathcal{F})$ with values in $\overline{\mathbb{R}}$. Then the functions $\sup_{n\in\mathbb{N}} u_n(x)$, $\inf_{n\in\mathbb{N}} u_n(x)$, $\limsup_{n\to\infty} u_n(x)$, $\liminf_{n\to\infty} u_n(x)$, $x \in \Omega$, are $\mathcal{F}$-measurable.*

The function $u : \Omega \to \overline{\mathbb{R}}$ on a measurable space $(\Omega, \mathcal{F})$ can be decomposed into its positive and negative parts

$$u(x) = u^+(x) - u^-(x), \quad x \in \Omega,$$

where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$. Note that u^+ and u^- are non-negative functions. If u is $\mathcal{F}$-measurable, then u^+ and u^- are $\mathcal{F}$-measurable. Hence, every measurable function can be written as the difference of two non-negative measurable functions. Note that $|u| = u^+ + u^-$. So, if u is $\mathcal{F}$-measurable, then $|u|$ is $\mathcal{F}$-measurable.

A.3.3 Integration

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

The *indicator function* of a set $A \in \mathcal{P}(\Omega)$ is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Note that χ_A is measurable if and only if $A \in \mathcal{F}$.

A measurable function $s : \Omega \to \mathbb{R}$ is called *simple function* if

$$s = \sum_{i=1}^{n} c_i \chi_{A_i},$$

for some $n \in \mathbb{N}$, where $c_i \in \mathbb{R}^d$ and $A_i \in \mathcal{F}$ for $1 \leq i \leq d$. The *Lebesgue integral* of s on $A \in \mathcal{F}$ is defined by

$$\int_A s \, d\mu = \sum_{i=1}^{n} c_i \mu(A \cap A_i).$$

Let $u : \Omega \to [0, \infty]$ be a $\mathcal{F}$-measurable function.

The *Lebesgue integral* of u on $A \in \mathcal{F}$ is

$$\int_A u \, d\mu = \sup \left\{ \int_A s \, d\mu : s \text{ simple measurable function and } 0 \leq s \leq f \right\}.$$

Finally if $u : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable and $A \in \mathcal{F}$, then the *Lebesgue integral* of u is defined by

$$\int_A u \, d\mu = \int_A u^+ \, d\mu - \int_A u^- \, d\mu,$$

provided at least one of the integrals on the right is finite.

Theorem A.3.6 (Monotone convergence theorem) *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative measurable functions on* $(\Omega, \mathcal{F}, \mu)$ *that is a.e. monotone increasing and converging pointwise to* u *a.e. Then*

$$\lim_{n\to\infty} \int_\Omega u_n \, d\mu = \int_\Omega u \, d\mu.$$

Theorem A.3.7 (Lebesgue's dominated convergence theorem) *If* $(u_n)_{n\in\mathbb{N}}$ *is a sequence of measurable functions on* $(\Omega, \mathcal{F}, \mu)$ *converging pointwise to* u *a.e. and* $g \geq 0$ *is an integrable function such that* $|u_n(x)| \leq g(x)$ *a.e. for all* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \int_\Omega u_n \, d\mu = \int_\Omega u \, d\mu.$$

Theorem A.3.8 (Fubini's theorem) *Let* $(\Omega_1, \mathcal{F}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{F}_2, \mu_2)$ *be measure spaces. If* $u : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *is* $\mathcal{F}_1 \otimes \mathcal{F}_2$*-measurable with*

$$\iint |u(x,y)| \, \mu_1(dx)\mu_2(dy) < \infty,$$

then

$$\begin{aligned}
\int_{\Omega_1\times\Omega_2} u(x,y)\,(\mu_1\times\mu_2)(dx,dy) &= \int_{\Omega_2}\left[\int_{\Omega_1} u(x,y)\,\mu_1(dx)\right]\mu_2(dy) \\
&= \int_{\Omega_1}\left[\int_{\Omega_2} u(x,y)\,\mu_2(dy)\right]\mu_1(dx).
\end{aligned}$$

The functions $y \to \int u(x,y)\,\mu_1(dx)$ *and* $x \to \int u(x,y)\,\mu_2(dy)$ *are defined* μ_2 *a.e. and* μ_1 *a.e., respectively.*

A.3.4 Random Varables. Expectation. Independence. Characteristic Function

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

A measurable mapping from Ω into $\mathbb{R}^d$ is called *random variable*. Random variables are usually denoted by $X, Y, Z, \ldots$.

Let $X : \Omega \to \mathbb{R}^d$ be a random variable. Then

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}\}$$

is a σ-algebra, called the *σ-algebra generated by X*. This is the smallest sub-σ-algebra of $\mathcal{F}$ with respect to which X is measurable. For a family of random variables

$(X_i)_{i\in I}$ on the same probability space $(\Omega, \mathcal{F})$, we denote by $\sigma(X_i, i \in I)$ the smallest σ-algebra contained in $\mathcal{F}$ with respect to which all the X_i are measurable.

If X is a random variable on $(\Omega, \mathcal{F}, P)$, its *distribution* or *law* is the probability measure P_X on $(\mathbb{R}^d, \mathcal{B})$ defined by

$$P_X(B) = P(X^{-1}(B)), \quad B \in \mathcal{B}.$$

In words, we pull $B \in \mathcal{B}$ back to $X^{-1}(B) \in \mathcal{F}$ and then take the probability measure P of that set. The associated *distribution function* $F_X : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$F_X(x) = P_X((-\infty, x_1] \times \ldots \times (-\infty, x_d]) = P\{\omega \in \Omega : X(\omega) \leq x\},$$

for each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Let (X, Y) be a random variable taking values in $\mathbb{R}^{2d}$. The distribution of (X, Y) is called the *joint distribution* of X and Y. The distributions P_X and P_Y are then called marginal distribution of (X, Y), where $P_X(A) = P_{(X,Y)}(A, \mathbb{R}^d)$ and $P_Y(A) = P_{(X,Y)}(\mathbb{R}^d, A)$, for every $A \in \mathcal{B}$.

A probability measure P_1 on $(\Omega, \mathcal{F})$ is said to be *absolutely continuous* with respect to probability measure P if $A \in \mathcal{F}$ and $P(A) = 0$ imply $P_1(A) = 0$. Then, we write $P_1 \ll P$.

Theorem A.3.9 (Radon-Nikodym theorem) *Let P and P_1 be two probability measures given on $(\Omega, \mathcal{F})$ such that $P_1 \ll P$. Then, there exists a unique measurable function $f : \Omega \to \mathbb{R}_+$ such that, for each $A \in \mathcal{F}$,*

$$P_1(A) = \int_A f \, dP.$$

Let X be a random variable with distribution P_X that is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. Then, the function appearing in the Radon-Nikodym theorem is denoted by f_X and it is called a *probability density function*. For example, the uniform distribution on $\Omega = [a, b]$ has the probability density function

$$f_X(x) = \frac{1}{b-a}, \quad x \in \mathbb{R}.$$

Two events A and B are *independent* if $P(A \cap B) = P(A) \cdot P(B)$. If A and B are independent events, then so are A^c and B, A and B^c, and A^c and B^c. Events A_i, $i \in \mathbb{N}$, are *independent* if

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \prod_{j=1}^{n} P(A_{i_j}),$$

for all choices of $1 \leq i_1 < i_2 < \ldots < i_k$. Let $(\mathcal{F}_i)_{i\in\mathbb{N}}$ be a sequence of sub-σ-algebras of $\mathcal{F}$. We say that a sequence $(\mathcal{F}_i)_{i\in\mathbb{N}}$ is *independent* if

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \prod_{j=1}^{n} P(A_{i_j}),$$

for all choices of $1 \leq i_1 < i_2 < \ldots < i_k$ and of $A_{i_j} \in \mathcal{F}_j$, $1 \leq j \leq k$. A sequence of random variables $(X_i)_{i\in\mathbb{N}}$ is said to be *independent* if $(\sigma(X_i))_{i\in\mathbb{N}}$ is independent.

The *expectation* of random variable X (when it exists) is given by

$$E(X) = \int_\Omega X\, dP(\omega) = \int_{\mathbb{R}^d} x\, dP_X(dx).$$

Note that if $A \in \mathcal{F}$, then we sometimes write $E(X, A) = E(X\chi_A)$.

For a random variable $X : \Omega \to \mathbb{R}^d$ and Borel measurable functions $u : \mathbb{R}^d \to \mathbb{R}^n$, it holds

$$E(u(X)) = \int_\Omega u(X(\omega))\, P(d\omega) = \int_{\mathbb{R}^n} u(x)\, P_X(dx),$$

if $u \circ X$ is integrable.

Theorem A.3.10 *If $X_i : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, n$ are independent random variables, with $E(|X_i|) < \infty$, $i = 1, 2, \ldots, n$, then $E(|X_1 \cdot X_2 \cdot \ldots \cdot X_n|) < \infty$ and*

$$E(X_1 \cdot \ldots \cdot X_n) = E(X_1) \cdot E(X_2) \cdot \ldots \cdot E(X_n).$$

The integral $E(X^p)$, $p \in \mathbb{N}$, is called the *pth moment* of X, when it exists. We say that X has moments to all orders if $E(|X|^p) < \infty$, for all $p \in \mathbb{N}$.

The set of all random variables with finite pth moment is denoted by $L^p(\Omega)$. The space $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(X, Y)_{L^2} = E(XY), \quad X, Y \in L^2(\Omega),$$

which induces the norm

$$\|X\|_{L^2}^2 = E(X^2), \quad X \in L^2(\Omega).$$

The set of random variables X such that $|X| \leq C$ a.s. for some constant $C \in \mathbb{R}$, is denoted by $L^\infty(\Omega)$. On the space $L^\infty(\Omega)$ we have the norm

$$\|X\|_\infty = \inf\{C \in \mathbb{R} : |X| \leq C \text{ a.e.}\}.$$

Theorem A.3.11 (Hölder's inequality) *Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $X \in L^p(\Omega)$ and $Y \in L^q(\Omega)$, then $XY \in L^1(\Omega)$ and*

$$\|XY\|_{L^1} \le \|X\|_{L^p} \cdot \|Y\|_{L^q}.$$

A special case of Hölder's inequality is the Cauchy-Schwarz inequality

$$\|XY\|_{L^1} \le \|X\|_{L^2} \cdot \|Y\|_{L^2}.$$

Theorem A.3.12 (Generalized Hölder's inequality) *Let $p_1, p_2, \ldots, p_n > 0$ be real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p}$. Let $X_i \in L^{p_i}(\Omega)$, $i = 1, 2, \ldots, n$. Then $\prod_{i=1}^{n} X_i \in L^p(\Omega)$ and*

$$\left\| \prod_{i=1}^{n} X_i \right\|_{L^p} \le \prod_{i=1}^{n} \|X_i\|_{L^{p_i}}.$$

Theorem A.3.13 (Minkowski's inequality) *Let $p \in [1, \infty]$ and $X, Y \in L^p$. Then*

$$\|X + Y\|_{L^p} \le \|X\|_{L^p} + \|Y\|_{L^p}.$$

Theorem A.3.14 (Minkowski's inequality in integral form) *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be measure spaces. If $u : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable, then*

$$\left(\int_{\Omega_2} \left| \int_{\Omega_1} u(x, y)\, d\mu_1(x) \right|^p d\mu_2(y) \right)^{\frac{1}{p}} \le \int_{\Omega_1} \left(\int_{\Omega_2} |u(x, y)|^p\, d\mu_2(y) \right)^{\frac{1}{p}} d\mu_1(x),$$

where $1 \le p < \infty$.

Let $X = (X_1, X_2, \ldots, X_d)$ and $Y = (Y_1, Y_2, \ldots, Y_d)$ be two d-dimensional random variables. The *covariance* of X and Y (when it exists) is the $d \times d$ matrix $B = Cov[X, Y] = [(Cov(X_i, Y_j))]_{d \times d}$, where

$$Cov(X_i, Y_j) = E[(X_i - E(X_i))(Y_j - E(Y_j))] = E(X_i Y_j) - E(X_i)E(Y_j).$$

The *variance* of a random variable X is defined by

$$Var(X) = Cov(X, X).$$

Let X be an $\mathbb{R}^d$-valued random variable on a probability space $(\Omega, \mathcal{F}, P)$ with distribution P_X. The *characteristic function* $L_X : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$L_X(t) = E(e^{i(t,X)}) = \int_{\Omega} e^{i(t,X(\omega))}\, dP(\omega) = \int_{\mathbb{R}^d} e^{i(t,x)} P_X(dx), \quad t \in \mathbb{R}^d,$$

and it has the following properties:

1. $|L_X(t)| \leq 1$;
2. L_X is hermitian, i.e. $L_X(-t) = \overline{L_X(t)}$ for all $t \in \mathbb{R}^d$;
3. If $X = (X_1, X_2, \ldots, X_d)$ and $E(|X_j^n|) < \infty$ for some $1 \leq j \leq d$ and $n \in \mathbb{N}$, then
$$E(X_j^n) = i^{-n} \frac{\partial^n}{\partial t_j^n} L_X(t)\Big|_{t=0}.$$

Note that the characteristic function of random variable X determines the distribution of X.

If μ is a probability measure on $\mathbb{R}^d$, then its characteristic function is the map
$$t \to \int_{\mathbb{R}^d} e^{i(t,x)}\, \mu(dx).$$
This mapping uniquely determines the probability measure μ.

Theorem A.3.15 (Bochner's theorem) *If $L : \mathbb{R}^d \to \mathbb{C}$ satisfies*

1. *the $d \times d$ matrix whose (i,j)th entry is $L(t_i - t_j)$ is positive definite for all $t_1, t_2, \ldots, t_d \in \mathbb{R}^d$,*
2. *$L(0) = 1$,*
3. *the map $t \to L(t)$ is continuous at the origin,*

then L is the characteristic function of a probability measure.

Proposition A.3.3 *If X_i, $i = 1, 2, \ldots, n$, are independent random variables, then*
$$L_{X_1+X_2+\ldots+X_n}(t) = L_{X_1}(t) \cdot L_{X_2}(t) \cdot \ldots \cdot L_{X_n}(t), \quad t \in \mathbb{R}^d.$$

A random varable X is a 1-*dimensional Gaussian (normal)* random varaible if its probability density function is
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right), \quad x \in \mathbb{R},$$
and we write $X : \mathcal{N}(m, \sigma^2)$. The expectation of X is
$$E(X) = m$$
and variance
$$Var(X) = \sigma^2.$$

The characteristic function of X is given by

$$f_X(t) = e^{itm} e^{-\frac{\sigma^2 t^2}{2}}, \quad t \in \mathbb{R}.$$

A d-dimensional random vector $X = (X_1, X_2, \ldots, X_d)$ has *multi-dimensional Gaussian (normal) distribution* , if there exists a vector $m = (m_1, m_2, \ldots, m_d) \in \mathbb{R}^d$ and a symmetric positive definite $d \times d$ matrix B such that X has a probability density function of the form

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^d \det B}} \exp\left(-\frac{1}{2}(x-m)^T B^{-1}(x-m)\right), \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d.$$

The vector m is the expectation of X and B is the covariance matrix. The characteristic function of X is given by

$$L_X(t) = \exp\left(i(t,m) - \frac{1}{2} t^T B t\right), \quad t = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d.$$

A.3.5 Convergence of Random Varaibles

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random variables on the probability space $(\Omega, \mathcal{F}, P)$. We say that

- $(X_n)_{n\in\mathbb{N}}$ converges to the random variable $X : \Omega \to \mathbb{R}^d$ *almost surely* if

  $$\lim_{n\to\infty} X_n(\omega) = X(\omega),$$

  for all $\omega \in \Omega \setminus N$, where $N \in \mathcal{F}$ satifies $P(N) = 0$.

- $(X_n)_{n\in\mathbb{N}}$ converges to the random variable $X : \Omega \to \mathbb{R}^d$ *in probability* if

  $$(\forall a > 0) \lim_{n\to\infty} P(|X_n - X| > a) = 0.$$

- $(X_n)_{n\in\mathbb{N}}$ converges to the random variable $X : \Omega \to \mathbb{R}^d$ *in distribution* if

  $$\lim_{n\to\infty} \int_{\mathbb{R}^d} u(x)\, P_{X_n}(dx) = \int_{\mathbb{R}^d} u(x)\, P_X(dx),$$

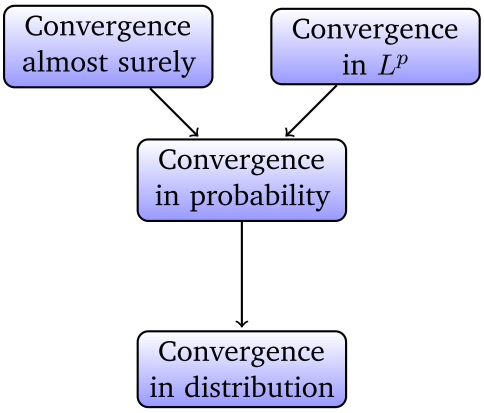
  for every continuous function u on $\mathbb{R}^d$ of compact support.

- $(X_n)_{n\in\mathbb{N}}$ converges to the random variable $X : \Omega \to \mathbb{R}^d$ *in L^p*, $p \in [1, \infty)$, if

  $$\lim_{n\to\infty} \|X_n - X\|_{L^p} = 0.$$

Figure A.1 summarizes how these types of convergence are related.

Figure A.1.: Relations between different types of convergence of random variables



A.3.6 Ordinary Stochastic Processes

This subsection is devoted to ordinary stochastic processes.

An *ordinary (classical) stochastic process* (OSP) is a parametrized family of random variables $\{X_t\}_{t\in T}$, $T \subseteq \mathbb{R}^d$, defined on the same probability space $(\Omega, \mathcal{F}, P)$ taking values in $\mathbb{R}^d$.

Notice that the index t is often interpreted as time, so we refer to X_t as the state of the OSP at time t. The set T is called the *index set* of the OSP. If the index set T is a set of non-negative integers, we have a *discrete time ordinary stochastic process*. If the index set T is an interval, $\mathbb{R}_+$ or $\mathbb{R}$, it is called a *continuous time ordinary stochastic process*.

Finite-dimensional distributions of an OSP $\{X_t\}_{t\in T}$ are given by

$$P_{t_1,t_2,\ldots,t_n}(B) = P((X_1(t_1), X_2(t_2), \ldots, X_n(t_n)) \in B), \quad B \in \mathcal{B}(\mathbb{R}^{dn}),$$

where $t_1, t_2, \ldots, t_n \in T$, $t_1 \neq t_2 \neq \ldots \neq t_n$, $n \in \mathbb{N}$. The family of finite-dimensional distributions of an OSP satisfies

(i) *consistency condition*:

$$P_{t_1,t_2,\ldots,t_n,t_{n+1}}(B_1 \times B_2 \times \ldots \times B_n \times \mathbb{R}^d) = P_{t_1,t_2,\ldots,t_n}(B_1 \times B_2 \times \ldots \times B_n),$$

for all $n \in \mathbb{N}$ and all $B_1, B_2, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$; and

(ii) *symmetry condition:*

$$P_{t_1,t_2,\ldots,t_n}(B_1 \times B_2 \times \ldots \times B_n) = P_{t_{\pi_1},t_{\pi_2},\ldots,t_{\pi_n}}(B_{\pi_1} \times B_{\pi_2} \times \ldots \times B_{\pi_n}),$$

for all $B_1, B_2, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$, all $n \in \mathbb{N}$ and all permutation π on $\{1, 2, \ldots, n\}$.

Theorem A.3.16 (Kolmogorov's existence theorem) *Given a family of probability measures* $(P_{t_1,t_2,\dots,t_n} : t_1, t_2, \dots, t_n \in T, t_1 \neq t_2 \neq \dots \neq t_n, n \in \mathbb{N})$ *satisfies the consistency condition and the symmetry condition, there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and an OSP* $\{X_t\}_{t\in T}$ *having the* $P_{t_1,t_2,\dots,t_n}$ *as its finite-dimensional distributions.*

We may regard the OSP $X_t(\omega) = X(t,\omega)$ as a function of two variables $\omega \in \Omega$ and $t \in T$. For each fixed $t \in T$ we can consider the random variable

$$X(t,\cdot) : \Omega \to \mathbb{R}^d, \quad \omega \mapsto X_t(\omega), \quad \omega \in \Omega.$$

For each fixed $\omega \in \Omega$ we obtain the function

$$X(\cdot,\omega) : T \to \mathbb{R}^d, \quad t \mapsto X_t(\omega), \quad t \in T,$$

which is called the *(sample) path* or *trajectory* of an OSP X.

We say that an OSP is continuous, if almost all of its paths are continuous.

Let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ be OSPs on the same probability space $(\Omega, \mathcal{F}, P)$. We say thet $\{Y_t\}_{t\in T}$ is a *version* or *modification* of $\{X_t\}_{t\in T}$, if for every $t \in T$

$$P(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\}) = 1.$$

It is clear that if $\{Y_t\}_{t\in T}$ is a version of $\{X_t\}_{t\in T}$, then $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ have the same finite-dimensional distributions. Note that such processes are the same, but they can have different path properties.

Theorem A.3.17 (Kolmogorov's continuity criterion) *Let* $\{X_t\}_t$ *be an OSP on* $\mathbb{R}^d$ *and assume that there exist constants* $a, b, C > 0$ *such that*

$$E|X_s - X_t|^a \leq C|s-t|^{d+b}, \quad s, t \in \mathbb{R}^d.$$

Then $\{X_t\}_t$ *has a pathwise continuous version.*

The following example shows that pathwise continuity does not imply L^2-continuity.

Example A.3.1 *Let* $\phi \in \mathcal{D}(\mathbb{R})$ *be a function such that*

- *supp* $\phi = [-1, 1]$,
- $\int \phi(x)\, dx = 0$ *and*
- $\int \phi^2(x)\, dx \neq 0$.

Take $\Omega = [-1, 1]$ *with the uniform probability distribution. Let*

$$X(t,\omega) = \begin{cases} \frac{1}{\sqrt{t}}\phi\left(\frac{\omega}{t}\right), & |\omega| < t, \\ 0, & |\omega| \geq t \text{ or } t \leq 0. \end{cases}$$

Then $E(X(t,\cdot)) = 0$. *Notice that all trajectories (except for* $\omega = 0$*) are smooth. But*

$$E((X(t,\cdot))^2) = \begin{cases} \frac{1}{2}\int \phi^2(x)\,dx, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and it does not converge to zero as $t \to 0$. *Therefore,* L^2*-continuity does not hold.* □

An OSP $\{N_t\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ taking values in $\mathbb{N}_0$ is a *Poisson process of intensity* $\lambda > 0$ if

(i) $N(0) = 0$,

(ii) for any $n \in \mathbb{N}$ and any $0 \leq t_1 < t_2 < \ldots < t_n$ the increments $N_{t_n} - N_{t_{n-1}}, \ldots,$ $N_{t_3} - N_{t_2}, N_{t_2} - N_{t_1}$ are independent random variables,

(iii) for any $0 \leq s < t$, the increment $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t-s)$, i.e.

$$P(N_t - N_s = k) = \frac{[\lambda(t-s)]^k}{k!}e^{-\lambda k}, \quad k = 0, 1, 2, \ldots.$$

The following example shows that L^2-continuity does not imply pathwise continuity.

Example A.3.2 *A Poisson process* $\{N_t\}_t$ *of intensity* $\lambda > 0$ *is* L^2*-continuous. Indeed,*

$$E((N_{t+s} - N_t)^2) = E(N_s^2) = \lambda s + (\lambda s)^2 \to 0 \text{ as } s \to 0.$$

Sample paths of Poisson process are not continuous. Notice that Poisson process violates the Kolmogorov continuity criterion.

A real-valued OSP $\{X_t\}_{t\in T}$ is called a *Gaussian (normal) ordinary stochastic process* if each of its finite-dimensional distributions is a multi-dimensional Gaussian random variable. Notice that the joint distributions of every Gaussian OSP are uniquely determined by its means and the covariance function.

A real-valued OSP $b = \{b_t\}_{t\geq 0}$ is called *(standard) Brownian motion* if the following holds:

(i) $b_0 = 0$ a.s.,

(ii) increments are independent, i.e. for all times $0 < t_1 < t_2 < \ldots < t_n$, the random variables $b_{t_1}, b_{t_2} - b_{t_1}, \ldots, b_{t_n} - b_{t_{n-1}}$ are independent,

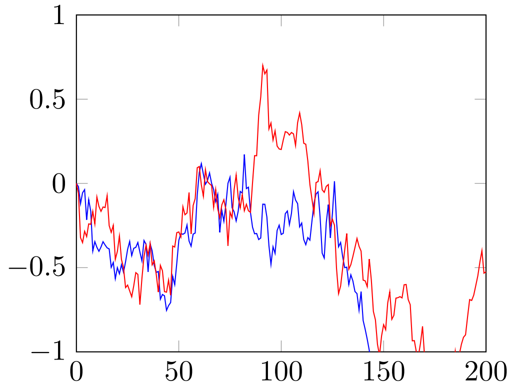
(iii) $b_t - b_s : \mathcal{N}(0, (t-s)\sigma^2)$ for each $0 \leq s \leq t$.

Note that

$$E(b_t) = 0, \quad E(b_t^2) = t \quad \text{for every time } t \geq 0.$$

A Brownian motion is a Gaussian OSP. The sample paths of Brownian motion are a.s. continuous, but nowhere differentiable functions. Therefore, OSP which is equal to the first derivative of Brownian motion does not exist.

Figure A.2.: Sample paths of Brownian motion



Notation and Abbreviations

B

In this appendix chapter we give the lists of notations and abbreviations used in the dissertation.

B.1 List of Notation

Sets, Fields, Rings, $\sigma-$algebras

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	the set of natural numbers with zero
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{R}^d$	the d-dimensional Euclidean space, $\mathbb{R}^d = \mathbb{R} \times \ldots \times \mathbb{R}$ (d-times)
$\mathbb{C}$	the set of complex numbers
$\mathbb{K}$	field
O	the open subset of $\mathbb{R}^d$
$K \Subset O$	K is a compact subset of O
D_O	the diagonal in $O \times O$
D	the diagonal in $\mathbb{R}^{2d}$
Q_O	the complement of the diagonal in $O \times O$
Q	the complement of the diagonal in $\mathbb{R}^{2d}$
$\mathcal{R}$	the ring of generalized real numbers
$\mathcal{R}_c$	the set of compactly supported generalized real numbers
$\tilde{O}$	the set of generalized points
$\tilde{O}_c$	the set of compactly supported generalized point
$\operatorname{supp} u$	the support of Colombeau generalized function u
$L((r_n)_n, k)$	the sharp open ball of $\mathcal{R}_c$
$\mathcal{B}(\mathcal{R}_c)$	$\sigma-$algebra generated by the sharp open balls of $\mathcal{R}_c$
$\mathcal{E}^k_{\mathcal{L}}(\Omega, O)$	the set of sequences $(u_n(\omega, x))_n$, $\omega \in \Omega$, $x \in \mathbb{N}$, $n \in \mathbb{N}$, such that $(u_n(\omega, \cdot))_n \in (\mathcal{C}^k(O))^{\mathbb{N}}$ for a.a. $\omega \in \Omega$, and for every $x \in O$, $(u_n(\cdot, x))_n$ is a sequence of measurable functions on Ω
$\mathcal{E}^k_{L^p}(\Omega, O)$	the set of sequences $(u_n(\omega, x))_n$, $\omega \in \Omega$, $x \in \mathbb{N}$, $n \in \mathbb{N}$, such that the mapping $x \mapsto u_n(\omega, \cdot)$ is in $\mathcal{C}^k(O)$ for a.a. $\omega \in \Omega$, and for every $x \in O$, $u_n(\cdot, x)$ is in $L^p(\Omega)$

$\mathcal{E}_{\mathcal{M}^\infty}(\Omega, O)$	the set of sequences $(u_n(\omega, x))_n$, $\omega \in \Omega$, $x \in \mathbb{N}$, $n \in \mathbb{N}$, such that the mapping $x \mapsto u_n(\omega, \cdot)$ is in $\mathcal{C}^k(O)$ for a.a. $\omega \in \Omega$, and for every $x \in O$, $u_n(\cdot, x)$ is in $\mathcal{M}^\infty(\Omega)$
$\mathcal{C}_{\mathcal{M}^\infty}(\Omega, O)$	the set of functions $f(\omega, x)$, $\omega \in \Omega$, $x \in O$, such that the mapping $x \mapsto f(\omega, x)$ is in $\mathcal{C}(O)$ for a.a. $\omega \in \Omega$, and for every $x \in O$, $f(\cdot, x)$ is in $\mathcal{M}^\infty(\Omega)$
$\mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}$	Borel σ-algebra of subsets of $\mathbb{R}^d$
$\mathcal{B}_\mathcal{M}$	σ-algebra generated by a family $\mathcal{M}$
$\sigma(X)$	σ-algebra generated by random variable X
$\overline{\mathbb{R}}$	extended real line, i.e. $\mathbb{R} \cup \{-\infty, +\infty\}$

Measures, Operators, Functions, Generalized Functions, Colombeau Generalized Functions

$dx, dy, \dots$	Lebesgue measure on $\mathbb{R}^d$
χ_A	the indicator function of a set A
μ	measure
P	probability measure
δ	Dirac delta distribution
$u = [(u_n)_n]$	Colombeau generalized function
$\mathcal{F}(u)$, $\hat{u}$	Fourier transform of $u \in \mathcal{S}(\mathbb{R}^d)$
$P(D)$	differential operator of order k with generalized constant coefficients
Δ	the Laplace operator
d_c	ultrapseudometric on $\mathcal{E}_M^c$
$\tilde{d}_c$	ultrametric on $\mathcal{R}_c$
$\cdot$	scalar product in $\mathbb{R}^d$
$(\cdot, \cdot)_{L^2}$	scalar product in $L^2(\Omega)$

Probability

$(\Omega, \mathcal{F}, P)$	the probability space
$X, Y, \dots$	random variables
P_X	distribution (law) of random variable X
F_X	distribution function of random variable X
$E(X)$	expectation of random variable X
$Var(X)$	variance of random variable X
$L_X(t)$	characteristic function of random variable X
$\mathcal{L}(\Omega)$	the space of real valued random variables endowed with almost sure convergence
$\mathcal{M}^\infty(\Omega)$	the space of random variables with finite seminorms $\|\| \cdot \|\|_s = \sup\{\| \cdot \|_{L^p}, 1 \le p \le s\}$, $s \in \mathbb{N}$

$L^p(\Omega)$	the space of random variables with finite pth moments
$\mathcal{E}_{M,\mathcal{L}}(\Omega)$	the space of moderate sequences of random variable with values in $\mathcal{L}(\Omega)$
$\mathcal{E}_{M,L^p}(\Omega)$	the space of moderate sequences of random variable with values in $L^p(\Omega)$
$\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega)$	the space of moderate sequences of random variable with values in $\mathcal{M}^\infty(\Omega)$
$\mathcal{N}_{\mathcal{L}}(\Omega)$	the space of negligible sequences of random variable with values in $\mathcal{L}(\Omega)$
$\mathcal{N}_{L^p}(\Omega)$	the space of negligible sequences of random variable with values in $L^p(\Omega)$
$\mathcal{N}_{\mathcal{M}^\infty}(\Omega)$	the space of negligible sequences of random variable with values in $\mathcal{M}^\infty(\Omega)$
ξ	distributional stochastic process
$Cd(\xi)$	embedded distributional stochastic process
$m = [(m_{u_n})_n]$	generalized expectation of Colombeau stochastic process $u = [(u_n)_n]$
$B = [(B_{u_n})_n]$	generalized correlation function of Colombeau stochastic process $u = [(u_n)_n]$
$C = [(C_{u_n})_n]$	generalized covariance function of Colombeau stochastic process $u = [(u_n)_n]$
$L_u(t,x)$	generalized characteristic function of Colombeau stochastic process u in $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ or $\mathcal{G}^k_{L^{kp}}(\Omega, O)$
$L_u(t,s;x,y)$	generalized characteristic function of the joint distribution of the random field $(u(\omega,x), u(\omega,y))$
$w = [(w_n)_n]$	white noise
$b = [(b_n)_n]$	Brownian motion

Spaces of Functions, Spaces of Sequences of Functions

$\mathcal{C}^k(O)$	the space of k times continuously differentiable functions
$\mathcal{C}^\infty(O)$	the space of the smooth functions
$\mathcal{D}(O)$	the space of the smooth test functions with compact support
$\mathcal{D}'(O)$	the space of Schwartz distributions on O
$\mathcal{S}(O)$	the Schwartz space of rapidly decreasing functions
$\mathcal{S}'(O)$	the space of tempered distributions
$\mathcal{E}(O)$	the space of sequences of smooth functions
$\mathcal{E}'(O)$	the space of Schwartz distribution with compact support
$\mathcal{O}_C(O)$	the space of smooth functions such that there exist $p \in \mathbb{N}$, for every $\alpha \in \mathbb{N}_0^d$, such that $\sup_{x\in O}(1+\lvert x\rvert)^{-p}\lvert\partial^\alpha f(x)\rvert < \infty$
$L^1_{loc}(O)$	the space of locally integrable functions on O

$\mathcal{O}_M(O)$	the space of functions such that for every $\alpha \in \mathbb{N}_0^d$, there exist $p \in \mathbb{N}$, such that $\sup\limits_{x\in O}(1+\lvert x\rvert)^{-p}\lvert\partial^\alpha f(x)\rvert < \infty$
$\mathcal{E}_M(O)$	the space of moderate sequences of functions
$\mathcal{N}(O)$	the space of negligible sequences of functions
$\mathcal{E}^k(O)$	the space of sequences of functions with continuous derivatives up to kth order
$\mathcal{E}^k_M(O)$	the subspace of $\mathcal{E}^k(O)$ of moderate sequence of functions
$\mathcal{N}^k(O)$	the subspace of $\mathcal{E}^k(O)$ of negligible sequence of functions
$\mathcal{E}^k_{M,\mathcal{L}}(\Omega, O)$	the space of moderate sequences of functions with values in $\mathcal{L}(\Omega)$
$\mathcal{N}^k_{\mathcal{L}}(\Omega, O)$	the space of negligible sequences of functions with values in $\mathcal{L}(\Omega)$
$\mathcal{E}^k_{M,L^p}(\Omega, O)$	the space of moderate sequences of functions with values in $L^p(\Omega)$
$\mathcal{N}^k_{L^p}(\Omega, O)$	the space of negligible sequences of functions with values in $L^p(\Omega)$
$\mathcal{E}_{M,\mathcal{M}^\infty}(\Omega, O)$	the space of moderate sequences of functions with values in $\mathcal{M}^\infty(\Omega)$
$\mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$	the space of negligible sequences of functions with values in $\mathcal{M}^\infty(\Omega)$
$\mathcal{E}_{\mathcal{L}}(\Omega)$	the space of sequences of measurable functions on Ω
$\mathcal{E}^k_{\tau,L^2}(\Omega, \mathbb{R}^d)$	the space of tempered moderate sequences of functions
$\mathcal{N}^k_{\tau,L^2}(\Omega, \mathbb{R}^d)$	the space of tempered negligible sequences of functions
$K(a)$	the space of all infinitely differentiable functions defined in $\mathbb{R}^d$ with support in domain $G_a = \{\lvert x_1\rvert \le a_1, \dots, \lvert x_d\rvert \le a_d\}$, $a = (a_1, \dots, a_d)$

Quotient Spaces, Algebras

$\mathcal{G}(O)$	Colombeau algebra on O
$\mathcal{G}^k(O)$	the quotient space $\mathcal{E}^k_M(O)/\mathcal{N}^k(O)$
$\mathcal{G}^k_{\mathcal{L}}(\Omega, O)$	the quotient space of Colombeau stochastic processes over O with values in $\mathcal{L}(\Omega)$
$\mathcal{G}^k_{L^p}(\Omega, O)$	the quotient space of Colombeau stochastic processes over O with values in $L^p(\Omega)$
$\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$	the quotient space of Colombeau stochastic processes over O with values in $\mathcal{M}^\infty(\Omega)$
$\mathcal{G}_{\mathcal{L}}(\Omega)$	the quotient space of generalized random variables with values in $\mathcal{L}(\Omega)$
$\mathcal{G}_{L^p}(\Omega)$	the quotient space of generalized random variables with values in $L^p(\Omega)$
$\mathcal{G}_{\mathcal{M}^\infty}(\Omega)$	the quotient space of generalized random variables with values in $\mathcal{M}^\infty(\Omega)$

$\mathcal{G}^k_{\tau,L^2}(\Omega,\mathbb{R}^d)$	the quotient space of tempered Colombeau stochastic processes over $\mathbb{R}^d$ with values in $L^2(\Omega)$

Matrices

A^{-1}	the inverse matrix of a matrix A
A^T	the transpose matrix of a matrix A
$\det(A)$	the determinant of a matrix A

Other

$L(\mathcal{D}(\mathbb{R}), L^2(\Omega))$	the space of linear continuous mappings of a test space $\mathcal{D}(\mathbb{R})$ into the space $L^2(\Omega)$ of random variables with finite second moments
$\approx$	association relation
$\tilde{x} = [(x_n)_n]$	generalized point
$\blacksquare$	end of proof
$\square$	end of example

Multi-index notation

A *multi-index* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is an d-tuple of non-negative integers.

The *length* (or *order*) of a multi-index α is defined by

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d.$$

The *factorial* of a multi-index α is defined by

$$\alpha! = \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_d! = \prod_{i=1}^{d} \alpha_i!.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$ be two multi-indices. Sum and difference of α and β are defined component-wise, i.e.

$$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \ldots, \alpha_d \pm \beta_d).$$

Thus, $|\alpha \pm \beta| = |\alpha| \pm |\beta|$.

For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ we define

$$x^\alpha = x_1^\alpha x_2^\alpha \cdot \ldots \cdot x_d^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}.$$

If $\beta_i \leq \alpha_i$ for all $i = 1, 2, \ldots, d$, then we write $\beta \leq \alpha$. For multi-indices α and β with $\beta \leq \alpha$, we define

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha - \beta)!\beta!}.$$

For $u : O \to \mathbb{R}$ we write

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_d}^{\alpha_d}}.$$

B.2 List of Abbreviations

ODE	ordinary differential equation
PDE	partial differential equation
SODE	stochastic ordinary differential equation
SPDE	stochastic partial differential equation
a.a.	almost all
a.e.	almost everywhere
a.s.	almost surely
OSP	ordinary stochastic process
GSP	generalized stochastic process
CSP	Colombeau stochastic process
GCSP	Gaussian Colombeau stochastic process

Biographical Index

C

René Louis Baire (1874-1932) was a French mathematician. His doctoral dissertation on the theory of functions of real variables applied concepts from set theory to categorize functions. The Baire category theorem was the main result in dissertation. He made significant contributions to the theory of irrational numbers.

Stefan Banach (1892-1945) was a Polish mathematician. He was the founder of modern functional analysis. Banach made major contributions to the theory of topological vector spaces. Also, he contributed to measure theory, the theory of sets and orthogonal series.

Salomon Bochner (1899-1982) was an American mathematician. Areas of his scientific interest are mathematical analysis, probability theory and differentail geometry.

Félix Édouard Justin Émile Borel (1871-1956) was a French mathematician and politician. He was among the pioneers of measure theory and its applications to probability theory. The concepts of a Borel σ-algebra, a Borel measure, a Borel set are named in his honor.

Robert Brown (1773-1858) was a Scottish botanist. He was a pioneer in the field of microscopy. Brown is known for his descriptions of the nucleus of cells. He also observed Brownian motion.

Augustin-Louis Cauchy (1789-1857) was a French mathematician, engineer and physicist. He was one of the greatest mathematicians during the 19th century. Cauchy was one of the first to state and prove theorems of calculus rigorously. He was a developer of the theory of functions of complex variable. Cauchy made contributions to the number theory, elasticity and wave theory. Algebra and mechanics are indebted to him for many improvements. Cauchy collected works were published in 27 volumes.

Albert Einstein (1879-1955) was a German theoretical physicist. He has developed the theory of relativity. He published more than 300 scientific papers.

Jean-Baptiste Joseph Fourier (1768-1830) was a French mathematician and physicist. He was a government administrator during the reign of Napoleon. The Fourier transform, the Fourier series and Fourier's law are named in his honour.

Guido Fubini (1879-1943) was an Italian mathematician. His research focused on differential equations, functional analysis, complex analysis, calculus of variations, group theory, non-Euclidean geometry, projective geometry. Fubini applied results from his work to problems in electrical circuits and acoustics. He is known for Fubini's theorem and Fubini-Study metric. A main belt asteroid is named in his honour.

Carl Friedrich Gauss (1777-1855) was a German mathematician and physicist, who made important contributions to many different areas of mathematics. In addition to mathematics, Gauss made significant contributions to astronomy and physics. He published over 150 papers.

Israel Moiseevich Gel'fand (1913-2009) was a Russian mathematician. He was a professor at Moscow State University and at Rutgers University. Gel'fand made significant contributions to the group theory, representation theory and functional analysis. He published over 800 papers and 30 books. Gel'fand is the recipient of many honors and awards, including the Order of Lenin and the Wolf Prize.

Walter Gordon (1893-1939) was a German theoretical physicist. Max Planck was his doctoral advisor.

Otto Ludwig Hölder (1859-1937) was a German mathematician. He was a student of Karl Weierstrass, Ernest Kummer and Leopold Kronecker. Hölder is noted for many theorems including: Hölder's theorem on the Gamma function, Hölder's inequality, the Jordan-Hölder theorem and many theorems from the theory of groups. He is known for the class of Hölder continuous functions.

Kiyosi Itô (1915-2008) was a Japanese mathematician. He was the founder of Itô calculus. Itô has received numerous awards and honors. He was awarded the Gauss Prize in 2006 for applications of mathematics.

Avul Pakir Jainulabdeen abdul Kalam (1931-2015) was elected as the 11th President of India. Also, he was a scientist. He studied physics and aerospace engineering.

Oskar Benjamin Klein (1894-1977) was a Swedish theoretical physicist. He was awarded the Max Planck Medal in 1959.

Andrey Nikolaevich Kolmogorov (1903-1987) was a Russian mathematician. He made contributions to almost all areas of mathematics. Kolmogorov was the founder of modern probability theory.

Pierre Simon Laplace (1749-1827) was a French mathematician, physicist and astronomer. He contributed in development of difference equations, differential equations, probability and statistics. The Laplace's equation, the Laplace transform and the Laplacian differential operator are named after him.

Gottfried Wilhelm Leibniz (1646-1716) was a German mathematician and philosopher. He is credited with the discovery of differential and integral calculus. Leibniz made important contributions to physics and technology, probability theory, medicine, biology.

Henri Léon Lebesgue (1875-1941) was a French mathematician. In 1902, Lebesgue defended his doctoral dissertation and in it he developed the theory of measure and integration, which was a generalization of the Riemann concept of integral. In addition, Lebesgue made important contribution to topology, Fourier analysis and potential theory.

Joseph Liouville (1809-1882) was a French mathematician. He made contribution to number theory, complex analysis, differential geometry, topology, mathematical physics and astronomy.

Stanisław Łojasiewicz (1926-2002) was a Polish mathematician. He solved the problem of distribution division by analytic functions.

Hermann Minkowski (1864-1909) was a German mathematician. He developed the geometrical theory of numbers. Minkowski made important contributions to number theory, mathemtical physics and the theory of relativity. Albert Einstein was his student. In 1883, Minkowski was awarded by the French Academy of Science for his manuscript on the theory of quadratic forms.

Augustus De Morgan (1806-1871) was a British mathematician and logician. He is known for formulation of De Morgan's law.

Steven Paul Jobs (1955-2011) was an American entrepreneur and business magnate. He was a co-founder of Apple Computers.

Otto Marcin Nikodym (1887-1974) was a Polish mathematician. He made important contribution in measure theory. The Radom-Nikodym theorem is named after Nikodym, who proved the general case in 1930. Nikodym worked on the theory of operators in Hilbert space, based on Boolean lattices. Nikodym was interested in teaching of mathematics.

Marc-Antoine Parseval (1755-1836) was a French mathematician. He is known for the Parseval's theorem. He was nominated to the French Academy of Science five times, but was never elected.

Michel Plancherel (1885-1967) was a Swiss mathematician. Areas of his scientific interest are mathematical analysis, mathematical physics and algebra. He is known for the Plancherel theorem in harmonic analysis.

Siméon Denis Poisson (1781-1840) was a French mathematician, engineer and physicist. He published more than 300 papers. The Poisson distribution and the Poisson process are named after him.

Johan Karl August Radon (1887-1956) was an Austrian mathematician. Radon received his doctoral degree at the University of Vienna in 1910. His doctoral dissertation was on calculus of variations. In 1947, Radon became a member of the Austrian Academy of Science. Radon is known for significant contributions, including: the Radon measure, the Radon-Nikodym theorem (In 1913, Radon proved the theorem for the special case where the underlying space is $\mathbb{R}^d$.), the Radon transform, the Radon-Hurwitz numbers.

Georg Friedrich Bernhard Riemann (1826-1866) was a German mathematician. He made major contributions to real and complex analysis, differential geometry and number theory.

Laurent-Moïse Schwartz (1915-2002) was a French mathematician. In 1950, Schwartz was awarded the Fields Medal for his work on the theory of distributions.

Bogoljub Stanković (1924-2018) was a Serbian mathematician. He was a doctoral student of Professor Jovan Karamata. Stanković has created mathematical school in Novi Sad. The fields of his scientific interest were functional analysis, the theory of generalized functions and their applications. He published 6 monographs and about 150 papers.

Ruslan Leont'evich Stratonovich (1930-1997) was a Russian physicist and engineer. He was mathematician specializing in probability theory. Stratonovich was one of the founders of the theory of stochastic differential equations. The Stratonovich calculus is an alternative to the Itô calculus. The Stratonovich integral is named after him.

Brook Taylor (1685-1731) was an English mathematician. He is known for Taylor's theorem and Taylor series.

Leo Tolstoy (1828-1910) was a Russian writer.

Naum Yakovievich Vilenkin (1920-1991) was a Russian mathematician. He was an expert in combinatorics. In 1976, Vilenkin was awarded the Ushinsky prize for his school mathematics textbooks.

Norbert Wiener (1894-1964) was an American mathematician. Wiener was a professor at the Massachusetts Institute of Technology (MIT). He remained on the MIT's mathematics department until his retirement. Wiener remained active as a professor, advisor and researcher until his death. The Norbert Wiener Prize in Applied Mathematics was endowed in 1967 in his honor by MIT's mathematics department. Wiener was an expert in stochastic processes and, in particular, on the theory of Brownian motion. He made significant contributions in the field of electronic engineering, telecommunications and control systems. Wiener wrote many books and hundreds of papers. He wrote science fiction book and two volumes of autobiography. The crater Wiener at the moon is named after him.

Akiva Moiseevich Yaglom (1921-2007) was a Soviet mathematician and physicist. He was known for his contributions to the theory of stochastic processes and the statistical theory of turbulence. He published 6 books and about 120 papers.

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Proširen izvod

> *U razvoju matematike, kao i svake druge nauke, smenjuju se periodi ekstenzivnog razvoja, koji slede nove značajne prodore matematičke misli, i periodi sinteze i analize postignutog. U prvom periodu preovlađuje kvalitet istraživanja; svaki istraživač ili ekipa istraživača sa svoje strane žure da što više iskoriste to novo. Kada se mogućnosti njegovog korišćenja uglavnom iscrpe, analizira se i ocenjuje postignuto; sa više simetričnosti i kritičnosti se odabira ono što treba da ostane stvarni doprinos matematičkoj nauci i njenim rezultatima.*
>
> — **Akademik Bogoljub Stanković (1924-2018)**
> (odlomak iz akademske besede)

Sredinom XX veka teoriju uopštenih procesa razvili su Ito (vidi [Itô54]) i Gelfand (vidi [GV64], nezavisno jedan od drugog. Uopšteni stohastički procesi javljaju se kao rešenja stohastičkih parcijalnih diferencijalnih jednačina, koje modeliraju mnoge prirodne pojave. Stoga su uopšteni stohastički procesi postali predmet istraživanja mnogih autora.

Sredinom devedesetih godina prošog veka stohastičke procese sa trajektorijama u Kolomboovoj algebri počinju da proučavaju Ruso (vidi [Rus94]) i Obergugenberger (vidi [Obe95]). Primene na rešavanje stohastičkih parcijalnih diferencijalnih jednačina podstakle su razvoj stohastičkog kalkulusa u Kolomboovoj algebri uopštenih funkcija.

Disertacija se bavi prostorima $\mathcal{G}^k_{L^p}(\Omega, O)$, $\mathcal{G}^k_{\mathcal{L}}(\Omega, O)$, $(k \in \mathbb{N} \cup \{\infty\})$ i $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$, čiji elementi se nazivaju *Kolomboovi stohastički procesi* sa vrednostima u $L^p(\Omega)$, $\mathcal{L}(\Omega)$ i $\mathcal{M}^\infty(\Omega)$, redom. Primetimo da je O otvoren skup u $\mathbb{R}^d$, $(\Omega, \mathcal{F}, P)$ je prostor verovatnoće. Ako je $k = \infty$, onda pišemo $\mathcal{G}_{L^p}(\Omega, O)$ i $\mathcal{G}_{\mathcal{L}}(\Omega, O)$.

Neka je $u = [(u_n)_n]$ iz $\mathcal{G}_{\mathcal{L}}(\Omega, O)$ ili $\mathcal{G}_{L^p}(\Omega, O)$ ili $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$. Tada, za fiksno $\tilde{x} = [(x_n)_n] \in \tilde{O}_c$, $u(\omega, \tilde{x}) = [(u_n(\omega, x_n)_n]$ je uopštena slučajna promenljiva u $\mathcal{G}_{\mathcal{L}}(\Omega)$ ili $\mathcal{G}_{L^p}(\Omega)$ ili $\mathcal{G}_{\mathcal{M}^\infty}(\Omega)$. (To znači da niz $(u_n)_n$ ne zavisi od $x \in O$.)

Sledeća teorema predstavlja glavni rezultat u disertaciji.

Teorema 1 *Neka je $u = [(u_n)_n]$ Kolomboov stohastički proces u $\mathcal{G}_{\mathcal{L}}(\Omega, O)$ ili $\mathcal{G}_{L^p}(\Omega, O)$ ili $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$. Za fiksno $\tilde{x} = [(x_n)_n] \in \tilde{O}_c$, preslikavanje*

$$(\Omega, \mathcal{F}) \ni \omega \mapsto u(\omega, \tilde{x}) \in (\mathcal{R}, \mathcal{B}(\mathcal{R}))$$

je merljivo, gde je $\mathcal{B}(\mathcal{R})$ σ-algebra generisana oštrim otvorenim loptama u $\mathcal{R}$.

Merljivost Kolomboovog stohastičkog procesa nam omogućava da proučavamo njegove probabilističke osobine.

Uopšteno očekivanje Kolomboovog stohastičkog procesa $u = [(u_n)_n] \in \mathcal{G}^k_{L^2}(\Omega, O)$ je element m iz $\mathcal{G}^{k-1}(O)$ sa reprezentacijom

$$m_{u_n}(x) = E(u_n(\cdot, x)) = \int_\Omega u_n(\omega, x)\, dP(\omega), \quad x \in O, \;\; n \in \mathbb{N}.$$

Uopštena korelacijska funkcija Kolomboovog stohastičkog procesa $u = [(u_n)_n] \in \mathcal{G}^k_{L^2}(\Omega, O)$ je element B iz $\mathcal{G}^{k-1}(O \times O)$ sa reprezentacijom

$$B_{u_n}(x, y) = E(u_n(\cdot, x) u_n(\cdot, y)), \quad x, y \in O, \;\; n \in \mathbb{N}.$$

Označimo sa Q_O komplement dijagonale, a sa D_O dijagonalu u $O \times O$. Ako je $O = \mathbb{R}^d$, onda koristimo oznake Q i D. U sledeća dva tvrđenja dajemo strukturnu karakteizaciju uopštene korelacijske funkcije.

Teorema 2 *Neka je $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ uopštena korelcijska funkcija Kolomboovog stohastičkog procesa $u = [(u_n)_n]$ na O sa vrednostima u $L^2(\Omega)$ koji je dobijen potapanjem distributivnog stohastičkog procesa ξ na O, odnosno $u = Cd(\xi)$.*

a) Neka je $F \in \mathcal{D}'(O \times O)$ korelacijski funkcional procesa ξ. Tada je $B = Cd(F)$.

b) $B(\tilde{x}, \tilde{y}) = 0$ za sve $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$ ako i samo ako supp $F \subseteq D_O$.

c) Ako je $B(\tilde{x}, \tilde{y}) = 0$ za sve $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$, onda je B asocirana sa uopštenom funkcijom koja ima reprezentaciju u sledećem obliku

$$B^*_n(x, y) = \int_O \sum_{j,k \in \mathbb{N}_0} R_{j,k}(s) \varphi^{(j)}_n(x - s) \varphi^{(k)}_n(y - s) ds, \quad x, y \in O, \qquad (1)$$

gde je za sve $n \in \mathbb{N}$ samo konačan broj neprekidnih funkcija $R_{j,k}$ različit od nule na svakom kompaktnom podskupu skupa O.

Propozicija 1 *Neka je* $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ *uopštena korelacijska funkcija Kolomboovog stohastičkog procesa* $u = [(u_n)_n]$ *na* O *sa vrednostima u* $L^2(\Omega)$*. Pretpostavimo da je* B *asocirana sa* $F \in \mathcal{D}'(O \times O)$*. Ako je* $B(\tilde{x}, \tilde{y}) = 0$ *za sve* $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$*, onda*

a) F *je asocirana na dijagonali* D_O*,*

b) B *je asocirana sa uopštenom funkcijom koja ima reprezentaciju oblika (1).*

Neka je $u = [(u_n)_n]$ Kolomboov stohastički proces u $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$. Tada se

$$L_u(t,x) = [(L_{u_n}(t,x))_n] = [(E(e^{itu_n(\cdot,x)}))_n] \in \mathcal{G}(\mathbb{R} \times O), \;\; t \in \mathbb{R}, \, x \in O,$$

naziva *uopštena karakteristična funkcija* procesa u.

Element ϕ prostora $\mathcal{C}^\infty_{\mathcal{M}^\infty}(\Omega, O)$ možemo potopiti u prostor $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ na dva načina: konvolucijom sa molifajerom ili kao konstantan niz. Postavlja se pitanje da li će uopštene karakteristične funkcije dobijenih elemenata u $\mathcal{G}_{\mathcal{M}^\infty}(\Omega, O)$ biti jednake. Odgovor je dat u sledećoj propoziciji.

Propozicija 2 *Neka je* $\phi \in \mathcal{C}^\infty_{\mathcal{M}^\infty}(\Omega, O)$*. Pretpostavimo da je*

$$\sup_{x \in K} \|\phi^{(\alpha)}(\cdot, x)\|_{L^p} < \infty$$

za sve $\alpha \in \mathbb{N}_0$ *i sve* $K \Subset O$*. Neka je*

$$\phi_n(\omega, x) = (\phi(\omega, \cdot) * \varphi_n(\cdot))(x), \;\; x \in O, \;\; \omega \in \Omega,$$

za dovoljno veliko n*. Tada je*

a) $(\phi_n(\omega, x))_n - (\phi(\omega, x))_n \in \mathcal{N}_{\mathcal{M}^\infty}(\Omega, O)$,

b) $(L_{\phi_n}(t, x))_n - (L_\phi(t, x))_n \in \mathcal{N}(\mathbb{R} \times O)$,

gde je $(\phi)_n$ *konstntan niz.*

Međutim, ako $\phi \in \mathcal{C}_{\mathcal{M}^\infty}(\Omega, O)$, onda su odgovarajuće uopštene karakteristične funkcije asocirane.

Propozicija 3 *Ako* $\phi \in \mathcal{C}_{\mathcal{M}^\infty}(\Omega, O)$*, onda je* $[(L_{\phi * \varphi_n}(t, x))_n]$ *asocirana sa Kolomboovom uopštenom funkcijom čija je reprezentacija* $(L_\phi(t, \cdot) * \varphi_n(\cdot))(x)$*,* $t \in \mathbb{R}$*,* $x \in O$*.*

Sledeća propozicija nam omogućava da definišemo uopštenu karakterističnu funkciju Kolomboovog stohastičkog procesa u $\mathcal{G}^k_{L^{kp}}(\Omega, O)$, $k \leq p$.

Propozicija 4 *Ako* $(u_n)_n \in \mathcal{E}^k_{M,L^{kp}}(\Omega, O)$*, onda* $(e^{itu_n(\omega,x)})_n \in \mathcal{E}^k_{M,L^p}(\Omega, \mathbb{R} \times O)$ *i* $(E(e^{itu_n(\cdot,x)}))_n \in \mathcal{E}^k_M(\mathbb{R} \times O)$*.*

Dakle, *uopštena karakteristična funkcija* Kolomboovog stohastičkog procesa $u = [(u_n)_n]$ u $\mathcal{G}^k_{L^{kp}}(\Omega, O)$ je

$$L_u(t,x) = [(L_{u_n}(t,x))_n] = [(E(e^{itu_n(\cdot,x)}))_n] \in \mathcal{G}^k(\mathbb{R} \times O), \;\; t \in \mathbb{R}, \, x \in O.$$

Kao u klasičnom slučaju, uopšteno očekivanje i uopštena korelacijska funkcija mogu se izračunati pomoću uopštene karakteristične funkcije.

Kolomboov stohastički proces u na O sa vrednostima u $L^p(\Omega)$ *ima nezavisne vrednosti* ako ima reprezentaciju $(u_n)_n$ takvu da sledeća dva uslova važe:

(NV1) za sve $n \in \mathbb{N}$, $u_n(\omega, x)$ i $u_n(\omega, y)$ su nezavisne slučajne promenljive za sve $(x, y) \in K$, $K \Subset Q_O$, odnosno za sve $n \in \mathbb{N}$,

$$P\{u_n(\omega,x) \in B_1 \cap u_n(\omega,y) \in B_2\} = P\{u_n(\omega,x) \in B_1\} P\{u_n(\omega,y) \in B_2\}$$

za sve $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ i $(x,y) \in K$, $K \Subset Q_O$,

(NV2) za $n \neq m$, $u_n(\omega, x)$ i $u_m(\omega, y)$ su nezavisne slučajne promenljive za sve $x, y \in O$, odnosno za $n \neq m$,

$$P\{u_n(\omega,x) \in B_1 \cap u_m(\omega,y) \in B_2\} = P\{u_n(\omega,x) \in B_1\} P\{u_m(\omega,y) \in B_2\}$$

za sve $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ i $x, y \in O$.

Neka je u Kolomboov stohastički proces sa nezavisnim vrednostima. Napomenimo da osobine (NV1) i (NV2) ne ispunjavaju sve reprezentacije procesa u. Reprezentaciju koja ispunjava uslove (NV1) i (NV2) nazivamo reprezentacija sa nezavisnim vrednostima.

Teorema 3 *Neka je u Kolomboov stohastički proces na O sa vrednostima u $L^p(\Omega)$ i neka u ima nezavisne vrednosti. Tada*

$$P\{u(\omega,\tilde{x}) \in \mathfrak{O}_1 \cap u(\omega,\tilde{y}) \in \mathfrak{O}_2\} = P\{u(\omega,\tilde{x}) \in \mathfrak{O}_1\} P\{u(\omega,\tilde{y}) \in \mathfrak{O}_2\}$$

za sve otvorene lopte $\mathfrak{O}_1, \mathfrak{O}_2$ u $\mathcal{R}_c$ i $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$.

U sledećoj propoziciji dajemo karakterizaciju Kolomboovog stohastičkog procesa sa nezavisnim vrednostima preko uopštene korelacijske funkcije.

Propozicija 5 *Neka je u Kolomboov stohastički proces na O sa vrednostima u $L^2(\Omega)$ i neka u ima nezavisne vrednosti. Tada je $B(\tilde{x}, \tilde{y}) = 0$ za sve $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_O)_c$.*

U disertaciji smo proučavali strogo stacionarne i slabo stacionarne Kolomboove stohastičke procese.

Kažemo da je Kolomboov stohastički proces u na O sa vrednostima u $L^p(\Omega)$ (*strogo*) *stacionaran* ako ima reprezentaciju $(u_n)_n$ takvu da za sve $n \in \mathbb{N}$, za sve $x_1, \ldots, x_m \in O$ i svako $h \in \mathbb{R}^d$ takve da $x_1 + h, \ldots, x_m + h \in O$, slučajne promenljive

$$(u_n(\cdot, x_1), \ldots, u_n(\cdot, x_m)) \quad \text{i} \quad (u_n(\cdot, x_1 + h), \ldots, u_n(\cdot, x_m + h))$$

su identički raspoređene.

Teorema 4 *Uopšteno očekivanje $m \in \mathcal{G}(O)$ stacionarnog Kolomboovog stohastičkog procesa na $O \subseteq \mathbb{R}^d$ je uopštena konstanta.*

Primetimo da ne moraju sve reprezentacije stacionarnog Kolomboovog stohastičkog procesa biti stacionarne, ali sve imaju konstantno očekivanje.

Teorema 5 *Neka je $O \subseteq \mathbb{R}^d$ centralno simetričan konveksan otvoren skup i neka je u stacionaran Kolomboov stohastički proces na $2O \cong O - O$ sa vrednostima u $L^2(\Omega)$. Ako je $B = [(B_n)_n] \in \mathcal{G}(O \times O)$ uopštena korelacijska funkcija procesa u, onda postoji pozitivno-definitna uopštena funkcija $B^* = [(B_n^*)_n] \in \mathcal{G}(2O)$ takva da je*

$$B_n(x, y) = B_n^*(x - y), \quad x, y \in O,\ n \in \mathbb{N}.$$

Kolomboov stohastički proces u na $2O$ sa vrednostima u $L^2(\Omega)$ se naziva *slabo stacionaran* ako su njegovo uopšteno očekivanje $m_u \in \mathcal{G}(O)$ i uopštena korelacijska funkcija $B_u \in \mathcal{G}(O \times O)$ translatorno invarijante, odnosno

$$m_u(x + h) = m_u(x)$$

za sve $h \in \mathbb{R}$ takve da $x, x + h \in O$, i

$$B_u(x, y) = B^*(x - y), \;\; x, y \in O,$$

za neku pozitivno-definitnu uopštenu funkciju $B^* \in \mathcal{G}(2O)$.

Teorema 6 *Neka je O otvoren konveksan skup u $\mathbb{R}^d$. Pretpostavimo da za sve $K \Subset O$ i svako $h \in \mathbb{R}$ takvo da $t \in K$ implicira $t + h \in O$, važi*

$$(\forall p \in \mathbb{N})(\exists n_p \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq n_p \Rightarrow \sup_{t \in K} n^p |u_n(t + h) - u_n(t)| \leq 1).$$

Tada je $[(u_n)_n]$ uopštena konstanta na O, odnosno postoji $(r_n)_n \in \mathbb{C}^{\mathbb{N}}$ takav da za sve $K \Subset O$ i svako $p > 0$ postoji $n_p > 0$ takvo da

$$\sup_{x \in K} n^{p-2} |u_n(x) - r_n| \leq 1, \;\; n > n_p.$$

Iz Teoreme 6 dobijamo da je uopšteno očekivanje $m \in \mathcal{G}(O)$ slabo stacionarnog Kolomboovog stohastičkog procesa na centralno simetričnom konveksnom otvorenom skupu $O \subseteq \mathbb{R}^d$ uopštena konstanta $m \in \mathcal{R}$.

Stroga stacionarnost implicira slabu stacionarnost Kolomboovog stohastičkog procesa. Obrnuto ne važi. Međutim, pošto je gausovski Kolomboov stohastički proces kompletno određen uopštenim očekivanjem i uopštenom korelacijskom funkcijom, sledi da je svaki slabo stacionaran gausovski Kolomboov stohastički proces strogo stacionaran. Izvodi (slabo) stacionarnog Kolomboovog stohastičkog procesa su (slabo) stacionarni.

Kolomboov stohastički proces u na O sa vrednostima u $L^p(\Omega)$ ima (*slabo*) *stacionarne priraštaje* ako je ∇u (slabo) stacionaran proces. Lako se proverava da Braunovo kretanje ima stacionarne priraštaje.

Na kraju disertacije je dat metod za rešavanje klase linearnih stohastičkih parcijalnih diferencijalnih jednačina u okviru stacionarnih gausovskih Kolomboovih stohastičkih procesa na $\mathbb{R}^d$ sa vrednostima u $L^2(\Omega)$. Pošto ćemo koristiti Furijeovu transformaciju potrebno je da pređemo na temperirane Kolomboove stohastičke procese na $\mathbb{R}^d$ sa vrednostima u $L^2(\Omega)$.

Neka je

$$P(D) = \sum_{|\alpha|\leq k} \tilde{a}_\alpha D_x^\alpha, \quad \tilde{a}_\alpha \in \mathcal{R}_c,$$

diferencijalni operator reda k sa uopštenim konstantnim koeficijentima. Predstavimo metod za rešavanje jednačine

$$P(D)u(\omega, x) = f(\omega, x), \ \ \omega \in \Omega, \ x \in \mathbb{R}^d, \tag{2}$$

gde je $f = [(f_n)_n]$ slabo stacionaran temperiran gausovski Kolomboov stohastički proces na $\mathbb{R}^d$ sa vrednostima u $L^2(\Omega)$ sa uopštenim očekivanjem $\tilde{m}_f = [(m_{f_n})_n] \in \mathcal{R}_c$ i uopštenom korelacijskom funkcijom $B_f = [(B_{f_n})_n] \in \mathcal{G}_\tau(\mathbb{R}^{2d})$. U sledećoj teoremi dat je potreban uslov za egzistenciju stacionarnog rešenja posmatrane jednačine.

Teorema 7 *Neka je $f = [(f_n)_n] \in \mathcal{G}_{\tau,L^2}(\Omega, \mathbb{R}^d)$ slabo stacionarni temperirani gausovski Kolomboov stohastički proces sa uopštenim očekivanjem $\tilde{m}_f = [(m_{f_n})_n]$ i uopštenom korelacijskom funkcijom $B_f = [(B_{f_n})_n]$.*

(a) Za uopšteno očekivanje $\tilde{m}_u = [(m_{u_n})_n] \in \mathcal{R}_c$ slabo stacionarnog rešenja jednačine (2) važi

$$\tilde{m}_u = \begin{cases} \frac{\tilde{m}_f}{\tilde{a}_0}, & \textit{ako je } \tilde{a}_0 \neq \tilde{0}, \\ \textit{proizvoljno}, & \textit{ako je } \tilde{a}_0 = \tilde{0} \textit{ i } \tilde{m}_f = \tilde{0}, \\ \textit{ne postoji}, & \textit{ako je } \tilde{a}_0 = \tilde{0} \textit{ i } \tilde{m}_f \neq \tilde{0}. \end{cases}$$

Specijalno, ako je $\tilde{a}_0 = \tilde{0}$ i $\tilde{m}_f \neq \tilde{0}$, onda jednačina (2) nema slabo stacionarno rešenje u $\mathcal{G}_{\tau,L^2}(\Omega, \mathbb{R}^d)$.

(b) Uopštena korelacijska funkcija $[(B_{u_n})_n] \in \mathcal{G}_\tau(\mathbb{R}^{2d})$ slabo stacionarnog rešenja jednačine (2) zadovoljava

$$P_n(D)P_n(-D)B_{u_n}(z) = B_{f_n}(z), \;\; z = x - y \in \mathbb{R}^d, \, n \in \mathbb{N}.$$

Specijalno, ako postoji otvoren skup $S \subset \mathbb{R}^d$ takav da je $\hat{B}_{f_n}(\xi) > 0$, za sve $\xi \in S, n \in \mathbb{N}$, i $P_n(\xi)P_n(-\xi) < 0$, za $\xi \in S, n \in \mathbb{N}$, za sve reprezentacije koeficijenata $(a_\alpha)_n$, onda B_{u_n} ne može biti pozitivno-definitna funkcija.

(c) Neka je $|P_n(\xi)| \geq Cn^{-r}(1 + |\xi|)^k$, $n \in \mathbb{N}$, $\xi \in \mathbb{R}^d$, za neke $C > 0$, $r > 0$, $k > 0$, za neku reprezentaciju koeficijenata $(a_\alpha)_n$. Tada jednačina (2) ima slabo stacionarno rešenje $u = [(u_n)_n] \in \mathcal{G}_{\tau,L^2}(\Omega, \mathbb{R}^d)$ i njegova uopštena korelacijska funkcija zadovoljava

$$P_n(\xi)P_n(-\xi)\hat{B}_{u_n}(\xi) = \hat{B}_{f_n}(\xi), \;\; \xi \in \mathbb{R}^d, \, n \in \mathbb{N}.$$

Kao iliustraciju, metod smo primenili na stacionarnu Klajn–Gordonovu jednačinu

$$(\tilde{1} - \Delta_x)u(\omega, x) = \tilde{c} + \tilde{f} \cdot \partial_x^k w(\omega, x), \;\; \omega \in \Omega, \, x \in \mathbb{R}^d,$$

gde su $\tilde{1} = (1, 1, 1, \ldots)$, $\tilde{c}, \tilde{f} \in \mathcal{R}_c$ uopštene konstante i $w = [(w_n)_n]$ je beli šum.

Biografija

Snežana Gordić je rođena 8. avgusta 1984. godine u Vlasenici. Osnovnu školu "Braća Jakšić" u Milićima završava kao đak generacije 1999. godine i iste godine upisuje opšti smer Gimnazije "Milutin Milanković" u Milićima. Gimnaziju je završila 2003. godine kao nosilac Vukove diplome.

Osnovne studije na Prirodno–matematičkom fakultetu u Novom Sadu, smer Diplomirani matematičar–profesor matematike, upisala je 2003. godine i završava ih 2007. godine sa prosečnom ocenom 9.79.

Nakon završetka osnovnih studija upisuje master studije na istom fakultetu i opredeljuje se za modul Teorijska matematika. Master studije završava 2009. godine sa prosečnom ocenom 10.00. Zimski semestar školske 2008/2009 godine je provela na Institutu za matematiku, Univerziteta u Beču, kao stipendista fondacije ÖAD.

Doktorske studije matematike na Prirodno–matematičkom fakultetu u Novom Sadu upisala je 2009. godine i zaključno sa aprilom 2016. godine položila je sve ispite predviđene planom i programom sa prosečnom ocenom 9.83.

Od 2007. do 2009. godine kao student master studija, a od 2009. do 2011. godine kao istraživač–pripravnik je držala vežbe na Prirodno–matematičkom fakultetu u Novom Sadu. Od februara 2011. do septembra 2016. godine radila je kao istraživač–saradnik na Prirodno–matematičkom fakultetu u Novom Sadu. Od septembra 2016. godine do danas radi kao asistent na Katedri za matematiku i metodiku nastave matematike na Pedagoškom fakultetu u Somboru.

Oblast njenog naučnog interesovanja su stohastički procesi i teorija uopštenih funkcija. Imala je izlaganja na konferencijama WING201, XIV srpskom matematičkom kongresu i GF2018. Koautor je tri naučna rada i jedne zbirke zadataka.

Novi Sad, 5. septembar 2018. godine

Snežana Gordić

Curriculum Vitae

Personal data

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Education

2003–2007	Department of Mathematics and Informatics Faculty of Sciences, University of Novi Sad Graduate mathematician–Secondary Teacher of Mathematics GPA: 9.79 (out of 10.00)
2007–2009	Department of Mathematics and Informatics Faculty of Sciences, University of Novi Sad Master in Mathematics Scientific area of studies: Mathematics - Theoretic mathematics GPA: 10.00 (out of 10.00)
2009–present	Department of Mathematics and Informatics Faculty of Sciences, University of Novi Sad Doctoral studies in Mathematics GPA: 9.83 (out of 10.00)

Work experience

2009–2010	Junior Researcher Department of Mathematics and Informatics Faculty of Sciences University of Novi Sad
2010–2016	Secondary Teacher of Geometry High School "Jovan Jovanović Zmaj" Novi Sad
2011–2016	Associate Resercher Department of Mathematics and Informatics Faculty of Sciences University of Novi Sad
2016–present	Teaching Assistant Chair of Mathematics and Methodology of Mathematics Faculty of Education in Sombor University of Novi Sad

Teaching

2007–present Conduction exercises for following courses:

- *For students of Mathematics:* Numerical Analysis, Analytic Geometry, History of Mathematics, Methods of Solving Operator Equations
- *For students of Informatics:* Analysis 2
- *For students of Physics:* Mathematics 1, Mathematics 3, Probability and Statistics
- *For students of Chemistry:* Statistics, General Mathematics, Selected Chapters of Mathematics, Software for Processing Experimental Data
- *For students of Tourism, Hotel Management, Gastronomy, Hunting Tourism, Science in Teaching Geography:* Business Statistics, Business Mathematics, Statistical Methods in Geography
- *For students of study programs Librarianship and Information Science, Media Design in Education, Preschool Teacher Education, Primary Teacher Education:* Mathematics 1, Mathematics 2, Introduction to Statistics, Games in Learning Math, Games in Mathematical Activities, Mathematics for Talented Pupils, Didactics of Mathematics 2, Modern Didactics of Mathematics 2

Study visits

- 1^{st} October 2008 – 28^{th} February 2009, $5-$months scholarship of the ÖAD for doing research at the Faculty of Mathematics, University of Vienna, Austria, supervised by Prof. Michael Kunzinger.

- 4^{th} December 2017 – 10^{th} December 2017, Unit for Engineering Mathematics, Department of Civil Engineering, University of Innsbruck, Austria, scientific host: prof. Michael Oberguggenberger

Research

2009–2010	Engaged at the project *Functional Analysis, ODEs and PDEs with Singularities, No. 144016,* financed by the Ministry of Science of Serbia and Montenegro.
2011–present	Engaged at the project *Methods of Functional and Harmonic Analysis and PDE with Singularities, No. 174024* financed by the Ministry of Education, Science and Technological Development of Serbia.
2015–2017	Engaged at the project *Solutions of Stochastic Equations Involving Differential and Pseudodifferential Operators in Algebras of Generalized Stochastic Processes* project 451-03-01039/2015-09/26 of the bilateral scientific and technological co–operation between Serbia and Austria
2018–2019	Engaged at the project *Functional analytic methods for models of wave propagation in viscoelastic media* project 451-03-02141/2017-09/12 of the bilateral scientific and technological co–operation between Serbia and Austria

Scientific impact

Attended several national and international conference, workshops, seminars, winter/summer school, apart from those listed below.

▷ **Presentations at Conferences, Workshops and Seminars**

- Gordić S., *Generalized Stochastic Processes in Algebras of Generalized Functions.* Workshop and Conference: Wien–Innsbruck–Novi Sad–Gent, June 29–July 3, 2016, University of Innsbruck, Austria.
- Gordić S., *Generalized Stochastic Processes with Applications in Equation Solving*. Seminar for Analysis and Foundation of Mathematics, Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, 25^{th} April 2016.
- Gordić S., *Mathematical games in science and teaching*. Seminar for Mathematics and Teaching of Mathematics, Faculty of Education in Sombor, University of Novi Sad, 23^{rd} May 2017.

- Gordić S., *Probabilistic properties of Colombeau stochastic processes.* XIV Serbian mathematical congress, May 16-19, 2018, Faculty of Science, University of Kragujevac, Serbia.

- Gordić S., *Stationary Colombeau Stochastic Processes.* International Conference on Generalized Functions, Dedicated to Professor Michael Oberguggenberger's 65th birthday, August 27-31, 2018, Faculty of Sciences, University of Novi Sad

▷ **List of publications**

Scientific papers:

- Zarin H., Gordić S., *Numerical solving of singularly perturbed boundary value problems with discontinuities.* Novi Sad Journal of Mathematics, Vol. 42, No. 1, 2012, 131-145

- Gordić, S., Oberguggenberger, M., Pilipović, S., Seleši, D., *Probabilistic properties of generalized stochastic processes in algebras of generalized functions*. Monatshefte für Mathematik, Vol. 189, No. 4, 2018, 609-633

- Gordić, S., Oberguggenberger, M., Pilipović, S., Seleši, D., *Generalized stochastic processes in algebras of generalized functions: independence, stationarity and SPDEs.* Preprint. 2018.

Textbooks:

- Oparnica Lj., Zobenica M., Gordić S., *A Collections of Exercises in Combinatorics and Probability* (in Serbian). Faculty of Eduaction in Sombor, University of Novi Sad, Sombor.

Membership

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U disertaciji se stohastički procesi posmatraju u okviru Kolomboove algebre uopštenih funkcija. Takve procese nazivamo Kolomboovi stohastički procesi.

Pojam vrednosti Kolomboovog stohastičkog procesa u tačkama sa kompaktnim nosačem je uveden. Dokazana je merljivost odgovarajuće slučajne promenljive sa vrednostima u Kolomboovoj algebri uopštenih konstanti sa kompaktnim nosačem, snabdevenom topologijom generisanom oštrim otvorenim loptama.

Uopštena korelacijska funkcija i uopštena karakteristična funkcija Kolomboovog stohastičkog procesa su definisane i njihove osobine su izučavane. Pokazano je da se karakteristična funkcija klasičnog stohastičkog procesa može potopiti u prostor uopštenih karakterističnih funkcija. Dati su primeri uopštenih karakterističnih funkcija gausovskih Kolomboovih stohastičkih procesa. Data je strukturna reprezentacija uopštene korelacijske funkcije sa nosačem na dijagonali. Kolomboovi stohastički procesi sa nezavisnim vrednostima su predstavljeni. Izučavani su strogo stacionarni i slabo stacionarni Kolomboovi stohastički procesi. Kolomboovi stohastički procesi sa stacionarnim priraštajima su okarakterisani preko stacionarnosti gradijenta procesa.

Gausovska stacionarna rešenja za linearnu stohastičku parcijalnu diferencijalnu jednačinu sa uopštenim konstantnim koeficijentima su analizirana u okvirima Kolomboovih stohastičkih procesa.

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In this dissertation stochastic processes are regarded in the framework of Colombeau-type algebras of generalized functions. Such processes are called Colombeau stochastic processes.

The notion of point values of Colombeau stochastic processes in compactly supported generalized points is established. The Colombeau algebra of compactly supported generalized constants is endowed with the topology generated by sharp open balls. The measurability of the corresponding random variables with values in the Colombeau algebra of compactly supported generalized constants is shown.

The generalized correlation function and the generalized characteristic function of Colombeau stochastic processes are introduced and their properties are investigated. It is shown that the characteristic function of classical stochastic processes can be embedded into the space of generalized characteristic functions. Examples of generalized characteristic function related to gaussian Colombeau stochastic

processes are given. The structural representation of the generalized correlation function which is supported on the diagonal is given. Colombeau stochastic processes with independent values are introduced. Strictly stationary and weakly stationary Colombeau stochastic processes are studied. Colombeau stochastic processes with stationary increments are characterized via their stationarity of the gradient of the process.

Gaussian stationary solutions are analyzed for linear stochastic partial differential equations with generalized constant coefficients in the framework of Colombeau stochastic processes.

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