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FACULTY OF SCIENCES
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Translation invariant Banach spaces of distributions and boundary values of integral transform

-doctoral dissertation-

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Транслационо инваријантни Банахови простори дистрибуција и граничне вредности преко интегралне трансформације

-докторска дисертација-

Novi Sad, 2014

Dedicated to my family

Contents

Абстракт

Користимо ознаку $*$ за дистрибуционо (Шварцово), (M_p) (Берлингово) и $\{M_p\}$ (Роумиеуово) окружење. Уводимо и проучавамо нове (ултра)дистрибуционе просторе, тест функцијске просторе $\mathcal{D}^*_E$ и њихове дуале $\mathcal{D}'^*_{E'_*}$. Ови простори уопштавају просторе $\mathcal{D}^*_{L^q}$, $\mathcal{D}'^*_{L^p}$, $\mathcal{B}'^*$ и њихове тежинске верзије. Конструкција наших нових (ултра)дистрибуционих простора је засновано на анализи одговарајучих транслационо-инваријантних Банахових простора (ултра)дистрибуције E која је конволуциони модул над Беурлинговом алгебром L^1_ω, где је тежина ω повезана са операторима транслације простора E. Банахов простор E'_* означава простор $L^1_{\check{\omega}} * E'$. Користечи добијених резултата проучавамо конволуција ултрадистрибуција. Простори конволутора $\mathcal{O}'^*_C(\mathbb{R}^n)$ темперираних ултрадистрибуција, анализирани су помочу дуалности тест функцијских простора $\mathcal{O}^*_C(\mathbb{R}^n)$, дефинисаних овом тезом. Користечи својства транслационо-инваријантних Банахових простора темпериране ултрадистрибуције E, добијамо карактеризацију конволуције Ромуиеу-ових ултрадистрибуција, преко интеграбилних ултрадистрибуција. Доказујемо да: Конволуција две Роумиеуових ултрадистрибуција $T, S \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ постоји ако и само ако $\left(\varphi * \check{S}\right) T \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ за сваки $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$. Ми проучавамо граничне вредности холомпрфних функција дефинисаних на тубама. Доказане су нове теореме клина. Резултати се затим користе за прецтављање $\mathcal{D}'_{E'_*}$ као фактор простор холоморфних функција. Такође, представљамо елементе $\mathcal{D}'_{E'_*}$ користечи хеат кернел методе.

Нови Сад, 2014 Павел Димовски

Abstract

We use the common notation $*$ for distribution (Schwartz), (M_p) (Beurling) and $\{M_p\}$ (Roumieu) setting. We introduce and study new (ultra)distribution spaces, the test function spaces $\mathcal{D}^*_E$ and their strong duals $\mathcal{D}'^*_{E'_*}$. These spaces generalize the spaces $\mathcal{D}^*_{L^q}$, $\mathcal{D}'^*_{L^p}$, $\mathcal{B}'^*$ and their weighted versions. The construction of our new (ultra)distribution spaces is based on the analysis of a suitable translation-invariant Banach space of (ultra)distributions E, which turns out to be a convolution module over the Beurling algebra L^1_ω, where the weight ω is related to the translation operators on E. The Banach space E'_* stands for $L^1_{\check{\omega}} * E'$. We apply our results to the study of the convolution of ultradistributions. The spaces of convolutors $\mathcal{O}'^*_C(\mathbb{R}^n)$ for tempered ultradistributions are analyzed via the duality with respect to the test function spaces $\mathcal{O}^*_C(\mathbb{R}^n)$, introduced in this thesis. Using the properties of the translation-invariant Banach space of ultradistributions E, we obtain a full characterization of the general convolution of Roumieu ultradistributions via the space of integrable ultradistributions. We show: The convolution of two Roumieu ultradistributions $T, S \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ exists if and only if $\left(\varphi * \check{S}\right) T \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$. In addition, we study boundary values of holomorphic functions defined in tube domains. New edge of the wedge theorems are obtained. The results are then applied to represent $\mathcal{D}'_{E'_*}$ as a quotient space of holomorphic functions. We also give representations of elements of $\mathcal{D}'_{E'_*}$ via the heat kernel method.

Novi Sad, 2014 Pavel Dimovski

Acknowledgments

I would like to express my deepest gratitude to my advisor Prof. Stevan Pilipović for the continuous support and immense knowledge. He has been a tremendous mentor for me. His advices on both research as on personal life have been priceless.

I am indebted to my other advisor Prof. Jasson Vindas for our collaboration, and all the knowledge and references I got from him. Without his assistance and dedicated involvement in every step throughout the process, this thesis would have never been accomplished.

Thanks to my friend and colleague Bojan Prangoski, for sharing his enthusiasm. He made me realize that mathematics is as Mell Gibson would say "Freedom!". Also thanks to Daniel Velinov, for being great colleague, roommate and an (onion free) chef. He showed me that Macedonian is really an exotic language.

Thanks to Svetlana Poznanoviḱ for providing me a lot of (for me priceless) papers.

My great appreciation for Nataša and Dragan for their affection and support during the hardest moments. No need to say more, they were here whenever I needed them and still are.

Now I am writing, I think not so bad in Latex, only because of one person. That is Vesna Andova. Maybe she could work with me on my English because I'm struggling to find the right words. Thanks to Vesna for her patients and understanding.

Finally I would like to thank my family for the support they provided me through everything.

Preface

Translation-invariant spaces of functions, distributions and ultradistributions are very important in mathematical analysis. They are connected with many central questions in harmonic analysis [5, 25, 26, 31, 86, 112].

This thesis introduces and studies new classes of translation-invariant distribution spaces, the test function space $\mathcal{D}_E$ and their duals, denoted as $\mathcal{D}'_{E'_*}$. The construction of these spaces is based upon the analysis of suitable translation-invariant Banach spaces of distributions E. As will be shown, our new spaces are useful in the analysis of boundary values of holomorphic functions in tube domains and solutions to the heat equation in the upper half-space.

The space E is a natural extension of a large class of weighted L^p spaces, while $\mathcal{D}'_{E'_*}$ generalizes the spaces $\mathcal{D}'_{L^p}$. The spaces $\mathcal{D}'_{L^p}$ were introduced by Schwartz [94, 93] as a major tool in the study of convolution within distribution theory and are still the subject of various modern investigations [71]. Ortner and Wagner [69] have considered weighted versions of the $\mathcal{D}'_{L^p}$ spaces, which have proved usefulness in the analysis of convolution semigroups associated to many PDE [70] and boundary values of harmonic functions [2]. It turns out that their spaces are also particular instances of our $\mathcal{D}'_{E'_*}$.

The study of boundary values of holomorphic functions in distribution and ultradistribution spaces has shown to be quite important for a deeper understanding of properties of generalized functions, which are of much relevance to the theory of PDE [39, 90]. There is a vast literature in the subject, we only mention a small part of it. The theory of analytic representation of distributions was initiated by Köthe [55] and Tillmann [102]. We also mention the influential works of Silva [97], Martineau [59, 61], and Vladimirov [106, 107]. The book by Carmichael and Mitrović [13] contains an overview of results concerning boundary values in distribution spaces. For ultradistributions and hyperfunctions, see the articles [24, 49, 64, 75] and the monographs [12, 44, 68].

The representation of the Schwartz spaces $\mathcal{D}'_{L^p}$ as boundary values of holomorphic functions has also attracted much attention. The problem has been treated by Tillmann [103], Łuszczki and Zieleźny [58], and Bengel [3]. More recently [32, 33], Fernández, Galbis, and Gómez-Collado have obtained various ultradistribution analogs of such results. All these works basically deal with holomorphic functions in tube domains whose bases are the orthants of $\mathbb{R}^n$. In a series of papers [8, 9, 10, 11], Carmichael has systematically studied boundary values in $\mathcal{D}'_{L^p}$ of holomorphic functions defined in more general tubes, namely, tube domains whose bases are open convex cones. The present work makes a thorough analysis of boundary values in the space $\mathcal{D}'_{E'_*}$. Many of the results we obtain in Section 2.1

are new or improve earlier results even for the special case $\mathcal{D}'_{E'_*} = \mathcal{D}'_{L^p}$.

In his seminal work [62, 63] Matsuzawa introduced the so-called heat kernel method in the theory of generalized functions. His approach consists in describing distribution and hyperfunction spaces in terms of solutions to the heat equation fulfilling suitable growth estimates. Several other authors have investigated characterizations of various distributions, ultradistributions, and hyperfunction spaces [16, 18, 46, 101]. Our results from Section 2.4 add new information to Matsuzawa's program by obtaining the description of $\mathcal{D}'_{E'_*}$ via the heat kernel method. In the case of $\mathcal{D}'_{L^p}$, this characterization reads as follows: $f \in \mathcal{D}'_{L^p}$ if and only if there is a solution U to the heat equation on $\mathbb{R}^n \times (0, t_0)$ such that $\sup_{t\in(0,t_0)} t^k \|U(\,\cdot\,, t)\|_{L^p} < \infty$ for some $k \geq 0$ and $f = \lim_{t\to 0^+} U(\,\cdot\,, t)$.

The second half of the thesis focuses on ultradistributions. In Section 3.4 we obtain results which characterize convolutors through the duality with respect to the space of test functions $\mathcal{O}_C^{\{M_p\}}$. Such results were recently published in our result paper [21]. Often, the Beurling case is not considered since it is simpler than the Roumieu one. An important achievement of the thesis is related to the existence of the general convolution of ultradistributions of Roumieu type. After the introduction of Schwartz' conditions for the general convolvability of distributions, many authors gave alternative definitions and established their equivalence. Notably, Shiarishi [99] found out that the convolution of two distributions $S, T \in \mathcal{D}'(\mathbb{R}^n)$ exists if and only if: $\left(\varphi * \check{S}\right) T \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$. The existence of the convolution for Beurling ultradistributions can be treated [41, 40, 78] analogously as for Schwartz distributions. In contrast, corresponding characterizations for the convolution of Roumieu ultradistributions has been a long-standing open question in the area. It was only until recently [80] that progress in this direction was made through the study of ε tensor products of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ and locally convex spaces. The following characterization of convolvability was shown in [80]: The convolution of two ultradistributions $T, S \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ exists if and only if $\left(\varphi * \check{S}\right) T \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ and for every compact subset K of $\mathbb{R}^n$, $(\varphi, \chi) \mapsto \left\langle\left(\varphi * \check{T}\right) S, \chi\right\rangle$, $\mathcal{D}_K^{\{M_p\}} \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}} \longrightarrow \mathbb{C}$, is a continuous bilinear mapping. The spaces $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ and $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ were introduced in [79]. In this thesis we make a significant improvement to this result, namely, we show the following more transparent version of Shiarishi's result for Roumieu ultradistributions: *The convolution of* $T, S \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ *exists if and only if* $\left(\varphi * \check{S}\right) T \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ *for every* $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$.

Our proof of the above-mentioned result about the general convolvability of Roumieu ultradistributions is postponed to the last section of the thesis and it is based upon establishing the topological equality $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1} = \mathcal{D}'^{\{M_p\}}_{L^1}$. This and other topological properties of the spaces of "integrable" ultradistributions can be better understood from a rather broader perspective. In this thesis we introduce and study new classes of translation-invariant ultradistribution spaces which are natural generalizations of the weighted $\mathcal{D}'^{*}_{L^p}$-spaces [12]. In the distribution setting, our recent work [19] (explained in detail in Chapter 1 of the present thesis) extends that of Schwartz on the $\mathcal{D}'_{L^p}$ spaces and that of Ortner and Wagner on their weighted versions [69, 111]; recent applications of those ideas to the study of boundary values of holomorphic functions and solutions to the heat equation can be found in our recent paper [20]; such applications are treated in detail in Chapter 2 of this thesis. The theory we present here is a generalization of that given in [19] for distributions. Although some results are analogous to those for distributions, it should be remarked that their proofs turn out to be much more complicated since they demand the use of more sophisticated techniques and new ideas adapted to the ultradistribution setting– especially in the Roumieu case.

This doctoral dissertation is organized in five chapters. Thematically, it is divided in two parts. Chapter 1 and Chapter 2 are devoted to distributions and Chapter 3 and Chapter 4 deal with ultradistributions.

Chapter 0 is devoted to introduction to the spaces of distributions and ultradistributions, the notation used in this thesis, the known facts and results.

In Chapter 1 we study a class of tempered translation-invariant Banach spaces of distributions on the Euclidean space $\mathbb{R}^n$. The class of Banach spaces in which we are interested are translation-invariant spaces E such that $\mathcal{D}(\mathbb{R}^n) \hookrightarrow E \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$, $T_h : E \to E$ for every $h \in \mathbb{R}^n$ and the growth function ω of its translation group (cf. Definition 1.1.1) is polynomially bounded. The symbol " $\hookrightarrow$ " stands for continuous dense inclusion. The space E carries a natural Banach convolution module structure over the Beurling algebra L^1_ω. Furthermore, it is shown that E possesses bounded approximations of the unity for this module structure. We also study properties of its dual space. Inspired by various results on factorization of Banach and Fréchet convolution algebras [47, 74, 87, 110], we introduce the Banach space $E'_* = L^1_{\check{\omega}} * E'$. The convolution module structures of E and E'_* are crucial for achieving the main results of this thesis. Our new distribution spaces are introduced in Section 1.2. The test function space $\mathcal{D}_E$ consists of tempered distributions for which all partial derivatives belong to E. We first show that $\mathcal{D}_E$ is a Fréchet space of smooth functions and actually the following inclusions between familiar test function spaces hold $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_E \hookrightarrow \mathcal{O}_C(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n)$. The space $\mathcal{D}'_{E'_*}$ is defined as the strong dual of $\mathcal{D}_E$. It satisfies $\mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{O}'_C(\mathbb{R}^n) \to \mathcal{D}'_{E'_*} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, the second inclusion becomes dense when E is reflexive. We study various structural and topological properties of $\mathcal{D}'_{E'_*}$ via Schwartz parametrix method [94]. In particular, it is proved the every $f \in \mathcal{D}'_{E'_*}$ is the finite sum of partial derivatives of elements of the Banach space E'_*. If E is reflexive, we prove that $\mathcal{D}_E$ is an FS^* space and $\mathcal{D}'_{E'_*}$ is a DFS^* space [48], so that they are reflexive in this case. Convolution and multiplicative products on $\mathcal{D}'_{E'_*}$ are also discussed. Our ideas are exemplified with the weighted spaces $\mathcal{D}'_{L^p_\eta}$, $1 \leq p < \infty$, and the space of η-bounded distributions $\mathcal{B}'_\eta$, where η is a polynomially bounded weight function.

Chapter 2 is devoted to the study of boundary values of holomorphic functions and analytic representations of $\mathcal{D}'_{E'_*}$. Our first main result (Theorem 2.1.1) characterizes those holomorphic function in truncated wedges which have boundary values in $\mathcal{D}'_{E'_*}$. It is worth pointing out that those two results improve earlier knowledge about boundary values in $\mathcal{D}'_{L^p}$; in fact, part of our conclusion is strong convergence in $\mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$. The strong convergence was only known for $1 < p < \infty$ and certain tubes [3, 8, 9, 11, 103]. Next, we consider extensions of Carmichael's generalizations of the H^p spaces [9, 10, 11]. We also provide in this section new edge of the wedge theorems. Our ideas are then applied to exhibit an isomorphism between $\mathcal{D}'_{E'_*}$ and a quotient space of holomorphic functions, this quotient space is constructed in the spirit of hyperfunction theory. Chapter 2 concludes with the heat kernel characterization of $\mathcal{D}'_{E'_*}$ in Section 2.4.

In the first part of Chapter 3 we analyze the space of convolutors, also called here ultratempered convolutors, for the space of tempered ultradistributions. Naturally, such an investigation would be of general interest as being part of the modern theory of multipliers. For tempered distributions this space was introduced already by Schwartz [94] and the full topological characterization was given in the book of Horváth [38]. The space of ultratempered convolutors $\mathcal{O}'^*_C(\mathbb{R}^n)$ was recently studied in [19]. We give structure theorems for the space of convolutors in the Roumieu case, as well as the completeness of $\mathcal{O}'^{(M_p)}_C(\mathbb{R}^n)$, resp. $\mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$. Also the space of multipliers $\mathcal{O}^{(M_p)}_M(\mathbb{R}^n)$, resp. $\mathcal{O}^{\{M_p\}}_M(\mathbb{R}^n)$ is considered. Characterization theorem for the space of multipliers in Roumieu case is given. The Fourier transform gives a topological isomorphism between the space of multipliers and the space of convolutors in Roumieu case.

In Section 3.3 we characterize tempered ultradistributions using the growth of its convolution averages. Following Komatsu approach [49], we describe $\mathcal{O}'^*_C(\mathbb{R}^n)$ through the duality with respect to the test function space $\mathcal{O}^*_C(\mathbb{R}^n)$, constructed in this thesis. The treatment of the Roumieu case is considerably more elaborated than the Beurling one, as it involves the use of dual Mittag-Leffler lemma arguments for establishing the sought duality. We also mention that the characterization of the spaces $\mathcal{O}'^*_C(\mathbb{R}^n)$ is given using weighted L_2 estimates.

Translation-invariant Banach spaces of tempered ultradistributions are considered in Chapter 4, see also [22]. The presented results are analogous to the results concerning translation-invariant Banach spaces of tempered distributions analyzed in Chapter 1 (see also our paper [21]), although, the proofs and technics used here are different or adapted in ultradistributional setting. We are interested in class of Banach spaces of ultradistributions that satisfy the conditions $\mathcal{D}^*(\mathbb{R}^n) \hookrightarrow E \hookrightarrow \mathcal{D}'^*(\mathbb{R}^n)$, translation operators $T_h : E \to E$ for every $h \in \mathbb{R}^n$ and has norm with ultrapolynomial growth denoted by $\omega(h)$. Such E becomes Banach module over the Beurling algebra L^1_ω and has nice regularizing properties. Using duality, we obtain some results concerning E' which turns out to be also Banach module over the Beurling algebra $L^1_{\check{\omega}}$. But E' lacks some of the properties that E has. That motivates the definition of a new closed subspace E'_* of E' which has better properties with respect to the translation group. We give a characterization of E'_* as the largest subspace of E' which satisfies $\lim_{h\to 0} \|T_h f - f\|_{E'} = 0$ for each $f \in E'$. In Section 4.2 we define our new test spaces $\mathcal{D}^{(M_p)}_E$ and $\mathcal{D}^{\{M_p\}}_E$ of Beurling and Roumieu type, respectively. In the Roumieu case we also consider another space $\tilde{\mathcal{D}}^{\{M_p\}}_E$. The test spaces satisfy the property $\mathcal{S}^*(\mathbb{R}^n) \hookrightarrow \mathcal{D}^*_E \hookrightarrow E \hookrightarrow \mathcal{S}'^*(\mathbb{R}^n)$ and $\mathcal{D}^*_E$ is a topological module over the Beurling algebra L^1_ω. The spaces $\mathcal{D}^*_E$ are continuously and densely embedded in the spaces $\mathcal{O}^*_C(\mathbb{R}^n)$ defined in the Section

3.4. In Section 4.3 we consider the strong dual of the spaces $\mathcal{D}^*_E$ denoted by $\mathcal{D}'^*_{E'_*}$ which has been defined in Section 4.2. A structural theorem for ultradistributions in the space $\mathcal{D}'^*_{E'_*}$ concerning convolution with elements of $\mathcal{D}^*(\mathbb{R}^n)$ and representation as finite sum of ultradifferential operators of elements in $E'_* \cap UC_\omega$ is obtained. Analogously to the distribution results considered in 1.3, we obtain results that enables us to embed the spaces $\mathcal{D}^*_E$ into the space of E'_* tempered ultradistributions $\mathcal{S}'^*(\mathbb{R}^n, E'_*)$. We prove that the spaces $\mathcal{D}^{\{M_p\}}_E$ and $\tilde{\mathcal{D}}^{\{M_p\}}_E$ are topologically isomorphic. When E is reflexive, $\mathcal{D}^{(M_p)}_E$ and $\mathcal{D}'^{\{M_p\}}_{E'}$ are (FS^*)-spaces, $\mathcal{D}^{\{M_p\}}_E$ and $\mathcal{D}'^{(M_p)}_E$ are (DFS^*)-spaces. Also, examples of spaces $\mathcal{D}^*_E$ and $\mathcal{D}'^*_{E'_*}$ are given. The techniques used in this thesis enable us to identify the spaces $\mathcal{D}^*_{C_\eta}$ with already known spaces $\dot{\mathcal{B}}^*_\eta$ [79]. This theory is applied in the Section 4.5 to the study of the convolution of Roumieu ultradistributions, namely we prove that the convolution of $T, S \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ exists if and only if $(\varphi * \check{S})\, T \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$.

I want to emphasize that already known results have citation next to them, in order to distinguish them from the new results.

Chapter 0

Preliminaries

0.1 Distributions and tempered distributions

We use the standard notation from Schwartz distribution theory. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. There exist a sequence of compact sets in $\mathbb{R}^n$, $\{K_j\}_{j=1}^{\infty}$ satisfying the condition

$$K_1 \subset \text{int}K_2 \subset K_2 \subset ...\text{int}K_j \subset K_j \subset K_{j+1} \text{ and } \Omega = \bigcup_{j=1}^{\infty} \text{int}K_j. \tag{1}$$

For arbitrary compact set K, consider the space

$$\mathcal{D}_K(\Omega) = \{\varphi \in C^{\infty}(\Omega) \mid \text{supp}\varphi \subset K\}$$

which is a Fréchet space when provided with the family of norms

$$p_{j,K}(\varphi) = \sup_{\substack{x \in K \\ |\alpha| \leq j}} |\varphi^{(\alpha)}(x)|, \quad j \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let the sequence $\{K_j\}_{j=1}^{\infty}$ satisfy condition (1). The *test function space* is the space

$$\mathcal{D}(\Omega) = \{\varphi \in C^{\infty}(\Omega) \mid \text{supp}\varphi \text{ is compact }\} = \bigcup_{j=1}^{\infty} \mathcal{D}_{K_j}(\Omega)$$

endowed with the inductive limit topology. The topology on $\mathcal{D}(\Omega)$ is the same independently of the choice of the sequence $\{K_j\}_{j=1}^{\infty}$ satisfying the condition (1). A *distribution* on Ω is a continuous linear functional on $\mathcal{D}(\Omega)$. The vector space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$.

Also we use, the space of *rapidly decreasing functions*,

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^{\infty} \,|\forall \alpha, \beta \in \mathbb{N}_0^n \,, \left|x^{\beta}\varphi^{(\alpha)}(x)\right| < \infty\}$$

which is a Fréchet space when equipped with topology defined via the family of norms

$$q_j(\varphi) = \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq j}} (1 + |x|)^j |\varphi^{(\alpha)}(x)|, \quad j \in \mathbb{N}_0.$$

A *tempered distribution* is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. Unless stated differently, $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are endowed with the strong topology.

The *Fourier transform* of $f \in L^1(\mathbb{R}^n)$ denoted as $(\mathcal{F}f)(\xi) = \hat{f}(\xi)$ is the integral

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

The Fourier transform is a topological endomorphism on $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined as

$$\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The Fourier transform is a topological endomorphism on $\mathcal{S}'(\mathbb{R}^n)$.

The function $\check{g}$ denotes the *reflection*, i.e., $\check{g}(x) = g(-x)$. Given $h \in \mathbb{R}^n$, we employ the notation T_h for the *translation operator*, that is, $(T_h g)(x) = g(x+h)$. Naturally, the translation and reflection operations are well-defined for distributions as well. A subspace $Y \subset \mathcal{D}'(\mathbb{R}^n)$ is called *translation-invariant* if $T_h(Y) = Y$ for all $h \in \mathbb{R}^n$.

The space of distributions with values in a (Hausdorff) locally convex space X is $\mathcal{D}'(\mathbb{R}^n, X) = L_b(\mathcal{D}(\mathbb{R}^n), X)$ [93, 98], the space of continuous linear mappings from $\mathcal{D}(\mathbb{R}^n)$ to X, equipped with the strong topology. Similarly, $\mathcal{S}'(\mathbb{R}^n, X)$ stands for the space of *X-valued tempered distributions.*

0.1.1 Some results on distributions

Using the following well known theorem of Schwartz [94], we will obtain results concerning distributions with values in particular Banach spaces of distributions. Also, analogously to the distribution theory, we obtain similar results in ultradistribution theory.

Theorem 0.1.1. *([94]) Let u be a continuous linear map from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(i) u commutes with all all partial derivatives.

(ii) u commutes with all translations.

(iii) u commutes with all convolutions.

(iv) There exists a distribution L on $\mathbb{R}^n$ such that

$$u(f) = L * f$$

for every $f \in \mathcal{D}(\mathbb{R}^n)$.

The *parametrix of Schwartz* is crucial for our observations.

Lemma 0.1.1. *([94]) Let K be compact symmetric neighborhood of* 0 *and $\chi \in \mathcal{D}_K$ be such that $\chi = 1$ near 0. Let F_l be* fundamental solution *of $\triangle^l$, i.e., $\triangle^l F_l = \delta$. Then, $\triangle^l(\chi F_l) - \delta = \varsigma_l \in \mathcal{D}(\mathbb{R}^n)$ and the following formula*

$$f = \triangle^l((\chi F_l) * f) - \varsigma_l * f,$$

holds for every $f \in \mathcal{D}'(\mathbb{R}^n)$.

We will frequently use the following Lemma:

Lemma 0.1.2. *([36]) Let $M \subset \mathbb{R}^n$ be measurable. Let $f(x,a)$ be a family of functions of $x \in M$ depending on the parameter $a \in B = B(a_0, r) = \{y \in \mathbb{R}^k | \|y - a_0\| < r\}$, such that for each $a \in B$, $f(x,a) \in L^1(M)$. Consider the function F on B defined by*

$$F(a) = \int_M f(x,a)dx \text{ for all } a \in B.$$

(1) Assume that for each $x \in M$, $f(x,a)$ is a continuous function of a at the point $a_1 \in B$ and that there is a function $g(x) \in L^1(M)$ such that

$$|f(x,a)| \leq g(x) \text{ for all } (x,a) \in M \times B.$$

Then $F(a)$ is continuous at the point a_1.

(2) Assume that $\frac{\partial}{\partial a} f(x,a)$ exists for all $(x,a) \in M \times B$ and that there is a function $g(x) \in L^1(M)$ such that

$$\left| \frac{\partial}{\partial a} f(x,a) \right| \leq g(x) \text{ for all } (x,a) \in M \times B.$$

Then $F(a)$ is a differentiable function of $a \in B$, and

$$\frac{dF(a)}{da} = \int_M \frac{\partial f(x,a)}{\partial a} dx \text{ for all } (x,a) \in M \times B.$$

We also use the concept of the $\phi-$*transform*[28, 29, 83, 105], which is defined as follows.

Definition 0.1.1. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. The $\phi-$transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ is the smooth function

$$F_\phi f(x,t) = \langle f(x + t\xi), \phi(\xi) \rangle = (f * \check{\phi}_t)(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+.$$

where $\phi_t(\cdot) = t^{-n}\phi(\cdot/t)$ and $t \in \mathbb{R}_+$.

Let (Ω, S, μ) be σ-finite measure space and X be a Banach space. The function, $\mathbf{x} : \Omega \to X$, is a simple function if it is of the form $\mathbf{x}(s) = \sum_{i=1}^k a_i \chi_{B_i}(s)$ where $a_i \in X$, $B_i \in S$ and $\mu(B_i) < \infty$ for each i. A function $\mathbf{x} : \Omega \to X$ is said to be *strongly measurable* if there exists a sequence of simple functions $\mathbf{x}_j$ converging pointwise to $\mathbf{x}$. A function $\mathbf{x} : \Omega \to X$ is said to be *weakly measurable* if, for each $f \in X'$, the function $f \circ \mathbf{x}$ is a measurable scalar valued function.

A result, due to Pettis, says that if the function $\mathbf{x}$ takes values in a separable Banach space, then $\mathbf{x}$ is weakly measurable if and only if $\mathbf{x}$ is strongly measurable.

Definition 0.1.2. A strongly measurable function $\mathbf{f}$ is *Bochner integrable* if there exist a sequence of simple functions $\mathbf{f}_j$ converging to $\mathbf{f}$ pointwise and satisfying $\int_\Omega \|\mathbf{f}_j(t) - \mathbf{f}_i(t)\| d\mu \to 0$ when $i, j \to \infty$. If $\mathbf{f}$ is Bochner integrable, the *Bochner integral* of $\mathbf{f}$ is $\lim_{j\to\infty} \int_\Omega \mathbf{f}_j(t) d\mu$.

Let E be a Banach subspace of a Hausdorff locally convex space X such that the inclusion $E \to X$ is linear and continuous. We need the following characterization of $\mathcal{S}'(\mathbb{R}^n, E)$.

Theorem 0.1.2. *([28, 83]) Let* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \phi(x)dx = 1$. *Necessary and sufficient conditions for* $\mathbf{f}$ *to be in* $\mathcal{S}'(\mathbb{R}^n, E)$ *are:*

i) $F_\phi \mathbf{f}(x, y)$ *takes values in* E *for almost all* $(x, y) \in \mathbb{R}^n \times (0, 1]$ *and is measurable as* E *valued function on* $\mathbb{R}^n \times (0, 1]$, *and*

ii) There exist constants $k, l \in \mathbb{N}$ *and* $C > 0$ *such that*

$$\|F_\phi \mathbf{f}(x, y)\| \leq C \frac{(1 + |x|)^l}{y^k} \text{ for almost all } (x, y) \in \mathbb{R}^n \times (0, 1].$$

The measurability of E valued functions is meant in the Bochner sense. Also, the integrals of E valued functions are in the Bochner sense.

We fix the notation concerning tubes and cones. Let $V \subseteq \mathbb{R}^n$ be an open subset. The *tube* domain $T^V \subseteq \mathbb{C}^n$, with base V, is defined as

$$T^V = \mathbb{R}^n + iV = \{x + iy \in \mathbb{C}^n : \, x \in \mathbb{R}^n, \, y \in V\}.$$

We always write $z = x + iy \in \mathbb{C}^n$ (and similarly for other complex variables), where $x, y \in \mathbb{R}^n$. We employ the notation $d_V(y) = \operatorname{dist}(y, \partial V)$ for $y \in V$. The convex hull of a set $A \subset \mathbb{R}^n$ is denoted by $\operatorname{ch}(A)$.

Let $C \subseteq \mathbb{R}^n$ be an open cone (with vertex at the origin hereafter). Note that C may be $\mathbb{R}^n$. If $r > 0$, we write in short $C(r) := C \cap \{y \in \mathbb{R}^n : |y| < r\}$. We denote by $\operatorname{pr} C$ the intersection of the cone C with the unit sphere of $\mathbb{R}^n$. We say that the subcone C' is compact in C and write $C' \Subset C$ if $\overline{\operatorname{pr} C'} \subset \operatorname{pr} C$. It should be noticed that d_C is homogeneous of degree 1, namely, $d_C(\lambda y) = \lambda d_C(y)$, for every $\lambda > 0$. Recall [107] that the *conjugate cone* of C is defined as $C^* := \{\xi \in \mathbb{R}^n : \, y \cdot \xi \geq 0, \, \forall y \in C\}$. Since C is open, one actually has $y \cdot \xi > 0$, for all $y \in C$ and $\xi \in C^*$. The cone C is called *acute* if $\operatorname{int} C^* \neq \emptyset$. For acute cones one has

$$d_C(y) = \min_{\xi \in \operatorname{pr} C^*} y \cdot \xi, \quad y \in C.$$

(This equality is well-known [107, p.61].) Given $a \geq 0$, we denote the closed Euclidean ball (centered at the origin) of radius a as $\overline{B}(a)$.
We will need the following results of Vladimirov:

Theorem 0.1.3. *([106]) Let* $f(z)$ *be holomorphic on the tube* $T^{C_r} = \mathbb{R}^n + iC_r$, $C_r = C \cap B(0, r)$, *where* C *is a connected cone. If for arbitrary number* $r' < r$ *and cone* $C' \Subset C$, *the estimate*

$$|f(x + iy)| \leq M(r', C') \frac{(1 + |x|)^\beta}{|y|^\alpha} \text{ for every } z \in \mathbb{R}^n + i(C' \cap B(0, r'))$$

holds, where α, β *do not depend on* C' *and* r', *then*

$$f(x) = \lim_{\substack{y \to 0 \\ y \in C}} f(x + iy) \in \mathcal{S}^{(m)'}(\mathbb{R}^n) \, , m = \alpha + \beta + n + 3.$$

The convergence is in $\mathcal{S}'(\mathbb{R}^n)$ *and independent of the way that* $y \to 0, y \in C$.

Theorem 0.1.4. *([107]) For a holomorphic function $f(z)$ on T^C belong to $H_a(C)$, sufficient condition are: for an arbitrary cone $C' \Subset C$ and arbitrary number $\varepsilon > 0$, there exist numbers $\alpha \geq 0, \beta \geq 0$ and $M > 0$ such that*

$$|f(z)| \leq Me^{(a+\varepsilon)|y|}\frac{(1+|z|)^{\alpha}}{|y|^{\beta}} \text{ for all } z \in T^{C'}.$$

We denote the *heat kernel* with $E(x,t) = (4\pi t)^{-n/2}e^{-|x|2/4t}, (x,t) \in \mathbb{R}^{n+1}_{+}$. The following result is due to Matsuzawa.

Theorem 0.1.5. *([62]) Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $U(x,t) = u_y(E(x-y,t)) \in C^{\infty}(\mathbb{R}^{n+1}_{+})$ and satisfies the following conditions:*

$$\left(\frac{\partial}{\partial t} - \triangle\right)U(x,t) = 0 \quad \text{in } \mathbb{R}^{n+1}_{+}. \tag{2}$$

There are positive constants C, M and N such that

$$|U(x,t)| < Ct^{-M}(1+|x|)^{N} \quad \text{in } \mathbb{R}^{n+1}_{+}, \;\; U(x,t) \to u \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \quad \text{as } t \to 0+\,, \tag{3}$$

i.e.,

$$\lim_{t\to 0+}\int U(x,t)\varphi(x)dx = u(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{4}$$

Conversely, every $C^{\infty}(\mathbb{R}^{n+1}_{+})$ function defined on $\mathbb{R}^{n+1}_{+}$ satisfying the conditions (2) and (3) can be expressed in the form $U(x,t) = u_y(E(x-y))$ with a unique element $u \in \mathcal{S}'(\mathbb{R}^n)$.

0.2 Ultradistributions and tempered ultradistributions

Let (M_p) be a sequence of positive numbers. Some of the following conditions will be assumed on (M_p):
$(M.1)$ *(Logarithmic convexity)* $M_p^2 \leq M_{p-1}M_{p+1}$ for $p \in \mathbb{N}$;
$(M.2)$ *(Stability under ultradifferential operators)* For some $A, H > 0$

$$M_p \leq AH^p \min_{0\leq q\leq p} M_{p-q}M_q, \quad p, q \in \mathbb{N};$$

$(M.3)$ *(Strong non-quasi-analyticity)*

$$\sum_{p=q+1}^{\infty}\frac{M_{p-1}}{M_p} \leq Aq\frac{M_q}{M_{q+1}}, \quad q \in \mathbb{N},$$

and weaker conditions on (M_p):
$(M.2)'$ *(Stability under differential operators)* For some $A, H > 1$

$$M_{p+1} \leq AH^{p+1}M_p, \quad p \in \mathbb{N};$$

$(M.3)'$ (Non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

The *Gevrey sequence* $M_p = p!^s$, $s > 1$ satisfies all of the above conditions. Here we always assume that $M_0 = 1$.

For a sequence (M_p), the *associate function* $M(\rho)$ on $(0, +\infty)$ is defined by

$$M(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

Unless stated differently, we assume that (M_p) satisfies $(M.1)$, $(M.2)$ and $(M.3)$. Next, we give the definition and several important properties of the spaces $\mathcal{D}_K^{M_p,r}, \mathcal{D}_K^{(M_p)}, \mathcal{D}_K^{\{M_p\}}, \mathcal{D}^{(M_p)}(\Omega), \mathcal{D}^{\{M_p\}}(\Omega), \mathcal{E}^{(M_p)}(\Omega), \mathcal{E}^{\{M_p\}}(\Omega)$, (see [52, 51, 12]). Let K be a regular compact set in $\mathbb{R}^n$ and let Ω be an open set in $\mathbb{R}^n$. Denote:

$$\mathcal{E}^{\{M_p\},r}(K) = \{\varphi \in C^\infty(K) | \frac{\|D^\alpha \varphi\|_\infty}{r^{|\alpha|} M_{|\alpha|}} < \infty, \forall \alpha \in \mathbb{N}_0^n\},$$

$$\mathcal{D}_K^{\{M_p\},r} = \{\varphi \in C^\infty(\mathbb{R}^n) | \operatorname{supp}\varphi \subseteq K, \frac{\|D^\alpha \varphi\|_\infty}{r^{|\alpha|} M_{|\alpha|}} < \infty, \forall \alpha \in \mathbb{N}_0^n\}.$$

Both spaces are Banach spaces with norm

$$\|\varphi\| = \sup_{x \in K, \alpha \in \mathbb{N}_0^n} \frac{|D^\alpha \varphi(x)|}{r^{|\alpha|} M_{|\alpha|}}.$$

Standard locally convex spaces, defined by Komatsu [52], that we are going to work with are

$$\begin{aligned}
\mathcal{E}^{(M_p)}(K) &= \varprojlim_{r \to 0} \mathcal{E}^{\{M_p\},r}(K) \ ; \mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}(K) \\
\mathcal{E}^{\{M_p\}}(K) &= \varinjlim_{r \to \infty} \mathcal{E}^{\{M_p\},r}(K); \mathcal{E}^{\{M_p\}}(\Omega) = \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}(K) \\
\mathcal{D}_K^{(M_p)} &= \varprojlim_{r \to 0} \mathcal{D}_K^{\{M_p\},r} \qquad ; \mathcal{D}^{(M_p)}(\Omega) = \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{(M_p)} \\
\mathcal{D}_K^{\{M_p\}} &= \varinjlim_{r \to \infty} \mathcal{D}_K^{\{M_p\},r} \qquad ; \mathcal{D}^{\{M_p\}}(\Omega) = \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{\{M_p\}}.
\end{aligned}$$

The strong duals of the spaces $\mathcal{D}^{(M_p)}(\Omega)$ and $\mathcal{D}^{\{M_p\}}(\Omega)$ are the so called *ultradistributions* on Ω of *Beurling* and *Roumieu* type, respectively.

Assume $(M.1)$, $(M.2)$ and $(M.3)$. We denote by $\mathcal{S}_2^{M_p,m}(\mathbb{R}^n)$, $m > 0$, the space of smooth functions φ which satisfy

$$\sigma_{m,2}(\varphi) := \Big(\sum_{p,q \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \Big| \frac{m^{p+q} \langle x \rangle^p \varphi^{(q)}(x)}{M_p M_q} \Big|^2 dx \Big)^{1/2} < \infty, \tag{5}$$

supplied with the topology induced by the norm $\sigma_{m,2}$. If we put instead of 2, $p \in [1, \infty]$ in (5), we obtain equivalent sequence of norms $\sigma_{m,p}$, $m > 0$. The spaces $\mathcal{S}'^{(M_p)}(\mathbb{R}^n)$ and $\mathcal{S}'^{\{M_p\}}(\mathbb{R}^n)$ of *tempered ultradistributions* of Beurling and Roumieu type respectively, are defined as the strong duals of the spaces

$$\mathcal{S}^{(M_p)}(\mathbb{R}^n) = \lim \operatorname{proj}_{m\to\infty} \mathcal{S}_2^{M_p,m}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{S}^{\{M_p\}}(\mathbb{R}^n) = \lim \operatorname{ind}_{m\to 0} \mathcal{S}_2^{M_p,m}(\mathbb{R}^n),$$

respectively. We use $*$ as a common notation for the symbols (M_p) and $\{M_p\}$.

0.2.1 Some results on ultradistributions

Theorem 0.2.1. *[52] $\mathcal{E}^{(M_p)}(K)$, $\mathcal{E}^{(M_p)}(\Omega)$ and $\mathcal{D}_K^{(M_p)}$ are (FS)-spaces, $\mathcal{E}^{\{M_p\}}(K)$, $\mathcal{D}_K^{\{M_p\}}$ and $\mathcal{D}^{\{M_p\}}(\Omega)$ are (DFS)-spaces and $\mathcal{D}^{(M_p)}(\Omega)$ is an (LFS)-space. In particular these spaces are separable complete bornological Montel and Schwartz spaces. Every bounded set in $\mathcal{D}_K^{\{M_p\}}$ or $\mathcal{D}^{(M_p)}(\Omega)$ ($\mathcal{E}^{\{M_p\}}(K)$) is a bounded set in some $\mathcal{D}_K^{\{M_p\},r}$ ($\mathcal{E}^{\{M_p\},r}(K)$). $\mathcal{E}^{\{M_p\}}(\Omega)$ is a complete Schwartz space. In particular, it is semi-reflexive. If (M_p) satisfies $(M.2)'$, then all the spaces defined above are nuclear.*

Theorem 0.2.2. *[52] A sequence of positive numbers (M_p), satisfies condition $(M.1)$ if and only if*

$$M_p = M_0 \sup_\rho \frac{\rho^p}{e^{M(\rho)}}.$$

Theorem 0.2.3. *[52] The sequence (M_p) satisfies $(M.2)$ if and only if for some $A, H > 0$,*

$$2M(\rho) \le M(H\rho) + \log(AM_0).$$

When (M_p) satisfies conditions $(M.1)$, $(M.2)$ and $(M.3)$ one defines ultradifferential operators as follows:
It is said that $P(\xi) = \sum_{\alpha\in\mathbf{N}_0^n} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n$, is an *ultrapolynomial* of the class (M_p) resp. $\{M_p\}$, whenever the coefficients a_α satisfy the estimate

$$|\, a_\alpha \,| \le \frac{CL^\alpha}{M_\alpha}, \quad \alpha \in \mathbb{N}_0^n, \tag{6}$$

for some $L > 0$ and $C > 0$ resp. for every $L > 0$ and some $C_L > 0$. The corresponding operator $P(D) = \sum_\alpha a_\alpha D^\alpha$ is an *ultradifferential operator* of the class (M_p), resp. $\{M_p\}$. By $\mathfrak{R}$ we denote the set of positive sequences which monotonically increases (not necessarily strictly) to infinity. Assume now $(M.1)$, $(M.2)$ and $(M.3)$ and put

$$\begin{aligned} P_r(\zeta) &= (1+\zeta^2)\prod_{p\in\mathbb{N}_0^n}\left(1+\frac{\zeta^2}{r^2 m_p^2}\right), \quad \textit{resp.} \\ P_{r_p}(\zeta) &= (1+\zeta^2)\prod_{p\in\mathbb{N}_0^n}\left(1+\frac{\zeta^2}{r_p^2 m_p^2}\right),\ \zeta\in\mathbb{C}^n, \end{aligned} \tag{7}$$

where $m_p = M_p/M_{p-1}$ and $r > 0$, resp. $(r_p) \in \mathfrak{R}$. Conditions $(M.1)$, $(M.2)$ and $(M.3)$ imply that P_r, resp. P_{r_p}, is an ultradifferential operator of the (M_p), resp. of $\{M_p\}$, class.

All the good properties of $\mathcal{S}^*(\mathbb{R}^n)$ and its strong dual follow from the equivalence of the sequence of norms $\sigma_{m,p},\ m>0,\ p\in[1,\infty]$ with the each of the following sequences of norms [54, 12]:

(a) $\sigma_{m,p},\ \ m>0,\ p\in[1,\infty]$ is fixed ;

(b) $s_{m,p},\ m>0,\ p\in[1,\infty]$ is fixed, where

$$s_{m,p}(\varphi):=\sum_{\alpha,\beta\in\mathbb{N}_0^n}\frac{m^{\alpha+\beta}\|x^\beta\varphi^{(\alpha)}\|_p}{M_\alpha M_\beta};$$

(c) $s_m,\ \ m>0$, where $s_m(\varphi):=\sup\limits_{\alpha\in\mathbb{N}_0^n}\dfrac{m^\alpha\|\varphi^{(\alpha)}e^{M(m|\cdot|)}\|_{L_\infty}}{M_\alpha}$;

In [12] it is proved that

$$\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)=\operatorname{proj}\lim_{(r_i),(s_j)\in\mathfrak{R}}S^{M_p}_{r_i,s_j}(\mathbb{R}^n),$$

$$S^{M_p}_{r_i,s_j}(\mathbb{R}^n)=\{\varphi\in C^\infty(\mathbb{R}^n);\ \gamma_{r_i,s_j}(\varphi)<\infty\},$$

$$\text{where } \gamma_{r_i,s_j}(\varphi):=\sum_{p,q\in\mathbb{N}_0^n}\{\frac{\|\langle x\rangle^p\varphi^{(q)}\|_{L^\infty}}{(\prod_{i=1}^p r_i)M_p(\prod_{j=1}^q s_j)M_q}\}\text{ for }\ (r_i),(s_j)\in\mathfrak{R}.$$

Note that $\mathcal{F}:\mathcal{S}^*(\mathbb{R}^n)\to\mathcal{S}^*(\mathbb{R}^n)$ is a topological isomorphism and that the Fourier transformation on $\mathcal{S}'^*(\mathbb{R}^n)$ is defined as usual via duality.

Lemma 0.2.1. *(Dual Mittag-Leffler [49]) Suppose that*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{i_1} & Y_1 & \xrightarrow{p_1} & Z_1 & \longrightarrow & 0\\
 & & \downarrow u_{2,1} & & \downarrow v_{2,1} & & \downarrow w_{2,1} & & \\
0 & \longrightarrow & X_2 & \xrightarrow{i_2} & Y_2 & \xrightarrow{p_2} & Z_2 & \longrightarrow & 0\\
 & & \downarrow u_{3,2} & & \downarrow v_{3,2} & & \downarrow w_{3,2} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_n & \xrightarrow{i_n} & Y_n & \xrightarrow{p_n} & Z_n & \longrightarrow & 0\\
 & & \downarrow u_{n+1,n} & & \downarrow v_{n+1,n} & & \downarrow w_{n+1,n} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

is an inductive sequence of short topologically exact sequences of Banach spaces. Then the sequence

$$0\longrightarrow\varinjlim X_n\xrightarrow{i}\varinjlim Y_n\xrightarrow{p}\varinjlim Z_n\longrightarrow 0$$

is topologically exact and p is open. If the sequence (Y_n) is regular and injective, the sequence (Z_n) is weakly compact, then

$$0 \longleftarrow \left(\varinjlim X_n\right)' \overset{i'}{\longleftarrow} \left(\varinjlim Y_n\right)' \overset{p'}{\longleftarrow} \left(\varinjlim Z_n\right)' \longleftarrow 0$$

is topologically exact. As a consequence, $\varinjlim X_n$ has the same strong dual as the closed subspace $i\left(\varinjlim X_n\right)$ of $\varinjlim Y_n$.

By $\Re$ we denote the set of positive sequences which monotonically increases (not necessarily strictly) to infinity. For $(r_p) \in \Re$ and K a compact set in $\mathbb{R}^n$, we denote by $\mathcal{D}_{K,r_p}^{\{M_p\}}$ the space of smooth functions φ on $\mathbb{R}^n$ supported by K such that

$$\|\varphi\|_{K,r_p} = \sup\left\{\frac{|\ D^p\varphi(x)\ |}{N_p};\ |p \in \mathbb{N}_0^n,\ \ x \in K\right\} < \infty,$$

where $N_p = M_p \prod_{i=0}^{|p|} r_i, \quad p \in \mathbb{N}_0^n$. Clearly, this is a Banach space. It is proved in [51] that

$$\mathcal{D}_K^{\{M_p\}} = \text{proj} \lim_{(r_p)\in\Re} \mathcal{D}_{K,r_p}^{\{M_p\}}.$$

If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $r > 0$, resp. $(r_p) \in \Re$, we put

$$\mathcal{D}_{\Omega,r}^{(M_p)} = \text{ind} \lim_{K\subset\subset\Omega} \mathcal{D}_{K,r}^{\{M_p\}}, \quad \mathcal{D}_{\Omega,r_p}^{\{M_p\}} = \text{ind} \lim_{K\subset\subset\Omega} \mathcal{D}_{K,r_p}^{\{M_p\}}.$$

The associated function for the sequence N_p is

$$N_{r_p}(\rho) = \sup\{\log_+ \frac{\rho^p}{N_p};\ \ p \in \mathbb{N}\}, \quad \rho > 0.$$

Note that for given (r_p) and every $k > 0$ there is $\rho_0 > 0$ such that

$$N_{r_p}(\rho) \le M(k\rho), \quad \rho > \rho_0. \tag{8}$$

Lemma 0.2.2. *([51]) Let (a_p) be a sequence of nonnegative numbers.*

(i) There are positive constants h and C such that

$$a_p \le Ch^p \ \ \text{for every nonegative integer } p$$

if and only if

$$\sup_p \frac{a_p}{\prod_{j=1}^p h_j} < \infty$$

for every sequence (h_p), $h_p > 0$, monotonously increasing to infinity.

(ii) There are a constant C and a sequence (h_p), $h_p > 0$, monotonously increasing to infinity such that

$$a_p \le \frac{C}{\prod_{j=1}^p h_j} \ \ \text{for every nonegative integer } p$$

if and only if

$$\sup_p h^p a_p < \infty$$

for any $h > 0$.

In this thesis we intensively use the following result of Komatsu. In the text we refer to it as the *parametrix of Komatsu.*

Lemma 0.2.3. *[52] Let K be a compact neighborhood of zero, $r > 0$, and $(r_p) \in \Re$.*

i) There are $u \in \mathcal{D}^{(M_p)}_{K,r/2}$ and $\psi \in \mathcal{D}^{(M_p)}_K$ such that

$$P_r(D)u = \delta + \psi\,, \tag{9}$$

where P_r is of form (7).

ii) There are $u \in C^\infty$ and $\psi \in \mathcal{D}^{\{M_p\}}_K$ such that

$$P_{r_p}(D)u = \delta + \psi\,, \tag{10}$$

$$\operatorname{supp} u \subset K, \quad \sup_{x\in K}\left\{\frac{|\partial^\alpha u(x)|}{\prod_{j=1}^{|\alpha|} r_j\, M_\alpha}\right\} \longrightarrow 0, \quad |\alpha| \to \infty\,, \tag{11}$$

where P_{r_p} is of form (7).

We end this Chapter by stating and proving two already known facts and an algebraic result which are used later in the thesis.

Lemma 0.2.4. *Let E_j, $j \in \mathbb{N}$, be reflexive Banach spaces, each with norm $\|\cdot\|_{E_j}$. Then the Banach space*

$$F = \left\{(e_j)_j \,\middle|\, e_j \in E_j,\ \|(e_j)_j\|_F = \left(\sum_{j=1}^{\infty} \|e_j\|^2_{E_j}\right)^{1/2} < \infty\right\}$$

is reflexive with its dual the Banach space

$$L = \left\{(e'_j)_j \,\middle|\, e'_j \in E'_j,\ \|(e'_j)_j\|_L = \left(\sum_{j=1}^{\infty} \|e'_j\|^2_{E'_j}\right)^{1/2} < \infty\right\}.$$

Proof. Every element $(e'_j)_j \in L$ generates a linear continuous functional on F by $T_{(e'_j)_j}((e_j)_j) = \sum_{j=1}^{\infty}\langle e'_j, e_j\rangle$ and obviously $\left\|T_{(e'_j)_j}\right\|_{F'} \le \|(e'_j)_j\|_L$. Since E_j are reflexive Banach spaces, there exist $e_j \in E_j$, $j \in \mathbb{N}$, such that $\|e_j\|_{E_j} \le 1$ and $\|e'_j\|_{E'_j} = \langle e'_j, e_j\rangle$. Put $f_j = e_j\|e'_j\|_{E'_j} \in E_j$. Then $\sum_{j=1}^{\infty}\|f_j\|^2_{E_j} \le \|(e'_j)_j\|^2_L$, i.e., $(f_j)_j \in F$ and $\|(f_j)_j\|_F \le \|(e'_j)_j\|_L$. Moreover

$$\begin{aligned}\left\|T_{(e'_j)_j}\right\|_{F'}\|(e'_j)_j\|_L &\ge \left\|T_{(e'_j)_j}\right\|_{F'}\|(f_j)_j\|_F \ge \left|T_{(e'_j)_j}((f_j)_j)\right| \\ &= \left|\sum_{j=1}^{\infty}\langle e'_j, f_j\rangle\right| = \sum_{j=1}^{\infty}\|e'_j\|^2_{E'_j} = \|(e'_j)_j\|^2_L.\end{aligned}$$

We obtain $\left\|T_{(e'_j)_j}\right\|_{F'} \geq \|(e'_j)_j\|_L$, hence $\left\|T_{(e'_j)_j}\right\|_{F'} = \|(e'_j)_j\|_L$.

Now, let $T \in F'$. For $k \in \mathbb{N}$, observe the mapping $e \mapsto S_k(e)$, $E_k \to F$, defined by $S_k(e) = (f_j)_j$ where $f_k = e$ and $f_j = 0$ for $j \neq k$. It is obviously a continuous linear functional, hence the composition $T \circ S_k$ is continuous linear functional on E_k. Hence, there exists $e'_k \in E'_k$ such that $T \circ S_k(e) = \langle e'_k, e\rangle$. For $k \in \mathbb{N}$ and $e_j \in E_j$ for $j = 1, ..., k$ and $e_j = 0$ for $j \geq k+1$, $(e_j)_j \in F$. We will denote this element of F by $(e_j)_j^{(k)}$. Observe that

$$T\left((e_j)_j^{(k)}\right) = T\left(\sum_{j=1}^{k} S_j(e_j)\right) = \sum_{j=1}^{k} T \circ S_j(e_j) = \sum_{j=1}^{k}\langle e'_j, e_j\rangle. \tag{12}$$

Since E_j, $j \in \mathbb{N}$, are reflexive Banach spaces there exist $g_j \in E_j$ such that $\|g_j\|_{E_j} \leq 1$ and $\|e'_j\|_{E'_j} = \langle e'_j, g_j\rangle$. Put $f_j = g_j\|e'_j\|_{E'_j}$ and note that $\|f_j\|_{E_j} \leq \|e'_j\|_{E'_j}$. For $k \in \mathbb{N}$, denote by $(f_j)_j^{(k)}$ the element of F which first k coordinates are precisely $f_1, ..., f_k$ and all other are zero. Now,

$$T\left((f_j)_j^{(k)}\right) = \sum_{j=1}^{k}\langle e'_j, f_j\rangle = \sum_{j=1}^{k} \|e'_j\|^2_{E'_j} \geq \sum_{j=1}^{k} \|f_j\|^2_{E_j}.$$

Since T is continuous, there exists $C > 0$ such that $|T((e_j)_j)| \leq C\|(e_j)_j\|_F$ for all $(e_j)_j \in F$. Hence, by the above inequality, we have $\sum_{j=1}^{k} \|f_j\|^2_{E_j} \leq C\left\|(f_j)_j^{(k)}\right\|_F$, i.e., $\left(\sum_{j=1}^{k} \|f_j\|^2_{E_j}\right)^{1/2} \leq C$. We obtain that $\sum_{j=1}^{\infty} \|f_j\|^2_{E_j}$ converges, i.e., $(f_j)_j \in F$. Again by the above inequality and the fact $\|f_j\|_{E_j} \leq \|e'_j\|_{E'_j}$ we have

$$\sum_{j=1}^{k} \|e'_j\|^2_{E'_j} \leq C\left(\sum_{j=1}^{k} \|f_j\|^2_{E_j}\right)^{1/2} \leq \left(\sum_{j=1}^{k} \|e'_j\|^2_{E'_j}\right)^{1/2}$$

hence $\left(\sum_{j=1}^{k} \|e'_j\|^2_{E'_j}\right)^{1/2} \leq C$, i.e., $(e'_j) \in L$. Moreover, from the continuity of T and (12), for $(e_j)_j \in F$, we have

$$T((e_j)_j) = \lim_{k\to\infty} T\left((e_j)_j^{(k)}\right) = \lim_{k\to\infty}\sum_{j=1}^{k}\langle e'_j, e_j\rangle = \sum_{j=1}^{\infty}\langle e'_j, e_j\rangle,$$

since $(e_j)_j^{(k)} \to (e_j)_j$, when $k \to \infty$ in F and $\sum_j\langle e'_j, e_j\rangle$ is absolutely convergent. Hence $T = T_{(e'_j)}$ and by the $\left\|T_{(e'_j)_j}\right\|_{F'} = \|(e'_j)_j\|_L$. Which proves that L is the strong dual of F. Since all E_j, $j \in \mathbb{N}$, are reflexive Banach spaces so are E'_j. Hence we can perform the same discussions as above with E'_j in place of E_j to obtain that the strong dual of L is F. Moreover by the proof it follows that the evaluation mapping $F \to F''(= F)$ is surjective, hence F is reflexive. $\square$

Lemma 0.2.5. *The composition of ultradifferential operators of $*$ type is an ultradifferential operator of $*$ type.*

Proof. Let $P_{r_1}(D) = \sum_\alpha a_\alpha D^\alpha$ and $P_{r_2}(D) = \sum_\beta b_\beta D^\beta$ be ultradifferential operators of $*$ type and $f \in \mathcal{D}^*(\mathbb{R}^n)$ be arbitrary. Applying Fourier transform to the composition $P_{r_2}(D)(P_{r_1}(D)f)$ one obtains

$$\mathcal{F}\left(P_{r_2}(D)\left(P_{r_1}(D)f\right)\right)(\xi) = P_{r_2}(\xi) \cdot P_{r_1}(\xi)\hat{f}(\xi). \tag{13}$$

Let $P(\xi) = P_{r_2}(\xi)P_{r_1}(\xi) = \sum_\gamma c_\gamma \xi^\gamma$. In (M_p) case there exist $B, C, m, \tilde{h} > 0$ such that for the coefficients c_γ we have the estimates

$$\begin{aligned} c_\gamma &= \frac{\partial^\gamma}{\gamma!}P(0) = \frac{\partial^\gamma}{\gamma!}\left(P_{r_2}(\xi)P_{r_1}(\xi)\right)|_0 = \frac{1}{\gamma!}\sum_{\delta\leq\gamma}\binom{\gamma}{\delta}\partial^\delta P_{r_2}(0)\partial^{\gamma-\delta}P_{r_1}(0) \\ &= \frac{1}{\gamma!}\sum_{\delta\leq\gamma}\binom{\gamma}{\delta}\delta!(\gamma-\delta)!a_\delta b_{\gamma-\delta} < BC\frac{1}{\gamma!}\sum_{\delta\leq\gamma}\binom{\gamma}{\delta}\delta!(\gamma-\delta)!\frac{\tilde{h}^\delta m^{\gamma-\delta}}{M_\delta M_{\gamma-\delta}} \\ &< \frac{ABC}{M_\gamma \gamma!}\sum_{\delta\leq\gamma}\binom{\gamma}{\delta}\delta!(\gamma-\delta)!\tilde{h}^\delta m^{\gamma-\delta}H^\gamma \leq \frac{AB}{M_\gamma}\left((m+\tilde{h})H\right)^\gamma \end{aligned}$$

where $A, H > 0$ are the constants from $(M.2)$. Then, choose $h = (m+\tilde{h})H$. In $\{M_p\}$ case, for $h > 0$ choose $m, \tilde{h} > 0$ such that $(m+\tilde{h})H < h$. There exist $B, C > 0$ such that the same estimates hold for the coefficients. In both cases we get that $P(\xi)$ is ultrapolynomial of $*$ type. Applying the inverse Fourier transform to equation (13) one obtains $P_{r_2}(D)\left(P_{r_1}(D)f\right) = P(D)f$. □

0.3 Factorization Theorem

A *left approximate identity* in a Banach algebra A is, by definition, a net $(e_\nu)_{\nu\in I}$ such that $e_\nu \in A$, for all $\nu \in I$, and $\lim_\nu \|e_\nu a - a\|_A = 0$ for every $a \in A$. A left approximate identity $(e_\nu)_{\nu\in I} \subset A$ is said to be *bounded* if $\sup_{\nu\in I}\|e_\nu\|_A < \infty$. If A is a Banach algebra with bounded left approximate identity $(e_\nu)_{\nu\in I} \subset A$ Ąand T is a *continuous representation* of A on a Banach space X, then $\lim_\nu \|T(e_\nu)y - y\|_X = 0$ for every $y \in \overline{span} T(A)X$. This follows from the fact that $((T(e_\nu)))_{\nu\in I}$ is a bounded net in $\mathcal{L}(X)$ such that $\lim_\nu \|T(e_\nu)y - y\| = 0$ for every $y \in T(A)X$.

Theorem 0.3.1. *(*The Cohen-Hewitt Factorization Theorem *[47]) If A is a Banach algebra with bounded left approximate identity $(e_\nu)_{\nu\in I}$ and T is a continuous representation of A on a Banach space X, then $T(A)X$ is a closed subspace of X. Furthermore, for every $y \in T(A)X$ and every $\varepsilon > 0$ there are $a \in A$ and $x \in T(A)y$ such that $T(a)x = y$, $\|x - y\| < \varepsilon$, and $a = \sum_{n=1}^\infty p_n e_{\nu_n}$ where $\nu_n \in I$ $p_n > 0$ for $n \in \mathbb{N}$ and $\sum_{n=1}^\infty p_n = 1$.*

Other known results used in the thesis and additional notation will be cited and introduced in the thesis when needed.

Chapter 1

New distribution spaces associated to translation-invariant Banach spaces

Translation-invariant spaces of functions and distributions are very important in mathematical analysis. They are connected with many central questions in harmonic analysis [5, 25, 26, 31, 86, 112]. This thesis introduces and studies new classes of translation-invariant distribution spaces, the test function space $\mathcal{D}_E$ and its dual, denoted as $\mathcal{D}'_{E'_*}$. This chapter investigate in detail their topological properties; in Chapter2 we will apply such properties in the study of boundary values of holomorphic functions. The construction of these spaces is based upon the analysis of suitable translation-invariant Banach space of tempered distributions E, which we carry out in Section 1.1.

1.1 On a class of translation-invariant Banach spaces

In this Section we study the class of translation-invariant Banach space of tempered distributions on the Euclidean space $\mathcal{R}^n$, introduced below. It should be mentioned that such Banach spaces have already been considered in one-dimension by Drozhzhinov and Zav'yalov in connection with Tauberian theorems and generalized Besov spaces [27]. It should also be remarked that these Banach spaces are not necessarily solid Banach spaces in the sense of [30, 31]; indeed, the elements of E may not be regular distributions and actually E needs not even be a module over $\mathcal{D}(\mathbb{R}^n)$ under pointwise multiplication.

The class of Banach spaces E of distributions in which we are interested are those satisfying the following three properties:

$(a)'$ $\mathcal{D}(\mathbb{R}^n) \hookrightarrow E \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$.

$(b)'$ $T_h : E \to E$ for every $h \in \mathbb{R}^n$ (i.e., E is translation-invariant).

$(c)'$ For any $g \in E$, there are $M = M_g > 0$ and $\tau = \tau_g \geq 0$ such that

$$\|T_h g\|_E \leq M(1 + |h|)^\tau, \quad \text{for all } h \in \mathbb{R}^n.$$

We shall call any such Banach space satisfying the conditions $(a)'$, $(b)'$, and $(c)'$ a *translation-invariant Banach space of tempered distributions*.

Remark 1.1.1. The conditions $(a)'$ and $(b)'$ imply that every translation operator $T_h : E \to E$ is continuous. Indeed, for every $h \in \mathbb{R}^n$, $T_h : E \to \mathcal{D}'(\mathbb{R}^n)$ since $T_h : E \xrightarrow{id} \mathcal{D}'(\mathbb{R}^n) \xrightarrow{T_h} \mathcal{D}'(\mathbb{R}^n)$ is continuous as a composition of continuous mappings. Hence it has a closed graph. Then the preimage of the graph via the continuous mapping $E \times E \xrightarrow{id \times id} E \times \mathcal{D}'(\mathbb{R}^n)$ is closed in $E \times E$. The closed graph argument implies the claim.

Our first important result tells us that $(a)'$, $(b)'$ and $(c)'$ may always be replaced by stronger conditions.

Theorem 1.1.1. *Let E be a translation-invariant Banach space of tempered distributions. The following properties hold:*

- (a) $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.
- (b) *The mappings $\mathbb{R}^n \to E$ given by $h \mapsto T_h g$ are continuous for each $g \in E$.*
- (c) *There are absolute constants $M > 0$ and $\tau \geq 0$ such that*

$$\|T_h g\|_E \leq M \|g\|_E (1 + |h|)^\tau, \quad \text{for all } g \in E \text{ and } h \in \mathbb{R}^n.$$

Proof. Let us first prove (c). Consider the following sets,

$$E_{j,\nu} = \{g \in E \ : \ \|T_h g\|_E \leq j(1 + |h|)^\nu \text{ for all } \ h \in \mathbb{R}^n\}, \quad j, \nu \in \mathbb{N}.$$

Because of $(c)'$, we have $E = \bigcup_{j,\nu \in \mathbb{N}} E_{j,\nu}$. Baire's Theorem implies that one of the sets E_{j_0,ν_0} contains a ball $\{f \in E : \|f - u\|_E < r\}$. If $g \in E$ is such that $\|g\|_E < r$, then

$$\|T_h g\|_E \leq \|T_h g + T_h u\|_E + \|T_h u\|_E \leq 2 j_0 (1 + |h|)^{\nu_0}$$

for all $h \in \mathbb{R}^n$. So, for arbitrary $g \in E$, we get $\|T_h g\|_E < (4 j_0 / r)(1 + |h|)^{\nu_0} \|g\|_E$.

The property (b) follows easily from $(a)'$, $(b)'$ and (c).

Let us now show (a). We first prove the embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E$. Since $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$, it is enough to prove that $\mathcal{S}(\mathbb{R}^n) \subset E$ and the continuity of the inclusion mapping. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We use a special partition of unity:

$$1 = \sum_{m \in \mathbb{Z}^n} \psi(x - m), \quad \psi \in \mathcal{D}_{[-1,1]^n}.$$

Hence, we get the representation $\varphi(x) = \sum_{m \in \mathbb{Z}^n} \psi(x - m) \varphi(x)$. We estimate each term in this sum. Because of (c),

$$\|\varphi \, T_{-m} \psi\|_E \leq \frac{M}{(1 + |m|)^{n+1}} \|(1 + |m|)^{n+\tau+1} \psi \, T_m \varphi\|_E, \tag{1.1}$$

where $|m|$ denotes Euclidean norm. We first prove that the multi-indexed sequence $\{\rho_m\}_{m \in \mathbb{Z}^n}$ is bounded in $\mathcal{D}_{[-1,1]^n}$, where

$$\rho_m = (1 + |m|)^{n+\tau+1} \psi \, T_m \varphi. \tag{1.2}$$

In fact, for any $j > n + \tau + 1$, we have

$$p_j(\rho_m) \leq p_j(\psi) \max_{|\alpha|\leq j} \sup_{|y-m|\leq 1} (1 + |m|)^j |\varphi^{(j)}(y)| \leq M_1 q_j(\varphi). \tag{1.3}$$

By the assumption $(a)'$, the mapping $\mathcal{D}_{[-1,1]^n} \to E$ is continuous. So, there are $M_2 > 0$ and $j \in \mathbb{N}_0$, such that $\|\phi\|_E \leq M_2 p_j(\phi)$, for every $\phi \in \mathcal{D}_{[-1,1]^n}$. We may assume that $j > n + \tau + 1$. Therefore, by (1.3),

$$\|\rho_m\|_E \leq M_1 M_2 q_j(\varphi), \quad \text{for all } m \in \mathbb{Z}^n. \tag{1.4}$$

Next, let $F(r)$ be the lattice counting function of points with integer coordinates inside the n-dimensional Euclidean closed ball of radius r. It is well known that $F(r)$ has asymptotics

$$F(r) = \sum_{m_1^2+\cdots+m_n^2\leq r^2} 1 \sim \frac{\pi^{n/2} r^n}{\Gamma(n/2+1)}, \quad r \to \infty.$$

In view of (1.1), (1.2), and (1.4), we obtain

$$\| \sum_{N'<|m|\leq N} \varphi\, T_{-m}\psi \|_E \leq M_3 q_j(\varphi) \int_{N'}^{N} \frac{dF(r)}{(1+r)^{n+1}} \leq \frac{M_4 q_j(\varphi)}{N'+1} \tag{1.5}$$

and thus $\left\{\sum_{|m|\leq N} \varphi T_{-m}\psi\right\}_{N=0}^{\infty}$ is a Cauchy sequence in E whose limit is $\varphi \in E$. Taking $N' = 0$ and $N \to \infty$ in (1.5), we get $\|\varphi\|_E \leq M_4 q_j(\varphi)$, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The continuity of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E$ has been established.

We now address $E \subset \mathcal{S}'(\mathbb{R}^n)$ and the continuity of the inclusion mapping. Let $g \in E$. Due to Schwartz' characterization of $\mathcal{S}'(\mathbb{R}^n)$ [94, Thm. VI, p. 239]: g belongs to $\mathcal{S}'(\mathbb{R}^n)$ if and only if $g * \varphi$ is a function of at most polynomial growth for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Let B be a bounded set in $\mathcal{D}(\mathbb{R}^n)$. The embedding $E \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ yields the existence of a constant $M_5 = M_5(B)$ such that $|\langle g, \check{\phi}\rangle| \leq M_5 ||g||_E$ for all $g \in E$ and $\phi \in B$. Therefore, by (c),

$$|(g * \phi)(h)| \leq M_5 ||T_h g||_E \leq M_5 M ||g||_E (1 + |h|)^{\tau}, \tag{1.6}$$

for all $g \in E$, $\phi \in B$, and $h \in \mathbb{R}^n$. This shows $E \subset \mathcal{S}'(\mathbb{R}^n)$. The continuity of the inclusion mapping would follow if we show that the unit ball of E is weakly bounded in $\mathcal{S}'(\mathbb{R}^n)$. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and write again $\varphi(x) = \sum_{m\in\mathbb{Z}^n} \psi(x-m)\varphi(x)$, where ψ is the partition of the unity used above. We use $\rho_m \in \mathcal{D}_{[-1,1]^n}$ as in (1.2). Taking (1.6) into account and the fact that $B = \{\check{\rho}_m : m \in \mathbb{Z}^n\}$ is a bounded subset of $\mathcal{D}(\mathbb{R}^n)$ (cf. (1.3)), we obtain, for all $g \in E$,

$$\begin{aligned} |\langle g, \varphi\rangle| &\leq \lim_{N\to\infty} \sum_{|m|\leq N} \frac{|(g * \check{\rho}_m)(m)|}{(1+|m|)^{n+\tau+1}} \\ &\leq M_5 M ||g||_E \lim_{N\to\infty} \sum_{|m|\leq N} \frac{1}{(1+|m|)^{n+1}} \leq M_6 ||g||_E. \end{aligned}$$

Finally, the density of E in $\mathcal{S}'(\mathbb{R}^n)$ follows from the dense inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E$. The proof of (a) is complete. □

Observe that condition (c) gives us the order of growth in h of the norms $||T_h||_{L(E)}$, where as usual $L(E)$ is the Banach algebra of continuous linear operators on E.

Definition 1.1.1. Let E be a translation-invariant Banach space of tempered distributions. The *growth function of the translation group* is defined as

$$\omega(h) := ||T_{-h}||_{L(E)}.$$

From now on in this Chapter, we shall *always* assume that E is a translation-invariant Banach space of tempered distributions with growth function ω. It is clear that ω is measurable, $\omega(0) = 1$, the function $\log \omega$ is subadditive, and by (c), it satisfies the estimate

$$\omega(h) \leq M(1 + |h|)^{\tau}, \quad h \in \mathbb{R}^n. \tag{1.7}$$

We now study various properties of E. We start with the convolution.

Lemma 1.1.1. *The convolution mapping $(\varphi, \psi) \in \mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n) \to \varphi*\psi \in \mathcal{S}(\mathbb{R}^n)$ extends to a continuous bilinear mapping $\mathcal{S}(\mathbb{R}^n) \times E \to E$. Furthermore, the following estimate holds*

$$\|\varphi * g\|_E \leq \|g\|_E \int_{\mathbb{R}^n} |\varphi(x)|\, \omega(x) dx. \tag{1.8}$$

Proof. Given $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ we can view $(\varphi * \psi)(x) = \int_{\operatorname{supp}\varphi} \varphi(t)\psi(x - t)dt$ as an integral in the Fréchet space $\mathcal{S}(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E$, the Riemann sums of this integral converge to $\varphi * \psi$ in the Banach space E. We have $\| \sum_j (t_{j+1} - t_j)\varphi(t_j)T_{-t_j}\psi\|_E \leq \|\psi\|_E \sum_j (t_{j+1} - t_j)|\varphi(t_j)|\, \omega(t_j)$. Passing to the limit of the Riemann sums, we obtain $\|\varphi * \psi\|_E \leq \|\psi\|_E \int_{\mathbb{R}^n} |\varphi(t)|\, \omega(t)dt$. By using a standard density argument, one obtains the desired extension and (1.8). □

The convolution of elements of E can actually be performed with more general functions. Let L^1_ω be the *Beurling algebra* [5, 86] with weight ω, i.e., the Banach algebra of measurable functions u such that $||u||_{1,\omega} := \int_{\mathbb{R}^n} |u(x)|\, \omega(x)dx < \infty$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in this Beurling algebra, we obtain the ensuing proposition, a corollary of Lemma 1.1.1.

Proposition 1.1.1. *The convolution extends to a mapping $* : L^1_\omega \times E \to E$ and E becomes a* Banach module *over the Beurling algebra L^1_ω, i.e., $||u * g||_E \leq ||u||_{1,\omega}||g||_E$.*

We shall denote this extension simply by $u * g = g * u$ whenever $u \in L^1_\omega$ and $g \in E$. We call L^1_ω the *associated Beurling algebra* to E.

Lemma 1.1.1 also allows us to consider approximations in E by smoothing with test functions.

Corollary 1.1.1. *Let $g \in E$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then,*

$$\lim_{\varepsilon\to 0^+} \|cg - \varphi_\varepsilon * g\|_E = 0,$$

where $c = \int_{\mathbb{R}^n} \varphi(x)dx$.

Proof. We first consider the case when $\varphi \in \mathcal{D}(\mathbb{R}^n)$. As in the proof of Lemma 1.1.1, for $g \in \mathcal{S}(\mathbb{R}^n)$ we can view $g * \varphi_\varepsilon = \int_{\mathbb{R}^n} (T_{-y} g)\, \varphi_\varepsilon(y) dy$, an E-valued integral. Thus, if $\varepsilon < 1$,

$$\begin{aligned}\|cg - \varphi_\varepsilon * g\| &= \left\| \int_{\mathbb{R}^n} (g - T_{-y} g) \frac{1}{\varepsilon^n} \varphi\left(\frac{y}{\varepsilon}\right) dy \right\|_E \\ &\leq \sup_{t \in \operatorname{supp} \varphi} \|g - T_{-\varepsilon t} g\|_E \int_{\operatorname{supp} \varphi} |\varphi(t)|\, dt.\end{aligned}$$

Due to the density of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E$, the above inequality remains true for $g \in E$. Hence, in view of condition (b), this gives the result when $g \in E$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. In the general case, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and let $\{\psi_j\}_{j=1}^{\infty} \in \mathcal{D}(\mathbb{R}^n)$ be a sequence such that $\psi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$. By Lemma 1.1.1 and (1.7), we have for $\varepsilon < 1$,

$$\|(\psi_j)_\varepsilon * g - \varphi_\varepsilon * g\|_E \leq M \|g\|_E \int_{\mathbb{R}^n} (1 + |x|)^\tau |\psi_j(x) - \varphi(x)| dx,$$

whence the result follows because $\int_{\mathbb{R}^n} \psi_j(x) dx \to c$. □

We now study the dual space of E.

Proposition 1.1.2. *The space E' satisfies*

$(a)''$ $\mathcal{S}(\mathbb{R}^n) \to E' \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, *where the embeddings are continuous.*

$(b)''$ *The mappings $\mathbb{R}^n \to E'$ given by $h \mapsto T_h f$ are continuous for the weak* topology.*

Moreover, the property (c) from Theorem 1.1.1 holds true when E is replaced by E'.

Proof. It follows from (a) that $\mathcal{S}(\mathbb{R}^n) \to E' \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Given $f \in E'$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$|\langle T_h f, \varphi \rangle| = |\langle f, T_{-h} \varphi \rangle| \leq \omega(h) ||f||_{E'} ||\varphi||_E \leq M \|f\|_{E'} \|\varphi\|_E (1 + |h|)^\tau.$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in E, $T_h f \in E'$ and $\|T_h f\|_{E'} \leq M \|f\|_{E'} (1 + |h|)^\tau$. By (b) applied to E, $\lim_{h \to h_0} \langle T_h f - T_{h_0} f, g \rangle = \langle f, \lim_{h \to h_0} (T_{-h} g - T_{-h_0} g) \rangle = 0$, for each $g \in E$. □

We can also associate a Beurling algebra to E'. Set

$$\check{\omega}(h) := ||T_{-h}||_{L(E')} = ||T_h^\top||_{L(E')} = \omega(-h).$$

The associated Beurling algebra to the dual space E' is $L^1_{\check{\omega}}$. We define the convolution $u * f = f * u$ of $f \in E'$ and $u \in L^1_{\check{\omega}}$ via transposition:

$$\langle u * f, g \rangle := \langle f, \check{u} * g \rangle, \quad g \in E. \tag{1.9}$$

In view of Proposition 1.1.1, this convolution is well-defined because $\check{u} \in L^1_\omega$.

Corollary 1.1.2. *We have* $||u * f||_{E'} \leq ||u||_{1,\check{\omega}}||f||_{E'}$ *and thus* E' *is a Banach module over the Beurling algebra* $L^1_{\check{\omega}}$. *In addition, if* φ_ε *and* c *are as in Corollary 1.1.1, then* $\varphi_\varepsilon * f \to cf$ *as* $\varepsilon \to 0$ *weakly* in* E' *for each fixed* $f \in E'$.

In general the embedding $\mathcal{S}(\mathbb{R}^n) \to E'$ is not dense (consider for instance $E = L^1$). However, E' inheres the three properties (a), (b), and (c) whenever E is reflexive.

Proposition 1.1.3. *If* E *is reflexive, then its dual space* E' *is also a translation-invariant Banach space of tempered distributions.*

Proof. By Proposition 1.1.2, it is enough to see that $\mathcal{S}(\mathbb{R}^n)$ is dense in E' and that E' satisfies (b). But if $g \in E'' = E$ is such that $\langle g, \varphi\rangle = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the property (a) of E implies that $g = 0$. Thus, $\mathcal{S}(\mathbb{R}^n)$ is dense in E'. For $\varepsilon > 0$, we pick $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f - \varphi\|_{E'} \leq \varepsilon$. Then $\|T_h f - f\|_{E'} \leq \|T_h\varphi - \varphi\|_{E'} + \varepsilon(1 + \omega(h))$. Observe that $T_h\varphi - \varphi \to 0$ in E' because it does in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is continuously embedded into E'. Therefore, $\lim_{h\to 0} \sup \|T_h f - f\|_{E'} \leq 2\varepsilon$. □

The fact that property (b) fails for E' in the non-reflexive case ($E = L^1$ is again an example) causes various difficulties when dealing with this space. We will often work with a certain closed subspace of E' rather than with E' itself. We denote the linear span of a set A as $\operatorname{span}(A)$.

Definition 1.1.2. The Banach space E'_* is defined as $E'_* = L^1_{\check{\omega}} * E'$.

That E'_* is a closed linear subspace of E' is a non-trivial fact. It follows from the celebrated Cohen-Hewitt Factorization Theorem [47], which asserts in this case the equality $L^1_{\check{\omega}} * E' = \overline{\operatorname{span}}(L^1_{\check{\omega}} * E')$ because the Beurling algebra $L^1_{\check{\omega}}$ possesses a bounded approximation unity (e.g., $\{\varphi_\varepsilon\}_{\varepsilon\in(0,1)}$ such that $c = 1$ with the notation of Corollary 1.1.1). The space E'_* will be of crucial importance throughout the rest of this work. It possesses richer properties than E' with respect to the translation group, as stated in the next theorem. The proof of the ensuing result makes use of an important property of the Fréchet algebra $\mathcal{S}(\mathbb{R}^n)$. Miyazaki [67, Lem. 1, p. 529] (cf. [74, 110])) has shown the factorization theorem $\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n)$ (the related result $\mathcal{D} = \operatorname{span}(\mathcal{D} * \mathcal{D})$ has been proved in [87]).

Theorem 1.1.2. *The space* E'_* *has the properties* $(a)''$, (b), *and* (c). *It is a Banach module over the Beurling algebra* $L^1_{\check{\omega}}$. *If* φ_ε *and* c *are as in Corollary 1.1.1, then, for each* $f \in E'_*$,

$$\lim_{\varepsilon\to 0^+} \|cf - \varphi_\varepsilon * f\|_{E'} = 0. \tag{1.10}$$

Furthermore, if E *is reflexive, then* $E'_* = E'$.

Proof. For $(a)''$, $\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n) \subset L^1_{\check{\omega}} * E' = E'_*$, whence the assertion follows. The property (c) for E'_* directly follows from Proposition 1.1.2. Observe that $\|T_h(u * f) - u * f\|_{E'} \leq \|f\|_{E'}\|T_h u - u\|_{1,\check{\omega}} \to 0$ as $h \to 0$ for $u \in L^1_{\check{\omega}}$ and $f \in E'$. The property (b) on E'_* then follows. Likewise, the approximation property (1.10) is easily established. Finally, if E is reflexive we have, by Proposition 1.1.3, that $\mathcal{S}(\mathbb{R}^n)$ is dense in E'; since $\mathcal{S}(\mathbb{R}^n) \subset E'_*$ and E'_* is closed, we must have $E'_* = E'$. □

We point out factorization properties of the Banach modules E and E'_* which also follow from the Cohen-Hewitt Factorization Theorem.

Proposition 1.1.4. *The factorizations $E = L^1_\omega * E$ and $E'_* = L^1_{\check{\omega}} * E'_*$ hold.*

Proof. The Cohen-Hewitt Factorization Theorem yields $L^1_\omega * E = \overline{\operatorname{span}(L^1_\omega * E)}$ and $L^1_{\check{\omega}} * E'_* = \overline{\operatorname{span}(L^1_{\check{\omega}} * E'_*)}$. By Corollary 1.1.1 and Theorem 1.1.2, $E = \overline{\mathcal{S}(\mathbb{R}^n) * E} \subseteq \overline{\operatorname{span}(L^1_\omega * E)} = L^1_\omega * E$ and $E'_* = \overline{\mathcal{S}(\mathbb{R}^n) * E'_*} \subseteq \overline{\operatorname{span}(L^1_{\check{\omega}} * E'_*)} = L^1_{\check{\omega}} * E'_*$, that is, $E = L^1_\omega * E$ and $E'_* = L^1_{\check{\omega}} * E'_*$. □

We now characterize E'_* by showing that it is the biggest subspace of E' where the property (b) holds.

Proposition 1.1.5. *We have that $E'_* = \{f \in E' : \lim_{h\to 0} \|T_h f - f\|_{E'} = 0\}$.*

Proof. Call momentarily $X = \{f \in E' : \lim_{h\to 0} \|T_h f - f\|_{E'} = 0\}$, it is clearly a closed subspace of E'. By the approximation property (1.10), it is enough to prove that $\mathcal{D}(\mathbb{R}^n) * E'$ is dense in X. For this, we will show that if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is positive and $\int_{\mathbb{R}^n} \varphi(y)dy = 1$, then $\lim_{\varepsilon\to 0} \|f * \varphi_\varepsilon - f\|_{E'} = 0$, for $f \in X$. We apply a similar argument to that used in the proof of Corollary 1.1.1. Take $\phi \in \mathcal{D}(\mathbb{R}^n)$, then

$$|\langle f * \varphi_\varepsilon - f, \phi\rangle| = \left|\left\langle f, \int_{\mathbb{R}^n} \varphi(y)(T_{\varepsilon y}\phi - \phi)dy\right\rangle\right| \leq \|\phi\|_E \sup_{y\in \operatorname{supp}\varphi} \|T_{-\varepsilon y} f - f\|_{E'},$$

which shows the claim. □

In view of property $(b)''$ from Proposition 1.1.2, we can naturally define a convolution mapping $E' \times \check{E} \to C(\mathbb{R}^n)$, where $\check{E} = \{g \in \mathcal{S}'(\mathbb{R}^n) : \ \check{g} \in E\}$ with norm $\|g\|_{\check{E}} := \|\check{g}\|_E$. We give a simple proposition that describes the mapping properties of this convolution. As usual, L^∞_ω, the dual of the Beurling algebra L^1_ω, is the Banach space of all measurable functions satisfying

$$||u||_{\infty,\omega} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \frac{|u(x)|}{\omega(x)} < \infty.$$

We need the following two closed subspaces of L^∞_ω,

$$UC_\omega := \left\{u \in L^\infty_\omega : \lim_{h\to 0} ||T_h u - u||_{\infty,\omega} = 0\right\} \tag{1.11}$$

and

$$C_\omega := \left\{u \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} \frac{u(x)}{\omega(x)} = 0\right\}. \tag{1.12}$$

Proposition 1.1.6. *$E' * \check{E} \subseteq UC_\omega$ and $* : E' \times \check{E} \to UC_\omega$ is continuous. If E is reflexive, then $E' * \check{E} \subset C_\omega$.*

Proof. The first assertion follows at once from the property $(b)''$. If E' is reflexive, Proposition 1.1.3 gives $\mathcal{S}(\mathbb{R}^n) \hookrightarrow E'$. Thus, $\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n)$ is dense in the closure of $\operatorname{span}(E' * \check{E})$ with respect to the norm $\| \ \|_{\infty,\omega}$. Since $\mathcal{S}(\mathbb{R}^n)$ is obviously dense in C_ω, we obtain $E' * \check{E} \subseteq C_\omega$. □

We end this section with the following remark.

Remark 1.1.2. The properties from Lemma 1.1.1 and Corollary 1.1.1 essentially characterize the class of translation-invariant Banach spaces of tempered distributions in the following sense. Let X be a Banach space that satisfies the condition $(a)'$ and let $\eta : \mathbb{R}^n \to (0,\infty)$ be a measurable function such that $\log \eta$ is subadditive, $\eta(0) = 1$, and η is polynomially bounded. Assume that $\|\varphi * g\|_X \leq \|\varphi\|_{1,\eta}\|g\|_X$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $g \in X$. The density of $\mathcal{D}(\mathbb{R}^n)$ in L^1_η automatically guarantees that X becomes a Banach convolution module over the Beurling algebra L^1_η and the convolution obviously satisfies $T_h(u * g) = (T_h u) * g = u * (T_h g)$. If we additionally assume that L^1_η possesses a bounded approximation of the unity for X, that is, there is a sequence $\{e_j\}_{j=0}^\infty \subset L^1_\eta$ such that $\sup_j \|e_j\|_{1,\eta} = M < \infty$, $\lim_j \|e_j * u - u\|_{1,\eta} = 0$, and $\lim_j \|e_j * g - g\|_X$ for all $u \in L^1_\eta$, $g \in X$, then the Cohen-Hewitt Theorem yields the factorization $X = L^1_\eta * X$. The latter factorization property implies that X satisfies the conditions $(b)'$ and $(c)'$.
In addition, its weight function ω satisfies $\omega(x) \leq M\eta(x)$. Indeed, let $\varepsilon > 0$. Then

$$\begin{aligned}\|T_{-h}\varphi\|_X &\leq \|T_{-h}e_j * \varphi\|_X + \varepsilon \leq \int_{\mathbb{R}^n} |e_j(x-h)|\eta(x)dx\|\varphi\|_X + \varepsilon \\ &= \int_{\mathbb{R}^n} |e_j(x)|\eta(x+h)dx\|\varphi\|_X + \varepsilon \leq \int_{\mathbb{R}^n} |e_j(x)|\eta(x)\eta(h)dx\|\varphi\|_X + \varepsilon \\ &\leq M\eta(h)\|\varphi\|_X + \varepsilon\end{aligned}$$

for j big enough.

1.2 The test function space $\mathcal{D}_E$

In this section we construct and study test function and distribution spaces associated to translation-invariant Banach spaces. We recall that throughout the rest of the paper E stands for a translation-invariant Banach space of tempered distributions whose growth function of its translation group is ω (cf. Definition 1.1.1). The Banach space $E'_* \subseteq E'$ was introduced in Definition 1.1.2.

We begin by constructing our space of test functions. Let $\mathcal{D}_E$ be the subspace of tempered distributions $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\varphi^{(\alpha)} \in E$ for all $\alpha \in \mathbb{N}_0^n$. We topologize $\mathcal{D}_E$ by means of the family of norms

$$\|\varphi\|_{E,N} := \max_{|\alpha| \leq N} \|\varphi^{(\alpha)}\|_E. \tag{1.13}$$

Proposition 1.2.1. *$\mathcal{D}_E$ is a Fréchet space and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_E \hookrightarrow E \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Moreover, $\mathcal{D}_E$ is a Fréchet module over the Beurling algebra L^1_ω, namely,*

$$||u * \varphi||_{E,N} \leq ||u||_{1,\omega}||\varphi||_{E,N}, \quad N \in \mathbb{N}_0. \tag{1.14}$$

Proof. $\mathcal{D}_E$ is a Fréchet space as a countable intersection of Banach spaces and $\mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{D}_E \hookrightarrow E \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. The relation (1.14) follows from Proposition 1.1.1 and the definition of the norms (1.13). It remains to show the density of the embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_E$. Let $\varphi \in \mathcal{D}_E$ and fix $N \in \mathbb{N}$. Find a sequence $\{\psi_j\}_{j=1}^\infty$ of functions from $\mathcal{S}(\mathbb{R}^n)$ such that $||\varphi-\psi_j||_E \leq j^{-N-1}$, for all j. Pick then $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$ and set $\phi_j(x) = j^n\phi(jx)$. We show that $\psi_j * \phi_j \to \varphi$ with respect to the norm $|| \ ||_{E,N}$; indeed, by Corollary 1.1.1 and Proposition 1.1.1,

$$\limsup_{j\to\infty} ||\varphi - \psi_j * \phi_j||_{E,N} \leq \limsup_{j\to\infty} j^N||\varphi - \psi_j||_E \max_{|\alpha|\leq N} \int_{\mathbb{R}^n} |\phi^{(\alpha)}(x)|\, \omega(x/j)dx = 0.$$

□

It turns out that all elements of our test function space $\mathcal{D}_E$ are smooth functions. We need a lemma in order to establish this fact.

Lemma 1.2.1. *Let $K \subset \mathbb{R}^n$ be compact. There is a positive integer j such that $\mathcal{D}^j_K \subset E \cap E'_*$ and the inclusion mappings $\mathcal{D}^j_K \to E$ and $\mathcal{D}^j_K \to E'_*$ are continuous.*

Proof. We may of course assume that K has non-empty interior. Let $\sigma > 0$ and set $K_\sigma = K + \{x \in \mathbb{R}^n : |x| \leq \sigma\}$. Since $\mathcal{D}(\mathbb{R}^n) \hookrightarrow E$ and $\mathcal{D}(\mathbb{R}^n) \to E'_*$ are continuous, there is $j = j_{K_\sigma} \in \mathbb{N}$ such that

$$||\varphi||_E \leq M_{K_\sigma} p_j(\varphi) \quad \text{and} \quad ||\varphi||_{E'} \leq M_{K_\sigma} p_j(\varphi) \tag{1.15}$$

for every $\varphi \in \mathcal{D}_{K_\sigma}$. Using a regularization argument, Corollary 1.1.1, and Theorem 1.1.2, we convince ourselves that (1.15) remains valid for all $\varphi \in \mathcal{D}^j_K$. □

We can now show that $\mathcal{D}_E \hookrightarrow \mathcal{E}(\mathbb{R}^n)$. More generally [38, 94], let $\mathcal{O}_C(\mathbb{R}^n)$ be the test function space corresponding to the space $\mathcal{O}'_C(\mathbb{R}^n)$ of convolutors of $\mathcal{S}'(\mathbb{R}^n)$, that is, $\varphi \in \mathcal{O}_C(\mathbb{R}^n)$ if there is $k \in \mathbb{N}$ such that $|\varphi^{(\alpha)}(x)| \leq M_\alpha(1 + |x|)^k$, for all α. It is topologized by a canonical inductive limit topology as in [38]. The spaces of continuous functions UC_ω and C_ω were introduced in (1.11) and (1.12). We have,

Proposition 1.2.2. *The embedding $\mathcal{D}_E \hookrightarrow \mathcal{O}_C(\mathbb{R}^n)$ holds. Furthermore, the partial derivatives of every $\varphi \in \mathcal{D}_E$ are elements of $C_{\check{\omega}}$, namely, they have decay*

$$\lim_{|x|\to\infty} \frac{\varphi^{(\alpha)}(x)}{\omega(-x)} = 0, \qquad \alpha \in \mathbb{N}^n. \tag{1.16}$$

Proof. We will employ the powerful Schwartz parametrix method Lemma 0.1.1, [94]. Let K be a compact symmetric neighborhood of 0 and find $\chi \in \mathcal{D}_K$ such that $\chi = 1$ near 0. Consider the Laplace operator Δ on $\mathbb{R}^n$. Let F_l be a fundamental solution of Δ^l, i.e., $\Delta^l F_l = \delta$. Then, $\triangle^l(\chi F_l) - \delta = \varsigma_l \in \mathcal{D}(\mathbb{R}^n)$, so that the parametrix formula

$$f = \triangle^l((\chi F_l) * f) - \varsigma_l * f \tag{1.17}$$

holds for every $f \in \mathcal{D}'(\mathbb{R}^n)$. By the Lemma 1.2.1, one can find $j \in \mathbb{N}$ for which $\mathcal{D}^j_K \subset E \cap E'$. Let $\varphi \in \mathcal{D}_E$. Since there is a sufficiently large $l \in \mathbb{N}$ such that $\chi F_l \in \mathcal{D}^j_K \subset E'$, we conclude from (1.17) and Proposition 1.1.6 that for each $\alpha \in \mathbb{N}^n$ one has $\varphi^{(\alpha)} = ((\chi F_l) * (\Delta^l \varphi^{(\alpha)})) - \varsigma_l * \varphi^{(\alpha)} \in \check{E}' * E \subset UC_{\check{\omega}}$ and, by Proposition 1.1.2 we actually obtain

$$|\varphi^{(\alpha)}(x)| \leq \omega(-x) \left(\|\check{\chi}\check{F}_l\|_{E'}\|\Delta^l\varphi^{(\alpha)}\|_E + \|\check{\varsigma}_l\|_{E'}\|\varphi^{(\alpha)}\|_E\right)$$

$$\leq M_l \omega(-x) \|\varphi\|_{E,2l+|\alpha|},$$

which also shows the embedding $\mathcal{D}_E \hookrightarrow \mathcal{O}_C(\mathbb{R}^n)$. Furthermore, if we set $\|\cdot\|_{\infty,\check{\omega},N} := \max_{|\alpha|\leq N} \|\cdot\|_{\infty,\check{\omega}}$, $N = 0, 1, 2, \ldots$, the above estimates imply

$$\|\varphi\|_{\infty,\check{\omega},N} \leq M_l \|\varphi\|_{E,N+2l}, \quad \varphi \in \mathcal{D}_E, \ N \in \mathbb{N}_0. \tag{1.18}$$

In order to show (1.16), we make use of the density $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_E$. Fix N. Given $\varepsilon > 0$, find $\rho \in \mathcal{D}(\mathbb{R}^n)$ such that $\|\varphi - \rho\|_{E,2l+N} < \varepsilon / M_l$. Choose also $\lambda > 0$ so large that $\rho(x) = 0$ for all $|x| \geq \lambda$. By (1.18), we obtain that $|\varphi^{(\alpha)}(x)| < \varepsilon\omega(-x)$ for all $|x| \geq \lambda$ and $|\alpha| \leq N$. □

Remark 1.2.1. For $u \in \mathcal{S}'(\mathbb{R}^n)$ with $u^{(\alpha)} \in L^1_\omega$, $|\alpha| \leq N$, set

$$\|u\|_{1,\omega,N} := \max_{|\alpha|\leq N} \|u^{(\alpha)}\|_{1,\omega}$$

and keep l as above. Note that $\chi F_l \in \mathcal{D}^j_K \subset E$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, Proposition 1.1.1 leads to

$$\|\varphi^{(\alpha)}\|_E \leq \|\chi F_l\|_E \|\Delta^l \varphi^{(\alpha)}\|_{1,\omega} + \|\varsigma_l\|_E \|\varphi^{(\alpha)}\|_{1,\omega}, \quad \alpha \in \mathbb{N}^n_0,$$

namely, E-norm bounds

$$\|\varphi\|_{E,N} \leq M'_l \|\varphi\|_{1,\omega,N+2l}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \ N \in \mathbb{N}_0. \tag{1.19}$$

The inequality (1.19) will be employed in Section 1.5 to study further properties of $\mathcal{D}_E$.

1.3 The distribution space $\mathcal{D}'_{E'_*}$

We can now define our new distribution space. We denote by $\mathcal{D}'_{E'_*}$ the strong dual of $\mathcal{D}_E$. When E is reflexive, we write $\mathcal{D}'_{E'} = \mathcal{D}'_{E'_*}$ in accordance with the last assertion of Theorem 1.1.2. In view of Proposition 1.2.1 and Proposition 1.2.2, we have the (continuous) inclusions $\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{D}'_{E'_*} \subset \mathcal{S}'(\mathbb{R}^n)$. In particular, every compactly supported distribution belongs to the space $\mathcal{D}'_{E'_*}$.

The notation $\mathcal{D}'_{E'_*} = (\mathcal{D}_E)'$ is motivated by the next structural theorem, which characterizes the elements of this dual space in two ways, in terms of convolution averages and as sums of derivatives of elements of E'_* (or E'). These characterizations play a fundamental role in our further considerations.

Theorem 1.3.1. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. The following statements are equivalent:*

(i) $f \in \mathcal{D}'_{E'_*}$.

(ii) $f * \psi \in E'$ *for all* $\psi \in \mathcal{D}(\mathbb{R}^n)$.

(iii) $f * \psi \in E'_*$ *for all* $\psi \in \mathcal{D}(\mathbb{R}^n)$.

(iv) f *can be expressed as* $f = \sum_{|\beta|\leq N} g_\beta^{(\beta)}$, *with* $g_\beta \in E'$.

(*v*) *There are* $f_\alpha \in E'_* \cap UC_\omega$ *such that*

$$f = \sum_{|\alpha| \le N} f_\alpha^{(\alpha)}. \tag{1.20}$$

Moreover, if E *is reflexive, we may choose* $f_\alpha \in E' \cap C_\omega$.

Remark 1.3.1. One can replace $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$ by $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ in the statement of Theorem 1.3.1. It follows from Theorem 1.3.1, since $E' \subset \mathcal{D}'_{E'_*}$, that every element of $f \in E'$ can be expressed as a sum of partial derivatives of elements of $E'_* \cap UC_\omega$ (or $E' \cap C_\omega$ in the reflexive case).

Proof. Clearly, $(v) \Rightarrow (i)$. We denote below $B_E = \{\varphi \in \mathcal{D}(\mathbb{R}^n) : \|\varphi\|_E \le 1\}$.

$(i) \Rightarrow (ii)$. Fix first $\psi \in \mathcal{D}(\mathbb{R}^n)$. By Proposition 1.1.1, the set $\check{\psi} * B_E = \{\check{\psi} * \varphi : \varphi \in B_E\}$ is bounded in $\mathcal{D}_E$.

Hence, $|\langle f * \psi, \varphi\rangle| = |\langle f, \check{\psi} * \varphi\rangle| < M_\psi$ for $\varphi \in B_E$. So, $|\langle f * \psi, \varphi\rangle| < M_\psi \|\varphi\|_E$, for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Using the fact that $\mathcal{D}(\mathbb{R}^n)$ is dense in E, the last inequality means that $f * \psi \in E'$, for every $\psi \in \mathcal{D}(\mathbb{R}^n)$.

$(ii) \Rightarrow (iii)$. We use the factorization property of $\mathcal{D}(\mathbb{R}^n)$ from [87] to write $\psi = \psi_1 * \phi_1 + \psi_2 * \phi_2 + \cdots + \psi_N * \phi_N \in \mathcal{D}(\mathbb{R}^n)$ with $\psi_j, \phi_j \in \mathcal{D}(\mathbb{R}^n)$. From (ii), we conclude $f * \psi = (f * \psi_1) * \phi_1 + \cdots + (f * \psi_N) * \phi_N \in \operatorname{span}(E' * \mathcal{D}(\mathbb{R}^n)) \subset E'_*$, for any $\psi \in \mathcal{D}(\mathbb{R}^n)$.

$(iii) \Rightarrow (iv)$. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ be arbitrary. Because $\langle f * \check{\varphi}, \check{\psi}\rangle = \langle f * \psi, \varphi\rangle$ we get that the set $\{\langle f * \check{\varphi}, \check{\psi}\rangle : \varphi \in B_E\}$ is bounded in $\mathbb{C}$. The Banach-Steinhaus Theorem implies that $\{f * \check{\varphi} : \varphi \in B_E\}$ is an equicontinuous subset of $\mathcal{D}'(\mathbb{R}^n)$. Namely, for any compact set $K \subset \mathbb{R}^n$ there exist $N = N_K \in \mathbb{N}_0$ and $M = M_K > 0$ such that $|\langle f * \rho, \varphi\rangle| < M p_N(\rho)$ for every $\varphi \in B_E$ and $\rho \in \mathcal{D}^N_K$. Hence, for all $\rho \in \mathcal{D}^N_K$ we have $f * \rho \in E'$.

Let K, $\chi \in \mathcal{D}_K$ and F_l be as in the proof of Proposition 1.2.2. Then $\chi F_l \in \mathcal{D}^N_K$ for sufficiently large l so that the parametrix formula (1.17) yields $f \in \triangle^l(E') + E' \subseteq \mathcal{D}'_{E'_*}$. In particular, one obtains the representation

$$f = \sum_{|\beta| \le 2l} g_\beta^{(\beta)}, \quad g_\beta \in E'. \tag{1.21}$$

$(iv) \Rightarrow (v)$. In order to improve the representation (1.21) to the one stated in (v), we apply the parametrix method again to each $g_\beta \in E'$, $|\beta| \le 2l$. Let K be a compact symmetric set as above. By Lemma 1.2.1, one can find $j = j_K$ such that $\mathcal{D}^j_K \subset E$. Choosing l' so large that $\chi F_{l'} \in \mathcal{D}^j_K$, the parametrix formula (1.17) yields

$$g_\beta = \sum_{|\nu| \le 2l'} (f_{\beta,\nu})^{(\nu)}, \tag{1.22}$$

where each $f_{\beta,\nu} \in L^1_{\check{\omega}} * E' \subset E'_*$. Furthermore, each $f_{\beta,\nu}$ is of the form $f_{\beta,\nu} = g_\beta * \check{\varrho}_\nu$ with $\varrho_\nu \in \mathcal{D}^j_K \subset E$. By Proposition 1.1.6, we have $f_{\beta,\nu} \in UC_\omega$ (resp., C_ω in the reflexive case). □

Let us recall that the translation of a vector-valued distribution $\mathbf{f}$ is defined in the standard way, i.e., $\langle T_h \mathbf{f}, \varphi\rangle := \langle \mathbf{f}, T_{-h}\varphi\rangle$. We also have,

Corollary 1.3.1. *Let* $\mathbf{f} \in \mathcal{D}'(\mathbb{R}^n, E'_{\sigma(E',E)})$*, that is, a continuous linear mapping* $\mathbf{f} : \mathcal{D}(\mathbb{R}^n) \to E'_{\sigma(E',E)}$*. If* $\mathbf{f}$ *commutes with every translation, i.e.,*

$$\langle T_h \mathbf{f}, \varphi \rangle = T_h \langle \mathbf{f}, \varphi \rangle, \quad \text{for all } h \in \mathbb{R}^n \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^n), \tag{1.23}$$

then, there exists $f \in \mathcal{D}'_{E'_*}$ *such that* $\mathbf{f}$ *is of the form*

$$\langle \mathbf{f}, \varphi \rangle = f * \check{\varphi}, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1.24}$$

Proof. The mapping $\mathbf{f} : \mathcal{D}(\mathbb{R}^n) \to E'_{\sigma(E',E)}$ is linear and continuous. Since $E'_{\sigma(E',E)} \to \mathcal{D}'(\mathbb{R}^n)$ is continuous, we obtain that $\mathbf{f} : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is also a continuous linear mapping. Due to the fact that $\mathbf{f}$ commutes with every translation, it follows from a well-known theorem (cf. [91, Thm. 5.11.3, p. 332]) that there exists $f \in \mathcal{D}'(\mathbb{R}^n)$ such that $\langle \mathbf{f}, \varphi \rangle = f * \check{\varphi} \in E'$, for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Theorem 1.3.1 yields $f \in \mathcal{D}'_{E'_*}$. □

Our results from above implicitly suggest to embed the distribution space $\mathcal{D}'_{E'_*}$ into the space of E'-valued tempered distributions as follows. Define first the continuous injection

$$\iota : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n, \mathcal{S}'(\mathbb{R}^n)), \tag{1.25}$$

where $\iota(f) = \mathbf{f}$ is given by (1.24). Now, the restriction of ι to $\mathcal{D}_{E'_*}$,

$$\iota : \mathcal{D}'_{E'_*} \to \mathcal{S}'(\mathbb{R}^n, E'), \tag{1.26}$$

is clearly continuous for the strong topologies. Furthermore, by (v) of Theorem 1.3.1, $\iota(\mathcal{D}'_{E'_*}) \subset \mathcal{S}'(\mathbb{R}^n, E'_*)$. Corollary 1.3.1 then tells us that $\iota(\mathcal{D}'_{E'_*})$ is precisely the subspace of $\mathcal{S}'(\mathbb{R}^n, E'_*)$ consisting of those $\mathbf{f}$ which commute with all translations in the sense of (1.23). Since the translations T_h are continuous operators on E'_*, we actually obtain that the range $\iota(\mathcal{D}'_{E'_*})$ is a closed subspace of $\mathcal{S}'(\mathbb{R}^n, E'_*)$. Indeed,

$$\iota(\mathcal{D}'_{E'}) = \bigcap_{h \in \mathbb{R}^n} X_h,$$

where X_h is the space of vector valued distributions such that (1.23) holds for a fixed $h \in \mathbb{R}^n$. The X_h is closed because it is the set where two continuous operators are equal to each other. Hence, being intersection of closed sets, $\iota(\mathcal{D}'_{E'})$ is closed. Note that we may consider $\mathcal{D}'(\mathbb{R}^n)$ instead of $\mathcal{S}'(\mathbb{R}^n)$ in these embeddings.

One can readily adapt the proof of Theorem 1.3.1 to show the following characterizations of bounded subsets and convergent sequences of $\mathcal{D}'_{E'_*}$. It is worth noticing that Corollary 1.3.3 implies that the inverse of (1.26), defined on $\iota(\mathcal{D}'_{E'_*})$, is sequentially continuous.

Corollary 1.3.2. *The following properties are equivalent:*

(i) B' *is a bounded subset of* $\mathcal{D}'_{E'_*}$.

(ii) $\iota(B')$ *is bounded in* $\mathcal{S}'(\mathbb{R}^n, E')$ *(or equivalently in* $\mathcal{S}'(\mathbb{R}^n, E'_*)$*).*

(*iii*) *There are* $C > 0$ *and* $N \in \mathbb{N}$ *such that every* $f \in B'$ *admits a representation (1.20) with continuous functions* $f_\alpha \in E'_* \cap UC_\omega$ *satisfying the uniform bounds* $||f_\alpha||_{E'} < M$ *and* $||f_\alpha||_{\infty,\omega} < M$ *(if* E *is reflexive, one may choose* $f_\alpha \in E' \cap C_\omega$*).*

Corollary 1.3.3. *Let* $\{f_j\}_{j=0}^\infty \subset \mathcal{D}'_{E'_*}$ *(or similarly, a filter with a countable or bounded basis). The following three statements are equivalent:*

(*i*) $\{f_j\}_{j=0}^\infty$ *is (strongly) convergent in* $\mathcal{D}'_{E'_*}$.

(*ii*) $\{\iota(f_j)\}_{j=0}^\infty$ *is convergent in* $\mathcal{S}'(\mathbb{R}^n, E')$ *(equiv. in* $\mathcal{S}'(\mathbb{R}^n, E'_*)$*).*

(*iii*) *There are* $N \in \mathbb{N}$ *and continuous functions* $f_{\alpha,j} \in E'_* \cap UC_\omega$ *such that* $f_j = \sum_{|\alpha|\leq N} f_{\alpha,j}^{(\alpha)}$ *and the sequences* $\{f_{\alpha,j}\}_{j=0}^\infty$ *are convergent in both* E'_* *and* L^∞_ω *(if* E *is reflexive one may choose* $f_{\alpha,j} \in E' \cap C_\omega$*).*

Concerning weak * *convergence of sequences, the following three properties are equivalent:*

$(i)^*$ $\{f_j\}_{j=0}^\infty$ *is weakly* * *convergent in* $\mathcal{D}'_{E'_*}$.

$(ii)^*$ $\{\iota(f_j)\}_{j=0}^\infty$ *converges in* $\mathcal{S}'(\mathbb{R}^n, E'_{\sigma(E',E)})$ *(equiv. in* $\mathcal{S}'(\mathbb{R}^n, (E'_*)_{\sigma(E'_*,E)})$*).*

$(iii)^*$ *There are* $N \in \mathbb{N}$ *and continuous functions* $f_{\alpha,j} \in E'_* \cap UC_\omega$ *such that* $f_j = \sum_{|\alpha|\leq N} f_{\alpha,j}^{(\alpha)}$, *the sequences* $\{f_{\alpha,j}\}_{j=0}^\infty$ *are uniformly convergent over compacts of* $\mathbb{R}^n$, *and the norms* $||f_{\alpha,j}||_{E'}$ *and* $||f_{\alpha,j}||_{\infty,\omega}$ *remain uniformly bounded (if* E *is reflexive, one may choose* $f_{\alpha,j} \in E' \cap C_\omega$*).*

Proof. We only show that (*ii*) in Corollary 1.3.3 implies (*iii*) (resp., $(ii)^*$ implies $(iii)^*$), the rest is left to the reader. Let $K \subset \mathbb{R}^n$ be a compact symmetric neighborhood of the origin. Since $\{\iota(f_j)\}_{j=0}^\infty$ converges in $\mathcal{S}'(\mathbb{R}^n, E')$ (resp., in $\mathcal{S}'(\mathbb{R}^n, E'_{\sigma(E',E)})$), there exists N such that $\{\iota(f_j)\}_{j=0}^\infty$ converges in $L_b(\mathcal{D}^N_K, E')$ (resp., in $L_b(\mathcal{D}^N_K, E'_{\sigma(E',E)})$); in particular, $\{\psi * f_j\}_{j=0}^\infty$ converges in E' (resp., weakly* E') for each fixed $\psi \in \mathcal{D}^N_K$. If we take l as in the proof of Theorem 1.3.1, the representation (1.17) gives $f = \sum_{|\beta|\leq 2l} g_{\beta,j}^{(\beta)}$ with each $\{g_{\beta,j}\}_{j=0}^\infty$ convergent in E' (resp., weakly* convergent in E') because $\{\varsigma_l * f_j\}_{j=0}^\infty$ and $\{(\chi F_l) * f_j\}_{j=0}^\infty$ are then convergent in E' (resp., weakly* convergent in E'). Likewise, another application of the parametrix method, as in the proof of Theorem 1.3.1, allows us to replace the sequences $\{g_{\beta,j}\}_{j=0}^\infty$ by sequences $\{f_{\alpha,j}\}_{j=0}^\infty$ having the claimed properties. □

Observe that Corollaries 1.3.2 and 1.3.3 are still valid if $\mathcal{S}'(\mathbb{R}^n)$ is replaced by $\mathcal{D}'(\mathbb{R}^n)$.

When E is reflexive, the space $\mathcal{D}_E$ is also reflexive. Furthermore, we have:

Proposition 1.3.1. *If* E *is reflexive, then* $\mathcal{D}_E$ *is an FS***-space and* $\mathcal{D}'_{E'}$ *is a DFS***-space. In addition,* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $\mathcal{D}'_{E'}$.

Proof. Let $\mathcal{D}^N_E$ be the Banach space of distributions such that $\varphi^{(\alpha)}\in E$ for $|\alpha|\leq N$ provided with the norm $||\ ||_{E,N}$ (cf. (1.13)). We then have the projective sequence

$$E\leftarrow \mathcal{D}^1_E\leftarrow\cdots\leftarrow\mathcal{D}^N_E\leftarrow\mathcal{D}^{N+1}_E\leftarrow\cdots\leftarrow\mathcal{D}_E, \tag{1.27}$$

where clearly $\mathcal{D}_E=\operatorname{proj}\lim_N\mathcal{D}^N_E$. Using the Hahn-Banach Theorem, one readily sees that every $f\in(\mathcal{D}^N_E)'$ is of the form $f=\sum_{|\alpha|\leq N}f_\alpha^{(\alpha)}$, with $f_\alpha\in E'$. Thus, each Banach space $\mathcal{D}^N_E$ is reflexive, or equivalently its closed unit ball is weakly compact. The latter implies that every injection in the projective sequence (1.27) is weakly compact. This implies all the assertions. □

It should be noticed that the convolution of $f\in\mathcal{D}'_{E'_*}$ and $u\in L^1_{\check{\omega}}$, defined as $\langle u*f,\varphi\rangle:=\langle f,\check{u}*\varphi\rangle$, $\varphi\in\mathcal{D}_E$, gives rise to a continuous bilinear mapping $*:L^1_{\check{\omega}}\times\mathcal{D}'_{E'_*}\to\mathcal{D}'_{E'_*}$, as follows from (1.14). We will show in Section 1.5 that the convolution of elements of $\mathcal{D}'_{E'_*}$ can be defined with distributions in a lager class than $L^1_{\check{\omega}}$, namely, with elements of the space $\mathcal{D}'_{L^1_{\check{\omega}}}$ to be introduced in Section 1.4.

We end this section with a third characterization of $\mathcal{D}'_{E'_*}$ in terms of norm growth bounds on convolutions with an approximation of the unity. For it, we employ the useful concept of the $\phi-$transform [28, 29, 83, 105], which is defined as follows. Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n}\phi(x)dx=1$. The $\phi-$transform of $f\in\mathcal{S}'(\mathbb{R}^n)$ is the smooth function

$$F_\phi f(x,t)=\langle f(x+t\xi),\phi\,(\xi)\rangle=(f*\check{\phi}_t)(x),\quad (x,t)\in\mathbb{R}^n\times\mathbb{R}_+.$$

Theorem 1.3.2. *A distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ belongs to $\mathcal{D}'_{E'_*}$ if and only if $F_\phi f(\cdot,t)\in E'$ for all $t\in(0,t_0)$ and there are constants $k\in\mathbb{N}$ and $M>0$ such that*

$$\|F_\phi f(\,\cdot\,,t)\|_{E'}\leq\frac{M}{t^k},\quad t\in(0,t_0). \tag{1.28}$$

In such a case,

$$\lim_{t\to0^+}F_\phi f(\,\cdot\,,t)=f\quad\textit{strongly in } \mathcal{D}'_{E'_*}. \tag{1.29}$$

Proof. The relation (1.29) follows by combining (v) of Theorem 1.3.1 with Theorem 1.1.2. Let $f\in\mathcal{D}'_{E'_*}$. Write f as in (1.20). By Corollary 1.1.3, for $t\in(0,t_0]$,

$$\begin{aligned}||F_\phi f(\,\cdot\,,t)||_{E'}&\leq\sum_{|\alpha|\leq N}||f_\alpha*(\check{\phi}_t)^{(\alpha)}||_{E'}\\&\leq\frac{M'}{t^N}\sum_{|\alpha|\leq N}||f_\alpha||_{E'}\int_{\mathbb{R}^n}|\phi^{(\alpha)}(x)|\,\omega(tx)dx\leq\frac{M}{t^N}.\end{aligned}$$

Conversely, assume (1.28). Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ be also such that $\int_{\mathbb{R}^n}\varphi(x)dx=1$. Setting $\phi_1=\phi*\varphi$, we have that $F_{\phi_1}f(x,t)=(F_\phi f(\cdot,t)*\check{\varphi}_t)(x)$, and so $F_{\phi_1}f(\cdot,t)\in E'*\mathcal{S}(\mathbb{R}^n)\subset E'_*$ for each $t\in(0,t_0)$. We will use the theory of (Tauberian) class estimates from [28, 83]. Set $\mathbf{f}=\iota(f)\in\mathcal{S}'(\mathbb{R}^n,\mathcal{S}'(\mathbb{R}^n))$ (cf. (1.25)). By Corollary 1.3.1, it is enough to show that $\mathbf{f}\in\mathcal{S}'(\mathbb{R}^n,E'_*)$. The $\mathcal{S}'(\mathbb{R}^n)$-valued ϕ_1-transform of $\mathbf{f}$ is the vector-valued distribution $F_{\phi_1}\mathbf{f}:\mathbb{R}^n\times\mathbb{R}_+\to\mathcal{S}'(\mathbb{R}^n)$

given by $F_{\phi_1}\mathbf{f}(x,t) = T_x F_{\phi_1} f(\cdot,t) \in \mathcal{S}'(\mathbb{R}^n)$. From what has been shown we have that $F_{\phi_1}\mathbf{f}(x,t) \in E'_* \subset \mathcal{S}'(\mathbb{R}^n)$ for all $(x,t) \in \mathbb{R}^n \times (0,t_0)$ and, by property (b) applied E'_* (cf. Theorem 1.1.2), we get that the the mapping $\mathbb{R}^n \to E'_*$ given by $x \mapsto F_{\phi_1}\mathbf{f}(x,t)$ is continuous for each fixed $t \in (0,t_0)$. Furthermore, using the fact that E'_* is a Banach modulo over $L^1_{\check{\omega}}$, we conclude that

$$\begin{aligned} ||F_{\phi_1}\mathbf{f}(x,t)||_{E'} &= ||T_x F_{\phi_1} f(\cdot,t)||_{E'} \\ &\leq \omega(x)||F_\phi f(\,\cdot\,,t)||_{E'} \int_{\mathbb{R}^n} |\varphi(\xi)|\omega(t\xi)d\xi \leq \tilde{M}\frac{(1+|x|)^\tau}{t^k}, \end{aligned}$$

for all $(x,t) \in \mathbb{R}^n \times (0,t_0)$. But, as shown in [28] (see also [83, Sect. 7]), the very last estimate is necessary and sufficient for $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E'_*)$. This completes the proof. □

We will apply Theorem 1.3.2 in Section 2.4 to characterize the elements of $\mathcal{D}'_{E'}$ via E'-norm estimates of solutions to the heat equation in the half-space $\mathbb{R}^n \times \mathbb{R}_+$.

1.4 Examples: L^p_η weighted spaces

In this section we discuss some important examples of the spaces $\mathcal{D}_E$ and $\mathcal{D}'_{E'_*}$. They extend the familiar Schwartz spaces $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^p}$. These particular instances are useful for studying properties of the general $\mathcal{D}'_{E'_*}$ (cf. section 1.5).

Let η be a *polynomially bounded weight*, that is, a measurable function $\eta : \mathbb{R}^n \to (0,\infty)$ that fulfills the requirement $\eta(x+h) \leq M\eta(x)(1+|h|)^\tau$, for some $M, \tau > 0$. We consider the norms

$$||g||_{p,\eta} = \left(\int_{\mathbb{R}^n} |g(x)\eta(x)|^p dx\right)^{\frac{1}{p}} \text{ for } p \in [1,\infty) \quad \text{and} \quad ||g||_{\infty,\eta} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{|g(x)|}{\eta(x)}\ .$$

Then the space L^p_η consists of those measurable functions such that $||g||_{p,\eta} < \infty$ (for $\eta = 1$, we write as usual L^p and $\|\ \|_p$). The number q always stands for $p^{-1} + q^{-1} = 1$ ($p \in [1,\infty]$). Of course $(L^p_\eta)' = L^q_{\eta^{-1}}$ if $1 < p < \infty$ and $(L^1_\eta)' = L^\infty_\eta$. The spaces $E = L^p_\eta$ are clearly translation-invariant Banach space of tempered distributions for $p \in [1,\infty)$. The case $p = \infty$ is an exception, because $\mathcal{D}(\mathbb{R}^n)$ fails to be dense in L^∞_η. In view of Theorem 1.1.2, the space E'_* corresponding to $E = L^p_{\eta^{-1}}$ is $E'_* = E' = L^q_\eta$ whenever $1 < p < \infty$. On the other hand, Proposition 1.1.5 gives that $E'_* = UC_\eta$ for $E = L^1_\eta$, where UC_η is defined as in (1.11) with ω replaced by η.

We can easily find the Beurling algebra of L^p_η.

Proposition 1.4.1. *Let*

$$\omega_\eta(h) := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{\eta(x+h)}{\eta(x)}.$$

Then

$$||T_{-h}||_{L(L^p_\eta)} = \begin{cases} \omega_\eta(h) & \text{if } p \in [1,\infty), \\ \omega_\eta(-h) & \text{if } p = \infty. \end{cases}$$

Consequently, the Beurling algebra associated to L^p_η *is* $L^1_{\omega_\eta}$ *if* $p = [1, \infty)$ *and* $L^1_{\check{\omega}_\eta}$ *if* $p = \infty$.

Proof. Assume first that $1 \leq p < \infty$. Clearly, $\omega_\eta(h) \geq ||T_{-h}||_{L(L^p_\eta)}$. Let $\varepsilon > 0$ and set

$$A = \{x \in \mathbb{R}^n : \omega_\eta(h) - \varepsilon \leq \eta(x + h)/\eta(x)\}.$$

The Lebesgue measure of A is positive. Find a compact subset $K \subset A$ with positive Lebesgue measure and let g be the characteristic function of K. Then

$$\int_{\mathbb{R}^n} |g(x)|^p \eta^p(x + h) dx \geq (\omega_\eta(h) - \varepsilon)^p ||g||^p_{p,\eta},$$

which yields $||T_{-h}||_{L(L^p_\eta)}(h) \geq (\omega_\eta(h) - \varepsilon)$. Since ε is arbitrary, we obtain $\omega_\eta(h) = ||T_{-h}||_{L(L^p_\eta)}$. The case $p = \infty$ follows by duality. □

We remark that when the logarithm of η is a positive measurable subadditive function and $\eta(0) = 1$, one easily obtains from Proposition 1.4.1 that $\omega_\eta = \eta$.

Consider now the spaces $\mathcal{D}_{L^p_\eta}$ for $p \in [1, \infty]$, defined as in Section 1.2 by taking $E = L^p_\eta$. Once again, the case $p = \infty$ is an exception because $\mathcal{D}(\mathbb{R}^n)$ is not dense $\mathcal{D}_{L^\infty_\eta}$. In analogy to Schwartz notation [94], we write $\mathcal{B}_\eta := \mathcal{D}_{L^\infty_\eta}$. Set further $\dot{\mathcal{B}}_\eta$ for the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{B}_\eta$. We immediately see that $\dot{\mathcal{B}}_\eta = \mathcal{D}_{C_\eta}$, where $C_\eta = \left\{g \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} g(x)/\eta(x) = 0\right\} \subset L^\infty_\eta$. Observe that the space E'_* for $E = C_\eta$ is $E'_* = L^1_\eta$. By Proposition 1.2.2, $\mathcal{D}_{L^p_\eta} \subset \dot{\mathcal{B}}_{\check{\omega}_\eta}$ for $p \in (1, \infty)$; using the parametrix formula (1.17), one also deduces that $\mathcal{D}_{L^1_\eta} \subset \dot{\mathcal{B}}_{\check{\omega}_\eta}$. Actually, the estimate (1.18) gives $\mathcal{D}_{L^p_\eta} \hookrightarrow \dot{\mathcal{B}}_{\check{\omega}_\eta}$ for every $p \in [1, \infty)$. It follows from Proposition 1.3.1 that $\mathcal{D}_{L^p_\eta}$ is an FS*-space and hence reflexive when $p \in (1, \infty)$.

In accordance to Section 1.3, the weighted spaces $\mathcal{D}'_{L^p_\eta}$ are defined as $\mathcal{D}'_{L^p_\eta} = (\mathcal{D}_{L^q_{\eta^{-1}}})'$ where $p^{-1} + q^{-1} = 1$ if $p \in (1, \infty)$; if $p = 1$ or $p = \infty$, we have $\mathcal{D}'_{L^1_\eta} = (\mathcal{D}_{C_\eta})' = (\dot{\mathcal{B}}_\eta)'$ and $\mathcal{D}'_{L^\infty_\eta} = (\mathcal{D}_{L^1_\eta})'$. We write $\mathcal{B}'_\eta = \mathcal{D}'_{L^\infty_\eta}$ and $\dot{\mathcal{B}}'_\eta$ for the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{B}'_\eta$. We call $\mathcal{B}'_\eta$ the space of η-bounded distributions. Observe that the $\mathcal{D}'_{L^p_\eta}$ are DFS* spaces and $(\mathcal{D}'_{L^p_\eta})' = \mathcal{D}_{L^q_{\eta^{-1}}}$ when $1 < p < \infty$. Theorem 1.3.1 gives that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{D}'_{L^1_\eta}$ and Corollaries 1.3.2 and 1.3.3 imply that $(\mathcal{D}'_{L^1_\eta})' = \mathcal{B}_\eta$. Using the parametrix method, one deduces as in the proof of Theorem 1.3.1, that every element of $\dot{\mathcal{B}}'_\eta$ is the sum of partial derivatives of elements of C_η and that $f \in \dot{\mathcal{B}}'_\eta$ if and only if $f * \psi \in C_\eta$; likewise analogs to Corollaries 1.3.2 and 1.3.3 hold for $\dot{\mathcal{B}}'_\eta$. The later implies that $(\dot{\mathcal{B}}'_\eta)' = \mathcal{D}_{L^1_\eta}$. Employing Theorem 1.3.1, Corollary 1.3.2 and Corollary 1.3.3, one sees that $\mathcal{D}'_{L^p_\eta} \subset \dot{\mathcal{B}}'_{\check{\omega}_\eta}$, $1 \leq p < \infty$, and that the inclusion is sequentially continuous. Summarizing, we have the embeddings $\mathcal{D}_{L^1_{\omega_\eta}} \hookrightarrow \mathcal{D}_{L^p_\eta} \hookrightarrow \dot{\mathcal{B}}_{\check{\omega}_\eta}$ and $\mathcal{D}'_{L^1_{\omega_\eta}} \hookrightarrow \mathcal{D}'_{L^p_\eta} \hookrightarrow \dot{\mathcal{B}}'_{\check{\omega}_\eta}$ for $1 \leq p < \infty$, and $\dot{\mathcal{B}}_\eta \hookrightarrow \dot{\mathcal{B}}_{\omega_\eta}$ and $\dot{\mathcal{B}}'_\eta \hookrightarrow \dot{\mathcal{B}}'_{\omega_\eta}$.

The multiplicative product mappings $\cdot : \mathcal{D}'_{L^p_\eta} \times \mathcal{B}_\eta \to \mathcal{D}'_{L^p}$ and $\cdot : \mathcal{B}'_\eta \times \mathcal{D}_{L^p_\eta} \to \mathcal{D}'_{L^p}$ are well-defined and hypocontinuous for $1 \leq p < \infty$. In particular, $f\varphi$ is an integrable distribution in the Schwartz sense [94] whenever $f \in \mathcal{B}'_\eta$ and $\varphi \in \mathcal{D}_{L^1_\eta}$ or $f \in \mathcal{D}'_{L^1_\eta}$ and $\varphi \in \mathcal{B}_\eta$. If $(1/r) = (1/p_1) + (1/p_2)$ with $1 \leq r, p_1, p_2 <$

∞, it is also clear that the multiplicative product $\cdot : \mathcal{D}'_{L^{p_1}_{\eta_1}} \times \mathcal{D}_{L^{p_2}_{\eta_2}} \to \mathcal{D}'_{L^r_{\eta_1\eta_2}}$ is hypocontinuous. Clearly, the convolution product can always be canonically defined as a hypocontinuous mapping in the following situations, $* : \mathcal{D}'_{L^p_\eta} \times \mathcal{D}'_{L^1_\omega} \to \mathcal{D}'_{L^p_\eta}$, $1 \leq p < \infty$, $* : \mathcal{B}'_\eta \times \mathcal{D}'_{L^1_{\check{\omega}_\eta}} \to \mathcal{B}'_\eta$, and $* : \dot{\mathcal{B}}'_\eta \times \mathcal{D}'_{L^1_{\check{\omega}_\eta}} \to \dot{\mathcal{B}}'_\eta$.

1.5 Relation between $\mathcal{D}'_{E'_*}$, $\mathcal{B}'_\omega$, and $\mathcal{D}'_{L^1_{\check{\omega}}}$ – Convolution and multiplication

Many of the properties of the $\mathcal{D}_{L^p_\eta}$ and $\mathcal{D}'_{L^p_\eta}$ extend to $\mathcal{D}_E$ and $\mathcal{D}'_{E'_*}$ for the general translation-invariant Banach space of tempered distributions E with Beurling algebra L^1_ω. The next theorem summarizes some of our previous results.

Theorem 1.5.1. *We have $\mathcal{D}_{L^1_\omega} \hookrightarrow \mathcal{D}_E \hookrightarrow \dot{\mathcal{B}}_{\check{\omega}}$ and hence the continuous inclusions $\mathcal{D}'_{L^1_{\check{\omega}}} \to \mathcal{D}'_{E'_*} \to \mathcal{B}'_\omega$. When E is reflexive $\mathcal{D}'_{L^1_{\check{\omega}}} \hookrightarrow \mathcal{D}'_{E'} \hookrightarrow \dot{\mathcal{B}}'_\omega$.*

Proof. Notice that Proposition 1.2.2 gives the inclusions $\mathcal{D}_E \subsetneq \dot{\mathcal{B}}_{\check{\omega}}$. We actually have $\mathcal{D}_E \hookrightarrow \dot{\mathcal{B}}_{\check{\omega}}$ because of (1.18). The dense embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_{L^1_\omega}$ and the inequality (1.19) from Remark 1.2.1 show that $\mathcal{D}_{L^1_\omega} \subseteq \mathcal{D}_E$ and that (1.19) remains true for all $\varphi \in \mathcal{D}_{L^1_\omega}$. Consequently, $\mathcal{D}_{L^1_\omega} \hookrightarrow \mathcal{D}_E$. By transposition of the latter two dense inclusion mappings, $\mathcal{D}'_{L^1_{\check{\omega}}} \to \mathcal{D}'_{E'_*} \to \mathcal{B}'_\omega$. In the reflexive case, Theorem 1.3.1 gives $\mathcal{D}'_{E'} \subseteq \dot{\mathcal{B}}'_\omega$ and therefore $\mathcal{D}'_{L^1_{\check{\omega}}} \hookrightarrow \mathcal{D}'_{E'} \hookrightarrow \dot{\mathcal{B}}'_\omega$. □

We can now define multiplication and convolution operations on $\mathcal{D}'_{E'_*}$.

Proposition 1.5.1. *The multiplicative products $\cdot : \mathcal{D}'_{E'_*} \times \mathcal{D}_{L^1_\omega} \to \mathcal{D}'_{L^1}$ and $\cdot : \mathcal{D}'_{L^1_{\check{\omega}}} \times \mathcal{D}_E \to \mathcal{D}'_{L^1}$ are hypocontinuous. The convolution products are continuous in the following two cases: $* : \mathcal{D}'_{E'_*} \times \mathcal{D}'_{L^1_{\check{\omega}}} \to \mathcal{D}'_{E'_*}$ and $* : \mathcal{D}'_{E'_*} \times \mathcal{O}'_C(\mathbb{R}^n) \to \mathcal{D}'_{E'_*}$. The convolution $* : \mathcal{D}'_{E'_*} \times \mathcal{D}_{\check{E}} \to \mathcal{B}_\omega$ is hypocontinuous; when the space E is reflexive, we have $* : \mathcal{D}'_{E'} \times \mathcal{D}_{\check{E}} \to \dot{\mathcal{B}}_\omega$.*

Proof. That these bilinear mappings have the range in the stated spaces follows from Theorem 1.3.1 and Theorem 1.5.1. The hypocontinuity of the multiplicative products is a consequence of Theorem 1.5.1. In fact, the bilinear mapping $\cdot : \mathcal{D}'_{E'_*} \times \mathcal{D}_{L^1_\omega} \to \mathcal{D}'_{L^1}$ is hypocontinuous as the composition of the continuous inclusion mapping $\mathcal{D}'_{E'_*} \times \mathcal{D}_{L^1_\omega} \to \mathcal{B}'_\omega \times \mathcal{D}_{L^1_\omega}$ and the hypocontinuous mapping $\cdot : \mathcal{B}'_\omega \times \mathcal{D}_{L^1_\omega} \to \mathcal{D}'_{L^1}$. Likewise, $\cdot : \mathcal{D}'_{L^1_{\check{\omega}}} \times \mathcal{D}_E \to \mathcal{D}'_{L^1}$ is hypocontinuous. It is clear that $* : \mathcal{D}'_{L^1_\omega} \times \mathcal{D}_E \to \mathcal{D}_E$ is hypocontinuous, which, together with Corollary 1.3.2, yields the hypocontinuity of $* : \mathcal{D}'_{E'_*} \times \mathcal{D}'_{L^1_{\check{\omega}}} \to \mathcal{D}'_{E'_*}$. Since $\mathcal{D}'_{E'_*}$ and $\mathcal{D}'_{L^1_{\check{\omega}}}$ are DF-spaces, it automatically follows that the bilinear mapping $* : \mathcal{D}'_{E'_*} \times \mathcal{D}'_{L^1_{\check{\omega}}} \to \mathcal{D}'_{E'_*}$ is continuous (cf. [57, p. 160]). The continuity of $* : \mathcal{D}'_{E'_*} \times \mathcal{O}'_C(\mathbb{R}^n) \to \mathcal{D}'_{E'_*}$ is a direct consequence of the embedding $\mathcal{O}'_C(\mathbb{R}^n) \hookrightarrow \mathcal{D}'_{L^1_\omega}$. Finally, $\mathcal{B}_\omega = (\mathcal{D}'_{L^1_\omega})'$ and $* : \mathcal{D}'_{E'_*} \times \mathcal{D}'_{L^1_{\check{\omega}}} \to \mathcal{D}'_{E'_*}$ and $* : \mathcal{D}'_{L^1_\omega} \times \mathcal{D}_E \to \mathcal{D}_E$ are hypocontinuous, whence the hypocontinuity of $* : \mathcal{D}'_{E'_*} \times \mathcal{D}_{\check{E}} \to \mathcal{B}_\omega$ follows. □

It is worth pointing out that, as a consequence of Proposition 4.5.2, $f\varphi$ is an integrable distribution in Schwartz' sense [94] if $f \in \mathcal{D}'_{E'_*}$ and $\varphi \in \mathcal{D}_{L^1_\omega}$ or if $f \in \mathcal{D}'_{L^1_{\check{\omega}}}$ and $\varphi \in \mathcal{D}_E$. We end this section with four remarks. In Remarks 1.5.2 and 1.5.4, two open questions are posed.

Remark 1.5.1. Let $(X, \{|\,|_j\}_{j\in\mathbb{N}_0})$ and $(Y, \{\|\,\|_j\}_{j\in\mathbb{N}_0})$ be two graded Fréchet spaces, namely, Fréchet spaces with fixed increasing systems of seminorms defining the topology. Recall that a continuous linear mapping $A : (X, \{||_j\}_{j\in\mathbb{N}_0}) \to (Y, \{\|\|_j\}_{j\in\mathbb{N}_0})$ is called tame if there are $\nu, j_0 \in \mathbb{N}$ such that for any $j \geq j_0$ there is $M_j > 0$ such that $\|A(f)\|_j \leq M_j |f|_{\nu j}$ for all $f \in X$.

If l is chosen as in the proof of Proposition 1.2.2, the inequalities (1.18) and (1.19) actually show that

$$(\mathcal{D}_{L^1_\omega}, \{\|\,\|_{1,\omega,N}\}_{N\in\mathbb{N}_0}) \hookrightarrow (\mathcal{D}_E, \{\|\,\|_{E,N}\}_{N\in\mathbb{N}_0}) \hookrightarrow (\dot{\mathcal{B}}_{\check{\omega}}, \{\|\,\|_{\infty,\check{\omega},N}\}_{N\in\mathbb{N}_0})$$

are tame dense embeddings between these graded Fréchet spaces. With the notation used in the proof of Proposition 1.3.1, we obtain in particular the "Sobolev embedding" type results $\mathcal{D}^{2l}_{L^1_\omega} \hookrightarrow E$ and $\mathcal{D}^{2l}_E \hookrightarrow C_{\check{\omega}}$.

Remark 1.5.2. When E is reflexive, the space $\mathcal{D}'_{E'}$ is barrelled, as follows from Proposition 1.3.1 because a reflexive space is barrelled. In the general case: Is the space $\mathcal{D}'_{E'_*}$ barrelled?

Remark 1.5.3. The spaces $\mathcal{D}_{L^p_\omega}$ (resp., $\mathcal{B}_\omega$ and $\dot{\mathcal{B}}_\omega$) are isomorphic to the Schwartz spaces $\mathcal{D}_{L^p}$ (resp., $\mathcal{B}$ and $\dot{\mathcal{B}}$). To construct isomorphisms, first note that the weight $\omega_0 = \omega * \psi$, where $\psi \in \mathcal{D}(\mathbb{R}^n)$ is a non-negative function, satisfies the bounds $M_1\omega(x) \leq \omega_0(x) \leq M_2\omega(x)$, $x \in \mathbb{R}^n$. Furthermore, $\omega_0 \in \mathcal{B}_\omega$. These two facts imply that the multiplier mapping $\varphi \to \varphi\omega_0$ is a Fréchet space isomorphism from $\mathcal{D}_{L^p_\omega}$ onto $\mathcal{D}_{L^p}$, $1 \leq p < \infty$. The same mapping provides isomorphisms $\mathcal{B} \to \mathcal{B}_\omega$ and $\dot{\mathcal{B}} \to \dot{\mathcal{B}}_\omega$.

Remark 1.5.4. Schwartz has pointed out [94, p. 200] that the spaces $\mathcal{D}_{L^p}$ are not Montel. Remark 1.5.3 then yields that $\mathcal{D}_{L^p_\omega}$ are not Montel either, $1 \leq p < \infty$. The spaces $\mathcal{B}_\omega$ and $\dot{\mathcal{B}}_\omega$ can never be Montel because they are not reflexive. When ω is bounded, it is easy to see that $\mathcal{D}_E$ is never Montel. In fact, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is such that $\varphi(x) = 0$ for $|x| \geq 1/2$ and $\theta \in \mathbb{R}^n$ is a unit vector, then $\{T_{-j\theta}\varphi\}_{j=0}^{\infty}$ is a bounded sequence in $\mathcal{D}_E$ without any accumulation point, as follows from the continuous inclusion $\mathcal{D}_E \to \mathcal{B}$. In general: Can $\mathcal{D}_E$ be Montel?

Chapter 2

Boundary values of holomorphic functions in translation-invariant distribution spaces

The study of boundary values of holomorphic functions in distribution and ultradistribution spaces has shown to be quite important for a deeper understanding of properties of generalized functions which are of great relevance to the theory of PDE [39, 90]. There is a vast literature on the subject, we only mention a small part of it. The theory of analytic representation of distributions was initiated by Köthe [55] and Tillmann [102]. We also mention the influential works of Silva [97], Martineau [59, 61], and Vladimirov [106, 107]. The book by Carmichael and Mitrović [13] contains an overview of results concerning boundary values in distribution spaces. For ultradistributions and hyperfunctions, see the articles [24, 49, 64, 75] and the monographs [12, 44, 68].

The representation of the Schwartz spaces $\mathcal{D}'_{L^p}$ as boundary values of holomorphic functions has also attracted much attention. The problem has been treated by Tillmann [103], Łuszczki and Zieleźny [58], and Bengel [3]. More recently [32, 33], Fernández, Galbis, and Gómez-Collado have obtained various ultradistribution analogs of such results. All these works basically deal with holomorphic functions in tube domains whose bases are the orthants of $\mathbb{R}^n$. In a series of papers [8, 9, 10, 11], Carmichael has systematically studied boundary values in $\mathcal{D}'_{L^p}$ of holomorphic functions defined in more general tubes, namely, tube domains whose bases are open convex cones. The present chapter makes a thorough analysis of boundary values in the space $\mathcal{D}'_{E'_*}$. Many of the results we obtain in Section 2.1 are new or improve earlier results even for the special case $\mathcal{D}'_{E'_*} = \mathcal{D}'_{L^p}$.

Section 2.1 is devoted to the study of boundary values of holomorphic functions and analytic representations of $\mathcal{D}'_{E'_*}$. Our first main result (Theorems 2.1.1) characterizes those holomorphic functions in truncated wedges which have boundary values in $\mathcal{D}'_{E'_*}$. It is worth pointing out that this result improves earlier knowledge about boundary values in $\mathcal{D}'_{L^p}$; in fact, part of our conclusion is strong convergence in $\mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$. The strong convergence was only known for $1 < p < \infty$ and for certain tubes [3, 8, 9, 11, 103]. Next, we consider extensions of Carmichael's generalizations of the H^p spaces [9, 10, 11]. We also provide in this section new edge of the wedge theorems. Our ideas are then applied to exhibit an isomorphism between $\mathcal{D}'_{E'_*}$ and a quotient space of holomorphic functions, this quotient space is constructed in the spirit of hyperfunction theory.

In his seminal work [62, 63] Matsuzawa introduced the so-called heat kernel method in the theory of generalized functions. His approach consists in describing distributions and hyperfunctions in terms of solutions to the heat equation fulfilling suitable growth estimates. Several authors have investigated characterizations of many others distribution, ultradistributions, and hyperfunction spaces [16, 18, 46, 101]. Our results from Section 2.4 add new information to Matsuzawa's program by obtaining the description of $\mathcal{D}'_{E'_*}$ via the heat kernel method. In the case of $\mathcal{D}'_{L^p}$, this characterization reads as follows: $f \in \mathcal{D}'_{L^p}$ if and only if there is a solution U to the heat equation on $\mathbb{R}^n \times (0, t_0)$ such that $\sup_{t\in(0,t_0)} t^k \|U(\,\cdot\,, t)\|_{L^p} < \infty$ for some $k \geq 0$ and $f = \lim_{t\to 0^+} U(\,\cdot\,, t)$.

2.1 Boundary values and analytic representations

In this section we study boundary values and analytic representations in the context of the space $\mathcal{D}'_{E'_*}$. Subsection 2.1.1 is dedicated to characterize those holomorphic functions on tube domains, whose bases are open convex cones, that have boundary values in the strong topology of $\mathcal{D}'_{E'_*}$.

As usual, ω stands for the growth function of the translation group of E. The numbers $\tau \geq 0$ and $M' \geq 1$ are fixed constants such that $\omega(x) \leq M'(1 + |x|)^\tau$ (cf. (1.7)).

2.1.1 Boundary values in $\mathcal{D}'_{E'_*}$

Our first goal in this subsection is to characterize those holomorphic functions defined on a truncated wedge that have boundary values in $\mathcal{D}'_{E'_*}$. We begin with a useful lemma.

Lemma 2.1.1. *Let $V \subsetneq \mathbb{R}^n$ be an open subset and let F be holomorphic on the tube T^V. Suppose that $F(\,\cdot\, + iy) \in E'$ for $y \in V$ and*

$$\sup_{y\in V} \frac{(d_V(y))^{\kappa_1}}{(1 + d_V(y))^{\kappa_2}} ||F(\,\cdot\, + iy)||_{E'} = M < \infty \qquad (\kappa_1, \kappa_2 \geq 0). \tag{2.1}$$

Then, for every $\alpha \in \mathbb{N}_0^n$ one has $F^{(\alpha)}(\,\cdot\, + iy) \in E'$ for all $y \in V$ and

$$\sup_{y\in V} \frac{(d_V(y))^{\kappa_1+|\alpha|}}{(1 + d_V(y))^{\kappa_2+\tau}} ||F^{(\alpha)}(\cdot+iy)||_{E'} \leq (2\pi)^{n/2} M M' \frac{(1+\lambda)^{\kappa_2}}{(1-\lambda)^{\kappa_1}} \left(\frac{\sqrt{n}}{\lambda}\right)^{|\alpha|} \alpha!, \quad \lambda \in (0,1). \tag{2.2}$$

Furthermore, the E'-valued mapping $\mathbf{F} : T^V \to E'$ is holomorphic, where

$$\mathbf{F}(x + iy) = T_x(F(\,\cdot\, + iy)). \tag{2.3}$$

Proof. The assumption $V \neq \mathbb{R}^n$ is only used to ensure that $d_V(y) < \infty$ for all $y \in V$. Fix $0 < \lambda < 1$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Let $\zeta = u+iv = (\zeta_1, \zeta_2, \dots, \zeta_n)$ be an arbitrary point in the distinguished boundary of the polydisc $\mathbb{D}^n$, that is, $|\zeta_1| = |\zeta_2| = \cdots = |\zeta_n| = 1$. We write $s = t + i\sigma \in \mathbb{C}$. For arbitrary $y \in V$, define the function $G(s) = G_{y,\zeta}(s) = \int_{\mathbb{R}^n} F(x + iy + s\zeta)\varphi(x)dx = \langle T_{tu-\sigma v} F(\,\cdot\, + i(y + tv + \sigma u)), \varphi\rangle$. It is clear that G is defined and holomorphic in the disc $\{s \in \mathbb{C} : |s| < d_V(y)/\sqrt{n}\}$. Note that

$$\begin{aligned} |G(s)| &\leq \frac{M\omega(tu - \sigma v)(1 + d_V(y + tv + \sigma u))^{\kappa_2}}{(d_V(y + tv + \sigma u))^{\kappa_1}} \|\varphi\|_E \\ &\leq \frac{(1+\lambda)^{\kappa_2} MM'(1 + d_V(y))^{\tau+\kappa_2}}{((1-\lambda)d_V(y))^{\kappa_1}} \|\varphi\|_E \quad \text{for } |s| \leq \frac{\lambda d_V(y)}{\sqrt{n}}. \end{aligned}$$

The Cauchy inequality for derivatives applied to circle $|s| \leq (\lambda/\sqrt{n})d_V(y)$ thus yields

$$|G^{(N)}(0)| \leq \frac{n^{N/2}(1+\lambda)^{\kappa_2} MM'(1 + d_V(y))^{\tau+\kappa_2}}{\lambda^N (1-\lambda)^{\kappa_1} (d_V(y))^{\kappa_1+N}} N! \|\varphi\|_E, \quad N = 0, 1, 2, \ldots.$$

One easily obtains using analyticity of $G_{y,\zeta}(s)$ for $|s| = (\lambda/\sqrt{n})d_V(y)$ fixed and ζ on the distinguished boundary that $\frac{\langle F^{(N)}(\cdot+iy),\varphi\rangle}{N!} = \sum_{|\alpha|=N} \frac{\langle F^{(\alpha)}(\cdot+iy),\varphi\rangle}{\alpha!}\zeta^\alpha$. From $\langle F^{(N)}(\cdot + iy), \varphi\rangle = G^{(N)}(0)$ we get

$$|P_N(\zeta)| \leq \frac{n^{N/2}(1+\lambda)^{\kappa_2} MM'(1 + |d_V(y)|)^{\tau+\kappa_2}}{\lambda^N (1-\lambda)^{\kappa_1} (d_V(y))^{\kappa_1+N}} \|\varphi\|_E, \quad N = 0, 1, 2, \ldots,$$

where $P_N(\zeta) = \sum_{|\alpha|=N} \zeta^\alpha \left\langle F^{(\alpha)}(\,\cdot\, + iy), \varphi \right\rangle / \alpha!$. Integrating $|P_N(\zeta)|^2$ over $(\partial\mathbb{D})^n$, we obtain

$$\int_{(\partial\mathbb{D})^n} |P_N(\zeta)|^2 d\zeta = \int_{(\partial\mathbb{D})^n} \left(\sum_{|\alpha|=N} \frac{\langle F^{(\alpha)}(\cdot + iy), \varphi\rangle}{\alpha!} \zeta^\alpha \right) \left(\sum_{|\beta|=N} \frac{\overline{\langle F^{(\beta)}(\cdot + iy), \varphi\rangle}}{\beta!} \overline{\zeta}^\beta \right) d\zeta$$

$$= \sum_{|\alpha|=1}^{N} \int_{(\partial\mathbb{D})^n} \frac{|\langle F^{(\alpha)}(\cdot + iy), \varphi\rangle|^2}{(\alpha!)^2} |\zeta|^{2\alpha} + \sum_{\alpha\neq\beta} \int_{(\partial\mathbb{D})^n} \frac{\langle F^{(\alpha)}(\cdot + iy), \varphi\rangle \overline{\langle F^{(\beta)}(\cdot + iy), \varphi\rangle}}{\alpha!\beta!} \zeta^\alpha \overline{\zeta}^\beta$$

$$= (2\pi)^n \sum_{|\alpha|=1}^{N} \frac{|\langle F^{(\alpha)}(\cdot + iy), \varphi\rangle|^2}{(\alpha!)^2}.$$

The second sum is 0 because for $\alpha \neq \beta$ there must exist $1 \leq i \leq n$ such that say $\alpha_i > \beta_i$ and so

$$\int_{(\partial\mathbb{D})^n} \zeta^\alpha \overline{\zeta}^\beta = \int \int \ldots \left(\int_{|\zeta_i|=1} |\zeta_i|^{2\beta_i} \overline{\zeta}_i^{\alpha_i - \beta_i} d\zeta_i \right) d\zeta_1 d\zeta_2 \ldots d\zeta_{i-1} d\zeta_{i+1} d\zeta_n = 0.$$

Finally,

$$\left|\left\langle F^{(\alpha)}(\,\cdot\, + iy), \varphi\right\rangle\right| \leq \frac{(1+\lambda)^{\kappa_2} MM' n^{|\alpha|/2} \alpha! (2\pi)^{n/2} (1 + |d_V(y)|)^{\tau+\kappa_2}}{\lambda^{|\alpha|} (1-\lambda)^{\kappa_1} (d_V(y))^{\kappa_1+|\alpha|}} \|\varphi\|_E,$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $y \in V$, and $\alpha \in \mathbb{N}_0^n$. The very last inequality is equivalent to (2.2). To show (2.3), it is enough to fix $z \in V$ and $\zeta \in (\partial\mathbb{D})^n$ and to verify that $\mathbf{F}(z+s\zeta)$ is holomorphic in $|s| < d_V(y)/\sqrt{n}$. Indeed, by the previous argument, $F(\,\cdot\, + iy + s\zeta) = \sum_{k=0}^{\infty} s^k g_k$, with $g_k = \sum_{|\alpha|=k} \zeta^\alpha F^{(\alpha)}(\,\cdot\, + iy)/\alpha!$, is a convergent power series in E' for $|s| < d_V(y)/\sqrt{n}$. Employing the continuity of T_x, we obtain $\mathbf{F}(z+s\zeta) = \sum_{k=0}^{\infty} s^k T_x g_k$ meaning analyticity in one direction.
Now chose $\zeta^1, \zeta^2, ..., \zeta^n$ on $(\partial D)^n$ to be basis of $\mathbb{C}^n$. We define the mapping $\widetilde{F}(\omega) = \mathbf{F}(H(\omega))$, $\mathbb{C}^n \to \mathbb{C}^n$, where $H(\omega) = z + \omega_1\zeta^1 + ...\omega_n\zeta^n$ is biholomorphic mapping from $\mathbb{C}^n \to \mathbb{C}^n$. Using the analyticity of $\mathbf{F}$ in the directions for every one of $\zeta^1, \zeta^2, ..., \zeta^n$ and Hartgots' Theorem we obtain analyticity of $\widetilde{F}$. Hence $\widetilde{F}(H^{-1}(z + \omega_1\zeta^1 + ...\omega_n\zeta^n)) = \mathbf{F}(\omega)$ is holomorphic. □

Lemma 2.1.1 has the ensuing consequence.

Corollary 2.1.1. *Let $V \subseteq \mathbb{R}^n$ and let F be holomorphic in T^V such that $F(\cdot + iy) \in E'$ for all $y \in V$ and $\sup_{y\in K} \|F(\,\cdot\, + iy)\|_{E'} < \infty$ for every compact subset $K \subset V$. Then $\lim_{y\to y_0} \|F(\,\cdot\, + iy) - F(\,\cdot\, + iy_0)\|_{E'} = 0$ for each $y_0 \in V$.*

Proof. The statement is local, so we may assume $V \neq \mathbb{R}^n$. The mapping (2.3) is continuous at $z_0 = iy_0$ and $F(\,\cdot\, + iy) = \mathbf{F}(iy)$. □

In the rest of the subsection we mainly focus our attention on tubes whose bases are cones.

Theorem 2.1.1. *Let C be an open convex cone and let $r > 0$. Suppose that F is holomorphic on the tube $T^{C(r)}$ and satisfies*

$$F(\,\cdot\, + iy) \in \mathcal{D}'_{E'_*}, \quad \text{for every } y \in C(r), \tag{2.4}$$

and the sets $\{F(\cdot + iy) : r' < |y| < r,\, y \in C\}$ are bounded in $\mathcal{D}'_{E'_*}$ for each $r' > 0$. Then, the following three statements are equivalent:

(i) F satisfies

$$F(\,\cdot\, + iy) \in E', \quad y \in C(r), \tag{2.5}$$

and the bound

$$\|F(\,\cdot\, + iy)\|_{E'} \leq \frac{M}{(d_{C(r)}(y))^\kappa}, \quad y \in C(r). \tag{2.6}$$

(ii) F has boundary values in $\mathcal{D}'_{E'_*}$, namely, there is $f \in \mathcal{D}'_{E'_*}$ such that

$$f = \lim_{\substack{y\to 0\\ y\in C}} F(\,\cdot\, + iy) \quad \text{strongly in } \mathcal{D}'_{E'_*}. \tag{2.7}$$

(iii) The set $\{F(\,\cdot\, + iy) : y \in C(r)\}$ is bounded in $\mathcal{D}'_{E'_*}$.

In addition, if any of these equivalent conditions is satisfied, then $F(\,\cdot\, + y) \in E'_*$ for every $y \in C(r)$.

Proof. The implication $(ii) \Rightarrow (iii)$ is obvious.

$(i) \Rightarrow (ii)$. Assume (2.5) and (2.6). If $C = \mathbb{R}^n$, the result follows from Corollary 2.1.1. Suppose then that $C \neq \mathbb{R}^n$ (i.e., $0 \notin C$). Applying the parametrix method used in the proof of the implication $(iv) \Rightarrow (v)$ from Theorem 1.3.1, we can write

$$F(z) = \sum_{|\alpha| \leq N} \partial_z^\alpha F_\alpha(z), \quad z \in T^{C(r)}, \tag{2.8}$$

where each F_α has the form

$$F_\alpha(z) = (F(\,\cdot\, + iy) * \check{\varrho}_\alpha)(x) = \int_{\operatorname{supp} \varrho_\alpha} F_\alpha(z + \xi) \varrho_\alpha(\xi) d\xi \quad (z = x + iy) \tag{2.9}$$

and each $\varrho_\alpha \in E$ is a continuous function of compact support. Thus, each F_α is also holomorphic on the tube $T^{C(r)}$, satisfies $F_\alpha(\,\cdot\, + iy) \in E'_*$ for every $y \in C(r)$, the E'-norm estimate

$$\|F_\alpha(\,\cdot\, + iy)\|_{E'} \leq \frac{M \|\varrho_\alpha\|_{1,\omega}}{(d_{C(r)}(y))^\kappa}, \quad y \in C(r), \tag{2.10}$$

and the pointwise estimate

$$|F_\alpha(x + iy)| \leq \frac{M \|\varrho_\alpha\|_E \omega(x)}{(d_{C(r)}(y))^\kappa}, \quad x + iy \in T^{C(r)}. \tag{2.11}$$

Making use of Corollary 2.1.1, the mappings $y \in C(r) \mapsto F_\alpha(\,\cdot\, + iy) \in E'_*$ are continuous. The pointwise estimate (2.11) implies that each F_α has boundary values in $\mathcal{S}'(\mathbb{R}^n)$ [13, 106]. Set

$$f_\alpha = \lim_{\substack{y \to 0 \\ y \in C}} F_\alpha(\,\cdot\, + iy) \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \quad |\alpha| \leq N. \tag{2.12}$$

In view of (2.8), it suffices to show that each $f_\alpha \in \mathcal{D}'_{E'_*}$ and that the limit (2.12) actually holds in $\mathcal{D}'_{E'_*}$. We may assume that $\kappa \in \mathbb{N}$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and write $\Psi(x, y) = \sum_{|\beta| \leq \kappa} \psi^{(\beta)}(x)(iy)^\beta/\beta!$. Pick $\theta \in C(r/4)$. Since $-\theta \notin C$, we can find M_1 such that $\lambda \leq M_1 d_C(y + \lambda\theta)$ for every $y \in C$ and $\lambda > 0$. In particular, $\lambda \leq M_1 d_{C(r)}(y + \lambda\theta)$ for $\lambda \in (0, 1)$ and $y \in C(r/4)$. As in [39, p. 67], we can write

$$f_\alpha * \psi = \Psi(\,\cdot\,, \theta) * F_\alpha(\cdot\, + i\theta) + \sum_{|\beta| = \kappa + 1} \frac{(i\theta)^\beta (\kappa + 1)}{\beta!} \int_0^1 \lambda^\kappa (F_\alpha(\,\cdot\, + i\lambda\theta) * \psi^{(\beta)})\, d\lambda.$$

and, for $y \in C(r/4)$,

$$F_\alpha(\cdot + iy) * \psi = \Psi(\cdot, \theta) * F_\alpha(\cdot + i\theta + iy) + \sum_{|\beta| = \kappa + 1} \frac{(i\theta)^\beta (\kappa + 1)}{\beta!} \int_0^1 \lambda^\kappa (F_\alpha(\cdot + i\lambda\theta + iy) * \psi^{(\beta)}) d\lambda,$$

where the integrals are interpreted as E'_*-valued integrals in the Bochner sense. By Theorem 1.1.2, the net $\Psi(\,\cdot\,, \theta) * F_\alpha(\cdot\, + i\theta + iy) \to \Psi(\,\cdot\,, \theta) * F_\alpha(\cdot\, + i\theta)$ in E'_*. Furthermore, using the estimate (2.10), we majorize

$$\lambda^\kappa \|F_\alpha(\,\cdot\, + i\lambda\theta + iy) * \psi^{(\beta)}\|_{E'} \leq (M_1)^\kappa M \|\psi^{(\beta)}\|_{E'} \|\varrho_\alpha\|_{1,\omega}$$

and the dominated convergence theorem for Bochner integrals thus yields

$$f_\alpha * \psi = \lim_{\substack{y\to 0\\ y\in C}} (F_\alpha(\cdot\ + iy) * \psi) \quad \text{in } E'_*.$$

Since this holds for every $\psi \in \mathcal{S}(\mathbb{R}^n)$, Corollary 1.3.3 implies

$$f_\alpha = \lim_{\substack{y\to 0\\ y\in C}} F_\alpha(\cdot\ + iy) \quad \text{strongly in } \mathcal{D}'_{E'_*}$$

and (2.7) follows at once.

$(iii) \Rightarrow (i)$. Using the parametrix method once again (see (iii) from Corollary 1.3.2), we can write F as in (2.8) where each F_α is holomorphic in T^{C_r}, $F_\alpha(\cdot + iy) \in E'_*$ and $\sup_{y\in C(r)} \|F_\alpha(\cdot + iy)\|_{E'} < \infty$. The assertion (i) is a consequence of Lemma 2.1.1. In addition, we get that the holomorphic function (2.3) actually takes values in E'_*. Thus $\mathbf{F}^{(\alpha)}_\alpha(z) \in E'_*$ for all $z \in T^{C(r)}$, whence $\mathbf{F} : T^{C(r)} \to E'_*$. □

Corollary 2.1.2. *Let $V \subseteq \mathbb{R}^n$ be an open set and let F be holomorphic in T^V. If $F(\ \cdot\ + iy) \in \mathcal{D}'_{E'_*}$ for all $y \in V$ and $\{F(\ \cdot\ + iy) :\ y \in K\}$ is bounded in $\mathcal{D}'_{E'_*}$ for every compact subset $K \subset V$, then actually $F(\ \cdot\ + iy) \in E'_*$ for all $y \in V$, $\sup_{y\in K} \|F(\ \cdot\ + iy)\|_{E'} < \infty$ for every compact $K \subset V$, and the E'_*-valued function (2.3) is holomorphic in T^V. If in addition $V \neq \mathbb{R}^n$ and the set $\{F(\cdot + iy) : y \in V\}$ is bounded in $\mathcal{D}'_{E'_*}$, then there is $\kappa \geq 0$ such that $\sup_{y\in V}(d_V(y))^\kappa(1+d_V(y))^{-\tau}\|F(\cdot + iy)\|_{E'} < \infty$.*

Proof. The first part of the corollary follows from the second one. Exactly the same argument from the proof of the implication $(iii) \Rightarrow (i)$ of Theorem 2.1.1 shows the second assertion. □

Using Theorem 2.1.1, we can derive the following result.

Corollary 2.1.3. *Let $X \subset \mathcal{S}'(\mathbb{R}^n)$ be a Banach space. Assume that the inclusion mapping $X \to \mathcal{S}'(\mathbb{R}^n)$ is continuous. Let C be an open convex cone and $r > 0$. If F is holomorphic on the tube $T^{C(r)}$ and satisfies*

$$F(\ \cdot\ + iy) \in X \quad \textit{and} \quad \|F(\ \cdot\ + iy)\|_X \leq \frac{M}{(d_{C(r)}(y))^\kappa}, \qquad y \in C(r),$$

then $\lim_{\substack{y\to 0\\ y\in C}} F(\ \cdot\ + iy)$ exists in $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Let $\mathcal{S}_j(\mathbb{R}^n)$ be the completion of $\mathcal{S}(\mathbb{R}^n)$ in the norm q_j (cf. Section 0). Notice that each $\mathcal{S}_j(\mathbb{R}^n)$ is a tempered translation-invariant Banach spaces of distributions. The embeddings $\mathcal{S}_{j+1}(\mathbb{R}^n) \hookrightarrow \mathcal{S}_j(\mathbb{R}^n)$ are compact, $\mathcal{S}(\mathbb{R}^n) = \operatorname{proj\,lim}_{j\in\mathbb{N}} \mathcal{S}_j(\mathbb{R}^n)$, and hence $\mathcal{S}'(\mathbb{R}^n) = \operatorname{ind\,lim}_{j\in\mathbb{N}} \mathcal{S}'_j(\mathbb{R}^n)$ is a regular inductive limit of Banach spaces. Thus, there are $M_1 > 0$ and $j_0 \in \mathbb{N}$ such that $\|f\|_{\mathcal{S}'_{j_0}(\mathbb{R}^n)} \leq M_1\|f\|_X$, for all $f \in X$. The assertion then follows by applying Theorem 2.1.1 with $E' = \mathcal{S}'_{j_0}(\mathbb{R}^n)$. □

Observe that Corollary 2.1.3 provides sufficient conditions for the existence of boundary values in $\mathcal{S}'(\mathbb{R}^n)$ in terms of rather general norms; however, in contrast with Theorem 2.1.1, very little can be said about the boundary distribution $f = \lim_{y\in C\to 0} F(\,\cdot\,+iy)$ unless the Banach space X possesses a richer structure. It should also be noticed that, as well-known, the holomorphic function F is uniquely determined by its distributional boundary values f.

We now turn our attention to holomorphic functions satisfying global estimates over a tube having an open acute convex cone as base. We need to introduce some notation in order to move further. Let $C \subset \mathbb{R}^n$ be an acute open convex cone. Set $\mathcal{S}'(C^* + \overline{B}(a)) = \{g \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp} g \subseteq C^* + \overline{B}(a)\}$. The *Laplace transform* of $g \in \mathcal{S}'(C^* + \overline{B}(a))$ is defined [107] as the holomorphic function

$$\mathcal{L}\{g; z\} = \left\langle g(\xi), e^{iz\cdot\xi} \right\rangle, \quad z \in T^C.$$

The above distributional evaluation is well-defined because $\mathcal{S}'(C^* + \overline{B}(a))$ is canonically isomorphic to the dual of the function space $\mathcal{S}(C^* + \overline{B}(a))$ (cf. [106, 108]).

We are interested in the class of holomorphic functions $F : T^C \to \mathbb{C}$ that satisfy the following two conditions:

$$F(\,\cdot\,+iy) \in E', \quad \text{for all } y \in C, \tag{2.13}$$

and the estimate (for some constants M, m, and k)

$$\|F(\,\cdot\,+iy)\|_{E'} \leq M(1+|y|)^m e^{a|y|} \left(1 + \frac{1}{d_C(y)}\right)^k, \quad y \in C. \tag{2.14}$$

Because of Corollary 2.1.2, the membership relation (2.13) is equivalent to $F(\,\cdot\,+iy) \in E'_*$.

We now show that these holomorphic functions are in one-to-one correspondence with those elements of $\mathcal{D}'_{E'_*}$ having Fourier transforms with supports in the set $C^* + \overline{B}(a)$. We work with the constants in the Fourier transform as

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The next theorem extends various results by Carmichael [9, 11] and Vladimirov [107] (obtained by them in the particular cases when $E' = L^p$ or when E' is an L^2 based Sobolev space).

Theorem 2.1.2. *Let $C \subset \mathbb{R}^n$ be an acute open convex cone and let $a \geq 0$. If $f \in \mathcal{D}'_{E'_*}$ is such that $\hat{f} \in \mathcal{S}'(C^* + \overline{B}(a))$, then the holomorphic function*

$$F(z) = (2\pi)^{-n}\mathcal{L}\{\hat{f}; z\}, \quad z \in T^C, \tag{2.15}$$

satisfies (2.13), (2.14) and (2.7).

Conversely, if F is a holomorphic function on T^C that satisfies the condition (2.13) and for every subcone $C' \Subset C$ and $\varepsilon > 0$ there are $M = M(C', \varepsilon), \kappa = \kappa(C', \varepsilon) > 0$ such that

$$\|F(\,\cdot\,+iy)\|_{E'} \leq M\frac{e^{(a+\varepsilon)|y|}}{|y|^\kappa}, \quad y \in C', \tag{2.16}$$

then there is $f \in \mathcal{D}'_{E'_*}$ with $\operatorname{supp} \hat{f} \subseteq C^* + \overline{B}(a)$ such that (2.15) holds.

Proof. Assume that $f \in \mathcal{D}'_{E'_*}$ is such that $\operatorname{supp} \hat{f} \subseteq C^* + \overline{B}(a)$. Set $\mathbf{f} = \iota(f) \in \mathcal{S}'(\mathbb{R}^n, E'_*)$ (cf. (1.26) and comments thereunder). Then, $\hat{\mathbf{f}} \in \mathcal{S}'(C^* + \overline{B}(a), E'_*) = \left\{\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, E'_*) : \operatorname{supp} \mathbf{g} \subseteq C^* + \overline{B}(a)\right\}$. The same procedure used to identify $\mathcal{S}'(C^*_a)$ with the dual of $\mathcal{S}(C^* + \overline{B}(a))$ [108] shows that $\mathcal{S}'(C^* + \overline{B}(a), E'_*)$ is canonically isomorphic to $L_b(\mathcal{S}(C^* + \overline{B}(a)), E'_*)$. So we identify the latter two spaces. This allows us to define the Laplace transform of the E'_*-valued distribution $(2\pi)^{-n}\hat{\mathbf{f}} \in \mathcal{S}'(C^* + \overline{B}(a), E'_*)$ as

$$\mathbf{F}(z) := (2\pi)^{-n}\mathcal{L}\{\hat{\mathbf{f}}; z\} = (2\pi)^{-n}\langle \hat{\mathbf{f}}, e^{iz\cdot\xi}\rangle \in E'_*, \text{ for every } z \in T^C.$$

Clearly, $\mathbf{F}$ is holomorphic in $z \in T^C$ with values in E'_* and $\mathbf{F}(z) \to \mathbf{f}$ as $z \to 0, z \in C0$ in $\mathcal{S}'(\mathbb{R}^n, E'_*)$. For the later statement we choose $\theta \in prC$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\varphi}(\xi) = e^{\theta\cdot\xi}$ for $\xi \in C^*$. Then $\mathbf{F}(x + it\theta) = 1/(2\pi)^n \mathcal{L}(\hat{f}, x + it\theta) = F_\varphi \mathbf{f}(x,t) \to \mathbf{f}(x)$ in $\mathcal{S}'(\mathbb{R}^n, E'_*)$. The second inequality in the previous relation follows from $\mathbf{F}(x + it\theta) = \mathcal{F}^{-1}_\xi\left(\hat{\mathbf{f}}\hat{\varphi}(t\xi)\right)(x) = \mathbf{f} * \mathcal{F}^{-1}_\xi\left(\hat{\varphi}(t\xi)\right)(x)$ and $\mathcal{F}^{-1}_\xi\left(\hat{\varphi}(t\xi)\right)(x) = 1/t^n\varphi\left(x/t\right)$.

It is easy to see that $\mathbf{F}(x + iy) = T_x F(\,\cdot\, + iy) \in E'_*$ and we obtain at once (2.13) by setting $x = 0$. Furthermore, $\iota(F(\,\cdot\, + iy)) = \mathbf{F}(\,\cdot\, + iy) \to \mathbf{f} = \iota(f)$ in E'_*; hence, Corollary 1.3.3 yields the limit relation (2.7). Next, one readily sees that $\mathbf{F}(z)$ satisfies the estimate

$$\|\mathbf{F}(z)\|_{E'} \leq M(1+|z|)^m e^{a|\Im m\, z|}\left(1 + \frac{1}{d_C\left(\Im m\, z\right)}\right)^k, \quad z \in T^C, \tag{2.17}$$

for some constants $m, k, M > 0$. The bound (2.14) follows by setting $z = iy$ in (2.17). The proof of (2.17) is exactly the same as in the scalar-valued case. We give it for the sake of completeness. Since $\hat{\mathbf{f}} : \mathcal{S}(C^* + \overline{B}(a)) \to E'_*$ is continuous, there are constants $k \in \mathbb{N}$ and $M_1 > 0$ such that

$$(2\pi)^{-n}\|\langle \hat{\mathbf{f}}, \phi\rangle\|_{E'} \leq M_1 \sup_{\substack{0\leq|\alpha|\leq k\\ \xi\in C^*+\overline{B}(a)}} (1+|\xi|)^k \left|\phi^{(\alpha)}(\xi)\right|, \quad \forall \phi \in \mathcal{S}(C^* + \overline{B}(a)).$$

Setting $\phi(\xi) = e^{iz\cdot\xi}$, $z = x + iy \in T^C$, in the above inequality, we obtain

$$\begin{aligned}\|\mathbf{F}(z)\|_{E'} &\leq M_1(1+|z|)^k \sup_{\substack{\xi_1\in C^*\\ |\xi_2|\leq a}} (1 + |\xi_1+\xi_2|)^k e^{-y\cdot\xi_1}e^{-y\cdot\xi_2}\\ &\leq (a+1)^k M_1(1+|z|)^k e^{a|y|}\sup_{\xi\in C^*}(1+|\xi|)^k e^{-|\xi| d_C(y)}\\ &\leq M e^{a|y|}(1+|z|)^k\left(1+\frac{1}{d_C\left(y\right)}\right)^k,\end{aligned}$$

which gives (2.17) with $M = M_1 \sup_{\xi\in C^*}(1+a)^k(1+|\xi|)^k e^{-|\xi|}$ and $m = k$.

Conversely, assume (2.13) and (2.16). As in the proof of Theorem 2.1.1, we express F as in (2.8), where each F_α is holomorphic in T^C and satisfies: $F_\alpha(\cdot + iy) \in E'_*$ for $y \in C$ and the estimates

$$\|F_\alpha(\,\cdot\, + iy)\|_{E'} \leq M_\alpha \frac{e^{(a+\varepsilon)|y|}}{|y|^\kappa} \text{ and } |F_\alpha(x+iy)| \leq M_\alpha \frac{\omega(x)e^{(a+\varepsilon)|y|}}{|y|^\kappa}, \quad x + iy \in T^{C'},$$

where the constants M_α and κ are only dependent on the subcone $C' \Subset C$ and ε. The pointwise estimate and Vladimirov's theorem [107, p. 167] imply that there are $f_\alpha \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\hat{f}_\alpha \subseteq C^* + \overline{B}(a)$ such that $F_\alpha(z) = (2\pi)^{-n}\mathcal{L}\{\hat{f}_\alpha; z\}$. Theorem 2.1.1 gives $f_\alpha \in \mathcal{D}'_{E'_*}$. Hence, (2.15) holds with $f = \sum_{|\alpha|\leq N} f_\alpha^{(\alpha)}$. This completes the proof. □

Theorem 2.1.2 leads to the following general criterion for concluding that a holomorphic function is the Laplace transform of a tempered distribution. The proof goes in the same lines as that of Corollary 2.1.3 and we therefore omit it.

Corollary 2.1.4. *Let $X \subset \mathcal{S}'(\mathbb{R}^n)$ be a Banach space for which the inclusion mapping $X \to \mathcal{S}'(\mathbb{R}^n)$ is continuous and let C be an acute open convex cone. If F is holomorphic on the tube T^C and satisfies*

$$F(\,\cdot\,+iy) \in X \quad \text{and} \quad \|F(\,\cdot\,+iy)\|_X \leq M(C',\varepsilon)\frac{e^{(a+\varepsilon)y}}{|y|^{\kappa(C',\varepsilon)}}, \quad y \in C', \tag{2.18}$$

for any subcone $C' \Subset C$ and $\varepsilon > 0$, then there is $g \in \mathcal{S}'(C^* + \overline{B}(a))$ such that $F(z) = \mathcal{L}\{g; z\}$.

We also obtain the following corollary, a result of Paley-Wiener type.

Corollary 2.1.5. *A necessary and sufficient condition for $f \in \mathcal{D}'_{E'_*}$ to have $\operatorname{supp}\hat{f} \subset \overline{B}(a)$ is that f is the restriction to $\mathbb{R}^n$ of an entire function F that satisfies $F(\,\cdot\,+iy) \in E'$ for all $y \in \mathbb{R}^n$ and the estimate $\sup_{y\in\mathbb{R}^n}(1+|y|)^{-m}e^{-a|y|}\|F(\,\cdot\,+iy)\|_{E'} < \infty$ for some $m \geq 0$ (or equivalently, $\sup_{y\in\mathbb{R}^n} e^{-(a+\varepsilon)|y|}\|F(\,\cdot\,+iy)\|_{E'} < \infty$ for each $\varepsilon > 0$).*

Proof. If $\sup_{y\in\mathbb{R}^n}(1+|y|)^{-m}e^{-a|y|}\|F(\,\cdot\,+iy)\|_{E'} < \infty$ for some $m \geq 0$, then $\operatorname{supp}\hat{f} \subset \overline{B}(a)+C^*$ for every acute open convex cone and $\bigcap_C(\overline{B}(a)+C^*) = \overline{B}(a)$. The other direction can be established as in the proof of Theorem 2.1.2. □

2.1.2 Analytic representations

The results from Subsection 1.5 and Subsection 2.1.1 enable us to obtain analytic representations of arbitrary elements of $\mathcal{D}'_{E'_*}$.

Let $C_1, C_2, \ldots, C_m$ be acute open convex cones of $\mathbb{R}^n$. We assume that $\mathbb{R}^n = \bigcup_{j=1}^m C_j^*$. For example, the C_j might be the 2^n pairwise disjoint open orthants of $\mathbb{R}^n$.

Lemma 2.1.2. *Given $a > 0$, there are convolutors $\chi_1, \chi_2, \ldots, \chi_m \in \mathcal{O}'_C(\mathbb{R}^n)$ such that $\delta = \sum_{j=1}^m \chi_j$ and $\operatorname{supp}\chi_j \subset C_j^* + \overline{B}(a)$.*

Proof. As in [107, p. 7], there are $\rho_1, \ldots, \rho_m \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\rho_j \subset C_j^* + \overline{B}(a)$, $0 \leq \rho_j \leq 1$, $\rho_j(x) = 1$ for $x \in C_j^*$, and $\sup_{x\in\mathbb{R}^n}|\rho_j^{(\alpha)}(x)| \leq M_\alpha a^{-|\alpha|}$, $j = 1, 2, \ldots, m$. The distributions χ_ν given in Fourier side as $\hat{\chi}_\nu = \rho_\nu/(\sum_{j=1}^m \rho_j) \in \mathcal{O}_C(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n)$, $\nu = 1, 2, \ldots, m$, satisfy the requirements.

□

We now show that every element of $\mathcal{D}'_{E'_*}$ can be represented as the sum of boundary values of holomorphic functions.

Theorem 2.1.3. *Every* $f \in \mathcal{D}'_{E'_*}$ *admits the boundary value representation*

$$f = \sum_{j=1}^{m} \lim_{\substack{y \to 0 \\ y \in C_j}} F_j(\,\cdot\, + iy) \quad \textit{strongly in } \mathcal{D}'_{E'_*}, \tag{2.19}$$

where each F_j *is holomorphic in the tube* T^{C_j}.

Proof. Set $f_j = \chi_j * f$ so that $f = \sum_{j=1}^{m} f_j$, where $\chi_1, \dots, \chi_m \in \mathcal{O}'_C(\mathbb{R}^n)$ are the distributions from Lemma 2.1.2. By Proposition 1.5.1, each $f_j \in \mathcal{D}'_{E'_*}$. In addition, $\operatorname{supp} \hat{f}_j \subset C_j^* + \overline{B}(a)$. Theorem 2.1.2 gives the representation (2.19) with $F_j(z) = (2\pi)^{-n} \mathcal{L}\{\hat{f}_j; z\}$. □

The analytic functions F_j from Theorem 2.1.3 of course have the properties (2.13) and (2.14) on the corresponding cone C_j.

2.2 Edge of the wedge theorems

Our next aim is to provide $\mathcal{D}'_{E'_*}$-versions of edge of the wedge theorems. Our first results are of Epstein and Bogoliubov type and they are related to the following classes of holomorphic functions on tubes, whose definitions are motivated by Corollary 2.1.2.

Definition 2.2.1. Let $V \subseteq \mathbb{R}^n$. The vector space $\mathcal{O}_{E'}(T^V)$ consists of all holomorphic functions F on the tube $T^V = \mathbb{R}^n + iV$ satisfying $F(\,\cdot\, + iy) \in E'$ for all $y \in V$ and $\sup_{y \in K} \|F(\,\cdot\, + iy)\|_{E'} < \infty$ for every compact $K \subset V$. The space $\mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^V)$ is defined as

$$\mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^V) = \{F \in \mathcal{O}_{E'}(T^V) : \{F(\,\cdot\, + iy) : y \in V\} \text{ is bounded in } \mathcal{D}'_{E'_*}\}.$$

It should be noticed that if $F \in \mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^V)$ and V is truncated cone, then Theorem 2.1.1 guarantees that $\lim_{y \to 0, y \in V} F(\,\cdot\, + iy)$ exists (strongly) in $\mathcal{D}'_{E'_*}$.

We need the following lemma, which is a variant of a result shown by Rudin in [88, Sect. 3].

Lemma 2.2.1. *Let* V_1 *and* V_2 *be open connected bounded subsets of* $\mathbb{R}^n$ *such that* $0 \in \partial V_1 \cap \partial V_2$. *Set* $V = V_1 \cup V_2$. *Then, any function* F *that is holomorphic on the tube* T^V, *continuous on* $T^V \cup \mathbb{R}^n$, *and satisfies* $\sup_{x+iy \in T^V} (1+|x|^2)^{-N/2} |F(x+iy)| < \infty$, *for some* $N \geq 0$, *extends to a function* $\tilde{F}$, *which is holomorphic in the tube* $T^{\mathrm{ch}(V)}$ *and satisfies*

$$\sup_{x+iy \in T^{\mathrm{ch}(V)}} \frac{|\tilde{F}(x+iy)|}{(1+|x|^2)^{nN/2}} \leq M_N \sup_{x+iy \in T^V} \frac{|F(x+iy)|}{(1+|x|^2)^{N/2}}, \tag{2.20}$$

where the constant M_N *does not depend on* F.

Remark 2.2.1. If V_1 and V_2 are truncated cones, then the holomorphic function $\tilde{F}$ continuously extends on $T^{\mathrm{ch}(V)} \cup \mathbb{R}^n$, as follows from Epstein's edge of the wedge theorem (cf. [88, Sect. 11]).

Proof. Applying exactly the same argument as in [100, Sect. 6.2, p. 122], one can show that any function G, holomorphic on T^V, that fulfills the L^2 conditions

$$\sup_{y\in V}\int_{\mathbb{R}^n}|G(x+iy)|^2dx<\infty \quad \text{and} \quad \lim_{\substack{y\to 0\\ y\in V_1}} G(\,\cdot\,+iy)=\lim_{\substack{y\to 0\\ y\in V_2}} G(\,\cdot\,+iy),\quad \text{in } L^2(\mathbb{R}^n),$$

admits a holomorphic extension $\tilde{G}$ to $T^{\mathrm{ch}(V)}$. Find r such that $|x|<r$ for all $x\in V$. Let $\lambda>r+1$ and set $Q_\lambda(z)=\Pi_{j=1}^n(z_j+i\lambda)^{N+2}$. The function $G(z)=F(z)/Q_\lambda(z)$ satisfies the above two L^2 conditions and so $\tilde{F}=Q_\lambda\tilde{G}$ is the desired holomorphic extension of F to $T^{\mathrm{ch}(V)}$. We first show (2.20) when $N=0$, which follows if we prove $\tilde{F}(T^{\mathrm{ch}(V)})\subseteq F(T^V\cup\mathbb{R}^n)$. Indeed, if $\zeta\in\tilde{F}(T^{\mathrm{ch}(V)})\setminus F(T^V\cup\mathbb{R}^n)$, then $J(z)=1/(F(z)-\zeta)$ would be continuous in $T^V\cup\mathbb{R}^n$ and holomorphic on T^V, but this would contradict the fact that J must have a holomorphic extension to the tube $T^{\mathrm{ch}(V)}$. For general N, take again $\lambda>r+1$ and define $F_\lambda(z)=F(z)/\Pi_{j=1}^n(z_j+i\lambda)^N$. Then, if $|F(x+iy)|\leq M(1+|x|^2)^{N/2}$ for all $x+iy\in T^V$, we obtain that $\sup_{x+iy\in\mathrm{ch}(V)}(1+|x|^2)^{-nN/2}|\tilde{F}(x+iy)|\leq(\lambda+r)^{nN}\sup_{x+iy\in T^{\mathrm{ch}(V)}}|\tilde{F}_\lambda(x+iy)|=(\lambda+r)^{nN}\sup_{x+iy\in T^V}|F_\lambda(x+iy)|\leq M(\lambda+r)^{nN}$ (the equality $\sup_{x+iy\in T^{\mathrm{ch}(V)}}|\tilde{F}_\lambda(x+iy)|=\sup_{x+iy\in T^V}|F_\lambda(x+iy)|$ follows from the case $N=0$), which in turn implies the claimed inequality with $M_N=(2r+1)^{nN}$. □

We have the following $\mathcal{D}'_{E'_*}$ edge of the wedge theorem of Epstein type.

Theorem 2.2.1. *Let V_1 and V_2 be open connected bounded subsets of $\mathbb{R}^n$ with $0\in\partial V_1\cap\partial V_2$. Set $V=V_1\cup V_2$. If $F_1\in\mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^{V_1})$ and $F_2\in\mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^{V_1})$ have distributional boundary values on $\mathbb{R}^n$ and*

$$\lim_{\substack{y\to 0\\ y\in V_1}} F_1(\,\cdot\,+iy)=\lim_{\substack{y\to 0\\ y\in V_2}} F_2(\,\cdot\,+iy)\qquad \textit{weakly in } \mathcal{D}'_{E'_*},$$

then, there is $F\in\mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^{\mathrm{ch}(V)})$ such that $F(z)=F_j(z)$ for $z\in T^{V_j}$, $j=1,2,$.

Remark 2.2.2. The existence of the limits $\lim_{y\to 0, y\in V_j}F_j(\,\cdot\,+iy)$ in $\mathcal{D}'_{E'_*}$, $j=1,2$, is part of the assumptions of Theorem 2.2.1; however, if V_1 and V_2 are truncated cones, such limits automatically exist and in particular $F(\cdot+iy)$ converges strongly in $\mathcal{D}'_{E'_*}$ to the common limit as $y\in\mathrm{ch}(V)$ tends to 0.

Proof. Reasoning as in the proof of Theorem 2.1.1 (via a parametrix argument), we may assume that F_j have continuous extensions to $T^{V_j}\cup\mathbb{R}^n$ with $F_1(x)=F_2(x)$ for $x\in\mathbb{R}^n$ and that there is M such that $\|F_j(\,\cdot\,+y)\|_{E'}\leq M$ and $|F_j(x+iy)|\leq\tilde{M}\omega(x)\leq M(1+|x|^2)^{\tau/2}$ for $x+iy\in T^{V_j}$, $j=1,2$. The pointwise estimate and Lemma 2.2.1 imply the existence of F, holomorphic in $T^{\mathrm{ch}(V)}$, such that $F(z)=F_j(z)$ for $z\in T^{V_j}$, $j=1,2$. It is remains to show that $F(\,\cdot\,+iy)\in E'$ for every $y\in\mathrm{ch}(V)$ and $\{F(\,\cdot\,+iy):\ y\in\mathrm{ch}(V)\}$ is bounded in $\mathcal{D}'_{E'_*}$. Let $\varphi\in\mathcal{D}(\mathbb{R}^n)$ with

$\|\varphi\|_E \leq 1$. Set $G(z) := \int_{\mathbb{R}^n} F(t+z)\varphi(t)dt$, $z \in T^{\mathrm{ch}(V)}$. Then the restriction of G to V extends continuously to $T^V \cup \mathbb{R}^n$ and $|G(x+iy)| \leq M(1+|x|^2)^{\tau/2}$ for $x+iy \in T^V$. The inequality (2.20) from Lemma 2.2.1 gives $|G(x+iy)| \leq MM_\tau(1+|x|^2)^{n\tau/2}$ for $x+iy \in T^{\mathrm{ch}(V)}$; in particular $|G(iy)| \leq MM_\tau$ for all $y \in \mathrm{ch}(V)$. Since φ is arbitrary and $\mathcal{D}(\mathbb{R}^n) \hookrightarrow E$, we obtain that $\sup_{y\in\mathrm{ch}(V)} \|F(\,\cdot\,+iy)\|_{E'} \leq MM_\tau$. □

In particular, we have the ensuing corollary on analytic continuation; the case $C_2 = -C_1$ is an edge of the wedge theorem of Bogoliubov type.

Corollary 2.2.1. *Let C_1 and C_2 be open cones such that $\mathrm{int}(C_1^*) \cap \mathrm{int}(C_2^*) = \emptyset$ and let $r_1, r_2 > 0$. Set $V = C_1(r_1) \cup C_2(r_2)$. If $F_j \in \mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^{C_j(r_j)})$, $j = 1, 2$, are such that*

$$\lim_{\substack{y\to 0\\ y\in C_1}} F_1(\,\cdot\,+iy) = \lim_{\substack{y\to 0\\ y\in C_2}} F_2(\,\cdot\,+iy) \quad \text{in } \mathcal{D}'_{E'_*},$$

then F_1 and F_2 can be glued together as a holomorphic function through $\mathbb{R}^n$; more precisely, the domain $T^{\mathrm{ch}(V)}$ of their holomorphic extension $F \in \mathcal{O}^b_{\mathcal{D}'_{E'_*}}(T^{\mathrm{ch}(V)})$ contains a tube $\mathbb{R}^n + i\{y \in \mathbb{R}^n : |y| < r\}$.

Proof. The condition implies that the cone $C = \mathrm{ch}(C_1 \cup C_2)$ contains a line, and therefore the origin as interior point. □

The last result of this section is an edge of the wedge theorem of Martineau type [61, 68], it is related to the classes of holomorphic functions on wedges introduced in the next definition.

Definition 2.2.2. Let C be an acute open convex cone and $a \geq 0$.

(*i*) We define $\mathcal{O}^{a,\exp}_{\mathcal{D}'_{E'_*}}(T^C)$ as the space of all holomorphic functions $F \in \mathcal{O}_{E'}(T^C)$ such that there is $\kappa \geq 0$ such that for every $\varepsilon > 0$

$$\sup_{y\in C} e^{-(a+\varepsilon)|y|}\left(1 + \frac{1}{d_C(y)}\right)^{-\kappa} \|F(\,\cdot\,+iy)\|_{E'} < \infty.$$

(*ii*) The space $\mathcal{O}^{a,\exp}_{E'}(\mathbb{C}^n)$ consists of all $F \in \mathcal{O}_{E'}(\mathbb{C}^n)$ such that for every $\varepsilon > 0$

$$\sup_{y\in\mathbb{R}^n} e^{-(a+\varepsilon)|y|}\|F(\,\cdot\,+iy)\|_{E'} < \infty.$$

We also use the notation $\mathcal{O}^{a,\exp}_{\mathcal{D}'_{E'_*}}(\mathbb{C}^n) := \mathcal{O}^{a,\exp}_{E'}(\mathbb{C}^n)$ for this space.

Observe that a parametrix argument allows us to conclude that $\mathcal{O}^{a,\exp}_{\mathcal{D}'_{E'_*}}(T^C) \subseteq \mathcal{O}^{a,\exp}_{\mathcal{B}'_\omega}(T^C)$. In particular, every element of $\mathcal{O}^{a,\exp}_{E'}(\mathbb{C}^n)$ is actually an entire function of exponential type.

According to Theorem 2.1.2 (cf. Corollary 2.1.5 for the case $C = \mathbb{R}^n$), every element $F \in \mathcal{O}^{\exp}_{\mathcal{D}_{E'_*}}(T^C)$ is completely determined by its boundary value distribution, which we denote by $\mathrm{bv}(F) := \lim_{y\to 0, y\in C} F(\,\cdot\,+iy) \in \mathcal{D}'_{E'_*}$. In the next theorem each C_j is either an acute open convex cones or $C_j = \mathbb{R}^n$. Note that it considerably improves earlier results by Carmichael [10]

Theorem 2.2.2. *Let* $F_j \in \mathcal{O}^{a,\,\exp}_{\mathcal{D}'_{E'_*}}(T^{C_j})$, $j = 1, 2, \ldots, k$, *and let* $\varepsilon > 0$. *Set* $C_{j,\nu} = \operatorname{ch}(C_j \cup C_\nu)$ *and* $\widetilde{C}_j = \bigcap_{\nu \neq j} C_{j,\nu}$. *If* $\sum_{j=1}^{k} \operatorname{bv}(F_j) = 0$, *then for each* j *there are* $G_{j,\nu} \in \mathcal{O}^{a+\varepsilon,\,\exp}_{\mathcal{D}'_{E'_*}}(T^{C_{j,\nu}})$ *such that* $F_j = \sum_{\nu=1}^{k} G_{j,\nu}$. *In particular, each* F_j *has a holomorphic extension that belongs to* $\mathcal{O}^{a+\varepsilon,\,\exp}_{\mathcal{D}'_{E'_*}}(T^{\widetilde{C}_j})$. *The* $G_{j,\nu}$ *may be chosen such that* $G_{\nu,j} = -G_{j,\nu}$.

Proof. If some of the C_ν are $\mathbb{R}^n$, the corresponding terms in the sum can be absorbed into others. We may therefore assume that all $C_1, \ldots, C_k$ are acute open convex cones. We can find g_ν such that $F_\nu(z) = \mathcal{L}\{g_\nu; z\}$, with $\operatorname{supp} g_\nu \subset C^*_\nu + \overline{B}(a)$ and $\hat{g}_\nu \in \mathcal{D}'_{E'_*}$. Find ρ_ν with bounded partial derivatives of any order such that $\operatorname{supp} \rho_\nu \subseteq C^*_\nu + B(a+\varepsilon)$ and $\rho_\nu(x) = 1$ for $x \in \operatorname{supp} g_\nu$. Then, $g_j = -\sum_{\nu \neq j} \rho_j g_\nu$. Setting $G_{j,\nu} \in \mathcal{O}^{a+\varepsilon,\,\exp}_{\mathcal{D}'_{E'_*}}(T^{C_{j,\nu}})$ as the Laplace transform of $-\rho_j g_\nu$, we obtain $F_j = \sum_{\nu \neq j} G_{j,\nu}$. It remains to be shown that the $G_{j,\nu}$ may be chosen such that $G_{\nu,j} = -G_{j,\nu}$. We proceed by induction over the number of summands. The cases $k = 1, 2$ are trivial. Assume that such a choice is possible for k. If $\sum_{j=1}^{k+1} \operatorname{bv}(F_j) = 0$, from what we have shown we can write $F_{k+1} = \sum_{j=1}^{k} G_{k+1,\nu}$ where $G_{k+1,\nu} \in \mathcal{O}^{a+\varepsilon,\,\exp}_{\mathcal{D}'_{E'_*}}(T^{C_{k+1,\nu}})$. Thus, $\sum_{j=1}^{k} \operatorname{bv}(G_{k+1,\nu} + F_\nu) = 0$. By the inductive hypothesis, we that there are $G_{j,\nu} \in \mathcal{O}^{a+\varepsilon,\,\exp}_{\mathcal{D}'_{E'_*}}(T^{C_{j,\nu}})$ such that $G_{j,\nu} = -G_{\nu,j}$, $1 \leq j, \nu \leq k$, and $F_j + G_{k+1,j} = \sum_{\nu=1}^{k} G_{j,\nu}$. The property is then satisfied if we define $G_{j,k+1} := -G_{k+1,j}$ and $G_{k+1,k+1} = 0$. □

2.3 The boundary value isomorphism $\mathcal{D}'_{E'_*} \cong \mathcal{D}b^{\exp}_{E'}(\mathbb{R}^n)$

We will use our results to represent the space $\mathcal{D}'_{E'_*}$ as a quotient space of analytic functions. We introduce suitable spaces of analytic functions. For an acute open convex cone C, we set (cf. Definition 2.2.1)

$$\mathcal{O}^{\exp}_{\mathcal{D}'_{E'_*}}(T^C) = \bigcup_{a \geq 0} \mathcal{O}^{a,\,\exp}_{\mathcal{D}'_{E'_*}}(T^C).$$

Define also

$$\mathcal{O}^{\exp}_{E'}(\mathbb{C}^n) = \mathcal{O}^{\exp}_{\mathcal{D}'_{E'_*}}(\mathbb{C}^n) = \bigcup_{a \geq 0} \mathcal{O}^{a,\,\exp}_{\mathcal{D}'_{E'_*}}(\mathbb{C}^n).$$

We consider $\bigoplus_C \mathcal{O}^{\exp}_{\mathcal{D}_{E'_*}}(T^C)$, where C is either an acute open convex cone or $C = \mathbb{R}^n$ (so that $\mathcal{O}^{\exp}_{E'}(\mathbb{C}^n)$ is a term of the direct sum), and its subspace $\mathcal{N}^{\exp}_{\mathcal{D}_{E'_*}}$ generated by all elements of the form $F_1 + F_2 - F_3$, where $F_j \in \mathcal{O}^{\exp}_{\mathcal{D}'_{E'}}(T^{C_j})$, $j = 1, 2, 3$, are such that $C_3 \subseteq C_1 \cap C_2$, and $F_1(z) + F_2(z) = F_3(z)$ for $z \in C_3$. We remark that some of the three functions may be identically zero. Next, we define the quotient vector space

$$\mathcal{D}b^{\exp}_{E'}(\mathbb{R}^n) = \left(\bigoplus_C \mathcal{O}^{\exp}_{\mathcal{D}_{E'_*}}(T^C) \right) / \mathcal{N}^{\exp}_{\mathcal{D}_{E'_*}}.$$

The equivalence class of $\sum_{j=1}^k F_j \in \bigoplus_C \mathcal{O}^{\exp}_{\mathcal{D}_{E'_*}}(T^C)$ is denoted by $[\sum_{j=1}^k F_j] = \sum_{j=1}^k [F_j]$.

The mappings bv : $\mathcal{O}^{\exp}_{\mathcal{D}_{E'_*}}(T^C) \to \mathcal{D}'_{E'_*}$ clearly induce a well-defined boundary value mapping

$$\mathrm{bv} : \mathcal{D}b^{\exp}_{E'}(\mathbb{R}^n) \to \mathcal{D}'_{E'_*}, \tag{2.21}$$

namely, $\mathrm{bv}(\sum_{j=1}^k [F_j]) = \sum_{j=1}^k \mathrm{bv}(F_j) \in \mathcal{D}'_{E'_*}$. Combining our previous results, we obtain:

Theorem 2.3.1. *The boundary value mapping* $\mathrm{bv} : \mathcal{D}b^{\exp}_{E'}(\mathbb{R}^n) \to \mathcal{D}'_{E'_*}$ *is a bijection.*

Indeed, that (2.21) is surjective follows at once from Theorem 2.1.3, whereas the injectivity is a consequence of Theorem 2.2.2.

2.3.1 The one-dimensional case

Assume that the dimension $n = 1$. The construction from the previous subsection significantly simplifies if we take into account the natural orientation of the real line. Consider first

$$\mathcal{O}^{\exp}_{\mathcal{D}'_{E'_*}}(\mathbb{C} \setminus \mathbb{R}) = \{F \in \mathcal{O}_{E'}(\mathbb{C} \setminus \mathbb{R}) : \ F_{|\mathbb{R} \pm i(0,\infty)} \in \mathcal{O}^{\exp}_{\mathcal{D}'_{E'_*}}(\mathbb{R} \pm i(0,\infty))\}.$$

If we replace the boundary value mapping by a jump across $\mathbb{R}$ mapping, we obtain that

$$\mathcal{D}'_{E'_*} \cong \mathcal{O}^{\exp}_{\mathcal{D}'_{E'_*}}(\mathbb{C} \setminus \mathbb{R}) / \mathcal{O}^{\exp}_{E'}(\mathbb{C}),$$

the isomorphism being realized by the mapping $\mathcal{O}^{\exp}_{\mathcal{D}'_{E'_*}}(\mathbb{C} \setminus \mathbb{R}) / \mathcal{O}^{\exp}_{E'}(\mathbb{C}) \to \mathcal{D}'_{E'_*}$ given by $[F] \mapsto \lim_{y \to 0^+} (F(\,\cdot\, + iy) - F(\,\cdot\, - iy))$.

We may also give another version of the quotient representation. Let Ω be an neighborhood of the real line of the form $\Omega = \mathbb{R} + iI$, where I is an open interval containing 0. Set

$$\mathcal{O}_{\mathcal{D}'_{E'_*}}(\Omega \setminus \mathbb{R}) = \{F \in \mathcal{O}_{E'}(\Omega \setminus \mathbb{R}) : \, (\forall I' \Subset I)(\exists \kappa)(\sup_{y \in I' \setminus \{0\}} |y|^\kappa \|F(\,\cdot\, + iy)\|_{E'} < \infty)\}.$$

Then, in view of Theorem 2.1.1 and the edge of the wedge theorem of Epstein type (Theorem 2.2.1), the jump across $\mathbb{R}$ mapping produces the isomorphism

$$\mathcal{D}'_{E'_*} \cong \mathcal{O}_{\mathcal{D}'_{E'_*}}(\Omega \setminus \mathbb{R}) / \mathcal{O}_{E'}(\Omega).$$

2.4 Heat kernel characterization

We now turn our attention to the characterization of elements $\mathcal{D}'_{E'_*}$ as boundary values of solutions to the heat equation on $\mathbb{R}^n \times (0, t_0)$. Given $f \in \mathcal{D}'(\mathbb{R}^n)$, we consider the Cauchy problem for the heat equation

$$\partial_t U - \Delta U = 0, \quad t \in \mathbb{R}^n \times (0, t_0), \tag{2.22}$$

with initial value

$$\lim_{t\to 0^+} U(\,\cdot\,,t) = f \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{2.23}$$

Observe that under certain bounds over U, such as [15]

$$|U(x,t)| \le C\exp\left(\left(\frac{a}{t}\right)^{\alpha} + a|x|^2\right) \quad (0<\alpha<1,\ a>0),$$

one can ensure uniqueness of the solution U and, in such a case, U is determined via convolution with the heat kernel: $U(x,t) = (4\pi t)^{-n/2}\left\langle f(\xi), e^{-\frac{|\xi-x|^2}{4t}}\right\rangle$.

Theorem 2.4.1. *Let $f\in\mathcal{D}'(\mathbb{R}^n)$. Then, $f\in\mathcal{D}'_{E'_*}$ if and only if there is a solution U to the Cauchy problem (2.22) and (2.23) that satisfies*

$$U(\,\cdot\,,t)\in E' \quad \textit{for all } t\in(0,t_0) \tag{2.24}$$

and there are constants $M>$ and $k\ge 0$ such that

$$\|U(\,\cdot\,,t)\|_{E'} \le \frac{M}{t^k}, \quad t\in(0,t_0). \tag{2.25}$$

In such a case,

$$\lim_{t\to 0^+} U(\,\cdot\,,t) = f \quad \textit{strongly in } \mathcal{D}'_{E'_*}. \tag{2.26}$$

Proof. If $f\in\mathcal{D}'_{E'}$, then $U(x,t)=F_\phi f(x,\sqrt{t})$ with $\phi(\xi)=(4\pi)^{-n/2}e^{-|\xi|^2/4}$ satisfies (2.22)–(2.26), as follows from Theorem 1.3.2. Conversely, assume that (2.22)–(2.25) hold for U. Applying the parametrix method from the proof of Theorem 1.3.1, we conclude that U can be written as

$$U(x,t) = \sum_{|\alpha|\le N} \partial_x^\alpha U_\alpha(x,t), \quad (x,t)\in\mathbb{R}^n\times(0,t_0), \tag{2.27}$$

where each U_α has the form $U_\alpha(x,t) = (U(\,\cdot\,,t)*\check{\varrho}_\alpha)(x)$, with $\varrho_\alpha\in E$ being compactly supported and continuous. Each U_α is also a solution to the heat equation on $\mathbb{R}^n\times(0,t_0)$, and it satisfies $U_\alpha(\,\cdot\,,t)\in E'_*$ for all $t\in(0,t_0)$, the E'-norm estimate

$$\|U_\alpha(\,\cdot\,,t)\|_{E'} \le \frac{M\|\varrho_\alpha\|_{1,\omega}}{t^k}, \quad t\in(0,t_0), \tag{2.28}$$

and the pointwise estimate

$$|U_\alpha(x,t)| \le M\|\varrho_\alpha\|_E \frac{\omega(x)}{t^k} \le M_\alpha\frac{(1+|x|)^\tau}{t^k}, \quad (x,t)\in\mathbb{R}^n\times(0,t_0). \tag{2.29}$$

Using the pointwise estimate (2.29) and applying Matsuzawa's heat kernel characterization of $\mathcal{S}'(\mathbb{R}^n)$ [62], one concludes the existence of $f_\alpha\in\mathcal{S}'(\mathbb{R}^n)$ such that $\lim_{t\to0^+} U_\alpha(\,\cdot\,,t) = f_\alpha$ in $\mathcal{S}'(\mathbb{R}^n)$, for each $|\alpha|\le N$. The uniqueness criterion for solutions to the heat equation [15] yields $U_\alpha(x,t) = F_\phi f_\alpha(x,\sqrt{t})$ with again $\phi(\xi) = (4\pi)^{-n/2}e^{-|\xi|^2/4}$. The E'-norm estimate (2.28) and Theorem 1.3.2 now imply that each $f_\alpha\in\mathcal{D}'_{E'_*}$. Finally, by (2.27), we get $f=\sum_{|\alpha|\le N} f_\alpha^{(\alpha)}\in\mathcal{D}'_{E'_*}$. □

Theorem 2.4.1 is complemented by the ensuing result, whose proof was already given within that of Theorem 2.4.1.

Corollary 2.4.1. *Let* U *be a solution to the heat equation (2.22) that satisfies (2.24) and the estimate (2.25). Then, there is a distribution* $f \in \mathcal{D}'_{E'_*}$ *such that (2.26) holds. Moreover,* U *is uniquely determined by* f*.*

We end this section with the following corollary, the proof is analogous to that of Corollary 2.1.4.

Corollary 2.4.2. *Let* $X \subset \mathcal{S}'(\mathbb{R}^n)$ *be a Banach space for which the inclusion mapping* $X \to \mathcal{S}'(\mathbb{R}^n)$ *is continuous. If* U *is a solution to the heat equation (2.22) that satisfies* $U(\cdot, t) \in X$ *for every* $t \in (0, t_0)$ *and the estimate* $\sup_{t\in(0,t_0)} t^k \|U(\cdot, t)\|_X < \infty$ *for some* $k \geq 0$*, then* $\lim_{t\to 0^+} U(\,\cdot\,, t)$ *exists in* $\mathcal{S}'(\mathbb{R}^n)$*.*

Chapter 3

Convolutors and multipliers in the space of tempered ultradistributions

In [94] and [38] convolution operators and multipliers of the space $\mathcal{S}(\mathbb{R}^n)$ were studied by L. Schwartz and J. Horvath. Later, G. Sampson, Z. Zielezny [89, 116] characterized convolution operators of the spaces $\mathcal{K}_p'$, $p \geq 1$. D. H. Pahk [72] considered convolution operators in $\mathcal{K}_e'$. The topological structure of the spaces of multipliers and convolutors in $\mathcal{K}_M'$ was studied by S. Abdulah [1]. The convolution in ultradistribution spaces were considered in [42] by S. Pilipović, A. Kaminski, D. Kovačević, while convolutors in the spaces of ultradistributions were investigated in [12, 42, 43, 53, 76, 77, 78].

The main interest in this chapter are convolutors and multipliers in the space of tempered ultradistributions of Beurling and Roumieu type and their characterization. To motivate the research on convolutors, consider the following example: Let $P(D) = \sum\limits_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ (with suitable assumptions on coefficients), then the equation $P(D)u = v$ can be rewritten in the form $P(\delta) * u = v$. Hence, considering equations of the type $S * u = v$ one generalizes the concept of ultradifferential operators with constant coefficients. In order to consider such equations, S must be an ultradistribution that has well-defined convolution with elements of $\mathcal{S}^{(M_p)}(\mathbb{R}^n)$, resp. $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$.

3.1 The space of convolutors

Assume that $(M.1)$, $(M.2)$ and $(M.3)$ hold.

Definition 3.1.1. The space of the *convolutors* $\mathcal{O}_C'^*(\mathbb{R}^n)$ of $\mathcal{S}'^*(\mathbb{R}^n)$ is the space of all $S \in \mathcal{S}'^*(\mathbb{R}^n)$ such that the convolution $S*\varphi$ is in $\mathcal{S}^*(\mathbb{R}^n)$, for every $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$, and the mapping

$$\varphi \longrightarrow S * \varphi\,, \quad \mathcal{S}^*(\mathbb{R}^n) \longrightarrow \mathcal{S}^*(\mathbb{R}^n) \quad \text{is continuous.}$$

We shall also refer to the elements of $\mathcal{O}_C'^*(\mathbb{R}^n)$ as ultratempered convolutors. Recall from [77] several results.

Proposition 3.1.1. *[77] If $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$ and $S \in \mathcal{S}'^*(\mathbb{R}^n)$ then,*

$$(S * \varphi)(x) = \langle S(t), \varphi(x-t) \rangle, \quad x \in \mathbb{R}^n,$$

is a smooth function which satisfies the following condition: There is $k > 0$, resp. there is $(k_p) \in \Re$, such that for every operator P of class $*$ and $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$

$$\begin{aligned} P(D)(S * \varphi)(x) &= O(e^{M(k|x|)}), \mid x \mid\to \infty, \quad \textit{resp.} \\ P(D)(S * \varphi)(x) &= O(e^{N_{k_p}(|x|)}), \quad \mid x \mid\to \infty. \end{aligned} \tag{3.1}$$

From the definition, for $S \in \mathcal{O}_C'^*(\mathbb{R}^n)$ the mapping

$$T \to S * T, \quad \mathcal{S}'^*(\mathbb{R}^n) \to \mathcal{S}'^*(\mathbb{R}^n), \quad \text{is continuous.}$$

Proposition 3.1.2. *Let $S \in \mathcal{S}'^*(\mathbb{R}^n)$. The following statements are equivalent .*

(a) S is a convolutor.

(b) For every $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$, $S * \varphi \in \mathcal{S}^*(\mathbb{R}^n)$.

(c) For every $r > 0$, resp. there exist $k > 0$ such that $\{e^{M(r|x|)}S(\cdot - x); \ x \in \mathbb{R}\}$, resp. $\{e^{M(k|x|)}S(\cdot - x); \ x \in \mathbb{R}\}$, is bounded in $\mathcal{D}'^*(\mathbb{R}^n)$.

(d) For every $r > 0$, resp. there exist $k > 0$, there is $l > 0$, resp. there is $(k_p) \in \Re$, and $L^\infty(\mathbb{R}^n)$ functions F_1 and F_2 such that

$$S = P_l(D)F_1 + F_2, \quad \textit{resp.} \quad S = P_{k_p}(D)F_1 + F_2,$$

and

$$\|e^{M(r|x|)}(\mid F_1(x) \mid + \mid F_2(x) \mid)\|_{L^\infty} < \infty$$

resp.

$$\|e^{M(k|x|)}(\mid F_1(x) \mid + \mid F_2(x) \mid)\|_{L^\infty} < \infty.$$

Proof. We will prove only the Roumieu case. The Beurling case is similar.
$(a) \Rightarrow (b)$ It is obvious. $(b) \Rightarrow (c)$. Let $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$.

$$\langle e^{M(k|x|)}T_{-x}S_t, \varphi(t) \rangle = \langle e^{M(k|x|)}S_t, \varphi(t+x) \rangle = e^{M(k|x|)}(S * \check{\varphi})(-x)\,.$$

Hence,

$$\mid e^{M(k|x|))}(S * \check{\varphi})(-x) \mid\leq C s_k(S * \check{\varphi}).$$

$(c) \Rightarrow (d)$. For this part we need the parametrix of Komatsu Lemma 0.2.3. Let Ω be a bounded open set in $\mathbb{R}^n$ which contains zero and $K = \overline{\Omega}$. Let B be a bounded set in $\mathcal{D}_K^{\{M_p\}}$. For $\varphi \in B$

$$\mid \langle e^{M(k|x|)}T_{-x}S_t, \varphi(t) \rangle \mid= e^{M(k|x|)} \mid (S * \check{\varphi})(-x) \mid\leq C\,, \tag{3.2}$$

for all $x \in \mathbb{R}^n$ where $C > 0$ does not depend on $\varphi \in B$. Denote by $L^1_{\exp(-M(k|\cdot|))}$ the space of locally integrable functions f on $\mathbb{R}^n$ such that $f(\cdot)e^{-M(k|\cdot|)} \in L^1(\mathbb{R}^n)$ supplied with the norm

$$\|f\|_{L^1,\exp(-M(k|\cdot|))} = \|f(\cdot)e^{-M(k|\cdot|)}\|_{L^1}\,.$$

Let B_1 be the closed unit ball in the space $L^1_{\exp(-M(k|\cdot|))}$, $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ and $\varphi \in B$. Then,

$$\begin{aligned} |\langle S * \psi, \varphi\rangle| &= |\langle (S * \check{\varphi})(-x), \psi\rangle| \leq \|S * \check{\varphi}(-x) \cdot e^{M(k|x|)}\|_{L^\infty} \|\psi\|_{L^1,\exp(-M(k|\cdot|))} \\ &\leq C\|\psi\|_{L^1,\exp(-M(k|\cdot|))} \leq C. \end{aligned}$$

Which means

$$|\langle S * \psi, \varphi\rangle| \leq C\|\psi\|_{L^1,\exp(-M(k|\cdot|))} \tag{3.3}$$

for all $\varphi \in B$ and $\psi \in \mathcal{D}^{\{M_p\}}$. From (3.3) it follows that

$$\{S * \psi |\ \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}\} \tag{3.4}$$

is bounded set in $\mathcal{D}'^{\{M_p\}}_K$, and because $\mathcal{D}'^{\{M_p\}}_K$ is barrelled, the set (3.4) is equicontinuous. Hence, there exist $(k_p) \in \Re$ and $\varepsilon > 0$ such that

$$|\langle S * \theta, \check{\psi}\rangle| \leq 1\,, \psi \in B_1 \cap \mathcal{D}^{\{M_p\}},\ \theta \in V_{k_p}(\varepsilon),$$

where

$$V_{k_p}(\varepsilon) = \{\chi \in \mathcal{D}^{\{M_p\}}_K |\ \|\chi\|_{K,k_p} \leq \varepsilon\}\,. \tag{3.5}$$

The same inequality holds for the closure $\overline{V_{k_p}(\varepsilon)}$ of $V_{k_p}(\varepsilon)$ in $\mathcal{D}^{\{M_p\}}_{K,k_p}$. If $\theta \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, then for some $L_\theta > 0$, $\|\theta/L_\theta\|_{K,k_p} < \varepsilon$. Hence, $\theta/L_\theta \in \overline{V_{k_p}(\varepsilon)}$ and $|\langle S*\theta, \check{\psi}\rangle| \leq L_\theta$ for $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$. It follows that for $\psi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$

$$|\langle S * \theta, \check{\psi}\rangle| \leq L_\theta\|\psi\|_{L^1,\exp(-M(k|\cdot|))}\,. \tag{3.6}$$

Because $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $L^1_{\exp(-M(k|\cdot|))}$ it follows that for every θ in $\mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, $S*\theta$ is a continuous functional on $L^1_{\exp(-M(k|\cdot|)}$. Thus $S*\theta$ belongs to $L^\infty_{\exp(M(k|\cdot|))} = \{f \in L_{1,loc}(\mathbb{R}^n) \mid \|f(\cdot)e^{M(k|\cdot|)}\|_{L_\infty} < \infty\}$, since the space $L^\infty_{\exp(M(k|\cdot|))}$ is the dual of the space $L^1_{\exp(-M(k|\cdot|)}$. Hence,

$$\|S * \theta(x)\|_{L^\infty,\exp(M(k|\cdot|)} \leq L_\theta\,,$$

where $L_\theta > 0$ is a constant which depends of θ. From Lemma 0.2.3 for the chosen $(k_p) \in \Re$ and Ω there exist $(\tilde{k_p})$ and $u \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$ and $\psi \in \mathcal{D}^{\{M_p\}}_\Omega$ such that

$$S = P_{\tilde{k_p}}(D)(u * S) + (\psi * S).$$

Now it is obvious that $F_1 = u * S$ and $F_2 = \psi * S$ satisfy the conditions in (d).

$(d) \Rightarrow (a)$ Assume that $F_2 = 0$. The general case can be proved analogously. We will prove that $\varphi \longrightarrow F * \varphi$ is a continuous mapping from $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ to $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. Then, (a) will hold because of the continuity of the operator $P_{k_p}(D)$ and the fact that $P_{k_p}(D)(S * \varphi) = P_{k_p}(D)S * \varphi$.

Observe that the continuity of the mapping $\varphi \longrightarrow F * \varphi$ will follow if we prove that for every r which is bigger than some fixed r_0, there exist l such that $\varphi \longrightarrow F * \varphi$ is a continuous mapping from $\mathcal{S}^{M_p,r}_\infty$ to $\mathcal{S}^{M_p,l}_\infty$ (because $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ is a inductive limit of $\mathcal{S}_{\infty}^{M_p,r}$). We will prove the later statement. For the k in the condition (d) we choose r_0, small enough such that for all $r \leq r_0$ the integral

$$\int\limits_{\mathbb{R}^n} e^{-M(k|t|)} e^{M(r|t|)} dt$$

converge. Fix r such that $r \leq r_0$. Note that

$$\frac{r^p \mid x \mid^p}{2^p M_p} \leq \frac{r^p \mid x-t \mid^p}{M_p} + \frac{r^p \mid t \mid^p}{M_p} \leq e^{M(r|x-t|)} + e^{M(r|t|)} \leq 2e^{M(r|x-t|)} e^{M(r|t|)} \quad (3.7)$$

and the last inequality holds since the function $M(\rho)$ is nonnegative. For the associated function there exist $\rho_0 > 0$ such that for $\rho \leq \rho_0$, $M(\rho) = 0$ and for $\rho > \rho_0$, $M(\rho) > 0$ (for the properties of the associated function we refer to [49]). If $\mid x \mid > \dfrac{2\rho_0}{r}$ then from the inequality (3.7) it follows that

$$e^{M(\frac{r}{2}|x|)} \leq 2e^{M(r|x-t|)} e^{M(r|t|)}.$$

If $\mid x \mid \leq \dfrac{2\rho_0}{r}$, there exist $c > 0$ such that $e^{M(\frac{r}{2}|x|)} \leq c$. Hence, it follows that for all $x \in \mathbb{R}^n$, the following inequality holds $e^{M(\frac{r}{2}|x|)} \leq 2(c+1)\, e^{M(r|x-t|)} e^{M(r|t|)}$ and obtain that

$$e^{-M(r|x-t|)} \leq C e^{M(r|t|)} e^{-M(\frac{r}{2}|x|)},$$

where $C = 2(c+1)$. Let $l < r/4$. Then,

$$\frac{l^\alpha \mid F * D^\alpha \varphi(x) \mid e^{M(l|x|)}}{M_\alpha} \leq \frac{l^\alpha}{M_\alpha} \int\limits_{\mathbb{R}^n} \mid F(t) \mid\mid D^\alpha \varphi(x-t) \mid dt\, e^{M(l|x|)}$$

$$= \left(\frac{l}{r}\right)^\alpha \frac{1}{M_\alpha} \int\limits_{\mathbb{R}^n} \mid F(t) \mid \frac{e^{M(k|t|)}}{e^{M(k|t|)}} \mid D^\alpha \varphi(x-t) \mid r^\alpha \frac{e^{M(r|x-t|)}}{e^{M(r|x-t|)}} dt\, e^{M(l|x|)}$$

$$\leq C' \left(\frac{l}{r}\right)^\alpha s_r(\varphi) \int\limits_{\mathbb{R}^n} e^{-M(k|t|)} e^{M(r|t|)} dt\, e^{-M(\frac{r}{2}|x|)}\, e^{M(l|x|)}.$$

Because of the way that l is chosen, it follows that

$$s_l(F * \varphi) = \sup_\alpha \frac{l^\alpha \| \mid F * D^\alpha \varphi(x) \mid e^{M(l|x|)} \|_{L^\infty}}{M_\alpha} \leq C'' s_r(\varphi),$$

where C'' is a constant which does not depend on φ. We have shown the continuity of the mapping $\varphi \longrightarrow F*\varphi$, from $\mathcal{S}_{\infty}^{M_p,r}$ to $\mathcal{S}_{\infty}^{M_p,l}$. Hence by the previous discussion, $\varphi \longrightarrow F * \varphi$ is continuous mapping from $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ to $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. □

It is clear that the ultratempered convolution of $S_1, S_2 \in \mathcal{O}_C'^{*}(\mathbb{R}^n)$ is in $\mathcal{O}_C'^{*}(\mathbb{R}^n)$ (see [42]). As well for any $T \in \mathcal{S}'^{*}$, and $\psi \in \mathcal{S}^{*}$,

$$\langle (S_1 * S_2) * T, \psi \rangle = \langle S_1 * T, \check{S}_2 * \psi \rangle = \langle T * S_2, \check{S}_1 * \psi \rangle = \langle T, (S_1 * S_2) * \psi \rangle. \quad (3.8)$$

Supply $\mathcal{O}_C'^{*}(\mathbb{R}^n)$ with the topology from $\mathcal{L}_b(\mathcal{S}^{*}(\mathbb{R}^n), \mathcal{S}^{*}(\mathbb{R}^n))$ and denote it by$\mathcal{O}_{C,b}'^{*}(\mathbb{R}^n)$. The same topology on this space is induced by $\mathcal{L}_b(\mathcal{S}'^{*}(\mathbb{R}^n), \mathcal{S}'^{*}(\mathbb{R}^n))$.

Proposition 3.1.3. *The strong topology on $\mathcal{L}(\mathcal{S}'^{*}(\mathbb{R}^n), \mathcal{S}'^{*}(\mathbb{R}^n))$ induces the same topology on $\mathcal{O}'^{*}_{C}(\mathbb{R}^n)$.*

Proof. Let U be a neighborhood of zero in $\mathcal{S}'^{*}(\mathbb{R}^n)$. Without loss of generality we can assume that

$$U = U(V'; B') = \{S \in \mathcal{O}'^{*}_{C}(\mathcal{S}'^{*}; \mathcal{S}'^{*}) \,|\, S * T \in V', \, for\, all\ T \in B'\},$$

where B' is a bounded subset in $\mathcal{S}'^{*}(\mathbb{R}^n)$ and V' is a neighborhood of zero in $\mathcal{S}'^{*}(\mathbb{R}^n)$. Assume that

$$V' = V'(B, \varepsilon) = \{T \in \mathcal{S}'^{*}(\mathbb{R}^n) \mid |\langle T, \varphi\rangle| < \varepsilon, \, for\, all\, \varphi \in B\},$$

where B is bounded in $\mathcal{S}^{*}(\mathbb{R}^n)$, and $\varepsilon > 0$. Let

$$V = \{\varphi \in \mathcal{S}^{*}(\mathbb{R}^n) \mid |\langle T, \varphi\rangle| < \varepsilon, \, for\, all\, T \in B'\}.$$

Since $\mathcal{S}^{*}(\mathbb{R}^n)$is barreled is follows that V is a neighborhood of zero in $\mathcal{S}^{*}(\mathbb{R}^n)$. Without loss of generality it can be assumed that $B = \check{B} = \{\check{\varphi} \mid \varphi \in B\}$ and $B' = \check{B}' = \{\check{T} \mid T \in B'\}$. Let

$$W = W(V, B) = \{S \in \mathcal{O}'^{*}_{C}(\mathcal{S}'^{*}; \mathcal{S}'^{*}) \,|\, S * \varphi \in V, \, for\, all\, \varphi \in B\}.$$

We will show that $W(V, B) \subset U(V', B')$. Let $S \in W(V, B)$, $T \in B'$ and $\varphi \in B$. Then

$$|\langle S * T, \varphi\rangle| = |\langle T, \check{S} * \varphi\rangle| < \varepsilon.$$

Hence $S*T \in V'$ for all $T \in B'$. So, the topology induced by $\mathcal{L}_b(\mathcal{S}'^{*}(\mathbb{R}^n), \mathcal{S}'^{*}(\mathbb{R}^n))$ is stronger than the topology induced by $\mathcal{L}_b(\mathcal{S}^{*}(\mathbb{R}^n), \mathcal{S}^{*}(\mathbb{R}^n))$. The other direction is similar and it is omitted. □

Proposition 3.1.4. *$\mathcal{O}'^{*}_{C,b}(\mathbb{R}^n)$ is complete.*

Proof. Let $\{S_\mu\}$ be a Cauchy net in $\mathcal{O}'^{*}_{C,b}(\mathbb{R}^n)$. Then the net $\{\tilde{S}_\mu\}$ is a Cauchy net in $\mathcal{L}_b(\mathcal{S}^{*}(\mathbb{R}^n), \mathcal{S}^{*}(\mathbb{R}^n))$, where $\tilde{S}_\mu : \mathcal{S}^{*}(\mathbb{R}^n) \longrightarrow \mathcal{S}^{*}(\mathbb{R}^n)$ are induced continuous linear operators by S_μ, $\tilde{S}_\mu(\varphi) = S_\mu * \varphi$. Since $\mathcal{S}^{*}(\mathbb{R}^n)$ is complete and bornological ([104, Cor.1 of Thm.32.2]), $\mathcal{L}_b(\mathcal{S}^{*}(\mathbb{R}^n), \mathcal{S}^{*}(\mathbb{R}^n))$ is complete, there exists $R \in \mathcal{L}_b(\mathcal{S}^{*}(\mathbb{R}^n), \mathcal{S}^{*}(\mathbb{R}^n))$, such that $\tilde{S}_\mu \longrightarrow R$. Define $T \in \mathcal{S}'^{*}(\mathbb{R}^n)$, by $\langle T, \varphi\rangle = R(\varphi)(0)$. For $\varphi \in \mathcal{S}^{*}(\mathbb{R}^n)$, $R(\varphi) = T * \varphi$, since for $x \in \mathbb{R}^n$

$$\begin{aligned} R(\varphi)(x) &= \lim_{\mu}(S_\mu * \varphi)(x) = \lim_{\mu}(S_\mu * (T_{-x}\varphi))(0) \\ &= R(T_{-x}\varphi)(0) = \langle T, T_{-x}\varphi\rangle = T * \varphi(x). \end{aligned}$$

Thus for $\varphi \in \mathcal{S}^{*}(\mathbb{R}^n)$, $T * \varphi \in \mathcal{S}^{*}(\mathbb{R}^n)$ and the map $\varphi \longrightarrow T * \varphi$ is continuous. It follows that $T \in \mathcal{O}'^{*}_{C}(\mathbb{R}^n)$, and moreover $S_\mu \longrightarrow T$ in $\mathcal{O}'^{*}_{C}(\mathbb{R}^n)$ since $\tilde{T} = R$. □

Proposition 3.1.5. *A sequence S_j from $\mathcal{O}'^{*}_{C,b}(\mathbb{R}^n)$ converges to zero in $\mathcal{O}'^{*}_{C,b}$ if and only if for every $k > 0$ resp. there exist $k > 0$, there exists $r > 0$, resp. there exists $(k_p) \in \Re$ and sequences of $L^\infty(\mathbb{R}^n)$ functions F_{1n} and F_{2n}, such that*

$$S_j = P_r(D)F_{1j} + F_{2j}, \quad resp.\ S_j = P_{k_p}(D)F_{1j} + F_{2j}, \tag{3.9}$$

$F_{1j}, F_{2j} \in \mathcal{O}'^*_C(\mathbb{R}^n)$,

$$\|e^{M(k|x|)}(| F_{1j} | + | F_{2j} |)\|_{L^\infty} < \infty$$

and

$$F_{1j} \longrightarrow 0, \quad F_{2j} \longrightarrow 0 \qquad \text{in } \mathcal{O}'^*_{C,b}. \tag{3.10}$$

Proof. The proof of the proposition is similar with the proof of the Proposition 3.1.2, but it will be given for the sake of completeness. Let S_j be a sequence in $\mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$ which converges to zero in $\mathcal{O}'^{\{M_p\}}_{C,b}(\mathbb{R}^n)$. Let Ω be a bounded open set in $\mathbb{R}^n$ which contains zero and $K = \overline{\Omega}$ and $\varphi \in \mathcal{D}^{\{M_p\}}_K$ be fixed. Then $S_j * \varphi \longrightarrow 0$ in $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. Because $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ is a (DFS) space, it follows that there exist $k > 0$ such that $S_j * \varphi \in \mathcal{S}^{M_p,k}_\infty$, and is bounded there, i.e.,

$$\sup_\alpha \frac{k^\alpha \|e^{M(k|x|)} D^\alpha (S_j * \varphi)(x)\|_{L^\infty}}{M_\alpha} \leq C_\varphi, \forall j \in \mathbb{N},$$

where C_φ is a constant which depends only on φ. So,

$$\|e^{M(k|x|)}(S_j * \varphi)(x)\|_{L^\infty} \leq C_\varphi\,, \forall j \in \mathbb{N}.$$

Let $\psi \in B_1 \cap \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, then

$$| \langle S_j * \psi, \check{\varphi} \rangle | = | \langle S_j * \varphi, \check{\psi} \rangle | \leq \|S_j * \varphi\|_{L^\infty_{\exp(M(k|\cdot|))}} \leq C_\varphi \tag{3.11}$$

for all $j \in \mathbb{N}$, where B_1 is the closed unit ball in $L^1_{\exp(-M(k|\cdot|))}$.

From (3.11) it follows that

$$\{S_j * \psi |\ \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}, j \in \mathbb{N}\} \tag{3.12}$$

is weakly bounded set in $\mathcal{D}'^{\{M_p\}}_K$, and because $\mathcal{D}^{\{M_p\}}_K$ is barreled, the set (3.12) is equicontinuous [91, Thm.5.2]. There exist $(k_p) \in \Re$ and $\delta > 0$ such that

$$| \langle S_j * \theta, \check{\psi} \rangle | \leq 1\,,\ \theta \in V_{k_p}(\delta)\,, \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)\ ,\ j \in \mathbb{N},$$

where $V_{k_p}(\delta) = \{\chi \in \mathcal{D}^{\{M_p\}}_K |\ \|\chi\|_{K,k_p} \leq \delta\}$. The same inequality holds for the closure $\overline{V_{k_p}(\delta)}$ of $V_{k_p}(\delta)$ in $\mathcal{D}^{\{M_p\}}_{K,k_p}$. If $\theta \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$, then for some $L_\theta > 0$, $\|\theta/L_\theta\|_{K,k_p} < \delta$, hence $\theta/L_\theta \in \overline{V_{k_p}(\delta)}$ and

$$| \langle S_j * \theta, \check{\psi} \rangle | \leq L_\theta\,,\ \psi \in B_1 \cap \mathcal{D}^{\{M_p\}}\,,\ j \in \mathbb{N}.$$

It follows that for $\psi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$

$$| \langle S_j * \theta, \check{\psi} \rangle | \leq L_\theta \|\psi\|_{L^1,\exp(-M(k|\cdot|))}. \tag{3.13}$$

Because $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $L^1_{\exp(-M(k|\cdot|))}$ it follows that for every θ in $\mathcal{D}^{\{M_p\}}_{\Omega,k_p}(\mathbb{R}^n)$, $S_j * \theta$ are continuous functionals on $L^1_{\exp(-M(k|\cdot|))}$ and uniformly bounded. Thus, $S_j * \theta$ belong to $L^\infty_{\exp(M(k|\cdot|))}$. Hence,

$$\|S_j * \theta(x)\|_{L^\infty_{\exp(M(k|\cdot|))}} \leq L_\theta\,, \forall j \in \mathbb{N},$$

where $L_\theta > 0$ is a constant which depends on θ. From the parametrix of Komatsu (Lemma 0.2.3), for the chosen $(k_p) \in \Re$ and Ω, there exist $(\tilde{k_p})$ and $u \in \mathcal{D}^{\{M_p\}}_{\Omega,k_p}$ and $\psi \in \mathcal{D}^{\{M_p\}}_{\Omega}$ such that

$$S_n = P_{\tilde{k_p}}(D)(S_n * u) + (S_n * \psi).$$

Let $F_{1j} = S_j * u$ and $F_{2j} = S_j * \psi$. It's obvious that $u \in \mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$, hence $F_{1j}, F_{2j} \in \mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$. Also $F_{1j} = S_n * u \longrightarrow 0$ and $F_{2j} = S_n * \psi \longrightarrow 0$ in $\mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$.

Conversely, let $F_j \longrightarrow 0$ and $F_{1j} \longrightarrow 0$ in $\mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$, $S_j = P_{k_p}(D)F_j + F_{1j}$, for some $(k_p) \in \Re$. Assume that $F_{1n} = 0$ for all $n \in \mathbb{N}$. The general case is proved similarly. Let $M(B,V)$ is a neighborhood of zero in $\mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$, where B is a bounded set in $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$, and V is an open neighborhood of zero in $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. Since, $P_{k_p}(D) : \mathcal{S}^{\{M_p\}}(\mathbb{R}^n) \longrightarrow \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ is continuous, there exist an open neighborhood V_0 such that $P_{k_p}(D)(V_0) \subset V$. Since $F_j \longrightarrow 0$ in $\mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$, and $M(B,V_0)$ is a neighborhood of zero, there exists j_0, such that for all $j \geq j_0$, $F_j \in M(B,V_0)$. Thus, $F_j * \varphi \in V_0$, for all $\varphi \in B$ and $j \geq j_0$, and it follows that

$$P_{k_p}(D)(F_j * \varphi) \subset P_{k_p}(D)(V_0) \subset V.$$

□

Remark 3.1.1. The inclusion $\mathcal{O}'^*_{C,b}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'^*(\mathbb{R}^n)$ is continuous since for arbitrary open neighborhood V in $\mathcal{S}'^*(\mathbb{R}^n)$ consider $\mathcal{O}'^*_C(\mathbb{R}^n)$:

$$W = \{S \in \mathcal{O}'^*_C(\mathbb{R}^n) \mid S * \delta \in V\}.$$

Then it is obvious that from $S \in W$, it follows that $S \in V$.

From the convergence of F_{1j}, F_{2j} to zero in $\mathcal{O}'^*_C(\mathbb{R}^n)$, in the above proposition, it follows convergence in $\mathcal{S}'^*(\mathbb{R}^n)$.

Denote by $\mathcal{ES}'^*(\mathbb{R}^n)$ the space of elements $f \in \mathcal{S}'^*(\mathbb{R}^n)$ such that for every $S \in \mathcal{O}'^*_C(\mathbb{R}^n)$, $S * f \in \mathcal{E}^*(\mathbb{R}^n)$ and the mapping

$$S \to S * f, \quad \mathcal{O}'^*_{C,b}(\mathbb{R}^n) \to \mathcal{E}^*(\mathbb{R}^n) \text{ is continuous.}$$

Proposition 3.1.6. *(i)* $\mathcal{ES}'^*(\mathbb{R}^n) \subset \mathcal{E}^*(\mathbb{R}^n) \cap \mathcal{S}'^*(\mathbb{R}^n)$.

(ii) If $f \in \mathcal{ES}'^*(\mathbb{R}^n)$ *and* $S \in \mathcal{O}'^*_C(\mathbb{R}^n)$ *then* $S * f \in \mathcal{ES}'^*(\mathbb{R}^n)$.

Proof. (i) It is clear from the definition of $\mathcal{ES}'^*(\mathbb{R}^n)$ and that if $f \in \mathcal{ES}'^*(\mathbb{R}^n)$, $f \in \mathcal{S}'^*(\mathbb{R}^n)$ and because δ is in $\mathcal{O}'^{\{M_p\}}_C(\mathbb{R}^n)$, $\delta * f = f$ is an element in $\mathcal{E}^*(\mathbb{R}^n)$.

(ii) From (i) it follows that $S * f \in \mathcal{S}'^*(\mathbb{R}^n)$. Let $T \in \mathcal{O}'^*_C(\mathbb{R}^n)$. So,

$$T * (S * f) = (T * S) * f$$

is in $\mathcal{E}^*(\mathbb{R}^n)$. It is obvious that the mapping $T \longrightarrow T * (S * f)$ is continuous, since the mappings $T \longrightarrow T * S \longrightarrow (T * S) * f = T * (S * f)$ are continuous. Hence, $S * f \in \mathcal{ES}'^*(\mathbb{R}^n)$. □

Note that $\mathcal{S}^*(\mathbb{R}^n)$ is subset of $\mathcal{ES}'^*(\mathbb{R}^n)$.

3.2 The space of multipliers

Assume $(M.1)$, $(M.2)$ and $(M.3)$ hold.

As in [53] and [79], the definition of the space of *multipliers* is given as follows.

Definition 3.2.1. The space $\mathcal{O}_M^*(\mathbb{R}^n)$ is the space of functions $\varphi \in \mathcal{E}^*(\mathbb{R}^n)$ such that $\varphi \in \mathcal{O}_M^*(\mathbb{R}^n)$ if and only if for every $\psi \in \mathcal{S}^*(\mathbb{R}^n)$, $\varphi\psi \in \mathcal{S}^*(\mathbb{R}^n)$ and the mapping $\psi \to \varphi\psi$, $\mathcal{S}^*(\mathbb{R}^n) \longrightarrow \mathcal{S}^*(\mathbb{R}^n)$ is continuous.

From the definition, for $\varphi \in \mathcal{O}_M^*(\mathbb{R}^n)$ the mapping

$$T \to \varphi T,\ \mathcal{S}'^*(\mathbb{R}^n) \to \mathcal{S}'^*(\mathbb{R}^n)$$

is continuous. In the proof of the next proposition the following function will be needed:

$$\psi(x) = \sum_{j=1}^{\infty} \frac{\rho(x-x_j)}{e^{M(k|x_j|)}}, \tag{3.14}$$

where the function $\rho \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, has values in $[0,1]$, and $\operatorname{supp}\rho \subset \{x : \mid x \mid \leq 1, x \in \mathbb{R}^n\}$, $\rho(x) = 1$, for $x \in \{x \mid \ \mid x \mid \leq 1/2\}$. Here (x_j) is a sequence of vectors of $\mathbb{R}^n$ such that $\mid x_j \mid > 2$ and $\mid x_{j+1} \mid \geq \mid x_j \mid +2$, $j \in \mathbb{N}$.

Since $\rho \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, there exist $h > 0$ and $C > 0$ such that $\sup\limits_x |D^\alpha \rho(x)| < Ch^\alpha M_\alpha$. We will show that $\psi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. Choose $r > 0$ such that $rh < 1/2$ and $r < k/(H4\sqrt{2})$. Using that $\dfrac{\mid x \mid}{\mid x_j \mid} \leq 2$, one easily obtains

$$\begin{aligned}
&\sum_{\alpha,\beta} \int_{\mathbb{R}^n} \frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta} \mid D^\alpha \psi(x) \mid^2}{M_\alpha^2 M_\beta^2} dx \\
&\leq \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \int_{|x-x_j|\leq 1} \frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta} C^2 h^{2\alpha} M_\alpha^2}{M_\alpha^2 M_\beta^2 e^{2M(k|x_j|)}} dx \\
&\leq \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \int_{|x-x_j|\leq 1} \frac{r^{2\alpha+2\beta}\langle x\rangle^{2\beta} C^2 h^{2\alpha}}{M_\beta^2 e^{2M(k|x_j|)}} dx \\
&\leq \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \int_{|x-x_j|\leq 1} \frac{r^{2\alpha} r^{2\beta} 2^\beta \mid x \mid^{2\beta} C^2 h^{2\alpha}}{M_\beta^2 e^{2M(k|x_j|)}} dx \\
&\leq C_1 \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \frac{(rh)^{2\alpha}(r\sqrt{2})^{2\beta}}{M_\beta^2 e^{2M(k|x_j|)}} \int_{|x-x_j|\leq 1} \mid x \mid^{2\beta} dx \\
&\leq C_2 \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \frac{(rh)^{2\alpha}(r\sqrt{2})^{2\beta} \mid x_j \mid^{2\beta}}{M_\beta^2 e^{2M(k|x_j|)}} \\
&\leq C_2 \sum_{\alpha,\beta} \sum_{j=1}^{\infty} \frac{(rh)^{2\alpha}(r2\sqrt{2})^{2\beta} \mid x_j \mid^{2\beta} M_{\beta+1}^2}{M_\beta^2 k^{2\beta+2} \mid x_j \mid^{2\beta+2}}
\end{aligned}$$

$$\leq \frac{C_2 A}{k^2} \sum_{\alpha,\beta} \sum_{j=1}^{\infty} (rh)^{2\alpha} \Big(\frac{2r\sqrt{2}H}{k}\Big)^{2\beta} \frac{1}{\mid x_j \mid^2} \leq C'.$$

The proof of the next proposition in (M_p)-case is given in [42] and [79].

Proposition 3.2.1. *Let $\varphi \in C^\infty(\mathbb{R}^n)$. The following statements are equivalent:*

(i) $\varphi \in \mathcal{O}^*_M(\mathbb{R}^n)$.

(ii) For every $h > 0$, resp. for every $k > 0$, there exists $k > 0$, resp. there exist $h > 0$,

$$\sup_{\alpha \in \mathbb{N}_0^n} \{ \frac{h^\alpha \| e^{-M(k|\cdot|)} \varphi^{(\alpha)} \|_{L^\infty}}{M_\alpha} \} < \infty.$$

(iii) For every $\psi \in \mathcal{S}^*(\mathbb{R}^n)$ and every $r > 0$, resp. for some $r > 0$

$$\sigma_{r,\psi}(\varphi) := \sigma_{r,\infty}(\psi\varphi) < \infty.$$

(iv) In Roumieu case, for every $\psi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ and for every $(r_i), (s_j) \in \Re$

$$\gamma_{r_i,s_j,\psi}(\varphi) := \gamma_{r_i,s_j}(\psi\varphi) < \infty.$$

Proof. Only the proof for the Roumieu case will be given.
$(iii) \Leftrightarrow (iv)$ It is obvious. We will prove $(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$.
$(iii) \Rightarrow (ii)$ First, $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$. Indeed, let K be a fixed compact set in $\mathbb{R}^n$ and choose $\chi \in \mathcal{D}^{\{M_p\}}$, with values in $[0,1]$ and $\chi(x) = 1$ on a neighborhood of K. Then there exists $r > 0$ such that

$$\sup_\alpha \frac{r^\alpha \| D^\alpha(\varphi(x)\chi(x)) \|_{L^\infty(K)}}{M_\alpha} \leq \sup_\alpha \frac{r^\alpha \| e^{M(r|x|)} D^\alpha(\varphi(x)\chi(x)) \|_{L^\infty(\mathbb{R}^n)}}{M_\alpha}$$
$$= C s_r(\varphi\chi) < \infty.$$

Then, $D^\alpha(\varphi(x)\chi(x)) = D^\alpha\varphi(x)$ for $x \in K$. Thus $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$.

Suppose that (ii) does not hold. Then there exist $k > 0$ such that for all $n \in \mathbb{N}$,

$$\sup_\alpha \frac{\| e^{-M(k|x|)} D^\alpha \varphi(x) \|_{L^\infty}}{n^\alpha M_\alpha} = \infty\,.$$

Since $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$ for every compact set K, there exist $C > 0$ and $n_K \in \mathbb{N}$ such that for $n \geq n_K$

$$\sup_\alpha \frac{\| e^{-M(k|x|)} D^\alpha (\varphi(x)) \|_{L^\infty(K)}}{n^\alpha M_\alpha} < C\,.$$

Hence, we can choose α_j and x_j, where $\mid x_{j+1} \mid > \mid x_j \mid +2$, such that

$$\frac{e^{-M(k|x_j|)} \mid D^{\alpha_j}\varphi(x_j) \mid}{j^{\alpha_j} M_{\alpha_j}} \geq 1.$$

Now take ψ as in (3.14), where k and the sequence (x_j) are the ones chosen here. Then $\varphi\psi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$, i.e., there exist l such that

$$\sup_\alpha \frac{l^\alpha \|e^{M(k|x|)} D^\alpha(\varphi(x)\psi(x))\|_{L^\infty}}{M_\alpha} < \infty.$$

Then, there exists j_0 such that for all $j \geq j_0$, $l > 1/j$ and

$$\begin{aligned} &\sup_\alpha \frac{l^\alpha \|e^{M(l|x|)} D^\alpha(\varphi(x)\psi(x))\|_{L^\infty}}{M_\alpha} \\ &\geq \frac{l^{\alpha_j} e^{M(l|x_j|)} \mid D^{\alpha_j}(\varphi(x_j)\psi(x_j)) \mid}{M_{\alpha_j}} \\ &\geq \frac{1}{j^{\alpha_j}} \frac{e^{M(l|x_j|)} \mid D^{\alpha_j}\varphi(x_j) \mid}{e^{M(k|x_j|)} M_{\alpha_j}} \\ &\geq e^{M(l|x_j|)}. \end{aligned}$$

This implies that $\varphi\psi$ is not in $\mathcal{S}_\infty^{M_p,l}$, which is a contradiction with the above assumption.
$(ii) \Rightarrow (i)$ From the condition (ii) it is obvious that $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$. It is enough to prove that for every $r > 0$ there is $l > 0$ such that the mapping $\psi \longrightarrow \varphi\psi$ from $\mathcal{S}_\infty^{M_p,r}$ to $\mathcal{S}_\infty^{M_p,l}$ is continuous.

Let $r > 0$ be fixed. Put $k = r/4$. By (ii), there exist $h > 0$ such that

$$\sup_\alpha \frac{h^\alpha \|e^{-M(k|x|)} D^\alpha\varphi(x)\|_{L^\infty}}{M_\alpha} < \infty.$$

If $l < h/4$ and $l < r/4$, then

$$\begin{aligned} &\frac{l^\alpha \|e^{M(l|x|)} D^\alpha(\varphi(x)\psi(x))\|_{L^\infty}}{M_\alpha} \\ &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{l^\alpha \|e^{M(l|x|)} D^\beta\varphi(x) D^{\alpha-\beta}\psi(x)\|_{L^\infty}}{M_{\alpha-\beta}M_\beta} \\ &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{(2l)^\alpha \|e^{M(l|x|)} D^\beta\varphi(x) e^{-M(k|x|)} e^{M(k|x|)} h^\beta}{2^\alpha h^\beta} \frac{D^{\alpha-\beta}\psi(x) e^{M(r|x|)} e^{-M(r|x|)} r^{\alpha-\beta}\|_{L^\infty}}{r^{\alpha-\beta} M_{\alpha-\beta} M_\beta} \\ &\leq C s_r(\psi) \|e^{-M(r|x|)} e^{M(k|x|)} e^{M(l|x|)}\|_{L^\infty} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{1}{2^\alpha} \left(\frac{2l}{h}\right)^\beta \left(\frac{2l}{r}\right)^{\alpha-\beta} \leq C' s_r(\psi), \end{aligned}$$

where the last inequality holds because of the way that l is chosen.
$(i) \Rightarrow (iii)$ It is obvious. □

Remark 3.2.1. If $\varphi \in \mathcal{O}_M^*(\mathbb{R}^n)$, then $\varphi \in \mathcal{S}'^*(\mathbb{R}^n)$.

Denote by $\mathcal{L}(\mathcal{S}^*(\mathbb{R}^n), \mathcal{S}^*(\mathbb{R}^n))$ the space of continuous linear mappings from $\mathcal{S}^*(\mathbb{R}^n)$ into $\mathcal{S}^*(\mathbb{R}^n)$. The space $\mathcal{O}_M^*(\mathbb{R}^n)$ is its subspace. We use the notation $\mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^n), \mathcal{S}^*(\mathbb{R}^n))$ for the space $\mathcal{L}(\mathcal{S}^*(\mathbb{R}^n), \mathcal{S}^*(\mathbb{R}^n))$ with the strong topology. Also, $\mathcal{O}_M^*(\mathbb{R}^n)$ can be equipped with the topology induced by $\mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^n), \mathcal{S}'^*(\mathbb{R}^n))$.

Similarly as in Proposition 3.1.3 it can be proved that the topologies induced by $\mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^n), \mathcal{S}^*(\mathbb{R}^n))$ and $\mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^n), \mathcal{S}'^*(\mathbb{R}^n))$ are the same. The space $\mathcal{O}^*_M(\mathbb{R}^n)$ equipped with this topology is denoted by $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$.

Proposition 3.2.2. *The Fourier transform is a topological isomorphism of* $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$ *onto* $\mathcal{O}'^*_{C,b}(\mathbb{R}^n)$.

Proof. Only the Roumieu case will be shown. Using (d) from Proposition 3.1.2, there exists $k > 0$ and there exist $(k_p) \in \Re$ such that $S = P_{k_p}(D)F + F_1$, where F and F_1 satisfy the growth condition given in Proposition 3.1.2. Without loss of generality it may be assumed that $F_1 = 0$. By $(M.2)$, the following estimates for the derivatives of the Fourier transform of F can be obtained:

$$\begin{aligned} \mid D^\alpha \mathcal{F}(F) \mid &= \mid \mathcal{F}(x^\alpha F) \mid = \Big| \int_{\mathbb{R}^n} e^{-ix\xi} x^\alpha F(x) dx \Big| \\ &\leq \int_{\mathbb{R}^n} \mid x \mid^\alpha \mid F(x) \mid dx \leq C_1 \int_{\mathbb{R}^n} \mid x \mid^\alpha e^{-M(k|x|)} dx \qquad (3.15) \\ &\leq C_2 \int_{\mathbb{R}^n} \frac{|x|^\alpha}{\langle x \rangle^{\alpha+n+1}} M_{\alpha+n+1} \Big(\frac{c}{k}\Big)^{|\alpha|+n+1} dx \leq C M_\alpha M_{n+1} \Big(\frac{Hc}{k}\Big)^{|\alpha|+n+1}. \end{aligned}$$

In [49] the following estimate of the analytic function $P_{k_p}(\zeta)$ is given: For every L, there is C such that

$$\mid P_{k_p}(\zeta) \mid \leq AC e^{M(\sqrt{n} L H |\zeta|)}, \zeta \in \mathbb{C}^n.$$

Using this result and the Cauchy integral formula, we obtain that for every $L > 0$ there exist $C > 0$ such that

$$\mid D^\alpha P_{k_p}(\xi) \mid \leq C \frac{\alpha!}{d^\alpha} \cdot e^{M(Lc'|\xi|)}, \qquad (3.16)$$

where $c' > 0$ is a constant that does not depend on L. It is also known that, for every $m > 0$,

$$\frac{m^k k!}{M_k} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \qquad (3.17)$$

Let $m > 0$ be arbitrary and L be a constant such that

$$\| e^{-M(m|\xi|)} e^{M(Lc'|\xi|)} \|_{L^\infty} < \infty,$$

and h is chosen such that $2h < 1$ and $2hHc < k$. From (3.15), (3.16), (3.17) and $(M.1)$ one obtains,

$$\begin{aligned} &\sup_\alpha \frac{h^\alpha \| e^{-M(m|\xi|)} D^\alpha (P_{r_p}(\xi) \hat{F}(\xi)) \|_{L^\infty}}{M_\alpha} \\ &\leq \sup_\alpha \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{(2h)^\alpha \| e^{-M(m|\xi|)} D^{\alpha-\beta} P_{r_p}(\xi) D^\beta \hat{F}(\xi) \|_{L^\infty}}{2^\alpha M_{\alpha-\beta} M_\beta} \\ &\leq C \sup_\alpha \frac{1}{2^\alpha} \| e^{-M(m|\xi|)} e^{M(Lc'|\xi|)} \|_{L^\infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{(\alpha-\beta)!}{M_{\alpha-\beta} d^{\alpha-\beta}} M_{n+1} (2h)^\beta \Big(\frac{Hc}{k}\Big)^{|\beta|+n+1} \end{aligned}$$

$$\le C_1 \sup_{\alpha} \|e^{-M(m|\xi|)}e^{M(Lc'|\xi|)}\|_{L^\infty} \frac{1}{2^\alpha} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \Big(\frac{2hHc}{k}\Big)^{|\beta|}$$
$$\le C_2 \|e^{-M(m|\xi|)}e^{M(Lc'|\xi|)}\|_{L^\infty} \le C_3.$$

By (*ii*) of Proposition 3.2.1, it follows that $\hat{S} \in \mathcal{O}_M^{\{M_p\}}(\mathbb{R}^n)$ and it is obvious that the mapping $S \longrightarrow \hat{S}$ is injective.

Now, we will prove that the Fourier transform from $\mathcal{O}_M^{\{M_p\}}(\mathbb{R}^n)$ to $\mathcal{O}_C'^{\{M_p\}}(\mathbb{R}^n)$ is an injective mapping. Let $\varphi \in \mathcal{O}_M^{\{M_p\}}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. The mappings

$$\hat{\psi} \longrightarrow \psi \longrightarrow \varphi\psi \longrightarrow \mathcal{F}(\varphi\psi) = \Big(\frac{1}{2\pi}\Big)^n \hat{\varphi} * \hat{\psi}$$

are continuous from $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ to $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. Hence, $\hat{\varphi} \in \mathcal{O}_C'^{\{M_p\}}(\mathbb{R}^n)$ and the mapping $\varphi \longrightarrow \hat{\varphi}$ is injective from $\mathcal{O}_M^{\{M_p\}}(\mathbb{R}^n)$ into $\mathcal{O}_C'^{\{M_p\}}(\mathbb{R}^n)$. Now it is enough to see that the same things hold for the $\bar{\mathcal{F}} = (2\pi)^n\mathcal{F}^{-1}$ and the fact that $\mathcal{F}$ is isomorphism on $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ and $\mathcal{S}'^{\{M_p\}}(\mathbb{R}^n)$ with an inverse $\mathcal{F}^{-1}$. Because $\mathcal{F}$: $\mathcal{S}^*(\mathbb{R}^n) \longrightarrow \mathcal{S}^*(\mathbb{R}^n)$ is a topological isomorphism it is obvious that it is also a topological isomorphism from $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$ to $\mathcal{O}'^*_{C,b}(\mathbb{R}^n)$. □

Proposition 3.2.3. *The bilinear mappings*

$$\mathcal{O}^*_{M,b}(\mathbb{R}^n) \times \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n), \quad (\alpha, \psi) \to \alpha\psi,$$

$$\mathcal{O}^*_{M,b}(\mathbb{R}^n) \times \mathcal{S}'^*(\mathbb{R}^n) \to \mathcal{S}'^*(\mathbb{R}^n), \quad (\alpha, f) \to \alpha f,$$

are hypocontinuous.

Proof. It is obvious that the bilinear mappings are separately continuous. We will prove only that the mapping $T : \mathcal{O}^*_{M,b}(\mathbb{R}^n) \times \mathcal{S}^*(\mathbb{R}^n) \longrightarrow \mathcal{S}^*(\mathbb{R}^n)$, defined by $T(\varphi, \psi) = \varphi\psi$ is hypocontinuous. Since $\mathcal{S}^*(\mathbb{R}^n)$ is a barreled space, from [91, Thm 5.2], it follows that for every open set V in $\mathcal{S}^*(\mathbb{R}^n)$, and every bounded set B in $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$, there is an open set W in $\mathcal{S}^*(\mathbb{R}^n)$ such that $T(B \times W) \subset V$. Now, let V_1 be an arbitrary open set in $\mathcal{S}^*(\mathbb{R}^n)$ and let B_1 be a bounded set in $\mathcal{S}^*(\mathbb{R}^n)$. Then, for the open set W_1 in $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$, where $W_1 = \{\psi \in \mathcal{O}^*_{M,b}(\mathbb{R}^n) \mid \varphi\psi \in V,\ \textit{for all}\ \psi \in B\}$, follows $T(W_1 \times B_1) \subset V_1$. □

Proposition 3.2.4. *The space* $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$ *is nuclear and reflexive.*

Proof. The space $\mathcal{S}(\mathbb{R}^n)$ is reflexive and the space $\mathcal{S}'(\mathbb{R}^n)$ is nuclear which imply nuclearity of the space $\mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^n), \mathcal{S}^*(\mathbb{R}^n))$. The space $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$ is reflexive as a closed subspace of the reflexive space $\mathcal{S}'^*(\mathbb{R}^n) \hat{\bigotimes}_\pi \mathcal{S}^*(\mathbb{R}^n)$. The space $\mathcal{O}^*_{M,b}(\mathbb{R}^n)$ is nuclear as a closed subspace of a nuclear space. □

Corollary 3.2.1. *The space* $\mathcal{O}'^*_{C,b}(\mathbb{R}^n)$ *is nuclear and reflexive.*

3.3 Characterization of $\mathcal{S}'^*(\mathbb{R}^n)$ through regularization

We present here an important characterization of tempered ultradistributions in terms of growth properties of convolution averages, an analog to this result for $\mathcal{S}'(\mathbb{R}^d)$ was obtained long ago by Schwartz (cf. [94, Thm. VI, p. 239])

Proposition 3.3.1. *Let $f \in \mathcal{D}'^*(\mathbb{R}^n)$. Then, f belongs to $\mathcal{S}'^*(\mathbb{R}^n)$ if and only if there exists $\lambda > 0$, resp. $(l_p) \in \mathfrak{R}$, such that for every $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$*

$$\sup_{x\in\mathbb{R}^n} e^{-M(\lambda|x|)}|(f*\varphi)(x)| < \infty, \ \ \textit{resp.} \ \sup_{x\in\mathbb{R}^n} e^{-N_{l_p}(|x|)}|(f*\varphi)(x)| < \infty. \tag{3.18}$$

Proof. Observe that if $f \in \mathcal{S}'^*(\mathbb{R}^n)$ then (3.18) obviously holds (one only needs to use the representation theorem for the elements of $\mathcal{S}'^*(\mathbb{R}^n)$, see [12]). We prove the converse part in the $\{M_p\}$ case. The (M_p) case is similar. Let Ω be an open bounded subset of $\mathbb{R}^n$ which contains 0 and it is symmetric (i.e., $-\Omega = \Omega$) and denote $\overline{\Omega} = K$. Let B_1 be the unit ball in the weighted Banach space $L^1_{\exp(N_{l_p}(|x|))}$. Fix $\varphi \in \mathcal{D}^{\{M_p\}}_K$. For every $\phi \in B_1 \cap \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, (3.18) implies

$$|\langle f*\phi, \varphi\rangle| = |\langle f*\check{\varphi}, \check{\phi}\rangle| \leq \left\|e^{-N_{l_p}(|\cdot|)} f*\check{\varphi}(\cdot)\right\|_{L^\infty} \|\phi\|_{L^1_{\exp(N_{l_p}(|x|))}} \leq C_\varphi.$$

We obtain that the set

$$\left\{f*\phi \,|\, \phi \in B_1 \cap \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)\right\}$$

is weakly bounded, hence equicontinuous in $\mathcal{D}'^{\{M_p\}}_K$ ($\mathcal{D}^{\{M_p\}}_K$ is barrelled). Hence, there exist $(k_p) \in \mathcal{R}$ and $\varepsilon > 0$ such that

$$|\langle f*\psi, \check{\phi}\rangle| \leq 1 \ \text{ for all } \psi \in V_{k_p}(\varepsilon) = \{\eta \in \mathcal{D}^{\{M_p\}}_K | \|\eta\|_{K,k_p} \leq \varepsilon\}$$

and $\phi \in B_1 \cap \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$.

Let $r_p = k_{p-1}/H$, for $p \in \mathbb{N}$, $p \geq 2$ and put $r_1 = \min\{1, r_2\}$. Then $(r_p) \in \mathfrak{R}$. Let $\psi \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$ and choose C_ψ such that $\|\psi/C_\psi\|_{K,(r_p)} \leq \varepsilon/2$. Let $\delta_1 \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ such that $\delta_1 \geq 0$, $\operatorname{supp}\delta_1 \subseteq \{x \in \mathbb{R}^n | \, |x| \leq 1\}$ and $\int_{\mathbb{R}^n} \delta_1(x)dx = 1$. Put $\delta_j(x) = j^n\delta_1(jx)$, for $j \in \mathbb{N}$, $j \geq 2$. Observe that for j large enough $\psi * \delta_j \in \mathcal{D}^{\{M_p\}}_K$. Also

$$|\partial^\alpha((\psi*\delta_j)(x) - \psi(x))| \leq \int_{\mathbb{R}^n} |\partial^\alpha(\psi(x-t) - \psi(x))|\delta_j(t)dt.$$

Using Taylor expansion of the function $\partial^\alpha\psi$ in the point $x - t$ we obtain

$$\begin{aligned} |\partial^\alpha(\psi(x) - \psi(x-t))| &\leq \sum_{|\beta|=1} \left|t^\beta\right| \int_0^1 \left|\partial^{\alpha+\beta}\psi(sx + (1-s)(x-t))\right| ds \\ &\leq C|t|M_{|\alpha|+1} \prod_{i=1}^{|\alpha|+1} r_i. \end{aligned}$$

So, for j large enough,

$$|\partial^\alpha((\psi * \delta_j)(x) - \psi(x))| \leq \frac{C_1}{j} M_\alpha \prod_{i=2}^{|\alpha|+1} Hr_i \int_{\operatorname{supp}\delta_j} \delta_j(t)dt = \frac{C_1}{j} M_\alpha \prod_{i=1}^{|\alpha|} k_i.$$

Hence $C_\psi^{-1}\psi*\delta_j \in V_{(k_p)}(\varepsilon)$ for all large enough j. We obtain $|\langle f*(\psi*\delta_j),\phi\rangle| \leq C_\psi$ and after passing to the limit $|\langle f*\psi,\phi\rangle| \leq C_\psi$. From the arbitrariness of ψ we have that for every $\psi \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$ there exists $C_\psi > 0$ such that $|\langle f*\psi,\phi\rangle| \leq C_\psi\|\phi\|_{L^1_{\exp(N_{l_p}(|x|))}}$, for all $\phi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$. Density of $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ in $L^1_{\exp(N_{l_p}(|x|))}$ implies that for every fixed $\psi \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$, $f*\psi$ is a continuous functional on $L^1_{\exp(N_{l_p}(|x|))}$, hence

$$\|f*\psi\|_{L^\infty_{\exp(-N_{l_p}(|x|))}} \leq C_{2,\psi}.$$

From the parametrix of Komatsu for the sequence (r_p), there are $u \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$, $\chi \in \mathcal{D}^{\{M_p\}}(\Omega)$ and ultradifferential operator of $\{M_p\}$ type such that $f = P(D)(u*f) + \chi * f$. Thus $f \in \mathcal{S}'^*(\mathbb{R}^n)$. □

3.4 Space $\mathcal{O}^*_C(\mathbb{R}^n)$ and the duality characterization of $\mathcal{O}'^*_C(\mathbb{R}^n)$

Our next concern is to define the test function spaces $\mathcal{O}^*_C(\mathbb{R}^n)$ corresponding to the spaces $\mathcal{O}'^*_C(\mathbb{R}^n)$. We first define for every $m,h>0$ the Banach spaces

$$\mathcal{O}^{M_p}_{C,m,h}(\mathbb{R}^n) = \left\{ \varphi \in C^\infty(\mathbb{R}^n) \,\middle|\, \|\varphi\|_{m,h} = \left(\sum_{\alpha\in\mathbb{N}^n} \frac{m^{2|\alpha|}}{M_\alpha^2} \left\|D^\alpha\varphi e^{-M(h|\cdot|)}\right\|^2_{L^2}\right)^{1/2} < \infty \right\}.$$

Observe that for $m_1 \leq m_2$ we have the continuous inclusion $\mathcal{O}^{M_p}_{C,m_2,h}(\mathbb{R}^n) \to \mathcal{O}^{M_p}_{C,m_1,h}(\mathbb{R}^n)$ and for $h_1 \leq h_2$ the inclusion $\mathcal{O}^{M_p}_{C,m,h_1}(\mathbb{R}^n) \to \mathcal{O}^{M_p}_{C,m,h_2}(\mathbb{R}^n)$ is also continuous. As l.c.s. we define

$$\mathcal{O}^{(M_p)}_{C,h}(\mathbb{R}^n) = \varprojlim_{m\to\infty} \mathcal{O}^{M_p}_{C,m,h}(\mathbb{R}^n) \quad , \quad \mathcal{O}^{(M_p)}_{C}(\mathbb{R}^n) = \varinjlim_{h\to\infty} \mathcal{O}^{(M_p)}_{C,h}(\mathbb{R}^n);$$
$$\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n) = \varinjlim_{m\to 0} \mathcal{O}^{M_p}_{C,m,h}(\mathbb{R}^n) \quad , \quad \mathcal{O}^{\{M_p\}}_{C}(\mathbb{R}^n) = \varprojlim_{h\to 0} \mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n).$$

Note that $\mathcal{O}^{(M_p)}_{C,h}(\mathbb{R}^n)$ is an Fréchet space and since all inclusions $\mathcal{O}^{(M_p)}_{C,h}(\mathbb{R}^n) \to \mathcal{E}^{(M_p)}(\mathbb{R}^n)$ are continuous (by the Sobolev imbedding theorem), $\mathcal{O}^{(M_p)}_{C}(\mathbb{R}^n)$ is indeed a (Hausdorff) l.c.s. Moreover, as an inductive limit of barreled and bornological spaces, $\mathcal{O}^{(M_p)}_{C}(\mathbb{R}^n)$ is barreled and bornological as well. Also $\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)$ is a (Hausdorff) l.c.s. since all inclusions $\mathcal{O}^{M_p}_{C,m,h}(\mathbb{R}^n) \to \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$ are continuous (again by the Sobolev imbedding theorem). Hence $\mathcal{O}^{\{M_p\}}_{C}(\mathbb{R}^n)$ is indeed a (Hausdorff) l.c.s. Moreover, $\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)$ is barreled and bornological (DF)-space, as inductive limit of Banach spaces. By these considerations it also follows that $\mathcal{O}^*_C(\mathbb{R}^n)$ is continuously injected into $\mathcal{E}^*(\mathbb{R}^n)$. One easily verifies that for each $h > 0$, $\mathcal{S}^{(M_p)}(\mathbb{R}^n)$, respectively $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$, is continuously injected into $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$, respectively into $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n)$. Moreover, one can also prove (by using cutoff functions) that $\mathcal{D}^{(M_p)}(\mathbb{R}^n)$, respectively $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, is sequentially dense in $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$, respectively in $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n)$, for each $h > 0$. Hence $\mathcal{S}^{(M_p)}(\mathbb{R}^n)$, respectively $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$, is continuously and densely injected into $\mathcal{O}_C^{(M_p)}(\mathbb{R}^n)$, respectively into $\mathcal{O}_C^{\{M_p\}}(\mathbb{R}^n)$. From this we obtain that the dual $(\mathcal{O}_C^*(\mathbb{R}^n))'$ can be regarded as vector subspace of $\mathcal{S}'^*(\mathbb{R}^n)$.

We will prove that the dual of $\mathcal{O}_C^*(\mathbb{R}^n)$ is equal as a set to $\mathcal{O}_C'^*(\mathbb{R}^n)$ (the general idea is similar to the one used by Komatsu in [49]). To do this, we need several additional spaces.

For $m, h > 0$ define

$$Y_{m,h} = \left\{(\psi_\alpha)_{\alpha\in\mathbb{N}^n} \middle| e^{-M(h|\cdot|)}\psi_\alpha \in L^2(\mathbb{R}^n), \right.$$
$$\left. \|(\psi_\alpha)_\alpha\|_{Y_{m,h}} = \left(\sum_{\alpha\in\mathbb{N}^n} \frac{m^{2|\alpha|}\left\|e^{-M(h|\cdot|)}\psi_\alpha\right\|_{L^2}^2}{M_\alpha^2}\right)^{1/2} < \infty \right\}.$$

One easily verifies that $Y_{m,h}$ is a Banach space, with the norm $\|\cdot\|_{Y_{m,h}}$.

Let $\tilde{U}$ be the disjoint union of countable number of copies of $\mathbb{R}^n$, one for each $\alpha \in \mathbb{N}^n$, i.e., $\tilde{U} = \coprod_{\alpha\in\mathbb{N}^n} \mathbb{R}^n_\alpha$. Equip $\tilde{U}$ with the disjoint union topology. Then $\tilde{U}$ is Hausdorff locally compact space. Moreover every open set in $\tilde{U}$ is σ-compact. On each $\mathbb{R}^n_\alpha$ we define Radon measure ν_α by $d\nu_\alpha = e^{-2M(h|x|)}dx$. One can define a Borel measure μ_m on $\tilde{U}$ by

$$\mu_m(E) = \sum_\alpha \frac{m^{2|\alpha|}}{M_\alpha^2}\nu_\alpha\left(E\cap\mathbb{R}^n_\alpha\right),$$

for E a Borel subset of $\tilde{U}$. It is obviously locally finite, σ-finite and $\mu_m(\tilde{K}) < \infty$ for every compact subset $\tilde{K}$ of $\tilde{U}$. By the properties of $\tilde{U}$ described above, μ_m is regular (both inner and outer regular). We obtained that μ_m is a Radon measure. For every $(\psi_\alpha)_\alpha \in Y_{m,h}$ there corresponds an element $\chi \in L^2(\tilde{U}, \mu_m)$ defined by $\chi_{|\mathbb{R}^n_\alpha} = \psi_\alpha$.

One easily verifies that the mapping

$$(\psi_\alpha)_\alpha \mapsto \chi,\ \ Y_{m,h} \to L^2(\tilde{U}, \mu_m)$$

is an isometry, i.e., $Y_{m,h}$ can be identified with $L^2(\tilde{U}, \mu_m)$. Also, observe that $\mathcal{O}_{C,m,h}^{M_p}(\mathbb{R}^n)$ can be identified with a closed subspace of $Y_{m,h}$ via the mapping

$$\varphi \mapsto ((-D)^\alpha\varphi)_\alpha,$$

hence it is a reflexive space as a closed subspace of a reflexive Banach space. We obtain that the linking mappings in the projective, respectively inductive, limit $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n) = \varprojlim_{m\to\infty} \mathcal{O}_{C,m,h}^{M_p}(\mathbb{R}^n)$, respectively $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n) = \varinjlim_{m\to 0} \mathcal{O}_{C,m,h}^{M_p}(\mathbb{R}^n)$, are weakly compact, whence $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$ is an (FS^*)-space, respectively $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n)$ is a (DFS^*)-space, in particular they are both reflexive and the inductive limit $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n) = \varinjlim_{m\to 0} \mathcal{O}_{C,m,h}^{M_p}(\mathbb{R}^n)$ is regular.

Theorem 3.4.1. *$T \in \mathcal{D}'^*(\mathbb{R}^n)$ belongs to $(\mathcal{O}_C^*(\mathbb{R}^n))'$ if and only if*

(i) in the (M_p) case, for every $h > 0$ there exist $F_{\alpha,h}$, $\alpha \in \mathbb{N}^n$ and $m > 0$ such that

$$\sum_\alpha \frac{M_\alpha^2 \left\|F_{\alpha,h} e^{M(h|\cdot|)}\right\|_{L^2}^2}{m^{2|\alpha|}} < \infty \tag{3.19}$$

and the restriction of T to $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$ is equal to $\sum_\alpha D^\alpha F_{\alpha,h}$, where the series is absolutely convergent in the strong dual of $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$;

(ii) in the $\{M_p\}$ case, there exist $h > 0$ and $F_{\alpha,h}$, $\alpha \in \mathbb{N}^n$, such that for every $m > 0$ (3.19) holds and T is equal to $\sum_\alpha D^\alpha F_{\alpha,h}$, where the series is absolutely convergent in the strong dual of $\mathcal{O}_C^{\{M_p\}}(\mathbb{R}^n)$.

Proof. We will consider first the Beurling case. Let $T \in \left(\mathcal{O}_C^{(M_p)}(\mathbb{R}^n)\right)'$ and $h > 0$ be arbitrary but fixed. Denote by T_h the restriction of T on $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$. By the definition of the projective limit topology, it follows that there exists $m > 0$ such that T_h can be extended to a continuous linear functional on $\mathcal{O}_{C,m,h}^{M_p}(\mathbb{R}^n)$. Denote this extension by $T_{h,1}$. Extend $T_{h,1}$, by the Hahn-Banach theorem, to a continuous linear functional $T_{h,2}$ on $Y_{m,h}$. Since $Y_{m,h}$ is isometric with $L^2(\tilde{U}, \mu_m)$, there exists $g \in L^2(\tilde{U}, \mu_m)$ such that

$$T_{2,h}\left((\psi_\alpha)_\alpha\right) = \int_{\tilde{U}} (\psi_\alpha)_\alpha g d\mu_m.$$

We define

$$F_{\alpha,h} = \frac{m^{2|\alpha|}}{M_\alpha^2} g_{|\mathbb{R}_\alpha^n} e^{-2M(h|\cdot|)},\ \alpha \in \mathbb{N}^n.$$

Obviously $e^{M(h|\cdot|)} F_{\alpha,h} \in L^2(\mathbb{R}^n)$ and $\sum_\alpha \frac{M_\alpha^2 \left\|F_{\alpha,h} e^{M(h|\cdot|)}\right\|_{L^2}^2}{m^{2|\alpha|}} = \|g\|_{L^2(\tilde{U},\mu_m)}^2 < \infty.$

For $\varphi \in \mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$,

$$\langle T, \varphi\rangle = T_{h,2}\left(((-D)^\alpha \varphi)_\alpha\right) = \sum_\alpha \int_{\mathbb{R}^d} F_{\alpha,h}(x)(-D)^\alpha \varphi(x) dx = \sum_\alpha \langle D^\alpha F_{\alpha,h}, \varphi\rangle.$$

Moreover, one easily verifies that the series $\sum_\alpha D^\alpha F_{\alpha,h}$ is absolutely convergent in the strong dual of $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$.

Conversely, let $T \in \mathcal{D}'^{(M_p)}(\mathbb{R}^n)$ be as in (i). Let $h > 0$ be arbitrary but fixed. One easily verifies that T is continuous functional on $\mathcal{D}^{(M_p)}(\mathbb{R}^n)$ supplied with the topology induced by $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$. Since $\mathcal{D}^{(M_p)}(\mathbb{R}^n)$ is dense in $\mathcal{O}_{C,h}^{(M_p)}(\mathbb{R}^n)$ we obtain the conclusion in (i).

Next, we consider the Roumieu case. Let $T \in \left(\mathcal{O}_C^{\{M_p\}}(\mathbb{R}^n)\right)'$. By the definition of the projective limit topology it follows that there exists $h > 0$ such that T can be extended to a continuous linear functional T_1 on $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n)$. For brevity in notation, put

$$X_{m,h} = \mathcal{O}_{C,m,h}^{M_p}(\mathbb{R}^n) \text{ and } Z_{m,h} = Y_{m,h}/X_{m,h}.$$

Since $Y_{m,h}$ are reflexive so are $X_{m,h}$ and $Z_{m,h}$ as closed subspaces, respectively quotient spaces, of reflexive Banach spaces. Moreover, observe that for $m_1 < m_2$ we have $X_{m_1,h} \cap Y_{m_2,h} = X_{m_2,h}$. Hence we have the following injective inductive sequence of short topologically exact sequences of Banach spaces:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_{1,h} & \longrightarrow & Y_{1,h} & \longrightarrow & Z_{1,h} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \iota_{1,1/2} & & \downarrow & & \\
0 & \longrightarrow & X_{1/2,h} & \longrightarrow & Y_{1/2,h} & \longrightarrow & Z_{1/2,h} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \iota_{1/2,1/3} & & \downarrow & & \\
0 & \longrightarrow & X_{1/3,h} & \longrightarrow & Y_{1/3,h} & \longrightarrow & Z_{1/3,h} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \iota_{1/3,1/4} & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where every vertical line is a weakly compact injective inductive sequence of Banach spaces (since $X_{m,h}$, $Y_{m,h}$, $Z_{m,h}$ are reflexive Banach spaces). The dual Mittag-Leffler lemma (Lemma 0.2.1, see [49]) yields the short topologically exact sequence:

$$0 \longleftarrow \left(\varinjlim_{m\to 0} X_{m,h}\right)' \longleftarrow \left(\varinjlim_{m\to 0} Y_{m,h}\right)' \longleftarrow \left(\varinjlim_{m\to 0} Z_{m,h}\right)' \longleftarrow 0.$$

Since $(X_{m,h})_m$, $(Y_{m,h})_m$ and $(Z_{m,h})_m$ are weakly compact injective inductive sequences, hence regular, we have the following isomorphisms of l.c.s.

$$\left(\varinjlim_{m\to 0} X_{m,h}\right)' = \varprojlim_{m\to 0} X'_{m,h}\, , \left(\varinjlim_{m\to 0} Y_{m,h}\right)' = \varprojlim_{m\to 0} Y'_{m,h} \text{ and } \left(\varinjlim_{m\to 0} Z_{m,h}\right)' = \varprojlim_{m\to 0} Z'_{m,h},$$

from what we obtain the following short topologically exact sequence:

$$0 \longleftarrow \varprojlim_{m\to 0} X'_{m,h} \longleftarrow \varprojlim_{m\to 0} Y'_{m,h} \longleftarrow \varprojlim_{m\to 0} Z'_{m,h} \longleftarrow 0.$$

Hence, there exists $T_2 \in \varprojlim_{m\to 0} Y'_{m,h}$ whose restriction to $\mathcal{O}_{C,h}^{\{M_p\}} = \varinjlim_{m\to 0} X_{m,h}$ is T_1. Now observe the projective sequence:

$$Y'_{1,h} \xleftarrow{{}^t\iota_{1,1/2}} Y'_{1/2,h} \xleftarrow{{}^t\iota_{1/2,1/3}} Y'_{1/3,h} \xleftarrow{{}^t\iota_{1/3,1/4}} \dots$$

where ${}^t\iota_{1/k,1/(k+1)}$ is the transposed mapping of the inclusion $\iota_{1/k,1/(k+1)}$. The mapping

$${}^t\iota_{1/k,1/(k+1)} : Y'_{1/(k+1),h} \to Y'_{1/k,h}$$

is given by

$$(\psi_\alpha)_\alpha \mapsto \left(\frac{k^{2|\alpha|}}{(k+1)^{2|\alpha|}}\psi_\alpha\right)_\alpha .$$

Indeed, for $(\psi_\alpha)_\alpha \in Y_{1/k,h}$ and $T \in Y'_{1/(k+1),h}$ there exists $g \in L^2(\tilde{U}, \mu_{1/(k+1)})$ such that

$$\begin{aligned}
&\langle {}^t\iota_{1/k,1/(k+1)}T, (\psi_\alpha)_\alpha\rangle = \langle T, (\psi_\alpha)_\alpha\rangle = \int_{\tilde{U}} g(\psi_\alpha)_\alpha d\mu_{1/(k+1)} \\
&= \sum_\alpha \int_{\mathbb{R}^n} \psi_\alpha g_\alpha d\mu_{1/(k+1)} = \sum_\alpha \frac{1}{(k+1)^{2|\alpha|}M_\alpha^2}\int_{\mathbb{R}^n} \psi_\alpha g_\alpha e^{-2M(h|x|)}dx \\
&= \sum_\alpha \frac{1}{k^{2|\alpha|}M_\alpha^2}\int_{\mathbb{R}^n} \psi_\alpha \frac{k^{2|\alpha|}}{(k+1)^{2|\alpha|}} g_\alpha e^{-2M(h|x|)}dx = \int_{\tilde{U}} (\chi_\alpha)_\alpha(\psi_\alpha)_\alpha d\mu_{1/k}
\end{aligned}$$

where $(\chi_\alpha)_\alpha = \left(\frac{k^{2|\alpha|}}{(k+1)^{2|\alpha|}} g_\alpha\right)_\alpha \in Y'_{1/k,h}$.
By definition, the projective limit $\varprojlim_{m\to 0} Y'_{m,h}$ is the subspace of $\prod_n Y'_{1/n,h}$ consisting of all elements $\left((\psi_\alpha^{(s)})_\alpha\right)_s \in \prod_k Y'_{1/k,h}$ such that for all $t, j \in \mathbb{Z}_+$, $t < j$, ${}^t\iota_{1/t,1/j}\left((\psi_\alpha^{(j)})_\alpha\right) = (\psi_\alpha^{(t)})_\alpha$ (where ${}^t\iota_{1/t,1/j} = {}^t\iota_{1/t,1/(t+1)}{}^t \circ ... \circ {}^t\iota_{1/(j-1),1/j}$). Hence, if we put $(\psi_\alpha)_\alpha = (\psi_\alpha^{(1)})_\alpha$, then $L^2(\tilde{U}, \mu_{1/k}) \ni (\psi_\alpha^{(s)})_\alpha = \left(s^{2|\alpha|}\psi_\alpha\right)_\alpha$ for all $s \in \mathbb{Z}_+$.

In other words, we can identify $\varprojlim_{m\to 0} Y'_{m,h}$ with the space of all $(\psi_\alpha)_\alpha$ such that for every $s > 0$, $\left(\sum_\alpha \frac{s^{2|\alpha|}}{M_\alpha^2}\left\|\psi_\alpha e^{-M(h|\cdot|)}\right\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} < \infty.$

Since $T_2 \in \varprojlim_{m\to\infty} Y'_{1/m,h}$, there exists such $(\psi_\alpha)_\alpha$ such that, for $m \in \mathbb{Z}_+$ and $(\chi_\alpha)_\alpha \in Y_{1/m,h}$, we have $T_2\left((\chi_\alpha)_\alpha\right) = \sum_\alpha \int_{\mathbb{R}^n_\alpha} m^{2|\alpha|}\psi_\alpha\chi_\alpha d\mu_{1/m}$. Put $F_{\alpha,h} = \dfrac{\psi_\alpha e^{-2M(h|\cdot|)}}{M_\alpha^2}$. Hence, for every $s > 0$, $\left(\sum_\alpha s^{2|\alpha|}M_\alpha^2\left\|F_{\alpha,h}e^{M(h|\cdot|)}\right\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} < \infty.$
Moreover, for $\varphi \in \mathcal{O}_C^{\{M_p\}}(\mathbb{R}^n)$, there exists $m \in \mathbb{Z}_+$ such that $\varphi \in \mathcal{O}^{M_p}_{C,1/m,h}(\mathbb{R}^n)$. We have

$$\langle T, \varphi\rangle = \sum_\alpha \int_{\mathbb{R}^n} F_{\alpha,h}(x)(-D)^\alpha\varphi(x)dx = \sum_\alpha \langle D^\alpha F_{\alpha,h}, \varphi\rangle.$$

Since $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n)$ is a (DFS^*)-space its strong dual $\left(\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n)\right)'_b$ is complete. If B is a bounded subset of $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n)$ then it must belong to some $\mathcal{O}^{M_p}_{C,m,h}(\mathbb{R}^n)$ and to be bounded there (the inductive limit $\mathcal{O}_{C,h}^{\{M_p\}}(\mathbb{R}^n) = \varinjlim_{m\to 0} \mathcal{O}^{M_p}_{C,m,h}(\mathbb{R}^n)$ is regular). One easily verifies that $\sum_\alpha \sup_{\varphi\in B}|\langle D^\alpha F_{\alpha,h}, \varphi\rangle| < \infty$, hence $\sum_\alpha D^\alpha F_{\alpha,h}$

converges absolutely in $\left(\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)\right)'_b$. Since $\mathcal{O}^{\{M_p\}}_C(\mathbb{R}^n)$ is continuously and densely injected into $\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)$ ($\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in these spaces) it follows that the series $\sum_\alpha D^\alpha F_{\alpha,h}$ converges absolutely in the strong dual of $\mathcal{O}^{\{M_p\}}_C(\mathbb{R}^n)$.

Conversely, let $T \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ be as in (ii). Then it is easy to verify that T is a continuous functional on $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ when we regard it as subspace of $\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)$, where h is the one from the condition in (ii). Since $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)$, T is continuous functional on $\mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)$ and hence on $\mathcal{O}^{\{M_p\}}_C(\mathbb{R}^n)$. □

The next theorem realizes our goal in this chapter: We may identify $\mathcal{O}'^*_C(\mathbb{R}^n)$ with the topological dual of $\mathcal{O}^*_C(\mathbb{R}^n)$.

Theorem 3.4.2. *The dual of $\mathcal{O}^*_C(\mathbb{R}^n)$ is algebraically isomorphic to $\mathcal{O}'^*_C(\mathbb{R}^n)$.*

Proof. Let $T \in (\mathcal{O}^*_C(\mathbb{R}^n))' \subseteq \mathcal{S}'^*(\mathbb{R}^n)$. To prove that $T \in \mathcal{O}'^*_C(\mathbb{R}^n)$, by Proposition 3.1.2, it is enough to prove that $T * \varphi \in \mathcal{S}^*(\mathbb{R}^n)$ for each $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$.

We consider first the (M_p) case. Let $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R}^n)$ and $m > 0$ be arbitrary but fixed. By Theorem 3.4.1, for $h \geq 2m$, there exist $m_1 > 0$ and $F_{\alpha,h}$, $\alpha \in \mathbb{N}^n$, such that (3.19) holds. Take $m_2 > 0$ such that $m_2 \geq Hm$ and $H/m_2 \leq 1/(2m_1)$. For this m_2 there exists $C' > 0$ such that $\left|D^\beta\varphi(x)\right| \leq C' M_\beta/m_2^{|\beta|}$. Using the inequality $e^{M(\rho+\lambda)} \leq 2e^{M(2\rho)}e^{M(2\lambda)}$, $\rho, \lambda > 0$, for $x, t \in \mathbb{R}^n$ one obtains

$$e^{M(m|x|)} \leq 2e^{M(h|x-t|)}e^{M(h|t|)}.$$

Then, we have

$$\begin{aligned}
&\frac{m^{|\beta|}\left|D^\beta(T*\varphi)(x)\right|e^{M(m|x|)}}{M_\beta}\\
&\leq \frac{m^{|\beta|}e^{M(m|x|)}}{M_\beta}\sum_\alpha\int_{\mathbb{R}^n}|F_{\alpha,h}(t)|\left|D^{\alpha+\beta}\varphi(x-t)\right|dt\\
&\leq \frac{m^{|\beta|}e^{M(m|x|)}}{M_\beta}\sum_\alpha\left\|F_{\alpha,h}e^{M(h|\cdot|)}\right\|_{L^2}\left(\int_{\mathbb{R}^n}\left|D^{\alpha+\beta}\varphi(x-t)\right|^2e^{-2M(h|t|)}dt\right)^{1/2}\\
&\leq 2\frac{m^{|\beta|}}{M_\beta}\sum_\alpha\left\|F_{\alpha,h}e^{M(h|\cdot|)}\right\|_{L^2}\left(\int_{\mathbb{R}^n}\left|D^{\alpha+\beta}\varphi(x-t)\right|^2e^{2M(h|x-t|)}dt\right)^{1/2}\\
&\leq C_1\sum_\alpha\frac{m^{|\beta|}M_{\alpha+\beta}}{M_\beta m_2^{|\alpha|+|\beta|}}\left\|F_{\alpha,h}e^{M(h|\cdot|)}\right\|_{L^2} \leq C_2\sum_\alpha\frac{(Hm)^{|\beta|}H^{|\alpha|}M_\alpha}{m_2^{|\alpha|+|\beta|}}\left\|F_{\alpha,h}e^{M(h|\cdot|)}\right\|_{L^2}\\
&\leq C_3\left(\frac{Hm}{m_2}\right)^{|\beta|}\sum_\alpha\frac{1}{2^{|\alpha|}} \leq C.
\end{aligned}$$

Since $m > 0$ is arbitrary, $T * \varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^n)$ and we obtain $T \in \mathcal{O}'^{(M_p)}_C(\mathbb{R}^n)$. In the $\{M_p\}$ case, there exist $m_2, C' > 0$ such that $\left|D^\beta\varphi(x)\right| \leq C' M_\beta/m_2^{|\beta|}$. Also, for

T there exist $h > 0$ and $F_{\alpha,h}$, $\alpha \in \mathbb{N}^n$ such that (3.19) holds for every $m_1 > 0$. Take $m > 0$ such that $m \leq h/2$ and $m \leq m_2/H$ and take $m_1 > 0$ such that $1/(2m_1) \geq H/m_2$. Then the same calculations as above give

$$\frac{m^{|\beta|} \left|D^{\beta}(T * \varphi)(x)\right| e^{M(m|x|)}}{M_{\beta}} \leq C,$$

i.e., $T * \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$. We obtain $T \in \mathcal{O}_C'^{\{M_p\}}(\mathbb{R}^n)$.

Conversely, let $T \in \mathcal{O}_C'^{*}(\mathbb{R}^n)$. In the (M_p) case, by Proposition 3.1.2 ([21, Prop. 2]) for every $r > 0$ there exist an ultradifferential operator $P(D)$ of class (M_p) and $F_1, F_2 \in L^{\infty}(\mathbb{R}^n)$ such that $T = P(D)F_1 + F_2$ and $\left\|e^{M(r|\cdot|)}(F_1 + F_2)\right\|_{L^{\infty}} \leq C$. Let $h > 0$ be arbitrary but fixed. Choose such a representation of T for $r \geq H^2 h$. For simplicity, we assume that $F_2 = 0$ and put $F = F_1$. The general case is proved analogously. Let $P(D) = \sum_{\alpha} c_{\alpha} D^{\alpha}$. Then, there exist $c, L \geq 1$ such that $|c_{\alpha}| \leq cL^{|\alpha|}/M_{\alpha}$. Let $F_{\alpha} = c_{\alpha} F$. By [49, Prop. 3.6] we have $e^{4M(h|x|)} \leq C_1 e^{M(H^2 h|x|)} \leq C_1 e^{M(r|x|)}$. We obtain

$$\sum_{\alpha} \frac{M_{\alpha}^2}{(2L)^{2|\alpha|}} \left\|e^{M(h|\cdot|)} F_{\alpha}\right\|_{L^2}^2 \leq C_1 \sum_{\alpha} \frac{M_{\alpha}^2}{(2L)^{2|\alpha|}} |c_{\alpha}|^2 \left\|e^{M(r|\cdot|)} F\right\|_{L^{\infty}}^2 \left\|e^{-M(h|\cdot|)}\right\|_{L^2}^2 < \infty.$$

So, for the chosen $h > 0$, (3.19) holds with $m = 2L$. Since $T = \sum_{\alpha} D^{\alpha} F_{\alpha}$, by Theorem 3.4.1 we have $T \in \left(\mathcal{O}_C^{(M_p)}(\mathbb{R}^n)\right)'$. In the $\{M_p\}$ case there exist $r > 0$, an ultradifferential operator $P(D)$ of class $\{M_p\}$ and L^{∞} functions F_1 and F_2 such that $T = P(D)F_1 + F_2$ and $\left\|e^{M(r|\cdot|)}(F_1 + F_2)\right\|_{L^{\infty}} \leq C$. For simplicity, we assume that $F_2 = 0$ and put $F = F_1$. The general case is proved analogously. Since $P(D) = \sum_{\alpha} c_{\alpha} D^{\alpha}$ is of class $\{M_p\}$ for every $L > 0$ there exists $c > 0$ such that $|c_{\alpha}| \leq cL^{|\alpha|}/M_{\alpha}$. Put $F_{\alpha} = c_{\alpha} F$. Take $h \leq r/H^2$. Let $m > 0$ be arbitrary but fixed. Then there exists $c > 0$ such that $|c_{\alpha}| \leq cm^{|\alpha|}/(2^{|\alpha|} M_{\alpha})$. Similarly as above

$$\sum_{\alpha} M_{\alpha}^2 m^{-2|\alpha|} \left\|e^{M(h|\cdot|)} F_{\alpha}\right\|_{L^2}^2 < \infty.$$

Since $T = \sum_{\alpha} D^{\alpha} F_{\alpha}$, by Theorem 3.4.1 we have $T \in \left(\mathcal{O}_C^{\{M_p\}}(\mathbb{R}^n)\right)'$. □

By this Proposition, from now we will denote the dual of $\mathcal{O}_C^{*}(\mathbb{R}^n)$ by $\mathcal{O}_C'^{*}(\mathbb{R}^n)$.

Chapter 4

Translation-invariant spaces of ultradistributions

In this chapter we obtain analogs of the results in Chapter 1(see also [21]) for ultradistributions. Note that new difficulties arise, mainly due to the different structure of the spaces used in the rest of the thesis such as $\mathcal{D}^*(\mathbb{R}^n), \mathcal{D}'^*(\mathbb{R}^n), \mathcal{S}^*(\mathbb{R}^n), \mathcal{S}'^*(\mathbb{R}^n)$ e.t.c., e specially in $\{M_p\}$ case. Also, there are less results available in the literature concerning ultradistributions than those from the classical theory of distributions that we can use. As expected, the proofs and technics used here are different and more complex. We mention that we always assume that the sequence (M_p) satisfies the assumptions $(M1)$, $(M2)$, and $(M3)$ (see Section 0.2).

4.1 Translation-invariant Banach spaces of tempered ultradistributions

We start by defining translation-invariant Banach spaces of ultradistributions.

Definition 4.1.1. A Banach space E is said to be a *translation-invariant Banach space of tempered ultradistributions* of class $*$ if it satisfies the following three axioms:

(I) $\mathcal{D}^*(\mathbb{R}^n) \hookrightarrow E \hookrightarrow \mathcal{D}'^*(\mathbb{R}^n)$.

(II) $T_h : E \to E$ for each $h \in \mathbb{R}^n$.

(III) For any $g \in E$ there exist $C = C_g > 0$ and $\tau = \tau_g > 0$, resp. for every $\tau > 0$ there exists $C = C_{g,\tau} > 0$, such that $\|T_h g\|_E \leq C e^{M(\tau|h|)}$, $\forall h \in \mathbb{R}^n$.

As in the distribution case (Remark 1.1.1), note that (I) and (II) imply the continuity of the operators $T_h : E \to E$ for each $h \in \mathbb{R}^n$. The weight function of E is the function $\omega : \mathbb{R}^n \to (0, \infty)$ given by $\omega(h) := \|T_{-h}\|_{\mathcal{L}(E)}$.

Throughout the rest of the chapter we assume that E is a translation-invariant Banach space of tempered ultradistributions. It is clear that $\omega(0) = 1$ and that $\log \omega$ is a subadditive function. We will prove that ω is measurable and locally bounded; this allows us to associate to E the Beurling algebra L^1_ω, i.e., the Banach algebra of measurable functions u such that $\|u\|_{1,\omega} := \int_{\mathbb{R}^n} |u(x)|\,\omega(x)dx < \infty$. The set of sequences $\mathfrak{R}$ and the associated function N_{r_p} for the sequence $N_p = M_p \prod_{i=0}^{|p|} r_i$, where $(r_p) \in \mathfrak{R}$, were introduced in Subsection 0.2.1.

The next theorem collects a number of important properties of E.

Theorem 4.1.1. *The following property hold for E and ω:*

(a) $\mathcal{S}^*(\mathbb{R}^n) \hookrightarrow E \hookrightarrow \mathcal{S}'^*(\mathbb{R}^n)$.

(b) For each $g \in E$, $\lim_{h\to 0} \|T_h g - g\|_E = 0$ (hence the mapping $h \mapsto T_h g$ is continuous).

(c) There are $\tau, C > 0$, resp. for every $\tau > 0$ there is $C > 0$, such that

$$\omega(h) \leq Ce^{M(\tau|h|)}, \quad \forall h \in \mathbb{R}^n.$$

(d) E is separable and ω is measurable.

(e) The convolution mapping $* : \mathcal{S}^*(\mathbb{R}^n) \times \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$ extends to $* : L^1_\omega \times E \to E$ and E becomes a Banach module over the Beurling algebra L^1_ω, i.e.,

$$\|u * g\|_E \leq \|u\|_{1,\omega} \|g\|_E. \tag{4.1}$$

Furthermore, the bilinear mapping $* : \mathcal{S}^*(\mathbb{R}^n) \times E \to E$ is continuous.

(f) Let $g \in E$ and $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$. Set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ and $c = \int_{\mathbb{R}^n} \varphi(x)dx$. Then, $\lim_{\varepsilon\to 0^+} \|cg - \varphi_\varepsilon * g\|_E = 0$.

Alternatively, in the $\{M_p\}$ case, the property (c) is equivalent to:

($\tilde{c}$) there exist $(l_p) \in \mathfrak{R}$ and $C > 0$ such that $\omega(h) \leq Ce^{N_{l_p}(|h|)}$, $\forall h \in \mathbb{R}^n$.

Proof. The property (b) follows directly from the axioms (I)–(III). For (d), notice that (I) yields at once the separability of E. On the other hand, if D is a countable and dense subset of the unit ball of E, we have $\omega(h) = \sup_{g\in D} \|T_{-h} g\|_E$, and so (b) yields the measurability of ω.

We now show (c). In the (M_p) case, consider the sets

$$E_{j,\nu} = \{g \in E \,|\, \|T_h g\|_E \leq je^{M(\nu|h|)}, \forall h \in \mathbb{R}^n\},\ j, \nu \in \mathbb{Z}_+.$$

Because of (III), $E = \bigcup_{j,\nu\in\mathbb{Z}_+} E_{j,\nu}$. Since $E_{j,\nu} = \bigcap_{h\in\mathbb{R}^n} E_{j,\nu,h}$, where $E_{j,\nu,h} = \left\{g \in E \,|\, \|T_h g\|_E \leq je^{M(\nu|h|)}\right\}$ and each of these sets is closed in E by the continuity of T_h, so are $E_{j,\nu}$. Now, a classical category argument gives the claim. In the $\{M_p\}$ case, for fixed $\tau > 0$, consider the sets

$$E_j = \left\{g \in E \,|\, \|T_h g\|_E \leq je^{M(\tau|h|)} \text{ for all } h \in \mathbb{R}^n\right\}, j \in \mathbb{Z}_+.$$

Obviously $E = \bigcup_{j\in\mathbb{Z}_+} E_j$. Again the Baire Category Theorem yields the claim.

Before proving that (c) is equivalent to $(\tilde{c})$ we state the following Lemma.

Lemma 4.1.1. *([85]) Let $g : [0,\infty) \to [0,\infty)$ be an increasing function that satisfies the following estimate: for every $L > 0$ there exists $C > 0$ such that $g(\rho) \leq M(L\rho) + \ln C$. Then, there exists a subordinate function $\epsilon(\rho)$ such that $g(\rho) \leq M(\epsilon(\rho)) + \ln C'$, for some constant $C' > 1$.*

Obviously $(\tilde{c}) \Rightarrow (c)$. Conversely, define $F : [0,\infty) \to [0,\infty)$ as

$$F(\rho) = \sup_{|h|\leq\rho} \sup_{\|g\|_E\leq 1} \ln_+ \|T_h g\|_E.$$

One easily verifies that $F(\rho)$ is increasing and satisfies the conditions of Lemma 4.1.1. Hence there exists an subordinate function $\epsilon(\rho)$ and $C' > 1$ such that $F(\rho) \leq M(\epsilon(\rho)) + \ln C'$. Hence we obtain $\sup_{\|g\|_E\leq 1} \|T_h g\|_E \leq C' e^{M(\epsilon(|h|))}$. Now, [49, Lemma 3.12] implies that there exists a sequence $\tilde{N}_p$ which satisfies $(M.1)$ such that $M(\epsilon(\rho)) \leq \tilde{N}(\rho)$ as $\dfrac{\tilde{N}_p M_{p-1}}{\tilde{N}_{p-1} M_p} \to \infty$ as $p \to \infty$. Set $l'_p = \dfrac{\tilde{N}_p M_{p-1}}{\tilde{N}_{p-1} M_p}$. Take $(l_p) \in \mathfrak{R}$ such that $l_p \leq l'_p$, for all $p \in \mathbb{Z}_+$. Then

$$\sup_{\|g\|_E\leq 1} \|T_h g\|_E \leq C' e^{\tilde{N}(|h|)} = C' \sup_{p\in\mathbb{N}} \frac{|h|^p}{M_p \prod_{j=1}^p l'_j} \leq C' \sup_{p\in\mathbb{N}} \frac{|h|^p}{M_p \prod_{j=1}^p l_j} = C' e^{N_{l_p}(|h|)},$$

whence $(\tilde{c})$ follows.

We now address the property (a). We first prove the embedding $\mathcal{S}^*(\mathbb{R}^n) \hookrightarrow E$. Since $\mathcal{D}^*(\mathbb{R}^n) \hookrightarrow \mathcal{S}^*(\mathbb{R}^n)$, it is enough to prove that $\mathcal{S}^*(\mathbb{R}^n)$ is continuously injected into E. Let $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$. We use a special partition of unity:

$$1 = \sum_{m\in\mathbb{Z}^n} \psi(x-m), \quad \psi \in \mathcal{D}^*_{[-1,1]^n}$$

and we get the representation $\varphi(x) = \sum_{m\in\mathbb{Z}^n} \psi(x-m)\varphi(x)$. We estimate each term in this sum. Because of (c), there exist constants $C > 0$ and $\tau > 0$, resp. for every $\tau > 0$ there exists $C > 0$, such that:

$$\|\varphi\, T_{-m}\psi\|_E \leq \frac{C}{e^{M(\tau|m|)}} \left\| e^{2M(\tau|m|)} \psi T_m \varphi \right\|_E . \tag{4.2}$$

We need to prove that the multi-sequence of operators $\{\rho_m\}_{m\in\mathbb{Z}^n} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{D}^*_{[-1,1]^n}$, defined as

$$\rho_m(\varphi) := e^{2M(\tau|m|)} \psi T_m \varphi, \tag{4.3}$$

is uniformly bounded. Let B be bounded set in $\mathcal{S}^*(\mathbb{R}^n)$. Then for each $h > 0$, resp. for some $h > 0$,

$$\sup_{\varphi\in B} \sup_{\alpha\in\mathbb{N}^n} \frac{h^{|\alpha|} \left\| e^{M(h|\cdot|)} D^\alpha \varphi \right\|_{L^\infty(\mathbb{R}^n)}}{M_\alpha} < \infty. \tag{4.4}$$

By [49, Lemma 3.6] we have $e^{2M(\tau|m|)} \leq c_0 e^{M(H\tau|m|)}$ and hence

$$e^{2M(\tau|m|)} \leq 2c_0 e^{M(2H\tau|m+x|)} e^{M(2H\tau|x|)} \leq C_1 e^{M(2H\tau|m+x|)}, \tag{4.5}$$

for all $x \in [-1,1]^n$ and for all $m \in \mathbb{Z}^n$. In the (M_p) case let $h_1 > 0$ be arbitrary but fixed. Choose $h > 0$ such that $h \geq 2h_1$ and $h \geq 2H\tau$. For this h, (4.4) holds and by (4.5) and the fact $\psi \in \mathcal{D}^{(M_p)}_{[-1,1]^n}$, one easily verifies

$$\begin{aligned} &\frac{h_1^{|\alpha|}\left|D^\alpha\left(\psi(x)T_m\varphi(x)\right)\right|}{M_\alpha} \qquad (4.6)\\ &\leq \frac{1}{2^{|\alpha|}}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{h^{|\alpha|}\left|D^\beta\psi(x)\right|\left|D^{\alpha-\beta}\left(T_m\varphi(x)\right)\right|}{M_\alpha}\\ &\leq \frac{C_1}{2^{|\alpha|}}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{h^{|\beta|}\left|D^\beta\psi(x)\right|h^{|\alpha|-|\beta|}\left|D^{\alpha-\beta}\varphi(x+m)\right|e^{M(h|x+m|)}}{M_\beta M_{\alpha-\beta}e^{2M(\tau|m|)}} \leq \frac{C''}{e^{2M(\tau|m|)}}, \end{aligned}$$

for all $\varphi \in B$, $m \in \mathbb{Z}^n$. Hence $\{\rho_m | \, m \in \mathbb{Z}^n\}$ is uniformly bounded on B. In the $\{M_p\}$ case, there exist $\tilde{h}, \tilde{C} > 0$ such that $|D^\alpha\psi(x)| \leq \tilde{C}M_\alpha/\tilde{h}^{|\alpha|}$. For the h for which (4.4) holds choose $h_1 > 0$ such that $h_1 \leq \min\{h/2, \tilde{h}/2\}$ and choose $\tau \leq h/(2H)$. Then, by using (4.5), similarly as in the (M_p) case, we obtain (4.6), i.e., $\{\rho_m | \, m \in \mathbb{Z}^n\}$ is uniformly bounded. By (I), the mapping $\mathcal{D}^*_{[-1,1]^n} \to E$ is continuous, hence $\|\rho_m(\varphi)\|_E \leq C_2$, for all $\varphi \in B$, $m \in \mathbb{Z}^n$.

In view of (4.2) and the later fact, we have that $\left\{\sum_{|m|\leq N}\varphi T_{-m}\psi\right\}_{N=0}^{\infty}$ is a Cauchy sequence in E whose limit is $\varphi \in E$; one also obtains $\|\varphi\|_E \leq C$ for all $\varphi \in B$. We proved that the inclusion $\mathcal{S}^*(\mathbb{R}^n) \to E$ maps bounded sets into bounded and, since $\mathcal{S}^*(\mathbb{R}^n)$ is bornological, it is continuous.

We now address $E \subseteq \mathcal{S}'^*(\mathbb{R}^n)$ and the continuity of the inclusion mapping. Let $g \in E$. We employ Proposition 3.3.1. Let B be a bounded set in $\mathcal{D}^*(\mathbb{R}^n)$. The inclusion $E \hookrightarrow \mathcal{D}'^*(\mathbb{R}^n)$ yields the existence of a constant $D = D(B)$ such that $\left|\langle g, \check{\phi}\rangle\right| \leq D\|g\|_E$ for all $g \in E$ and $\phi \in B$. Therefore, by (c), there exist $\tau, C > 0$, resp. for every $\tau > 0$ there exists $C > 0$, such that

$$|(g*\phi)(h)| \leq D\|T_h g\|_E \leq CD\|g\|_E e^{M(\tau|h|)},$$

for all $g \in E$, $\phi \in B$, $h \in \mathbb{R}^n$. In the (M_p) case, Proposition 3.3.1 implies that $E \subseteq \mathcal{S}'^{(M_p)}(\mathbb{R}^n)$. In the $\{M_p\}$ case, the property $(\tilde{c})$, together with Proposition 3.3.1, implies $E \subseteq \mathcal{S}'^{\{M_p\}}(\mathbb{R}^n)$. Since $E \to \mathcal{D}'^*(\mathbb{R}^n)$ is continuous it has a closed graph, hence so does the inclusion $E \to \mathcal{S}'^*(\mathbb{R}^n)$ ($\mathcal{S}'^*(\mathbb{R}^n)$ is continuously injected into $\mathcal{D}'^*(\mathbb{R}^n)$). Since $\mathcal{S}'^*(\mathbb{R}^n)$ is a (DFS)-space, resp. (FS)-space, it is Ptak space (cf. [91, Sect. 8]). Thus, the continuity of $E \to \mathcal{S}'^*(\mathbb{R}^n)$ follows from the Ptak Closed Graph Theorem (cf. [91, Thm 8.5, p. 166]). The proof of (a) is complete.

We now show that E is a Banach modulo over L^1_ω. Let $\varphi, \psi \in \mathcal{D}^*(\mathbb{R}^n)$ and denote $K = \operatorname{supp}\varphi$. We prove that

$$\|\varphi * \psi\|_E \leq \|\psi\|_E \int_{\mathbb{R}^n} |\varphi(x)|\,\omega(x)dx. \qquad (4.7)$$

The Riemann sums

$$L_\varepsilon(\cdot) = \varepsilon^n \sum_{n\in\mathbb{Z}^n,\,\varepsilon n\in K}\varphi(\varepsilon n)\psi(\cdot - \varepsilon n) = \varepsilon^n \sum_{n\in\mathbb{Z}^n,\,\varepsilon n\in K}\varphi(\varepsilon n)T_{-\varepsilon n}\psi$$

converge to $\varphi * \psi$ in $\mathcal{S}^*(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. By (a) they also converge in E to the same element, i.e., $L_\varepsilon \to \varphi * \psi$ as $\varepsilon \to 0^+$ in E. Set $\omega_\psi(t) = \|T_{-t}\psi\|_E$. Then ω_ψ is continuous by (b). Observe that

$$\|L_\varepsilon\|_E \leq \sum_{y\in\mathbb{Z}^n,\, \varepsilon y\in K} |\varphi(\varepsilon y)| \|T_{-\varepsilon y}\psi\|_E \varepsilon^n = \sum_{y\in\mathbb{Z}^n,\, \varepsilon y\in K} |\varphi(\varepsilon y)| \omega_\psi(\varepsilon y)\varepsilon^n \tag{4.8}$$

and the last term converges to $\int_K |\varphi(y)|\omega_\psi(y)dy$. Since $\omega_\psi(t) = \|T_{-t}\psi\|_E \leq \|\psi\|_E\omega(t)$, if we let $\varepsilon \to 0^+$ in (4.8) we obtain (4.7). Using (I) and a standard density argument, the convolution can be extended to $* : L^1_\omega \times E \to E$ and (4.7) leads (4.1). The continuity of the convolution as a bilinear mapping $\mathcal{S}^*(\mathbb{R}^n) \times E \to E$ in the (M_p) case is an easy consequence of (4.1). In the $\{M_p\}$ case, we can conclude separate continuity from (4.1), but then, [104, Thm 41.1, p. 421] implies the desired continuity. This shows (e).

The proof of (f). We first consider the case when $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ and $g \in \mathcal{S}^*(\mathbb{R}^n)$. Then $g * \varphi \in \mathcal{S}^*(\mathbb{R}^n) \subseteq E$. For $\varepsilon < 1$, similarly as in Theorem 4.1.1 (e) we have

$$\begin{aligned} \|cg - \varphi_\varepsilon * g\|_E &= \left\| \int_{\operatorname{supp}\varphi_\varepsilon} (g - T_{-y}g)\frac{1}{\varepsilon^n}\varphi\left(\frac{y}{\varepsilon}\right) dy \right\|_E \\ &\leq \sup_{t\in\operatorname{supp}\varphi} \|g - T_{-\varepsilon t}g\|_E \int_{\operatorname{supp}\varphi} |\varphi(t)|\, dt, \end{aligned}$$

which proves the corollary in this case. Due to the density of $\mathcal{S}^*(\mathbb{R}^n) \hookrightarrow E$, the above inequality remains true for $g \in E$. Indeed, let $g_n \to g$ in E and $g_n \in \mathcal{S}^*(\mathbb{R}^n)$. Then, using the estimate (4.7) we get that $\varphi_\varepsilon * g_n \to \varphi * g$ in E, which proves the corollary when $g \in E$ and $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$.

In the general case, let $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$ and let $\{\psi_j\}_{j=1}^\infty \in \mathcal{D}^*(\mathbb{R}^n)$ be a sequence such that $\psi_j \to \varphi$ in $\mathcal{S}^*(\mathbb{R}^n)$. By Proposition 3.3.1 there exist $C > 0$ and $\tau > 0$, resp. for every $\tau > 0$ there exist $C > 0$, such that for $0 < \varepsilon < 1$, we have

$$\|(\psi_j)_\varepsilon * g - (\psi_k)_\varepsilon * g\|_E \leq C\|g\|_E \int_{\mathbb{R}^n} \exp(M(\tau|x|))|\psi_j(x) - \psi_k(x)|dx.$$

In the (M_p) case, it is easy to see that this inequality implies that $(\psi_j)_\varepsilon * g$ is a Cauchy sequence in E hence it must converge. In the $\{M_p\}$ case, there exists $m > 0$ such that $\psi_j \to \varphi$ in $\mathcal{S}^{(M_p),m}_\infty$, hence by choosing appropriate $\tau > 0$ we obtain that $(\psi_j)_\varepsilon * g$ is a Cauchy sequence in E. We obtain that $(\psi_j)_\varepsilon * g$ must converge in E. Observe that the limit in E of $(\psi_j)_\varepsilon * g$ must be $\varphi_\varepsilon * g$ since this sequence converges to $\varphi_\varepsilon * g$ in $\mathcal{S}'^*(\mathbb{R}^n)$ and $E \hookrightarrow \mathcal{S}'^*(\mathbb{R}^n)$. Now, the claim in the lemma easily follows.

□

Remark 4.1.1. To produce another proof of the continuity of the inclusion $E \subseteq \mathcal{S}'^*(\mathbb{R}^n)$, it is enough to prove that the unit ball E is weakly bounded in $\mathcal{S}'^*(\mathbb{R}^n)$. Indeed, a Banach space is bornological, so the continuity of the inclusion would follow if we show that the unit ball of E is bounded in $\mathcal{S}'^*(\mathbb{R}^n)$. But bounded subsets of $\mathcal{S}'^*(\mathbb{R}^n)$ are the same for the weak and strong topology, since $\mathcal{S}^*(\mathbb{R}^n)$ is barrelled. Fix $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$ and write $\varphi(x) = \sum_{m\in\mathbb{Z}^n} \psi(x-m)\varphi(x)$, where ψ is the partition of unity used above. Again, $\rho_m \in \mathcal{D}^*_{[-1,1]^n}$ as in (4.3). Taking (4.7) into account and the fact that $B = \{\check{\rho}_m(\varphi) |\, m \in \mathbb{Z}^n\}$ is a bounded subset of $\mathcal{D}^*(\mathbb{R}^n)$, we obtain, for all $g \in E$,

$$|\langle g, \varphi\rangle| \leq C \lim_{N\to\infty} \sum_{|m|\leq N} |(g * \check{\rho}_m(\varphi))(m)|\, e^{-2M(\tau|m|)}$$
$$\leq C'\|g\|_E \lim_{N\to\infty} \sum_{|m|\leq N} e^{-M(\tau|m|)} \leq C''\|g\|_E,$$

which completes the proof.

As done in (e), one can also extend the convolution as a mapping $* : E \times L^1_\omega \to E$ and obviously $u * g = g * u$.

We now discuss some properties that automatically transfer to the dual space E' by duality.

Proposition 4.1.1. *The space E' satisfies*

$(a)''$ $\mathcal{S}^*(\mathbb{R}^n) \to E' \hookrightarrow \mathcal{S}'^*(\mathbb{R}^n)$*, with continuous imbeddings.*

$(b)''$ *The mappings $\mathbb{R}^n \to E'$ given by $h \mapsto T_h f$ are continuous for the weak* topology.*

Proof. It follows from (a) that $\mathcal{S}^*(\mathbb{R}^n) \to E' \hookrightarrow \mathcal{S}'^*(\mathbb{R}^n)$. Let $f \in E'$ and $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$. Then there exist constants $C > 0$ and $\tau > 0$, resp. for every $\tau > 0$ there exists $C > 0$, such that

$$|\langle T_h f, \varphi\rangle| = |\langle f, T_{-h}\varphi\rangle| \leq \omega(h)||f||_{E'}||\varphi||_E \leq C\|f\|_{E'}\|\varphi\|_E \exp(M(\tau|h|)).$$

Since $\mathcal{S}^*(\mathbb{R}^n)$ is dense in E, $T_h f \in E'$ and the same inequality holds for $\varphi \in E$. Moreover, there exist constants $C > 0$ and $\tau > 0$, resp. for every $\tau > 0$ there exists $C > 0$, such that $\|T_h f\|_{E'} \leq C\|f\|_{E'} \exp(M(\tau|h|))$. On the other hand, by (b) applied to E, $\lim_{h\to h_0} \langle T_h f - T_{h_0} f, g\rangle = \left\langle f, \lim_{h\to h_0} (T_{-h} g - T_{-h_0} g)\right\rangle = 0$, for each $g \in E$. □

The condition (II) from Definition 4.1.1 remains valid for E'. We define the weight function of E' as

$$\check{\omega}(h) := \|T_{-h}\|_{\mathcal{L}(E')} = \|T_h^\top\|_{\mathcal{L}(E')} = \omega(-h),$$

where one of the equalities follows from the well known Bipolar Theorem (cf. [91, p. 160]). Thus (c) and $(\tilde{c})$ from Theorem 4.1.1 hold for the weight function $\check{\omega}$ of E'. In particular, the axiom (III) holds for E'. In general, however, E' may fail to be an translation-invariant Banach space of tempered ultradistributions because (I) may not be any longer true for it. Note also that E' can be non-separable. In addition, the property (b) from Theorem 4.1.1 may also fail for E'. The associated

Beurling algebra to E' is $L^1_{\check{\omega}}$. We define the convolution $u * f = f * u$ of $f \in E'$ and $u \in L^1_{\check{\omega}}$ via transposition:

$$\langle u * f, g\rangle := \langle f, \check{u} * g\rangle \,, \; g \in E.$$

In view of (e) from Theorem 4.1.1, this convolution is well defined because $\check{u} \in L^1_{\omega}$. It readily follows that (e) holds when E and ω are replaced by E' and $\check{\omega}$; so E' is a Banach module over the Beurling algebra $L^1_{\check{\omega}}$, i.e.,

$$\|u * f\|_{E'} \leq \|u\|_{1,\check{\omega}} \|f\|_{E'}.$$

Concerning the property (f) from Theorem 4.1.1, it may not be any longer satisfied by E' (for instance consider $E = L^1(\mathbb{R}^n)$). But, when E is a reflexive space then E' inheres the properties $(I), (II)$ and (III).

Summing up, E' might not be as rich as E. We introduce the following space that enjoys better properties than E' with respect to the translation group.

Definition 4.1.2. The Banach space E'_* stands for $E' = L^1_{\check{\omega}} * E'$.

Note that E'_* is a closed linear subspace of E', due to the Cohen-Hewitt Factorization Theorem [47] and the fact that $L^1_{\check{\omega}}$ possesses bounded approximation unities. The ensuing theorem shows that E'_* possesses many of the properties that E' lacks. It also gives a characterization of E'_* and tells us that the property (I) holds for E' when E is reflexive.

Theorem 4.1.2. *The Banach space E'_* satisfies:*

(i) $\mathcal{S}^*(\mathbb{R}^n) \to E'_* \to \mathcal{S}'^*(\mathbb{R}^n)$ *and E'_* a is Banach modulo over $L^1_{\check{\omega}}$.*

(ii) The properties (II) from Definition 4.1.1 and (b) and (f) from Theorem 4.1.1 are valid when E is replaced by E'_*.

(iii) $E'_* = \left\{ f \in E' | \lim_{h \to 0} \|T_h f - f\|_{E'} = 0 \right\}.$

(iv) If E is reflexive, then $E'_* = E'$ and E' is also an translation-invariant Banach space of tempered ultradistributions.

Proof. Call momentarily $X = \{f \in E' : \lim_{h\to 0} \|T_h f - f\|_{E'} = 0\}$. Since X is closed subspace of E' it is enough to show that $\mathcal{S}^*(\mathbb{R}^n) * E'$ is dense in X. For this, we will show that if $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ is positive and $\int_y \varphi(y)dy = 1$, then $\lim_{\varepsilon \to 0^+} \|f * \varphi_\varepsilon - f\|_{E'} = 0$, for each $f \in X$. We apply a similar argument to that used in the proof of (f) in Proposition 4.1.1. Let $f \in X$ and take $\phi \in \mathcal{D}^*(\mathbb{R}^n)$. For $0 < \varepsilon < 1$, we have

$$\begin{aligned} &|\langle f * \varphi_\varepsilon - f, \phi\rangle| \\ &= \left| \left\langle f(t), \int_{\mathbb{R}^n} \varphi(y)\psi(t + \varepsilon y)dy - \phi(t) \right\rangle \right| = |\langle f(t) \otimes 1_y, \varphi(y)\left(\phi(t+\varepsilon y) - \phi(t)\right)\rangle| \\ &\leq \int_{\mathbb{R}^n} \varphi(y) \left|\langle T_{-\varepsilon y} f - f, \phi\rangle\right| dy \leq \|\phi\|_E \sup_{y \in \operatorname{supp} \varphi} \|T_{-\varepsilon y} f - f\|_{E'}, \end{aligned}$$

which shows the claim. Except for the inclusion $\mathcal{S}^*(\mathbb{R}^n) \subseteq E'_*$, the rest of the assertions can be proved in exactly the same way as for the distribution case; we therefore omit details and refer to Section 1.1. To show the inclusion $\mathcal{S}^*(\mathbb{R}^n) \subseteq E'_*$, note $\mathcal{S}^*(\mathbb{R}^n) = \overline{\operatorname{span}(\mathcal{S}^*(\mathbb{R}^n) * \mathcal{S}^*(\mathbb{R}^n))}$ (this follows easily by using an approximation of the unity). Hence $\mathcal{S}^*(\mathbb{R}^n)$ is a subset of the closure of $\operatorname{span}(\mathcal{S}^*(\mathbb{R}^n) * \mathcal{S}^*(\mathbb{R}^n))$ in E', and so the inclusion $\mathcal{S}^*(\mathbb{R}^n) \subseteq E'_*$ must hold. □

It is worth noticing that E' carries another useful convolution structure. In fact, we can define the convolution mapping $* : E' \times \check{E} \to L^\infty_\omega$ by

$$(f * g)(x) = \langle f(t), g(x-t)\rangle = \langle f(t), T_{-x}\check{g}(t)\rangle,$$

where $\check{E} = \{g \in \mathcal{S}'^*(\mathbb{R}^n) |\, \check{g} \in E\}$ with norm $\|g\|_{\check{E}} := \|\check{g}\|_E$ and L^∞_ω is the dual of the Beurling algebra L^1_ω, i.e., the Banach space of all measurable functions satisfying $\|u\|_{\infty,\omega} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |g(x)|/\omega(x) < \infty$. As in Section 1.1 we consider the following two closed subspaces of L^∞_ω:

$$UC_\omega = \left\{u \in L^\infty_\omega \middle|\, \lim_{h\to 0} \|T_h u - u\|_{\infty,\omega} = 0\right\}$$
$$C_\omega = \left\{u \in C(\mathbb{R}^n) \middle|\, \lim_{|x|\to\infty} \frac{u(x)}{\omega(x)} = 0\right\}.$$

The first part of the next proposition is a direct consequence of (b) from Theorem 4.1.1. The range refinement in the reflexive case follows from the density of $\mathcal{S}^*(\mathbb{R}^n)$ in E' (part (iv) of Theorem 4.1.2).

Proposition 4.1.2. *$E' * \check{E} \subseteq UC_\omega$ and $* : E' \times \check{E} \to UC_\omega$ is continuous. If E is reflexive, then $E' * \check{E} \subseteq C_\omega$.*

4.2 The test function space $\mathcal{D}^*_E$

We begin by constructing our test function space $\mathcal{D}^*_E$, where E is translation-invariant Banach space of tempered ultradistribution. Let $m > 0$ and

$$\mathcal{D}^{\{M_p\},m}_E = \left\{\varphi \in E \middle|\, D^\alpha\varphi \in E, \forall \alpha \in \mathbb{N}^n, \|\varphi\|_{E,m} = \sup_{\alpha\in\mathbb{N}^n} \frac{m^\alpha \|D^\alpha\varphi\|_E}{M_\alpha} < \infty\right\}.$$

The space $\mathcal{D}^{\{M_p\},m}_E$ is a Banach space. One easily verifies that none of these spaces is trivial. To see this in the (M_p) case one only needs to use the continuity of the inclusion $\mathcal{S}^{(M_p)} \to E$, to obtain that $\mathcal{S}^{(M_p)} \subseteq \mathcal{D}^{\{M_p\},m}_E$ for each $m > 0$. In the $\{M_p\}$ case observe that $\mathcal{S}^{(M_p)}$ is continuously injected into $\mathcal{S}^{\{M_p\}}$, hence we have the continuous inclusions $\mathcal{S}^{(M_p)} \to E$. Now, similarly one proves that $\mathcal{S}^{(M_p)} \subseteq \mathcal{D}^{\{M_p\},m}_E$ for each $m > 0$. Also, $\mathcal{D}^{\{M_p\},m_1}_E \subseteq \mathcal{D}^{\{M_p\},m_2}_E$ for $m_2 < m_1$ with continuous inclusion mapping. As l.c.s. we define

$$\mathcal{D}^{(M_p)}_E = \varprojlim_{m\to\infty} \mathcal{D}^{\{M_p\},m}_E, \ \ \mathcal{D}^{\{M_p\}}_E = \varinjlim_{m\to 0} \mathcal{D}^{\{M_p\},m}_E.$$

Since $\mathcal{D}_E^{\{M_p\},m}$ is continuously injected in E for each $m > 0$, $\mathcal{D}_E^{\{M_p\}}$ is indeed a (Hausdorff) l.c.s. Moreover $\mathcal{D}_E^{\{M_p\}}$ is barreled, bornological (DF)-space as an inductive limit of Banach spaces. Obviously $\mathcal{D}_E^{(M_p)}$ is a Fréchet space. Of course $\mathcal{D}_E^{(M_p)}$, resp. $\mathcal{D}_E^{\{M_p\}}$, is continuously injected into E.

Additionally, in the $\{M_p\}$ case, for each fixed $(r_p) \in \mathfrak{R}$ we define the Banach space

$$\mathcal{D}_E^{\{M_p\},(r_p)} = \left\{\varphi \in E \,\middle|\, D^\alpha\varphi \in E, \forall \alpha \in \mathbb{N}^n, \|\varphi\|_{E,(r_p)} = \sup_\alpha \frac{\|D^\alpha\varphi\|_E}{M_\alpha \prod_{j=1}^{|\alpha|} r_j} < \infty\right\},$$

with norm $\|\cdot\|_{E,(r_p)}$. Since for $k > 0$ and $(r_p) \in \mathfrak{R}$, there exists $C > 0$ such that $k^{|\alpha|} \geq C/\left(\prod_{j=1}^{|\alpha|} r_j\right)$, $\mathcal{D}_E^{\{M_p\},k}$ is continuously injected into $\mathcal{D}_E^{\{M_p\},(r_p)}$. Define as l.c.s.

$$\tilde{\mathcal{D}}_E^{\{M_p\}} = \varprojlim_{(r_p)\in\mathfrak{R}} \mathcal{D}_E^{\{M_p\},(r_p)}.$$

Then $\tilde{\mathcal{D}}_E^{\{M_p\}}$ is complete l.c.s. and $\mathcal{D}_E^{\{M_p\}}$ is continuously injected into it.

Proposition 4.2.1. *The space $\mathcal{D}_E^{\{M_p\}}$ is regular, i.e., every bounded set B in $\mathcal{D}_E^{\{M_p\}}$ is bounded in some $\mathcal{D}_E^{\{M_p\},m}$. In addition $\mathcal{D}_E^{\{M_p\}}$ is complete.*

Proof. For $(r_p) \in \mathfrak{R}$ denote by R_α the product $\prod_{j=1}^{|\alpha|} r_j$. Let B be a bounded set in $\mathcal{D}_E^{\{M_p\}}$. Then B is bounded in $\tilde{\mathcal{D}}_E^{\{M_p\}}$, hence for each $(r_p) \in \mathfrak{R}$ there exists $C_{(r_p)} > 0$ such that $\sup_\alpha \frac{\|D^\alpha\varphi\|_E}{R_\alpha M_\alpha} \leq C_{(r_p)}$, for all $\varphi \in B$. By Lemma 3.4 of [51] we obtain that there exist $m, C_2 > 0$ such that $\sup_\alpha \frac{m^{|\alpha|}\|D^\alpha\varphi\|_E}{M_\alpha} \leq C_2$, $\forall \varphi \in B$, which proves the regularity of $\mathcal{D}_E^{\{M_p\}}$.

It remains to prove the completeness. Since $\mathcal{D}_E^{\{M_p\}}$ is a (DF)-space it is enough to prove that it is quasi-complete (see [57, p. 402, Theorem 3]). Let φ_ν be a bounded Cauchy net in $\mathcal{D}_E^{\{M_p\}}$. Hence there exist $m, C > 0$ such that $\|\varphi_\nu\|_{E,m} \leq C$ and since the inclusions $\mathcal{D}_E^{\{M_p\}} \to \mathcal{D}_E^{\{M_p\},(r_p)}$ are continuous it follows that φ_ν is a Cauchy net in $\mathcal{D}_E^{\{M_p\},(r_p)}$ for each $(r_p) \in \mathfrak{R}$. It is obvious that without losing generality we can assume that $m \leq 1$. Fix $m_1 < m$. Let $\varepsilon > 0$. There exists $p_0 \in \mathbb{Z}_+$ such that $(m_1/m)^p \leq \varepsilon/(2C)$ for all $p \geq p_0$, $p \in \mathbb{N}$. Let $r_p = p$. Obviously $(r_p) \in \mathfrak{R}$. Since φ_ν is a Cauchy net in $\mathcal{D}_E^{\{M_p\},(r_p)}$, there exists ν_0 such that for all $\nu, \lambda \geq \nu_0$ we have $\|\varphi_\nu - \varphi_\lambda\|_{E,(r_p)} \leq \varepsilon/(p_0!)$. Hence, for $|\alpha| < p_0$

$$\frac{m_1^{|\alpha|}\|D^\alpha\varphi_\nu - D^\alpha\varphi_\lambda\|_E}{M_\alpha} \leq \frac{\|D^\alpha\varphi_\nu - D^\alpha\varphi_\lambda\|_E}{M_\alpha} \leq \varepsilon$$

and for $|\alpha| \geq p_0$

$$\frac{m_1^{|\alpha|}\|D^\alpha\varphi_\nu - D^\alpha\varphi_\lambda\|_E}{M_\alpha} \leq 2C\left(\frac{m_1}{m}\right)^{|\alpha|} \leq \varepsilon.$$

We obtain that for $\nu,\lambda\geq\nu_0$, $\|\varphi_\nu-\varphi_\lambda\|_{E,m_1}\leq\varepsilon$, i.e., φ_ν is a Cauchy net in the Banach space $\mathcal{D}_E^{\{M_p\},m_1}$, hence it converges to $\varphi\in\mathcal{D}_E^{\{M_p\},m_1}$ in it and thus also in $\mathcal{D}_E^{\{M_p\}}$. □

Similarly as in the first part of the proof of this proposition one can prove, by using Lemma 3.4 of [51], that $\mathcal{D}_E^{\{M_p\}}$ and $\tilde{\mathcal{D}}_E^{\{M_p\}}$ are equal as sets, i.e., the canonical inclusion $\mathcal{D}_E^{\{M_p\}}\to\tilde{\mathcal{D}}_E^{\{M_p\}}$ is surjective.

Proposition 4.2.2. *The following dense inclusions hold $\mathcal{S}^*(\mathbb{R}^n)\hookrightarrow\mathcal{D}^*_E\hookrightarrow E\hookrightarrow\mathcal{S}'^*(\mathbb{R}^n)$ and $\mathcal{D}^*_E$ is a topological module over the Beurling algebra L^1_ω, i.e., the convolution $*:L^1_\omega\times\mathcal{D}^*_E\to\mathcal{D}^*_E$ is continuous. Moreover in the (M_p) case the following estimate*

$$\|u*\varphi\|_{E,m}\leq\|u\|_{1,\omega}\|\varphi\|_{E,m},\ \ m>0 \tag{4.9}$$

holds. In the $\{M_p\}$ case, for each $m>0$ the convolution is also continuous bilinear mapping $L^1_\omega\times\mathcal{D}_E^{\{M_p\},m}\to\mathcal{D}_E^{\{M_p\},m}$ and the inequality (4.9) holds.

Proof. Clearly $\mathcal{D}^*_E$ is continuously injected in E. We will consider the $\{M_p\}$ case. We will prove that for every $h>0$, $\mathcal{S}_\infty^{\{M_p\},h}(\mathbb{R}^n)$ is continuously injected into $\mathcal{D}_E^{\{M_p\},h/H}$. From this it readily follows that $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ is continuously injected into $\mathcal{D}_E^{\{M_p\}}$. Denote by σ_h the norm in $\mathcal{S}_\infty^{\{M_p\},h}(\mathbb{R}^n)$ (see [12]). Since $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)\to E$, it follows that $\mathcal{S}_\infty^{\{M_p\},h/H}(\mathbb{R}^n)\to E$. Hence there exists $C_1>0$ such that $\|\varphi\|_E\leq C_1\sigma_{h/H}(\varphi)$, $\forall\varphi\in\mathcal{S}_\infty^{\{M_p\},h/H}(\mathbb{R}^n)$. Let $\psi\in\mathcal{S}_\infty^{\{M_p\},h}(\mathbb{R}^n)$. It is easy to verify that for every $\beta\in\mathbb{N}^n$, $D^\beta\psi\in\mathcal{S}_\infty^{\{M_p\},h/H}(\mathbb{R}^n)$. We have

$$\begin{aligned}\frac{h^{|\alpha|}\left\|D^\alpha\psi\right\|_E}{H^{|\alpha|}M_\alpha}&\leq C_1\frac{h^{|\alpha|}}{H^{|\alpha|}M_\alpha}\sup_\beta\frac{h^{|\beta|}\left\|e^{M(\frac{h}{H}|\cdot|)}D^{\alpha+\beta}\psi\right\|_{L^\infty(\mathbb{R}^n)}}{H^{|\beta|}M_\beta}\\&\leq c_0C_1\sup_\beta\frac{h^{|\alpha|+|\beta|}\left\|e^{M(h|\cdot|)}D^{\alpha+\beta}\psi\right\|_{L^\infty(\mathbb{R}^n)}}{M_{\alpha+\beta}}\leq c_0C_1\sigma_h(\psi),\end{aligned}$$

which proofs the continuity of the inclusion $\mathcal{S}_\infty^{\{M_p\},h}(\mathbb{R}^n)\to\mathcal{D}_E^{\{M_p\},h/H}$. The proof that $\mathcal{S}^{(M_p)}(\mathbb{R}^n)$ is continuously injected into $\mathcal{D}_E^{(M_p)}$ is similar and we omit it. We have shown that $\mathcal{S}^*(\mathbb{R}^n)\to\mathcal{D}^*_E\hookrightarrow E\hookrightarrow\mathcal{S}'^*(\mathbb{R}^n)$.
To prove that $\mathcal{D}^*_E$ is a module over the Beurling algebra L^1_ω we first consider the (M_p) case. For $u\in\mathcal{D}^{(M_p)}(\mathbb{R}^n)$, $\varphi\in\mathcal{D}_E^{(M_p)}$ and $m>0$ we have

$$\frac{m^{|\gamma|}}{M_\gamma}\left\|D^\gamma(u*\varphi)\right\|_E=\left\|u*\frac{m^{|\gamma|}}{M_\gamma}D^\gamma\varphi\right\|_E\leq\|u\|_{1,\omega}\|\varphi\|_{E,m}.$$

By density argument, the same inequality holds for $u\in L^1_\omega$ and $\varphi\in\mathcal{D}_E^{(M_p)}$. After taking supremum over $\gamma\in\mathbb{N}^n$ we obtain (4.9). In the $\{M_p\}$ case, by similar calculation as above we again obtain (4.9) for $\varphi\in\mathcal{D}_E^{\{M_p\},m}$ and $u\in L^1_\omega$. Hence the convolution is continuous bilinear mapping $L^1_\omega\times\mathcal{D}_E^{\{M_p\},m}\to\mathcal{D}_E^{\{M_p\},m}$. From this we obtain that the convolution is separately continuous mapping $L^1_\omega\times\mathcal{D}_E^{\{M_p\}}\to\mathcal{D}_E^{\{M_p\}}$

and since L^1_ω and $\mathcal{D}_E^{\{M_p\}}$ are barreled (DF)-spaces it follows that it is continuous.

It remains to prove the density of the injection $\mathcal{S}^*(\mathbb{R}^n) \hookrightarrow \mathcal{D}_E^*$. Let $\varphi \in \mathcal{D}_E^*$. Pick then $\phi \in \mathcal{D}^*(\mathbb{R}^n)$ with support in the unit ball of $\mathbb{R}^n$ with center at the origin such that $\phi(x) \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$ and set $\phi_j(x) = j^n\phi(jx)$. We consider the $\{M_p\}$ case, the (M_p) case is similar. There exists $m > 0$ such that $\phi, \varphi \in \mathcal{D}_E^{\{M_p\},m}$ and $|D^\alpha\phi(x)| \leq \tilde{C}M_\alpha/m^{|\alpha|}$, for some $\tilde{C} > 0$. Let $0 < m_1 < m$ be arbitrary but fixed. We will prove that

$$\|\varphi - \varphi * \phi_j\|_{E,m_1} \to 0.$$

Let $\varepsilon > 0$. Observe that there exists $C_1 \geq 1$ such that $\|\phi_j\|_{1,\omega} \leq C_1$, $\forall j \in \mathbb{Z}_+$ and $\|\phi\|_{1,\omega} \leq C_1$. Choose $p_0 \in \mathbb{Z}_+$ such that $(m_1/m)^p \leq \varepsilon/(2C_2)$ for all $p \geq p_0$, $p \in \mathbb{N}$, where $C_2 = C_1(1 + \|\varphi\|_{E,m}) \geq 1$. By (f) of Theorem 4.1.1 we can choose $j_0 \in \mathbb{Z}_+$ such that $\frac{m_1^{|\alpha|}}{M_\alpha}\left\|D^\alpha\varphi - D^\alpha\varphi * \phi_j\right\|_E \leq \varepsilon$ for all $|\alpha| \leq p_0$ and all $j \geq j_0$, $j \in \mathbb{N}$. Observe that if $|\alpha| \geq p_0$ we have

$$\begin{aligned} \frac{m_1^{|\alpha|}}{M_\alpha}\left\|D^\alpha\varphi - D^\alpha\varphi * \phi_j\right\|_E &\leq \frac{m_1^{|\alpha|}}{M_\alpha}\left\|D^\alpha\varphi\right\|_E + \frac{m_1^{|\alpha|}}{M_\alpha}\left\|D^\alpha\varphi\right\|_E\|\phi_j\|_{1,\omega} \\ &\leq \left(\frac{m_1}{m}\right)^{|\alpha|}\|\varphi\|_{E,m} + C_1\left(\frac{m_1}{m}\right)^{|\alpha|}\|\varphi\|_{E,m} \leq \varepsilon. \end{aligned}$$

Hence, for $j \geq j_0$, $\|\varphi - \varphi*\phi_j\|_{E,m_1} \leq \varepsilon$, so $\varphi*\phi_j \to \varphi$ in $\mathcal{D}_E^{\{M_p\},m_1}$ and consequently also in $\mathcal{D}_E^{\{M_p\}}$.

Let V neighborhood of 0 in $\mathcal{D}_E^{\{M_p\}}$. Choose a neighborhood W of 0 in $\mathcal{D}_E^{\{M_p\}}$ such that $W + W \subseteq V$. Then $W_{m_1} = W \cap \mathcal{D}_E^{\{M_p\},m_1}$ is a neighborhood of 0 in $\mathcal{D}_E^{\{M_p\},m_1}$, hence there exists $j_1 \in \mathbb{Z}_+$ such that $\varphi * \phi_{j_1} - \varphi \in W_{m_1} \subseteq W$. Choose $m_2 > 0$ such that $m_2 < m_1/j_1$. Then $W_{m_2} = W \cap \mathcal{D}_E^{\{M_p\},m_2}$ is a neighborhood of 0 in $\mathcal{D}_E^{\{M_p\},m_2}$. So there exists $\varepsilon > 0$ such that $\left\{\chi \in \mathcal{D}_E^{\{M_p\},m_2}\middle|\, \|\chi\|_{E,m_2} \leq \varepsilon\right\} \subseteq W_{m_2}$. Since $j_1m_2 < m$, $|D^\alpha\phi(x)| \leq \tilde{C}M_\alpha/(j_1m_2)^{|\alpha|}$. Pick $\psi \in \mathcal{S}^{\{M_p\}}$ such that $\|\varphi - \psi\|_E \leq \varepsilon/(\tilde{C}C')$ where $C' = \sup_{j\in\mathbb{Z}_+}\int_{|x|\leq 1}\omega(x/j)dx$ which is finite by the growth estimate for ω. Now we have

$$\begin{aligned} \frac{m_2^{|\alpha|}}{M_\alpha}\|(\varphi - \psi) * D^\alpha\phi_{j_1}\|_E &\leq \|\varphi - \psi\|_E\int_{\mathbb{R}^n}\frac{j_1^d(j_1m_2)^{|\alpha|}}{M_\alpha}\left|D^\alpha\phi(j_1x)\right|\omega(x)dx \\ &\leq \tilde{C}\|\varphi - \psi\|_E\int_{|x|\leq 1}\omega(x/j_1)dx \leq \varepsilon. \end{aligned}$$

We obtain that $\psi * \phi_{j_1} - \varphi * \phi_{j_1} \in W_{m_2} \subseteq W$. Hence $\psi * \phi_{j_1} - \varphi = \psi * \phi_{j_1} - \varphi * \phi_{j_1} + \varphi * \phi_{j_1} - \varphi \in W + W \subseteq V$. Since $\psi * \phi_j \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ we conclude that $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $\mathcal{D}_E^{\{M_p\}}$. □

Lemma 4.2.1. *If $P(D)$ is ultradifferential operator of $*$ type, then $P(D) : \mathcal{D}_E^* \to \mathcal{D}_E^*$ is continuous.*

Proof. Let $P(D)=\sum_{\alpha\in\mathbb{N}^d}c_\alpha D^\alpha$. There exist $h,C'>0$, resp. for every $h>0$ there exists $C'>0$, such that $|c_\alpha|\leq C'h^{|\alpha|}/M_\alpha$. For $\varphi\in\mathcal{D}^{(M_p)}$ we have

$$\begin{aligned}\frac{m^{|\beta|}}{M_\beta}\|D^\beta P(D)\varphi\|_E &\leq \frac{m^{|\beta|}}{M_\beta}\sum_{\alpha\in\mathbb{N}^d}|c_\alpha|\|D^{\alpha+\beta}\varphi\|_E\\ &\leq C_1\sum_{\alpha\in\mathbb{N}^d}\frac{(H\max\{h,m\})^{|\alpha|+|\beta|}}{M_{\alpha+\beta}}\|D^{\alpha+\beta}\varphi\|_E\\ &\leq C_2\|\varphi\|_{E,\{M_p\},2H\max\{m,h\}}.\end{aligned}$$

In the $\{M_p\}$ case, similarly as above one proves that $P(D)$ is continuous as a mapping from $\mathcal{D}_E^{\{M_p\},2Hm}\to\mathcal{D}_E^{\{M_p\},m}$. □

In order to prove that ultradifferential operators of $\{M_p\}$ class act continuously on $\tilde{\mathcal{D}}_E^{\{M_p\}}$, we need the following technical result from [85].

Let $(k_p)\in\mathfrak{R}$. There exists $(k'_p)\in\mathfrak{R}$ such that $k'_p\leq k_p$ and

$$\prod_{j=1}^{p+q}k'_j\leq 2^{p+q}\prod_{j=1}^{p}k'_j\cdot\prod_{j=1}^{q}k'_j,\ \text{for all } p,q\in\mathbb{Z}_+. \tag{4.10}$$

Proposition 4.2.3. *Every ultradifferential operator of $\{M_p\}$ class acts continuously on $\tilde{\mathcal{D}}_E^{\{M_p\}}$.*

Proof. Since $P(D)=\sum_\alpha c_\alpha D^\alpha$ is of $\{M_p\}$ class for every $L>0$ there exists $C>0$ such that $|c_\alpha|\leq CL^{|\alpha|}/M_\alpha$. Lemma 3.4 of [51] implies that there exist $(r_p)\in\mathfrak{R}$ and $C_1>0$ such that $|c_\alpha|\leq C_1/\left(M_\alpha\prod_{j=1}^{|\alpha|}r_j\right)$. Let $(l_p)\in\mathfrak{R}$ be arbitrary but fixed. Define $k_p=\min\{r_p,l_p\}$, $p\in\mathbb{Z}_+$. Then $(k_p)\in\mathfrak{R}$ and for this (k_p) take $(k'_p)\in\mathfrak{R}$ as in (4.10). Then, one can prove that there exist $C'>0$ and $H>0$ such that $\|P(D)\varphi\|_{E,(l_p)}\leq C'\|\varphi\|_{E,(k'_p/(4H))}$ for all $\varphi\in\tilde{\mathcal{D}}_E^{\{M_p\}}$, which proves the continuity of $P(D)$. □

In fact all elements of our test space $\mathcal{D}^*_E$ are ultradifferentiable functions of class $*$. To establish this fact we need the following lemma.

Lemma 4.2.2. *Let $K\subseteq\mathbb{R}^n$ be compact. There exists $m>0$, resp. there exists $(l_p)\in\mathfrak{R}$, such that $\mathcal{D}_{K,m}^{\{M_p\}}\subseteq E\cap E'_*$, resp. $\mathcal{D}_{K,(l_p)}^{\{M_p\}}\subseteq E\cap E'_*$. Moreover, the inclusion mappings $\mathcal{D}_{K,m}^{\{M_p\}}\to E$ and $\mathcal{D}_{K,m}^{\{M_p\}}\to E'_*$, resp. $\mathcal{D}_{K,(l_p)}^{\{M_p\}}\to E$ and $\mathcal{D}_{K,(l_p)}^{\{M_p\}}\to E'_*$, are continuous.*

Proof. We will give the proof in the Roumieu case, the Beurling case is similar. Let U be a bounded open subset of $\mathbb{R}^n$ such that $K\subset\subset U$ and put $K_1=\overline{U}$. Since the inclusion $\mathcal{D}_{K_1}^{\{M_p\}}\to E$ is continuous and $\mathcal{D}_{K_1}^{\{M_p\}}=\varprojlim_{(r_p)\in\mathfrak{R}}\mathcal{D}_{K_1,(r_p)}^{\{M_p\}}$ there exist $C>0$ and $(r_p)\in\mathfrak{R}$ such that

$$\|\varphi\|_E\leq C\|\varphi\|_{K_1,(r_p)}.$$

Let χ_m, $m \in \mathbb{Z}_+$, be a δ-sequence from $\mathcal{D}^{\{M_p\}}$ such that $\operatorname{diam}(\operatorname{supp}\chi_m) \leq \operatorname{dist}(K, \partial U)/2$, for $m \in \mathbb{Z}_+$. Take $l_p = r_{p-1}/(2H)$, $p \geq 2$ and $l_1 = r_1/(2H)$. Then $(l_p) \in \mathfrak{R}$. Let $\psi \in \mathcal{D}^{\{M_p\}}_{K,(l_p)}$. Then $\psi * \chi_m \in \mathcal{D}^{\{M_p\}}_{K_1}$ and one easily obtains that $\psi * \chi_m \to \psi$ in $\mathcal{D}^{\{M_p\}}_{K_1,(r_p)}$. We have $\|\psi * \chi_m\|_E \leq C\|\psi * \chi_m\|_{K_1,(r_p)}$, hence $\psi * \chi_m$ is a Cauchy sequence in E, so it converges. Since $\psi * \chi_m \to \psi$ in $\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ and E is continuously injected into $\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ the limit of $\psi * \chi_m$ in E must be ψ. If we let $m \to \infty$ in the last inequality we have $\|\psi\|_E \leq C\|\psi\|_{K_1,(r_p)}$. Observe that $\|\psi\|_{K_1,(r_p)} \leq \|\psi\|_{K,(l_p)}$ (since $\psi \in \mathcal{D}^{\{M_p\}}_{K,(l_p)}$, $\operatorname{supp}\psi \subseteq K$). Hence,

$$\|\psi\|_E \leq C\|\psi\|_{K,(l_p)},$$

which gives the desired continuity of the inclusion $\mathcal{D}^{\{M_p\}}_{K,(l_p)} \to E$. Similarly, one obtains the continuous inclusion $\mathcal{D}^{\{M_p\}}_{K,(l'_p)} \to E'_*$ possibly with another $(l'_p) \in \mathfrak{R}$.

The conclusion of the lemma now follows with $(\tilde{l}_p) \in \mathfrak{R}$ defined by $\tilde{l}_p = \min\{l_p, l'_p\}$, $p \in \mathbb{Z}_+$. □

Proposition 4.2.4. *The embedding* $\mathcal{D}^*_E \hookrightarrow \mathcal{O}^*_C(\mathbb{R}^n)$ *holds. Furthermore, for* $\varphi \in \mathcal{D}^*_E$, $D^\alpha\varphi \in C_{\check{\omega}}$ *for all* $\alpha \in \mathbb{N}^n$ *and they satisfy the following growth condition: for every* $m > 0$, *resp. for some* $m > 0$,

$$\sup_{\alpha\in\mathbb{N}^n} \frac{m^{|\alpha|}}{M_\alpha} \|D^\alpha\varphi\|_{L^\infty_{\check{\omega}}(\mathbb{R}^n)} < \infty. \tag{4.11}$$

Proof. Let U be the open unit ball in $\mathbb{R}^n$ with center at 0 and $K = \overline{U}$. Let $r > 0$, resp. $(r_p) \in \mathfrak{R}$ be as in Lemma 4.2.2, i.e.,

$$\mathcal{D}^{\{M_p\}}_{K,r} \subseteq E \cap E'_*, \text{ resp. } \mathcal{D}^{\{M_p\}}_{K,(r_p)} \subseteq E \cap E'_*$$

and the inclusion mappings

$$\mathcal{D}^{\{M_p\}}_{K,r} \to E \text{ and } \mathcal{D}^{\{M_p\}}_{K,r} \to E'_*, \text{ resp. } \mathcal{D}^{\{M_p\}}_{K,(r_p)} \to E \text{ and } \mathcal{D}^{\{M_p\}}_{K,(r_p)} \to E'_*,$$

are continuous. By the parametrix of Komatsu, there exist $u \in \mathcal{D}^{(M_p)}_{U,r}$, $\psi \in \mathcal{D}^{(M_p)}(U)$ and $P(D)$ of type (M_p), resp. $u \in \mathcal{D}^{\{M_p\}}_{U,(r_p)}$ such that $\dfrac{\|D^\alpha u\|_{L^\infty}}{R_\alpha M_\alpha} \to 0$ when $|\alpha| \to \infty$, $\psi \in \mathcal{D}^{\{M_p\}}(U)$ and $P(D)$ of type $\{M_p\}$, such that $P(D)u = \delta + \psi$.

Let $f \in \mathcal{D}^*_E$. Then $f = u * P(D)f - \psi * f$. Observe that $\psi * f \in \mathcal{E}^*(\mathbb{R}^n)$. For $\beta \in \mathbb{N}^n$, $D^\beta P(D)f \in \mathcal{D}^*_E$. By Proposition 4.2.2, $\check{u} \in \mathcal{D}^{\{M_p\}}_{K,(r_p)} \subseteq E'$ and so $u \in (E')\check{} = \check{E}'$. Hence, by the discussion before Proposition 4.1.2, all ultradistributional derivatives of $u * P(D)f$ are continuous functions on $\mathbb{R}^n$. From this we obtain that $u * P(D)f \in C^\infty(\mathbb{R}^n)$. Indeed, this result is local in nature, so it is enough to use the Sobolev imbedding theorem on an open disk V of arbitrary point $x \in \mathbb{R}^n$ and the fact that $\mathcal{D}^*(V)$ is dense in $\mathcal{D}(V)$. Hence $f \in C^\infty(\mathbb{R}^n)$. For $\beta \in \mathbb{N}^n$,

$$D^\beta f(x) = u * D^\beta P(D)f(x) - \psi * D^\beta f(x) = F_1(x) - F_2(x).$$

By the above discussion, the last equality and Proposition 4.1.2 it follows that $D^\beta f \in UC_{\check{\omega}}$.

To prove the inclusion $\mathcal{D}^*_E \to \mathcal{O}^*_C(\mathbb{R}^n)$, we consider first the (M_p) case. Let $m>0$ be arbitrary but fixed. Since $P(D)=\sum_\alpha c_\alpha D^\alpha$ is of (M_p) type, there exist $m_1, C'>0$ such that $|c_\alpha| \leq C' m_1^{|\alpha|}/M_\alpha$. Let $m_2 = 4\max\{m,m_1\}$. For F_1, Since $P(D)$ acts continuously on $\mathcal{D}^*_E$, we have

$$|F_1(x)| \leq \|u\|_{\check{E}'} \left\|D^\beta P(D) f(x)\right\|_E \omega(-x) \leq C_2 \omega(-x) \|\check{u}\|_{E'} \|f\|_{E,m_2H} \frac{M_\beta}{(2m)^{|\beta|}}$$

and similarly

$$|F_2(x)| \leq C_3 \omega(-x) \|\check{\psi}\|_{E'} \|f\|_{E,2m} \frac{M_\beta}{(2m)^{|\beta|}} \leq C_3 \omega(-x) \|\check{\psi}\|_{E'} \|f\|_{E,m_2H} \frac{M_\beta}{(2m)^{|\beta|}}.$$

Hence

$$\frac{(2m)^{|\beta|}\left|D^\beta f(x)\right|}{M_\beta w(-x)} \leq C''\left(\|\check{u}\|_{E'} + \|\check{\psi}\|_{E'}\right)\|f\|_{E,m_2H}. \tag{4.12}$$

Since there exist $\tau, C'''>0$ such that $\omega(x) \leq C''' e^{M(\tau|x|)}$, by using Proposition 3.6 of [49] we obtain $\omega(-x)e^{M(\tau|x|)} \leq C_4 e^{M(\tau H|x|)}$. Hence

$$\begin{aligned}\left(\sum_\alpha \frac{m^{2|\alpha|}}{M_\alpha^2}\left\|D^\alpha f e^{-M(\tau H|\cdot|)}\right\|^2_{L^2}\right)^{1/2} &\leq C_5 \left(\sum_\alpha \frac{m^{2|\alpha|}}{M_\alpha^2}\left\|\frac{D^\alpha f}{\omega(-\cdot)}\right\|^2_{L^\infty}\right)^{1/2}\\ &\leq C\left(\|\check{u}\|_{E'} + \|\check{\psi}\|_{E'}\right)\|f\|_{E,m_2H},\end{aligned}$$

which proves the continuity of the inclusion $\mathcal{D}^{(M_p)}_E \to \mathcal{O}^{(M_p)}_{C,\tau H}(\mathbb{R}^n)$ and hence also the continuity of the inclusion $\mathcal{D}^{(M_p)}_E \to \mathcal{O}^{(M_p)}_C(\mathbb{R}^n)$.

In order to prove that $\mathcal{D}^{\{M_p\}}_E \to \mathcal{O}^{\{M_p\}}_C(\mathbb{R}^n)$ is continuous inclusion it is enough to prove that for each $h>0$, $\mathcal{D}^{\{M_p\}}_E \to \mathcal{O}^{\{M_p\}}_{C,h}(\mathbb{R}^n)$ is continuous inclusion. Now, it is enough to prove that for every $m>0$ there exists $m'>0$ such that we have the continuous inclusion $\mathcal{D}^{\{M_p\},m}_E \to \mathcal{O}^{\{M_p\}}_{C,m',h}(\mathbb{R}^n)$. So, let $h,m>0$ be arbitrary but fixed. Take $m' \leq m/(4H)$. For $f \in \mathcal{D}^{\{M_p\},m}_E$, using the same technique as above, we have

$$\frac{(2m')^{|\beta|}\left|D^\beta f(x)\right|}{M_\beta w(-x)} \leq C''\left(\|\check{u}\|_{E'} + \|\check{\psi}\|_{E'}\right)\|f\|_{E,m}. \tag{4.13}$$

For the fixed h take $\tau>0$ such that $\tau H \leq h$. Then there exists $C'''>0$ such that $\omega(x) \leq C''' e^{M(\tau|x|)}$ and by using Proposition 3.6 of [49] we obtain $\omega(x)e^{M(\tau|x|)} \leq C_4 e^{M(\tau H|x|)}$. Similarly as above, we have

$$\left(\sum_\alpha \frac{m'^{2|\alpha|}}{M_\alpha^2}\left\|D^\alpha f e^{-M(h|\cdot|)}\right\|^2_{L^2}\right)^{1/2} \leq C\left(\|\check{u}\|_{E'} + \|\check{\psi}\|_{E'}\right)\|f\|_{E,m},$$

which proves the continuity of the inclusion $\mathcal{D}^{\{M_p\},m}_E \to \mathcal{O}^{\{M_p\}}_{C,m',h}(\mathbb{R}^n)$.

Observe that (4.11) follows by (4.12), resp. (4.13).

It remains to prove that $D^\alpha f \in C_{\check{\omega}}$. We will prove this in the $\{M_p\}$ case, the (M_p) case is similar. Using Proposition 4.2.3, with similar technique as above one can prove that for every $(k_p) \in \mathfrak{R}$ there exists $(l_p) \in \mathfrak{R}$ such that for $f \in \mathcal{D}_E^{\{M_p\}}$ we have

$$\frac{\left|D^\beta f(x)\right|}{w(-x) M_\beta \prod_{j=1}^{|\beta|} k_j} \leq C'' \left(\|\check{u}\|_{E'} + \|\check{\psi}\|_{E'}\right) \|f\|_{E,(l_p)}. \tag{4.14}$$

Let $\varepsilon > 0$. Since $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $\mathcal{D}_E^{\{M_p\}}$ (Proposition 4.2.2) it is dense in $\tilde{\mathcal{D}}_E^{\{M_p\}}$. Pick $\chi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ such that $\|f - \chi\|_{E,(l_p)} \leq \varepsilon / \left(C'' \left(\|\check{u}\|_{E'} + \|\check{\psi}\|_{E'}\right)\right)$. Then, by (4.14), for $x \in \mathbb{R}^n \backslash \operatorname{supp} \chi$ we have

$$\frac{\left|D^\beta f(x)\right|}{w(-x) M_\beta \prod_{j=1}^{|\beta|} k_j} = \frac{\left|D^\beta \left(f(x) - \chi(x)\right)\right|}{w(-x) M_\beta \prod_{j=1}^{|\beta|} k_j} \leq \varepsilon,$$

which proves that $D^\beta f \in C_{\check{\omega}}$. □

Remark 4.2.1. If $f \in \mathcal{S}^*(\mathbb{R}^n)$, by the proof of this proposition (and (4.1)) we have

$$\left\|D^\beta f\right\|_E \leq \|u\|_E \left\|D^\beta P(D) f\right\|_{1,\omega} + \|\psi\|_E \left\|D^\beta f\right\|_{1,\omega},$$

since $u, \psi \in E$ (by the way we obtained them). Also, one easily verifies that (cf. the proof of Proposition 4.2.3) for every $m > 0$ there exist $\tilde{m} > 0$ and $C_1 > 0$, resp. for every $(k_p) \in \mathfrak{R}$ there exist $(l_p) \in \mathfrak{R}$ and $C_1 > 0$, such that

$$\|f\|_{E,m} \leq C_1 \sup_\alpha \frac{\tilde{m}^{|\alpha|} \left\|D^\alpha f\right\|_{1,\omega}}{M_\alpha}, \text{ resp. } \|f\|_{E,(k_p)} \leq C_1 \sup_\alpha \frac{\left\|D^\alpha f\right\|_{1,\omega}}{M_\alpha \prod_{j=1}^{|\alpha|} l_j}. \tag{4.15}$$

4.3 The ultradistribution space $\mathcal{D}'^*_{E'_*}$

We denote by $\mathcal{D}'^*_{E'_*}$ the strong dual of $\mathcal{D}^*_E$. Then, $\mathcal{D}'^{(M_p)}_{E'_*}$ is a complete (DF)-space since $\mathcal{D}_E^{(M_p)}$ is a Fréchet space. Also, $\mathcal{D}'^{\{M_p\}}_{E'_*}$ is a Fréchet space as the strong dual of a (DF)-space. When E is reflexive, we write $\mathcal{D}'^*_{E'} = \mathcal{D}'^*_{E'_*}$ in accordance with the last assertion of Theorem 4.1.2. The notation $\mathcal{D}'^*_{E'_*} = (\mathcal{D}^*_E)'$ is motivated by the next structural theorem.

Theorem 4.3.1. */ Let $f \in \mathcal{D}'^*(\mathbb{R}^n)$. The following statements are equivalent:*

(i) $f \in \mathcal{D}'^*_{E'_*}$.

(ii) $f * \psi \in E'$ *for all* $\psi \in \mathcal{D}^*(\mathbb{R}^n)$.

(iii) $f * \psi \in E'_*$ *for all* $\psi \in \mathcal{D}^*(\mathbb{R}^n)$.

(iv) *f can be expressed as $f = P(D) g + g_1$, where $P(D)$ is ultradifferential operator of $*$ type with $g, g_1 \in E'$.*

(*v*) *There exist ultradifferential operators* $P_k(D)$ *of* $*$ *type and* $f_k \in E'_* \cap UC_\omega$ *for* k *in finite set* N *such that*

$$f = \sum_{k \in N} P_k(D) f_k. \tag{4.16}$$

Moreover, if E *is reflexive, we may choose* $f_k \in E' \cap C_\omega$.

Remark 4.3.1. One can replace $\mathcal{D}'^*(\mathbb{R}^n)$ and $\mathcal{D}^*(\mathbb{R}^n)$ by $\mathcal{S}'^*(\mathbb{R}^n)$ and $\mathcal{S}^*(\mathbb{R}^d)$ in the statement of Theorem 4.3.1.

Proof. We denote $B_E = \{\varphi \in \mathcal{D}^*(\mathbb{R}^n) |\, \|\varphi\|_E \leq 1\}$.

(*i*) $\Rightarrow$ (*ii*). Fix first $\psi \in \mathcal{D}^*(\mathbb{R}^n)$. By (4.1) the set $\check{\psi} * B_E = \{\check{\psi} * \varphi : \varphi \in B_E\}$ is bounded in $\mathcal{D}^*_E$. Hence, $|\langle f * \psi, \varphi\rangle| = |\langle f, \check{\psi} * \varphi\rangle| \leq C_\psi$ for $\varphi \in B_E$. So, $|\langle f * \psi, \varphi\rangle| \leq C_\psi \|\varphi\|_E$, for all $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$. Since $\mathcal{D}^*(\mathbb{R}^n)$ is dense in E, we obtain $f * \psi \in E'$, for each $\psi \in \mathcal{D}^*(\mathbb{R}^n)$.

(*ii*) $\Rightarrow$ (*iv*). Let Ω be a bounded open symmetric neighborhood of 0 in $\mathbb{R}^n$ and put $K = \overline{\Omega}$. For arbitrary but fixed $\psi \in \mathcal{D}^*_K$ we have $\langle f * \check{\varphi}, \check{\psi}\rangle = \langle f * \psi, \varphi\rangle$. We obtain that the set $\{\langle f * \check{\varphi}, \check{\psi}\rangle |\, \varphi \in B_E\}$ is bounded in $\mathbb{C}$, i.e., $\{f * \check{\varphi} |\, \varphi \in B_E\}$ is weakly bounded in $\mathcal{D}'^*_K$, hence it is equicontinuous. Using the same technique as in Proposition 3.3.1 we obtain that there exists $r > 0$, resp. there exists $(r_p) \in \mathfrak{R}$, such that for each $\rho \in \mathcal{D}^{(M_p)}_{\Omega,r}$, resp. for each $\rho \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$, there exists $C_\rho > 0$ such that $|\langle f * \rho, \varphi\rangle| \leq C_\rho$ for all $\varphi \in B_E$. The density of $\mathcal{D}^*(\mathbb{R}^n)$ in E implies that $f * \rho \in E'$ for each $\rho \in \mathcal{D}^{(M_p)}_{\Omega,r}$, resp. for each $\rho \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$. By the parametrix of Komatsu we obtain that there exist $u \in \mathcal{D}^{(M_p)}_{\Omega,r}$, $\psi \in \mathcal{D}^{(M_p)}(\Omega)$ and ultradifferential operator $P(D)$ of class (M_p), resp. there exist $u \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$, $\psi \in \mathcal{D}^{\{M_p\}}(\Omega)$ and ultradifferential operator $P(D)$ of class $\{M_p\}$, such that $f = P(D)(u * f) + \psi * f$. This gives the desired representation.

(*iv*) $\Rightarrow$ (*i*) is obvious.

(*ii*) $\Rightarrow$ (*v*). Proceed as in (*ii*) $\Rightarrow$ (*iv*) to obtain $f = P(D)(u*f) + \psi*f$ for some $u \in \mathcal{D}^{(M_p)}_{\Omega,r}$, $\psi \in \mathcal{D}^{(M_p)}(\Omega)$ and ultradifferential operator $P(D)$ of class (M_p), resp. some $u \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$, $\psi \in \mathcal{D}^{\{M_p\}}(\Omega)$ and ultradifferential operator $P(D)$ of class $\{M_p\}$. Moreover, by using Lemma 4.2.2, one can easily see from the proof of (*ii*) $\Rightarrow$ (*iv*) that we can choose r, resp. (r_p), such that $\mathcal{D}^{(M_p)}_{\Omega,r} \subseteq \check{E}$, resp. $\mathcal{D}^{\{M_p\}}_{\Omega,(r_p)} \subseteq \check{E}$. Because the composition of ultradifferential operators of class * is again ultradifferential operator of class *, Lemma 0.2.5, we obtain

$$\begin{aligned} f &= P(D)(u * (P(D)(u * f) + \psi * f)) + \psi * (P(D)(u * f) + \psi * f) \\ &= P(D)(P(D)(u * (u * f))) + P(D)(u * (\psi * f)) \\ &+ P(D)(\psi * (u * f)) + \psi * (\psi * f) \end{aligned}$$

and $u * (u * f), u * (\psi * f), \psi * (u * f), \psi * (\psi * f) \in E'_* \cap UC_\omega$ by the definition of E'_* and Proposition 4.1.2. If E is reflexive, all of these are in fact elements of C_ω by the same proposition.

(*v*) $\Rightarrow$ (*i*), (*iv*) $\Rightarrow$ (*iii*) and (*iii*) $\Rightarrow$ (*ii*) are obvious. $\square$

Proposition 4.3.1. *Let* $\mathbf{f} : \mathcal{D}^*(\mathbb{R}^n) \to \mathcal{D}'^*(\mathbb{R}^n)$ *be linear and continuous. The following statements are equivalent:*

(*i*) $\mathbf{f}$ *commutates with every translation, i.e.,* $\langle \mathbf{f}, T_{-h}\varphi\rangle = T_h \langle \mathbf{f}, \varphi\rangle$, *for all* $h \in \mathbb{R}^n$ *and* $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$.

(*ii*) $\mathbf{f}$ *commutates with every convolution, i.e.,* $\langle \mathbf{f}, \psi * \varphi\rangle = \check{\psi} * \langle \mathbf{f}, \varphi\rangle$, *for all* $\psi, \varphi \in \mathcal{D}^*(\mathbb{R}^n)$.

(*iii*) *There exists* $f \in \mathcal{D}'^*(\mathbb{R}^n)$ *such that* $\langle \mathbf{f}, \varphi\rangle = f * \check{\varphi}$ *for every* $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$.

Proof. (*i*) $\Rightarrow$ (*ii*) Let $\varphi, \psi \in \mathcal{D}^*(\mathbb{R}^n)$ and denote $K = \operatorname{supp}\psi$. Then the Riemann sums

$$L_\varepsilon(\cdot) = \sum_{y\in\mathbb{Z}^n,\, \varepsilon y\in K} \psi(\varepsilon y)\varphi(\cdot - \varepsilon y)\varepsilon^n = \sum_{y\in\mathbb{Z}^n,\, \varepsilon y\in K} \psi(\varepsilon y) T_{-\varepsilon y}\varphi \varepsilon^n$$

converge to $\psi * \varphi$ in $\mathcal{D}^*(\mathbb{R}^n)$, when $\varepsilon \to 0^+$. The continuity of $\mathbf{f}$ implies

$$\langle \mathbf{f}, \psi * \varphi\rangle = \lim_{\varepsilon\to 0^+} \sum_{y\in\mathbb{Z}^n,\, \varepsilon y\in K} \psi(\varepsilon y)\langle \mathbf{f}, T_{-\varepsilon y}\varphi\rangle \varepsilon^d = \lim_{\varepsilon\to 0^+} \sum_{y\in\mathbb{Z}^d,\, \varepsilon y\in K} \psi(\varepsilon y) T_{\varepsilon y}\langle \mathbf{f}, \varphi\rangle \varepsilon^d,$$

in $\mathcal{D}'^*(\mathbb{R}^n)$. Let $\chi \in \mathcal{D}^*(\mathbb{R}^n)$. Then

$$\left\langle \lim_{\varepsilon\to 0^+} \sum_{y\in\mathbb{Z}^n,\, \varepsilon y\in K} \psi(\varepsilon y) T_{\varepsilon y}\langle \mathbf{f}, \varphi\rangle \varepsilon^n, \chi \right\rangle = \langle\langle \mathbf{f}, \varphi\rangle, \psi * \chi\rangle = \langle \check{\psi} * \langle \mathbf{f}, \varphi\rangle, \chi\rangle.$$

(*ii*) $\Rightarrow$ (*iii*). Let Ω be an arbitrary symmetric bounded open neighborhood of 0 in $\mathbb{R}^n$ and put $K = \overline{\Omega}$. Take $\delta_m \in \mathcal{D}^*(\mathbb{R}^n)$ as in the proof of Proposition 3.3.1. For every $\psi \in \mathcal{D}^*(\mathbb{R}^n)$ we have that $\psi * \delta_m \to \psi$ in $\mathcal{D}^*(\mathbb{R}^n)$ when $m \to \infty$. Also,

$$\check{\psi} * \langle \mathbf{f}, \delta_m\rangle = \langle \mathbf{f}, \psi * \delta_m\rangle \to \langle \mathbf{f}, \psi\rangle \text{ when } m \to \infty. \tag{4.17}$$

First we will prove that the set

$$\{\langle \mathbf{f}, \delta_m\rangle | m \in \mathbb{Z}_+\}$$

is equicontinuous subset of $\mathcal{D}'^*(\mathbb{R}^n)$, or equivalently, bounded in $\mathcal{D}'^*(\mathbb{R}^n)$ (since $\mathcal{D}^*(\mathbb{R}^n)$ is barreled). By (4.17), for each fixed $\psi \in \mathcal{D}^*(\mathbb{R}^n)$, the set $\{\psi * \langle \mathbf{f}, \delta_m\rangle | m \in \mathbb{Z}_+\}$ is bounded in $\mathcal{D}'^*(\mathbb{R}^n)$. Denote by T_m the bilinear mapping

$$\begin{aligned} T_m : \mathcal{D}^*_K \times \mathcal{D}^*_K &\to C(K) \\ (\varphi, \psi) &\mapsto \langle \mathbf{f}, \delta_m\rangle * \varphi * \psi|_K \end{aligned}$$

For fixed $\psi \in \mathcal{D}^*_K$, the mappings $T_{m,\psi}$ defined by $\varphi \mapsto \langle \mathbf{f}, \delta_m\rangle * \varphi * \psi|_K$, $\mathcal{D}^*_K \to C(K)$ are linear and continuous and the set $\{T_{m,\psi} | m \in \mathbb{Z}_+\}$ is pointwise bounded in $\mathcal{L}(\mathcal{D}^*_K, C(K))$. Since $\mathcal{D}^*_K$ is barreled, this set is equicontinuous. Similarly, for each fixed $\varphi \in \mathcal{D}^*_K$, the mappings $\psi \mapsto \langle \mathbf{f}, \delta_m\rangle * \varphi * \psi|_K$, $\mathcal{D}^*_K \to C(K)$ form an equicontinuous subset of $\mathcal{L}(\mathcal{D}^*_K, C(K))$. We obtain that the set of bilinear mappings $\{T_m | m \in \mathbb{Z}_+\}$ is separately equicontinuous and since $\mathcal{D}^{(M_p)}_K$ is a Fréchet

space, resp. $\mathcal{D}_K^{\{M_p\}}$ is a barrelled (DF)-space, it is equicontinuous ([57, Thm. 2]) for the case of Fréchet spaces and ([57, Thm. 11]) for the case of barreled (DF)-spaces).
We will continue the proof considering only the $\{M_p\}$ case, the (M_p) case is similar. By the equicontinuity of the mappings T_m, $m \in \mathbb{Z}_+$, there exist $C > 0$ and $(k_p) \in \mathfrak{R}$ such that for all $\varphi, \psi \in \mathcal{D}_K^{\{M_p\}}$, $m \in \mathbb{Z}_+$, we have

$$\|T_m(\varphi, \psi)\|_{L^\infty(K)} \leq C\|\varphi\|_{K,(k_p)}\|\psi\|_{K,(k_p)}.$$

Let $r_p = k_{p-1}/H$, for $p \in \mathbb{N}$, $p \geq 2$ and put $r_1 = \min\{1, r_2\}$. Then $(r_p) \in \mathfrak{R}$. For $\chi \in \mathcal{D}_{\Omega,(r_p)}^{\{M_p\}}$, for large enough j, $\chi * \delta_j \in \mathcal{D}_K^{\{M_p\}}$ and by similar technique as in the proof of Proposition 3.3.1 one can prove that $\chi * \delta_j \to \chi$ in $\mathcal{D}_{K,(k_p)}^{\{M_p\}}$, where $\delta_j \in \mathcal{D}^*(\mathbb{R}^n)$, $j \in \mathbb{Z}_+$, is the same sequence used in the proof of Proposition 3.3.1. Let $\varphi, \psi \in \mathcal{D}_{\Omega,(r_p)}^{\{M_p\}}$ and put $\varphi_j = \varphi * \delta_j$, $\psi_j = \psi * \delta_j$. Since

$$\begin{aligned}
&\|T_m(\varphi_j, \psi_j) - T_m(\varphi_s, \psi_s)\|_{L^\infty(K)}\\
&\qquad \leq \|T_m(\varphi_j, \psi_j - \psi_s)\|_{L^\infty(K)} + \|T_m(\varphi_j - \varphi_s, \psi_s)\|_{L^\infty(K)}\\
&\qquad \leq C\left(\|\varphi_j\|_{K,(k_p)}\|\psi_j - \psi_s\|_{K,(k_p)} + \|\varphi_j - \varphi_s\|_{K,(k_p)}\|\psi_s\|_{K,(k_p)}\right),
\end{aligned}$$

it follows that for each fixed m, $T_m(\varphi_j, \psi_j)$ is a Cauchy sequence in $C(K)$, hence it must converge.

On the other hand, $\langle \mathbf{f}, \delta_m\rangle * \varphi_j * \psi_j \to \langle \mathbf{f}, \delta_m\rangle * \varphi * \psi$ in $\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ and since $C(K)$ is continuously injected into $\mathcal{D}_K'^{\{M_p\}}$ it follows that $T_m(\varphi_j, \psi_j)$ converges to $\langle \mathbf{f}, \delta_m\rangle * \varphi * \psi|_K$ in $\mathcal{D}_K'^{\{M_p\}}$ (here the restriction to K is in fact the transposed mapping of the inclusion $\mathcal{D}_K^{\{M_p\}} \to \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$). Thus, $T_m(\varphi_j, \psi_j) \to \langle \mathbf{f}, \delta_m\rangle * \varphi * \psi|_K$ in $C(K)$. By arbitrariness of $\varphi, \psi \in \mathcal{D}_{\Omega,(r_p)}^{\{M_p\}}$ and by passing to the limit in the inequality $\|T_m(\varphi_j, \psi_j)\|_{L^\infty(K)} \leq C\|\varphi_j\|_{K,(k_p)}\|\psi_j\|_{K,(k_p)}$, we have

$$\|\langle \mathbf{f}, \delta_m\rangle * \varphi * \psi|_K\|_{L^\infty(K)} \leq C\|\varphi\|_{K,(k_p)}\|\psi\|_{K,(k_p)}$$

for all $m \in \mathbb{Z}_+$, $\varphi, \psi \in \mathcal{D}_{\Omega,(r_p)}^{\{M_p\}}$. For the fixed $(r_p) \in \mathfrak{R}$, by the parametrix of Komatsu, there exist ultradifferential operator $P(D)$ of class $\{M_p\}$, $u \in \mathcal{D}_{\Omega,(r_p)}^{\{M_p\}}$ and $\psi \in \mathcal{D}^{\{M_p\}}(\Omega)$ such that $\langle \mathbf{f}, \delta_m\rangle = P(D)\left(\langle \mathbf{f}, \delta_m\rangle * u\right) + \langle \mathbf{f}, \delta_m\rangle * \psi$. Applying again the parametrix we have

$$\langle \mathbf{f}, \delta_m\rangle = P(D)P(D)\left(\langle \mathbf{f}, \delta_m\rangle * u * u\right) + 2P(D)\left(\langle \mathbf{f}, \delta_m\rangle * \psi * u\right) + \langle \mathbf{f}, \delta_m\rangle * \psi * \psi.$$

Since each of the sets $\{\langle \mathbf{f}, \delta_m\rangle * u * u|_K \,|\, m \in \mathbb{Z}_+\}$, $\{\langle \mathbf{f}, \delta_m\rangle * \psi * u|_K \,|\, m \in \mathbb{Z}_+\}$ and $\{\langle \mathbf{f}, \delta_m\rangle * \psi * \psi|_K \,|\, m \in \mathbb{Z}_+\}$ is bounded in $\mathcal{D}_K'^{\{M_p\}}$ hence also in $\mathcal{D}'^{\{M_p\}}(\Omega)$ we obtain that $\{\langle \mathbf{f}, \delta_m\rangle|_\Omega \,|\, m \in \mathbb{N}\}$ is bounded in $\mathcal{D}'^{\{M_p\}}(\Omega)$. By the arbitrariness of Ω it follows that this set is bounded in $\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$. Hence it is relatively compact ($\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ is Montel), thus there exists subsequence $\langle \mathbf{f}, \delta_{m_s}\rangle$ which converges to f in $\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$. Since $\langle \mathbf{f}, \delta_{m_s} * \chi\rangle = \langle \mathbf{f}, \delta_{m_s}\rangle * \check{\chi}$ for each $\chi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, after passing to the limit we have $\langle \mathbf{f}, \chi\rangle = f * \check{\chi}$.

$(iii) \Rightarrow (i)$ is obvious. $\square$

We also have the following interesting corollary.

Corollary 4.3.1. *Let* $\mathbf{f} \in \mathcal{D}'^{*}(\mathbb{R}^n, E'_{\sigma(E',E)})$*, that is, a continuous linear mapping* $\mathbf{f} : \mathcal{D}^{*}(\mathbb{R}^n) \to E'_{\sigma(E',E)}$*. If* $\mathbf{f}$ *commutes with every translation in sense of Proposition 4.3.1 then there exists* $f \in \mathcal{D}'^{*}_{E'_*}$ *such that* $\mathbf{f}$ *is of the form*

$$\langle \mathbf{f}, \varphi \rangle = f * \check{\varphi}, \quad \varphi \in \mathcal{D}^{*}(\mathbb{R}^n). \tag{4.18}$$

Proof. Since $E'_{\sigma(E',E)} \to \mathcal{D}'^{*}_{\sigma}(\mathbb{R}^n)$ is continuous, $\mathbf{f} : \mathcal{D}^{*}(\mathbb{R}^n) \to \mathcal{D}'^{*}_{\sigma}(\mathbb{R}^n)$ is also continuous. For B be bounded in $\mathcal{D}^{*}(\mathbb{R}^n)$, $\mathbf{f}(B)$ is bounded in $\mathcal{D}'^{*}_{\sigma}(\mathbb{R}^n)$ and hence bounded in $\mathcal{D}'^{*}(\mathbb{R}^n)$. Since $\mathcal{D}^{*}(\mathbb{R}^n)$ is bornological, $\mathbf{f} : \mathcal{D}^{*}(\mathbb{R}^n) \to \mathcal{D}'^{*}(\mathbb{R}^n)$ is continuous. Now the claim follows from Proposition 4.3.1 and Theorem 4.3.1. □

If F is complete l.c.s. we define $\mathcal{S}'^{*}(\mathbb{R}^n, F) = \mathcal{S}'^{*}(\mathbb{R}^n)\varepsilon F$ and since $\mathcal{S}'^{*}(\mathbb{R}^n)$ is nuclear it satisfies the weak approximation property we obtain $\mathcal{L}_b(\mathcal{S}^{*}(\mathbb{R}^n), F) \cong \mathcal{S}'^{*}(\mathbb{R}^n)\varepsilon F \cong \mathcal{S}'^{*}(\mathbb{R}^n)\hat{\otimes} F$ (for the definition of the ε tensor product, the definition of the weak approximation property and their connection we refer to [93] and [51]).

Our results from above implicitly suggest to embed the ultradistribution space $\mathcal{D}'^{*}_{E'_*}$ into the space of E'-valued tempered ultradistributions as follows. Define first the continuous injection

$$\iota : \mathcal{S}'^{*}(\mathbb{R}^n) \to \mathcal{S}'^{*}(\mathbb{R}^n, \mathcal{S}'^{*}(\mathbb{R}^n)) \text{ , where } \iota(f) = \mathbf{f}$$

is given by (4.18). Observe the restriction of ι to $\mathcal{D}'^{*}_{E'_*}$,

$$\iota : \mathcal{D}'^{*}_{E'_*} \to \mathcal{S}'^{*}(\mathbb{R}^n, E')$$

(the range of ι is subset of $\mathcal{S}'^{*}(\mathbb{R}^n, E')$ by Theorem 4.3.1 and the remark after it).

Let B_1 be arbitrary bounded subset of $\mathcal{S}^{*}(\mathbb{R}^n)$. The set $B = \{\psi * \varphi | \varphi \in B_1, \|\psi\|_E \leq 1\}$ is bounded in $\mathcal{D}^{*}_{E}$ (by (e) of Theorem 4.1.1). For $f \in \mathcal{D}'^{*}_{E'_*}$,

$$\sup_{\varphi \in B_1} \|\langle \mathbf{f}, \varphi \rangle\|_{E'} = \sup_{\varphi \in B_1} \|f * \check{\varphi}\|_{E'} = \sup_{\varphi \in B_1} \sup_{\|\psi\|_E \leq 1} |\langle f, \psi * \varphi \rangle| = \sup_{\chi \in B} |\langle f, \chi \rangle|.$$

Hence, the mapping ι is continuous. Furthermore, by (iii) of Theorem 4.3.1, $\iota(\mathcal{D}'_{E'_*}) \subseteq \mathcal{S}'^{*}(\mathbb{R}^n, E'_*)$ and Proposition 4.3.1 tells us that $\iota(\mathcal{D}'^{*}_{E'_*})$ is precisely the subspace of $\mathcal{S}'(\mathbb{R}^n, E'_*)$ consisting of those $\mathbf{f}$ which commute with all translations in the sense of Proposition 4.3.1. Since the translations T_h are continuous operators on E'_*, we actually obtain that the range $\iota(\mathcal{D}'^{*}_{E'_*})$ is a closed subspace of $\mathcal{S}'^{*}(\mathbb{R}^n, E'_*)$ (see the comments after Corollary 1.3.1). Note that we may consider $\mathcal{D}'^{*}(\mathbb{R}^n)$ instead of $\mathcal{S}'^{*}(\mathbb{R}^n)$ in these embeddings.

Corollary 4.3.2. *Let* $B' \subseteq \mathcal{D}'^{*}_{E'_*}$*. The following properties are equivalent:*

- (i) *B' is a bounded subset of* $\mathcal{D}'^{*}_{E'_*}$*.*
- (ii) *$\iota(B')$ is bounded in* $\mathcal{S}'^{*}(\mathbb{R}^n, E')$ *(or equivalently in* $\mathcal{S}'^{*}(\mathbb{R}^n, E'_*)$*).*
- (iii) *There exist a bounded subset* $\tilde{B}$ *of* E' *and an ultradifferential operator* $P(D)$ *of class* $*$ *such that each* $f \in B'$ *can be represented as* $f = P(D)g + g_1$ *for some* $g, g_1 \in \tilde{B}$*.*

(*iv*) *There are* $C > 0$ *and a finite set* N *such that every* $f \in B'$ *admits a representation (4.16) with continuous functions* $f_k \in E'_* \cap UC_\omega$ *satisfying the uniform bounds* $\|f_k\|_{E'} \leq C$ *and* $\|f_k\|_{\infty,\omega} \leq C$ *(if* E *is reflexive one may choose* $f_k \in E' \cap C_\omega$*).*

Proof. $(i) \Rightarrow (ii)$ Follows from continuity of the mapping ι.

$(ii) \Rightarrow (iii)$. Let Ω be bounded open symmetric neighborhood of 0 in $\mathbb{R}^n$ and put $K = \overline{\Omega}$. Let $\iota(B')$ be bounded in $\mathcal{S}'^*(\mathbb{R}^n, E') = \mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^n), E')$. Then it is equicontinuous subset of $\mathcal{L}_b(\mathcal{D}^*_K, E')$.
We will continue the proof in the $\{M_p\}$ case, the (M_p) case is similar. There exist $(k_p) \in \mathfrak{R}$ and $C > 0$ such that $\|\langle \mathbf{f}, \varphi\rangle\|_{E'} \leq C\|\varphi\|_{K,(k_p)}$ for all $\mathbf{f} \in \iota(B')$ and $\varphi \in \mathcal{D}^{\{M_p\}}_K$, i.e.,

$$\|f * \check{\varphi}\|_{E'} \leq C\|\varphi\|_{K,(k_p)}$$

for all $f \in B$ and $\varphi \in \mathcal{D}^{\{M_p\}}_K$. Let $r_p = k_{p-1}/H$, for $p \in \mathbb{N}$, $p \geq 2$ and put $r_1 = \min\{1, r_2\}$. Then $(r_p) \in \mathfrak{R}$. For $\chi \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$, for large enough j, $\chi * \delta_j \in \mathcal{D}^{\{M_p\}}_K$ and by similar technic as in the proof of Proposition 3.3.1 one can prove that $\chi * \delta_j \to \chi$ in $\mathcal{D}^{\{M_p\}}_{K,(k_p)}$. Let $\varphi \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$ and put $\varphi_j = \varphi * \delta_j$. $f * \check{\varphi}_j$ is a Cauchy sequence in E', hence it must converge. But $f * \check{\varphi}_j$ converges to $f * \check{\varphi}$ in $\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$, hence $f * \check{\varphi}_j \to f * \check{\varphi}$ in E'. By arbitrariness of $\varphi \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$ and by passing to the limit in the inequality $\|f * \check{\varphi}_j\|_{E'} \leq C\|\varphi_j\|_{K,(k_p)}$, we have

$$\|f * \check{\varphi}\|_{E'} \leq C\|\varphi\|_{K,(k_p)}$$

for all $f \in B'$, $\varphi \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$. For the fixed $(r_p) \in \mathfrak{R}$, by the parametrix of Komatsu, there exist ultradifferential operator $P(D)$ of class $\{M_p\}$, $u \in \mathcal{D}^{\{M_p\}}_{\Omega,(r_p)}$ and $\psi \in \mathcal{D}^{\{M_p\}}(\Omega)$ such that $f = P(D)(f * u) + f * \psi$. By what we proved above $\{f * u | \, f \in B'\}$ and $\{f * \psi | \, f \in B'\}$ are bounded in E' and (iii) follows.

$(ii) \Rightarrow (iv)$ Proceed as in is $(ii) \Rightarrow (iii)$ and then use the same technique as in the proof of $(ii) \Rightarrow (v)$ of Theorem 4.3.1.

$(iii) \Rightarrow (i)$ and $(iv) \Rightarrow (i)$ are obvious. □

Corollary 4.3.3. *Let* $\{f_j\}_{j=0}^\infty \subseteq \mathcal{D}'^*_{E'_*}$ *(or similarly, a filter with a countable or bounded basis). The following three statements are equivalent:*

(*i*) $\{f_j\}_{j=0}^\infty$ *is (strongly) convergent in* $\mathcal{D}'^*_{E'_*}$.

(*ii*) $\{\iota(f_j)\}_{j=0}^\infty$ *is convergent in* $\mathcal{S}'^*(\mathbb{R}^n, E')$ *(or equivalently in* $\mathcal{S}'^*(\mathbb{R}^n, E'_*)$*).*

(*iii*) *There exist convergent sequences* $\{g_j\}_j, \{\tilde{g}_j\}_j$ *in* E' *and an ultradifferential operator* $P(D)$ *of class* $*$ *such that each* $f_j = P(D)g_j + \tilde{g}_j$.

(*iv*) *There exist* $N \in \mathbb{Z}_+$*, sequences* $\{g_j^{(k)}\}_j$*,* $k = 1, ..., N$*, in* $E'_* \cap UC_\omega$ *each convergent in* E'_* *and in* L^∞_ω *and ultradifferential operators* $P_k(D)$*,* $k = 1, ..., N$*, of class* $*$ *such that* $f_j = \sum_{k=1} P_k(D)g_j^{(k)}$ *(if* E *is reflexive one may choose* $g_j^{(k)} \in E' \cap C_\omega$*).*

Proof. The proof is similar to the proof of the above corollary and we omit it. □

Observe that Corollaries 4.3.2 and 4.3.3 are still valid if $\mathcal{S}'^{*}(\mathbb{R}^n)$ is replaced by $\mathcal{D}'^{*}(\mathbb{R}^n)$.

In the beginning of Section 4.2, we defined the spaces $\tilde{\mathcal{D}}_E^{\{M_p\},(r_p)}$ and $\tilde{\mathcal{D}}_E^{\{M_p\}}$. As we saw there, $\mathcal{D}_E^{\{M_p\}}$ and $\tilde{\mathcal{D}}_E^{\{M_p\}}$ are equal as sets and the former has a stronger topology than the latter. We will prove that these spaces are also topologically isomorphic.

Theorem 4.3.2. *The spaces* $\mathcal{D}_E^{\{M_p\}}$ *and* $\tilde{\mathcal{D}}_E^{\{M_p\}}$ *are isomorphic as l.c.s.*

Proof. By the above considerations its enough to prove that the topology of $\tilde{\mathcal{D}}_E^{\{M_p\}}$ is stronger than the topology of $\mathcal{D}_E^{\{M_p\}}$. Let V be a neighborhood of zero in $\mathcal{D}_E^{\{M_p\}}$. Since $\mathcal{D}_E^{\{M_p\}}$ is complete and barreled, its topology is in fact the topology $b\left(\mathcal{D}'^{\{M_p\}}_{E'_*}, \mathcal{D}_E^{\{M_p\}}\right)$. Hence we can assume that $V = B^\circ$, for a bounded set B in $\mathcal{D}'^{\{M_p\}}_{E'_*}$ (B° is the polar of B), i.e., $V = \left\{\varphi \in \mathcal{D}_E^{\{M_p\}} \middle| \sup_{T\in B} |\langle T,\varphi\rangle| \leq 1\right\}$. By Corollary 4.3.2 there exists $C > 0$ and a finite set N such that every $T \in B$ admits a representation (4.16) with continuous functions $f_k \in E'_* \cap UC_\omega$ satisfying the uniform bounds $\|f_k\|_{E'} \leq C$. Since $P_k(D)$ are continuous on $\tilde{\mathcal{D}}_E^{\{M_p\}}$ (Proposition 4.2.3), there exists $(r_p) \in \mathfrak{R}$ and $C_1 > 0$ such that $\|P_k(-D)\varphi\|_E \leq C_1\|\varphi\|_{E,(r_p)}$ for all $k \in N$, $\varphi \in \tilde{\mathcal{D}}_E^{\{M_p\}}$. Let $W = \left\{\varphi \in \tilde{\mathcal{D}}_E^{\{M_p\}} \middle| \|\varphi\|_{E,(r_p)} \leq 1/(CC_1N)\right\}$ be a neighborhood of zero in $\tilde{\mathcal{D}}_E^{\{M_p\}}$. If $\varphi \in W$, then for $T \in B$,

$$|\langle T,\varphi\rangle| \leq \sum_{k\in N} |\langle f_k, P(-D)\varphi\rangle| \leq \sum_{k\in N} \|f_k\|_{E'}\|P(-D)\varphi\|_E \leq 1,$$

i.e., $\varphi \in V$. Hence we obtain the desired result. □

When E is reflexive, the space $\mathcal{D}_E^*$ is also reflexive. Furthermore, we have:

Proposition 4.3.2. *If E is reflexive, then* $\mathcal{D}_E^{(M_p)}$ *and* $\mathcal{D}'^{\{M_p\}}_{E'}$ *are* (FS^*)*-spaces,* $\mathcal{D}_E^{\{M_p\}}$ *and* $\mathcal{D}'^{(M_p)}_E$ *are* (DFS^*)*-spaces. Consequently, they are reflexive. In addition,* $\mathcal{S}^*(\mathbb{R}^n)$ *is dense in* $\mathcal{D}'^{*}_{E'}$.

Proof. Let $\tilde{\tilde{\mathcal{D}}}_E^{\{M_p\},m}$ be the Banach space of all $\varphi \in \mathcal{D}'^{*}(\mathbb{R}^n)$ such that $D^\alpha\varphi \in E$, $\forall\alpha \in \mathbb{N}^n$ and

$$\|\varphi\|_{E,m} = \left(\sum_\alpha \frac{m^{2|\alpha|}}{M_\alpha^2}\|D^\alpha\varphi\|_E^2\right)^{1/2} < \infty.$$

Then we have the following obvious continuous inclusions $\tilde{\tilde{\mathcal{D}}}_E^{\{M_p\},m} \to \mathcal{D}_E^{\{M_p\},m}$ and $\mathcal{D}_E^{\{M_p\},2m} \to \tilde{\tilde{\mathcal{D}}}_E^{\{M_p\},m}$. Hence $\mathcal{D}_E^{(M_p)} = \varprojlim_{m\to\infty} \tilde{\tilde{\mathcal{D}}}_E^{\{M_p\},m}$ and $\mathcal{D}_E^{\{M_p\}} = \varinjlim_{m\to 0} \tilde{\tilde{\mathcal{D}}}_E^{\{M_p\},m}$. If $l_m^2(E)$ is the Banach space of all $(\psi_\alpha)_{\alpha\in\mathbb{N}^n}$ with $\psi_\alpha \in E$ and norm $\|(\psi_\alpha)_\alpha\|_{l_m^2(E)} =$

$\left(\sum_{\alpha\in\mathbb{N}^n} \frac{m^{2|\alpha|}}{M_\alpha^2}\|\psi_\alpha\|_E^2\right)^{1/2}$, then $l_m^2(E)$ is reflexive since E is ([56, Thm 2 p.360] or Lemma 0.2.4). Observe that $\tilde{\tilde{\mathcal{D}}}_E^{\{M_p\},m}$ is isometrically injected onto a closed subspace of $l_m^2(E)$ by the mapping $\varphi \mapsto (D^\alpha\varphi)_\alpha$, hence $\tilde{\tilde{\mathcal{D}}}_E^{\{M_p\},m}$ is reflexive. Thus $\mathcal{D}_E^{(M_p)}$ is an (FS^*)-space and $\mathcal{D}_E^{\{M_p\}}$ is a (DFS^*)-space. In particular, they are reflexive and $\mathcal{D}_E'^{(M_p)}$ is a (DFS^*)-space and $\mathcal{D}_E'^{\{M_p\}}$ is an (FS^*)-space. Now, the denseness of $\mathcal{S}^*(\mathbb{R}^n)$ in $\mathcal{D}'^*_{E'}$ is an easy consequence of the Hahn-Banach Theorem cf. Proposition 1.1.3. □

4.4 The weighted spaces $\mathcal{D}^*_{L^p_\eta}$ and $\mathcal{D}'^*_{L^p_\eta}$

As examples, in this section we discuss the weighted spaces $\mathcal{D}^*_{L^p_\eta}$ and $\mathcal{D}'^*_{L^p_\eta}$, which are particular examples of the spaces $\mathcal{D}^*_E$ and $\mathcal{D}'^*_{E'_*}$. They turn out to be important in the study of properties of the general $\mathcal{D}'^*_{E'_*}$ and general convolution in $\mathcal{D}'^*(\mathbb{R}^n)$ (cf. Section 4.5.2).

Let η be an *ultrapolynomially bounded weight*, that is, a measurable function $\eta : \mathbb{R}^n \to (0,\infty)$ that fulfills the requirement

$$\eta(x+h) \leq C\eta(x)e^{M(\tau|h|)},$$

for some $C, \tau > 0$, resp. for every $\tau > 0$ there exists $C > 0$.

An interesting nontrivial example in the (M_p) case is given by the following function

$$\eta(x) = e^{\tilde{\eta}(|x|)}$$

where

$$\tilde{\eta} : [0,\infty) \to [0,\infty), \quad \tilde{\eta}(\rho) = \rho\int_\rho^\infty \frac{M(s)}{s^2}ds.$$

To see this, observe that $\tilde{\eta}$ is differentiable function with nonnegative monotonically decreasing derivative. Hence $\tilde{\eta}$ is concave monotonically increasing function and $\tilde{\eta}(0) = 0$. Also, it is easy to see that $M(\rho) \leq \tilde{\eta}(\rho)$ and $\tilde{\eta}(\rho+\lambda) \leq \tilde{\eta}(\rho)+\tilde{\eta}(\lambda)$, for all $\rho, \lambda > 0$. By $(M.3)$ and Proposition 4.4 of [49] there exist $C, C_1 > 0$ such that $\tilde{\eta}(\rho) \leq M(C\rho) + C_1$, for all $\rho > 0$. For the $\{M_p\}$ case take $(r_p) \in \mathfrak{R}$ and perform the same construction with the sequence N_p defined by $N_0 = 1$ and $N_p = M_p\prod_{j=1}^p r_j$, $p \in \mathbb{Z}_+$, which obviously satisfies $(M.1)$ and $(M.3)$ since M_p does.

For $1 \leq p < \infty$ we denote with L^p_η the measurable functions g such that $\|\eta g\|_p < \infty$. Clearly L^p_η are translation-invariant Banach space of tempered ultradistributions for $p \in [1,\infty)$ and the space L^∞_η is an exception since $\mathcal{D}^*(\mathbb{R}^n)$ is not dense in L^∞_η. In the next considerations the number q always stands for $p^{-1}+q^{-1} = 1$ ($p \in [1,\infty]$). Of course $(L^p_\eta)' = L^q_{\eta^{-1}}$ if $1 < p < \infty$ and $(L^1_\eta)' = L^\infty_\eta$. In view of Proposition 4.1.2, the space E'_* corresponding to $E = L^p_{\eta^{-1}}$ is $E'_* = L^q_\eta$ whenever $1 < p < \infty$. On the other hand, (iii) of Theorem 4.1.2 gives that $E'_* = UC_\eta$ for $E = L^1_\eta$, where UC_η is defined as in (1.11) with ω replaced by η.

We can easily find the Beurling algebra of L^p_η.

Proposition 4.4.1. *Let* $\omega_\eta(h) := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \eta(x+h)/\eta(x)$. *Then*

$$\|T_{-h}\|_{\mathcal{L}(L^p_\eta)} = \begin{cases} \omega_\eta(h) & \text{if } p \in [1,\infty), \\ \omega_\eta(-h) & \text{if } p = \infty. \end{cases}$$

Consequently, the Beurling algebra associated to L^p_η *is* $L^1_{\omega_\eta}$ *if* $p = [1,\infty)$ *and* $L^1_{\check{\omega}_\eta}$ *if* $p = \infty$.

Proof. The proof is identically the same as that of Proposition 1.4.1. □

Observe that when the logarithm of η is a continuous subadditive function and $\eta(0) = 1$, one easily obtains from Proposition 4.4.1 that $\omega_\eta = \eta$.

Consider now the spaces $\mathcal{D}^*_{L^p_\eta}$ for $p \in [1,\infty]$ and $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ defined as in Section 4.2 by taking $E = L^p_\eta$. Once again, the case $p = \infty$ is an exception since $\mathcal{D}^*(\mathbb{R}^n)$ is not dense in $\mathcal{D}^*_{L^\infty_\eta}$ nor in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$. Nonetheless, we can repeat the proof of Proposition 4.2.1 to prove that $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ is regular and complete. One can proof that each ultradifferential operator of $*$ class acts continuously on $\mathcal{D}^*_{L^\infty_\eta}$ and each ultradifferential operator of $\{M_p\}$ class acts continuously on $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ (cf. the proof of Proposition 4.2.3). Obviously $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ is injected continuously into $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ and by using Lemma 3.4 of [51] and employing similar technique as in the proof of Proposition 4.2.1, one can prove that this inclusion is in fact surjective. As usual, we denote by $\mathcal{B}^*_\eta$ the space $\mathcal{D}^*_{L^\infty_\eta}$ and by $\dot{\mathcal{B}}^*_\eta$ the closure of $\mathcal{D}^*(\mathbb{R}^n)$ in $\mathcal{B}^*_\eta$. We denote by $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ the closure of $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$. It is important to note that in the case $\eta = 1$ these spaces were considered in [79] (see also [12]).

We immediately see that $\dot{\mathcal{B}}^{(M_p)}_\eta = \mathcal{D}^{(M_p)}_{C_\eta}$, where we denoted $C_\eta = \{g \in C(\mathbb{R}^n) |\ \lim_{|x|\to\infty} g(x)/\eta(x) = 0\} \subseteq L^\infty_\eta$. In the $\{M_p\}$ case this is not trivial. The following theorem gives that result.

Theorem 4.4.1. *The spaces* $\mathcal{D}^{\{M_p\}}_{C_\eta}$, $\dot{\mathcal{B}}^{\{M_p\}}_\eta$ *and* $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ *are isomorphic among each other as l.c.s..*

Proof. By Proposition 4.2.1, $\mathcal{D}^{\{M_p\}}_{C_\eta}$ is complete barreled (DF)-space. First we prove that $\mathcal{D}^{\{M_p\}}_{C_\eta}$ and $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ are isomorphic l.c.s. Observe that $\mathcal{D}^{\{M_p\}}_{C_\eta} \subseteq \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$. Moreover, by Theorem 4.3.2, the topology of $\mathcal{D}^{\{M_p\}}_{C_\eta}$ is the same as the induced topology on $\mathcal{D}^{\{M_p\}}_{C_\eta}$ by $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$. Since $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $\mathcal{D}^{\{M_p\}}_{C_\eta}$ and $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ is the closure of $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ in the complete l.c.s. $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$, $\mathcal{D}^{\{M_p\}}_{C_\eta}$ and $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ are isomorphic l.c.s. and the canonical inclusion $\mathcal{D}^{\{M_p\}}_{C_\eta} \to \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ gives the isomorphism. Now, observe that the inclusion $\mathcal{D}^{\{M_p\}}_{C_\eta} \to \mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ is continuous. Since $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $\mathcal{D}^{\{M_p\}}_{C_\eta}$ and $\dot{\mathcal{B}}^{\{M_p\}}_\eta$, $\mathcal{D}^{\{M_p\}}_{C_\eta} \subseteq \dot{\mathcal{B}}^{\{M_p\}}_\eta$ and the inclusion is continuous. Also, since the inclusion $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta} \to \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ is continuous and $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ is dense in $\dot{\mathcal{B}}^{\{M_p\}}_\eta$ and $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$, we obtain that $\dot{\mathcal{B}}^{\{M_p\}}_\eta \subseteq \dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ and the inclusion is continuous. But, since we already proved that the inclusion $\mathcal{D}^{\{M_p\}}_{C_\eta} \to \dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ is a topological isomorphism onto, we obtain that so is the inclusion $\mathcal{D}^{\{M_p\}}_{C_\eta} \to \dot{\mathcal{B}}^{\{M_p\}}_\eta$. □

By Proposition 4.2.4 and estimate (4.12), resp. (4.13), one easily sees that $\mathcal{D}^*_{L^p_\eta} \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}_\eta}$ for every $p \in [1,\infty)$. It follows from Proposition 4.3.2 that $\mathcal{D}^*_{L^p_\eta}$ is reflexive when $p \in (1,\infty)$.

In accordance to Section 4.3, the weighted spaces $\mathcal{D}'^*_{L^p_\eta}$ are defined as $\mathcal{D}'^*_{L^p_\eta} = (\mathcal{D}^*_{L^q_{\eta^{-1}}})'$ where $p^{-1}+q^{-1}=1$ if $p \in (1,\infty]$; if $p=1$, $\mathcal{D}'^*_{L^1_\eta} = (\mathcal{D}^*_{C_\eta})' = (\dot{\mathcal{B}}^*_\eta)'$. We write $\mathcal{B}'^*_\eta = \mathcal{D}'^*_{L^\infty_\eta}$ and $\dot{\mathcal{B}}'^*_\eta$ for the closure of $\mathcal{D}^*(\mathbb{R}^n)$ in $\mathcal{B}'^*_\eta$.

When η is continuous the dual of $E = C_\eta$ is the space $\mathcal{M}^1_\eta$ consisting of all elements $\nu \in (C_c(\mathbb{R}^n))'$ which are of the form $d\nu = \eta^{-1} d\mu$, for $\mu \in \mathcal{M}^1$ and the norm is $\|\nu\|_{\mathcal{M}^1_\eta} = \|\mu\|_{\mathcal{M}^1}$. Observe that then $E'_* = L^1_\eta$. In this case, by using Theorem 4.3.1, similarly as in the case of distributions (see [92], [93]), one can prove that the bidual of $\dot{\mathcal{B}}^{(M_p)}_\eta$ is isomorphic to $\mathcal{D}^{(M_p)}_{L^\infty_\eta}$ as l.c.s. and that $\dot{\mathcal{B}}^{(M_p)}_\eta$ is a distinguished Fréchet space, i.e., $\mathcal{D}'^{(M_p)}_{L^1_\eta}$ is barreled and bornological. In the $\{M_p\}$ case, observe that $\mathcal{D}'^{\{M_p\}}_{L^1_\eta}$ is a Fréchet space as the strong dual of a barreled (DF)-space. Moreover, we have the following theorem.

Theorem 4.4.2. *The bidual of $\dot{\mathcal{B}}^{\{M_p\}}_\eta$ is isomorphic to $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ as l.c.s. Moreover $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ and $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ are isomorphic l.c.s.*

Proof. We already saw that $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ and $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ are equal as sets. First we prove that the bidual of $\dot{\mathcal{B}}^{\{M_p\}}_\eta$ is isomorphic to $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$. Since $\mathcal{E}'^{\{M_p\}}(\mathbb{R}^n)$ is continuously and densely injected into $\mathcal{D}'^{\{M_p\}}_{L^1_\eta}$ (the denseness can be proved by using cut-off functions and Theorem 4.3.1) we have the continuous inclusion $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'_b \to \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$ (b stands for the strong topology). Let $(r_p) \in \mathfrak{R}$ and put $R_\alpha = \prod_{j=1}^{|\alpha|} r_j$. Observe the set

$$B = \left\{ \frac{(\eta(a))^{-1} D^\alpha \delta_a}{M_\alpha R_\alpha} \middle| \, a \in \mathbb{R}^n,\ \alpha \in \mathbb{N}^n \right\}.$$

One easily proves that it is a bounded subset of $\mathcal{D}'^{\{M_p\}}_{L^1_\eta}$. Hence if $\psi \in \left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'_b$, $\psi(B)$ is bounded in $\mathbb{C}$ and hence

$$\sup_{a,\alpha} \frac{|(\eta(a))^{-1} D^\alpha \psi(a)|}{M_\alpha R_\alpha} = \sup_{T\in B} |\langle \psi, T\rangle| < \infty.$$

We obtain that $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)' \subseteq \mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ and the inclusion $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'_b \to \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ is continuous.

Let $\psi \in \mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$. If $T \in \mathcal{D}'^{\{M_p\}}_{L^1_\eta}$, by Theorem 4.3.1 there exist an ultradifferential operator $P(D)$ of $\{M_p\}$ class and $f, f_1 \in \mathcal{M}^1_\eta$ such that $T = P(D)f + f_1$. Let $df = \eta^{-1} dg$ and $df_1 = \eta^{-1} dg_1$ for $g, g_1 \in \mathcal{M}^1$. Define S_ψ by

$$S_\psi(T) = \int_{\mathbb{R}^n} \frac{P(-D)\psi(x)}{\eta(x)} dg + \int_{\mathbb{R}^n} \frac{\psi(x)}{\eta(x)} dg_1.$$

Obviously, the integrals on the right hand side are absolutely convergent. We will prove that S_ψ is well defined element of $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'$. Let $\tilde{P}(D)$, $\tilde{f}, \tilde{f}_1 \in \mathcal{M}^1_\eta$ be such that $T = \tilde{P}(D)\tilde{f} + \tilde{f}_1$ and let $d\tilde{f} = \eta^{-1}d\tilde{g}$ and $d\tilde{f}_1 = \eta^{-1}d\tilde{g}_1$ for $\tilde{g}, \tilde{g}_1 \in \mathcal{M}^1$. Chose $\chi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$ to be a function such that $\chi = 1$ on the closed unit ball with center at 0 and $\chi = 0$ on $\{x \in \mathbb{R}^n | \, |x| > 2\}$. Put $\psi_n(x) = \chi(x/n)\psi(x)$, $n \in \mathbb{Z}_+$. Then Lebesque Dominated Convergence Theorem implies

$$\int_{\mathbb{R}^n} \frac{P(-D)\psi_n(x)}{\eta(x)} dg \to \int_{\mathbb{R}^n} \frac{P(-D)\psi(x)}{\eta(x)} dg,$$
$$\int_{\mathbb{R}^n} \frac{\psi_n(x)}{\eta(x)} dg_1 \to \int_{\mathbb{R}^n} \frac{\psi(x)}{\eta(x)} dg_1,$$
$$\int_{\mathbb{R}^n} \frac{\tilde{P}(-D)\psi_n(x)}{\eta(x)} d\tilde{g} \to \int_{\mathbb{R}^n} \frac{\tilde{P}(-D)\psi(x)}{\eta(x)} d\tilde{g},$$
$$\int_{\mathbb{R}^n} \frac{\psi_n(x)}{\eta(x)} d\tilde{g}_1 \to \int_{\mathbb{R}^n} \frac{\psi(x)}{\eta(x)} d\tilde{g}_1,$$

when $n \to \infty$. Also, observe that for each $n \in \mathbb{Z}_+$

$$\int_{\mathbb{R}^n} \frac{P(-D)\psi_n(x)}{\eta(x)} dg + \int_{\mathbb{R}^n} \frac{\psi_n(x)}{\eta(x)} dg_1 = \int_{\mathbb{R}^n} \frac{\tilde{P}(-D)\psi_n(x)}{\eta(x)} d\tilde{g} + \int_{\mathbb{R}^n} \frac{\psi_n(x)}{\eta(x)} d\tilde{g}_1,$$

since both of the terms are equal to $\langle T, \psi_n\rangle$ in the sense of the duality $\langle \mathcal{D}^{\{M_p\}}(\mathbb{R}^n), \mathcal{D}'^{\{M_p\}}(\mathbb{R}^d)\rangle$. Hence, S_ψ is well defined mapping $\mathcal{D}'^{\{M_p\}}_{L^1_\eta} \to \mathbb{C}$, since it does not depend on the representation of T.

To prove that it is continuous it is enough to prove that it maps bounded sets into bounded sets, since $\mathcal{D}'^{\{M_p\}}_{L^1_\eta}$ is a Fréchet space. Let B be a bounded set in $\mathcal{D}'^{\{M_p\}}_{L^1_\eta}$. By Corollary 4.3.2, there exist an ultradifferential operator $P(D)$ of class $\{M_p\}$ and bounded subset B_1 of $\mathcal{M}^1_\eta$ such that each $T \in B$ can be represented by $T = P(D)f + f_1$ for some $f, f_1 \in B_1$. By the way we defined S_ψ, it is easy to verify that $S_\psi(B)$ is bounded in $\mathbb{C}$, so $S_\psi \in \left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'$. We obtained that $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)' = \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ as sets and $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'_b$ has stronger topology than the latter.

Let $V = B^\circ$ be a neighborhood of zero $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'_b$ for B be a bounded subset of $\mathcal{D}'^{\{M_p\}}_{L^1_\eta}$. By Corollary 4.3.2, there exist an ultradifferential operator $P(D)$ of class $\{M_p\}$ and bounded subset B_1 of $\mathcal{M}^1_\eta$ such that each $T \in B$ can be represented by $T = P(D)f + f_1$ for some $f, f_1 \in B_1$. There exists $C_1 \geq 1$ such that $\|\tilde{g}\|_{\mathcal{M}^1_\eta} \leq C_1$ for all $\tilde{f} \in B_1$. Also, since $P(D) = \sum_\alpha c_\alpha D^\alpha$ is of $\{M_p\}$ class, there exist $(r_p) \in \mathfrak{R}$ and $C_2 \geq 1$ such that $|c_\alpha| \leq C_2/(M_\alpha R_\alpha)$ (see the proof of Proposition 4.2.3). Observe the neighborhood of zero $W = \left\{\psi \in \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta} \middle| \sup_{x,\alpha} \frac{|(\eta(x))^{-1}D^\alpha\psi(x)|}{M_\alpha \prod_{j=1}^{|\alpha|}(r_j/2)} \leq \frac{1}{2C_1C_2C_3}\right\}$ in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$, where we put $C_3 = \sum_\alpha 2^{-|\alpha|}$. One easily verifies that $W \subseteq V$. We obtain that $\left(\mathcal{D}'^{\{M_p\}}_{L^1_\eta}\right)'_b$ and $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ are isomorphic l.c.s.

Hence $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ is a complete (DF)-space (since $\mathcal{D}'^{\{M_p\}}_{L^1_\eta}$ is a Fréchet space). Obviously, the identity mapping $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta} \to \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ is continuous and bijective.
Since $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ is a (DF)-space, to prove the continuity of the inverse mapping it is enough to prove that its restriction to every bounded subset of $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ is continuous [91, Cor. 6.7, p. 155]). If B is a bounded subset of $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ then for every $(r_p) \in \mathfrak{R}$, $\sup\limits_{\psi\in B}\sup\limits_{\alpha} \dfrac{\|D^\alpha\psi\|_{L^\infty_\eta(\mathbb{R}^n)}}{M_\alpha R_\alpha} < \infty$. Hence, by [51, Lemma 3.4], there exists $h > 0$ such that $\sup\limits_{\psi\in B}\sup\limits_{\alpha} \dfrac{h^{|\alpha|}\,\|D^\alpha\psi\|_{L^\infty_\eta(\mathbb{R}^n)}}{M_\alpha} < \infty$, i.e., B is bounded in $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$. Since every bounded subset of $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ is obviously bounded in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$, $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ and $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ have the same bounded sets. Let ψ_λ be bounded net in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ which converges to ψ in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$. Then there exist $0 < h \leq 1$ and $C > 0$ such that

$$\sup_{\lambda}\sup_{\alpha} \frac{h^{|\alpha|}\,\|D^\alpha\psi_\lambda\|_{L^\infty_\eta}}{M_\alpha} \leq C \text{ and } \sup_{\alpha} \frac{h^{|\alpha|}\,\|D^\alpha\psi\|_{L^\infty_\eta}}{M_\alpha} \leq C.$$

Fix $0 < h_1 < h$. Let $\varepsilon > 0$ be arbitrary but fixed. Take $p_0 \in \mathbb{Z}_+$ such that $(h_1/h)^{|\alpha|} \leq \varepsilon/(2C)$ for all $|\alpha| \geq p_0$.
Since $\psi_\lambda \to \psi$ in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$, for the sequence $r_p = p$, $p \in \mathbb{Z}_+$, there exists λ_0 such that for all $\lambda \geq \lambda_0$ we have $\sup\limits_{\alpha} \dfrac{\|D^\alpha(\psi_\lambda - \psi)\|_{L^\infty_\eta}}{M_\alpha R_\alpha} \leq \dfrac{\varepsilon}{p_0!}$. Then for $|\alpha| < p_0$, we have

$$\frac{h_1^{|\alpha|}\,\|D^\alpha(\psi_\lambda - \psi)\|_{L^\infty_\eta}}{M_\alpha} \leq \varepsilon.$$

For $|\alpha| \geq p_0$, we have

$$\frac{h_1^{|\alpha|}\,\|D^\alpha(\psi_\lambda - \psi)\|_{L^\infty_\eta}}{M_\alpha} \leq 2C\left(\frac{h_1}{h}\right)^{|\alpha|} \leq \varepsilon.$$

It follows that $\psi_\lambda \to \psi$ in $\mathcal{D}^{\{M_p\},h_1}_{L^\infty_\eta}$ and hence in $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$. We obtain that the induced topology by $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ on every bounded subset of $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ is stronger than the induced topology by $\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$. Hence the identity mapping $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta} \to \mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ is continuous. □

4.5 Convolution of ultradistributions

We now apply our results to the study of the convolution of ultradistributions. As a corollary of the last Theorem of the previous section for $\eta = 1$ we give an improvement of the following theorem from [80] for existence of convolution of Roumieu ultradistributions.

4.5.1 On the general convolution of Romieu ultradistributions

Theorem 4.5.1. *([80]) Let $S, T \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$. The following statements are equivalent:*

i) the convolution of S and T exists;

ii) $S \otimes T \in \left(\dot{\mathcal{B}}_{\Delta}^{\{M_p\}}\right)'$;

iii) for all $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, $\left(\varphi * \check{S}\right) T \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ and for every compact subset K of $\mathbb{R}^n$, $(\varphi, \chi) \mapsto \left\langle \left(\varphi * \check{S}\right) T, \chi\right\rangle$, $\mathcal{D}_K^{\{M_p\}} \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}} \longrightarrow \mathbb{C}$, is a continuous bilinear mapping;

iv) for all $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, $\left(\varphi * \check{T}\right) S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ and for every compact subset K of $\mathbb{R}^n$, $(\varphi, \chi) \mapsto \left\langle \left(\varphi * \check{T}\right) S, \chi\right\rangle$, $\mathcal{D}_K^{\{M_p\}} \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}} \longrightarrow \mathbb{C}$, is a continuous bilinear mapping;

v) for all $\varphi, \psi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, $\left(\varphi * \check{S}\right)(\psi * T) \in L^1(\mathbb{R}^n)$.

Corollary 4.5.1. *Let $S, T \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$. Then the following conditions are equivalent*

i) the convolution of S and T exists;

iii)′ for all $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, $\left(\varphi * \check{S}\right) T \in \mathcal{D}'^{\{M_p\}}_{L^1}$;

iv)′ for all $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$, $\left(\varphi * \check{T}\right) S \in \mathcal{D}'^{\{M_p\}}_{L^1}$.

Proof. We will prove that $iii) \Leftrightarrow iii)'$, the prove that $iv) \Leftrightarrow iv)'$ is similar. Observe that $iii) \Rightarrow iii)'$ is trivial. Let $iii)'$ holds. Then, by Theorem 4.4.1, $\mathcal{D}'^{\{M_p\}}_{L^1}$ is a Fréchet space as the strong dual of a (DF)-space. The mapping

$$\chi \mapsto \left\langle \left(\varphi * \check{S}\right) T, \chi\right\rangle, \ \dot{\mathcal{B}}^{\{M_p\}} \to \mathbb{C}$$

is continuous for each fixed $\varphi \in \mathcal{D}_K^{\{M_p\}}$ since $\left(\varphi * \check{S}\right) T \in \mathcal{D}'^{\{M_p\}}_{L^1}$. Fix $\chi \in \dot{\mathcal{B}}^{\{M_p\}}$. Then the mapping

$$\varphi \mapsto \left(\varphi * \check{S}\right) T, \ \mathcal{D}_K^{\{M_p\}} \to \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$$

is continuous, hence it has a closed graph. But $\left(\varphi * \check{S}\right) T \in \mathcal{D}'^{\{M_p\}}_{L^1}$ and $\mathcal{D}'^{\{M_p\}}_{L^1}$ is continuously injected into $\mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$, which implies the mapping

$$\varphi \mapsto \left(\varphi * \check{S}\right) T, \ \mathcal{D}_K^{\{M_p\}} \to \mathcal{D}'^{\{M_p\}}_{L^1}$$

has a closed graph. $\mathcal{D}_K^{\{M_p\}}$ is barreled (in fact it is a (DFS)-space). Since $\mathcal{D}'^{\{M_p\}}_{L^1}$ is a Fréchet space it is Ptak space hence this mapping is continuous by the Ptak

Closed Graph Theorem ([91, Thm. 8.5, p. 166]). We obtain that for each fixed $\chi \in \dot{\mathcal{B}}^{\{M_p\}}$, the mapping

$$\varphi \mapsto \left\langle \left(\varphi * \check{S}\right) T, \chi\right\rangle, \ \mathcal{D}_K^{\{M_p\}} \to \mathbb{C}$$

is continuous. Hence, the bilinear mapping

$$(\varphi, \chi) \mapsto \left\langle \left(\varphi * \check{S}\right) T, \chi\right\rangle, \ \mathcal{D}_K^{\{M_p\}} \times \dot{\mathcal{B}}^{\{M_p\}} \to \mathbb{C}$$

is separately continuous. Since $\mathcal{D}_K^{\{M_p\}}$ and $\dot{\mathcal{B}}^{\{M_p\}}$ are barreled (DF)-spaces, this mapping is continuous. □

4.5.2 Relation between $\mathcal{D}'^*_{E'_*}$, $\mathcal{B}'^*_{\omega}$, and and $\mathcal{D}'^*_{L^1_{\check{\omega}}}$ – Convolution and multiplication

Some of the properties of the $\mathcal{D}^*_{L^p_\eta}$ and $\mathcal{D}'^*_{L^p_\eta}$ extend to $\mathcal{D}^*_E$ and $\mathcal{D}'^*_{E'_*}$ for the general translation-invariant Banach space of tempered ultradistributions E with Beurling algebra L^1_ω.

Proposition 4.5.1. *The following dense and continuous inclusions hold:* $\mathcal{D}^*_{L^1_\omega} \hookrightarrow \mathcal{D}^*_E \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}}$ *and the inclusions are continuous* $\mathcal{D}'^*_{L^1_{\check{\omega}}} \to \mathcal{D}'^*_{E'_*} \to \mathcal{B}'^*_{\omega}$. *If E is reflexive, one has* $\mathcal{D}'^*_{L^1_{\check{\omega}}} \hookrightarrow \mathcal{D}'^*_{E'} \hookrightarrow \dot{\mathcal{B}}'^*_{\omega}$.

Proof. The proof follows the same lines as in the distribution case treated in Theorem 1.5.1(by using the analogous results for ultradistributions).

□

By the above proposition and by the fact $\mathcal{D}^*(\mathbb{R}^n) \hookrightarrow \mathcal{D}'^*_{L^1_\eta}$ (which is easily obtainable by direct inspection) we have $\mathcal{D}^*_{L^1_{\omega_\eta}} \hookrightarrow \mathcal{D}^*_{L^p_\eta} \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}_\eta}$ and $\mathcal{D}'^*_{L^1_{\check{\omega}_\eta}} \hookrightarrow \mathcal{D}'^*_{L^p_\eta} \hookrightarrow \dot{\mathcal{B}}'^*_{\omega_\eta}$ for $1 \leq p < \infty$.

Also, direct consequence of this Proposition is that the spaces $\mathcal{D}^*_E$ are never Montel spaces when ω is bounded weight. In fact, if $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ is non-negative with $\varphi(x) = 0$ for $|x| \geq 1/2$ and $\theta \in \mathbb{R}^n$ is a unit vector, then $\{(T_{-j\theta}\varphi)/\omega(j\theta)\}_{j=0}$ is a bounded sequence in $\mathcal{D}^*_{L^1_\omega}$ hence in $\mathcal{D}^*_E$ without any accumulation point.

It is also easy to verify that $\dot{\mathcal{B}}^*_\eta \hookrightarrow \dot{\mathcal{B}}^*_{\omega_\eta}$ and $\dot{\mathcal{B}}'^*_\eta \hookrightarrow \dot{\mathcal{B}}'^*_{\omega_\eta}$.

The multiplicative product mappings $\cdot : \mathcal{D}'^*_{L^p_\eta} \times \mathcal{B}^*_\eta \to \mathcal{D}'^*_{L^p}$ and $\cdot : \mathcal{B}'^*_\eta \times \mathcal{D}^*_{L^p_\eta} \to \mathcal{D}'^*_{L^p}$ are well-defined and hypocontinuous for $1 \leq p < \infty$. In particular, $f\varphi$ is an integrable ultradistribution whenever $f \in \mathcal{B}'^*_\eta$ and $\varphi \in \mathcal{D}^*_{L^1_\eta}$ or $f \in \mathcal{D}'^*_{L^1_\eta}$ and $\varphi \in \mathcal{B}^*_\eta$. If $(1/r) = (1/p_1) + (1/p_2)$ with $1 \leq r, p_1, p_2 < \infty$, it is also clear that the multiplicative product $\cdot : \mathcal{D}'^*_{L^{p_1}_{\eta_1}} \times \mathcal{D}^*_{L^{p_2}_{\eta_2}} \to \mathcal{D}'^*_{L^r_{\eta_1\eta_2}}$ is hypocontinuous. Clearly, the convolution product can always be canonically defined as a hypocontinuous mapping in the following situations, $* : \mathcal{D}'^*_{L^p_\eta} \times \mathcal{D}'^*_{L^1_\omega} \to \mathcal{D}'^*_{L^p_\eta}$, $1 \leq p \leq \infty$, and $* : \dot{\mathcal{B}}'^*_\eta \times \mathcal{D}'^*_{L^1_\omega} \to \dot{\mathcal{B}}'^*_\eta$. Furthermore, such convolution products are continuous bilinear mappings. In fact, in the Roumieu case these spaces are (F)-spaces and therefore continuity is equivalent to separate continuity; for the Beurling case, it follows from the equivalence between hypocontinuity and continuity for bilinear mappings on (DF)-spaces (cf. [57, p. 160]).

We can now define multiplication and convolution operations on $\mathcal{D}'^{*}_{E'_{*}}$. In the next proposition we denote by $\mathcal{O}'^{*}_{C,b}(\mathbb{R}^n)$ the space $\mathcal{O}'^{*}_{C}(\mathbb{R}^n)$ equipped with the strong topology from the duality $\langle \mathcal{O}^{*}_{C}(\mathbb{R}^n), \mathcal{O}'^{*}_{C}(\mathbb{R}^n)\rangle$. Using Proposition 4.5.1 and Proposition 4.2.4 for $E = C_\eta$ we obtain the next proposition.

Proposition 4.5.2. *The convolution mappings* $* : \mathcal{D}'^{*}_{E'_{*}} \times \mathcal{D}'^{*}_{L^1_{\check{\omega}}} \to \mathcal{D}'^{*}_{E'_{*}}$ *and* $* : \mathcal{D}'^{*}_{E'_{*}} \times \mathcal{O}'^{*}_{C,b}(\mathbb{R}^n) \to \mathcal{D}'^{*}_{E'_{*}}$ *are continuous. The convolution and multiplicative products are hypocontinuous in the following cases:* $\cdot : \mathcal{D}'^{*}_{E'_{*}} \times \mathcal{D}^{*}_{L^1_{\omega}} \to \mathcal{D}'^{*}_{L^1}$, $\cdot : \mathcal{D}'^{*}_{L^1_{\check{\omega}}} \times \mathcal{D}^{*}_{E} \to \mathcal{D}'^{*}_{L^1}$, *and* $* : \mathcal{D}'^{*}_{E'_{*}} \times \mathcal{D}^{*}_{\check{E}} \to \mathcal{B}^{*}_{\omega}$. *If* E *is reflexive, we have* $* : \mathcal{D}'^{*}_{E'} \times \mathcal{D}^{*}_{\check{E}} \to \dot{\mathcal{B}}^{*}_{\omega}$.

Proof. The first two mappings are continuous because of [57, p. 160]. Using Proposition 4.5.1 and Proposition 4.2.4 for $E = C_\eta$ we obtain the continuity of the mappings $* : \mathcal{D}'^{*}_{E'_{*}} \times \mathcal{O}'^{*}_{C,b}(\mathbb{R}^n) \to \mathcal{D}'^{*}_{E'_{*}}$. The rest of the proof follows the same lines as that of Proposition 1.5.1. □

Observe that, as a consequence of Proposition 4.5.2, $f\varphi$ is an integrable ultradistribution (i.e., an element of $\mathcal{D}'^{*}_{L^1}$) if $f \in \mathcal{D}'^{*}_{E'_{*}}$ and $\varphi \in \mathcal{D}^{*}_{L^1_{\omega}}$ or if $f \in \mathcal{D}'^{*}_{L^1_{\check{\omega}}}$ and $\varphi \in \mathcal{D}^{*}_{E}$.

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Библиографија-радови:

P. Dimovski, B. Prangoski, D. Velinov, *On the space of multipliers and convolutors in the space of tempered ultradistributions*, NSJOM Vol. 44 (2), 2014

P. Dimovski, S. Pilipović, J. Vindas, *New distribution spaces associated to translation-invariant Banach spaces*, to appear in Monatsh. Math.

P. Dimovski, S. Pilipović, J. Vindas, *Boundary values of holomorphic functions in translation-invariant distribution spaces*, submitted.

P. Dimovski, S. Pilipović, B. Prangoski, J. Vindas, *Convolution of ultradistributions and ultradistribution spaces associated to translation-invariant Banach spaces*, submitted.

Нови Сад, 2014 Павел Димовски

University of Novi Sad
Faculty of Sciences
Key Words Documentation

Accession number:
ANO
Identification number:
INO
Document type: Monograph type
DT
Type of record: Printed text
TR
Content Code: PhD Dissertation
CC
Author: Pavel Dimovski
AU
Mentor: Academician Professor Stevan Pilipović, PhD
and Professor Jasson Vindas, PhD
MN
Title: Translation invariant Banach spaces of distributions and boundary values of integral transform
XI
Language of text: English
LT
Language of abstract: English/Serbian
LA
Country of publication: Serbia
CP
Locality of publication: Vojvodina
LP
Publication year: 2014
PY
Publisher: Author's reprint
PU
Publ. place: University of Novi Sad, Faculty of Sciences, Department of mathematics and informatics, Trg Dositeja Obradovića 4
PP

Physical description: (chapters/pages/literature/tables/images/graphs/appendices)
(6, 131, 116, 0, 0, 2, 0)
PD
Scientific field: Mathematics
SF
Scientific discipline: Functional analysis
SD
Key words: Tempered distributions; Tempered ultradistributions; Translation-invariant; Convolution of distributions; Convolution of ultradistributions; Convolutors; Beurling algebra; Parametrix
KW
Holding data: Library of Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad
HD
Note:

Abstract: We use the common notation $*$ for distribution (Schwartz), (M_p) (Beurling) and $\{M_p\}$ (Roumieu) setting. We introduce and study new (ultra)distribution spaces, the test function spaces $\mathcal{D}^*_E$ and their strong duals $\mathcal{D}'^*_{E'_*}$. These spaces generalize the spaces $\mathcal{D}^*_{L^q}$, $\mathcal{D}'^*_{L^p}$, $\mathcal{B}'^*$ and their weighted versions. The construction of our new (ultra)distribution spaces is based on the analysis of a suitable translation-invariant Banach space of (ultra)distributions E with continuous translation group, which turns out to be a convolution module over the Beurling algebra L^1_ω, where the weight ω is related to the translation operators on E. The Banach space E'_* stands for $L^1_{\check{\omega}} * E'$. We apply our results to the study of the convolution of ultradistributions. The spaces of convolutors $\mathcal{O}'^*_C(\mathbb{R}^n)$ for tempered ultradistributions are analyzed via the duality with respect to the test function spaces $\mathcal{O}^*_C(\mathbb{R}^n)$, introduced in this thesis. Using the properties of translation-invariant Banach space of ultradistributions E we obtain a full characterization of the general convolution of Roumieu ultradistributions via the space of integrable ultradistributions is obtained. We show: The convolution of two Roumieu ultradistributions $T, S \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ exists if and only if $\left(\varphi * \check{S}\right) T \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$. We study boundary values of holomorphic functions defined in tube domains. New edge of the wedge theorems are obtained. The results are then applied to represent $\mathcal{D}'_{E'_*}$ as a quotient space of holomorphic functions. We also give representations of elements of $\mathcal{D}'_{E'_*}$ via the heat kernel method.

AB

Accepted by the Scientific Board on: 21.12.2012
Defended:
Thesis defend board:

President:

Member:

Member:

Универзитет у Новом Саду
Природно-Математички факултет
Кључна Документацијска Информација

Редни број:
Идентификациони број:
ИБР
Тип документације: Монографска документација
ТД
Тип записа: Текстуални штампани материјал
ТЗ
Врста рада: Докторска дисертација
ВР
Аутор: Павел Димовски
АУ
Ментор: Академик Професор др Стеван Пилиповић
Професор др Јассон Виндас
МН
Наслов рада: Транслационо инваријантни Банахови простори дистрибуција и граничне вредности преко интегралне трансформације
МР
Језик публикације: Енглески
ЈП
Језике извода: Енглески/Српски
ЈИ
Земља публиковања: Република Србија
ЗП
Уже географско подручје: Војводина
УГП
Година: 2014
ГО
Издавач: Ауторски репринт
ИЗ
Место и адреса: Универзитет у Новом Саду, Природно-математички факултет, Департман за математику и информатику, Трг Доситеја Обра-

довића 4
МА
Физички опис рада: (број поглавља, број страна, број лит. цитата, број табела, број слика, број графика, број прилога)
(6, 131, 116, 0, 0, 2, 0)
ФО
Научна област: Математика
НО
Научна дисциплина: Функционална анализа
НД
Кључне речи: Темпериране дистрибуције; Темпериране ултрадистрибуције; Транслационо-инваријантан; Конволуција дистрибуција; Конволуција ултрадистрибуција; Конволуторе; Берлинг алгебра; Параметрикс
ПО
УДК:
Чува се: Библиотека Департмана за математику и информатику, Природно-математички факултет, Универзитет у Новом Саду
ЧУ
Важна напомена:
ВН

Извод: Користимо ознаку $*$ за дистрибуционо (Сварцово), (M_p) (Берлингово) и $\{M_p\}$ (Роумиеуово) окружење. Уводимо и проуавамо нове (ултра)дистрибуционе просторе, тест функцијске просторе $\mathcal{D}^*_E$ и њихове дуале $\mathcal{D}'^*_{E'_*}$. Ови простори уопштавају просторе $\mathcal{D}^*_{L^q}$, $\mathcal{D}'^*_{L^p}$, $\mathcal{B}'^*$ и њихове тежинске верзије. Конструкција наших нових (ултра)дистрибуционих простора је заснована на анализи одговарајућих транслационо - инваријантних Банахових простора (ултра)дистрибуцијз које ознаавамо са E. Ови простори имају непрекидну групу транслација, која је конволуциони модул над Беурлинговом алгебром L^1_ω, где је теина ω повезана са операторима транслације простора E. Банахов простор E'_* означава простор $L^1_{\check{\omega}} * E'$. Користечи добијене резултата проучавамо конволуцију ултрадистрибуција. Простори конволутора $\mathcal{O}'^*_C(\mathbb{R}^n)$ темперираних ултрадистрибуција, анализирани су помочу дуалности тест функцијских простора $\mathcal{O}^*_C(\mathbb{R}^n)$, дефинисаних у овој тези. Користечи својства транслационо - инваријантних Банахових простора темперираних ултрадистрибуција, опет ознаених са E, добијамо карактеризацију конволуције Ромуиеу-ових ултрадистрибуција, преко интеграбилних ултрадистрибуција. Доказујемо да: конволуција две Роумиеу-ове ултрадистрибуција $T, S \in \mathcal{D}'^{\{M_p\}}(\mathbb{R}^n)$ постоји ако и само ако $(\varphi * \check{S})\, T \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^n)$ за сваки $\varphi \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$. Такође, проучавамо граничне вредности холомпрфних функција дефинисаних на тубама. Доказане су нове теореме "отрог клина". Резултати се затим користе за презентацију $\mathcal{D}'_{E'_*}$ преко фактор простора холоморфних функција. Такође, дата је презентација елементе $\mathcal{D}'_{E'_*}$ користечи хеат кернел методе.

ИЗ

Датум прихватања теме од стране НН већа: 21.12.2012
ДП
Датум одбране:
ДО
Чланови комисије:
КО

Председник:

Члан:

Члан: