UNIVERZITET U NOVOM SADU

FAKULTET TEHNIČKIH NAUKA

Strong positional games

-doctoral dissertation-

Jake pozicione igre

-doktorska disertacija-

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Novi Sad, 2023.

УНИВЕРЗИТЕТ У НОВОМ САДУ **ОБРАЗАЦ – 5а**
ФАКУЛТЕТ ТЕХНИЧКИХ НАУКА

КЉУЧНА ДОКУМЕНТАЦИЈСКА ИНФОРМАЦИЈА[1]

Врста рада:	Докторска дисертација
Име и презиме аутора:	Јелена Стратијев
Ментор (титула, име, презиме, звање, институција)	др Милош Стојаковић, редовни професор, Природно-Математички факултет, Нови Сад
Наслов рада:	Јаке позиционе игре
Језик публикације (писмо):	Енглески (латиница)
Физички опис рада:	Унети број: Страница 129 Поглавља 6 Референци 111 Табела 0 Слика 26 Графикона 0 Прилога 0
Научна област:	Примењена математика
Ужа научна област (научна дисциплина):	Дискретна математика
Кључне речи / предметна одредница:	Позиционе игре на графовима, јаке игре, Авојдер-Авојдер игре, Мејкер-Мејкер игре, ачивмент број, игре сатурације

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Аутор докторске дисертације потписао је и приложио следеће Обрасце:

5б – Изјава о ауторству;

5в – Изјава о истоветности штампане и електронске верзије и о личним подацима;

5г – Изјава о коришћењу.

Ове Изјаве се чувају на факултету у штампаном и електронском облику и не кориче се са тезом.

Резиме на језику рада:	У овој тези проучавамо комбинаторне игре на графовима које играју 2 играча. Посебну пажњу посвећујемо јаким позиционим играма, у којима оба играча имају исти циљ. Прво, посматрамо такозвану јаку Авојдер-Авојдер игру са задатим фиксним графом у којој два играча, Црвени и Плави наизменично селектују гране комплетног графа K_n, а играч који први селектује копију фиксног графа F губи игру. Ако ниједан од играча не садржи копију од F у свом графу и сви елементи табле су селектовани, игра се проглашава нерешеном. Иако су ове игре проучаване деценијама, врло је мало познатих резултата. Ми смо направили корак напред доказавши да Плави има победничку стратегију у две различите игре ове врсте. Такође, уводимо јаке ЦАвојдер-ЦАвојдер F игре у којима граф сваког играча мора остати повезан током игре. Ово је природно проширење јаких Авојдер-Авојдер игара, са ограничењем повезаности. Доказујемо да Плави може да победи у три стандардне ЦАвојдер-ЦАвојдер F игре. Затим проучавамо јаке Мејкер-Мејкер F игре, у којима је играч који први селектује копију од F победник. Познато је да исход ових игара уколико оба играча играју оптимално може бити или победа првог играча или нерешено. Циљ нам је да пронађемо ачивмент број $a(F)$ јаке Мејкер-Мејкер F игре, односно најмање n за које Црвени има победничку стратегију. Дајемо тачну вредност $a(F)$ за неколико графова F, укључујући путеве, циклусе, савршене мечинге и поткласу стабала са n чворова. Такође, дајемо горње и доње ограничење ачивмент броја за звезде и стабла. Коначно, уводимо уопштене игре сатурације као природно проширење две различите врсте комбинаторних игара, игара сатурације и Конструктор-Блокер игара. У уопштеној игри сатурације унапред су дата два графа H и F. Два играча по имену Макс и Мини наизменично селектују слободне гране комплетног графа K_n и заједно постепено граде граф игре G, који се састоји од свих грана које су селектовала оба играча. Граф G не сме да садржи копију од F, а игра се завршава када више нема потеза, односно када је G сатуриран граф који не садржи F. Занима нас резултат ове игре, односно, број копија графа H у G на крају игре. Макс жели да максимизира овај резултат, док Мини покушава да га минимизира. Игра се под претпоставком да оба играча играју оптимално. Проучавамо неколико уопштених игара сатурације за природне изборе F и H, у настојању да што прецизније одредимо резултат игре.

Датум прихватања теме од стране надлежног већа:	26.01.2023.
Датум одбране: (Попуњава одговарајућа служба)	
Чланови комисије: (титула, име, презиме, звање, институција)	Председник: др Силвиа Гилезан, редовни професор, Факултет техничких наука, Нови Сад Члан: др Јелена Иветић, ванредни проесор, Факултет техничких наука, Нови Сад Члан: др Мирјана Микалачки, ванредни професор, Природно-математички факултет, Нови Сад Члан: др Марко Радовановић, ванредни професор, Математички факултет, Београд Члан, ментор: др Милош Стојаковић, редовни професор, Природно-математички факултет, Нови Сад
Напомена:	

UNIVERSITY OF NOVI SAD
FACULTY OF TECHNICAL SCIENCES

KEY WORD DOCUMENTATION[2]

Document type:	Doctoral dissertation
Author:	Jelena Stratijev
Supervisor (title, first name, last name, position, institution)	PhD Miloš Stojaković, Full Professor, Faculty of Sciences, Novi Sad
Thesis title:	Strong positional games
Language of text (script):	English
Physical description:	Number of: Pages 129 Chapters 6 References 111 Tables 0 Illustrations 26 Graphs 0 Appendices 0
Scientific field:	Applied mathematics
Scientific subfield (scientific discipline):	Discrete mathematics
Subject, Key words:	Positional games on graphs, Strong games, Avoider-Avoider games, Maker-Maker games, achievement number, Saturation games
Abstract in English language:	In this thesis, we study 2-player combinatorial games on graphs. We devote a lot of attention to strong positional games, where both players have the same goal. First, we consider the so-called fixed graph strong Avoider-Avoider game in which two players called Red and Blue alternately claim edges of the complete graph

2 The author of doctoral dissertation has signed the following Statements:
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	K_n, and the player who first completes a copy of a fixed graph F loses the game. If neither of the players claimed a copy of F in his graph and all the elements of the board are claimed, the game is declared a draw. Even though these games have been studied for decades, there are very few known results. We make a step forward by proving that Blue has a winning strategy it two different games of this kind. Furthermore, we introduce strong CAvoider-CAvoider F games where the claimed edges of each player must form a connected graph throughout the game. This is a natural extension of the strong Avoider-Avoider games, with a connectedness constraint. We prove that Blue can win in three standard CAvoider-CAvoider F games. Next, we study strong Maker-Maker F games, where now, the player who first occupies a copy of F is the winner. It is well-known that the outcome of these games when both players play optimally can be either the first player's win or a draw. We are interested in finding the achievement number a(F) of a strong Maker-Maker F game, that is, the smallest n for which Red has a winning strategy. We can find the exact value a(F) for several graphs F, including paths, cycles, perfect matchings, and a subclass of trees on n vertices. We also give the upper and lower bounds for the achievement number of stars and trees. Finally, we introduce generalized saturation games as a natural extension of two different types of combinatorial games, saturation games and Constructor-Blocker games. In the generalized saturation game, two graphs H and F are given in advance. Two players called Max and Mini alternately claim unclaimed edges of the complete graph K_n and together gradually building the game graph G, the graph that consists of all edges claimed by both players. The graph G must never contain a copy of F, and the game ends when there are no more moves, i.e. when G is a saturated F-free graph. We are interested in the score of this game, that is, the number of copies of the graph H in G at the end of the game. Max wants to maximize this score, whereas Mini tries to minimize it. The game is played under the assumption that both players play optimally. We study several generalized saturation games for natural choices of F and H, in an effort to locate the score of the game as precisely as possible.
Accepted on Scientific Board on:	26.1.2023

Defended: (Filled by the faculty service)	
Thesis Defend Board: (title, first name, last name, position, institution)	President: Dr. Silvia Gilezan, Full Professor, Faculty of Technical Sciences, Novi Sad Member: Dr. Jelena Ivetić, Associate Professor, Faculty of Technical Sciences, Novi Sad Member: Dr. Mirjana Mikalački, Associate Professor, Faculty of Sciences, Novi Sad Member: Dr. Marko Radovanović, Associate Professor, Faculty of Mathematics, Belgrade Member, Mentor: Dr. Miloš Stojaković, Full Professor, Faculty of Sciences, Novi Sad
Note:	

Preface

In this thesis, we study 2-player combinatorial games on graphs, which means that they are played with perfect information, no chance moves and sequentially play. Structurally, a positional game consists of a board on which the game is played (a finite set X), and a family of target sets (a family $\mathcal{F}$ of subsets of X). When the game is played, the players alternately claim unclaimed elements of the board until all the elements are claimed. When it comes to determining the winner of the game, there are several conventions of positional games, some of which are Maker-Breaker games, Avoider-Enforcer games, strong Maker-Maker games, and strong Avoider-Avoider games. We devote a lot of attention to strong positional games. In these games, both players have the same goal, and generally speaking, that makes them more complex and harder to analyze.

First, we consider the so-called fixed graph strong Avoider-Avoider game in which two players called Red and Blue alternately claim edges of the complete graph K_n, and the player who completes a copy of a fixed graph F first loses the game. If neither of the players claimed a copy of F in his graph and all the elements of the board are claimed, the game is declared a draw. Even though these games have been studied for decades, they turned out to be notoriously hard to analyze and consequently, there are very few known results. It is proven that Blue wins for F a 2-path in [62], and that Blue wins for $F = S_k$, where S_k represents a star on k vertices, see [17]. We make a step forward by analyzing two more games, proving that Blue has a winning strategy for F a 3-path and for $\mathcal{CC}_{>3}$, where $\mathcal{CC}_{>3}$ is a family of inclusion-minimal connected graphs on more than three vertices. Furthermore, we introduce strong CAvoider-CAvoider F games where the claimed edges of each player must form a connected graph throughout the game. This is a natural extension of the strong Avoider-Avoider games, with a connectedness constraint analog to the ones introduced in [82] and similar to [42, 50]. We prove that Blue can win in three standard CAvoider-CAvoider F games.

Next, we study strong Maker-Maker F games, where now, the player who occupies a copy of F first is the *winner*. It is well-known that the outcome of these games when both players play optimally can be either the first player's win or a draw. We are interested in finding the achievement number $a(F)$ of a strong Maker-Maker F game, that is, the smallest n for which Red has a winning strategy, introduced by Harary in [61]. We can find the exact value $a(F)$ for several graphs F, including paths, cycles, perfect matchings, and a subclass of trees on n vertices. We also give the upper and lower bounds for the achievement number of stars and trees.

Finally, we investigate a problem laid out by Patkós, Stojaković and Vizer in [93], where the concept of generalized saturation games is first introduced as a natural extension of two different types of combinatorial games – saturation games and Constructor-Blocker games. In the generalized saturation game, two graphs H and F are given in advance. Two players called Max and Mini alternately claim unclaimed edges of the complete graph K_n and together gradually build the game graph G, the graph that consists of all edges claimed by both players. The graph G must never contain a copy of F, and the game ends when there are no more moves, i.e. when G is a saturated F-free graph. We are interested in the score of this game, that is, the number of copies of the graph H in G at the end of the game. Max wants to maximize this score, whereas Mini tries to minimize it. The game is played under the assumption that both players play optimally. We study several generalized saturation games for natural choices of F and H, in an effort to locate the score of the game as precisely as possible.

The thesis is organized as follows.

In Chapter 1 we introduce the basic notions and theoretic concepts that lie in the background of the obtained results. Our goal is to make a solid foundation for a formal introduction of combinatorial games, various conventions of positional games and saturation games.

In Chapter 2 we list our main results that will be proven in the rest of this thesis.

In Chapter 3 we take a closer look at the strong Avoider-Avoider games, giving a Blue's winning strategy for two different games. We also introduce strong CAvoider-CAvoider games and present a winning strategy for Blue in a number of well-studied fixed graph games.

The results of this chapter are published in:

- M. Stojaković and J. Stratijev, On strong avoiding games, *Discrete Mathematics*, **346** (2023), 113270 [103].

In Chapter 4 we study the achievement number $a(F)$ in strong Maker-Maker games, and we find that value for several graphs F.

The results of this chapter will be published in:

- M. Stojaković and J. Stratijev, On achievement number in strong Maker-Maker games, in preparation [104].

In Chapter 5 we introduce generalized saturation games, and we find the score for a number of pairs of graphs H and F, many of which were previously studied in similar settings.

The results of this chapter will be published in:

- M. Stojaković and J. Stratijev, On generalized saturation games, in preparation [105].

Finally, in Chapter 6 we give some open problems and concluding remarks.

Novi Sad, 2023. Jelena Stratijev

Acknowledgement

I would like to express my sincere gratitude to my advisor Professor Miloš Stojaković, for his guidance, support, and encouragement throughout my doctoral studies. His expertise and knowledge in the field have been invaluable in shaping my research.

I would also like to thank the members of my dissertation committee, Professor Silvia Gilezan, Professor Jelena Ivetić, Professor Mirjana Mikalački, and Professor Marko Radovanović, for their valuable feedback, insights, and suggestions that helped me to improve my work.

I extend my appreciation to my family for their love, support, and patience during this challenging journey. Their encouragement and belief in me have been my driving force.

Finally, I would like to express my sincere appreciation and gratitude to my husband and daughter for their unwavering love, support, and encouragement. They have been my pillars of strength through thick and thin, and their presence in my life has made it all the more meaningful and fulfilling.

Thank you all for your invaluable contributions to my academic and personal growth.

Contents

Prošireni izvod

Igre

U ovoj tezi su proučavane igre koje igraju dva igrača i za čiju analizu koristimo različite matematičke alate. Posmatrane su igre *savršenih informacija*, što znači da svaki igrač pre nego što odigra svoj potez zna sve prethodne poteze ostalih igrača. Mi smo zainteresovani za igre kod kojih nema slučajnih poteza i koje se igraju *sekvencijalno*, što znači da igrači povlače svoje poteze naizmenično.

Osim pomenutih igara, o kojima će biti više reči kasnije, postoji mnogo različitih vrsta igara koje izučavamo koristeći matematičke alate. Pre nego što nastavimo dalje, reći ćemo nekoliko reči i o njima. Naučna grana koja u poslednje vreme privlači puno pažnje je teorija igara.

Teorija igara proučava matematičke modele strateških interakcija između racionalnih agenata. Uopšteno govoreći, uključuje igre na sreću, igre nesavršenog znanja i igre u kojima igrači povlače poteze istovremeno. Ove igre imaju tendenciju predstavljanja situacije donošenja odluka u stvarnom životu. To je veoma primenljiva grana nauke, posebno u ekonomiji, logici, informatici, itd. Jednu od prvih knjiga iz teorije igara napisao je Neumann 1944. godiine, videti [88], nakon koje kreće nagli razvoj ove nauke.

Za razliku od igara savršenih informacija koje smo prethodno pomenuli, *igre nesavršenih informacija* igraju igrači koji ne znaju sve poteze koje su odigrali njihovi protivnici. Većina igara koje se proučavaju u teoriji igara su igre nesavršenih informacija, kao što su igre simultanih poteza, i većina igara sa kartama.

Igre sa nultom sumom (engl. *zero-sum games*) su one u kojima izbori igrača ne mogu ni povećati niti smanjiti ukupne raspoložive resurse – jedan igrač može dobiti samo onoliko resursa koliko drugi igrač izgubi. Jedan primer igre sa nultom sumom je poker jer jedan igrač može da osvoji tačno onoliko koliko drugi igrač izgubi.

Sa druge strane, u *igrama bez nulte sume*, dobitak jednog igrača ne mora nužno da odgovara gubitku drugog. Mnoge igre koje proučava teorija igara su ove vrste, a mi ćemo dati jedan primer koji se zove zatvorenička dilema, videti [2].

Primer 0.0.1. Vlasti su uhvatile dva zatvorenika. Oni su razdvojeni i svakome je dat izbor između priznanja i ćutanja. Dogodiće se jedan od četiri moguća ishoda. Ako zatvorenik A prizna dok drugi ćuti, zatvorenik A izlazi na slobodu. Ako obojica ćute, svako dobija po godinu dana zatvora. Ako obojica priznaju, svako dobija po pet godina zatvora. Ako zatvorenik A ćuti dok drugi priznaje, zatvorenik A se suočava sa desetogodišnjom kaznom, dok zatvorenik B izlazi na slobodu.

Ispostavlja se da svaki zatvorenik dobija veću nagradu za izdaju onog drugog. Pretpostavimo da smo na mestu zatvorenika A. U slučaju da je zatvorenik B priznao, zatvorenik A može ćutati i dobiti desetogodišnju kaznu ili izdati i dobiti kaznu od pet godina. U suprotnom, ako je zatvorenik B ćutao, zatvorenik A takođe može da ćuti i dobije kaznu od godinu dana ili prizna i ode kući. Stoga, bez obzira na to šta zatvorenik B odluči, za zatvorenika A najbolja opcija je izdaja.

Za skup strategija kažemo da je *Nešova ravnoteža* (engl. *Nash equilibrium*) ako nijedan igrač ne može učiniti svoju strategiju boljom. Zamislimo da svaki igrač zna strategije drugih. Imajući sve te strategije na umu svaki igrač se pita: „Mogu li imati koristi od promene strategije?“ Ako bi bilo koji igrač mogao da odgovori sa „Da”, onda taj skup strategija nije Nešova ravnoteža. U suprotnom, ako ni jedan igrač ne želi da menja svoju strategiju, onda taj skup strategija jeste Nešova ravnoteža.

Posmatrajući primer 0.0.1 shvatamo da je Nešova ravnoteža (izdaja, izdaja), dok je društveni optimum (ćutati, ćutati).

Kombinatorne igre

Kombinatornu igru igraju dva igrača, u pitanju je igra savršenih informacija bez slučajnih poteza, pogledati npr. [51]. U poređenju sa klasičnom teorijom igara, glavna razlika je u tome što igrači igraju naizmenično, a ne istovremeno, tako da nema skrivenih informacija. Kombinatorne igre uključuju dobro poznate igre kao što su šah, dame i Go. Mogu biti igrane i na beskonačnoj tabli. Svi mogući potezi u kombinatornoj igri mogu se predstaviti stablom igre. Jedna od najproučavanijih igara ove vrste je Nim za čije se korene veruje da potiču iz Kine, dok je pod današnjim imenom prvi put uvedena u [21].

Primer 0.0.2. Nim je igra koju igraju dva igrača koji naizmenično uklanjaju predmete sa različitih gomila, formiranih na početku igre. Igrač bira jednu gomilu i uklanja najmanje jedan predmet sa nje u svakom svom potezu. Postoje dve verzije igre, u prvoj je cilj da se izbegne uzimanje poslednjeg predmeta, dok je u drugoj cilj da se uzme poslednji predmet.

Pozicione igre

Pozicione igre su kombinatorne igre. To su konačne igre savršenih informacija bez slučajnih poteza. Primeri pozicionih igara su Iks-Oks, Heks, Sim, igra red-kolona, itd.

Istorijski prvi radovi o pozicionim igrama pojavili su se 1963. godine, kada su Hales i Jewett napisali [60], a zatim su 1973. godine Erdős i Selfridge publikovali rad [40]. Kasnije je József Beck doneo mnoge nove ideje, pitanja i otvorene probleme u ovoj oblasti. Objavio je brojne radove i knjigu [9] koja pokriva širok spektar oblasti o pozicionim igrama. U ovoj knjizi može se naći mnogo teorema, primera, rešenih i nerešenih problema, kao i zanimljivih metoda za analizu novih problema. Nedavno se pojavila knjiga koju su napisali Hefetz, Krivelevich, Stojaković i Szabó [69], koja predstavlja sjajnu bazu za izučavanje ovih igara dajući uvid u aktuelne rezultate kao i otvorene probleme.

Analizirajući ove igre pretpostavljamo da oba igrača igraju sledeći svoju optimalnu strategiju. Strategiju igrača možemo zamisliti kao „knjigu" u kojoj možemo pronaći odgovor za svaki protivnikov potez, šta god da protivnik odigra. Cilj nam je da odredimo kakav će biti krajnji ishod igre, a moguće su tri opcije: pobeda prvog igrača, pobeda drugog igrača ili nerešeno. Ako je ishod igre *igrač A pobeđuje*, kažemo da *igrač A ima pobednički strategiju.*

Pitanje koje se samo nameće je da li bi korišćenjem moćnog računara mogli odrediti ishod igre. Međutim, ispostavlja se da su čak i današnji računari ograničeni po tom pitanju. Neophodno je iscrpno pretražiti celo stablo igre koje je obično eksponencijalno veliko. To je razlog zašto su matematički alati i metode najvažniji u analizi pozicionih igara.

Sada ćemo dati formalnu definiciju pozicionih igara. Poziciona igra je uređen par $(X, \mathcal{F})$, gde je X konačan skup koji se zove *tabla*, a $\mathcal{F}$ je familija *ciljnih skupova*. Igru igraju dva igrača koji naizmenično biraju slobodne elemente table X dok god tabla ne ostane prazna. Uređen par $(X, \mathcal{F})$ takođe nazivamo hipergrafom igre, čiji čvorovi su elementi table X, a hipergrane su elementi familije $\mathcal{F}$.

Kada u svakom svom potezu igrač bira tačno jedan element table onda se igra naziva *fer*. Takođe možemo definisati i *asimetričnu igru* $(a : b)$ (engl. biased game), u kojoj igrač koji prvi počinje igru bira a elemenata table po potezu, dok drugi bira b elemenata po potezu.

Kada je reč o pravilima za određivanje pobednika igre u pozicionoj igri, postoji nekoliko varijanti. Dve osnovne klase pozicionih igara su *jake* igre i *slabe* igre i svaka od njih ima svoje potklase. Više o ovim varijantama biće rečeno u narednim odeljcima.

Jake igre

U jakim pozicionim igrama, oba igrača imaju isti cilj. Među ovim igrama su veoma popularni primeri, jedan od njih je nadaleko poznata igra Iks-Oks. Možemo ih podeliti u dve velike potklase, a to su *jake Mejker-Mejker igre* i *jake Avojder-Avojder igre*.

Jake Mejker-Mejker igre

U *jakoj Mejker-Mejker igri* $(X, \mathcal{F})$ (engl. *strong Maker-Maker game*), dva igrača, koje zovemo Crveni i Plavi, naizmenično biraju slobodne elemenate table X, tako da Crveni počinje igru. Igrač koji prvi selektuje sve elemente nekog $F \in \mathcal{F}$ je pobednik. Ako nijedan od igrača ne pobedi i nema više slobodnih elemenata table, igra se proglašava *nerešenom*. Ovde se ciljni skupovi familije $\mathcal{F}$ nazivaju *pobednički skupovi*.

Primer 0.0.3. Najistaknutiji primer ove klase pozicionih igara je široko popularna igra Iks-Oks. Ova igra se igra na tabli predstavljenoj kvadratnom mrežom 3×3. Tabla se sastoji od devet elemenata, dok se familija $\mathcal{F}$ sastoji od osam pobedničkih skupova: tri horizontalne linije, tri vertikalne linije i dve dijagonale. Pobednik je prvi igrač koji poseduje celu pobedničku liniju (skup). Poznato je da ukoliko oba igrača igraju optimalno, ova igra se završava nerešenim ishodom.

Uopšteno govoreći, određivanje ishoda u jakoj Mejker-Mejker igri se pokazalo kao prilično izazovno, i jedva da postoje opšti matematički alati na raspolaganju. Jedan od njih je *krađa strategije* (engl. *strategy stealing*) koju možemo koristiti da pokažemo da Crveni može da garantuje makar nerešeno u bilo kojoj igri.

Teorema 0.0.1. *[9] U jakoj Mejker-Mejker igri $(X, \mathcal{F})$, prvi igrač može garantovati najmanje nerešeno.*

Dokaz. Pretpostavimo suprotno, da drugi igrač (Plavi) ima pobedničku strategiju S. Prvi igrač (Crveni) igra svoj prvi potez proizvoljno, a zatim krade strategiju S. Nakon prvog poteza Plavog, Crveni zamišlja da je drugi igrač i odgovara na svaki potez Plavog kako strategija S nalaže. Ako ova strategija predlaže da Crveni izabere element koji je već uzeo u svom prvom potezu, onda on igra proizvoljno. Primetimo da ovde jedan dodatni potez ne može naštetiti igraču. Dakle, prateći strategiju S, Crveni takođe pobeđuje, što je kontradikcija. $\square$

Ovaj argument potvrđuje da prvi igrač zaista ima prednost. Zaključujemo da su u ovakvoj igri moguća samo dva ishoda: pobeda Crvenog i nerešeno. Ponekad proučavamo igre u kojima nerešeno nije moguće, u tom slučaju koristimo krađu strategije da zaključimo da Crveni ima pobedničku strategiju. Loša strana ovog argumenta je činjenica da ne znamo ništa o pobedničkoj strategiji, a pravljenje eksplicitne pobedničke strategije vrlo često deluje beznadežno.

Kao što smo već videli, u jakoj Mejker-Mejker igri, dovoljno je znati da nerešeno nije opcija, pa da znamo pobednika. Matematički alat koji možemo koristiti u ovoj situaciji je Remzijeva osobina hipergrafa igre. Ako igra $(X, \mathcal{F})$ ima Remzijevu osobinu, to znači da svako bojenje table u dve boje, crvenu i plavu, daje monohromatski skup $F \in \mathcal{F}$. Prema tome, ako igra ima Remzijevu osobinu, nerešen ishod je nemoguć.

Skoro na kraju liste alata je *strategija uparivanja* (engl. *pairing strategy*), koju možemo koristiti da pokažemo da Plavi može da garantuje nerešeno.

Zaista, ako možemo da napravimo disjunktno uparivanje elemenata table X, tako da svaki pobednički skup sadrži jedan element nekog od parova, onda Plavi uvek može da izabere drugi element iz para koji je Crveni odabrao, što sprečava Crvenog da pobedi.

Nije previše iznenađujuće da se u literaturi može naći tako malo rezultata o jakim Mejker-Mejker igrama.

Nedavno se u radu [44] pojavila zanimljiva ideja, do koje su došli Ferber i Hefetz. Oni su dokazali da igrajući na skupu grana kompletnog grafa K_n, za dovoljno veliko n, Crveni može da pobedi u igri savršenog mečinga (engl. *perfect matching game*) i igri Hamiltonove konture (engl. *Hamilton cycle game*), dok su u radu [45] isti autori dokazali da, za dovoljno veliko n i svaki pozitivan ceo broj k, prvi igrač može pobediti u igri k-povezanosti (engl. *k-Connectivity game*). Oba rada se oslanjaju na strategiju brze pobede u slabim igrama.

Jake Avojder-Avojder igre

Druga vrsta jakih igara su *jake Avojder-Avojder igre* $(X, \mathcal{F})$ (engl. *strong Avoider-Avoider games*). Ovu igru ponovo igraju dva igrača Crveni i Plavi, ali sada igrač koji prvi potpuno selektuje neki $F \in \mathcal{F}$ *gubi igru.* Ako nijedan od igrača ne izgubi i svi elementi table su selektovani, igra se proglašava *nerešenom.* Ovde se ciljni skupovi familije $\mathcal{F}$ nazivaju *gubitnički skupovi.*

Primer 0.0.4. Prvu takvu igru, poznatu pod imenom *Sim*, uveo je 1961. Simmons [99]. Tabla ove igre je skup grana kompletnog grafa K_6, a igrač koji prvi selektuje trougao u svom grafu gubi. Iako je jasno da je nerešeno nemoguće (koristeći Remzijevo svojstvo table), a pritom je tabla relativno mala (ima samo petnaest grana), analiziranje ove igre je izazovno, a dokaz da Plavi pobeđuje je izveden uz pomoć kompjutera.

U [100] Slany je dao metodološku studiju o kompleksnosti određivanja pobednika za nekoliko igara sličnih igri Sim. Zatim su Mead, Rosa i Huang u [87] dali eksplicitnu pobedničku strategiju za Plavog u igri Sim, a nedavno je u [110] Wrzos-Kaminska dala jednostavnu pobedničku strategiju koju može da igra čovek bez pomoći računara. Druge varijante jakih Avojder-Avojder igara proučavao je Harary u [61], koji je posmatrao konačne igre na grafovima do šest čvorova.

Na prvi pogled može izgledati da u jakim Avojder-Avojder igrama, za razliku od jakih Mejker-Mejker igara, Plavi uvek ima prednost, a Crveni kao prvi igrač ne može očekivati pobedu igrajući optimalno. Ispostaviće se da ovo nije tačno!

Na primer, u d-dimenzionalnoj igri Iks-Oks n^d (pogledati [9] za više detalja), gde je n neparno, Crveni ima eksplicitnu pobedničku strategiju: U svom prvom potezu bira centralni element, označimo ga sa C. Nakon toga, kad god Plavi odabere element P, Crveni bira P' koji je simetričan u odnosu na C. Pretpostavimo suprotno, da Crveni gubi, tj. da njegov graf ima crvenu liniju L (primetimo da nije moguće da C pripada L), tada je L', njena simetrična slika u odnosu na centar kocke, plava linija koja je selektovana pre L, što je kontradikcija.

Znajući da se igra 3^3 ne može završiti nerešenim rezultatom (koristeći Remzijevo svojstvo [9]), možemo zaključiti da Crveni pobeđuje. Johnson, Leader i Walters su dokazali da postoje tranzitivne igre u kojima Crveni pobeđuje, za sve veličine table osim prostog broja i stepena dvojke [77].

Mejker-Brejker igre

Za razliku od jakih pozicionih igara u kojima se dva igrača takmiče u želji da postignu isti cilj, u *slaboj igri* igrači imaju različite ciljeve. Prvi igrač želi da ostvari svoj cilj, dok drugi igrač samo pokušava da spreči prvog igrača u postizanju cilja.

U *Mejker-Brejker pozicionoj igri* $(X, \mathcal{F})$ (engl. *Maker-Breaker game*), koju takođe zovemo i „slaba igra”, dva igrača se zovu Mejker i Brejker, a elementi familije $\mathcal{F}$ se nazivaju pobednički skupovi. Mejker pobeđuje u igri ako do kraja igre selektuje sve elemente nekog $F \in \mathcal{F}$, dok u suprotnom pobeđuje Brejker.

Kao i u jakim Mejker-Mejker igrama, Mejker želi da selektuje sve elemente nekog pobedničkog skupa, ali sada to ne mora da uradi prvi. Štaviše, sada je potpuno nebitno da li će Plavi selektovati pobednički set ili ne. Ova igra ne može biti završena nerešenim rezultatom, Mejker pobeđuje ako do kraja igre selektuje neki pobednički skup, a Brejker pobeđuje ako uzme bar jedan element u svakom od pobedničkih skupova.

Primer 0.0.5. Posmatramo igru Iks-Oks igranu po pravilima Mejker-Brejker igre. Mejker želi da stavi tri svoja znaka u istu pobedničku liniju, ali sada ne mora da razmislja da li će i Brejker staviti tri svoja znaka u neku pobedničku liniju. Imajući ovo na umu, lako se može pronaći Mejkerova pobednička strategija.

Obično posmatramo igre koje se igraju na kompletnom grafu K_n, što znači da je tabla igre skup grana kompletnog grafa sa n čvorova. Neke od najproučavanijih igara ove vrste su: *igra savršenog mečinga* $\mathcal{M}_n$, u kojoj su pobednički skupovi svi savršeni mečinzi grafa K_n, zatim *igra povezanosti* $\mathcal{C}_n$ (engl. *Connectivity game*), gde su pobednički skupovi sva pokrivajuća stabla od K_n, i igra *Hamiltonove konture* $\mathcal{H}_n$, gde su pobednički skupovi svi Hamiltonovi ciklusi od K_n.

U nekim Mejker-Brejker igrama nije teško dokazati da Mejker može da pobedi, za te igre se prirodno pojavilo pitanje „Koliko brzo to može da uradi?". Pokazalo se da je ovo pitanje važno jer njegov odgovor ima brojne primene u drugim vrstama pozicionih igara. Videli smo u prethodnom odeljku da je jake igre teško analizirati, ono što je interesantno je da brza pobeda Mejkera podrazumeva pobedu Crvenog u jakoj Mejker-Mejker igri. Zaista, pošto je Mejker prvi igrač, ako uspe da selektuje ceo pobednički skup A u $|A|$ poteza, gde je A najmanji pobednički skup, igra se završava i pre nego što Plavi ima priliku da odigra svoj $|A|$-ti potez. Dakle, Crveni ne mora da razmišlja o tome kako da spreči Plavog, nego samo kako da selektuje ceo pobednički skup za sebe.

U sledećem primeru biće data eksplicitna pobednička strategija za Mejkera, za $n \geq 6$.

Primer 0.0.6. [69] Igra trougla se igra na skupu grana kompletnog grafa K_n, pobednički skupovi su kopije grafa K_3 u grafu K_n. Mejker započinje igru tako što bira proizvoljnu granu uv, a zatim u svom prvom potezu Brejker bira granu xy. Označimo sa vw drugi potez Mejkera, takav da $w \notin \{u, v, x, y\}$. Ako drugi potez Brejker-a nije grana uw, onda Mejker bira tu granu i pobeđuje u svom trećem potezu. U suprotnom, ako Brejker izabere baš uw, onda Mejker bira granu vz za neki čvor $z \notin \{u, v, x, y, w\}$. Sada postoje dve slobodne grane čijim izborom Mejker pravi trougao u svom grafu (zw i zu), pošto Brejker ne može da selektuje obe u svom sledećem potezu, Mejker pravi trougao u četvrtom potezu i pobeđuje.

Mejker-Brejker asimetrične igre

Kao što se može videti u strategiji iz primera 0.0.6, Mejker lako pobeđuje u ovoj igri, dok su šanse Brejkera jednake nuli. Prirodno se nameće pitanje kako malom promenom pravila možemo Brejkeru dati veću moć. Jedan standardni način da se to uradi po prvi put su predložili Chvátal i Erdős u [28]. Ovu vrstu igre nazivamo *asimetrična* igra.

Definicija 0.0.7. [69] Neka su p i q pozitivni celi brojevi, neka je X konačan skup, i neka je $\mathcal{F} \subseteq 2^X$ familija podskupova od X. U *asimetričnoj igri* $(p : q)$ Mejker-Brejker $(X, \mathcal{F})$, Mejker selektuje p slobodnih elemenata table po potezu i Brejker selektuje q slobodnih elemenata table po potezu. Cele brojeve p i q nazivamo *biasom* Mejkera i Brejkera, redom. Ako u poslednjem potezu igrač nema dovoljno slobodnih elemenata na tabli, on selektuje sve slobodne elemente table i igra se završava.

Kada su Chvátal i Erdős uveli $(1 : b)$ Mejker-Brejker igre [28], primetili su da su one *bias monotone*. Da budemo precizniji, ako Brejker može da pobedi u igri $(1 : b)$, može da pobedi i u igri $(1 : b + 1)$. Primetimo da ako je tabla igre X, onda Brejker pobeđuje u igri $(1 : |X|)$, osim u slučaju kada familija pobedničkih skupova $\mathcal{F}$ sadrži prazan ili jednočlan skup. Prema tome, možemo definisati *granični bias* igre $(X, \mathcal{F})$ kao jedinstveni pozitivan ceo broj $b_{\mathcal{F}}$ takav da je $(1 : b)$ igra pobeda Brejkera ako i samo ako je $b > b_{\mathcal{F}}$, pod pretpostavkom da su $\mathcal{F} \neq \emptyset$ i $min\{|A| : A \in \mathcal{F}\} \geq 2$.

Asimetrična Mejker-Brejker $(a : b)$ igra $(X, \mathcal{F})$ je takođe bias monotona, tj. ako Mejker može da pobedi u igri $(a : b)$, onda može da pobedi i u igri $(a + 1 : b)$ kao i $(a : b - 1)$. Ista stvar važi i za Brejkera, ako je $(a : b)$ Mejker-Brejker $(X, \mathcal{F})$ igra Brejkerova pobeda, onda Brejker takođe može da pobedi u $(a : b + 1)$ i $(a - 1 : b)$ igri. Generalno, ako igrač selektuje više elemenata u bilo kom trenutku igre, to mu ne može naškoditi.

Avojder-Enforser igre

Avojder-Enforser igre (engl. *Avoider-Enforcer*) su *misère* verzija Mejker-Brejker igara, sa dva igrača po imenu Avojder i Enforser. Pravila u ovim igrama su na neki način suprotna pravilima u Mejker-Brejker igrama. Zaista, dok je cilj Mejkera da selektuje ceo pobednički skup i Brejker želi da ga spreči u tome, Avojderov cilj je da *izbegne* da selektuje ceo pobednički skup, dok

Enforser želi da *natera* Avojdera da to uradi. Preciznije, Enforser pobeđuje u igri $(X, \mathcal{F})$ ako na kraju igre Avojder selektuje sve elemente nekog $F \in \mathcal{F}$, u suprotnom Avojder pobeđuje.

Asimetrična verzija ove igre definiše se na isti način kao u Mejker-Brejker igrama. U Avojder-Enforser igri $(a : b)$ koja se igra na tabli X sa datom familijom *gubitničkih skupova* $\mathcal{F}$, Avojder selektuje a elemenata table po potezu, dok Enforser selektuje b elemenata table po potezu. Ako u poslednjem potezu ima manje slobodnih elemenata na tabli od biasa igrača koji treba da igra, on selektuje sve preostale slobodne elemente table.

Kao što smo već videli, Mejker-Brejker igre su bias monotone i prirodno je zapitati se da li to važi i za Avojder-Enforser igre. Ako razmišljamo kao Avojder, odnosno cilj je da izbegnemo nešto, na prvi pogled izgleda da selektovanje manje elemenata ne može da naškodi igraču, međutim ispostavlja se da to nije tačno. Avojder-Enforser igre u opštem slučaju nisu bias monotone. Ovo implicira da nije moguće definisati granični bias na isti način kao u igrama Mejker-Brejker. Imajući to u vidu, Hefetz, Krivelevich i Szabó su u [70] uveli sledeću definiciju.

Definicija 0.0.8. Za igru $(1 : b)$ Avojder-Enforser $(X, \mathcal{F})$ definišemo *donji granični bias* kao najveći ceo broj $f_{\mathcal{F}}^-$ tako da za svaki $b \leq f_{\mathcal{F}}^-$ Enforser pobeđuje. Analogno tome, *gornji granični bias* je najmanji nenegativan ceo broj $f_{\mathcal{F}}^+$ takav da za svaki $b > f_{\mathcal{F}}^+$ Avojder pobeđuje. Gornji i donji granični bias uvek postoje (osim u nekim trivijalnim slučajevima) i znamo da $f_{\mathcal{F}}^- \leq f_{\mathcal{F}}^+$ važi. U specijalnom slučaju kada je $f_{\mathcal{F}}^- = f_{\mathcal{F}}^+$, ovaj broj se naziva *granični bias* igre.

Odsustvo graničnog biasa inspirisalo je autore [64] da prilagode ova pravila tako da obezbede njegovo postojanje. Definisali su takozvana *monotona pravila*, gde je svakom od igrača dozvoljeno da zahteva više elemenata table po potezu. Preciznije, neka je $(X, \mathcal{F})$ monotona $(a : b)$ Avojder-Enforser igra. U svakom svom potezu, Avojder selektuje *najmanje* a elementa table, dok Enforser selektuje *najmanje* b elementa table. Sa ovim pravilima igra zaista postaje monotona. Ako u igri $(a : b)$ Avojder pobeđuje, onda je to takođe slučaj i u igrama $(a - 1 : b)$ i $(a : b + 1)$.

Definicija 0.0.9. Neka je $(X, \mathcal{F})$ $(1 : b)$ Avojder-Enforser igra. Definišemo jedinstveni *monotoni granični bias* $f_{\mathcal{F}}^{mon}$ kao najveći nenegativan ceo broj takav da Enforser pobeđuje u igri ako i samo ako je $b \leq f_{\mathcal{F}}^{mon}$.

Originalna pravila, gde svaki od igrača selektuje tačno onoliko elemenata koliko bias sugeriše, nazivamo *striktna pravila*. Svaka Avojder-Enforser igra se može igrati na oba skupa pravila, pa uvek naglašavamo na kom skupu pravila igramo igru, striknom ili monotonom. Iako ova izmena pravila deluje neznatna, njihovom zamenom moguće je drastično promeniti ishod igre.

Jake CAvojder-CAvojder igre

U poslednjih nekoliko godina pojavilo se nekoliko varijanti pozicionih igara, poput igre *PrimMejker-Brejker* (engl. *PrimMaker-Breaker*) uvedene u [82] gde podgraf koji se sastoji od Mejkerovih grana mora ostati povezan tokom čitave igre. U *Voker-Brejker* (engl. *Walker-Breaker*) igrama koje su uveli Espig, Frieze, Krivelevich i Pegden [42], Mejker je prinuđen da selektuje grane šetnje ili puta. Slično, u igrama *VokerMejker–VokerBrejker* [50], oba igrača imaju ograničenje da selektuju grane šetnje. Na sličan način, *Jake CAvojder-CAvojder igre* (engl. *strong CAvoider-CAvoider*) su nastale kao prirodno proširenje jakih Avojder-Avojder igara.

Jaku CAvojder-CAvojder igru $(E(K_n), \mathcal{F})$ igraju dva igrača Crveni i Plavi, pri čemu igrač koji prvi u potpunosti selektuje neki $F \in \mathcal{F}$ gubi. Igrači naizmenično selektuju grane kompletnog grafa K_n, tako da graf oba igrača mora ostati povezan tokom čitave igre. Ako nijedan od igrača ne izgubi i svi elementi table su selektovani, igra se proglašava nerešenom. Ciljni skupovi familije $\mathcal{F}$ nazivaju se gubitnički skupovi. Ove igre su uvedene u [103] gde se familija $\mathcal{F}$ sastoji od kopija nekog zadatog grafa F.

Igre saturacije sa dva igrača

Igre saturacije

Za dati graf $G = (V, E)$ se kaže da je *saturiran* u odnosu na monotono rastuću osobinu grafa $\mathcal{P}$, ako G nema svojstvo $\mathcal{P}$, ali $G \cup \{e\} \in \mathcal{P}$ za svaku granu $e \in \binom{V}{2} \setminus E$.

Za dati prazan graf sa n čvorova $\overline{K_n}$ i osobinu $\mathcal{P}$, dva igrača Maks i Mini progresivno grade graf $G \subseteq K_n$ takav da G ne zadovoljava osobinu $\mathcal{P}$. Primetimo da oba igrača selektuju grane iste boje, tj. oni grade isti graf G. Igra se završava kada više nema slobodnih grana koje se mogu dodati u G, pa je G saturiran graf. Maksov cilj je da igra što duže traje, dok Mini želi da minimizira dužinu igre. Označimo sa $s(n,\mathcal{P})$ *rezultat* igre, to je broj grana grafa G na kraju igre. Cilj je naći rezultat igre u dva slučaja, kada je Maks prvi igrač i kada je Mini prvi igrač.

Definisaćemo dva broja koja su povezana sa rezultatom ove igre. Prvi je *broj saturiranosti grafa* (engl. *saturation number*), označavamo ga sa $sat(n,\mathcal{P})$, što je minimalna veličina saturiranog grafa sa n čvorova u odnosu na osobinu $\mathcal{P}$. Drugi je maksimalna moguća veličina saturiranog grafa sa n čvorova u odnosu na osobinu $\mathcal{P}$, označavamo ga sa $ex(n,\mathcal{P})$. Iz svega navedenog zaključujemo da je $sat(n,\mathcal{P}) \leq s(n,\mathcal{P}) \leq ex(n,\mathcal{P})$.

Igre saturacije na grafovima prvi put su u [52] predstavili Füredi, Reimer i Seress. Oni su posmatrali broj $s(n,\mathcal{K}_3)$ gde je $\mathcal{K}_3$ osobina grafa da sadrži trougao. Iz Mantelove teoreme [109] znamo da je $ex(n,\mathcal{K}_3) = \lfloor \frac{n^2}{4} \rfloor$. Sa druge strane, $sat(n,\mathcal{K}_3) = n-1$, jer je zvezda saturiran graf u odnosu na osobinu $\mathcal{K}_3$. Prema tome, znamo da je $s(n,\mathcal{K}_3)$ negde između ovih granica. U [52] autori daju donje ograničenje reda $n \ln n$. Kasnije su Biró, Horn i Vildstrom [18] poboljšali gornje ograničenje na $\frac{26}{121}n^2 + o(n^2)$.

Carraher *et al.* u [27] su pronašli rezultate nekih određenih igara, u kojima je osobina $\mathcal{P}$ pripadanje familiji neparnih ciklusa $\mathcal{O}$, familiji pokrivajućih stabala $\mathcal{T}_n$, sadržati graf $K_{1,3}$ i sadržati graf P_4. U istom radu su takođe posmatrane igre saturacije koje se igraju na bipartitnim grafovima. Kasnije u [63] Hefetz, Krivelevich, Naor i Stojaković su našli donje i gornje ograničenje rezultata u igrama u kojima je svojstvo $\mathcal{P}$ biti k-povezan, imati hromatski broj najmanje k i postojanje mečinga date veličine.

Konstruktor-Bloker igre

Constructor-Blocker igre su nedavno uveli Patkós, Stojaković i Vizer [93]. Ove igre su spoj dve dobro poznate igre: Mejker-Brejker igre i igre saturacije.

Neka su H i F dva fiksna grafa. Dva igrača, koja se zovu Konstruktor i Bloker, naizmenično selektuju slobodne grane kompletnog grafa K_n. Konstruktor je ograničen da traži samo grane čijom selekcijom njegov graf ne sadrži kopiju od F. Sa druge strane, Bloker može selektovati bilo koju granu bez ograničenja. Igra je gotova kada Konstruktor ne može da povuče nijedan dalji potez, tj. njegov graf je saturiran ili su sve grane selektovane. *Rezultat* ove igre je broj kopija grafa H u Konstruktorovom grafu na kraju igre. Cilj

Konstruktora je da maksimizira rezultat, dok Bloker želi da rezultat bude što manji. Rezultat igre kada oba igrača igraju optimalno označavamo sa $g(n, H, F)$.

Označimo sa $\mathcal{N}(H, G)$ broj kopija grafa H u grafu G, a sa $ex(n, H, F) = \max_G\{\mathcal{N}(H, G) : G$ je graf bez F sa n čvorova$\}$. Lako se može videti da je $ex(n, F) = ex(n, K_2, F)$ jer u tom slučaju brojimo broj grana u grafu G. Nedavno u [1] autori su proučavali funkciju $ex(n, H, F)$. Jasno je, da je nejednakost $g(n, H, F) \leq ex(n, H, F)$ uvek tačna.

U igrama Konstruktor-Bloker, za razliku od saturacionih igara, oba igrača grade sopstvene grafove, a broji se broj kopija H samo u grafu Konstruktora. U [93] autori su došli do rezultata za nekoliko različitih igara: kada su i F i H zvezde, $F = P_4$ i $H = P_3$, F je zvezda i H je drvo, $F = P_5$ i $H = K_3$, i dali su gornje i donje ograničenje za $g(n, P_4, P_5)$.

Uopštena igra saturacije

Uopštenu igru saturacije (engl. *Generalized saturation game*) uvodimo kao prirodno proširenje dve različite vrste igara, igara saturacije i Konstruktor-Bloker igara.

Neka su H i F grafovi koji su unapred dati. Dva igrača, Maks i Mini, naizmenično selektuju grane K_n tako da graf igre G ne sadrži kopiju od F. Igra se završava kada igrači ne mogu da selektuju ni jednu granu, tj. graf G je saturiran ili više nema slobodnih grana. Primetimo da ovde, kao i u saturacionim igrama, oba igrača zajedno grade isti graf $G \subseteq K_n$.

Rezultat igre je broj kopija od H u G na kraju igre. Maks želi da maksimizira rezultat, dok Mini pokušava da napravi rezultat što manjim. Kada je graf $H = K_2$, ove igre postaju igre saturacije, tako da možemo da kažemo da su igre saturacije specijalan slučaj uopštenih igara saturacije.

Rezultati

Jake Avojder-Avojder igre

Jake Avojder-Avojder igre na $E(K_n)$

U ovoj tezi proučavamo Jake Avojder-Avojder F igre koje se igraju na granama kompletnog grafa K_n, gde su elementi familije $\mathcal{F}$ skupovi koji sadrže kopiju grafa F. Dakle, igrač koji prvi u potpunosti selektuje kopiju od F gubi igru. O ovim igrama se ne zna puno, dok je otvorenih problema mnogo. U [62] je pokazano da Plavi ima pobedničku strategiju u igri P_3, gde je zabranjeni graf put sa samo dve grane. Nedavno je Beker [17] generalizovao ovaj rezultat na sve zvezde, dokazujući da je za svako fiksno k jaka Avojder-Avojder igra zvezda S_{k+1} pobeda drugog igrača za svako n dovoljno veliko. Dokaz je koncipiran više na pravljenju nego na izbegavanju – pokazujući da Plavi može da napravi graf maksimalne veličine koji ne sadrži S_{k+1} brzo, bez gubljenja poteza, čime se automatski obezbeđuje pobeda. Zapravo, jedina netrivijalna jaka Avojder-Avojder igra na granama kompletnog grafa čiji je ishod poznat je igra zvezda.

Koristimo skraćenicu $\mathcal{CC}_{>3}$ za kolekciju inkluzivno-minimalnih povezanih grafova sa više od tri čvora. Neka je S_4 zvezda sa četiri čvora, dok P_4 predstavlja put sa četiri čvora. Cilj je naći ishod za igre P_4 i $\mathcal{CC}_{>3}$.

Teorema 0.0.2. *Plavi ima pobedničku strategiju u jakoj Avojder-Avojder P_4 igri, igranoj na K_n, gde je $n \geq 8$.*

U sledećoj teoremi, razmatramo igru u kojoj igrač gubi igru čim napravi povezanu komponentu sa više od tri čvora.

Teorema 0.0.3. *Plavi ima pobedničku strategiju u jakoj Avojder-Avojder $\mathcal{CC}_{>3}$ igri, igranoj na K_n, gde je $n \geq 5$.*

Neka je $R(F)$ dijagonalni Remzijev broj, tako da svako bojenje grana kompletnog grafa sa najmanje $R(F)$ čvorova u dve boje daje monohromatsku kopiju od F. Ako $n \geq R(F)$ znamo da jaka Avojder-Avojder F igra na $E(K_n)$ ne može da se završi nerešeno. Za igru P_4 kao i za igru $\mathcal{CC}_{>3}$ odavde sledi da nema nerešenog ishoda ako je $n \geq 5$.

Jake CAvojder-CAvojder igre

Proučavamo *jake CAvojder-CAvojder F igre* u kojima graf oba igrača mora ostati povezan tokom igre. Tabla je i dalje skup grana od K_n, a igrači ne smeju da selektuju kopiju zabranjenog grafa F.

U nastavku dokazujemo da Plavi može da pobedi u tri različite jake CAvojder-CAvojder igre.

Teorema 0.0.4. *Plavi ima pobedničku strategiju u jakoj CAvojder-CAvojder S_4 igri, igranoj na K_n, gde je $n \geq 7$.*

Teorema 0.0.5. *Plavi ima pobedničku strategiju u jakoj CAvojder-CAvojder P_4 igri, igranoj na K_n, gde je $n \geq 5$.*

U sledećoj teoremi posmatramo Ciklus igru u kojoj igrač koji prvi selektuje ciklus gubi.

Teorema 0.0.6. *Plavi ima pobedničku strategiju u jakoj CAvojder-CAvojder Ciklus igri, igranoj na K_n, gde je $n \geq 6$.*

Primetimo da ako je $F \in \{S_4, K_3\}$, onda je dijagonalni Remzijev broj $R(F)$ 6, pa nerešeno nije moguće ni u jednoj od gore pomenute tri igre.

Jake CAvojder-CAvojder igre igrane sa pozicije

Ovde proučavamo jake CAvojder-CAvojder igre koje počinju sa određene pozicije na grafu, odnosno možemo pretpostaviti da je u ovoj igri već odigrano nekoliko poteza, i pritom znamo kako graf igre izgleda u tom trenutku, i zatim nastavljamo da igramo. Ovakav način igre je proučavan u [86].

Graf K_5 čije su grane obojene u dve boje, tako da se sastoji od dva ciklusa C_5, jednog u plavoj a drugog u crvenoj boji, nazivamo „obojen K_5" (engl. "drawn K_5"). Pozicija T je konfiguracija grafa G, takva da se G sastoji od jednog „obojenog K_5" i $n-5$ izolovanih čvorova. Igra neparnog (odnosno, parnog) ciklusa je igra u kojoj je zabranjeni graf svaki graf koji sadrži neparan (odnosno, paran) ciklus.

U sledećoj teoremi pretpostavljamo da se posle prvih pet rundi graf igre sastoji od „obojenog K_5" i izolovanih čvorova.

Teorema 0.0.7. *Igrajući od pozicije T, Plavi ima pobedničku strategiju u jakoj CAvojder-CAvojder C_3 igri, C_4 igri, C_6 igri, igri neparnog ciklusa i igri parnog ciklusa.*

Dokazi za rezultate date iz ovog odeljka mogu se naći u glavi 3.

Ačivment broj u jakim Mejker-Mejker igrama

Proučavamo jaku Mejker-Mejker igru igranu na skupu grana kompletnog grafa K_n, sa unapred zadatim grafom F. Prvi igrač koji selektuje kopiju od F u svojoj boji je pobednik. Definišemo *ačivment broj* (engl. *Achievement number*) od F kao najmanje n za koje Crveni može da pobedi u ovoj igri i označavamo ga sa $a(F)$. Označimo sa $R(F)$ dijagonalni Remzijev broj, kao najmanji ceo broj takav da svako bojenje table u dve boje daje monohromatski graf F. Koristeći krađu strategije znamo da je $a(F) \leq R(F)$.

Želimo da damo odgovore na neka od pitanja koje je Harary postavio u [61]. U tom radu data je tabela sa ačivment brojevima za nekoliko malih grafova F čije su vrednosti izračunate pomoću računara. Kasnije je u [41] isti autor proširio ovu tabelu sa još dve vrednosti dobijene na isti način. Naredni problem i pretpostavka dati su u [61].

Problem 0.0.10. [61] Odrediti $a(F)$ za različite familije grafova. (Deluje teško čak i za stabla.)

Pretpostavka 0.0.11. [61] Minimalna vrednost $a(T)$ među svim stablima T veličine n se ostvaruje kada je $T = P_n$, put. Maksimum od $a(T)$ se postiže kada je T zvezda $K_{1,n-1}$.

Pobednička strategija Crvenog za male grafove

U dokazima sledećih tvrdnji, dajemo eksplicitnu pobedničku strategiju za Crvenog počevši od $a(F)$ izolovanih čvorova za neke grafove F date u [61], a takođe posmatramo broj poteza koji je Crvenom neophodan da bi pobedio.

Opservacija 0.0.12. *Ačivment brojevi za K_2 i P_3 su 2 i 3, redom, tj. $a(K_2) = 2$ i $a(P_3) = 3$.*

Propozicija 0.0.1. *Ačivment broj za $2K_2$ je 5, tj. $a(2K_2) = 5$.*

Propozicija 0.0.2. *Ačivment broj za P_4 je 5, tj. $a(P_4) = 5$.*

Propozicija 0.0.3. *Ačivment broj za* $K_{1,3}$ *je* 5*, tj.* $a(K_{1,3}) = 5$.

Propozicija 0.0.4. *Ačivment broj za* K_3 *je* 5*, tj.* $a(K_3) = 5$.

Propozicija 0.0.5. *Ačivment broj za* $K_{1,3} + e$ *je* 5*, tj.* $a(K_{1,3} + e) = 5$.

Propozicija 0.0.6. *Crveni može da pobedi u jakoj Mejker-Mejker igri* K_4-e *igranoj na* K_7*, tj.* $a(K_4 - e) \leq 7$.

Ačivment broj za puteve, cikluse, zvezde i savršen mečing

Bavimo se pronalaženjem ačivment broja za puteve, cikluse, zvezde i savršen mečing, istovremeno dajući neke od odgovora za problem 0.0.10.

Propozicija 0.0.7. *Ačivment broj za put sa* n *čvorova je* $a(P_n) = n$*, za* n *dovoljno veliko.*

Propozicija 0.0.8. *Gornje ograničenje ačivment broja za zvezdu sa* n *čvorova je* $a(S_n) \leq 2n - 3$*, za sve* $n \geq 3$ *.*

Propozicija 0.0.9. *Ačivment broj za ciklus sa* n *čvorova je* $a(C_n) = n$*, za* n *dovoljno veliko.*

Propozicija 0.0.10. *Ačivment broj za savršen mečing* M_{2n} *na* $2n$ *čvorova je* $a(M_{2n}) = 2n$*, za* n *dovoljno veliko.*

Primetimo da je ovde data tačna vrednost za puteve, cikluse i savršen mečing, dok je za zvezde data gornja granica. Očigledna donja granica za $a(S_n)$ je n, pošto igranjem na n čvorova Plavi može da dodirne sve čvorove (praveći savršen mečing) pre nego što Crveni odigra $n - 1$ poteza.

Ačivment broj za stabla

U ovom delu cilj je pronalaženje gornjeg i donjeg ograničenja ačivment broja za stabla sa n čvorova T_n. U pretpostavci 0.0.11 se pitamo da li je za fiksno drvo T_n, vrednost $a(T_n)$ negde između $a(P_n)$ i $a(S_n)$. Koristeći propozicije 0.0.7 i 0.0.8 znamo da je to negde između n i broja koji nije veći od $2n - 3$. Za koreno stablo T_n označavamo sa $\underline{d}(u)$ izlazni stepen čvora $u \in V(T_n)$.

Teorema 0.0.8. *Neka je T_n fiksno stablo sa n čvorova i v_0' čvor maksimalnog stepena u T_n. Ako je čvor v_0' koren od T_n, onda važi sledeće*

$$a(T_n) \leq max\{n + 4\sqrt{n}, \sum_{\substack{u \in V(T_n) \\ \underline{d}(u) \neq 0}} (2\underline{d}(u) - 1)\},$$

za n dovoljno veliko.

U narednoj propoziciji dat je ačivment broj za specijalnu klasu stabala, a to su stabla sa ograničenim stepenom koja sadrže dugačak ogoljen put čiji je jedan kraj istovremeno i list stabla. Put unutar stabla T naziva se *ogoljen* (engl. *bare*) ako su svi njegovi unutrašnji čvorovi stepena dva u T.

Propozicija 0.0.11. *Neka je Δ pozitivan ceo broj. Postoje celi brojevi $m = m(\Delta)$ i $n_0 = n_0(\Delta, m)$ takavi da za svako stablo T_n gde je $n \geq n_0$ i $\Delta(T_n) \leq \Delta$, važi sledeće. Ako T_n sadrži ogoljen put dužine m, a pri tome je jedan od njegovih krajeva istovremeno i list od T_n, onda je $a(T_n) = n$.*

Dokazi za tvrđenja data u ovom odeljku mogu se naći u glavi 4.

Uopštena igra saturacije

Cilj ovog odeljka je da nađemo rezultat igre za neke unapred date grafove H i F. Označavamo sa $s_1(n, H, F)$ rezultat igre kada oba igrača igraju optimalno i Maks počinje, a sa $s_2(n, H, F)$ kada Mini počinje. Ako je $s_1 = s_2$ koristimo s umesto $s = s_1 = s_2$. Ponekad koristimo s umesto $s(n, H, F)$ kada je jasno o kojoj igri je reč. Ukoliko je rezultat u oba slučaja između a i b, koristimo notaciju $a < s < b$ umesto $a < s_1 < b$ i $a < s_2 < b$.

Zabranjen je ili P_5 ili svi ciklusi

Prvo ćemo tražiti rezultat igre u kojoj je zabranjeni graf put sa 5 čvorova P_5 i na kraju igre brojimo trouglove, pretpostavljajući da oba igrača igraju optimalno.

Teorema 0.0.9. $\dfrac{n-4}{3} \leq s(n, K_3, P_5) \leq \dfrac{n-4}{3} + 4.$

Zatim, pronalazimo rezultat dve različite igre u kojima je zabranjeni graf F ciklus. U narednoj teoremi, brojimo zvezde sa k čvorova na kraju igre.

Teorema 0.0.10. $\binom{\lfloor \frac{n}{2} \rfloor}{k-1} \leq s(n, S_k, Ciklus) \leq \binom{\lceil \frac{n}{2} \rceil}{k-1}$, $k > 3$.

Zatim, brojimo puteve P_4 na kraju igre.

Teorema 0.0.11. $\frac{n^2}{16} + O(n) \leq s(n, P_4, Ciklus) \leq \frac{n^2}{16} + O(n)$.

Zabranjen je P_4

Prvo posmatramo igru u kojoj određujemo broj puteva. Jasno, jedini put koji se može pojaviti je P_3.

Teorema 0.0.12. $n - 4 \leq s(n, P_3, P_4) \leq n$.

Sada posmatramo igru u kojoj brojimo zveze dok je zabranjeni graf isti.

Opservacija 0.0.13. $s(n, S_k, P_4) = 0$, $k \geq 4$.

Ostaje igra u kojoj brojimo cikluse. Jasno je da je jedini ciklus koji može postojati C_3. Dakle, $s(n, C_k, P_4) = 0$, kada je $k > 3$.

Teorema 0.0.13. $s(n, C_3, P_4) = 1$ *ako je* n *neparan, i* $s(n, C_3, P_4) = 0$ *ako je* n *paran.*

Svi neparni ciklusi su zabranjeni

U naredna dva tvrđenja, posmatramo uopštene saturacione igre, gde su zabranjeni grafovi svi neparni ciklusi $\mathcal{O}$. Drugim rečima, u ovim igrama graf igre ostaje bipartitan.

Opservacija 0.0.14. $s(2n, P_{2k}, \mathcal{O}) = \left(\frac{n!}{(n-k)!}\right)^2$ *i* $s(2n, C_{2k}, \mathcal{O}) = \frac{1}{k}\left(\frac{n!}{(n-k)!}\right)^2$.

Zatim, brojimo zvezde u bipartitnom grafu sa $2n$ čvorova.

Opservacija 0.0.15. $s(2n, S_k, \mathcal{O}) = 2n\binom{n}{k-1}$.

T_n je zabranjen

U ovom odeljku interesuju nas uopštene igre saturacije, gde su zabranjeni grafovi sva pokrivajuća stabla $\mathcal{T}_n$. Na kraju igre graf mora biti disjunktna unija dva kompletna grafa K_r i K_{n-r}.

Opservacija 0.0.16. $s(n, P_k, \mathcal{T}_n) = \dfrac{k!}{2}\dbinom{n-2}{k}$,
$s(n, C_k, \mathcal{T}_n) = \dfrac{(k-1)!}{2}\dbinom{n-2}{k}$ *i* $s(n, S_k, \mathcal{T}_n) = (n-2)\dbinom{n-3}{k-1}$.

S_4 je zabranjen

U nastavku posmatramo uopštene igre saturacije gde je zabranjen graf zvezda S_4. Prvo, računamo broj puteva na kraju igre kada oba igrača igraju optimalno.

Teorema 0.0.14. $n - 1 \leq s(n, P_3, S_4) \leq n$ *za* $n \geq 3$. *Štaviše,* $s_1(n, P_3, S_4) = n$ *ako je* n *paran i* $s_2(n, P_3, S_4) = n$ *ako je* n *neparan.*

Teorema 0.0.15. $n - 3 \leq s(n, P_4, S_4) \leq n$ *za* $n \geq 4$.

Teorema 0.0.16. $s(n, P_5, S_4) \leq 6$. *Pored toga,* $s_2(n, P_5, S_4) \geq 5$ *za* $n = 4k$ *i* $k \geq 2$ *ili* $n = 4k + 1$.

Teorema 0.0.17. $s(n, P_k, S_4) = 0$ *gde je* $k \geq 6$.

Sledeće što želimo da odredimo je broj zvezda na kraju igre. Kako je S_4 zabranjeni graf, može se pojaviti samo zvezda $S_3 = P_3$, što je već urađeno u teoremi 0.0.14.

Na kraju, brojimo cikluse na kraju igre.

Teorema 0.0.18. $s(n, C_k, S_4) \leq 1$, *kada je* $3 \leq k \leq 5$ *i* $s(n, C_k, S_4) = 0$, *kada je* $k \geq 6$.

Dokazi tvrđenja iz ovog odeljka su izloženi u glavi 5

Chapter 1

Introduction

1.1 Games

In this thesis, we study games played by two players that we can analyze using different mathematical tools. We observe games of *perfect information*, which means that each player before each of his moves knows the previous moves made by all other players. Also, we are interested in games that are *sequential*, which means that players make their moves alternately. Finally, there are no chance moves.

Apart from the aforementioned games, which will be discussed much more later, there are many different types of games mathematically analyzed. We will say a few words about them before we continue with our prime interest. A scientific branch that has attracted a lot of attention recently is game theory.

Game theory is the study of mathematical models of strategic interactions among rational agents. In general, it includes games of chance, games of imperfect knowledge, and games in which players can move simultaneously, and they tend to represent real-life decision-making situations. It is a very applicable branch of science, especially in economics, logic, computer science, etc.

The modern game theory began with the idea of mixed-strategy equilibria in a two-person zero-sum game and its proof by Neumann, see [88]. This book was published in 1944, and later in the 1950s, this field of science start developing extremely fast.

To make these games more understandable we give some of their variants and examples. Unlike the games with perfect information that we mentioned above, *imperfect information games* are played by players who do not know all moves played by their opponents. Most games studied in game theory are imperfect-information games, such as simultaneous move games and most card games.

Zero-sum games (more generally, constant-sum games) are games in which choices by players can neither increase nor decrease the available resources. One player can gain just the amount of resources that another player loses. One example of a zero-sum game is poker because one player can win exactly the amount another player loses. Another example follows.

Example 1.1.1. Matching pennies is a game played by two players, Even and Odd. Both players have a penny and must secretly turn it to the side they want. After that, at the same time, they show each other what they have chosen. If the pennies match (both heads or both tails), the Even takes both pennies for himself (Even gains 1, and Odd loses 1). Otherwise, Odd takes both pennies (Odd gains 1, and Even loses 1).

On the other hand, in *non-zero-sum games*, a gain by one player does not necessarily correspond with a loss by another. Many games studied in game theory are of this kind.

If the identities of the players can be changed without changing the payoff to the strategies, then a game is *symmetric*. We give one example of a symmetric and non-zero-sum game, called prisoner's dilemma, see [2].

Example 1.1.2. Two prisoners have been captured by the authorities. They are separated and each is given the choice between confessing and remaining silent. One of four possible outcomes will occur. If prisoner A talk while the other one remains silent, prisoner A go free. If both of them remain silent, each receives one year in prison. If both confess, each receives a five-year sentence. If prisoner A remain silent while the other one confesses, prisoner A face a ten-year sentence while prisoner B goes free.

It turns out that each prisoner gets a higher reward for betraying the other. Suppose that we think as a prisoner A. In case prisoner B betrayed, prisoner A can stay silent and get a ten-year sentence or betray and get a five-year sentence. Otherwise, if prisoner B stayed silent, prisoner A can stay silent too and get one year sentence or betray and go home. Therefore, regardless of what prisoner B decides, for the prisoner A the best option is to betray.

A strategy profile is a *Nash equilibrium* if no player can do better by unilaterally changing their strategy. To see what this means, imagine that each player knows the strategies of the others. Having all those strategies in mind each player asks themselves: "Can I benefit by changing my strategy?" If any player could answer "Yes", then that set of strategies is not a Nash equilibrium. Otherwise, if every player prefers not to switch then the set of strategies is a Nash equilibrium.

Observing Example 1.1.2 we realize that the Nash equilibrium is (betray, betray), while the social optimum is (stay silent, stay silent).

1.2 Combinatorial game theory

A combinatorial game is defined to be a two-player, perfect-information game with no chance elements, see e.g. [51]. This is the class of games that we are going to talk about in this thesis. Compared to classical game theory the main difference is that players move in sequence and not simultaneously, so there is no hidden information. Combinatorial games include well-known games such as chess, checkers, and Go. They can also be played on the infinite board. Moves in combinatorial games can be represented by a game tree. One of the most studied combinatorial games is Nim.

Example 1.2.1. Nim is a game played by two players who take turns in removing objects from different piles. A player chooses one pile and removes at least one object from it in each of his moves. There are two versions of the game, and in the first one, the goal is to avoid taking the last object, whereas, in the second one, the goal is to take the last object.

It is believed that the origins of this game go back to China, but under this name, it was introduced for the first time by Bouton in [21].

1.3 Positional games

Positional games are combinatorial games. They are finite, perfect information games with no chance moves. Examples of positional games include

Tic-Tac-Toe, Hex, Sim, Row-column game, etc.

Historically first papers about positional games appeared in 1963, by Hales and Jewett [60], and in 1973, by Erdős and Selfridge [40]. Later, the man who brought many new ideas, questions, and open problems to this field was Józef Beck. He published numerous papers and a book [9] that covers a lot about positional games, containing many theorems, examples, solved and unsolved problems on this subject, and interesting methods for analyzing new problems. Recently, the book written by Hefetz, Krivelevich, Stojaković and Szabó [69] has appeared, which is a great study base with numerous up-to-date results and open problems.

While analyzing these games we assume that both players play according to their optimal strategy. We can imagine a strategy of the player as a "book" in which we can find response for every move of the opponent, regardless of his choice. We want to determine what will be the outcome of the game, and there are three options: the first player's win, the second player's win, or a draw. If the outcome of the game is *player A wins*, we say that *player A has a winning strategy*.

One can ask about using a powerful computer for determining the outcome of the game. However, it turns out that even today's computers are limited to help. It is necessary to exhaustively search the whole game tree which is usually exponentially large. That is the reason why mathematical tools and methods are the most important in analyzing positional games.

One of the surprising facts is that the probabilistic method can be used for analyzing these games. At first sight it may look implausible since in these games there is no hidden information. This method involves computing the probability of a certain move leading to a win or a draw and then choosing the move with the highest probability of success. The probabilistic method can also be used to estimate the expected outcome of a game, such as the probability of winning or drawing, given a particular position. This phenomenon was extensively studied both in [9] and [69].

Now we give a formal definition of positional games. A positional game is a pair $(X, \mathcal{F})$, where X is a finite set called the *board*, and $\mathcal{F}$ is the family of *target sets*. The game is played by two players who alternately claim previously unclaimed elements of X until all the elements of the board are claimed. $(X, \mathcal{F})$ is also called the hypergraph of the game, whose vertices are the elements of X and hyperedges are the elements of the family $\mathcal{F}$.

When in each of the rounds each player claims exactly one element of the board we call the game *unbiased*. But, we can also define the biased $(a : b)$ game, where the player who is first to move claims a elements of the board per round, while the other one claims b elements per round.

When it comes to the rules for determining the game winner in a positional game, there are several variants. The largest classes of positional games are certainly strong games and weak games and each of them has its subclasses. More about each variant of positional game is given in the following sections.

1.3.1 Strong games

In Strong positional games, both players have the same goal. These games have some popular examples, one of them is the widely known game Tic-Tac-Toe. We can divide these games into two big subclasses which are *strong Maker-Maker games* and *strong Avoider-Avoider games*.

Strong Maker-Maker game

In the *strong Maker-Maker game* $(X, \mathcal{F})$, two players called Red and Blue take turns in claiming previously unclaimed elements of X, with Red going first. The player who first fully occupies some $F \in \mathcal{F}$ is the winner. If neither of the players wins and all the elements of the board are claimed, the game is declared a *draw*. Here, the target sets of the family $\mathcal{F}$ are called *winning sets*.

Example 1.3.1. The most notable example of this class of positional games is the widely popular game Tic-Tac-Toe. This game is played on the board represented with a 3×3 grid square. The board consists of nine elements, while the family $\mathcal{F}$ consists of eight winning sets which are: three horizontal lines, three vertical lines, and two diagonals, see Figure 1.1. The winner is the first player who possesses a whole winning line (set). We know that if both players play optimally this game ends in a draw.

We give another example of a Strong Maker-Maker game that is often studied and represents the generalized version of the Tic-Tac-Toe game.

Example 1.3.2. The generalized version of the Tic-Tac-Toe game is the so-called n^d game. The board of this game is $X = [n]^d$, the d-dimensional

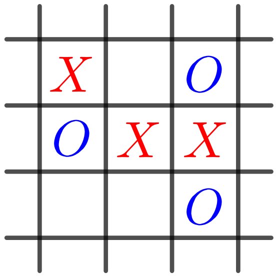


Figure 1.1. Tic-Tac-Toe game.

cube. A winning set (line) is an n-tuple $(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, ..., \boldsymbol{a}^{(n)})$, such that the sequence a_j^k, $1 \leq j \leq d$, is either increasing, decreasing or constant, where $1 \leq k \leq n$, and not all sequences are constant. The winner is the player who occupies an entire winning set (line) first. The classical Tic-Tac-Toe game is the 3^2 game. We do not know much about the n^d game, and the reason lies in the fact that analyzing this game in full generality is out of reach. What we do know is that for fixed n the first player wins if d is large enough. However, if the situation is the opposite, i.e. d is fixed and n is large enough, the game ends in a draw.

Generally speaking, determining the outcome in a strong Maker-Maker game proves to be challenging, and there are hardly any general tools at disposal. One such tool is the *strategy stealing argument* which we can use to show that Red can guarantee at least a draw in any game.

Theorem 1.3.3. *[9] In the strong Maker-Maker positional game* $(X, \mathcal{F})$*, the first player can guarantee at least a draw.*

Proof. We will assume to the contrary that the second player–Blue has a winning strategy S. The first player–Red plays his first move arbitrarily and then steals the strategy S. After the first move of Blue, Red imagines that he is the second player and responds to each of Blue's moves as the strategy S dictates. If this strategy proposes that Red claims the edge that he has already claimed in his first move, then he plays arbitrarily. Note that here one extra move cannot harm the player. Therefore, following the strategy S, Red can also win, which is a contradiction. $\square$

This argument affirms that the first player indeed has an advantage. We also conclude that in this kind of game, only two outcomes are possible: Red's win and a draw. Sometimes we work with games where a draw is not an option, in that case, using strategy stealing argument we conclude that Red has a winning strategy. One downside of this argument is the fact that we do not know anything about the winning strategy, and finding an explicit winning strategy very often looks hopeless.

As we saw above in the strong Maker-Maker game it is enough just to know that a draw is not an option to know the winner. A mathematical tool that we can use in this situation is the *Ramsey property of the game hypergraph* (see [9]). If a game has a Ramsey property, that means that every 2-coloring of the board in red and blue gives a monochromatic set $F \in \mathcal{F}$. Therefore, if a game has a Ramsey property draw is impossible.

Almost at the end of the list of tools is a *pairing strategy*, which we can use to show that Blue can guarantee a draw. Indeed, if we can make a disjoint pairing of the elements of the board X, such that each winning set contains one of the pairs, then Blue can always claim the other element from the pair Red has chosen and thus, prevent Red from winning.

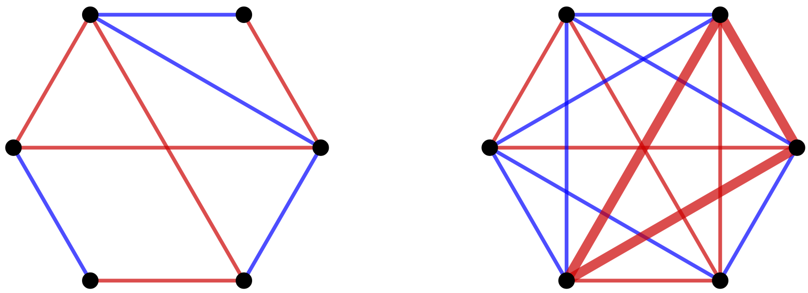
Having in mind how limited the set of tools is, it is not too surprising that so few results on strong Maker-Maker games can be found in the literature.

Recently, an interesting idea appeared in paper [44], by Ferber and Hefetz. They proved that playing on the edge set of K_n, for sufficiently large n, Red can win perfect matching and Hamilton cycle game, and in [45] the same authors proved that, for sufficiently large n and every positive integer k, the first player can win k-vertex-connectivity game. Both papers rely on fast winning strategies for weak games.

Strong Avoider-Avoider game

Another type of strong games is the *Strong Avoider-Avoider game* $(X, \mathcal{F})$. This game is again played by two players Red and Blue, but now the player who first fully occupies some $F \in \mathcal{F}$ *loses the game.* If neither of the players loses and all the elements of the board are claimed, the game is declared a *draw.* Here, the target sets of the family $\mathcal{F}$ are called *losing sets.*

Example 1.3.4. The first such game, widely known as *Sim*, was introduced in 1961 by Simmons [99]. The board of Sim is the edge set of K_6, and a player who first claims a triangle loses, see Figure 1.2. Even though it is immediate that draw is impossible (using the Ramsey property of the board), and the board is reasonably small (it has just fifteen edges), analyzing it is challenging, and the proof that Blue wins is performed with the help of a computer.



(a) (b)

Figure 1.2. The game Sim: (a) the graph after Red's fifth move. (b) the graph at the end of the game (Red claims a triangle and loses in his last move).

In [100] Slany gave a methodical study of the hardness of determining the winner for several games similar to Sim. Further, Mead, Rosa and Huang in [87] gave an explicit winning strategy for Blue in Sim, and recently in [110] Wrzos-Kaminska gave a simple human-playable winning strategy. Other variants of strong Avoider-Avoider games were studied by Harary in [61], who introduced several finite games on graphs on up to six vertices.

At first sight, it may seem that in strong Avoider-Avoider games, in contrast to the strong Maker-Maker games, Blue always has an upper edge, and Red as the first player cannot expect to win under optimal play. This, however, turns out not to be true!

For example, in d-dimensional Tic-Tac-Toe game n^d (see [9] for details), where n is odd, Red has an explicit drawing strategy: In his first move, he chooses the central element, let us denote it by C. After that, whenever Blue chooses an element P Red chooses P' that is symmetrical with respect to C. If we suppose for a contradiction that Red loses, i.e. that his graph has a red line L (note that it is not possible that C belongs to L), then

L', its mirror image over the cube's center, is a blue line and has been fully occupied before L, a contradiction.

Now, as game 3^3 cannot end in a draw (using the Ramsey property [9]), we can conclude that it is a Red's win. In [77] Johnson, Leader, and Walters proved that there are transitive games that are a Red's win, for all board sizes which are not a prime, or a power of 2.

1.3.2 Maker-Breaker games

In contrast to the strong positional games where the two players compete for achieving the same objective, the *weak games* are asymmetrical. The first player is given a goal while the second one just tries to prevent the first player from achieving his goal.

In *Maker-Breaker positional game* $(X, \mathcal{F})$ (also called "the weak game"), two players are called Maker and Breaker, and the elements of the family $\mathcal{F}$ are called the winning sets. Maker wins the game if, by the end of the game, he claims all elements of some $F \in \mathcal{F}$, otherwise, Breaker wins the game.

As in strong Maker-Maker games, Maker wants to claim all the elements of some winning set, but he does not have to do it first. Furthermore, whether Blue has occupied a winning set for himself or not is of no interest. Note that this game cannot end in a draw, Maker wins if by the end of the game, he occupies some winning set, and Breaker wins if he occupies an element in each of the winning sets.

Example 1.3.5. Observe the game Tic-Tac-Toe under the Maker-Breaker rules. Here Maker wants to put three of his marks in the same winning line, and he does not care about three Breaker marks in a winning line. Having this in mind, one can easily find a winning strategy for Maker.

We give another example of a Maker-Breaker game called the Row-Column Game.

Example 1.3.6. In the Row-Column Game, the board is $n \times n$ square and the players alternately claim its elements. Winning sets (lines) are all rows and columns, so together there are $2n$ winning sets. Maker wants to achieve an advantage in some winning set, and if he succeeds in doing that, the game is Maker's win, otherwise, it is Breaker's win.

Besides determining the winner of the game, one equally interesting question we can find in the literature about this game is how large the Maker's advantage can be. It is clear that in case Breaker wins this game, the advantage is 0. If we reduce the family of winning sets just to rows (or columns), then we can use the pairing strategy and conclude that for even n there is no advantage, otherwise, the advantage is 1.

Taking into account both rows and columns, the situation is substantially different. Beck proved in [8] that Maker can achieve at least $\frac{n}{2} + 32\sqrt{n}$ elements of some winning set. For the upper bound Székely in [108] showed that the second player has a strategy to limit the number of opponent's elements in each winning set to at most $\frac{n}{2} + O(\sqrt{n \log n})$.

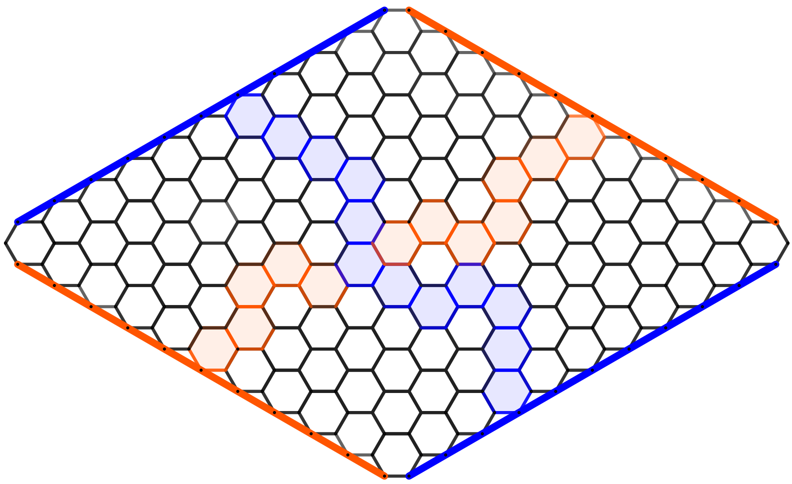


Figure 1.3. The game Hex on the 11×11 board: A Blue winning path.

Example 1.3.7. The game Hex is played on a board that is a rhombus of hexagons of size $n \times n$ (the well-known versions are when $n = 11$ or $n = 13$). Two opposite sides of the rhombus are colored red and blue, respectively, see Figure 1.3. Two players, Maker and Breaker alternately color uncolored hexagons of the board red and blue, respectively. The goal of each player is to connect the opposite sides of the board in his color first, i.e. to make a path in his color from one side to the other.

At first glance, it may seem that this is a strong game. However, this is not true, because the winning sets of the players are not the same. We can observe the family of winning sets as all red paths between two opposite red sides of the board, and reformulate Maker's goal to win, if by the end of the game he owns one of these paths. On the other hand, Breaker wins if he prevents Maker from making such a path. That this setup is equivalent to the original game we can conclude from the well-known Hex theorem of Nash [53], which says that every red/blue coloring of the Hex board results in a monochromatic path that connects two opposite sides of the rhombus of the same color.

In weak games, we will assume that Maker is the first player, unless it is stated differently. Being the first player is an advantage as one extra move cannot harm a player. Indeed, we know that if Maker can win as the second player, he can also win as the first player [69]. The same holds for Breaker.

One of the most applied results for Maker-Breaker games is the following criterion that gives us a general condition for Breaker's win.

Theorem 1.3.8. *([40], Erdős-Selfridge Theorem). Let $\mathcal{F}$ be a hypergraph. If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2}$$

then Breaker has the winning strategy as the second player. If Breaker plays first, then he wins if

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < 1.$$

This theorem has many applications in analyzing these games, one of the reasons is that the condition for Breaker's win can be easily checked. When we work with games where the hypergraph of the game is k-uniform, which means that each of the winning sets has exactly k elements, Erdős-Selfridge criterion becomes true if $|\mathcal{F}| < 2^{k-1}$.

Beck in [9] gave a similar criterion for Maker's win. Denote with $\Delta_2(\mathcal{F})$ the max-pair degree of $\mathcal{F}$, that is, $max\{|\{A \in \mathcal{F} : \{u, v\} \subseteq A\}| : u, v \in X\}$.

Theorem 1.3.9. *[9] Let $\mathcal{F}$ be a hypergraph of the game. If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} > \frac{1}{8}\Delta_2(\mathcal{F})|X|,$$

then Maker has a winning strategy.

Usually, we observe games that are played on K_n, which means that the board of the game is an edge set of the complete graph on n vertices. One of the most studied games of this kind is *Perfect Matching game* $\mathcal{M}_n$, in which the winning sets are all perfect matchings of K_n. The second one is *Connectivity game* $\mathcal{C}_n$, where the winning sets are all spanning trees of K_n. For *Hamiltonocity game* $\mathcal{H}_n$, the winning sets are all Hamilton cycles of K_n.

In [81] Lehman showed that Maker can win in the Connectivity game played on $E(K_n)$. Moreover, he proved that Maker needs just $n-1$ moves to make a spanning tree, which is the best possible.

An attempt was made to prove similar results for Hamiltonocity game. The first attempt can be found in [28] where Chvátal and Erdős proved that Maker can win Hamiltonocity game played on $E(K_n)$ for sufficiently large n. Then, Papaioannou in [92] improved this result proving that Maker can win this game for $n \geq 600$. Later, Hefetz and Stich in [74] showed that Maker wins Hamiltonocity game when $n \geq 29$. Eventually, Stojaković and Trkulja showed in [107] that Maker can win this game if and only if $n \geq 8$.

We now move on to study the number of moves Maker needs to play to ensure his win. This problem for the first time was considered in [65] by Hefetz, Krivelevich, Stojaković, and Szabó. They showed that the minimum number of moves Maker needs to play to win in Hamiltonocity game is $n+1$ or $n+2$, with the assumption that n is large enough. A few years later Hefetz and Stich in [74] showed that this number is exactly $n+1$.

Maker-Breaker games on a complete bipartite graph $K_{n,n}$ were studied by Lu in [85] and [83]. Here, Maker and Breaker, alternately take previously untaken edges of $K_{n,n}$, one edge per move, with Breaker going first. The game ends when all edges have been taken. Let us denote by M the Maker's graph at the end of the game. Maker wants as many edge-disjoint Hamilton cycles (respectively, Perfect matchings) as possible. He proved that Maker can achieve n edge-disjoint Hamilton cycles (respectively, $(\frac{1}{2}-\epsilon)n$ edge-disjoint perfect matchings, $\epsilon > 0$) for large n. The same author in [84] observed Hamilton cycle game played on the edge set of the complete graph K_n and proved that here Maker can claim $(116-o(1))n$ edge-disjoint Hamilton cycles.

For the Perfect matching game it is proven in [65] that Maker builds a perfect matching in $\frac{n}{2}+1$ moves when n is even, and in $\lfloor \frac{n}{2} \rfloor$ moves for odd n.

In some Maker-Breaker games it is not hard to prove that Maker can win, for those games the question "How fast he can do it?" appeared naturally. It turned out that this question is important because its answer has numerous applications in other types of positional games. We saw in the previous section that strong games are hard to analyze, what is interesting is that Maker's fast win implies Red's win in strong Maker-Maker game. Indeed, as Maker is the first player, if he can occupy a whole winning set A in $|A|$ moves, where A is the smallest winning set, the game is finished before Blue has a chance to play his $|A|$-th move. Therefore, Red does not have to think about how to prevent Blue, but only how to occupy a winning set for himself.

Let us mention another game where the question of Maker's quick win was observed. For a given tree T on n vertices, we play the Maker-Breaker game on the edge set of a complete graph on n vertices. Maker wins this game at the moment when his graph contains a copy of T. It is proven in [29] that if T is a tree with a bounded maximum degree and n is large enough, then Maker can win this game in at most $n+1$ moves. Notice that this is just 2 away from the best possible.

In the following example, we give an explicit winning strategy for Maker, when $n \geq 6$.

Example 1.3.10. [69] The triangle game is played on the edge set of K_n and the winning sets are all copies of K_3 in K_n. As Maker is first to move, he claims an arbitrary edge uv, then in his first move Breaker claims an edge, we denote it by xy. Let us denote by vw the second move of Maker, such that $w \notin \{u, v, x, y\}$. If the second move of Breaker is not the edge uw, then Maker claims that edge and wins in his third move. Otherwise, if Breaker claims uw, then Maker claims an edge vz for some vertex $z \notin \{u, v, x, y, w\}$. Now there exist two free edges that make a triangle in Maker's graph (zw and zu), as Breaker cannot claim both of them in his next move, Maker makes a triangle in his fourth move and wins.

Biased Maker-Breaker game

As can be seen, in the strategy in Example 1.3.10, Maker can easily win this game, while the Breaker's chances are equal to zero. One can ask what are the possibilities for changing this game to give more power to Breaker. One standard way of doing that was for the first time suggested by Chvátal and

Erdős in [28]. This type of game we call a *biased* game.

Definition 1.3.11. [69] Let p and q be positive integers, let X be a finite set, and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. In the *biased* $(p : q)$ *Maker-Breaker game* $(X, \mathcal{F})$ the only difference compared to Maker-Breaker games is that Maker claims p free elements of the board per move and Breaker claims q free elements of the board per move. The integers p and q we call the *biases* of Maker and Breaker, respectively. If in the last move the player does not have enough free elements of the board, he claims all the free elements of the board and the game ends.

The Maker-Braker games that we saw earlier are a special case when $p = q = 1$, that we call *unbiased* or *fair*. We can now see what happens when the triangle game from Example 1.3.10 is played under biased rules. In [69] it is proved that Maker has a winning strategy in the $(1 : b)$ triangle game for every $b \leq \frac{\sqrt{n}}{2}$, and Breaker has a winning strategy if $b \geq 2\sqrt{n}$. A slightly better strategy for Breaker given in [6] gives that Breaker wins for every $b \geq (2 - \frac{1}{24})\sqrt{n}$. Recently, in [56] Glazik and Srivastav significantly reduced the gap towards the lower bound, proving that Breaker wins for $b \geq \sqrt{(\frac{8}{3} + o(1))n} \approx 1.633\sqrt{n}$ and n sufficiently large.

When Chvátal and Erdős introduced $(1 : b)$ Maker-Breaker game in [28], they noticed that they are *bias monotone*. To be more precise, if Breaker can win in the $(1 : b)$ game, he can also win in the $(1 : b + 1)$ game. Also, note that if the board of the game is X then Breaker wins in $(1 : |X|)$ game, except in the case when the family of winning sets $\mathcal{F}$ contains an empty set or a set of cardinality one. Therefore, we can define the *threshold bias* of the game $(X, \mathcal{F})$ as a unique positive integer $b_{\mathcal{F}}$ such that the $(1 : b)$ game is a Breaker's win if and only if $b > b_{\mathcal{F}}$, assuming that $\mathcal{F} \neq \emptyset$ and $min\{|A| : A \in \mathcal{F}\} \geq 2$, see Figure 1.4.

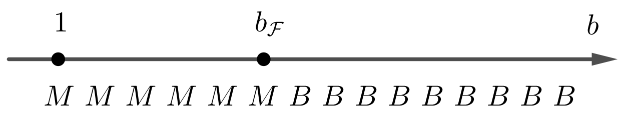


Figure 1.4. Threshold bias $b_{\mathcal{F}}$.

We can also conclude that general biased $(a : b)$ Maker-Breaker $(X, \mathcal{F})$ game is bias monotone, i.e. if Maker can win $(a : b)$ game, then he can also win in $(a + 1 : b)$ and $(a : b - 1)$ game. The same thing works for Breaker, if $(a : b)$ Maker-Breaker $(X, \mathcal{F})$ game is Breaker's win, then Breaker can also win in $(a : b + 1)$ and $(a - 1 : b)$ game. And more generally, if a player claims more elements at any point of the game, that cannot harm him. As in unbiased games, we have general theorems that are applicable to biased Maker-Breaker games.

Theorem 1.3.12. *([11], Biased Erdős-Selfridge Theorem) Let $(X, \mathcal{F})$ be a positional game, and let p and q be positive integers. If*

$$\sum_{A \in \mathcal{F}} (1 + q)^{\frac{-|A|}{p}} < \frac{1}{1 + q},$$

then Breaker (as the second player) wins (p, q) Maker-Breaker $(X, \mathcal{F})$ game. If Breaker is the first player, then the condition for Breaker's win is

$$\sum_{A \in \mathcal{F}} (1 + q)^{\frac{-|A|}{p}} < 1.$$

In the following theorem, we give a condition for Maker's win in the biased game.

Theorem 1.3.13. *[9] Let $(X, \mathcal{F})$ be a positional game, and let p and q be positive integers. If*

$$\sum_{A \in \mathcal{F}} \left(\frac{p + q}{p}\right)^{-|A|} > \frac{p^2 q^2}{(p + q)^3} \Delta_2(\mathcal{F}) |X|,$$

then Maker has a winning strategy in the biased $(p : q)$ Maker-Breaker game.

Here, we single out some results about the threshold bias in Maker-Breaker $(1 : b)$ games. For the Connectivity game, Chvátal and Erdős [28] proved that the threshold bias is between $(\frac{1}{4} - \epsilon)\frac{n}{\ln n}$ and $(1 + \epsilon)\frac{n}{\ln n}$, when $\epsilon > 0$. Later, Beck [11] improved the lower bound to $(\ln 2 - \epsilon)\frac{n}{\ln n}$. Finally, Gebauer and Szabó [55] showed that the threshold bias for the Connectivity game is asymptotically equal to $\frac{n}{\ln n}$, and solved the long-standing open problem.

When it comes to Hamilton Cycle game, Bollobás and Papaioannou [19] showed that Maker can win $(1 : b)$ Hamilton Cycle game if $b \leq O\Big(\frac{\ln n}{\ln \ln n}\Big)$.

Then, Beck [12] gave an explicit winning strategy for Maker, when $b \leq \left(\frac{\ln 2}{27} - o(1)\right)\frac{n}{\ln n}$ and n is sufficiently large. Later, Krivelevich and Szabó in [79] improved this bound to $(\ln 2 - o(1))\frac{n}{\ln n}$. Finally, Krivelevich in [78] showed that the threshold bias for Hamilton Cycle game is asymptotically equal to $\frac{n}{\ln n}$, finally settling this question.

Biased version of the Maker-Breaker game with biases a and b is also called *doubly biased* Maker-Breaker game. Some of the results about these games can be found in [73] where the authors studied the Connectivity game, determining the winner for almost all a and b. Results about the (a, b) Maker-Breaker Clique-game can be found in [54], whereas for the results about the planarity game, the k-colorability game, and the K_t-minor game, see [68].

1.3.3 Avoider-Enforcer games

Avoider-Enforcer games are the *misère* version of Maker-Breaker games, with two players named Avoider and Enforcer. The rules in these games are, in a way, opposite to the rules in Maker-Breaker games. Indeed, while the goal of Maker is to claim a whole winning set for himself and Breaker wants to prevent him from doing that, Avoider's goal is to *avoid* claiming a whole winning set, while Enforcer wants to *force Avoider* to claim a whole winning set. More precisely, Enforcer wins the game $(X, \mathcal{F})$ if, by the end of the game, Avoider claimed all elements of some $F \in \mathcal{F}$, otherwise Avoider wins.

We observe the biased version of this game in the same manner as in the Maker-Breaker games. Namely, in an $(a : b)$ Avoider-Enforcer game played on the board X with the given family of *losing sets* $\mathcal{F}$, Avoider claims a elements of X per turn, and Enforcer claims b elements of X per turn. If in the last move, there are fewer free elements of the board than a bias of the player that is to play, he claims all the remaining free elements of the board.

As we saw in the previous section Maker-Breaker games are bias monotone, and one can naturally wonder if this is true for Avoider-Enforcer games. If we think like Avoider, i.e. the goal is to avoid something, at first sight, it looks like claiming less elements cannot harm the player. It turns out that this is not true.

Example 1.3.14. Observe the Avoider-Enforcer $(a : b)$ game played on a hypergraph whose edge set contains two disjoint sets and each of them has two elements, see Figure 1.5. For $a = b = 1$ Avoider starts the game claiming one arbitrary element x. Then Enforcer claims one element y. If both of them are in the same set, Avoider plays arbitrarily in his following move and wins. Otherwise, Avoider claims another element from the set that contains y and wins. Therefore, the $(1 : 1)$ game is Avoider's win. When $a = 1$ and $b = 2$, whatever Avoider plays in his first move, Enforcer claims both elements from the other set and manages to force Avoider to lose in his second move. Hence, the $(1 : 2)$ game is Enforcer's win. If $a = b = 2$ Avoider starts the game claiming one element from both sets, after the first move of Enforcer, the game is over and Avoider wins.

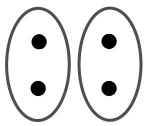


Figure 1.5. The hypergraph of the game from the example 1.3.14.

As we can conclude from Example 1.3.14, Avoider-Enforcer games in general are not bias monotone. This implies that it is not possible to define the threshold bias in the same way as in Maker-Breaker games. Having that in mind Hefetz, Krivelevich, and Szabó in [70] introduced the following definition.

Definition 1.3.15. For an $(1 : b)$ Avoider-Enforcer $(X, \mathcal{F})$ game we define the *lower threshold bias* as the largest integer $f_{\mathcal{F}}^-$ such that for every $b \leq f_{\mathcal{F}}^-$ Enforcer wins. Analogously, *upper threshold bias* is the smallest non-negative integer $f_{\mathcal{F}}^+$ such that for every $b > f_{\mathcal{F}}^+$ Avoider wins. Upper and lower threshold biases always exist (except in some trivial cases) and we know that $f_{\mathcal{F}}^- \leq f_{\mathcal{F}}^+$ holds, see Figure 1.6. In the special case when $f_{\mathcal{F}}^- = f_{\mathcal{F}}^+$, this number is called the *threshold bias* of the game.

The absence of the threshold bias inspired the authors of [64] to adjust these rules such that ensure the existence of bias monotonicity. They defined the so-called *monotone rules*, where each of the players is allowed to claim more elements of the board per move. More precisely, let $(X, \mathcal{F})$ be a monotone $(a : b)$ Avoider-Enforcer game. In each of his turns, Avoider claims *at least* a elements of the board, while Enforcer claims *at least* b elements of the board.

Figure 1.6. Upper and lower threshold bias.

With these rules the threshold bias becomes well-defined. If an $(a : b)$ game is Avoider's win, then so are these games $(a - 1 : b)$ and $(a : b + 1)$.

Definition 1.3.16. Let $(X, \mathcal{F})$ be an monotone $(1 : b)$ Avoider-Enforcer game. We define the unique *monotone threshold bias* $f_{\mathcal{F}}^{mon}$ as the largest non-negative integer such that Enforcer wins the game if and only if $b \leq f_{\mathcal{F}}^{mon}$, see Figure 1.7

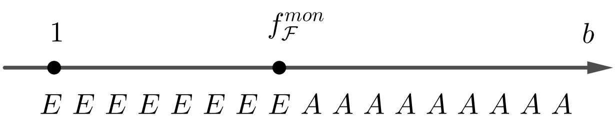


Figure 1.7. Monotone threshold bias.

Original rules, where each of the players claims exactly as many elements as the bias suggests we call the *strict rules*. Each Avoider-Enforcer game can be played under both sets of rules. Therefore, we emphasize which system of rules the game is based on, strict or monotone. Even though this adjustments in rules look minor, they can drastically change the outcome of the game. We compare as an example results related to the Connectivity game.

In the $(1 : b)$ Connectivity game played on $E(K_n)$ under strict rules, it is interesting to observe that the threshold bias in this game exists. Indeed, in [70] the authors showed that $f_{\mathcal{C}}^{-} = f_{\mathcal{C}}^{+} = \lfloor \frac{n-1}{2} \rfloor$. On the other hand, if the game is played under monotone rules, combining results from [64] and [79] we get that the threshold bias for the Connectivity game is $(1 + o(1))\frac{n}{\ln n}$, very far from $f_{\mathcal{C}}^{-}$ and $f_{\mathcal{C}}^{+}$.

As in the Maker-Breaker games, we also have two important winning criteria for Avoider's win in the $(a : b)$ game $(X, \mathcal{F})$.

Theorem 1.3.17. *[70] If*

$$\sum_{A\in\mathcal{F}}(1+\frac{1}{a})^{-|A|+a}<1,$$

then Avoider wins the biased $(a:b)$ game $(X,\mathcal{F})$, both strict and monotone, for every $b\geq 1$.

One limitting factor of this criterion is the fact that it does not consider the Enforcer's bias, hence it is rarely effective when b is large. This lack was partially improved by Bednarska-Bzdęga in [14], who gave another criterion for games played on the hypergraphs with small rank, where the *rank* of $\mathcal{F}$ is the cardinality of the largest set $X\in\mathcal{F}$.

Theorem 1.3.18. *Let $\mathcal{F}$ be a positional game of rank r. If*

$$\sum_{A\in\mathcal{F}}(1+\frac{b}{ar})^{-|A|+a}<1,$$

then Avoider wins the biased $(a:b)$ game $(X,\mathcal{F})$, both strict and monotone.

There is a number of results on Avoider-Enforcer games in the last few decades, we mention some of them. Anuradha, Jain, Snoeyink and Szabó [3] looked at the unbiased non-planarity game. This game ends when the edges chosen by Avoider form a non-planar subgraph. They showed that Avoider can play for $3n-26$ turns. The same game was studied by Hefetz, Krivelevich, Stojaković and Szabó in [66] where they were interested in the question how fast Enforcer can win. They estimate this quite precisely, giving the minimum number of moves Enforcer has to play in order to win in the non-planarity game, the connectivity game and the non-bipartite game.

Balogh and Martin [4] studied the Avoider-Enforcer game in which Enforcer wins this game if Avoider's graph has the property $\mathcal{P}$. In this paper $\mathcal{P}$ is the property that a member of $\mathcal{F}$ is a subgraph or an induced subgraph, and they tried to find the smallest number of moves Enforcer needs to play to win.

Barat and Stojaković in [7] analyzed the duration of the unbiased Avoider-Enforcer game for three different positional games in which the Avoider's goal is to keep his graph outerplanar, diamond-free and k-degenerate, respectively. Later in [46] the authors studied Avoider–Enforcer games played on edge-disjoint hypergraphs for both sets of rules, strict and monotone, and gave a sufficient condition to win for each player. This provides the analog to the well-known Box games.

Grzesik et al. in [59] studied $(1 : b)$ Avoider-Enforcer games played on K_n, where Avoider's goal is to avoid claiming a copy of some small fixed graph H. They gave the explicit winning strategy for both players when $H = K_{1,l}$, for both strict and monotone rules.

1.3.4 Client-Waiter (and Waiter-Client) games

Client-Waiter games were introduced by Beck [10] under the name *Chooser-Picker* games. In the unbiased game, the player called Waiter selects 2 unclaimed elements of the board X and offers them to Client, then Client takes one of them while the other element goes back to Waiter. Client wins in this game if he occupies a whole winning set, and Waiter wins if he can prevent Client's win. When $|X|$ is odd, the last element goes to Client.

In the counter version of this game called *Waiter-Client* game, everything stays the same except that the winning conditions are swapped, and when $|X|$ is odd, the last element goes to Waiter. Therefore, Client has the same goal as Avoider. Waiter's goal is to force Client to claim all elements of some winning set and Client tries to avoid it.

Csernenszky, Mándity and Pluhár [35] give winning conditions for Waiter in some Client–Waiter games, and extend the results of Beck. Csernenszky in [33] confirms Beck's conjecture that Waiter has better chances than Maker for the diameter-2 game, that was studied by Balogh et al. [5].

Later in [34] Csernenszky et al. looked at the complexity of these games, and proved that both versions are NP-hard, which gives support to the paradigm that the games behave similarly while being quite different in definition. They also investigate the pairing strategies for Maker–Breaker games in the same paper.

Bednarska in [13] studied a biased version of the Client-Waiter game, where she proved two weight-function-based winning criteria for Waiter and showed that the Erdős–Selfridge winning criterion for Breaker's win is also the winning criterion for Waiter in the unbiased game. She improved previous results from [35] and estimated the threshold bias for Waiter in some Client-Waiter games.

Dean and Krivelevich [36] studied several Client-Waiter games on the edge set of the complete graph, and the H-game on the edge set of the random graph. Hefetz et al. [71] observed the both Waiter–Client and Client–Waiter versions of the non-planarity, K_t-minor and non-k-colorability games, and for each of them gave the precise estimate of the unique integer at which the outcome of the game changes from Client's win to Waiter's win.

Another variant of the Waiter-Client game is introduced in [15]. For a given graph H and a positive integer n, a Waiter-Client H-game is a biased game played on the edge set of the complete graph in which Waiter is trying to force many copies of H and Client is trying to prevent him from doing so. They proved that the value of the game is roughly the same as the expected number of copies of H in the random graph when the graph H is a complete graph or a tree.

Hefetz, Krivelevich and Tan [72] studied Waiter–Client and Client–Waiter Hamiltonicity games on random graphs and found the smallest edge probabilities p_1 and p_2 for which a.a.s. (asymptotically almost surely) Waiter has a winning strategy for the $(1 : q)$ Waiter–Client and Client-Waiter Hamiltonicity game, for any q.

One more version of this game is introduced recently by Krivelevich and Trumer [80] called the Waiter-Client Maximum Degree game. Waiter and Client play on the edge set of the complete graph such that Waiter offers $q + 1$ free edges in each turn. Client claims one of them and all of the remaining edges go to Waiter. When less than $q + 1$ edges that have not been offered remain, Waiter claims them all and the game ends. Client wins if in his graph at the end of the game there is no vertex of degree at least D and Waiter wins otherwise. They studied the maximum degree of the Client's graph obtained by optimal play. For $q = 1$ they obtained that $D = \frac{n}{2} + \Theta(\sqrt{n \ln n})$.

Recently in [31] the authors studied fast strategies for several Waiter-Client games played on the edge set of the complete graph in which the winning sets are perfect matchings, Hamilton cycles, pancyclic graphs, fixed spanning trees or factors of a given graph.

1.3.5 Walker-Breaker games

Walker-Breaker games are another version of Maker-Breaker games. In fact, the only difference compared to Maker-Breaker games is that Maker is forced to play such that his graph must stay a walk or a path of a given graph G. These games are recently introduced by Espig, Frieze, Krivelevich, and Pegden in [42]. At any moment of the game, Walker is positioned at some vertex v on his turn (In his first move he chooses this vertex arbitrarily). Then, in his i-th move he claims an edge $e = vu_i$ that is incident with v and it is not previously claimed by Breaker. After that, the vertex u_i becomes the new position, i.e. $v = u_i$. Breaker plays as usual, without any restriction, so this setup brings more power to Breaker.

Example 1.3.19. As an example of this game, we illustrate the Breaker's strategy for isolating a vertex in Walker's graph, and that implies that Walker is not able to make a spanning graph in the unbiased version of this game. After Walker claimed his first edge, Breaker chooses one vertex x that is not touched by Walker. Then in each of his moves, Breaker claims the edge that is incident with x and Walker's current position v, which secures a Breaker's win.

One of the first questions about these games was studied in [42] and that is: What is the largest number of vertices Walker can visit in a given graph? They proved that the answer to this question in $(1 : b)$ biased games is $n - 2b + 1$. Then, in [32] the authors observed how large cycle can Walker make. Later, Forcan and Mikalački in [48] studied the $(2 : b)$ version of this game and showed that Walker can win Connectivity and Hamilton cycle game even when playing against Breaker whose bias is of the order $\frac{n}{\ln n}$. Several different game variants arose from this one such as *WalkerMaker-WalkerBreaker games*, see [50] and [47], where both players have the constraint to claim edges of a walk.

1.3.6 Toucher-Isolator game

Toucher-Isolator game is another version of Maker-Breaker games played on the edge set of a given graph G. This is the game played by two players called Toucher and Isolator. They alternately claim unclaimed edges of the board, and Toucher plays first. The aim of Toucher is to 'touch' as many vertices as possible (i.e. to maximize the number of vertices that are incident to at least one of his chosen edges), and the aim of Isolator is to minimise the number of such vertices.

This game was introduced recently in [38] where the authors observe the number of untouched vertices $u(G)$ at the end of the game when both players play optimally. They proved that for any graph G, the number of untouched vertices is bounded between $d_0+\frac{1}{2}d_1-1 \leq u(G) \leq d_0+\frac{3}{4}d_1+\frac{1}{2}d_2+\frac{1}{4}d_3$, where d_i represents the number of vertices with degree exactly i. Interestingly, from this result, we can see that it is enough to consider just the vertices with a degree at most three. In the same paper, they studied this game for some special classes of graphs, such as cycles, paths, trees, and k-regular graphs, and get the following results: $\frac{3}{16}(n-3) \leq u(C_n) \leq \frac{n}{4}$ for cycles, $\frac{3}{16}(n-2) \leq u(P_n) \leq \frac{n+1}{4}$ for paths, $\frac{n-3}{6} \leq u(G) \leq \frac{n}{4}$ for any 2 regular graph G with n vertices and $\frac{n+2}{8} \leq u(T) \leq \frac{n-1}{2}$ for any tree with n vertices.

Later, Räty in [96] improved the lower bound for trees to $u(T) \geq \lfloor \frac{n+3}{5} \rfloor$, which is sharp. The same author in [97] showed that $u(C_n) = \lfloor \frac{n+1}{5} \rfloor$ and $u(C_n) = \lfloor \frac{n+4}{5} \rfloor$ when both players play optimally. Recently, Boriboon and Kittipassorn in [20] gave the simpler proof for the Räty's lower bound for trees.

1.3.7 Maker-Breaker domination game

This game was for the first time introduced by Duchêne, Gledel, Parreau, and Renault in [39] as another variant of the Maker-Breaker game. Let a graph $G = (V, E)$ be given, the board of the game X is the vertex set V, and $\mathcal{F}$ is the set of all the dominating sets of G. Two players called Dominator and Staller alternately occupy unoccupied vertices of G. Dominator wins if he manages to build a *dominating set* of G, that is, a set D such that every vertex not in D has a neighbor in D. Otherwise, Staller wins, i.e. he wins if he manages to occupy a vertex and all its neighbors.

Domination game in a different setting was for the first time introduced in [22] by Brešar, Klavžar and Rall, and more about this game can be found in [24, 37, 90, 98, 111]

Studying Maker-Breaker domination game it turns out that it is hard to control the sizes of the winning sets. Hence, in [39] the authors proposed the reversed version of this game, i.e. Staller takes over the role of Maker in the neighborhood game, and Dominator becomes Breaker. In this paper, they observed which player has a winning strategy. Another interesting question is what is the minimum number of moves needed for Dominator to win provided that he has a winning strategy, that was studied in [58].

Recently Gledel, Henning, Iršič, and Klavžar proposed a natural extension of these games called Maker-Breaker total domination game, see [57]. Just as the total domination game [75] (see also [23],[25],[76]) was a generalization of the domination game, they introduced the *Maker–Breaker total domination game*. The first player, Dominator, wins on G if he can select a total dominating set of G, that is, a set D such that every vertex of G has a neighbor in D, and Staller wins if he can select all the vertices from the open neighborhood of some vertex.

Although the definition for these two games looks very similar, it turns out that they can be significantly different. Indeed, there are examples such as the recent characterization of perfect graphs for the domination game, which can confirm this, see [26]. Forcan and Mikalački [49] observed Maker-Breaker total domination game on the connected cubic graphs.

1.3.8 Strong CAvoider-CAvoider game.

In the last few years, several variants of positional games have emerged, like the *PrimMaker-Breaker game* introduced in [82] where the subgraph induced by Maker's edges must stay connected throughout the game. In the *Walker-Breaker games* introduced by Espig, Frieze, Krivelevich, and Pegden [42], Maker is constrained to choose edges of a walk or a path. Similarly, in the *WalkerMaker–WalkerBreaker games*, see [50], both players have the constraint to claim edges of a walk. Following this line of research *Strong CAvoider-CAvoider games* can be seen as a natural extension of the strong Avoider-Avoider games.

The Strong CAvoider-CAvoider game $(E(K_n), \mathcal{F})$ is played by two players Red and Blue, where the player who first fully occupies some $F \in \mathcal{F}$ loses the game. Players alternately claim the edges of the complete graph K_n, such that the graph of each player must stay connected throughout the game. If neither of the players loses and all the elements of the board are claimed, the game is declared a draw. The target sets of the family $\mathcal{F}$ are called losing sets. We introduced these games in [103] where the family $\mathcal{F}$ consists of all copies of some given graph F.

1.3.9 Games on random boards

As we can see in the previous sections, the fair Maker-Breaker game played on the edge set of a complete graph K_n for many standard graph properties is Maker's win. To give more power to Breaker, one approach was to introduce the biased version of these games. Here, we give another approach, randomly making the board sparser before the game starts. Therefore, some of the winning sets vanish, Maker gets fewer chances for a win, and Breaker is given more power. This kind of games were introduced by Stojaković and Szabó [106].

Definition 1.3.20. Let $(X, \mathcal{F})$ be a positional game and $p \in [0, 1]$. The *game on the random board* $(X_p, \mathcal{F}_p)$ is a probability space of games, where

1. each $x \in X$ is included in X_p with probability p and
2. $\mathcal{F}_p = \{A \in \mathcal{F} : A \subseteq X_p\}$.

As the probability p decreases, it becomes increasingly difficult for the Maker to win, and at some point the Maker is no longer expected to win. As "being a Maker's win in $\mathcal{F}$" is an increasing graph property [102], there must exist a threshold probability $p_{\mathcal{F}}$ for this property, and we search for it in the unbiased version of the game in such a way that the following holds, see Figure 1.8:

- $\Pr[\text{Breaker wins } \mathcal{F}_p] \to 1$ for $p \ll p_{\mathcal{F}}$,
- $\Pr[\text{Maker wins } \mathcal{F}_p] \to 1$ for $p \gg p_{\mathcal{F}}$.

This question was studied for different types of games. Denote by $\mathbb{G}(n, p)$ a graph on n vertices where each of the edges of the complete graph K_n is included independently with probability p.

For well-known games such as connectivity and perfect matching game it has been proven in [106] that the threshold probability is $\frac{\ln n}{n}$. In the same paper the authors showed that the threshold probability for Hamilton Cycle game is between $\frac{\ln n}{n}$ and $\frac{\ln n}{\sqrt{n}}$. Later, Stojaković in [101] showed that $p_{\mathcal{HAM}}$

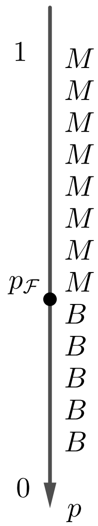


Figure 1.8. Probability threshold.

is of order $\theta(\frac{\ln n}{n})$. Finally in [67] the authors showed that the property that Maker can win Hamilton cycle game, has a sharp threshold at $(1+o(1))\frac{\ln n}{n}$. The Maker-Breaker k-clique game played on the edge set of the random graph $\mathbb{G}(n,p)$ was studied by Müller and Stojaković in [89]. They gave the threshold probability for the graph property that Maker wins this game and that is $p_{\mathcal{K}_k} = \Theta\left(n^{\frac{-2}{k+1}}\right)$, for all $k > 3$.

Another interesting fact about this game has been realized by comparing the last result with the result of Bednarska and Łuczak [16]. They found the threshold bias for the biased $(1:b)$ fixed graph game on $E(K_n)$. It turns out that for the K_k game the inverse threshold bias in $(1:b)$ game is of the same order of magnitude as threshold probability $p_{\mathcal{K}_k}$ in the unbiased version of the game. This surprising result was for the first time found in [106] for the perfect matching and connectivity game.

For the unbiased fixed graph game played on the random board, Nenadov, Steger and Stojaković in [91] obtained that except trees and triangles, the threshold for an H-game is given by the threshold of the corresponding Ramsey property of $\mathbb{G}(n,p)$ with respect to the graph H.

Combining two settings, the biased game and the game played on a random board, a new interesting question appeared. For every $p > p_{\mathcal{F}}$, how large can $b^p_{\mathcal{F}}$ be so that Maker wins $\mathcal{F}_p$ with bias $(1 : b^p_{\mathcal{F}})$ asymptotically almost surely? The connecivity and perfect matching game were considered in [106], where the authors obtained the smallest bias $b^p_{\mathcal{F}}$, for which Breaker can win the $(1 : b^p_{\mathcal{F}})$ game $(X_p, \mathcal{F}_p)$ almost surely. The result they got is that there exist constants C_1 and C_2 such that $b^p_{\mathcal{C}} = \Theta(pb_{\mathcal{C}}) = \Theta(p\frac{n}{\ln n})$, when $p \geq C_1\frac{1}{b_{\mathcal{C}}}$, and $b^p_{\mathcal{M}} = \Theta(pb_{\mathcal{M}}) = \Theta(p\frac{n}{\ln n})$, when $p \geq C_2\frac{1}{b_{\mathcal{M}}}$.
Later, in [43] Ferber et al. analyze biased Maker-Breaker games and Avoider-Enforcer games, both played on the edge set of a random graph $\mathbb{G}(n, p)$. More precisely, they proved that for every $p = \omega(\frac{\ln n}{n})$ random graph $\mathbb{G}(n, p)$ is such that the asymptotic threshold bias for perfect matching, Hamiltonocity and k-vertex-connectivity game is $\frac{np}{\ln n}$. This result resolved the conjecture given in [106]. For Avoider-Enforcer games, they proved that for $p = \Omega(\frac{\ln n}{n})$ the threshold bias for all the aforementioned games is $\frac{np}{\ln n}$.
Fast-winning strategies can also be studied on random boards. Clemens, Ferber, Krivelevich and Liebenau [30] observed perfect matching, Hamiltonocity and k-vertex-connectivity game. They proved that for $p = \frac{\ln n^K}{n}$ and $K > 100$ the graph $\mathbb{G}(n, p)$ is typically such that Maker can win all the aforementioned games as fast as possible, i.e. within $\frac{n}{2} + o(n)$, $n + o(n)$ and $\frac{kn}{2} + o(n)$ moves, respectively.

1.4 Saturation two-player games on graphs

1.4.1 Saturation games.

For a given graph $G = (V, E)$ is said to be *saturated* with respect to a monotone increasing graph property $\mathcal{P}$, if G does not have the property $\mathcal{P}$, but $G \cup \{e\} \in \mathcal{P}$ for every edge $e \in \binom{V}{2} \backslash E$.
Given an empty graph on n vertices $\overline{K}_n$ and a property $\mathcal{P}$, two players Max and Mini progressively build a graph $G \subseteq K_n$ such that G does not satisfy the property $\mathcal{P}$. Note that here both players claim edges of the same color, i.e. they build the same graph G. The game finishes when there are no more free edges that can be added to G, so G is a saturated graph. Max's goal is to maximize the length of the game, while Mini wants to minimize it. Denote by $s(n, \mathcal{P})$ the *score* of the game as the number of the edges of a

graph G at the end of the game. The goal is to determine the score of the game in two cases, when Max is the first player and when Mini is the first player.

We define the two numbers that are related to the score of this game. The first one is *saturation number*, denoted by $sat(n, \mathcal{P})$, that is the minimum size of a saturated graph on n vertices with respect to $\mathcal{P}$. The second one is the maximum possible size of a saturated graph on n vertices with respect to $\mathcal{P}$, denoted by $ex(n, \mathcal{P})$. From everything above mentioned we conclude that $sat(n, \mathcal{P}) \leq s(n, \mathcal{P}) \leq ex(n, \mathcal{P})$.

Saturation games on graphs were first introduced in [52] by Füredi, Reimer, and Seress. They observed the number $s(n, \mathcal{K}_3)$ where $\mathcal{K}_3$ is the property of containing a triangle. From Mantel's theorem [109] we know that $ex(n, \mathcal{K}_3) = \lfloor \frac{n^2}{4} \rfloor$. On the other hand, $sat(n, \mathcal{K}_3) = n - 1$, because the star is a saturated graph with respect to $\mathcal{K}_3$. Therefore, we know that $s(n, \mathcal{K}_3)$ lies between these bounds. In [52] authors give a lower bound of order $n \ln n$. Later, Biró, Horn and Wildstrom [18] improved the upper bound to $\frac{26}{121} n^2 + o(n^2)$.

Carraher et al. in [27] found the scores of some particular games, in which the property $\mathcal{P}$ is the family of odd cycles $\mathcal{O}$, the family of spanning trees $\mathcal{T}_n$, containing a $K_{1,3}$, containing a P_4. In the same paper they also observed the saturation games played on bipartite graphs. Later in [63] Hefetz, Krivelevich, Naor and Stojaković proved lower and upper bounds of the score in the games where the property $\mathcal{P}$ is being k-connected, having chromatic number at least k and admitting a matching of a given size.

Patkós and Vizer in [94] considered another version of this game in which the board is the edge-set of the complete k-graph $X_{n,k}$ on n vertices and $\mathcal{P} = \mathbb{I}_{n,k}$ is the set of intersecting families such that the following holds $\mathbb{I}_{n,k} = \{\mathcal{F} \subseteq X_{n,k},\ F \cap G \neq \emptyset,\ \forall F, G \in \mathcal{F}\}$.

1.4.2 Constructor-Blocker game.

Constructor-Blocker games are recently introduced by Patkós, Stojaković and Vizer [93]. These games are a merge of two well-known games: Maker-Breaker positional games and saturation games.

Let H and F be two fixed graphs. Two players, called Constructor and Blocker, alternately claim unclaimed edges of the complete graph K_n. Constructor is constrained to claiming only edges such that his graph does not contain a copy of F, i.e. his graph must remain F-free. On the other hand, Blocker can claim unclaimed edges without restrictions. The game is over when Constructor cannot make any further moves, i.e. his graph is F saturated or all edges have been claimed. The *score* of this game is the number of copies of H in the Constuctor's graph at the end of the game. The goal of Constructor is to maximize the score, whereas Blocker aims to keep the score as low as possible. The score of the game when both players play optimally we denote by $g(n, H, F)$.

Let us denote by $\mathcal{N}(H, G)$ the number of copies of H in the graph G, and by $ex(n, H, F) = \max_G\{\mathcal{N}(H, G) : G$ is an F-free graph on n vertices$\}$. It can be easily seen that $ex(n, F) = ex(n, K_2, F)$ because in that case, we count the number of edges in the graph G. Recently in [1] the authors studied the function $ex(n, H, F)$. It is clear that inequality $g(n, H, F) \leq ex(n, H, F)$ is always true.

In Constructor-Blocker games unlike saturation games, both players build their own graphs, and we count the number of copies of H just in the Constructor's graph. In [93] the authors obtain the score for several different games: when both F and H are stars, $F = P_4$ and $H = P_3$, F is a star and H is a tree, $F = P_5$ and $H = K_3$, and they gave upper and lower bounds on $g(n, P_4, P_5)$.

1.4.3 Generalized saturation game

Generalized saturation games are introduced as a natural extension of two different types of saturation games and Constructor-Blocker games.

Let H and F be graphs that are given in advance. Two players, Max and Mini, alternately claim unclaimed edges of K_n such that the graph of the game G does not contain a copy of F. The game ends when the players cannot claim further edges, i.e. the graph G is F-saturated or there are no more unclaimed edges. Note that here, as in the saturation games, both players together build the same graph $G \subseteq K_n$.

The *score* of the game is the number of copies of H in G at the end of the game. Max wants to maximize the score, while Mini tries to keep the score as low as possible. When the graph $H = K_2$, these games become saturation games, so we could say that saturation games are the special case of generalized saturation games.

Chapter 2

Main results

In this chapter, we list all the main results that will be proven in the rest of this text. As we study three main topics in this thesis, this chapter consists of three sections. Section 2.1 is devoted to Strong Avoider-Avoider positional games. In Section 2.2 we give the main results for the achievement number in strong Maker-Maker games. Finally, in Section 2.3 we list all the main results we obtained for the generalized saturation games.

2.1 Strong Avoider-Avoider games

We will take a closer look at strong Avoider-Avoider games. Even though their definition is natural and many questions about them have been asked, very few of them have been answered. To offer some intuition behind this phenomenon, we should keep in mind that the players in strong games have the same goal and the only thing that makes a difference is who goes first, we call this the "half-a-move advantage". Informally speaking, depending on the structure of the board there are different ways things can play out, but that half-a-move eventually decides the game. So the player that can win should propagate his (in most cases, comparatively small) advantage from the beginning to the end of the game, knowing that one wrong move may take the edge away from him. In contrast to this, in weak games, we have more freedom when designing a winning strategy, as players have different goals. This further allows the introduction of bias, first time introduced in [28], which gives us more room to spare. Hence, in most of the weak games studied in the literature, we are not that close to the breaking point at which a player stops winning and starts losing.

2.1.1 Strong Avoider-Avoider games on $E(K_n)$

As we have already seen, when we play a strong Avoider-Avoider game $(X, \mathcal{F})$, we are given a board X and a family of losing sets $\mathcal{F}$. Two players called Red and Blue in each of their moves claim one element of the board X each, and the player who first fully occupies some $F \in \mathcal{F}$ loses the game.

We are interested in Strong Avoider-Avoider F games played on the edges of the complete graph K_n, where the members of the family $\mathcal{F}$ are all sets that contain a copy of F. Hence, the player who first fully occupies a copy of F loses the game.

Not much is known about these games, while there are many open problems. In [62] it was shown that Blue has a winning strategy in the P_3 game, where the forbidden graph is the path with just two edges. Recently, Beker [17] generalized this result to all stars, proving that for each fixed k the Strong Avoider-Avoider star S_{k+1} game is a win for the second player for all n sufficiently large. The proof is performed by actually building rather than avoiding – showing that Blue can build a S_{k+1}-free graph of maximum size fast, without wasting any moves, thus automatically securing a win.

The main idea of the proof is the following. Note that during this game, both players must maintain the maximum degree in their graphs lower than $k + 1$. We call the graph that satisfies this condition "valid". Then, a straightforward way for Blue to win would be to build a "valid" graph with $\mathrm{ex}(n, S_{k+1}) = \lfloor \frac{nk}{2} \rfloor$ edges. This proof is based on the idea that Blue can build a "valid" graph of size $\lfloor \frac{nk-1}{2} \rfloor$, such that if nk is even, there exists an unclaimed edge which, when added to it, maintains the property that this graph is valid. Then, using the fact that Red's last move is uniquely determined, Blue can force a win. In order to prove that this is possible the author defined an auxiliary game played on a general graph, where the goal is to make a layers of almost perfect matchings until Blue's goal is reached. For this, a fast winning strategy of Maker in the Maker-Breaker perfect matching game is used.

Malekshahian [86] studied the possibility of Blue's win in the triangle game with the assumption that the game starts on several special mid-game positions, without any definite implications on the outcome of the triangle game itself. Hence, the only non-trivial Strong Avoider-Avoider game played on $E(K_n)$ for which the outcome is previously known is the star game.

Let S_4 be a star on four vertices. We use the abbreviation $\mathcal{CC}_{>3}$ for the collection of inclusion-minimal connected graphs on more than three vertices and P_4 represents a path on four vertices. Our goal is to determine the outcome for the P_4 game and the $\mathcal{CC}_{>3}$ game.

Theorem 2.1.1. *Blue has a winning strategy in the Strong Avoider-Avoider P_4 game, played on K_n, where $n \geq 8$.*

In the following theorem, we consider the game where a player loses the game as soon as he creates a connected component on more than three vertices.

Theorem 2.1.2. *Blue has a winning strategy in the Strong Avoider-Avoider $\mathcal{CC}_{>3}$ game, played on K_n, where $n \geq 5$.*

Let $R(F)$ be a diagonal Ramsey number, so every 2-coloring of edges of a complete graph on at least $R(F)$ vertices gives a monochromatic F. If $n \geq R(F)$ we know that the strong Avoider-Avoider F game on $E(K_n)$ cannot end in a draw. For both the P_4 game and the $\mathcal{CC}_{>3}$ game this readily implies that there is no draw for $n \geq 5$.

2.1.2 Strong CAvoider-CAvoider games

We have already seen in the previous chapter that intuition for these games came naturally from similar games in which one or both players are restricted to make a connected graph throughout the game.

We study *Strong CAvoider-CAvoider F games* in which the graph of each player must stay connected throughout the game. The board is still the edge set of K_n, and the players should not claim a copy of the forbidden graph F. This is a natural extension of the strong Avoider-Avoider games, with a connectedness constraint analog to the ones introduced in [82].

In the following, we prove that Blue can win in three different strong CAvoider-CAvoider games.

Theorem 2.1.3. *Blue has a winning strategy in the Strong CAvoider-CAvoider S_4 game, played on K_n, where $n \geq 7$.*

Theorem 2.1.4. *Blue has a winning strategy in the Strong CAvoider-CAvoider P_4 game, played on K_n, where $n \geq 5$.*

In the following theorem, we observe the Cycle game where the player who first claims a cycle loses.

Theorem 2.1.5. *Blue has a winning strategy in the Strong CAvoider-CAvoider Cycle game, played on K_n, where $n \geq 6$.*

Note that if $F \in \{S_4, K_3\}$, then diagonal Ramsey number $R(F)$ is 6, hence draw is not possible in any of the aforementioned three games.

2.1.3 Strong CAvoider-CAvoider games from a position

Here, we study strong CAvoider-CAvoider games that start from a particular position on the graph, i.e. we can suppose that in this game several moves have already been played, and we know how the graph of the game looks like at that moment, and then we continue to play. This kind of game was used by Malekshahian in [86].

A graph K_5 whose edges are colored in two colors, such that it consists of two C_5's, one in blue and the other one in red color we call a "drawn K_5", see Figure 2.1. Position T will be the configuration of the graph G, such that G consists of one "drawn K_5" and $n-5$ isolated vertices. Odd (respectively, Even) cycle game is the game where the forbidden graph is any graph that contains an odd (respectively, even) cycle.

In the following theorem we suppose that after the first five rounds, the graph of the game consists of a "drawn K_5" and some isolated vertices.

Theorem 2.1.6. *Playing from position T, Blue has a winning strategy in the Strong CAvoider-CAvoider C_3 game, C_4 game, C_6 game, Odd cycle game, and Even cycle game.*

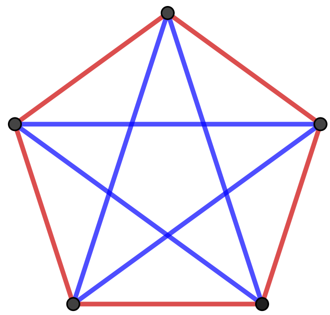


Figure 2.1. Position T.

In Section 3.1 we give notation and preliminaries. In Section 3.2 we give the proof of Theorem 2.1.1. In Section 3.3 we prove Theorem 2.1.2. Then, in Section 3.4 we prove Theorem 2.1.3, Theorem 2.1.4, Theorem 2.1.5. Finally, in Section 3.5 we prove Theorem 2.1.6.

2.2 Achievement number in strong Maker-Maker games

Let us recall that in a *strong Maker-Maker game* $(X, \mathcal{F})$, two players called Red and Blue take turns in claiming previously unclaimed elements of X, with Red going first. The player who first fully occupies some $F \in \mathcal{F}$ is the winner. If neither of the players wins and all the elements of the board are claimed, the game is declared a *draw*.

2.2.1 Opening remarks

We observe a strong Maker-Maker game played on the edge set of the complete graph K_n, with the graph F given in advance. The first player who completes a copy of F in his color is the winner. We define the *achievement number* of F as the smallest n for which Red can win this game, and denote it by $a(F)$. Let us denote by $R(F)$ a diagonal Ramsey number, that is the smallest integer such that every 2-coloring of the board gives a monochromatic F. Strategy stealing argument implies that $a(F) \leq R(F)$.

In the misère version of this game, called the strong Avoider-Avoider game, the first player who completes a copy of F in his color loses. Here, we define the *avoidance number* of F as the smallest n for which one of the players can force the other one to make a copy of F and denote it by $\overline{a}(F)$.

These numbers are for the first time introduced by Harary [61] where he studied them for several small fixed graphs F. Later, the same author [41] introduced the bipartite achievement number of F, denoted by $ba(F)$, which is the minimum n for which Red wins playing on the complete bipartite graph $K_{n,n}$. In this version, the fixed graph F is a bipartite graph with no isolated vertices. In the same paper, he determined the bipartite achievement number for stars, matchings, paths and cycles.

Achievement numbers were also studied by Beck in [9], where he was interested in finding this value for a clique K_q. Here the Erdős-Szekeres Theorem comes to light when we need to determine the size of the board for which a draw is not possible. It states that in any 2-coloring of the edges of the complete graph K_N with $N \geq \binom{2q-2}{q-1}$ vertices, there must be a monochromatic copy of K_q.

We are interested in several questions about the achievement numbers asked by Harary in [61] and we are going to answer some of them. First, he gave a table with the achievement numbers for several small graphs F whose values were calculated using a computer, see Figure 2.2, with some unknown values left blank. Later, in [41] Harary extended this table with two more values calculated in the same way, $a(K_4 - e) = 7$ and $a(K_4) = 10$. He spelled out the following problem and added a conjecture.

Problem 2.2.1. [61] Determine $a(F)$ and $\overline{a}(F)$ for various families of graphs. (This appears hard even for trees.)

Conjecture 2.2.2. [61] The minimum value of $a(T)$ among all trees T of order n is realized when $T = P_n$, the path. The maximum of $a(T)$ is attained for T the star $K_{1,n-1}$.

	K_2	P_3	$2K_2$	P_4	$K_{1,3}$	K_3	C_4	$K_{1,3}+e$	K_4-e	K_4
a	2	3	5	5	5	5	6	5	?	?
$\overline{a}$	2	3	5	5	5	6	6	5	?	?
R	2	3	5	5	6	6	6	7	10	18

Figure 2.2. [61] The known achievement and avoidance numbers for graphs, with the corresponding Ramsey numbers.

2.2.2 Red's winning strategy for small graphs

As we mentioned above, the values of the achievement numbers given in the table, see Figure 2.2, were calculated with the help of a computer, hence we do not know anything about Red's winning strategy. In the proofs of the following assertions, we give an explicit winning strategy for Red starting on $a(F)$ isolated vertices for some graphs F given in this table, and we also observe the number of moves Red needs in order to win.

Observation 2.2.3. *Achievement numbers for K_2 and P_3 are 2 and 3, respectively, i.e. $a(K_2) = 2$ and $a(P_3) = 3$.*

Proposition 2.2.4. *The achievement number for $2K_2$ is 5, i.e. $a(2K_2) = 5$.*

Proposition 2.2.5. *The achievement number for P_4 is 5, i.e. $a(P_4) = 5$.*

Proposition 2.2.6. *The achievement number for $K_{1,3}$ is 5, i.e. $a(K_{1,3}) = 5$.*

Proposition 2.2.7. *The achievement number for K_3 is 5, i.e. $a(K_3) = 5$.*

Proposition 2.2.8. *Achievement number for the $K_{1,3}+e$ is 5, i.e. $a(K_{1,3}+e) = 5$.*

We use notation $K_4 - e$ for a complete graph on four vertices minus one edge.

Proposition 2.2.9. *Red can win in a strong Maker-Maker $K_4 - e$ game played on K_7, i.e. $a(K_4 - e) \leq 7$.*

2.2.3 Achievement number for paths, cycles, stars and perfect matchings

Some of the most important and oftentimes studied graphs, definitely are paths, cycles, stars and perfect matchings. Hence, we are interested in finding achievement numbers of paths, cycles, stars and perfect matchings, and concurrently giving some of the answers for the Problem 2.2.1.

Proposition 2.2.10. *Achievement number for a path on n vertices is $a(P_n) = n$, for n sufficiently large.*

Note that in Proposition 2.2.10 we suppose that n is sufficiently large, but going through the proof we conclude that n must be bigger than 16. Hence, the question if this is true for $n \leq 16$ still remains open. It is clear that for $n \leq 3$ this is true, but $a(P_4) = 5$, so for $n = 4$ it is not. It can be proven that this is true for $n = 5, 6, 7$, so it suggests that it might be $a(P_n) = n$, for all $n \in \mathbb{N} \backslash \{4\}$.

Proposition 2.2.11. *Upper bound for the achievement number of a star on n vertices is $a(S_n) \leq 2n - 3$, for all $n \geq 3$.*

We have already seen that $a(F) \leq R(F)$ is always true, and it is interesting to see for which graphs F is true that $a(F) < R(F)$. Knowing Ramsey numbers for a path and a star on n vertices which are $R(P_n) = n + \lfloor \frac{n}{2} \rfloor - 1$ and $R(S_n) = 2(n-1) - \epsilon$, where $\epsilon = 1$ or $\epsilon = 0$ for n even or odd, respectively (see [95]), and the results of Proposition 2.2.10 and Proposition 2.2.11, we conclude that $a(P_n) < R(P_n)$ is true for all n, and $a(S_n) < R(S_n)$ is true when n is odd.

Proposition 2.2.12. *Achievement number for a cycle on n vertices is $a(C_n) = n$, for n sufficiently large.*

Comparing this result with the Ramsey number of the same graph, which is $R(C_n) = n + \frac{n}{2} - 1$ for n even and bigger than 4 or $R(C_n) = 2n - 1$ for n odd and bigger than 3 (see [95]), we conclude that $a(C_n) < R(C_n)$.

Proposition 2.2.13. *Achievement number for the perfect matching M_{2n} on $2n$ vertices is $a(M_{2n}) = 2n$, for n sufficiently large.*

Note that we gave here the exact value for paths, cycles, and perfect matchings, whereas for stars we gave an upper bound. An obvious lower bound for $a(S_n)$ is n, as playing on n vertices Blue can touch all vertices (making a perfect matching) before Red can play $n-1$ moves.

2.2.4 Achievement number for trees

In this segment we are interested in finding upper and lower bounds for the achievement number of a tree on n vertices T_n. In Conjecture 2.2.2 we wonder if for a fixed tree on n vertices T_n, the value $a(T_n)$ is somewhere between $a(P_n)$ and $a(S_n)$. Using Propositions 2.2.10 and 2.2.11 we know that it is somewhere between n and a number that is at most $2n-3$. For a rooted tree T_n we denote by $\underline{d}(u)$ a down degree of $u \in V(T_n)$.
The following theorem shows that the upper bound for the achievement number of any tree on n vertices is the same as for a star on n vertices.

Theorem 2.2.14. *Let T_n be a fixed tree on n vertices and v_0' be a vertex of maximum degree in T_n. If T_n is rooted at v_0', then the following holds*

$$a(T_n) \leq max\{n + 4\sqrt{n}, \sum_{\substack{u \in V(T_n) \\ \underline{d}(u) \neq 0}} (2\underline{d}(u) - 1)\},$$

for n sufficiently large.

Theorem 2.2.14 gives an upper bound for $a(T_n)$ for every T_n, we believe that for some subclasses of T_n the actual value is much closer to n.
The next proposition gives the achievement number for a special class of trees, the bounded degree trees which admit a long bare path whose one endpoint is at the same time a leaf of the tree. A path of a tree T is called *bare* if all of its interior vertices are of degree two in T.

Proposition 2.2.15. *Let Δ be a positive integer. There exists an integer $m = m(\Delta)$ and an integer $n_0 = n_0(\Delta, m)$ such that, for every tree T_n with $n \geq n_0$ and $\Delta(T_n) \leq \Delta$, the following holds. If T_n admits a bare path of length m, and one of its endpoints is a leaf of T_n, then $a(T_n) = n$.*

Proofs for the results given in this section together with the preliminaries can be found in Chapter 4.

2.3 Generalized saturation game

Let H and F be graphs that are given in advance. Two players, Max and Mini, alternately claim unclaimed edges of K_n such that the graph consisting of the claimed edges G does not contain a copy of F. The game ends when the players cannot claim further edges. Sometimes, instead of just one graph, we will have a family of graphs, where all graphs that belong to that family are forbidden.

The *score* of the game is the number of copies of H in G at the end of the game. Max wants to maximize the score, while Mini tries to keep the score as low as possible.

We denote by $s_1(n, H, F)$ the score of the game when both players play optimally and Max starts, and by $s_2(n, H, F)$ when Mini starts. If $s_1 = s_2$ we use s instead of $s = s_1 = s_2$. We sometimes use s instead of $s(n, H, F)$ when it is clear which game we are talking about. When we have that score in both cases is between a and b, we use the notation $a < s < b$ instead of $a < s_1 < b$ and $a < s_2 < b$.

2.3.1 Either P_5 or all cycles are forbidden

We prove that the score of the game where the forbidden graph is a path on 5 vertices P_5 and at the end of the game we count the number of triangles, while both players play optimally, is bounded as follows.

Theorem 2.3.1. $\dfrac{n-4}{3} \leq s(n, K_3, P_5) \leq \dfrac{n-4}{3} + 4.$

Then, we are interested in finding the score of two different games where the forbidden graph F is a cycle. First, we count the number of stars on k vertices at the end of the game.

Theorem 2.3.2. $\dbinom{\lfloor \frac{n}{2} \rfloor}{k-1} \leq s(n, S_k, Cycle) \leq \dbinom{\lceil \frac{n}{2} \rceil}{k-1}$, $k > 3$.

Then, we count the number of P_4's at the end of the game.

Theorem 2.3.3. $\frac{n^2}{16} + O(n) \leq s(n, P_4, Cycle) \leq \frac{n^2}{16} + O(n)$.

In the following we observe generalized saturation games where the forbidden graphs are the same as in [109], and for each of those forbidden graphs we count the number of the following graphs on k vertices P_k, S_k and C_k.

2.3.2 P_4 is forbidden

First, we observe the game where we count the number of paths. Clearly, the only path that can occur is P_3.

Theorem 2.3.4. $n - 4 \leq s(n, P_3, P_4) \leq n$.

Now we observe the game where we count the number of stars with the same forbidden graph. As $S_3 = P_3$, that case is already covered by Theorem 2.3.4. It turns out that we can derive the score for all larger stars by using the same strategy used to prove that result.

Observation 2.3.5. $s(n, S_k, P_4) = 0$, $k \geq 4$.

It remains to look at the game where we count the number of cycles. It is clear that the only cycle that can exist is a C_3. Hence, $s(n, C_k, P_4) = 0$, when $k > 3$.

Theorem 2.3.6. *$s(n, C_3, P_4) = 1$ if n is odd, and $s(n, C_3, P_4) = 0$ if n is even.*

2.3.3 All odd cycles are forbidden

In the following two observations, we look at the generalized saturation games where the forbidden graphs are all odd cycles $\mathcal{O}$. In other words, in these games, the game graph remains bipartite.

Observation 2.3.7.

$$s(2n, P_{2k}, \mathcal{O}) = \left(\frac{n!}{(n-k)!}\right)^2 \quad \text{and} \quad s(2n, C_{2k}, \mathcal{O}) = \frac{1}{k}\left(\frac{n!}{(n-k)!}\right)^2.$$

Then, we count the number of S_k in the bipartite graph on $2n$ vertices.

Observation 2.3.8. $s(2n, S_k, \mathcal{O}) = 2n\binom{n}{k-1}$.

2.3.4 T_n is forbidden

In the following observation, we look at generalized saturation games where the forbidden graphs are all spanning trees $\mathcal{T}_n$. Note that at the end of the game, the graph must be a disjoint union of two complete graphs K_r and K_{n-r}.

Observation 2.3.9. $s(n, P_k, \mathcal{T}_n) = \frac{k!}{2}\binom{n-2}{k}$,
$s(n, C_k, \mathcal{T}_n) = \frac{(k-1)!}{2}\binom{n-2}{k}$ *and* $s(n, S_k, \mathcal{T}_n) = (n-2)\binom{n-3}{k-1}$.

2.3.5 S_4 is forbidden

In the following we observe generalized saturation games where the forbidden graph is a star S_4.

First, we count the number of paths at the end of the game when both players play optimally. As counting P_2's is the same as counting edges, that has already been done in Theorem 5.1.3. Note that here the score of the game is bounded from above by n.

Theorem 2.3.10. *$n-1 \leq s(n, P_3, S_4) \leq n$ for $n \geq 3$. Moreover, $s_1(n, P_3, S_4) = n$ if n is even and $s_2(n, P_3, S_4) = n$ if n is odd.*

In the following theorem, we count the number of P_4's at the end of the game when both players play optimally.

Theorem 2.3.11. *$n-3 \leq s(n, P_4, S_4) \leq n$ for $n \geq 4$.*

Theorem 2.3.12. *$s(n, P_5, S_4) \leq 6$. Additionally, $s_2(n, P_5, S_4) \geq 5$ for $n = 4k$ and $k \geq 2$ or $n = 4k+1$.*

Theorem 2.3.13. *$s(n, P_k, S_4) = 0$ where $k \geq 6$.*

Then we want to see what is the number of stars at the end of the game when both players play optimally. As S_4 is the forbidden graph, the only star that can appear in the graph is $S_3 = P_3$, but we have already proven that in Theorem 2.3.10.

Finally, we want to count the number of cycles at the end of the game.

Theorem 2.3.14. *$s(n, C_k, S_4) \leq 1$, when $3 \leq k \leq 5$ and $s(n, C_k, S_4) = 0$, when $k \geq 6$.*

Proofs for the results given in this section together with the preliminaries can be found in Chapter 5.

Chapter 3

Strong Avoider-Avoider games

The aim of this chapter is to give a solid base understanding the assertions related to the strong Avoider-Avoider games, and then to present the proofs of the theorems given in Section 2.1.

In Section 3.1 we give notation and preliminaries. In Section 3.2 we give the proof of Theorem 2.1.1. In Section 3.3 we prove Theorem 2.1.2. Then, in Section 3.4 we prove Theorem 2.1.3, Theorem 2.1.4, Theorem 2.1.5. In Section 3.5 we prove Theorem 2.1.6. Finally in Section 3.6 we give concluding remarks and open problems.

3.1 Preliminaries

During a game, we say that the vertices that are touched by Red are *red vertices*, the ones touched by Blue are *blue vertices*, and the others, that are not touched by any of the players, are *black vertices*. If a vertex is touched just by Red and not by Blue, we call it a *pure red vertex*, and if the situation is opposite it is a *pure blue vertex*.

By a player's graph we consider the graph with all edges he claimed on the vertex set $V = [n]$.

A *star* is the complete bipartite graph $K_{1,k}$, where $k \geq 0$. We denote a star on n vertices by S_n, and by P_n a path on n vertices. We will refer to the star centered in v as a v-*star*. When we say that a player *star-adds* a vertex x to a v-star, this means that he claims the edge vx. An H-free graph is a graph that does not contain a copy of H.

We will use the abbreviation RC for non-trivial red components, i.e. connected components in Red's graph, where we do not count isolated vertices as RC. We will say that a connected component is *pure red* (respectively, *pure blue*) if all its vertices are pure red (respectively, pure blue).

We will make use of the following facts about the P_4-free graphs.

Observation 3.1.1. *For every graph that does not contain a P_4 as a subgraph, its connected components can be stars and triangles (where isolated edges and vertices are observed as stars).*

Observation 3.1.2. *A P_4-free graph on n vertices with the maximum number of edges is a disjoint union of triangles, when $n = 3k$, for some integer k, and otherwise a disjoint union of one star and a number (possibly zero) of triangles. The number of edges in that graph is n, if $n = 3k$, and $n - 1$ otherwise.*

Observation 3.1.3. *If in a maximal P_4-free graph there are k disjoint stars, then it has $n - k$ edges.*

We also need the following facts about graphs that do not have connected components on more than three vertices.

Observation 3.1.4. *For every graph that does not contain a $\mathcal{CC}_{>3}$ as a subgraph, its connected component can be a triangle, a path on three vertices, an isolated edge, or an isolated vertex.*

Observation 3.1.5. *A $\mathcal{CC}_{>3}$-free graph with the maximum number of edges is a disjoint union of triangles, when $n = 3k$, a disjoint union of triangles and one isolated vertex, when $n = 3k + 1$, or a disjoint union of triangles and one isolated edge, when $n = 3k + 2$, for some integer k.*
The number of edges in that graph is n, if $n = 3k$, and $n - 1$ otherwise.

3.2 Strong Avoider-Avoider P_4 game

Proof of Theorem 2.1.1: We will describe a winning strategy for Blue. Note that by definition of a RC and by Observation 3.1.1, Red is not allowed to claim any edge between two RC at any point of the game, as otherwise he would create a P_4 in his graph.

In the beginning, we have a graph G with n isolated vertices, and Red claims an edge, let us denote it by rt. Then Blue claims an edge that is not adjacent to the red one, we denote it by uv. In the following move Red has four options, up to isomorphism, for choosing an edge, and those four moves will make our four cases. For each of these cases we will show that Blue can win. Let us denote the second move of Red by $e = xy$.

In the first three cases we use the idea of strategy stealing: we will suppose that at this point of the game (after Red played two moves and Blue played one) Red has a strategy to finish the game and win. Then we will show how Blue can use this strategy to win the game. That will lead to a contradiction, implying that our assumption was wrong and Blue can win the game.

Case 1. Vertex x is red and y is black.

Suppose that Red has a strategy S to win the game. W.l.o.g. let $x = t$. After Red plays ty it is Blue's turn. The graph of the game consists of two adjacent red edges and one isolated blue edge. We denote the vertices as depicted in Figure 3.1a. Before his next move, Blue imagines that he has already claimed the edge yu and that Red has not claimed the edge ty, see Figure 3.1b. Note that the edge yu will remain free throughout the game, as otherwise Red would create a P_4 in his graph.

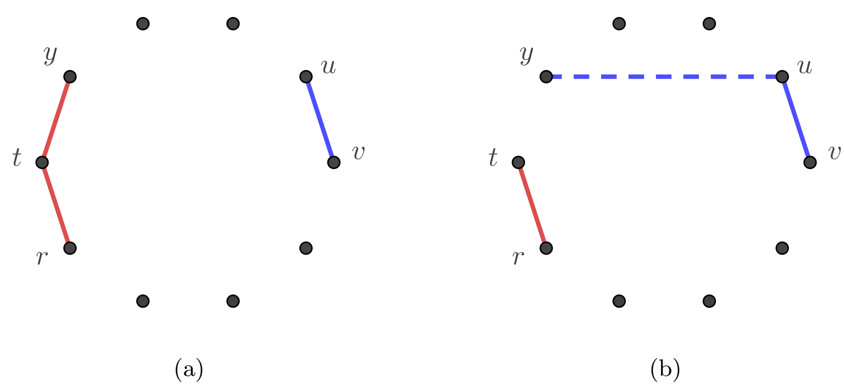
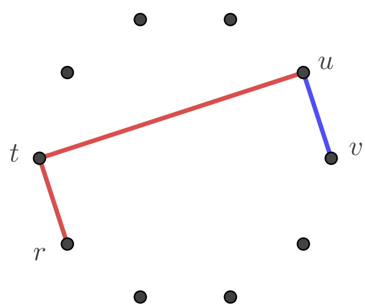


Figure 3.1. Case 1: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

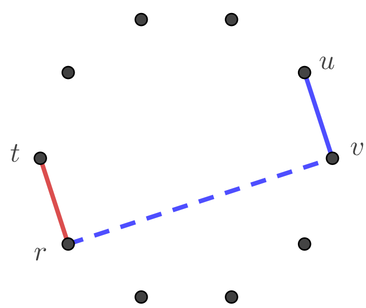
The imagined graph is isomorphic to the graph, where the roles of the players are swapped. Blue imagines that he is the first player, he further imagines that Red claims the edge ty as his second move, and from now on responds as advised by the winning strategy S. Because this is a winning strategy, Blue wins the game, a contradiction.

Case 2. Vertex x is red and y is blue.

Similar to Case 1, we suppose that Red has a strategy S to win the game. W.l.o.g. let $x = t$ and $y = u$. After Red plays tu it is Blue's turn. The graph of the game consists of one P_4 with two adjacent red edges and one blue edge, see Figure 3.2a. Before his next move, Blue imagines that he has already claimed the edge rv and that Red has not claimed the edge tu, see Figure 3.2b. Note that the edge rv will remain free throughout the game, as otherwise Red would create a P_4 in his graph.



(a)



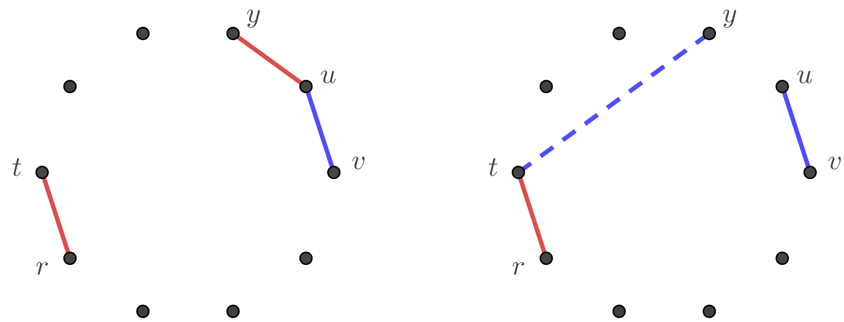
(b)

Figure 3.2. Case 2: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

The imagined graph is isomorphic to the graph, where the roles of the players swapped. Blue imagines that he is the first player and that Red claims the edge tu as his second move, and from now on Blue responds as advised by S winning the game, a contradiction.

Case 3. Vertex x is blue and y is black.

We again suppose that Red has a strategy S to win the game. W.l.o.g. let $x = u$. After Red plays uy the graph of the game consists of one isolated red edge and one P_3 with two edges of different colours, see Figure 3.3a. Before his next move, Blue imagines that he has already claimed the edge ty and that Red has not claimed uy, see Figure 3.3b. Note that the edge ty will remain free throughout the game.



(a) (b)

Figure 3.3. Case 3: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

The imagined graph is isomorphic to the graph, where the roles of the players are swapped. Blue imagines that he is the first player and that Red claims the edge uy as his second move. From now on, Blue responds as advised by S thus winning the game, a contradiction.

Case 4. Both vertices x and y are black.

The Red's graph at this moment has two isolated edges that make the first two RC. Let us denote by C_1 the component $\{r, t\}$, and by C_2 the component $\{x, y\}$. For the reminder of the game, we will dynamically update C_1 and C_2 as they grow. Note that C_1 will remain a different RC from C_2.

Blue is the second player, so his graph cannot have more edges than the Red's graph. Having that in mind, as well as Observation 3.1.3, we will describe a strategy for Blue to keep the number of stars in his graph less then or equal to the same number in the Red's graph throughout the game.

After Red claims the edge xy, Blue responds by claiming the edge vr, as depicted in Figure 3.4a. Note that at this moment the Blue's graph consists of one v-star. In the rest of the game Blue will enlarge this v-star, and possibly create isolated triangles.

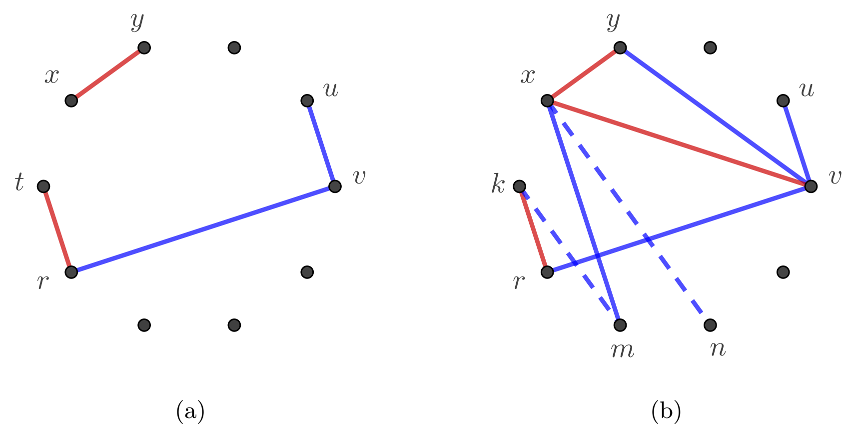


Figure 3.4. Case 4: (a) the graph after the second move of Blue. (b) The possible moves of Blue if the rule 1 of Stage 1 is in order, shown as dashed lines.

During the game, every vertex k that is not blue, for which it applies that kv is free and adding the edge kv to the Red's graph will not make a P_4 will be called a *dangerous vertex*. All the other vertices will be called *safe*. A pure red vertex j that is adjacent to the vertex v in Red's graph will be called *inaccessible*.

If in his third move Red claims the edge yv, Blue responds by claiming the edge xv, otherwise he claims the edge yv. W.l.o.g. we will suppose that Blue has claimed the edge yv in his third move.

Note that at this point there are only two vertices in $C_1 \cup C_2$, namely x and t, that can be dangerous. Let $S_1 := \{x, t\}$. During the game, whenever a vertex from S_1 becomes blue we remove it from S_1.

Now we give a strategy for Blue that he follows from his fourth move on.

Stage 1. While there are at least two black vertices in the game, Blue repeatedly plays by the first rule in this list that is applicable.

1. If Red has claimed one of the edges vx or vt;
   in his following three moves, Blue will claim edges that close a triangle incident with the vertex that just become inaccessible, w.l.o.g. let it be the vertex x. By m and n we denote two arbitrary black vertices and by k the other vertex from S_1. Note that the only case when k is not vertex t is when rule 4 has been played before. Blue starts by

claiming the edge xm. Then, if it is unclaimed he claims the edge km, otherwise the edge xn, and in the following move Blue closes either the triangle xmk or xmn, see Figure 3.4b. Then, he star-adds all the remaining vertices of the base graph that are not blue to the v-star.

2. If Red claims an edge creating a RC that is a star on three vertices with the vertex v as a leaf, and if the edge incident with v and the other leaf is unclaimed;
   Blue claims it. We denote by r' the inaccessible vertex, the center of the red star. In his following three moves Blue claims the edges of the triangle $r'xt$. Then, he star-adds all the remaining vertices of the base graph that are not blue to the v-star.

3. If Red claims an edge creating a RC that is a v-star on three vertices, and if the edge incident with both leaves is unclaimed;
   Blue claims that edge. That isolated blue edge we call a *cover-edge.*

4. If there is exactly two black vertices, and there is no pure red vertex that is not in $C_1 \cup C_2$, and there is no cover-edge, and one of $\{C_1, C_2\}$ is an isolated edge while the other one is a star with at least three edges that does not have v as a leaf;
   then Blue claims the edge incident with the center of that star and v. Now, we remove the blue vertex from S_1 and add a safe pure red vertex from the same RC to S_1.

5. Otherwise;
   Blue claims an edge incident with v and one black vertex.

Now, we will prove that if it is Blue's turn to play Stage 1, he can follow it. First, note that if in his third move Red claimed the edge yv, the game would be finished in step 1 of Stage 1.

If Red did not claim yv in his third move, in the beginning of Stage 1 Blue's graph consists of the v-star and isolated vertices, and he will continue claiming the edges of the v-star using rules 4 and 5, until a condition of one of the rules 1, 2, 3 is fulfilled. Note that Blue can use exactly one of the rules 1, 2, 3 at most once until the end of the game, so when it is Blue's turn to play one of them, his graph consists of the v-star and isolated vertices. Clearly, Blue can always claim an edge between v and a black vertex.

When Blue is to play by rule 1, we know that none of rules 1, 2 and 3 have been used before. Therefore Blue's graph consists of the v-star and isolated vertices. Also, there are two black vertices and Red has made one inaccessible vertex x. The edge vx will be the only red edge incident with v until the end of game, because vertex v is a leaf of a red star and claiming another edge incident with v would make a P_4 in Red's graph. So, all the remaining vertices that are not blue are safe. It is clear that k is pure red and not in the same RC as x. Hence, Blue can follow rule 1 and play up to $n-1$ moves, thus winning.

When it is Blue's turn to play by rule 2, first unclaimed edge advised by the strategy must be available for him because his graph has the v-star and every vertex not adjacent to v is isolated in Blue's graph. In his following three moves Blue can claim the edges between r', x and t, because each of them is in a different RC. Note that x and t are pure red because rule 4 could not have happened before and Blue could use only rule 5. For the same reason as above, all the remaining vertices that are not blue are safe. Now, it is clear that Blue can follow rule 2 and play up to $n-1$ moves, thus winning.

When it is Blue's turn to play by rule 3 none of rules 1, 2 and 3 have happened before, so every vertex not adjacent to v is isolated in Blue's graph and he can claim the cover-edge as advised by the strategy. Note that Red cannot ever claim any edge adjacent to the cover-edge.

For further analysis we need to verify the following claim.

Claim 3.2.1. *From the moment in the game when there is no more than two black vertices, until the first Blue's move after Stage 1, if there is no pure red vertex that is not in $C_1 \cup C_2$ and there is no cover-edge, Red's graph has at least four edges and one safe pure red vertex in $C_1 \cup C_2$, and at least one pure red vertex in each of these components.*

Proof. Before Blue had played his fourth move, his graph consisted of the v-star on 4 vertices, where v was pure blue. At that moment Red's graph had four edges and all of them had to be in $C_1 \cup C_2$, otherwise there would be at least one pure red vertex in the third RC, which we assumed was not the case. Therefore there are three options for C_1 and C_2:

- An isolated edge and a triangle.
  In this case there were at least two pure red vertices, one in C_1 and

the other one in C_2, where one of them was incident with a triangle, so it must have been safe.

- An isolated edge and a star on four vertices.
  In this case there were at least three pure red vertices, one of which was incident with the isolated edge, and all the others with the star. Therefore, there were at least one safe pure red vertex as a leaf of the red star.

- Both of them are a P_3.
  In this case there were at least three pure red vertices, at most two of them were dangerous, so there must have been one safe. At least one vertex in each component was pure red.

If the assumption of the claim holds, the only rules that Blue could have applied in the meantime are rules 4 and 5. The last one does not have any influence on pure red vertices, and rule 4 can just swap one pure red dangerous vertex with a pure red safe vertex in the same component. Therefore, the assertion of the claim is proven. □

When Blue is to play by rule 4, it is clear that he can claim that edge. Note that if that edge is not free it has to be blue, otherwise rule 1 would be achieved. Using Claim 3.2.1 we know that a pure red vertex incident with the star exists.

Note that during Stage 1, if Blue has not already won (rule 1 and 2), his graph consists of the v-star (rule 4 and 5), possibly one isolated cover-edge (rule 3) and isolated vertices. Also, S_1 consists of two pure red vertices where one belongs to C_1, and the other one to C_2.

When Stage 1 is finished, there is at most one black vertex. We then move on to Stage 2, distinguishing two cases.

If there is at least one pure red vertex that is not in $C_1 \cup C_2$ or there is a cover-edge, we proceed to Stage 2a.

Stage 2a. Before Blue plays his first move in Stage 2a, we add all inaccessible vertices and the ends of the cover-edge to S_1. If a pure red vertex that is not in $C_1 \cup C_2$ exists, we denote it by w. If $|S_1| < 3$ (there was not an inaccessible vertex nor a cover-edge) then we add w to S_1.

Blue repeatedly plays by the first rule that is applicable in this list and if before the move of Blue there is a new inaccessible vertex, we add it to S_1.

1. If the conditions of rule 2 or rule 3 from Stage 1 are fulfilled,
   Blue claims the next edge in the same way as that rule suggests.

2. If there is a black vertex,
   Blue star-adds it to the v-star.

3. If there is a cover-edge,
   then if $|S_1|$ is not divisible by three, we will make it by removing one or two vertices from $C_1 \cup C_2$. In his following two moves Blue claims a triangle using the cover-edge and one more vertex from S_1. Then, until S_1 is not empty, he chooses three vertices from S_1 and makes a triangle claiming all edges between them. At the end he star-adds all the remaining vertices of the graph that are not blue to the v-star.

4. If there is an inaccessible vertex r',
   then if $|S_1| = 4$ we remove w from S_1. In his following three moves Blue makes the triangle claiming edges between vertices from S_1. Then he star-adds all the remaining vertices that are not blue to the v-star.

5. If there is an unclaimed edge incident with v and one pure red dangerous vertex that is not in S_1,
   Blue claims it.

6. Otherwise,
   in his following three moves Blue makes the triangle claiming edges between the remaining three vertices from S_1. Then he star-adds all the remaining vertices that are not blue to the v-star.

Otherwise (there is neither a pure red vertex that is not in $C_1 \cup C_2$ nor a cover-edge), we proceed to Stage 2b.

Stage 2b. If there is one black vertex, we denote it by j. Depending on the types of the components C_1 and C_2, we have three conditions and Blue chooses the first one which is satisfied.

1. At least one of C_1 and C_2 is a star with more than two edges, and it is disjoint from v.

Let us denote by c the center of that star. If it is unclaimed, Blue claims the edge vc. Then, he star-adds all the remaining vertices that he can to the v-star.

2. Each of C_1 and C_2 is a star with at least two edges.

   (a) If v is red,
   we denote by c the center of the star incident with v, and by k a pure red vertex that is not in the same RC as c. We know that these vertices exists by Claim 3.2.1. Blue claims the edge cj, after that if it is free he claims the edge kj, and then the edge kc. Then he star-adds all the remaining vertices that are not blue to the v-star.

   (b) Otherwise, v is blue,
   we denote by w a safe pure red vertex and by c the center of the star of the same RC, and with k a pure red vertex from the other RC. We know that these vertices exists by Claim 3.2.1.
   If it is unclaimed, Blue claims the edge cv. Then, if kj is unclaimed Blue claims it. In his following three moves, he claims the triangle kjw and star-adds all the remaining vertices that are not blue to the v-star, if any. Otherwise, if kj is not unclaimed, he claims the edge kv and star-adds all the remaining vertices that are not blue to the v-star.

3. At least one of C_1 and C_2 is a triangle.
   We denote by k a pure red vertex incident with the triangle, and with c a pure red vertex from the other component, where if there are more than one such vertex the dangerous one has an advantage. We know that these vertices exists by Claim 3.2.1.

   Blue claims the edge cj, if it is unclaimed, and then creates the triangle cjk, otherwise he claims the edge cv. Then he star-adds all the remaining vertices that are not blue to the v-star.

Now let us first show that when it is Blue's turn to play Stage 2a, he can follow it and win. Note that there are no red edges between any two vertices of S_1 because they are in two different RC or they are leaves of the same red star, and all vertices in S_1 are pure red. Also, when it is Blue's turn to play rules 3-6 there are no more black vertices.

When it is Blue's turn to play rule 1, rules 1, 3, 4 and 6 could not have been activated before, so his graph consists of the v-star and isolated vertices. For the same reason as in rules 2 and 3 from Stage 1 he can claim his next edge.

When it is Blue's turn to play rule 2, he can obviously follow it.

When it is Blue's turn to play by rule 3, there are no more black vertices, so red star centered in v cannot spread any more as all pure blue vertices are in the v-star, so all the remaining vertices that are not blue have to be safe. Now it is evident that Blue can follow his strategy as described in rule 3. Here, Blue wins by playing $n-1$ edges.

When it is Blue's turn to play by rule 4, we know that v is a leaf of a red star. At this moment Blue's graph consists of the v star and isolated vertices, and all the remaining vertices that are not blue have to be safe. Now it is clear that Blue can follow his strategy and win by playing $n-1$ edges.

It is obvious that if it is Blue's turn to play by rule 5, he can claim as advised due to the definition of a dangerous vertex. Note that here v is blue.

If nothing above mentioned happened, v is still blue and the Blue's graph consists of the v star and isolated vertices. S_1 consists of three vertices, where each of them is in a different RC. Obviously, Blue can make the triangle described in 6, and because these were the last dangerous vertices, he can star-add all of the remaining vertices to the v-star and win with $n-1$ edges. Note that it is not possible that Red claims a triangle incident with v in this step because v is blue, and the vertices from S_1 cannot be adjacent in Red's graph.

Taking into account that Red's graph cannot have a triangle incident with v (considering rule 1 and the above mentioned), Red's graph cannot have more than $n-1$ edges, so Blue wins the game.

It remains to show that when it is Blue's turn to play Stage 2b, he can follow it and win. Note that all pure red vertices are in $C_1 \cup C_2$, so there are at most three dangerous vertices and each of them has to belong to the set $\{x, t, j\}$. Likewise, all the vertices that are not in $C_1 \cup C_2$ are blue, except j which is black (if it exists). Each of the blue vertices is a leaf of the v-star, therefore it is not possible that Red claims a triangle incident with v.

When it is Blue's turn to play by rule 1, it is clear that Blue can claim the edge vc if it is free, and then Blue can star-add to the v-star all the

remaining vertices but possibly one. In that case Red has at least two stars (one c-star and the other one incident with v) in his graph and he cannot have more than $n-2$ edges, by Observation 3.1.3, so Blue wins with $n-2$ edges.
Otherwise, if vc is not free it has to be blue (condition of this step), so Blue just skips this move and wins in the same way as argued above.

When it is Blue's turn to play by rule 2:

- If v is red, that happened in the last move, otherwise the game would have be finished in Stage 1, so there has to exist j, and Blue can claim cj. After that all the vertices that are not blue are safe.
  Then, if the edge kj has been claimed Red will have at least two stars at the end of game, so he can have at most $n-2$ edges, by Observation 3.1.3. Therefore, Blue can follow rule 2 to the end and win with $n-2$ edges.
  Else, if kj is unclaimed, Blue claims the triangle cjk, and wins with $n-1$ edges.

- If v is blue, each of C_1 and C_2 is a P_3, as otherwise it would be rule 1. There has to exist j, otherwise if Red took it in his last move, before that move one of C_1, C_2 was an isolated edge, and the other one P_3, and that is not possible because of Claim 3.2.1.
  If the edge cv has been already claimed, it has to be blue. It is clear that in his following move he can claim one of the edges kj or kv, and then all the remaining pure red vertices are safe, so he can follow his strategy until the end of the game and win with $n-1$ edges.

When it is Blue's turn to play by rule 3, we know that his previous move was in Stage 1, so after that move there was at least one black vertex and v was blue. Red could not claim both of the edges cj and cv in his following move, so one of them is unclaimed and Blue can claim it. All the remaining pure red vertices are safe, so he can follow his strategy until the end of the game and win with $n-1$ edges.

Note that Claim 3.2.1 guarantee that all cases are covered by Stage 2b except the case when one component is an isolated edge and the other one a star with at least four edges that are incident with v. That cannot happen because in his previous move Blue played in Stage 1 and the conditions of rule 4 had to be fulfilled, but then Blue would make the center of that star adjacent to v.

This concludes the proof for Case 4, and also the proof of the theorem. $\square$

3.3 Strong Avoider-Avoider $\mathcal{CC}_{>3}$ game

Proof of Theorem 2.1.2: We describe a winning strategy for Blue. In the beginning, we have a graph G with n isolated vertices, and Red claims an edge, let us denote it by uv. Then Blue claims an edge that is adjacent to the red one, let us denote it by vi.

In the following move Red has five options, up to isomorphism, for choosing an edge, and those five moves will make our five cases. For each of these cases we will show that Blue can win, in the first four cases we will use the idea of Strategy stealing, and in Case 5 we will design an explicit strategy. Let us denote the second move of Red by $e = xy$.

Case 1. Vertex x is pure red and vertex y is black, i.e. $x = u$.

Suppose that Red has a strategy S to win the game. After Red plays uy it is Blue's turn. The graph of the game has one P_4 with two adjacent red edges and one blue edge. We denote the vertices as depicted in Figure 3.5a.

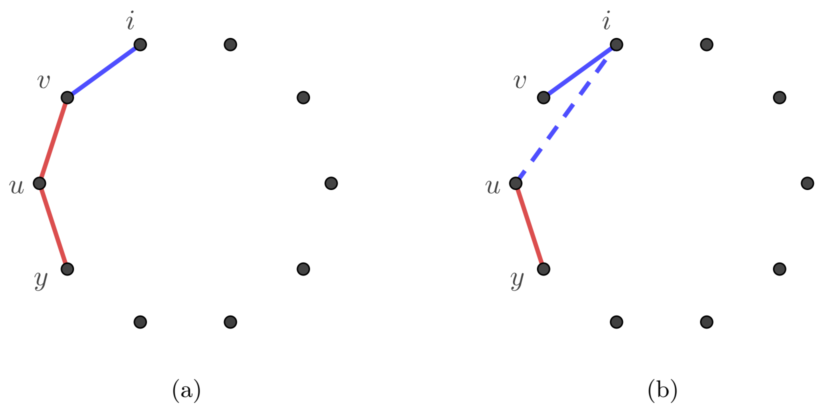


Figure 3.5. Case 1: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

Before his next move, Blue imagines that he has already claimed the edge ui, and that Red has not claimed the edge uv, see Figure 3.5b. Note that the edge ui will remain free throughout the game, as otherwise Red would create a $\mathcal{CC}_{>3}$ in his graph.

The imagined graph is isomorphic to the graph where the roles of the players are swapped. Blue imagines that he is the first player, he further imagines that Red claims the edge uv as his second move, and from now on responds as advised by the winning strategy S and wins the game, a contradiction.

Case 2. Vertex x is both red and blue and vertex y is black, i.e. $x = v$.

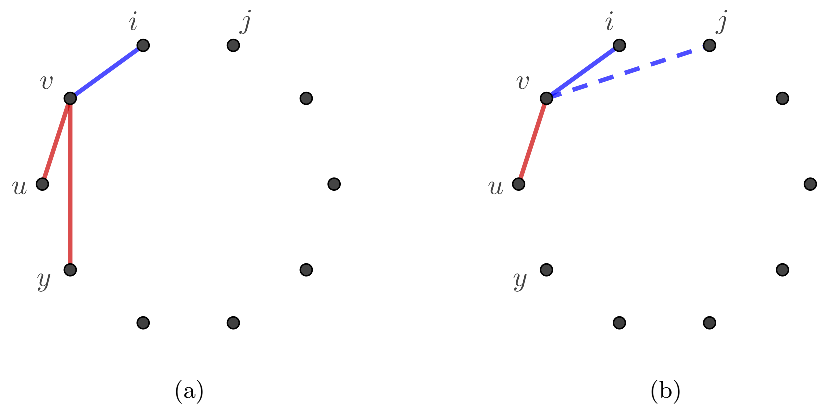


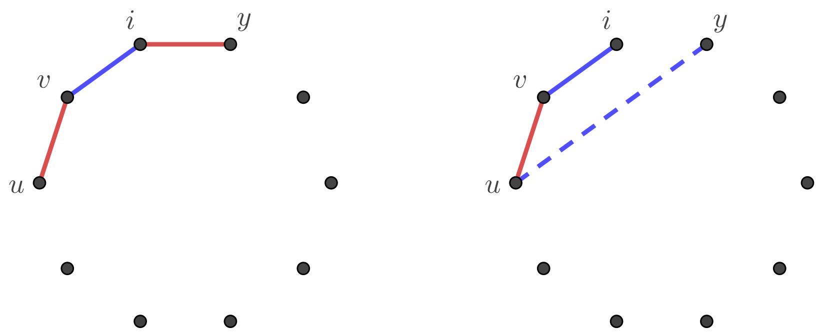
Figure 3.6. Case 2: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

Suppose that Red has a strategy S to win the game. After Red plays vy, the graph of the game has one star on three edges where two of them are red and one is blue. We denote the vertices as depicted in Figure 3.6a. Before his next move, Blue imagines that he has already claimed the edge vj, where j is a black vertex, and that Red has not claimed the edge vy, see Figure 3.6b. Note that the edge vj will remain free throughout the game, as otherwise Red would create a $\mathcal{CC}_{>3}$ in his graph.

The imagined graph is isomorphic to the graph where the roles of the players are swapped. Blue imagines that he is the first player, and that Red claims the edge vy as his second move. From now on Blue responds as advised by the winning strategy S and wins the game, a contradiction.

Case 3. Vertex x is pure blue and vertex y is black, i.e. $x = i$.

Suppose that Red has a strategy to win the game. After Red plays iy, the graph of the game has one P_4 which two non-adjacent edges are red, and the third one is blue. We denote the vertices as depicted in Figure 3.7a. Before his next move, Blue imagines that he has already claimed the edge uy, and that Red has not claimed the edge iy, see Figure 3.7b. Note that the edge uy will remain free throughout the game, as otherwise Red would create a $\mathcal{CC}_{>3}$ in his graph.



(a) (b)

Figure 3.7. Case 3: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

The imagined graph is isomorphic to the graph where the roles of the players are swapped. Blue imagines that he is the first player, and that Red claims the edge iy as his second move. Hereafter, Blue responds as advised by the winning strategy S and wins the game, a contradiction.

Case 4. Both x and y are black.

Suppose that Red has a strategy S to win the game. After Red plays xy, the graph of the game has one P_3, where one edge is red and the other one blue, and one isolated red edge. We denote the vertices as depicted in Figure 3.8a. Before his next move, Blue imagines that he has already claimed the edge ux, and that Red has not claimed the edge uv, see Figure 3.8b. Note that the edge ux will remain free throughout the game, as otherwise Red would create a $\mathcal{CC}_{>3}$ in his graph.

The imagined graph is isomorphic to the graph where the roles of the players

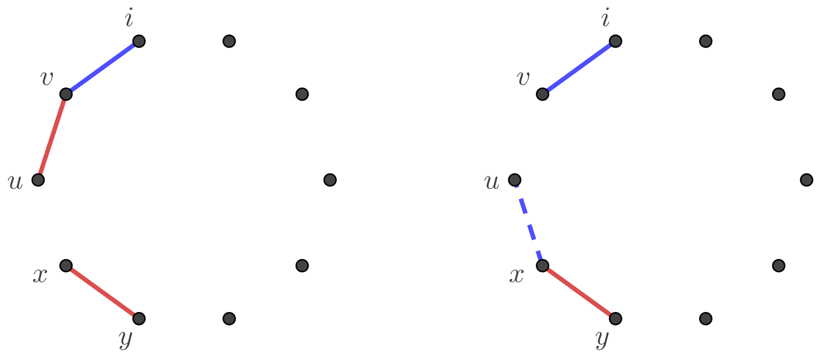


Figure 3.8. Case 4: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

are swapped. Blue imagines that he is the first player, and that Red claims the edge uv as his second move. Hereafter, Blue responds as advised by the winning strategy S and wins the game, a contradiction.

Case 5. Vertex x is pure red and vertex y is pure blue, i.e. $x = u$ and $y = i$.

Blue is the second player, so his graph can never have more edges than the Red's graph. Having that in mind, as well as Observation 3.1.5, we will describe an explicit strategy for Blue to claim disjoint triangles until the very end, which will enable him to win. Note that in this case the first RC is a P_3, so using Observation 3.1.5, it is not possible for Red to have more than $n-1$ edges if $n = 3m$, or $n-2$ otherwise.

Let us introduce some terminology. During the game, whenever a blue edge is added to a pure red P_3 so that it completes a triangle, we call it a *nice edge*. Furthermore we call the pure red vertex that is incident with that triangle a *nice vertex*.

First we will give a strategy for Blue that he follows from his second move on, and afterwards we will show that Blue can follow it.

Stage 1. While there is at least one black vertex in the game, Blue repeatedly plays by the first rule in this list that is applicable. Let k denote the number of nice edges in Blue's graph.

1. If there is a pure red P_3,
   Blue claims the isolated edge that completes a triangle when added to the pure red P_3. In other words, Blue claims a nice edge.

2. If $k = 0$,

   (a) Pure red vertices are in at least three different RC.
   Blue chooses two pure red vertices from two different RC, say x and y and claims an edge between them. We denote by X_3 the third RC. Then, Blue repeatedly plays by the first rule in the following list that is applicable until he claims all three edges of a new triangle xyz.

      i. If there is a pure red P_3 anywhere in the graph,
      Blue claims the isolated edge that added to the pure red P_3 completes a triangle, i.e. a nice edge.

      ii. If degree of the vertex x in Blue's graph is one,
      Blue claims the edge xz, where z is a pure red vertex in X_3.

      iii. Otherwise (the degree of the vertex x in Blue's graph is two),
      Blue claims the edge yz, where y and z are endpoints of the blue edges incident to x.

   (b) The pure red vertices are in exactly two RC.
   Denote by X_1 the RC that has two pure red vertices, and the other one with X_2, later we will see that X_1 exists. Blue claims an edge between pure red vertex from X_2 and one black vertex, denote it by w. Then, Blue repeatedly plays by the first rule in the following list that is applicable until he claims all three edges of a new triangle.

      i. If there is a pure red P_3 anywhere in the graph,
      Blue claims the isolated edge that added to the pure red P_3 completes a triangle, i.e. a nice edge.

      ii. If the degree of the vertex w in Blue's graph is one,
      Blue claims an edge between a pure red vertex from X_1 and w.

      iii. Otherwise (the degree of the vertex w in Blue's graph is two),
      Blue claims the edge that completes the triangle.

3. If $k = 1$,
   Blue claims an edge adjacent to the nice edge and incident with a black

vertex. Then, Blue repeatedly plays by the first rule in the following list that is applicable until he claims a new blue triangle.

(a) If there is a pure red P_3 anywhere in the graph,
Blue claims the isolated edge that added to the pure red P_3 completes a triangle, i.e. a nice edge.

(b) Otherwise,
Blue claims the edge that completes the blue triangle.

4. If $k > 1$,
Blue chooses a nice vertex and then a nice edge that are not in the same RC, and then claims an edge between them. Then, Blue repeatedly plays by the first rule in the following list that is applicable until he claims a new blue triangle.

(a) If there is a pure red P_3 anywhere in the graph,
Blue claims the isolated edge that added to the pure red P_3 completes a triangle, i.e. a nice edge.

(b) Otherwise,
Blue claims the edge that completes the blue triangle.

Now we will prove that when it is Blue's turn to play Stage 1, he can follow it.

When it is Blue's turn to play by rule 1, it is clear that he can claim that edge. Since the strategy of Blue is to make disjoint triangles and nice edges, before he plays any of the rules, his graph consists of t disjoint triangles, k nice edges and isolated vertices, where t or k are possibly zero. Obviously, these isolated vertices have to be black or pure red.

Claim 3.3.1. *When it is Blue's turn to play and his graph consists of disjoint triangles and isolated vertices, there are at least three pure red vertices.*

Proof. Blue's graph is a disjoint union of t triangles and isolated vertices, so it has $3t$ blue vertices and $3t$ edges. At this moment Red's graph has $3t+1$ edges. The first RC is a P_3 and the Red's graph without that component has $3t-1$ edges. The extremal graph that is described in Observation 3.1.5 gives the smallest number of vertices for a fixed number of edges. Therefore, a $\mathcal{CC}_{>3}$-free graph with $3t-1$ edges has at least $3t$ vertices. Then the Red's graph has at least $3t + 3$ red vertices three more than the number of blue vertices, so at least three of them must be pure red. $\square$

Claim 3.3.2. *When it is Blue's turn to start playing by rule 2 (his graph consists of disjoint triangles and isolated vertices), there are at most two pure red vertices in any RC.*

Proof. Suppose there is a RC with three pure red vertices.
It cannot be a pure red triangle, otherwise one move before Red made a pure red triangle, he had to have a pure red P_3. Realising that just one pure red P_3 can be made per move, Blue responds by claiming the nice edge following his strategy.
Therefore, the RC has to be a pure red P_3, but than rule 1 would be activated and Blue would claim a nice edge and that leads to a contradiction. $\square$

Note that Red cannot claim any edge adjacent to a nice edge or incident to a nice vertex, also he cannot claim any edge between two RC, otherwise he would create a $\mathcal{CC}_{>3}$. Every RC has to be an isolated edge, a P_3 or a triangle, by Observation 3.1.4.

Now we will show that at the moment when Blue has to play by one of the rules 2, 3 or 4 he can do that, and in particular he claims a new triangle.

1. $k = 0$.
   Note that in this moment there are at least three pure red vertices by Claim 3.3.1. If Blue claimed a nice edge in the middle of rule 2, in RC where Blue has made a nice edge, there is still one pure red vertex, precisely a nice vertex.
   (a) There are pure red vertices in at least three RC.
   It is clear that Blue can claim the edge xy. When it is Blue's turn to claim the edge xz, it is available for him because, as mentioned above, z must exists even if Blue has made a nice edge in the meantime. Clearly, Blue can follow rule 2 to the end and claim the triangle xyz.
   (b) The pure red vertices are in two RC.
   We know that in one RC there can be at most two pure red vertices by Claim 3.3.2. Also, there are at least three of them as proven in Claim 3.3.1, therefore one RC has to have precisely two

pure red vertices, so X_1 exists. Then X_2 has at least one pure red vertex and it is clear that Blue can follow rule 2 to the end and claim a new triangle, by same reasoning as above.

Note that it is not possible that all pure red vertices are in one RC because of Claims 3.3.1 and 3.3.2.

2. $k = 1$. Clearly, Blue can follow rule 3.

3. $k > 1$. First we prove the following claim needed to complete the proof for Stage 1.

Claim 3.3.3. *If there are $k > 1$ nice edges, then there are at least $k - 1$ nice vertices.*

Proof. We will go through the whole strategy of Blue to determine when the numbers of nice vertices and nice edges are changing.
In rule 1 both numbers increase by one.
In rule 2 it can happen that just the number of nice edges increases by one, that both numbers increase by one or that there is no change.
In rule 3 the number of nice edges decreases by one.
In rule 4 both numbers decrease by one.
Therefore, the only way to make the number of nice vertices less than the number of nice edges for one is using rule 2. That can happen only once because Blue will not use rule 2 again as long as $k \neq 0$. □

Claim 3.3.3 ensures that Blue can follow rule 4 to the end and claim a new triangle.

Note that during Stage 1, Blue's graph consists of t disjoint triangles, k nice edges and some isolated vertices.
When there are no more moves in Stage 1, clearly there are no more black vertices in the game.

Stage 2. In this stage we keep the structure and all the rules from Stage 1, we just change the following two rules:

2(b) If the pure red vertices are in exactly two RC and $k = 0$,
Blue claims a P_3 taking just pure red vertices.

3. If $k = 1$,
   Blue claims an edge adjacent to the nice edge and incident with a pure red vertex, that is not the nice vertex from the same RC. Then, in the following move he claims the edge that together with the nice edge and the edge that he just claimed closes a triangle.

We will prove that when Blue can no longer follow his strategy, he has already won. Note that at that point every isolated vertex in Blue's graph is pure red.

When the Blue's strategy tells him to play by rule 1, that means that in his last move Red joined an isolated edge and a black vertex and made a pure red P_3. Therefore, this move can only be the first move of Blue in Stage 2, because at that point there are no more black vertices. Therefore, we conclude that it is not possible for Red to claim a pure red triangle in this stage, and Blue will never claim a nice edge while performing the rules 2, 3 or 4.

Claim 3.3.4. *As long as Red has not already lost, Blue can play by one of the rules 2, 3 or 4.*

Proof. When the Blue's strategy tells him to play by rule 2a,
Blue can claim a new triangle in the following three moves, the argument is the same as in Stage 1. After that, his graph will have the same structure as in the beginning of Stage 2 and he continues to play.

When the Blue's strategy tells him to play by rule 2b,
note that it is not possible to have less than three pure red vertices here, because in that case Blue would have already won with at most $n-2$ edges in his graph, $3m+1 \leq n \leq 3m+2$, for some integer m (Blue's graph consists of triangles and isolated vertices). Red's graph cannot have more than $n-2$ edges, because of his first component and Observation 3.1.5.

These pure red vertices are the only vertices that are not blue, and by Claim 3.3.2 we know that in two RC we have at most four pure red vertices. Clearly, Blue can claim a P_3 in the following two moves. Now, we show that after these moves Blue wins.

- If there are four pure red vertices, we know that $n = 3m + 1$, for some integer m, and after making a P_3 Blue's graph has $n - 2$ edges. At the same time Red's graph cannot have more than $n-2$ edges, because of his first component and Observation 3.1.5, so Blue wins.

- If there are three pure red vertices. We know that $n = 3k$, for some integer k, and after making a P_3 Blue's graph has $n-1$ edges. At the same time Red's graph cannot have more than $n-1$ edges, because of his first component and Observation 3.1.5, so Blue wins.

When Blue's strategy tells him to play by rule 3,
there is exactly one nice edge and if there is a pure red vertex, that is not the nice vertex from the same RC, in his following two moves Blue can claim a triangle as advised by the strategy. Then, his graph has the same structure as in the beginning of Stage 2 and he continues to play.

Otherwise, if the only vertex that is not blue is the nice vertex from the same RC, then Blue has already won. This is true because $n = 3m$, for some integer m, and the Blue's graph has $n-2$ edges ($3t+1$, where t is the number of blue triangles), while Red's graph cannot have more edges because it contains at least two P_3.

Note that it is not possible that there is no pure red vertices, because in that case Blue's graph would have $n-1$ edges, and Red's graph cannot have more then $n-3$ edges, for the same reason as above ($n = 3m+2$, for some integer m).

When Blue's strategy tells him to play by rule 4,
the argument is the same as in Stage 1, and in his following two moves Blue can claim a triangle. After that, his graph has the same structure as in the beginning of Stage 2 and he continues to play. □

This concludes the proof of Case 5, and also the proof of the theorem. □

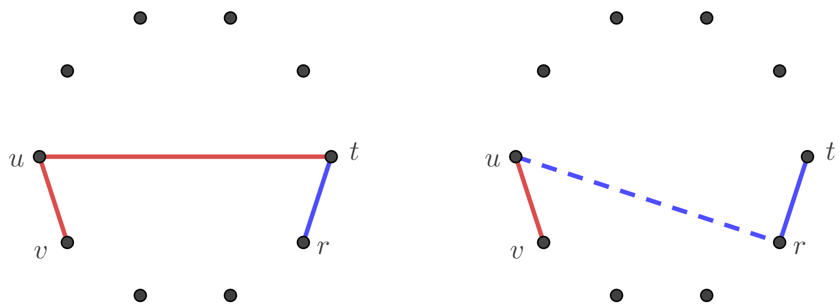
3.4 Strong CAvoider-CAvoider games

Proof of Theorem 2.1.3: In order not to lose each player must keep the maximal degree in his graph at most two. Furthermore, the rules of the game dictate that both players must maintain their respective graphs connected throughout the game. Hence, as long as no one loses the game, the graph of each player must be a path or a cycle. Note that if Blue can claim a Hamiltonian cycle he will win, because Red will lose in the following move.

We will show that Blue can win. In the beginning, we have a graph G with n isolated vertices, and Red claims an edge, let us denote it by uv. Then Blue claims an edge that is not adjacent to the red one, let us denote it by rt. In the following move Red has two options, up to isomorphism, for choosing an edge $e = xy$, which will be our two cases.

Case 1. Vertex x is red and y is blue, w.l.o.g $x = u$ and $y = t$.

We again apply the strategy stealing argument, assuming that after his second move Red has a strategy S to win the game.



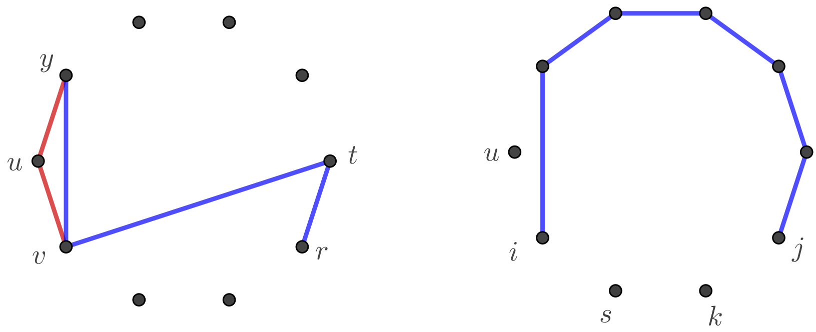
(a) (b)

Figure 3.9. Case 1: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

After Red plays ut it is Blue's turn. The graph of the game consists of one P_4, with two adjacent red edges and one blue edge, and isolated vertices, see Figure 3.9a. Before his next move, Blue imagines that he has already claimed the edge ur, and that Red has not claimed the edge ut, see Figure 3.9b. Note that the edge ur will remain free throughout the game, as otherwise Red would create a S_4 in his graph.

The imagined graph is isomorphic to the graph where the roles of the players are swapped. Blue imagines that he is the first player, and that Red claims the edge ut as his second move. From now on, Blue responds as advised by the strategy S. Because this is a winning strategy, Blue wins the game, a contradiction.

Case 2. Vertex x is red and y is black, w.l.o.g. $x = u$.
We will first describe a strategy for Blue, and then we will show that he can follow it and win. The following move of Blue is the edge tv. In his third move, if it is unclaimed, Blue claims the edge vy, see Figure 3.10a.



(a) (b)

Figure 3.10. Case 2: (a) Blue's graph after his third move. (b) Blue's graph after his $(n-4)th$ move.

If Blue's graph is not a path on $n-3$ vertices, then in his following move, Blue claims an edge incident with y and one black vertex. From that point on, the strategy of Blue will be to make a Hamiltonian path on the vertex set $V \backslash \{u\}$ and then to complete it to a Hamiltonian cycle by connecting the vertex u to both ends of the blue path.

While there are more than two vertices in $V \backslash \{u\}$ that are not in the blue path, Blue extends his path by adding one of the vertices from $V \backslash \{u\}$.
Then, we denote the last two isolated vertices besides u in the Blue's graph with s and k, and the ends of the Blue's path with i and j, see Figure 3.10b.

We distinguish these cases:

2.1 If just one of the edges $\{is, ik, js, jk\}$ is free, or two of them are, but both red edges are incident with one of the vertices $\{s, k\}$, see Figure 3.11a, 3.11b,
Blue will claim a free edge from the set $\{is, ik, js, jk\}$, and then the edge sk.

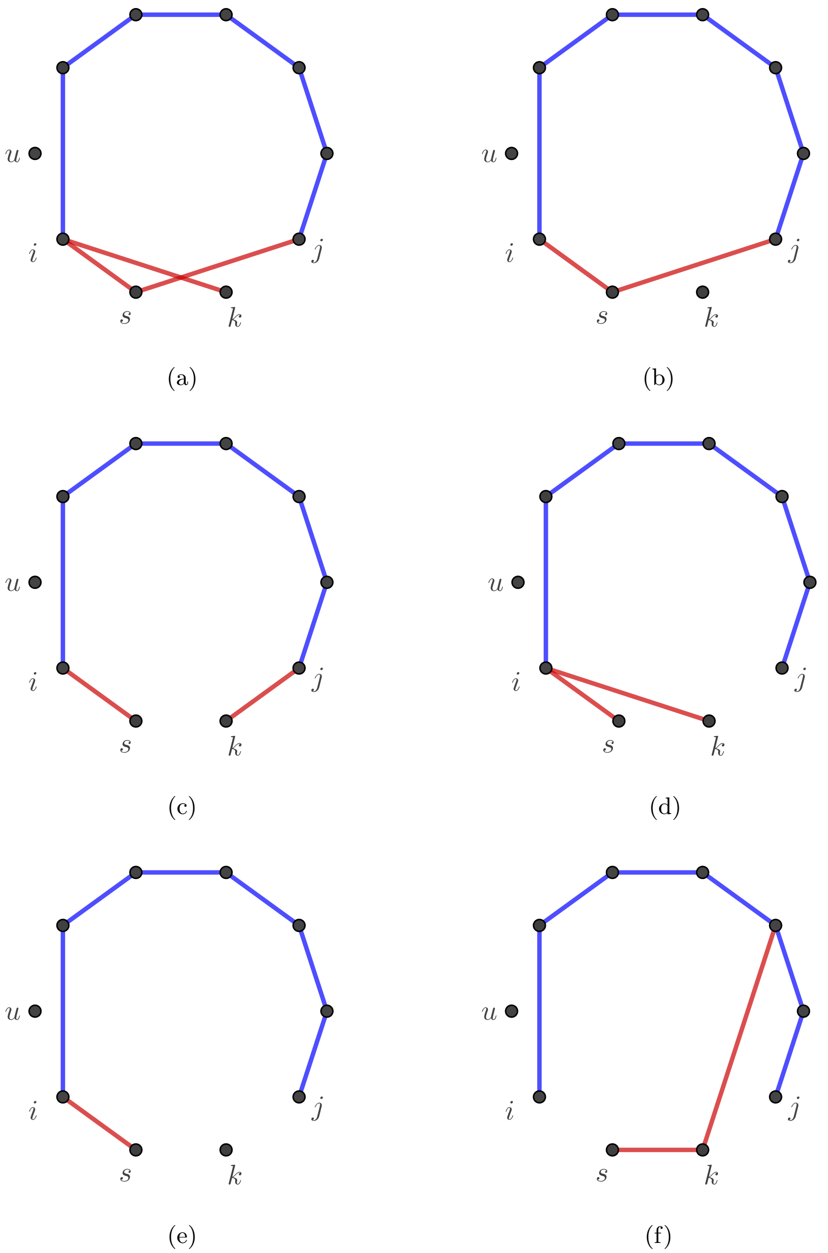


Figure 3.11. Case 2: different possibilities for the graph when it is Blue's turn to play and his graph consists of a blue path on $n-3$ edges and three isolated vertices.

2.2 If two of the edges $\{is, ik, js, jk\}$ are red and exactly one of those edges is incident with vertex s, see Figure 3.11c and 3.11d,
Blue will claim a free edge from the set $\{is, ik, js, jk\}$, and then the edge sk if it is free, otherwise the remaining one from the set of edges $\{is, ik, js, jk\}$.

2.3 If exactly one of the edges $\{is, ik, js, jk\}$ is red,
w.l.o.g. let us assume that the edge is is red, see Figure 3.11e. Blue claims the edge ik, and then in the following move he claims one of the edges $\{ks, js\}$.

2.4 If there are no red edges in $\{is, ik, js, jk\}$, we have two subcases.

(a) If the edge sk is free,
Blue claims the edge ik. In his following move he claims one of the edges $\{ks, js\}$.

(b) If the edge sk is red,
at least one of the vertices $\{s, k\}$ has to have degree two in the Red's graph, w.l.o.g. let us assume that vertex is k.
Then, Blue claims the edge is and in the following move he claims the edge jk, see Figure 3.11f.

2.5 Else, Blue claims any free edge that does not make an S_4 in his graph.

In his last two moves Blue claims the edges that complete the Hamiltonian cycle on the vertex set V.

Now we will show that when it is Blue's turn to play Case 2, Blue can follow his strategy and win.
It is clear that Blue can claim an edge in his second move. If in his third move the edge vy has been already claimed, that means that Red's graph has a red triangle and he will lose in his following move, so Blue can skip this move and claim the next edge and win. Otherwise, Blue claims the edge vy and his graph at this moment consists of a P_4 and isolated vertices.

Claim 3.4.1. *If Blue's graph consists of a path disjoint from u and more than three isolated vertices, Blue can extend his path by adding one of the isolated vertices from $V \setminus \{u\}$.*

Proof. Let as denote by P that blue path. There are at least three vertices from $V\backslash\{u\}$ that are not in P. Suppose that there is no edge such that Blue can extend his path with vertex from $V\backslash\{u\}$. That means that both ends of P are incident with at least three red edges and that leads to a contradiction because there are two vertices of red degree three. □

According to Claim 3.4.1 Blue can follow his strategy while there are more than two vertices in $V\backslash\{u\}$ that are not in the blue path. After that, one of the following cases happens:

Case 2.1 It is obvious that the edge sk cannot be red as otherwise Red would have a vertex of degree three. Therefore, Blue can claim his following two edges and make a path on $n-1$ vertices.

Case 2.2 We can have two different options as depicted in Figure 3.11c and 3.11d. For the first one, obviously, Blue can claim two of the edges ik, js, sk and make a path on $n-1$ vertices. For the second one, if sk has become red Blue can take the last edge from $\{js, jk\}$ and win because Red has made a triangle. Otherwise, Blue claims the edge sk and makes a path on $n-1$ vertices.

Case 2.3 It is evident that Blue can claim his following two moves and make a path on $n-1$ vertices.

Case 2.4 In case that sk is free it is obvious.
Otherwise if the edge sk is red, it cannot be an isolated edge in the Red's graph because his graph is connected. Therefore at least one of the vertices $\{s, k\}$ has to have degree two in Red's graph. Now, it is clear that Blue can follow his strategy.

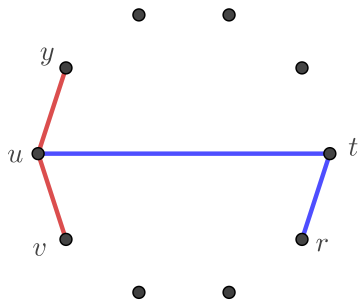
Case 2.5 In this case each of the edges $\{is, ik, js, jk\}$ is red and Red has a C_4 in his graph. Therefore, Blue can take the edge iu and win.

If Blue has not already won, at this moment his graph consists of a path on $n-1$ vertices and one isolated vertex u. Both edges that connect the vertex u with ends of the blue path are free and Red cannot claim them. Therefore, Blue can claim these two edges in the following two moves and create the Hamiltonian cycle on the vertex set V and win the game. □

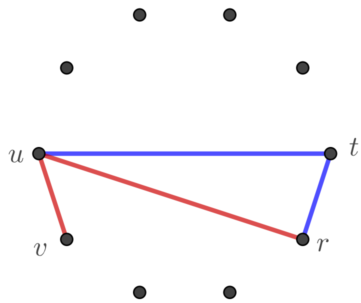
Proof of Theorem 2.1.4: First we will describe a strategy for Blue and then we will show that he can follow it and thus win. In the beginning, we have a graph G with n isolated vertices. After Red claims an edge, let us denote it by uv, Blue claims an edge that is not adjacent to the red one, let us denote it by rt. Because they have to play on connected graphs, in the following move Red, up to isomorphism, has two options to choose the following edge $e = xy$, and these will be our two cases.

Case 1. Vertex x is red and y is black, w.l.o.g. $x = u$.

The following move of Blue is the edge ut, see Figure 3.12a.



(a)



(b)

Figure 3.12. The graph after the second move of Blue: (a) Case 1. (b) Case 2.

Until the end of the game, Blue will star-add vertices that are not blue to the t-star.

Case 2. Vertex x is red and y is blue, w.l.o.g. $x = u$, $y = r$.

The following move of Blue is the edge ut, see Figure 3.12b. Until the end of the game, Blue will star-add vertices that are not blue to the t-star.

Now we will prove that Blue can follow his strategy and thus wins the game. In both cases it is evident that Blue can play first two moves and after that the graph of the game consists of two stars on three vertices (one red u-star and one blue t-star), and isolated vertices.

Note that Red cannot claim any edge incident with leaves of the u-star, not even close a red triangle, because his graph has to stay connected during the game, and if he claims a cycle he will inevitably lose the game. Therefore, only allowed moves for him are to star-add more vertices to the u-star. It is clear that Red cannot colour the vertex t in red, hence Blue can claim each of $n-1$ edges of t-star. At the end of the game Red will claim at most $n-2$

edges, and then he has to make a P_4 in his following move. Therefore, Blue wins the game in $n-1$ rounds. □

We proved Theorem 2.1.4 giving an explicit strategy for Blue. Alternatively, it is straightforward to check that we can use a similar argument of strategy stealing as in Case 1 and Case 2 in the proof of Theorem 2.1.1, assuming that the graphs of both players stay connected throughout the game.

Proof of Theorem 2.1.5: As each player maintains his graph connected, it has to be a tree. Therefore, the game can last for at most $n-1$ rounds. If after the $(n-1)$-st round Blue's graph is a tree, than Red will lose in his following move.

We will show that Blue has a winning strategy. In the beginning, we have a graph G with n isolated vertices. After Red claims an edge, let us denote it by uv, Blue claims an edge that is not adjacent to the red one, let us denote it by rt. In the following move Red has two options, up to isomorphism, for choosing an edge $e = xy$, and those two moves will make our two cases.

Case 1. Vertex x is red and y is blue, w.l.o.g. $x = u$ and $y = t$.
We apply the strategy stealing argument, assuming that after his second move Red has a winning strategy S.

The graph of the game consists of one P_4, with 2 adjacent red edges and one blue edge, and isolated vertices, see Figure 3.13a. Before his next move, Blue imagines that he has already claimed the edge vt and that Red has not claimed the edge ut, see Figure 3.13b. Note that the edge vt will remain free throughout the game, as otherwise Red would claim a triangle.

The imagined graph is isomorphic to the graph where the roles of players are swapped. Blue imagines that he is the first player and that Red claims the edge ut as his second move. From now on, Blue responds as advised by the winning strategy S, and wins the game, a contradiction.

Case 2. Vertex x is red and y is black, w.l.o.g. $x = u$.

After Red claims the edge uy it is Blue's turn, so he claims the edge tu, see Figure 3.9a. Then, no matter what Red plays, Blue claims an edge incident with t and one black vertex. Then repeats that move for as long as possible. When there are no more black vertices, Blue will claim the edge incident with the pure red vertex of maximum degree and a blue vertex. He will continue doing that until all vertices in the graph are blue.

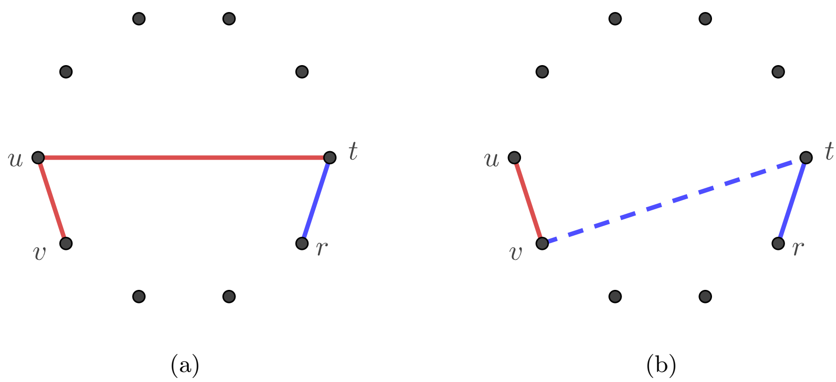


Figure 3.13. Case 1: (a) the graph before the second move of Blue. (b) The imagined graph before the second move of Red.

Obviously, if Blue can follow his strategy, his graph will be a tree on $n-1$ edges and he will win. It remains to show that he can follow his strategy.

While there is at least one black vertex, it is evident that Blue can make it adjacent to t by claiming the edge incident to a black vertex and t. When there are no more black vertices in the graph, we will denote the pure red vertex of maximum degree by m, and by i the number of edges in the Blue's graph. Note that at this moment Blue's graph consists of a t-star with $i \geq \left\lfloor \frac{n-5}{2} \right\rfloor + 2$ edges, and isolated vertices.

Assume for a contradiction that there is no free edge between m and any blue vertex. Note that $m \neq u$ because u is blue. That means that m is adjacent to each of the blue vertices in the Red's graph. Therefore, Red must have a star with $i+1$ edges in his graph and at least one more edge from the beginning of the game. Hence, Red would have two edges more than Blue, a contradiction. $\square$

3.5 Strong CAvoider-CAvoider games from a position

Proof of Theorem 2.1.6: Each of these games starts from position T. By cause of playing on the connected graph the only move that Red can make is to join a vertex of the "drawn K_5" to an isolated vertex, we will denote these vertices by r and v, respectively. We prove that using the strategy stealing argument Blue can win. Assume that Red has a winning strategy for each of the following games.

Blue imagines that Red did not claim the edge rv and that he claimed the edge vu, where u is chosen as described below depending on the type of the game.

1. If he plays C_3 game or C_6 game,
   u is a neighbour of the vertex r in the Red's graph, see Figure 3.14a.

2. If he plays C_4 game,
   u is not a neighbour of the vertex r in the Red's graph, see Figure 3.14b.

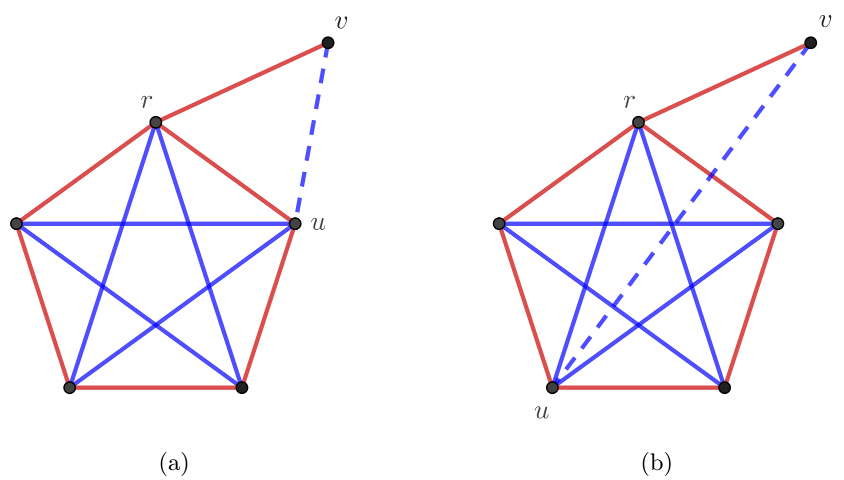


Figure 3.14. (a) The imagined blue edge in the C_3 or C_6 game depicted by a dashed line. (b) The imagined blue edge in the C_4 game is depicted by a dashed line.

Note that in both cases it is not possible that Red claims the edge vu, because he would create a forbidden subgraph. Then, the imagined graph is isomorphic to the graph of the game, where the roles of the players are swapped. Therefore, Blue can follow a corresponding winning strategy to the end of the game as a first player, starting with a response to the red edge rv and win, a contradiction.

In case we play the Even cycle or the Odd cycle game, Blue follows the same strategy as for the C_6 game and the C_3 game, respectively and wins. This concludes the proof of the theorem. □

3.6 Concluding remarks and open problems

As we have already seen, strong games are hard to analyze. Reasons for that are the lack of mathematical tools and the fact that both players have the same goal.

In this chapter, we saw an application of strategy stealing in proving that Blue has a winning strategy for two different strong Avoider-Avoider F games. We are interested in applying this tool to more such games, and also finding more tools that could apply to them.

When we look at strong Avoider-Avoider F games played on the edge set of the complete graph, through the whole literature the outcome for just two games was previously known, then we added two new results. Therefore, the vast majority of all other games remain open.

In particular, is it true that Blue has a winning strategy in a strong Avoider-Avoider F game, where the forbidden graph F is: a triangle, a path on k vertices P_k, a cycle on k vertices C_k or a tree on k vertices T_k?

Chapter 4

Achievement number in strong Maker-Maker games

This chapter is devoted to strong Maker-Maker games, more precisely to finding the achievement number for some graphs F. Here we are going to give definitions and notations needed for further understanding the text. Then, we give the proofs for all theorems presented in Section 2.2.

In Section 4.1 we give notation and preliminaries. In Section 4.2 we give the explicit Red's winning strategy playing on the edge set of the complete graph on $a(F)$ vertices, for some small graphs F. In Section 4.3 we find the value of the achievement number for paths, cycles, stars and perfect matchings. Then, in Section 4.4 we give an upper bound for $a(T_n)$. Finally, in Section 4.5 we give concluding remarks and open problems.

4.1 Preliminaries

During the game, we say that the vertices that are touched by Red are *red vertices*, the ones touched by Blue are *blue vertices*, and the others, that are not touched by any of the players, are *black vertices*. Denote the set of isolated vertices in Red's graph by I. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. Let us denote with S_n a star on n vertices.

Let $V(T_n)$ be the set of all vertices of a tree T_n, and let $S(T_n)$ be the set of all vertices of the tree T_n that are not leaves. If T_n is a rooted tree, denote by $\underline{d}(u)$ a down degree of $u \in V(T_n)$. A path of a tree T_n is called a *bare path* if all of its interior vertices have degree two in T_n.

For a given hypergraph $\mathcal{F}$, denote by $\tau(\mathcal{F})$ the smallest integer t such that Maker can win the Maker-Breaker game played on $\mathcal{F}$ within t moves.

We give the following theorems that we are going to use in our proofs. The first result refers to Maker-Breaker games.

Theorem 4.1.1 ([65]). *For sufficiently large n, we have $\tau(\mathcal{HP}_n) = n - 1$, where $\mathcal{HP}_n$ is the hypergraph whose hyperedges are all the Hamilton paths of K_n.*

The following two results are about strong Maker-Maker games.

Theorem 4.1.2 ([44]). *For sufficiently large n, Red has a winning strategy for the strong perfect matching game M_n. Moreover, he can win this game within $\lfloor \frac{n}{2} \rfloor$ moves if n is odd and within $\frac{n}{2} + 2$ moves if n is even.*

Theorem 4.1.3 ([44]). *For sufficiently large n, Red has a winning strategy for the strong Hamilton cycle game H_n. Moreover, he can win this game within $n + 2$ moves.*

In the following theorem we observe the Maker-Breaker tree game, providing us with a fast Maker's strategy for certain trees.

Theorem 4.1.4 ([29]). *Let Δ be a positive integer. Then there exists an integer $m = m(\Delta)$ and an integer $n_0 = n_0(\Delta, m)$ such that the following holds for every $n \geq n_0$ and for every tree $T = (V, E)$ with $|V| = n$ and $\Delta(T) < \Delta$. If T admits a bare path of length m, such that one of his endpoints is a leaf of T, then Maker has a strategy to win the game $(E(K_n), \mathcal{T}_n)$ in $n-1$ moves.*

Now, we single out some values for the diagonal Ramsey number of certain graphs, given in [95].

First, observe the diagonal Ramsey number of a path and a star on n vertices. For a path $R(P_n) = n + \lfloor \frac{n}{2} \rfloor - 1$, for all $n \geq 2$, whereas for a star $R(S_n) = 2(n-1) - \epsilon$, where $\epsilon = 1$ for even n and $\epsilon = 0$ otherwise.

Depending on the parity of n for a cycle on n vertices we distinguish two cases: $R(C_n) = 2n - 1$, for $n > 3$ and odd, and $R(C_n) = n + \frac{n}{2} - 1$, for $n > 4$ and even.

Ramsey number of a fixed tree on n vertices is bounded with $\lfloor\frac{4n-1}{3}\rfloor \leq R(T_n) \leq 4n+1$, and in particular if n is even there is another result for the upper bound $R(T_n) \leq 2n-2$.

4.2 Red's winning strategy for small graphs

Proof of Observation 2.2.3: It is clear that Red can win in his first, respectively, second move, regardless of the edge selection. □

Proof of Proposition 2.2.4: Red cannot win starting on K_4, because, in his first move, Blue can claim the edge that is not incident with the red one. However, if the game is played on K_5, after his first move, Red has three winning edges, and Blue can claim just one of them. Therefore, Red wins after his second move. □

Proof of Proposition 2.2.5: Again, Red cannot win starting on K_4, for the same reason as above. However, if the game is played on K_5, after the first move of Blue, Red makes a P_3 by adding a black vertex to his graph. At this moment there are at least three winning edges for Red, so he completes a P_4 in his next move. Therefore, Red wins after three moves. □

Proof of Proposition 2.2.6: Red cannot win starting on K_4, because after Red makes a P_3 with central vertex v, Blue can claim the third edge incident with v and prevent Red from completing a $K_{1,3}$. However, if the game is played on K_5, after the first move of Blue, Red makes a P_3 with central vertex v, such that v is pure red, and in his next move there is at least one free edge that completes a $K_{1,3}$, Red claims it and wins in his third move. □

Proof of Proposition 2.2.7: Note that Red cannot win starting on K_4, because after Red makes a P_3 with central vertex v, Blue can claim the third edge that completes a red triangle and prevent Red from completing a K_3. On the other hand, if the game is played on K_5, Red claims an arbitrary edge xy, and after the first move of Blue, we distinguish two different cases, up to isomorphism, and give the winning strategy for Red:

Case 1: Blue claimed an edge incident with x or y, w.l.o.g., assume that he claimed x.
Red in his second move claims an edge incident with x and a black vertex.

Then, in his following move either he wins or again claims the edge incident with x and the black vertex, making a red $K_{1,3}$. Two edges that complete a red triangle are free, so Red wins in his fourth move.

Case 2: Blue claimed an edge that is incident neither with x nor y.
Red in his second move claims an edge incident with x and the black vertex. Then, in his following move either he wins or claims an edge that completes a red $K_{1,3}$. After this move, two edges that complete a red triangle are free, hence Red wins in his fourth move.

Note that in both cases Blue cannot complete a triangle in three moves because his graph consists of at least four blue vertices. Therefore, Red wins after four moves. $\square$

Proof of Proposition 2.2.8: First note that $a(K_{1,3} + e) \geq 5$ because $K_{1,3} \subset K_{1,3} + e$. Now, we give an explicit winning strategy for Red in a strong Maker-Maker game played on K_5.

First edge Red claims arbitrarily, then after Blue's first move, Red claims an edge such that his graph is a P_3 with central vertex v and the graph of the game is a P_4. After the second move of Blue, Red claims the edge that completes the red triangle if possible, otherwise, he claims the edge incident with v and the black vertex. In his fourth move, Red makes a $K_{1,3} + e$ and wins.

Now we prove that Red can follow his strategy.
In the first case, Red's graph after the third move is a triangle, there are six edges such that each of them completes a $K_{1,3} + e$. Blue claimed at most three of them, so Red wins in his fourth move.

Otherwise, note that the black vertex must exist before Red's third move because Blue's graph is a P_3 with two vertices in both colors. Hence, Red's graph is a $K_{1,3}$ and there are two edges such that each of them completes a $K_{1,3} + e$. Blue claimed at most one of them, hence, Red wins in his fourth move. $\square$

Proof of Proposition 2.2.9: We give an explicit winning strategy for Red. First three moves of Red are the same as in the proof of Proposition 2.2.8. Before Red plays his fourth move he chooses one of the following two cases.

Case 1: Red's graph is a triangle.
Red chooses a vertex that is isolated in Red's graph and is not adjacent with any of red vertices in Blue's graph, let us denote it by s. In his two following

moves Red makes s adjacent to two different red vertices, and completes a $K_4 - e$.

Case 2: Red's graph is a $K_{1,3}$.
We denote the vertices as depicted in Figure 4.1.

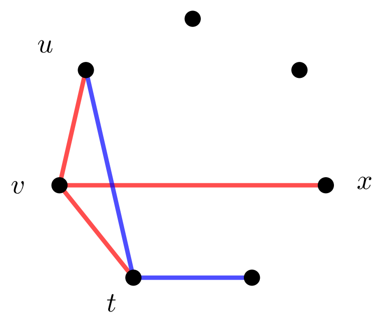


Figure 4.1. Case 2: the graph after Red claimed his third edge.

In his fourth move Red completes a triangle. When it is Red's turn to play his fifth move, if he can complete a $K_4 - e$, he claims that edge and wins. Otherwise, he chooses between two options depending on the Blue's graph.

1. Blue needs just one free edge to complete a $K_4 - e$.
   Red claims it. In his sixth move, Red chooses a vertex y such that each of the three edges incident with y, and a vertex from the red triangle is free. Then, in his following two moves, Red claims two of these edges and wins.

2. Otherwise, there are at least five blue vertices.
   In his fifth move, Red chooses a vertex y, such that all three edges incident with y and one vertex from the red triangle are free. In his following two moves, Red claims two of those edges and wins.

Here, we prove that Red can follow the strategy from Case 1. When it is Red's turn to play his fourth move, four vertices are isolated in Red's graph, and Blue's graph has three edges. Therefore, there must be at least one vertex that is not adjacent to any vertex of the red triangle in Blue's graph. Red chooses that vertex and in his two following moves makes it adjacent with two different red vertices, i.e. he completes a $K_4 - e$ in five moves and wins.

Now we need to show that Red can indeed follow the strategy from Case 2. Note that in both cases, before Red plays his fifth move, Blue's graph contains four edges including exactly one of the $\{ux, tx\}$. Otherwise, Red would complete a $K_4 - e$ in his fifth move. Moreover, if all of Blue's four edges are played on the four vertices, than Red follows rule 1. Otherwise, Blue touched at least 5 vertices, so Red follows rule 2.

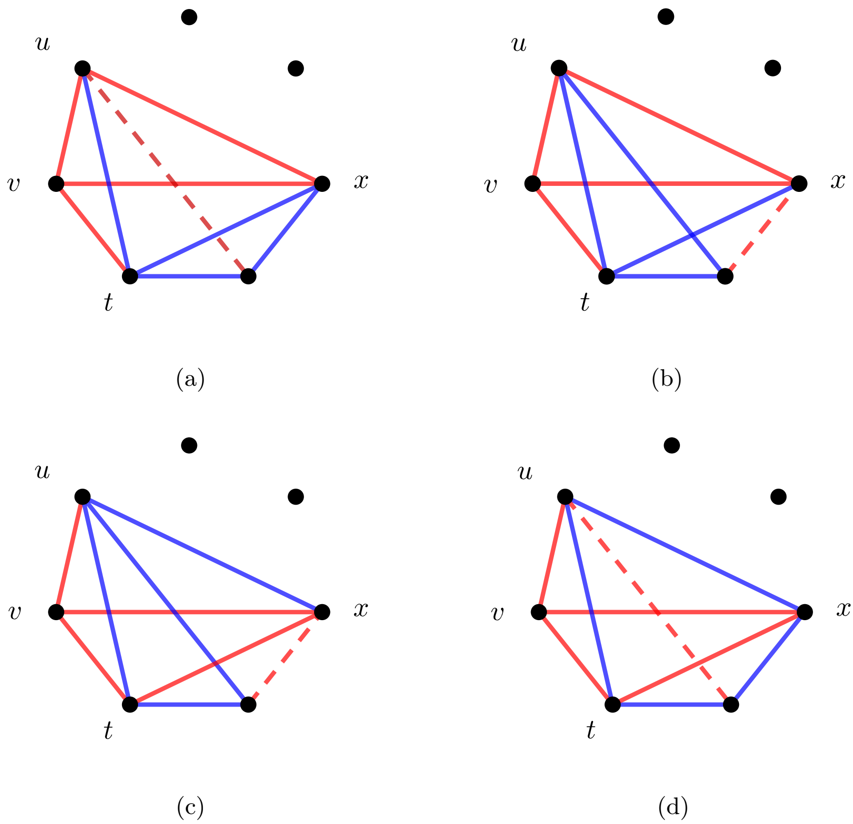


Figure 4.2. Case 2: four options for the graph after Red's fifth move.

If Red played his fifth move by rule 1, in Figure 4.2 we have depicted four options for the game graph after that move. In his fifth move, Blue has to claim the edge that prevents Red from making a $K_4 - e$ in all cases except the last one, see Figure 4.2d, but in that case, Blue's graph is a C_4, so he needs at least three more edges to finish the game. Therefore, after Blue's fifth move, both players need at least two more edges to complete a $K_4 - e$. When it is Red's turn to play his sixth move, there must be at least one vertex y, such that all three edges incident with y and one vertex from the red triangle are free. In his following two moves, Red claims two of these edges and wins in seven moves.

If it is Red's turn to play his fifth move by rule 2, Blue cannot win in five moves, because his graph has at least five vertices. When it is Red's turn to play his fifth move, Blue's graph contains four edges, and at most two of them are incident with isolated vertices in Red's graph ($V \backslash \{x, t, v, u\}$). Therefore, there must be at least one vertex y, such that all three edges incident with y and one vertex from the red triangle are free. In his following two moves Red can claim two of these edges and win in six moves. □

4.3 Achievement number for paths, cycles, stars and perfect matchings

Proof of Proposition 2.2.10: The proof is a direct consequence of Theorem 4.1.1. In [65] it is proved that Maker can complete a Hamilton path on n vertices in $n - 1$ moves. Hence, starting on n isolated vertices, Red can use the same strategy and make a path on n vertices before Blue is able to do the same. Therefore, the smallest number of vertices required for Red's win is n. □

Proof of Proposition 2.2.11: We prove that Red can complete a star on n vertices S_n starting on K_{2n-3}. His first move Red chooses arbitrarily, then after the move of Blue, Red claims the edge incident with one pure red and one vertex from I, and makes a S_3 with the central vertex v. Now, there are $2n - 6$ vertices isolated in Red's graph and none of them is adjacent to v. Hence, it is clear that in each of his following $n - 3$ moves Red can claim one vertex from I and make it adjacent to v. □

Proof of Proposition 2.2.12: The proof goes directly from Theorem 4.1.3. Starting on a complete graph K_n, Red has a strategy to complete a C_n first in $n + 2$ moves. □

Proof of Proposition 2.2.13: The proof goes directly from Theorem 4.1.2. Starting on K_{2n}, Red has a strategy to complete a perfect matching first in $n+2$ moves. $\square$

4.4 Achievement number for trees

Before proving Theorem 2.2.14 we show the following.

Proposition 4.4.1. *For every tree on n vertices T_n, the following inequality is true:*

$$\sum_{\substack{u \in V(T_n) \\ \underline{d}(u) \neq 0}} (2\underline{d}(u) - 1) \leq 2n - 3.$$

Proof. Let $|S(T_n)| = k$ where $S(T_n)$ is the set of all vertices of the tree T_n that are not leaves. Then

$$\sum_{u \in S(T_n)} \underline{d}(u) = \sum_{v \in V(T_n)} \underline{d}(v) = n - 1.$$

$$\sum_{\substack{u \in V(T_n) \\ \underline{d}(u) \neq 0}} (2\underline{d}(u)-1) = \sum_{u \in S(T_n)} (2\underline{d}(u)-1) = 2 \sum_{u \in S(T_n)} \underline{d}(u) - \sum_{u \in S(T_n)} 1 = 2 \sum_{u \in S(T_n)} \underline{d}(u) - k$$

$$= 2 \sum_{v \in V(T_n)} \underline{d}(v) - k = 2(n-1) - k = 2n - 2 - k.$$

Knowing that $k \geq 1$ for every tree T_n, we conclude that the assertion is true. Equality applies only when $T_n = S_n$. $\square$

Proof of Theorem 2.2.14: Let

$$n_1 = max\{n + 4\sqrt{n}, \sum_{\substack{u \in V(T_n) \\ \underline{d}(u) \neq 0}} (2\underline{d}(u) - 1)\}.$$

At the beginning of the game, the graph G is an empty graph K_{n_1}. We prove that following his strategy Red can make a copy of T_n in $n-1$ moves. Let $y = n_1 - n$. At any given moment during the game, we denote the graph spanned by Red's edges by R, and the graph spanned by Blue's edges by B. At the end of the game, we will have an injective function $f : V(T_n) \to V(G)$, such that $f(x)f(y) \in E(R)$ whenever $x, y \in V(T_n)$ and $xy \in E(T_n)$, and this will be a witness that Red's graph contains a copy of T_n. The function f will be gradually defined during the play. Initially, no value is assigned to f, as vertices of G gradually become red, each red vertex will be given its value of f, one at a time (except in the first round, when two vertices at the same time become red). For a vertex $v \in V(R)$, we denote by $f^{-1}(v) = v'$ its pre-image under f, and by $\underline{d}(v')$ the down degree of vertex v' with respect to T_n.

Throughout the game exactly one red vertex v_i is *active*, and when it reaches the down degree $\underline{d}(v_i')$ in red edges, then it becomes *passive*. A red vertex v that is at that point a leaf in R, and at the same time $f^{-1}(v) = v'$ belongs to $S(T_n)$, is called a *sleeping* vertex. All the remaining red vertices are *passive*. During the game, when one vertex v_i stops being active, an arbitrary sleeping vertex is elected to be the new active vertex v_{i+1}, and so on.

The strategy of Red:

Red progressively makes a copy of T_n, adding one edge to the graph R in each of his moves. His first edge is chosen arbitrarily, rendering two vertices red. Then after the first move of Blue, Red chooses a pure red vertex for the root v_0 and makes it active. The other red vertex we denote by r, and update $f^{-1}(v_0) = v_0'$ (where v_0' is the root of T_n), whereas $f^{-1}(r)$ is an arbitrary neighbour of v_0' in T_n. From now on Red claims an edge incident to the active vertex v_i and one vertex $x \in I$ which he chooses by following the first satisfied rule of the following four, and after each move he updates that x' is a neighbour of v_i' in T_n, such that $x' \notin f^{-1}(V(R))$. Denote by uv the last edge that Blue has claimed.

1. If $u, v \in I$, Red chooses x to be u or v, if possible. Otherwise, if $u, v, s \in I$ completes a blue P_3, Red chooses x to be s, if possible.

2. If there is a blue P_3 on vertices u, v, s, where v is the central vertex that is sleeping, while $u, s \in I$, Red chooses x to be one of the vertices $\{u, s\}$, if possible.

3. A blue vertex from I is chosen as x.

4. A black vertex is chosen as x.

We show that Red can follow this strategy through $n-1$ rounds, hence, makes a copy of T_n and wins.
It is clear that Red can complete the first level, i.e. make a star with central vertex v_0 and $\underline{d}(v_0')$ edges. When Red reaches a new active vertex v_i and it is his turn, we need to have at least $2\underline{d}(v_i')-1$ free edges, each of them between v_i and a vertex from I.

Suppose to the contrary that Red cannot follow his strategy, i.e. he reached a new active vertex v_t, with $\underline{d}(v_t')=r$. Let M be the set of vertices from I that are adjacent to v_t in B, and $|M|=m$. Then $m \geq |I|-2r+2$, as otherwise, Red could claim r edges incident with v_t. Hence, it remains to show that $|I|>m+2r-2$ to arrive to a contradiction.

Note that y is either $4\sqrt{n}$ or $n-2-k$, where $|S(T_n)|=k$, and each vertex from $S(T_n)\setminus\{v_0'\}$, with down degree d contributes with $d-1$ to y (and v_0' contributes with $\underline{d}(v_0')-2$ to y). Knowing that $d(v_0')$ is maximum in T_n, we conclude that $\underline{d}(v_0')\geq r+1$. Therefore, by counting just extra vertices obtained from v_0 and v_t, we get that $y\geq 2r-2$. Now, we conclude that $m\geq 2\sqrt{n}+1$, using that $|I|\geq r+y$ and $y\geq 4\sqrt{n}$. Indeed,

$$m \geq |I|-2r+2 \geq y-r+2 \geq \frac{y}{2}+1 \geq 2\sqrt{n}+1.$$

Now, we show that $|I|>m+2r-2$. We know that $|I|\geq r+y$ and $y\geq 2r-2$, therefore, $|I|\geq 3r-2$, i.e. we need to show that there are at least $m+1-r\leq m$ more vertices in $|I|$. If there is a vertex $v_i'\in S(T_n)$, $0<i<t$ with $\underline{d}(v_i')=m+1$ then the assertion is true.

Otherwise $\underline{d}(v_i')\leq m$, for each $v_i'\in S(T_n)$, $0<i<t$. Each of the vertices from M is adjacent to v_t in Blue's graph, more precisely they form a blue star with central vertex v_t and m edges, let us denote it by $S^{(t)}$. W.l.o.g. we suppose that the last of these edges Blue claimed in his previous move. After Blue claimed $(m-1)$-st edge of $S^{(t)}$, Red could not make the active vertex adjacent to any of the $m-1$ leaves of $S^{(t)}$ following his strategy. Therefore, the active vertex v_x must have already been incident with at least $m-1$ blue edges which form a blue star $S^{(x)}$ (otherwise, Red would color one vertex from M red following rule 2). Note that $x\neq 0$, otherwise all $2(m-1)$ edges of $S^{(x)}$ ($S^{(x)}=S^{(0)}$) and $S^{(t)}$ Blue claimed while v_0 was active vertex. Hence, $\underline{d}(v_0')\geq 2m-1$ and $y\geq r-1+2m-3$ and we conclude that $|I|>2r+2m-4>m+2r-2$.

During the game, let E be the subset of $2m-2$ blue edges of $S^{(x)}$ and $S^{(t)}$ that have been claimed by Blue up to that point of the game, and let M' be a subset of the vertices from M, that are incident to the edges of E. Denote by $A = \{v_j, 0 < j < t\}$ the set of vertices that were active while Blue was claiming $2m-2$ blue edges of $S^{(x)}$ and $S^{(t)}$, and $|A| = a$. Let a_i, $1 < i < m$, be the number of vertices from A such that $a_i = |\{v_j \in A, \underline{d}(v'_j) = i\}|$.

At the latest when Blue claimed the third edge of E, Red could not choose any of the vertices from M'. Therefore, the active vertex at that moment must have been incident with each of the vertices from M' in Blue's graph. When the next vertex from A became active, it must have been adjacent to at least as many vertices of M' as the previous one. Hence, we conclude that no matter what the value of a is, Blue's graph consists of more edges.

We suppose for a contradiction that all vertices from A together contribute less than m vertices to y, and then prove that B has too many edges. To prove this we will choose such B that has the smallest possible number of edges. For this reason, we choose the smallest possible a.

Now, we need to find the minimum of the function $f(a_1, \ldots, a_m) = a_1 + \ldots + a_m$ under two constraints. The first constraint is that the number of edges that Red claimed while Blue was claiming the edges of E from the third edge to the $(2m-2)$-nd one is $2m-4$, i.e. $1a_1 + 2a_2 + 3a_3 + ... + ma_m = 2m-4$. The second one is that all vertices from A together, contribute less than m vertices to y, i.e. $0a_1 + 1a_2 + 2a_3 + ... + (m-1)a_m < m$.

$$a_1 + ... + a_m = a_1 + 2a_2 + ... + ma_m - (a_2 + 2a_3 + ... + (m-1)a_m) \geq m-3.$$

Hence, from this point on we suppose that $a = m-3$. Afterward, the smallest number of blue edges we get when vertices of A are chosen in ascending order by the down degree of its pre-image with respect to f, and when M' grows as slowly as possible.

Finally, we conclude that Blue will have the fewest edges if $a = m-3$, the first $m-4$ vertices from A have down degree 1 and the last one has down degree m. Nevertheless, even in this worst-case scenario when we count the number of edges in Blue's graph, we get:

$$\begin{aligned} e &\geq 2m - 1 + 2 + 2 + 3 + 3 + 4 + 4 + ... + \lceil \frac{m-4}{2} \rceil + 1 + m - 1 \\ &\geq 3m - 2 + \frac{\lceil \frac{m-4}{2} \rceil (\lceil \frac{m-4}{2} \rceil + 1)}{2} 2 - 2 \end{aligned}$$

$$\geq 3m - 4 + \frac{(m-4)^2}{4} + \frac{m-4}{2} = \frac{m^2}{4} + \frac{3m}{2} - 2$$
$$\geq n + 3\sqrt{n} - 2,$$

and that leads to a contradiction because less than $n-1$ moves have been played. The values from the first inequality were obtained by counting the number of blue edges for each of the blue stars $S^{(v_j)}$, $v_j \in A$. $\square$

Note that using Proposition 4.4.1 and Theorem 2.2.14 we conclude that an upper bound for $a(T_n)$ is $2n-3$, for n sufficiently large (or more precisely, for $n \geq 22$).

Proof of Proposition 2.2.15: This proof follows directly from Theorem 4.1.4. If the conditions given in this proposition are satisfied then we know that Maker can build a tree T_n without wasting any moves, i.e. he can do it in $n-1$ moves. Therefore, using this strategy Red can win in a strong Maker-Maker version of this game played on K_n. $\square$

4.5 Concluding remarks and open problems

We gave Red's winning strategy in the strong Maker-Maker game played on $a(F)$ isolated vertices for some small graphs F. Then we found the achievement number for some particular graphs, such as paths, cycles, and perfect matchings. We also gave an upper bound for the achievement number for the star S_n, which is $2n-3$. As for the lower bound, we know that it is greater than n, and it would be interesting to find the exact value of the star achievement number, or at least to improve these bounds.

We observed the achievement number for trees, and we obtained an upper bound, whereas the lower bound is n, which has been reached for paths and one particular class of trees. Can the upper bound be improved? We believe that for all trees with a bounded maximum degree, the achievement number is at most $n+2$.

Finally, Harary's conjecture is still open. Finding the answer to this question needs a better understanding of the game in question. In particular, we would profit from an upper bound for the achievement number for the star on n vertices or a proof that this upper bound is the best possible.

Chapter 5

Generalized saturation game

In this chapter, we introduce the generalized saturation game. In this game, two graphs H and F are given in advance, and we are interested in the score of this game, that is, the number of copies of the graph H in the game graph at the end of the game. First, we give notations, definitions, and theorems from the literature, that are essential for further study. Then, we give the proofs for all assertions presented in Section 2.3.

In Section 5.1 we give notation and preliminaries. In Section 5.2 we determine the score of the game where the forbidden graph F is a path on 5 vertices and the graph H is a triangle. In Section 5.3 we find the score for games in which the forbidden graphs are all cycles. Then, in Section 5.4 we determine the score for games in which the forbidden graph is a P_4. Section 5.5 reveals the score for games in which the forbidden graphs are all odd cycles. Section 5.6 gives the score for several games in which the forbidden graph is a T_n, and Section 5.7 is devoted to games in which $F = S_4$. Finally, in Section 5.8 we give concluding remarks and open problems.

5.1 Preliminaries

During the game, we say that a legal move is claiming a free edge that added to the game graph does not create a copy of the forbidden graph. Sometimes, instead of just one graph, we will have a family of graphs, where all graphs that belong to that family are forbidden.

Let us denote with S_k a star on k vertices. We denote by G_4^1 a graph on four vertices, where three of them form a triangle and the fourth vertex is adjacent to one of the remaining three. Disjoint union of graphs G and H is denoted by $G + H$. Let us denote by $\mathcal{O}$ the family of all odd cycles and by $\mathcal{T}_n$ the family of all trees on n vertices.

We denote by $s_1(n, H, F)$ the score of the game when both players play optimally and Max is the first player, and by $s_2(n, H, F)$ when Max is the second player. We sometimes use s instead of $s(n, H, F)$ when it is clear which game we are talking about. If $s_1 = s_2$ we use s instead of $s = s_1 = s_2$. As well when we have that score in both cases is between a and b, we use the notation $a < s < b$ instead of $a < s_1 < b$ and $a < s_2 < b$.

We will need several results from [109].

Theorem 5.1.1 ([109])**.** $s_1(2k, K_2, \mathcal{O}) = s_2(2k, K_2, \mathcal{O}) = k^2$.

Theorem 5.1.2 ([109])**.** *If* $n \geq 3$, *then* $s_1(n, K_2, \mathcal{T}_n) = s_2(n, K_2, \mathcal{T}_n) = \binom{n-2}{2} + 1$, *except that* $s_1(5, K_2, \mathcal{T}_5) = 6$ *and* $s_2(4, K_2, \mathcal{T}_4) = 3$.

Theorem 5.1.3 ([109])**.**

$$s_1(n, K_2, K_{1,3}) = \begin{cases} n, & \textit{when } n \in \{3, 7\} \cup 2\mathbb{N} - \{2\} \\ n - 1, & \textit{otherwise.} \end{cases}$$

$$s_2(n, K_2, K_{1,3}) = \begin{cases} n - 1, & \textit{when } n \in 2\mathbb{N} - \{4\} \\ n, & \textit{otherwise.} \end{cases}$$

5.2 Determining the score $s(n, K_3, P_5)$

Proof of Theorem 2.3.1:

First, note that the only way to have more than one triangle in a connected component C of a graph that does not have a copy of P_5 is if C is a subgraph of K_4.

Now, we give a strategy for Max. Assume first that he is the second player. If there is at least one isolated vertex after the move of Mini, he follows Stage 1, otherwise, he proceeds to Stage 2.

Stage 1. Max chooses the first of the following three rules that is satisfied, depending on the type of the connected component C Mini played in her last move:

1. C is a K_2. Max tries to claim an isolated K_2. If that is not possible, he creates a P_3 connecting C with one isolated vertex and then proceeds to Stage 2.

2. C is a P_3. Max claims the edge that completes the triangle.

3. C has four vertices:

   (a) If C is a P_4, Max claims the edge that completes the C_4.

   (b) Else, if there is a free edge in that component Max claims it.

   (c) Else, if there are at least two isolated vertices Max claims an isolated K_2.

   (d) Else, if there is one isolated vertex, and on top of that either an isolated edge or an isolated triangle, Max claims the edge that creates a P_3 or a G_4^1, respectively, and proceeds to Stage 2.

   (e) Otherwise, Max proceeds to Stage 2.

Stage 2. Max repeatedly plays any legal move to the end of the game.

We now analyze the given strategy. It is clear that Max can follow his strategy in Stage 2.

Claim 5.2.1. *During Stage 1, after Max has finished his move, every non-trivial connected component is one of the following:* $K_2, C_3, C_4, K_4 - e$ *or* K_4.

Proof. We prove this by using mathematical induction. After the first move of Max, the graph consists of two isolated edges. Now, we suppose that the assertion is true after k rounds. Depending on Mini's $(k+1)$-st move we have the following options:

- If Max applied rule 1, there are two more isolated edges in the graph.

- If Max applied rule 2, there is one isolated edge less and one triangle more than in the previous round.

- If Max applied rule 3a, there are two isolated edges less and one C_4 more than in the previous round. Note that Mini could not make a P_4 adding an edge on P_3 because of the strategy of Max.

- If Max applied one of the rules 3b, 3c, 3d or 3e, that means Mini added an edge to one of the following connected components: C_3, C_4 or $K_4 - e$ and after Max's response we get $K_4 - e$, K_4 or $K_4 + K_2$ in that order.

This proves the assertion of the claim. □

Note that during Stage 1 it is not possible that Mini creates a connected component that is a triangle, because of rule 2. For the same reason, it is not possible to have a connected component $K_{1,3}$ in the graph. Moreover, the only connected components on three or four vertices that can be transformed into a connected component on five vertices by adding one edge (without making a copy of P_5) are $P_3, P_4, K_{1,3}$ and G_4^1. Following the strategy from Stage 1 it is not possible to have any of these components when Mini is to move, as we can see from the proof of Claim 5.2.1. Therefore, during Stage 1 there is no option for Mini to make a connected component on more than four vertices. Now it is clear that Stage 1 covers each of Mini's moves.

It remains to count the number of triangles at the end of the game.
At the moment Stage 2 is triggered, besides components provided by Claim 5.2.1, there can be at most one of the following three components: P_3, G_4^1, or an isolated vertex.

- In case there is an isolated vertex, that means Max played by rule 3e and each of the remaining connected components in the graph is one of the following: C_4, $K_4 - e$ or K_4. Hence, by the end of the game, there will be $4\frac{n-1}{4} = n - 1$ triangles.
- In case there is one G_4^1, there are no isolated vertices in the graph. Therefore, this component must become a K_4 in the rest of the game.
- In case there is one P_3, that means that Max played by rule 3d or rule 1 and there are no more isolated vertices in the graph. Here, Mini can make a connected component on five vertices making the central vertex of this path adjacent to one end of an isolated edge. If that happens, just one edge can be added to this component, the one that closes a triangle.

Later in Stage 2, the only components that can change the number of vertices are the pairs of isolated K_2 that can be transformed into a K_4 by the end

of the game. Therefore, the worst scenario for Max is that at the end of the game, there is one component on 5 vertices, one isolated edge that cannot be added to any of the components, and all of the remaining components are triangles. This implies $s_2(n, K_3, P_5) \geq \dfrac{n-4}{3}$.
Note that the same strategy works when Max is the first player, hence $s_1(n, K_3, P_5) \geq \dfrac{n-4}{3}$.

Now, we give a strategy for Mini. Let us first suppose that she is the first player.

If there is at least one isolated vertex before the move of Mini, she follows Stage 1, otherwise, she proceeds to Stage 2.

Stage 1. Mini chooses the first satisfied of the following three conditions.

1. If there is a connected component on four vertices in the graph,
   Mini chooses a vertex of maximum degree in this component and connects it to an isolated vertex. (Mini makes a connected component on five vertices.)

2. If there is a K_2 component in the graph,
   Mini claims an edge that connects it with an isolated vertex, creating a P_3.

3. If there are two isolated vertices, Mini connects them, otherwise, she proceeds to Stage 2.

Stage 2. Mini repeatedly plays any legal move to the end of the game.

Now we analyze the given strategy. First, we give the following claim that will be used in this proof.

Claim 5.2.2. *During Stage 1, after Mini has finished her move, the graph consists of the following connected components: at most one K_2, a number of P_3, a number of K_3 and a number of connected component on at least five vertices.*

Proof. We prove this by using mathematical induction. After the first move of Mini, the graph consists of one isolated edge. Now, we suppose that the assertion is true after k Mini's moves. After k moves of Max, the graph consists of the following connected components: at most two K_2, a number of P_3, a number of K_3, at most one component from the set $\{K_{1,3}, P_4, G_4^1\}$ and a number of connected components on at least five vertices. Now we compare the number of components after k-th and $(k+1)$-st Mini's move. In his k-th move Max played in a connected component C, where:

- If C is a K_2, using rule 2, Mini ensures that after her $(k+1)$-st move the number of K_2 components is at most one, the number of P_3 components is larger by one, while the remaining components stay the same.
- If C is a P_3, after the move of Mini played by rule 3 she ensures that only the number of P_3 components increased by one.
- If C is a K_3, after Mini applied one of the rules 2 or 3 she ensures that only the number of components K_2, P_3 and K_3 can be changed by at most one, provided that the number of K_2 components is at most one.
- If C is in $\{K_{1,3}, P_4, G_4^1\}$, after the move of Mini played by rule 1, she ensures that there is no connected component on four vertices. Therefore, the number of connected components on at least five vertices is increased by one, while the number of components P_3 or C_3 is decreased by one.

This proves the assertion of the claim. $\square$

Note that each of the connected components on four vertices that cannot be transformed into a bigger one as described in rule 1 (these are the components that belong to the set $\{C_4, K_4 - e, K_4\}$) was already a connected component on four vertices before the last edge was added. Using the proof of Claim 5.2.2, we conclude that one of these components cannot occur when Mini is to move.

At the moment Stage 2 is triggered, there are two options:

1. There is no isolated vertex in the graph.
2. There is one isolated vertex, but there is neither an isolated edge nor a connected component on four vertices.

If there is one isolated vertex that means that each of the connected components is on three vertices or at least five vertices. The worst case for Mini is when each of the connected components is on three vertices, then it is possible that at most one of them becomes K_4, by the end of the game. That implies $s_2(n, K_3, P_5) \leq \dfrac{n-4}{3} + 4$.

Otherwise, if there is no isolated vertex, the graph consists of the following connected components: at most two K_2, a number of P_3, a number of K_3, at most one component from the set $\{K_{1,3}, P_4, K_3 + e\}$ and a number of connected components on at least five vertices. Note that if there are two isolated edges then there is no connected component on four vertices and vice versa. Since connected components on three vertices cannot be extended by one vertex, there can be at most one K_4 by the end of the game. Therefore, $s_2(n, K_3, P_5) \leq \dfrac{n-4}{3} + 4$. This concludes the proof of the theorem.

As the strategy is the same when Mini is the second player, the same upper bound works for s_1. □

5.3 All cycles are forbidden

In the following two theorems, the forbidden graphs are all cycles. That means that during the game the graph is a forest and when there are no more legal moves the graph must be a spanning tree.

Before we prove Theorem 2.3.2 we give the following Lemma that we will use in the proofs of this section.

Lemma 5.3.1. *Let T be a tree on $\lfloor \frac{n}{2} \rfloor + 1$ vertices. If the players alternately claim edges of K_n and neither is allowed to play a cycle, a player can build T.*

Proof. During the game, a player that we call TreeBuilder builds a connected component C such that at the end of the game this component will contain a tree isomorphic to T. After each of his moves, C will contain a subgraph of T, let us call it T', which is by at least one vertex bigger than it was after the previous round. The game begins with n components, each of them a vertex. Before TreeBuilder plays his first move, he chooses a vertex that will serve as a root r of T'. If he is the first player, r is an arbitrary vertex and $C = r$, otherwise, r is a vertex incident to the claimed isolated edge e and

$C = e$. Then, in each of his moves, TreeBuilder adds one component to C thus adding one edge and one vertex to the growing tree in C, and decreases the number of components by one. Each of the Opponent moves is joining two components into one, so he also decreases the number of components by one in each move.

TreeBuilder needs at most $\lfloor \frac{n}{2} \rfloor$ rounds to build a tree that is isomorphic to T. When he has to play the last move ($\lfloor \frac{n}{2} \rfloor$-th or $(\lfloor \frac{n}{2} \rfloor - 1)$-st), by that time Opponent has played $\lfloor \frac{n}{2} \rfloor - 1$ moves and decreased the number of components by the same value. Based on the number of moves it follows that there must be at least one component different from C that TreeBuilder can join to C, thus completing a tree isomorphic to T in his last move. □

Proof of Theorem 2.3.2:

First, we look at a strategy for Max. Using Lemma 5.3.1 we know that Max can build a star on $\lfloor \frac{n}{2} \rfloor + 1$ vertices, regardless of whether he is the first or the second player. Therefore, $s(n, S_k, Cycle) \geq \binom{\lfloor \frac{n}{2} \rfloor}{k-1}$.

Now, we give a strategy for Mini. First assume that Mini is the second player. After Max plays his first edge $e_1 = xy$, Mini claims an isolated edge e_2. Then we have two cases depending on the second move of Max.

1. Max claimed an edge that completes a P_4. Mini chooses the end of e_2 with degree 2 to be the root r and then follows the strategy for TreeBuilder (as a second player) from Lemma 5.3.1 on $V \backslash \{x, y\}$ to build a path P on at least $\lfloor \frac{n-2}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor$ vertices.

2. Otherwise, Mini chooses one end of an isolated edge for root r, w.l.o.g. assume that edge is e_2. Let us denote by C_1 a connected component that contains e_1. Applying Lemma 5.3.1 on $V \backslash \{x, y\}$ Mini can build a path P on at least $\lfloor \frac{n}{2} \rfloor$ vertices, such that the component C_1 joins the last one, adding one vertex of degree one to P if needed.

It is clear that Mini can follow her strategy. Now we have to count the number of S_k at the end of the game. Denote by S the biggest star at the end of the game.

If Mini applied rule 1, there are at most $\lceil \frac{n}{2} \rceil$ vertices that are not in P. The worst case for Mini is when all vertices that are not in P belong to S, then

r must be a leaf of S and the number of leaves is at most $\lceil \frac{n}{2} \rceil$. Otherwise, if P and S share 2 or 3 common vertices, one or two vertices from the set $\{x, y\}$ is neither in P nor S, respectively. Therefore, again there are at most $\lceil \frac{n}{2} \rceil$ leaves in S. Hence, $s_1(n, S_k, Cycle) \leq \binom{\lceil \frac{n}{2} \rceil}{k-1}$.

Otherwise, if Mini applied rule 2, at the end of the game, path P has at least $\lfloor \frac{n}{2} \rfloor$ vertices. As in the previous case the worst scenario for Mini is that all the remaining vertices are in S, then P and S share at most one common vertex, so the number of leaves in S is at most $\lceil \frac{n}{2} \rceil$. Otherwise, if P and S share 2 or 3 common vertices, there are at least two vertices that are neither in P nor S. Hence, again the number of leaves in S is at most $\lceil \frac{n}{2} \rceil$, and $s_1(n, S_k, Cycle) \leq \binom{\lceil \frac{n}{2} \rceil}{k-1}$.

When Max is the second player strategy is analogue, so we again get that
$$s_2(n, S_k, Cycle) \leq \binom{\lceil \frac{n}{2} \rceil}{k-1}.$$
□

Proof of Theorem 2.3.3: First, we define a double star as a graph that is formed by two stars, each of them with at least three edges and their centers are joined by an edge. If the difference between the degree of those centers is at most one, we say that the double star is *balanced*.

Now, we look at a strategy for Max. Using Lemma 5.3.1 we know that Max can build a balanced double star on $\lfloor \frac{n}{2} \rfloor + 1$ vertices, regardless of whether he is the first or the second player. Therefore, the number of leaves in that double star is $\lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n-2}{2} \rfloor$.

Claim 5.3.2. *Suppose that during the game the graph contains a balanced double star $S_{x,x}$ with $2x$ leaves, and y isolated vertices. The game ends at the moment when the graph becomes a spanning tree. At the end of the game, the smallest number of P_4's that are not contained in $S_{x,x}$ is $x + y - 1$.*

Proof. Note that an edge adjacent to a leaf of the double star $S_{x,x}$ increases the number of P_4's by x. Let us denote with N the number of P_4's that are not contained in $S_{x,x}$ at the end of the game.

If all these y vertices form one path that is adjacent to a leaf of the double star $S_{x,x}$, then $N = x+y-1$. Otherwise, if we have a leaf of the double star $S_{x,x}$ as a center of a new star on y leaves, then $N = xy$. In all the remaining cases where y vertices do not form a new double star $N = x+y-1$, otherwise when a new double star appears $N > x + y - 1$, that proves the assertion of the claim. □

Now, we count the number of P_4's at the end of the game. The number of P_4's in the double star is at least $\left\lfloor \frac{\lfloor \frac{n-2}{2} \rfloor}{2} \right\rfloor \left\lceil \frac{\lfloor \frac{n-2}{2} \rfloor}{2} \right\rceil$. Using Claim 5.3.2 we know that the smallest number of P_4's in the rest of the graph is $x + y - 1$, where $x = \left\lfloor \frac{\lfloor \frac{n-2}{2} \rfloor}{2} \right\rfloor$ and $y = \lceil \frac{n-2}{2} \rceil$. Calculating this we get that $s(n, P_4, Cycle) \geq \dfrac{n^2}{16} + \dfrac{n}{8} - \dfrac{39}{16}$.

Next, we look at Mini's strategy. Using TreeBuilder's strategy from Lemma 5.3.1 Mini can build a path P on $\lfloor \frac{n}{2} \rfloor + 1$ vertices, regardless of whether she is the first or the second player. Therefore, the number of vertices that do not belong to P is $\lceil \frac{n}{2} \rceil - 1$. The worst scenario for Mini is when all the remaining vertices are in the same balanced double star $S_{\lfloor x \rfloor, \lceil x \rceil}$ (It is easy to verify that the graph with the largest number of P_4's is a balanced double star). There can be at most four vertices that are at the same time in both P and $S_{\lfloor x \rfloor, \lceil x \rceil}$. Therefore, the double star $S_{\lfloor x \rfloor, \lceil x \rceil}$ can have at most $\lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n+2}{2} \rceil$ leaves. Hence, the number of P_4's at the end of the game is at most $\lfloor x \rfloor \lceil x \rceil + 2\lceil x \rceil + y - 1$, where $x = \dfrac{\lceil \frac{n+2}{2} \rceil}{2} = \left\lceil \frac{n+2}{4} \right\rceil \leq \frac{n}{4} + \frac{5}{4}$ and $y = \lfloor \frac{n}{2} \rfloor - 3 \leq \frac{n}{2} - 3$. Calculating this we get that $s(n, P_4, Cycle) \leq \dfrac{n^2}{16} + \dfrac{15n}{8} + \dfrac{21}{16}$. □

5.4 P_4 is forbidden

First, we observe what happens when the forbidden graph is P_4.

Note that here the graph at the end of the game must be a disjoint union of stars with at least three leaves, triangles, and isolated edges. There can be at most one isolated vertex in case all the remaining nontrivial connected components are triangles. Moreover, during the game, if two components are combined into one, one of them must be an isolated vertex. Hence, when there are no isolated vertices, the only legal moves are edges within the component.

We observe the game where we count the number of paths. It is clear that the only path that can occur is a P_3.

Proof of Theorem 2.3.4:

Note that, informally speaking, in this game Max desires a "big star" with as many leaves as he can achieve, while Mini wants as many isolated edges and triangles. The following strategy applies regardless of who starts the game.

Mini's strategy.

Stage 1

1. If there is a P_3 component, Mini claims the edge that closes a triangle.
2. Otherwise, Mini tries to claim an isolated K_2, if that is not possible, she creates a P_3 connecting an isolated edge with an isolated vertex.

Note that when there are no more isolated vertices, the only legal move is turning a P_3 into a triangle. Following her strategy Mini closes triangles or claims an isolated edges, and in her last move, she can make a P_3 component. If that happens, there are no more isolated vertices, so Max has to claim the edge that closes the triangle and the game is over. Playing this way Mini ensures that there is no connected component on more than three vertices. Therefore, at the end of the game, the graph is a disjoint union of triangles and isolated edges, or a disjoint union of one isolated vertex and triangles. Hence, $s(n, P_3, P_4) \leq n$.

Max's strategy.

Stage 1 If there are more than 12 isolated vertices, Max chooses the first satisfied rule of the following three rules:

1. If there is a component that is an $S_k, k \geq 3$, Max claims an edge incident with the central vertex of that star and one isolated vertex;
2. If there is a component that is a K_2, Max creates a P_3 connecting that edge with an isolated vertex.
3. Otherwise, Max claims an isolated K_2.

Stage 2 If there is at least one isolated vertex, Max chooses the first satisfied of the following three rules:

1. If there is a component that is an $S_k, k \geq 4$, Max claims an edge incident with the central vertex of that star and one isolated vertex;

2. If there is a component that is a K_2, Max creates a P_3 connecting that edge with an isolated vertex.

3. Otherwise, Max tries to claim an isolated K_2, or if that is not possible, proceeds to Stage 3.

Stage 3 Max repeatedly plays any legal move to the end of the game.

It is clear that Max can follow his strategy.

Now we have to count the number of P_3's at the end of the game. Before that, notice that if we have a star S_{l+1} and l isolated edges on one side, and on the other side a disjoint union of l triangles and one isolated vertex (the same number of vertices on both sides), there are more P_3's in the first case when $\binom{l}{2} > 3l$, i.e. $l > 7$.

When it is Max's turn to play in Stage 2 for the first time and he skips rule 1 that means that during Stage 1, rule 1 has not been played. Hence, every P_3 component created by Max, Mini turned into a triangle. Therefore, after the last move of Max, the graph was a union of disjoint triangles and a P_3 component or K_2 component (or both of them). Hence, following his strategy Max ensures that at the end of the game, there are either at most two isolated edges and all the remaining connected components are triangles, or there is at most one isolated vertex and all the remaining connected components are triangles, or there is a star with l leaves (the biggest one) and at most $l-2$ isolated edges (where $l \leq 8$). Hence, $s(n, P_3, P_4) \geq n-4$.

Otherwise, it is Max's turn to play in Stage 2 for the first time and he has to play by rule 1, he will repeat that rule until there are no more isolated vertices. Hence, there has to be one star S_l, $l \geq 8$, and at most l isolated edges. As we explained above, there has to be more P_3's than in case the graph was a union of disjoint triangles. Hence, $s(n, P_3, P_4) \geq n$. □

Proof of Observation 2.3.5: *Mini's strategy:* Using Mini's strategy from Theorem 2.3.4, Mini ensures that at the end of the game, the only non-trivial connected components in the graph are triangles and isolated edges. Therefore, $s(n, S_k, P_4) = 0$, $k \geq 4$. □

Proof of Theorem 2.3.6: In this game, Max wants as many triangles as possible, while Mini wants as many isolated edges or a star with many leaves (at least 3). The following strategy applies regardless of who is the first player.

Mini's strategy when n is even.

- If Max played in a component that is an S_l, where $l \geq 2$ (after his move), Mini turns it into an S_{l+1} adding one isolated vertex as a leaf.
- Otherwise, if Max claims an isolated K_2, Mini does the same.

Note that one round (move of Max followed by the move of Mini) decreases the number of isolated vertices by 2 or 4. Hence the number of isolated vertices after each move of Mini is even and the strategy can be followed until the end of the game. Therefore, there is no triangle in the graph and $s(n, C_3, P_4) = 0$.

Mini's strategy when n is odd.

Stage 1 If there is at least one isolated vertex, Mini chooses the first satisfied of the following two rules:

1. If there is a P_3 component, Mini claims an edge connecting its central vertex with an isolated vertex creating an S_4.
2. Otherwise, Mini tries to claim an isolated K_2. If that is not possible and there is an S_l, $l \geq 4$ she claims an edge incident with the central vertex of that star and the isolated vertex (making an S_{l+1}), otherwise she proceeds to Stage 2.

Stage 2 Mini repeatedly plays any legal move to the end of the game.

It is clear that Mini can follow her strategy. During Stage 1 she ensures that there is no triangle in the graph. When it is Mini's turn to play in Stage 2 and there is one isolated vertex, all the remaining components are isolated edges. Then, Mini has to complete a P_3 and the last move is reserved for Max to claim the edge that closes a triangle, hence $s(n, C_3, P_4) \leq 1$. Otherwise, if Stage 2 is triggered and there are no more isolated vertices, the graph is a disjoint union of stars S_l, $l \geq 3$, isolated edges, and at most one P_3 component. If the last component occurs, Mini has to claim the edge that closes a triangle, so again $s(n, C_3, P_4) \leq 1$.

Max's strategy when n is odd.

- If Mini played in a component that is a P_3 after his move, Max claims the edge that closes a triangle.

- Otherwise, if Mini claims an isolated K_2, Max tries to do the same, or if that is not possible plays any legal move.

If during the game Mini completes a P_3 component, following his strategy Max completes a triangle. Otherwise, the graph is a matching and at the very end of the game, there is one isolated vertex. Regardless of which player is to move, he has to make a P_3 component. That component has to become a triangle in the following move. Therefore, $s(n, C_3, P_4) \geq 1$, and that proves the assertion. $\square$

5.5 All odd cycles are forbidden

First, we count the number of even paths P_{2k} and cycles C_{2k} in the complete bipartite graph on $2n$ vertices. The largest number of P_{2k} and C_{2k} in a complete bipartite graph has been reached when both partitions are equal. That can be verified by calculating these values using combinatorial counting and verifying that these expressions are maximal when both partitions are equal, and that is $\left(\frac{n!}{(n-k)!}\right)^2$ and $\frac{1}{k}\left(\frac{n!}{(n-k)!}\right)^2$, respectively.

Proof of Observation 2.3.7: In these games, Max wants to get as close as possible to $K_{n,n}$, while Mini wants an asymmetric bipartite graph. Max has a strategy to create $K_{n,n}$ following the strategy from the proof of Theorem 5.1.1, regardless of who is the first player. Therefore,
$s(2n, P_{2k}, \mathcal{O}) = \left(\frac{n!}{(n-k)!}\right)^2$ and $s(2n, C_{2k}, \mathcal{O}) = \frac{1}{k}\left(\frac{n!}{(n-k)!}\right)^2$. $\square$

Then, we count the number of S_k in the bipartite graph on $2n$ vertices. Note that the number of S_k in the bipartite graph with partitions of x and $2n-x$ vertices is

$$x\binom{2n-x}{k-1} + (2n-x)\binom{x}{k-1}, \text{ where we suppose that } x < n.$$

This function reaches its maximum for $x = 1$ if $k > n$, or when x is an integer between $\frac{2n}{k-1} - 1 \leq x < \frac{2n}{k-1}$, otherwise. That can be verified by calculating these values using combinatorial counting. Since n is big enough, we are primarily interested in the case when $k \leq n$. The minimum of this function is reached when $x = n$.

Proof of Observation 2.3.8: In this game, Mini is the player who wants balanced partitions. Therefore, she can use Max's strategy from Theorem 5.1.1 and make a $K_{n,n}$ regardless of who plays first. Therefore, $s(2n, S_k, \mathcal{O}) = 2n\binom{n}{k-1}$. □

5.6 T_n is forbidden

In the following observation, we look at generalized saturation games where the forbidden graphs are all spanning trees $\mathcal{T}_n$. Note that at the end of the game, the graph must be a disjoint union of two complete graphs K_r and K_{n-r}.

We count the number of paths P_k, cycles C_k and stars S_k in $K_r + K_{n-r}$ and we get

$$\frac{k!}{2}\left(\binom{r}{k} + \binom{n-r}{k}\right), \quad \frac{(k-1)!}{2}\left(\binom{r}{k} + \binom{n-r}{k}\right),$$

and

$$r\binom{r-1}{k-1} + (n-r)\binom{n-r-1}{k-1},$$

respectively. Each of these functions reaches its maximum for $r = 1$, and its minimum for $r = \lfloor \frac{n}{2} \rfloor$. That can be verified by calculating these values using combinatorial counting.

Proof of Observation 2.3.9:

In each of these games, Max wants to achieve $K_1 + K_{n-1}$, while Mini wants a union of two complete graphs on $\approx \frac{n}{2}$ vertices. From the proof of Theorem 5.1.2 we know that Max has a strategy to make $K_2 + K_{n-2}$ and Mini has a strategy to prevent an isolated vertex at the end of the game. Therefore, $s(n, P_k, \mathcal{T}_n) = \frac{k!}{2}\binom{n-2}{k}$, $s(n, C_k, \mathcal{T}_n) = \frac{(k-1)!}{2}\binom{n-2}{k}$ and $s(n, S_k, \mathcal{T}_n) = (n-2)\binom{n-3}{k-1}$. □

5.7 S_4 is forbidden

In the following, we observe generalized saturation games where the forbidden graph is a star S_4. Note that at the end of the game, the graph is a union of disjoint cycles and possibly either one isolated vertex or an edge (but not both).

First, we count the number of paths at the end of the game when both players play optimally. As counting P_2's is the same as counting edges, that has already been done in Theorem 5.1.3. Note that here the score of the game is bounded from above by n. We say that a player closes a cycle when he claims the edge that connects both ends of a path on at least three vertices.

We give the following strategy that we will use later.

Path extension strategy:
If Max is the first player he plays an isolated K_2, otherwise he completes a P_3 component. Then, if Mini closes a cycle, Max claims an isolated K_2 (starts a new path). Otherwise, Max claims an edge such that the path is by 2 vertices longer than it was after the last Max's move.

Proof of Theorem 2.3.10: As we know the structure of the graph at the end of the game, it is clear that
$s(n, P_3, S_4) \geq n - 2$. Therefore, Max wants to finish the game without an isolated vertex or an edge, while Mini wants an isolated edge or at least an isolated vertex.

We give the score for this game when $n \leq 7$, and for greater n we give the strategy for Max which reduces the problem to the cases when n is small.

When $n = 3$, the score is $s = 3$.

When $n = 4$, we have $s = 4$. If Max is the first player he completes a P_4 in his second move, otherwise, he completes a $2K_2$ in his second move, and the graph at the end of the game is a C_4.

When $n = 5$, if Max plays first $s_1 = 4$. Mini completes a P_3 in her first move, and in the following move, she closes a cycle. Otherwise, if Max is the second player he completes a $2K_2$ in his second move, and after his following move he completes a P_5, hence in this case $s_2 = 5$.

When $n = 6$, if Max plays first $s_1 = 6$. Max completes a triangle or $3K_2$ in his second move. Otherwise, if Max is the second player, Mini completes a P_4 in her second move, and after her following move she closes a cycle, hence in this case $s_2 = 5$.

When $n = 7$, we have $s = 7$. Indeed if Max is the first player, he completes a triangle or a $3K_2$ in his second move. Otherwise, if Mini plays first, after Max's second move, the graph is either a $K_3 + K_2$ or a C_4.

When $n \geq 8$, Max follows the Path extension strategy while there are at least 7 isolated vertices after his move. When this condition is not satisfied he chooses one of the following two options.

1. There are 5 isolated vertices.
    - If Mini closes a cycle, the game is reduced to Max as the first player on 5 vertices.
    - If Mini claims an isolated K_2, Max closes a cycle and reduces the game to Max as the first player on 5 vertices. If that is not possible (the graph is a disjoint union of cycles, $2K_2$, and three isolated vertices), the game is reduced to Max as the first player on 7 vertices.
    - If Mini adds a new vertex to the path, Max closes a cycle and reduces the game to Max as the second player on 4 vertices.
2. There are 6 isolated vertices.
    - If Mini closes a cycle, the game is reduced to Max as the first player on 6 vertices.
    - If Mini claims an isolated K_2, Max closes a cycle and reduces the game to Max as the first player on 6 vertices. If that is not possible (the graph is a disjoint union of cycles, $2K_2$, and four isolated vertices), Max completes a P_4 connecting two isolated edges, and after his following move he turns it into either a C_4+K_2 or a C_5 and the game is reduced to Max plays first on 4 vertices or Max plays first on 3 vertices, respectively.
    - If Mini adds a new vertex to the path, Max closes a cycle and reduces the game to Max as the second player on 5 vertices.

It is clear that Max can follow the Path extension Strategy. Considering this strategy, we conclude that the number of isolated vertices is decreased by 2 per round, except the first one (the move of Mini followed by the move of Max). Therefore, Max finishes following this strategy when there are 5 or 6 isolated vertices and it is Mini's turn to play. At that moment except for isolated vertices, the graph contains disjoint cycles and one path. For the same reason, if Max applied Case 2 that means he is the first player and n is even or he is the second player and n is odd. As we can see in this case the score is n regardless of Mini's choice, and that proves the second part of the theorem. If Max applied Case 1, $s \geq n-1$ and the assertion is proved. □

Proof of Theorem 2.3.11:

In this game, informally speaking, Max desires "big cycles" C_l where $l \geq 4$, while Mini wants triangles. The idea of the proof is the same as in the previous theorem.

Cases $n = 4$ and $n = 5$ are the same.

When $n = 6$, the score is $s = 4$. If Max is the first player Mini completes a P_3 in her second move and in his following move she closes a cycle. Therefore, the best scenario for Max is that the graph at the end of the game is $C_4 + K_2$. Otherwise, if Max is the second player, Mini completes either a triangle or $3K_3$ in her second move, whereas in the latter case, Mini completes a C_4 in her following move. Hence, again for Max the best option is the graph $C_4 + K_2$ at the end of the game.

When $n = 7$, the score is $s = 4$. Mini's strategy is analog to the previous case.

Now, we give a strategy for Max when $n \geq 8$. First, we consider the case where Max is the first player. He follows the Path extension Strategy while there are at least 7 isolated vertices after his move.

1. There are 5 isolated vertices.
    - If Mini closes a cycle, the game is reduced to Max as the first player on 5 vertices.
    - If Mini claims an isolated K_2, Max closes a cycle and reduces the game to Max as the first player on 5 vertices. If that is not possible (the graph is a disjoint union of cycles, $2K_2$, and three isolated vertices), the game is reduced to Max as the first player on 7 vertices.

- If Mini adds a new vertex to the path, Max closes a cycle and reduces the game to Max is the second player on 4 vertices.

2. There are 6 isolated vertices.

   - If Mini closes a cycle, the game is reduced to Max as the first player on 6 vertices.
   - If Mini claims an isolated K_2, Max closes a cycle and reduces the game to Max as the first player on 6 vertices. If that is not possible (the graph is a disjoint union of cycles, $2K_2$, and four isolated vertices), Max completes a P_4 connecting two isolated edges, and after his following move, he turns it into either C_4+K_2 or P_6. The game is reduced to Max plays first on 4 vertices in the first case, or in the latter case, he plays any legal move until the end of the game.
   - If Mini adds a new vertex to the path, Max closes a cycle and reduces the game to Max as the second player on 5 vertices.

Analogously to the proof of Theorem 2.3.10, the Path extension Strategy ends when there are 5 or 6 isolated vertices. At that moment the graph is a disjoint union of cycles on more than three vertices and one path (it cannot be a P_3, because the path is on an even number of vertices after Max's move). Now it is clear that Max can follow his strategy until the end of the game and ensures that $s_1(n, P_4, S_4) \geq n - 3$.

Now we suppose that Max is the second player. In his first move he completes a $2K_2$, then depending on the move of Mini there are two cases for his second move:

1. If Mini claims an isolated K_2, Max claims the edge such that the graph is $P_4 + K_2$ after his second move, and proceeds to Stage 1.

2. Otherwise, Max completes a P_5 in his second move and then follows the same strategy as in the case when Max is the first player to the end of the game.

Stage 1: While there are at least 5 isolated vertices after Max's move, he follows these rules:

a) If Mini closes a C_4, the problem is reduced to Max as the second player (with 4 vertices less).

b) If after Mini's move, the graph is $P_4 + 2K_2$, Max claims an edge such that the graph is $C_4 + 2K_2$ after his move and the problem is again reduced to Max as the second player (from his second move and on 4 vertices less).

c) Otherwise, Max completes a P_7 and then follows the same strategy as in the case when Max is the first player to the end of the game.

If Max at any moment of the game proceeds to the strategy when Max is the first player, the graph at that moment contains a number of C_4 and one path, and the game has been finished with the same score as there, i.e $s_2(n, P_4, S_4) \geq n - 3$.

If that is not the case, Stage 1 has been finished after the move of Max when there are 3 or 4 isolated vertices. The reason for this is that the only possible rules that have been applied during this stage were a), b), or 1, and after each of them there were two isolated vertices less than after the previous round. Additionally, the graph at that moment is a disjoint union of C_4 and either $2K_2$ or $P_4 + K_2$.

For each of these cases, we can prove that the score is at least $n - 3$, these proofs are similar to those from the previous case (Max is the first player), so we omit them. Hence, $s_2(n, P_4, S_4) \geq n - 3$. □

Proof of Theorem 2.3.12: In this game, Max wants big cycles C_l where $l \geq 5$, while Mini wants a disjoint union of triangles and C_4's. Without making difference in who is the first player, we give a strategy for Mini.

Stage 1: We look at the graph and observe the set S of nontrivial connected components that are not cycles. While there is at least one isolated vertex, before each of her moves Mini chooses the first satisfied of the following three rules:

1. There is a P_i component in S, where $i \in \{3, 4\}$. Mini closes a C_i.

2. S is a disjoint union of isolated edges. Mini tries to claim an isolated K_2. If that is not possible, Mini connects one isolated edge with an isolated vertex creating a P_3.

3. $S = \emptyset$. Mini plays an isolated K_2.

Stage 2: There is no isolated vertex. Mini chooses one of the following options depending on the type of connected components that are contained in the graph. Except for cycles and a number of K_2 the remaining components are:

1. One P_3 and one P_4. Here Mini closes a C_4 and after the following Max's move the game is reduced to one of the cases: 1, 4 or 5.
2. One P_3. Mini closes a triangle and then until the end of the game she closes a C_4.
3. One P_4. Until the end of the game Mini closes a C_4.
4. One P_5. Mini closes a C_5 and then the game is reduced to 3.
5. None. Mini connects two isolated edges and completes a P_4 and then until the end of the game, she closes a cycle. If there are two paths, she closes one arbitrarily.

Now we prove that Mini can follow her strategy. It is clear that she can follow rules 1 and 2 from Stage 1. If it is Mini's turn to play by rule 3 and there is just one isolated vertex, all the remaining components are cycles, so the game is over. Therefore, it is clear that Mini can follow Stage 1.

After the last Mini's move played in Stage 1 the graph is a disjoint union of the following components, a number of C_3, a number of C_4, a number of K_2, and at most one P_3. Since a cycle cannot be connected to any other component, each possible scenario before the first Mini's move in Stage 2 is shown in the five cases given above.

If Mini applies one of the rules 2 or 3, it is clear that she can close the cycle and after that, only the pairs of isolated edges can be connected into one component, hence the only possible move for Max is to complete a P_4, and then Mini closes a C_4 and these moves will be repeated to the end of the game. Here the graph at the end of the game does not contain a P_5, hence the score is $s = 0$.

If Mini applies rule 4, the score is $s = 5$.

If Mini applies rule 5, that is the only possible move for her and after that, Max can complete another P_4 component or create a P_6. In both of these cases, Mini will close a cycle in her following move. If she closes a C_6 (that is possible just once), the game is reduced to 3. Otherwise, Mini closes a C_4 and repeats closing a cycle such that at the end of the game the graph contains at most one C_6 and all the remaining cycles have at most 4 vertices. Therefore, the score of the game is $s \leq 6$.

This proves the first part of the theorem.

Now we look at Max's strategy when he is the second player and $n = 4k$ or $n = 4k + 1$. The game stops at the moment when the first cycle on at least 5 vertices is made, showing that the score is at least 5.

While there is no cycle in the graph:

1. If there is a P_i component, where $i \geq 5$, Max closes a C_i and the game is over.

2. If Mini plays an isolated K_2, Max does the same.

3. Else, if Mini plays in a component that is a P_i after her move, Max adds one isolated edge to this path, making a P_{i+2}, or if there is no isolated edge he adds an isolated vertex, making a P_{i+1}.

When Max has to apply rule 3 for the first time, Mini has completed either a P_3 or a P_4 component. In case this component is a P_3, there has to be at least one isolated edge because the first round has been finished with two isolated edges. Otherwise, if the component is a P_4 there must be either an isolated edge or vertex. Therefore, after this move of Max, there must be a P_i component, where $i \geq 5$, so the game will be over in the next round with $s_2 \geq 5$.

If the graph is matching and there is at most one isolated vertex, it must be Mini's turn (because of the condition for n). Therefore, Mini has to complete either a P_3 or P_4, and in his following move, Max completes a P_i where $i \geq 5$, following rule 3, so the game will be finished in the next round with $s_2 \geq 5$. □

Before proving Theorem 2.3.13, we prove the following lemma.

Lemma 5.7.1. $s(n, P_6, S_4) = 0$.

Proof. In this game, Max desires big cycles C_l where $l \geq 6$, while Mini wants a disjoint union of triangles, C_4's and C_5's. Without making a difference in who is the first player, we give a strategy for Mini.

We look at the graph and observe just the set S of nontrivial connected components that are not cycles. Before each of her moves, Mini chooses the first satisfied of the following three rules:

1. There is a P_i component in S, where $i \in \{3, 4, 5\}$. Mini closes a C_i.
2. $S = 2K_2$. Mini completes a P_4, connecting these isolated edges.
3. $S = K_2$. Mini completes a P_3, connecting one isolated vertex to K_2.
4. $S = \emptyset$. Mini plays an isolated K_2.

The only possible cases where Mini cannot answer using her strategy are rules 3 and 4, but then the game is over because $S = K_2$ and there is no isolated vertex or $S = \emptyset$ and there is at most one isolated vertex and all the remaining connected components are cycles on 3,4 or 5 vertices. Therefore, when Mini cannot follow this strategy, the game is over. At the moment when Mini has finished her move, the graph contains, a number of C_3, a number of C_4, a number of C_5, and at most one component from the set $\{K_2, P_3, P_4\}$. Therefore, after the Max's move the set S is exactly one of the $\{\emptyset, K_2, 2K_2, P_3, P_4, P_5, P_3 + K_2, P_4 + K_2\}$. Following her strategy, Mini ensures that there is no cycle on more than 5 vertices in the graph. Hence, $s(n, P_6, S_4) = 0$. $\square$

Proof of Theorem 2.3.13: In the proof of Lemma 5.7.1 we saw that Mini has a strategy to finish the game without a cycle on more than 5 vertices in the graph. That proves the assertion. $\square$

Before we prove Theorem 2.3.14 we give several lemmas.

Lemma 5.7.2. $s(n, C_3, S_4) \leq 1$.

Proof. In this game Mini's strategy is to avoid a triangle. In the game where we count a P_4 strategy of Max is also to avoid triangles. Therefore, we can use Max's strategy from the Theorem 2.3.11. If Max is the first player, Mini follows the strategy when Max is the second player from the Theorem 2.3.11 and vice versa. At the end of the game, there will be at most one triangle and our assertion is proven. □

Lemma 5.7.3. $s(n, C_4, S_4) \leq 1$.

Proof. Mini's goal in this game is to avoid a C_4, i.e. she does not want to complete a P_4 component. We give Mini's strategy which is divided into two stages and works both if Max is the first or the second player.

Stage 1. We look at the graph and observe just the set S of all nontrivial connected components that are not cycles. Here we denote by P_2 an isolated edge and by P_0 an empty set. Before each of her moves, Mini chooses one suitable of the following five rules:

1. $S = \emptyset$. Mini claims an isolated P_2.

2. $S = P_k$, where $k = 2$ or $k \geq 4$. Mini tries to complete a P_{k+1} by adding one isolated vertex to the path. If this is not possible Mini proceeds to Stage 2.

3. $S = P_3$. Mini closes a C_3.

4. $S = P_k + P_j + P_r$, where $k \geq 3$, $j \in \{2, 3\}$ and $r \in \{0, 2\}$. Mini claims an edge that connects two of the longest paths making a $P_{k+j} + P_r$.

5. $S = 2P_2$. Mini tries to claim an edge such that after her move $S = P_3 + P_2$. If this is not possible Mini proceeds to Stage 2.

Stage 2. Mini repeatedly plays any legal move to the end of the game.

If Mini cannot fulfill rule 1, that means all connected components are cycles and there is at most one isolated vertex, so the game is over. Hence, Mini can follow her strategy. After each of her moves during Stage 1 in the set $V \backslash S$, there is no C_4 and the remaining nontrivial connected components in S can be one of the following, P_k where $k \geq 2$ and $k \neq 4$ or $P_k + P_2$ where $k \geq 3$ and $k \neq 4$. Then, after the move of Max one of the 5 rules given above will be satisfied, and because there is no P_4 component, she cannot make a C_4. If Stage 2 is triggered, that means either $S = P_k$ and there is no isolated vertex, hence Mini plays the last move and closes the cycle C_k, or

$S = 2P_2$, and Mini has to complete a P_4 component in her following move. Therefore, at the end of the game, the graph contains at most one C_4 and the assertion of the theorem is proven. ☐

Lemma 5.7.4. $s(n, C_5, S_4) \leq 1$.

Proof. In this game, Mini's strategy is to avoid a C_5. Therefore, we can use Mini's strategy from the Theorem 2.3.12 and finish the game with at most one C_5. That proves the assertion. ☐

Lemma 5.7.5. $s(n, C_k, S_4) = 0$*, where* $k \geq 6$*.*

Proof. Following Mini's strategy from the Lemma 5.7.1 this game can be finished without making any component that is a cycle on more than 5 vertices. Therefore, the score is $s(n, C_k, S_4) = 0$, when $k \geq 6$. ☐

Proof of Theorem 2.3.14: Combining lemmas 5.7.2, 5.7.3, 5.7.4 and 5.7.5 the proof of this theorem goes directly. ☐

5.8 Concluding remarks and open problems

Here we introduced generalized saturation games and found very interesting results for the score of several different games. We observed the games where the forbidden graph is one of the following, a cycle, a tree, a path on five or four vertices, and a star on four vertices. Therefore, all the remaining games are still open. Particularly, we are interested in finding the score of the game where the forbidden graph is a star on k vertices, and we count the number of stars on l vertices at the end of the game, where $l < k$.

Chapter 6

Concluding remarks

Throughout this thesis, we studied Combinatorial games on graphs and paid special attention to their subvariants, positional games, and saturation games. From a wide range of positional games, we were particularly interested in strong games. As we could see from the previous sections this thesis is divided into three major parts, strong Avoider-Avoider games, achievement number in Maker-Maker games, and generalized saturation games. Here, we provide concluding observations for each of these parts, along with open problems interesting for further study.

First, we studied strong Avoider-Avoider F games played on the edge set of the complete graph K_n. As we could already see, these games are often mentioned in the literature, and a lot of questions were asked about them, but only a few of them have been answered. The reasons for their inaccessibility lie in the lack of mathematical tools as well as in the fact that both players have the same goal, resulting in a small margin by which a game is won. It was previously known that Blue has a winning strategy in only two games of this type. In this thesis, we provide two new results, by giving the existence of a winning strategy for Blue.

We proved our results by using strategy stealing to verify that Blue can win. It would be interesting to develop other new mathematical tools that can be used for this kind of game. As we saw earlier, the outcome of four different games is known up to now, so finding the outcome for any other strong Avoider-Avoider F game would be interesting. In particular, is it true that Blue has a winning strategy in a strong Avoider-Avoider F game, where the forbidden graph F is: a triangle, a path on k vertices P_k, a cycle on k vertices C_k or a tree on k vertices T_k? Additionally, it would be interesting to either find a game of this kind where Red can win or to prove that Blue can always win.

We also introduced strong CAvoider-CAvoider F games, as a variant of the above-mentioned games, which were created as their natural extension. We prove that Blue can win in three different games of this kind, which means that all the remaining games still stay open.

Secondly, we observed the achievement number in strong Maker-Maker games played on the edge set of the complete graph K_n. More precisely, we wanted to find the smallest integer $a(F)$ such that Red can win this game played on the complete graph on $a(F)$ vertices. Since we know that $a(F) \leq R(F)$ holds, where $R(F)$ is the Ramsey number, it is interesting to determine for which graphs F we have $a(F) < R(F)$?

We gave a Red's winning strategy in the strong Maker-Maker game played on $a(F)$ isolated vertices, for some small graphs F. Then, we found the achievement number for several particular graphs, such as paths, cycles, and perfect matchings. We also gave an upper bound for the achievement number for the star S_n, which is $2n - 3$. As for the lower bound, we know that it is greater than n, and it would be interesting to find the exact value of the star achievement number, or at least to improve these bounds.

Moreover, we were interested in finding the achievement number for trees, and we managed to get an upper bound, whereas the lower bound is n, which has been reached for paths and one particular class of trees. We believe that the upper bound can be improved for a class of trees with a bounded maximum degree. Furthermore, Harary's conjecture is still open, and the response to this question requires more analysis of the achievement number for stars.

Finally, we studied generalized saturation games, and managed to find the score in several different games. We observed the games where the forbidden graph is one of the following, a cycle, a tree, a path on five or four vertices, and a star on four vertices, and we found their scores.

Taking into account that saturation games are a special case of the generalized saturation games (where $H = K_2$), as well as the games studied in this thesis, all the remaining games are still open. One of the open problems that looks particularly enticing is finding the score of the game where the forbidden graph is a star on k vertices, while the graph we count at the end of the game is a star on l vertices, where $l < k$.

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Овај Образац чини саставни део докторске дисертације, односно докторског уметничког пројекта који се брани на Универзитету у Новом Саду. Попуњен Образац укоричити иза текста докторске дисертације, односно докторског уметничког пројекта.

План третмана података

Назив пројекта/истраживања
Јаке позиционе игре / Strong positional games
Назив институције/институција у оквиру којих се спроводи истраживање
а) Факултет техничких наука, Универзитет у Новом Саду
Назив програма у оквиру ког се реализује истраживање
Истраживање се врши у оквиру израде докторске дисертације на студијском програму Математика у техници.
1. Опис података
1.1 Врста студије *Укратко описати тип студије у оквиру које се подаци прикупљају* **У овој студији нису прикупљани подаци.** 1.2 Врсте података а) квантитативни б) квалитативни

1.3. Начин прикупљања података

а) анкете, упитници, тестови

б) клиничке процене, медицински записи, електронски здравствени записи

в) генотипови: навести врсту ________________________________

г) административни подаци: навести врсту _______________________

д) узорци ткива: навести врсту_________________________________

ђ) снимци, фотографије: навести врсту_____________________________

е) текст, навести врсту ____

ж) мапа, навести врсту ______________________________________

з) остало: описати ____

1.3 Формат података, употребљене скале, количина података

1.3.1 Употребљени софтвер и формат датотеке:

a) Excel фајл, датотека __________________

b) SPSS фајл, датотека __________________

c) PDF фајл, датотека ___________________

d) Текст фајл, датотека __________________

e) JPG фајл, датотека ___________________

f) Остало, датотека ____________________

1.3.2. Број записа (код квантитативних података)

а) број варијабли ____

б) број мерења (испитаника, процена, снимака и сл.) ______________________

1.3.3. Поновљена мерења

а) да

б) не

Уколико је одговор да, одговорити на следећа питања:

а) временски размак између поновљених мера је ___________________________

б) варијабле које се више пута мере односе се на ____________________________

в) нове верзије фајлова који садрже поновљена мерења су именоване као _____________

Напомене: ____

Да ли формати и софтвер омогућавају дељење и дугорочну валидност података?

а) Да

б) Не

Ако је одговор не, образложити

____

2. Прикупљање података

2.1 Методологија за прикупљање/генерисање података

2.1.1. У оквиру ког истраживачког нацрта су подаци прикупљени?

а) експеримент, навести тип ____

б) корелационо истраживање, навести тип ____

ц) анализа текста, навести тип ____

д) остало, навести шта

____

2.1.2 Навести врсте мерних инструмената или стандарде података специфичних за одређену научну дисциплину (ако постоје).

____

____

2.2 Квалитет података и стандарди

2.2.1. Третман недостајућих података

а) Да ли матрица садржи недостајуће податке? Да Не

Ако је одговор да, одговорити на следећа питања:

а) Колики је број недостајућих података? __________________________

б) Да ли се кориснику матрице препоручује замена недостајућих података? Да Не

в) Ако је одговор да, навести сугестије за третман замене недостајућих података

____

2.2.2. На који начин је контролисан квалитет података? Описати

____

2.2.3. На који начин је извршена контрола уноса података у матрицу?

____

3. Третман података и пратећа документација

3.1. Третман и чување података

3.1.1. Подаци ће бити депоновани у ______________________________________ репозиторијум.

3.1.2. URL адреса

____

3.1.3. DOI

____

3.1.4. Да ли ће подаци бити у отвореном приступу?

а) Да

б) Да, али после ембарга који ће трајати до

____

в) Не

Ако је одговор не, навести разлог ____

3.1.5. Подаци неће бити депоновани у репозиторијум, али ће бити чувани.

Образложење

____

3.2 Метаподаци и документација података

3.2.1. Који стандард за метаподатке ће бити примењен? _________________________

3.2.1. Навести метаподатке на основу којих су подаци депоновани у репозиторијум.

____

Ако је потребно, навести методе које се користе за преузимање података, аналитичке и процедуралне информације, њихово кодирање, детаљне описе варијабли, записа итд.

____

3.3 Стратегија и стандарди за чување података

3.3.1. До ког периода ће подаци бити чувани у репозиторијуму? ________________

3.3.2. Да ли ће подаци бити депоновани под шифром? Да Не

3.3.3. Да ли ће шифра бити доступна одређеном кругу истраживача? Да Не

3.3.4. Да ли се подаци морају уклонити из отвореног приступа после извесног времена?

Да Не

Образложити

____

4. Безбедност података и заштита поверљивих информација

Овај одељак МОРА бити попуњен ако ваши подаци укључују личне податке који се односе на учеснике у истраживању. За друга истраживања треба такође размотрити заштиту и сигурност података.

4.1 Формални стандарди за сигурност информација/података

Истраживачи који спроводе испитивања с људима морају да се придржавају Закона о заштити података о личности (*https://www.paragraf.rs/propisi/zakon_o_zastiti_podataka_o_licnosti.html*) и одговарајућег институционалног кодекса о академском интегритету.

4.1.2. Да ли је истраживање одобрено од стране етичке комисије? Да Не

Ако је одговор Да, навести датум и назив етичке комисије која је одобрила истраживање

____

4.1.2. Да ли подаци укључују личне податке учесника у истраживању? Да Не

Ако је одговор да, наведите на који начин сте осигурали поверљивост и сигурност информација везаних за испитанике:

а) Подаци нису у отвореном приступу

б) Подаци су анонимизирани

ц) Остало, навести шта

____

5. Доступност података

5.1. Подаци ће бити

а) јавно доступни

б) доступни само уском кругу истраживача у одређеној научној области

ц) затворени

Ако су подаци доступни само уском кругу истраживача, навести под којим условима могу да их користе:

____

____

Ако су подаци доступни само уском кругу истраживача, навести на који начин могу приступити подацима:

____

____

5.4. Навести лиценцу под којом ће прикупљени подаци бити архивирани.

____

6. Улоге и одговорност

6.1. Навести име и презиме и мејл адресу власника (аутора) података

6.2. Навести име и презиме и мејл адресу особе која одржава матрицу с подацима

6.3. Навести име и презиме и мејл адресу особе која омогућује приступ подацима другим истраживачима
