UNIVERSITY OF NOVI SAD
FACULTY OF SCIENCES
DEPARTMENT OF
MATHEMATICS AND INFORMATICS

Nevena Dugandžija

Nonlinear Schrödinger equation with singularities

-doctoral dissertation-

Nelinearna Šredingerova jednačina sa singularitetima

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Preface

The nonlinear Schrödinger equation (NLS) is a model for various physical phenomena. For example, the cubic Schrödinger equation is a model for propagation of pulses in optical fibers. In three dimensions it represents the dynamics of interacting Bose gases. Other applications are related to gravitational small amplitude waves and the dynamics of quantum plasma. This is an important equation of quantum physics, so it is natural to examine singular initial conditions, such as the Dirac delta function. The cubic equation with the delta potential is a model for Bose – Einstein condensates.

The topic of the research is the cubic defocusing equation in two and three dimensions, with and without potential. The equation without potential is studied primarily in Sobolev spaces, where it has the property of energy conservation. In the dissertation we will deal with singular initial conditions and examine the existence and uniqueness in the Colombeau algebra. The equation with the delta potential is not studied in the classical sense, but its significance is seen in a large number of papers on solitons and explicit solutions. We will also study the Hartree equation with the delta potential in three dimensions in the Colombeau algebra and compare results with the existing ones.

The Colombeau algebra is suitable for examining nonlinear phenomena. Also, the delta function makes this problem difficult to observe in the classical Sobolev space. Introducing a net of solutions gives a tool for studying different kinds of convergence, so it can be useful in connecting singular and less singular solutions.

This dissertation is based on the results from [DN19]. We will demonstrate existence and uniqueness in the Colombeau algebra, also compatibility with the H^2 solution for the equation without potential. Specifically, if we have an initial condition in the Sobolev space (and here we know that there is well – posedness), we can construct a regularized equation. We prove that the net of solutions of this regularized equation converges to a H^2 solution. For the equation with the delta potential we further show existence and uniqueness in the appropriate Colombeau algebra. Since well – posedness in a Sobolev space is not known, we do not have a candidate for the limit of the net of the regularized equation. Finally, for the Hartree equation with the delta potential we have well - posedness in the fractional Sobolev space, so the goal is to investigate well – posedness in the Colombeau algebra and then to examine whether there is compatibility, that is convergence of the net of solutions towards this “classical” solution.

Predgovor

Nelinearna Šredingerova jednačina (NLS) je model za različite fizičke fenomene. Na primer, kubna Šredingerova jednačina je model za propagaciju pulseva u optičkim vlaknima. U tri dimenzije, ona oslikava dinamiku interakcije Boze gasova. Druge primene su povezane sa gravitacionim talasima male amplitude i dinamikom kvantne plazme. Ovo je važna jednačina kvantne fizike, te je prirodno ispitati singularne početne uslove, kao što je Dirakova delta funkcija. Kubna jednačina sa delta potencijalom je model za Boze - Ajnštajnove kondenzate.

Tema ovog istraživanja je kubna defokusirajuća jednačina u dve i tri dimenzije, sa i bez potencijala. Jednačina bez potencijala je proučavana primarno u prostorima Soboljeva, gde ima svojstvo očuvanja energije. U disertaciji ćemo se baviti singularnim početnim uslovima i ispitati postojanje i jedinstvenost u Kolomboovoj algebri. Jednačina sa delta potencijalom nije proučavana u klasičnom smislu, ali njen značaj ogleda se u velikom broj u radova na temu solitona i eksplicitnih rešenja. Takodje ćemo proučiti Hartrijevu jednačinu sa delta potencijalom u tri dimenzije u Kolombo algebri i uporediti rezultate sa postojećim na tu temu.

Kolombo algebra je pogodna za ispitivanje nelinearnih fenomena. Takođe, delta funkcija čini ovaj problem teškim za posmatranje u klasičnim prostorima Soboljeva. Uvođenje mreže rešenja daje alat za proučavanje različitih vrsta konvergencije, te može biti korisno u povezivanju singularnih i manje singularnih rešenja.

Disertacija je bazirana na rezultatima iz [DN19]. Pokazaćemo postojanje i jedinstvenost rešenja u Kolombo algebri, kao i kompatibilnost sa H^2 rešenjem za jednačinu bez potencijala. Preciznije, ako ima početni uslov u prostoru Soboljeva (gde znamo da važi dobra postavljenost problema), možemo konstruisati regularizovanu jednačinu. Dokazaćemo da mreža rešenja regularizovane jednačine konvergira ka H^2 rešenju. Za jednačinu sa delta potencijalom ćemo predstaviti dokaz postajanja i jedinstvenosti rešenja u odgovarajućoj Kolombo algebri. Pošto je dobra postavljenost ovog problema u prostorima Soboljeva nepoznanica, nemamo kandidata za graničnu vrednost mreže rešenja regularizovane jednačine. Konačno, za Hartrijevu jednačinu sa delta potencijalom imamo dobru postavljenost u frakcionom prostoru Soboljeva, te je cilj ispitati dobru postavljenost u Kolomboovoj algebri, a zatim i pokazati kompatibilnost, to jest konvergenciju mreže rešenja ka ovom "klasičnom" rešenju.

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Contents

1 Introduction

In this work we concentrate on partial differential equations related to the equation that made Erwin Schrödinger famous and earned him the Nobel prize in 1933. Schrödinger (1887-1961) was an Austrian physicist and one of the several individuals who have been called "the father of quantum mechanics". In 1926 he published a paper in which he presented the linear equation, often written as

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi.$$

This paper was very influential in most areas of quantum mechanics. He went on to write four papers in a series and these papers were his central achievement.
Schrödinger is also famous for devising a thought experiment - the Schrödinger's cat, during a course of discussions with Albert Einstein. The scenario describes a paradox of a cat that can simultaneously be alive and dead and is a problem related to interpretation of quantum mechanics. It remains useful as a tool to compare and evaluate modern interpretations of quantum mechanics.

Today, a large body of theory exists on various types of Schrödinger equations. We are interested in the nonlinear ones and specifically in the theory of well - posedness. An interesting question is what will happen if an initial condition is very singular, or if the equation itself contains singular terms? Is there existence and uniqueness in these cases in certain spaces? Also, can these singular solutions be approximated with functions that are more regular? We hope to answer affirmatively to these questions.

1.1 Motivation and Problem Statement

We shall consider three Cauchy problems, the cubic equation:

$$\begin{aligned} iu_t + \triangle u &= u|u|^2, \\ u(0) &= a \end{aligned} \tag{1.1}$$

the cubic equation with the delta potential:

$$\begin{aligned} iu_t + \triangle u &= u|u|^2 + \delta u, \\ u(0) &= a \end{aligned} \tag{1.2}$$

and the Hartree equation with a delta potential:

$$\begin{aligned} iu_t + \triangle u &= (w * |u|^2)u + \delta u, \\ u(0) &= a \end{aligned} \tag{1.3}$$

The solution is a complex function of x and t: $u = u(x,t)$, where $t \in \mathbb{R}$, representing time, and $x \in \mathbb{R}^n$, where we consider mainly $n = 3$, but in some cases also $n = 2$. Also, $w : \mathbb{R}^n \to \mathbb{R}$ is a measurable function.

These equations are considered dispersive: intuitively, different frequencies tend to propagate at different velocities, thus dispersing the solution over time. In contrast to this, in the wave equation all frequencies move with the same velocity whereas the heat equation is considered dissipative, frequencies do not propagate but instead simply attenuate to zero. A solution to a linear Schrödinger equation $iu_t + \triangle u = 0$ is in the form

$$u(x,t) = Ae^{i\kappa x - iwt}$$

where the coefficients satisfy the *dispersion relation*

$$w = \kappa^2,$$

see [Tao06] and [Whi11] for more details.

Classical solutions of equation (1.1) have been studied extensively in the framework of Sobolev H^s spaces, where s is at least 0. For a summary of these results see [Bou99]. This equation is called defocusing, whereas the equation $iu_t + \triangle u + u|u|^2 = 0$ is called focusing. The critical regularity for global existence of solutions of (1.1) in three dimensions is in H^s for $s > \frac{4}{5}$, as is shown in [Col+04], for two dimensions it is $s \geq \frac{1}{2}$, see [FG07]. Also, it was shown in [KPV+01] that the one-dimensional cubic Schrödinger equation with the delta function as initial data is ill-posed in the class $L^\infty([0,\infty), \mathcal{S}'(\mathbb{R}))$.
On the other hand, there are no classical results in dimensions higher than one for the equation (1.2), but its significance as a model for Bose-Einstein condensates with a well potential is reflected in the large amount of papers regarding solitons, bound states and approximate solutions of (1.2), see for example [GHW04], [Le +08] and [HMZ07]. This motivates our study of the problem of singular solutions.

There are several papers dealing with the Schrödinger equation in the setting of the Colombeau algebra of generalized functions. In [Hör11], Hörmann solved the Cauchy problem in $\mathbb{R}^n$ for the linear Schrödinger equation with variable coefficients, provided the coefficients and initial data are generalized functions. In [Hör16], the convergence properties of regularized solutions to the linear equation were studied. In [Bu96], Bu showed that the cubic one-dimensional Schrödinger equation has a unique generalized solution.

Recently in [MOS18], well - posedness of the problem (1.3) in fractional Sobolev spaces was shown. It is of interest to us to see how this translates to a different type of setting, namely the Colombeau algebra which we will introduce in Chapter 2.

Equations (1.2) and (1.3) contain a product of the delta distribution and a function u. This product is a distribution if u is a smooth function. It is not defined for general distributions u, and this is one of the reasons of using a Colombeau type algebra. The delta function was first introduced by Paul Dirac in 1930. It is used to model the density of an idealized point mass or point charge as a function equal to zero everywhere except for zero and whose integral over the entire real line is equal to one

1.2 Thesis Structure

In **Chapter 2** we present the basic definitions, inequalities important for our work and also the function spaces needed for the analysis. This chapter includes the description of the Colombeau algebra - the setting we later use for the well-posedness problem.

In **Chapter 3** we present the theory of semilinear Schrödinger equations, following the works of Cazenave, Bourgain and many other authors that contributed to the field. This includes basics of semigroups of operators, Strichartz inequalities, and well - posedness of these equations in various spaces. We also give a description of previous results related to the three equations of interest.

In **Chapter 4** we present original results published in the paper [DN19] and concerning the existence and uniqueness of the solution for the two cubic equations (1.1) and (1.2). We show that the singular solution of (1.1) is compatible with the classical H^2 solution.

In **Chapter 5** we state the theory of singular (fractional) Sobolev spaces and results of well - posedness of the Hartree equation shown in [MOS18]. We prove a similar result in the setting of Colombeau algebra and discuss connections between solutions, these are the results from [DI21].

Chapter 6 is a summary of all of the results from the thesis. We discuss possible future work and future tasks.

2 Notation, definitions, function spaces

2.1 Notation and basic definitions

With $\mathbb{R}$, $\mathbb{N}$ i $\mathbb{C}$ we denote the set of real, natural and complex numbers, respectively. With $\mathbb{N}_0$ we denote the set $\mathbb{N} \cup \{0\}$.

For $x \in \mathbb{R}^n$ and multi - indices $\alpha = (\alpha_1, \dots, \alpha_n)$ i $\beta = (\beta_1, \dots, \beta_n)$ we use the standard notation:

- $|\alpha| = \alpha_1 + \cdots + \alpha_n$,
- $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$,
- $\alpha \leq \beta \iff \alpha_i \leq \beta_i$, $1 \leq i \leq n$.
- $x^\alpha = (x_1^{\alpha_1}, ..., x_n^{\alpha_n})$.

Then, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $1 \leq i \leq n$. If there is no risk of confusion we use just ∂^α for the derivative in $x \in \mathbb{R}^n$. Derivative in the time variable of the function u is often denoted by u_t, otherwise we use $\frac{\partial}{\partial t}$. Scalar product of vectors x and ξ is given by $x \cdot \xi = x_1\xi_1 + \dots x_n\xi_n$.

Let $\Omega \subset \mathbb{R}^n$ open. For $f : \Omega \to \mathbb{C}$ we define support in the following way

$$\text{supp } f := \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

We further list basic function spaces used throughout the thesis.

- $C(\Omega)$ is the space of continuous functions on Ω.
- $C^k(\Omega)$, $k \in \mathbb{N}$ is the space of k-times continuously differentiable functions on Ω.
- $C^\infty(\Omega)$ is the space of smooth functions Ω, that is $C^\infty(\Omega) = \bigcap_{k\in\mathbb{N}_0} C^k(\Omega)$.
- $C_0(\Omega)$ is the space of continuous functions with compact support.

- $C_0^\infty(\Omega)$ (or $\mathcal{D}(\Omega)$) is the space of C^∞ compactly supported functions $f : \Omega \to \mathbb{C}$.

- $\mathcal{L}(X, Y)$ is the space of linear, continuous mappings (operators) from X to Y and $\mathcal{L}(X)$ the space of linear operators from X to X.

- By X' we denote the dual of X, i.e. the space of linear mappings $f : X \to \mathbb{C}$. For $u \in X$ the action of the linear functional $f \in X'$ is denoted by $\langle f, u\rangle$.

Furthermore, a linear **unbounded** operator on a Banach space X is a pair $(D(A), A)$, where $D(A)$ is a linear subspace of X (the domain) and A is a linear mapping $D(A) \to X$. We say that A is **bounded** if there exists $c > 0$ such that $\|Ax\| \leq c\|x\|$, $x \in D(A)$. Otherwise, it is not bounded. Note that a linear unbounded operator can be either bounded or not bounded. If A is a linear operator with dense domain ($\overline{D(A)} = X$) and X is a Hilbert space, then

$$G(A^*) = \{(v, \varphi) \in X \times X;\ \langle \varphi, u\rangle = \langle v, f\rangle\ \forall (u, f) \in G(A)\}$$

defines A^* – the adjoint. Its domain is

$$D(A^*) = \{v \in X :\ \exists c < \infty, |\langle Au, v\rangle| \leq C\|u\|,\ \forall u \in D(A)\},$$

and A^* satisfies

$$\langle A^* v, u\rangle = \langle v, Au\rangle\ \forall u \in D(A).$$

We say that $f(\varepsilon) \sim g(\varepsilon)$ if $\lim_{\varepsilon\to 0} \dfrac{f(\varepsilon)}{g(\varepsilon)} = c > 0$. We use $\lesssim$ when inequality holds up to a positive constant:

$$f(\varepsilon) \lesssim g(\varepsilon) \quad \text{if} \quad f(\varepsilon) \leq c g(\varepsilon), \quad c > 0$$

and c does not depend on ε.

The big O notation is also used. One writes

$$f(\varepsilon) = \mathcal{O}(g(\varepsilon)),\ \varepsilon \to 0,$$

if there exists $M > 0$ and $\varepsilon_1 > 0$ such that

$$|f(\varepsilon)| \leq M g(\varepsilon), \quad \forall \varepsilon \leq \varepsilon_1.$$

2.2 Space of distributions and Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ open. For $1 \leq p < \infty$ define

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C} \mid \int_\Omega |f(x)|^p dx < \infty\},$$

with the norm

$$\|f\|_{L^p} = (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}}.$$

For $p = \infty$ define

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is Lebesgue measurable and there is a constant } C \\ \text{such that } |f(x)| \leq C \text{ for almost all } x \in \Omega \end{array} \right\},$$

with the norm

$$\|f\|_\infty = \inf\{C : |f(x)| \leq C \text{ for almost all } x \in \Omega\}.$$

If there is no risk of confusion we denote $\|\cdot\|_p = \|\cdot\|_{L^p}$. For all $1 \leq p \leq \infty$ spaces $L^p(\Omega)$ are Banach; they are reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$. As usual, we identify two functions that coincide a.e. on Ω.

We say that a function is **locally integrable** ($u \in L^1_{loc}(\Omega)$) if its Lebesgue integral is finite for any compact subset of Ω.

By $\mathcal{D}'(\Omega)$ we denote the space of **distributions**: linear functions $u : \mathcal{D}(\Omega) \to \mathbb{C}$, that is $u : \varphi \mapsto \langle u, \varphi\rangle$ such that for every compact set $K \subset \Omega$ there exist $m \in \mathbb{N}$ and $C > 0$ so that

$$|\langle u, \varphi\rangle| \leq C \sup_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha \varphi(x)|,$$

for all $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{supp}\varphi \subset K$. We have that $u_n \to u$ in $\mathcal{D}'(\Omega)$ if and only if the weak star convergence holds:

$$\langle u_n, \varphi\rangle \to \langle u, \varphi\rangle \text{ in } \mathbb{C}, \; \forall \varphi \in \mathcal{D}(\Omega).$$

Every $u \in L^1_{loc}(\Omega)$ is a *regular* distribution, meaning that $\int_\Omega u\varphi$, for $\varphi \in \mathcal{D}(\Omega)$ is a distribution. The derivative of a distribution is defined in the following way:

$$\langle \partial^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, \partial^\alpha \varphi\rangle.$$

Any distribution has derivatives of arbitrary order in $\mathcal{D}'(\Omega)$ and moreover ∂^α is a continuous operator $\mathcal{D}'(\Omega) \mapsto \mathcal{D}'(\Omega)$. This is a useful fact when solving approximately linear differential equations, since it means that if a sequence of solutions converges in the space $\mathcal{D}'(\Omega)$ then the limit is also a solution of the equation.

Let $u \in L^1_{loc}(\Omega)$. If there exist $v_\alpha \in L^1_{loc}(\Omega)$ such that $v_\alpha = \partial^\alpha u$ in $\mathcal{D}'(\Omega)$, then v_α is the called the **weak derivative** of u and is denoted by $\partial^\alpha u$.

Let $n \in \mathbb{N}$. We define the space of **rapidly decreasing functions**:

$$S(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \ \|\phi\|_{k,l} < \infty \quad \forall k \in \mathbb{N}_0, l \in \mathbb{N}_0\}$$

where

$$\|\phi\|_{k,l} = \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{k/2} \sum_{|\alpha|\le l} |D^\alpha \phi(x)|$$

is a semi-norm. A sequence $\{\phi_j\}_{j=1}^\infty \subset S$ converges in S to $\phi \in S$ iff

$$\|\phi_j - \phi\|_{k,l} \to 0, \text{ for } j \to \infty \text{ and all } k, l \in \mathbb{N}_0.$$

By $S'(\mathbb{R}^n)$ we denote the space of linear, continuous maps $u : S(\mathbb{R}^n) \mapsto \mathbb{C}$, also called the space of *tempered* distributions. We further define the **Fourier transform** for $\phi \in S(\mathbb{R}^n)$:

$$\hat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\phi(x)dx, \quad \xi \in \mathbb{R}^n.$$

Also

$$\check{\phi}(\xi) = (\mathcal{F}^{-1}\phi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\phi(x)dx, \quad \xi \in \mathbb{R}^n$$

is the inverse transform for ϕ. The Fourier transform is a bijective, linear and continuous map from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ and from $S'(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$. It is also unitary on $L^2(\mathbb{R}^n)$. For a tempered distribution T it is defined in the following way

$$\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle, \quad \varphi \in S(\mathbb{R}^n).$$

Fourier transform is a linear operation and some other important properties are

i) $\mathcal{F}(f * g) = \hat{f} \cdot \hat{g}$ and $\mathcal{F}(f \cdot g) = \hat{f} * \hat{g}$;

ii) $\mathcal{F}(\partial^\alpha f) = (i\xi)^\alpha \hat{f}$ and $\mathcal{F}((-x)^\beta f) = \partial^\beta \hat{f}$;

iii) $\hat{\delta}(\xi) = 1$ and as a consequence $S * \delta = S$ for any $S \in S'(\mathbb{R}^n)$.

We now give the definition and present some properties of Sobolev [1] spaces. For proofs of the theorems, see [AF03].

[1] Sergei Sobolev (1908-1989), a Soviet mathematician

Definition 2.2.1. *Let $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. The space*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \text{ for } 0 \leq \alpha \leq m\}$$

where ∂^α is the weak derivative, is called the Sobolev space.

This is a normed vector space, with the norm given by:

$$\begin{aligned} \|u\|_{p,m,\Omega} &:= (\sum_{|\alpha|\leq m} \|\partial^\alpha u\|_p^p)^{1/p} \quad \mathit{for}\ 1 \leq p < \infty, \\ \|u\|_{m,\infty} &:= \max_{0\leq|\alpha|\leq m} \|\partial^\alpha u\|_\infty. \end{aligned} \tag{2.1}$$

An equivalent norm is

$$\begin{aligned} \|u\|_{p,m,\Omega} &:= \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_p \quad \mathit{for}\ 1 \leq p < \infty, \\ \|u\|_{m,\infty} &:= \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_\infty. \end{aligned} \tag{2.2}$$

Definition 2.2.2. *$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ with respect to the norm* (2.1).

$W_0^{m,p}(\Omega)$ is sometimes referred to as the Sobolev space of zero boundary values. Indeed, under some additional assumptions, functions from $W_0^{m,p}(\Omega)$ are zero on the boundary $\partial\Omega$. For example, we have the following theorem (Theorem 9.17. from [Bre10]):

Theorem 2.2.3. *Suppose Ω is open and of class C^1. Let*

$$u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}), \quad 1 \leq p < \infty.$$

Then the following properties are equivalent

- *$u = 0$ on $\partial\Omega$,*
- *$u \in W_0^{1,p}(\Omega)$.*

Theorem 2.2.4. *The space $W^{m,p}(\Omega)$ is a Banach space for every $1 \leq p \leq \infty$. $W^{m,p}(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.*

On the space $H^m(\Omega) = W^{m,2}(\Omega)$ we can define a scalar product:

$$(u, v)_{H^m} = \sum_{0 \le |\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2} = \sum_{0 \le |\alpha| \le m} \int_\Omega D^\alpha u \, \overline{D^\alpha v} dx,$$

and therefore $H^m(\Omega)$ is a Hilbert space. There also holds

Theorem 2.2.5. *$W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n)$ for $1 \le p < \infty$.*

Furthermore, we have several useful embedding theorems and we will use the following (Corollary 9.13. from [Bre10]):

Theorem 2.2.6. *Let $m \ge 1$ be an integer and $p \in [1, \infty)$. We have*

$$\begin{aligned}
&W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{n} \text{ if } \frac{1}{p} - \frac{m}{n} > 0, \\
&W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, \infty) \quad \text{if } \frac{1}{p} - \frac{m}{n} = 0, \\
&W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \quad \text{if } \frac{1}{p} - \frac{m}{n} < 0
\end{aligned}$$

and all these injections are continuous. Moreover, if $k = [m - \frac{n}{p}]$, where $[]$ denotes the integer part, we have for all $u \in W^{m,p}(\mathbb{R}^n)$,

$$\|\partial^\alpha u\|_{L^\infty} \le C \|u\|_{W^{m,p}} \quad \forall |\alpha| \le k. \tag{2.3}$$

In particular, $W^{m,p}(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.

We will mostly use that in 3 dimensions and for $p = 2$ we have $\frac{1}{2} - \frac{m}{3} < 0 \Leftrightarrow m > \frac{3}{2}$, for $m \ge 2$, and that the functions in $W^{m,2}(\mathbb{R}^3)$ are bounded. In this case, there also holds that these functions tend to zero when $|x| \to \infty$. This is due to the fact that the space $C_0^\infty(\mathbb{R}^3)$ is dense in $W^{m,p}(\mathbb{R}^3)$ w.r.t. the Sobolev norm, but because of the continuous injection it is also dense w.r.t. the supremum norm, meaning that a function from $W^{m,2}(\mathbb{R}^3)$ is a uniform limit of a sequence from $C_0^\infty(\mathbb{R}^3)$ and hence has to tend to zero, when x tends to infinity.

We get an alternative definition of $H^m = H^m(\mathbb{R}^n)$ via the definition of these spaces for real indices.

Definition 2.2.7. *For arbitrary $s \in \mathbb{R}$ by $H^s(\mathbb{R}^n)$ we denote the space of tempered distributions u for which*

$$u \in H^s(\mathbb{R}^n) \iff (1 + |y|^2)^{s/2} \hat{u} \in L^2, \quad y \in \mathbb{R}^n.$$

Let $H^s = H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$. There holds

$$H^{s_1} \subset H^{s_2}, \quad \text{for } -\infty < s_2 \leq s_1 < \infty$$

and particularly

$$H^{s_1} \subset H^{s_2} \subset L^2, \quad \text{for } 0 \leq s_2 \leq s_1 < \infty.$$

The norm in the space H^s is given by

$$|u|_s = \|(1+|y|^2)^{s/2}\hat{u}\|_{L^2}, \quad y \in \mathbb{R}^n \tag{2.4}$$

and the scalar product by

$$[u,v]_s = \int_{\mathbb{R}^n} \hat{u}(y)\bar{\hat{v}}(y)(1+|y|^2)^s dy, \quad u,v \in H^m, \quad y \in \mathbb{R}^n. \tag{2.5}$$

Again,

Definition 2.2.8. *$H_0^s(\mathbb{R}^n)$ is the closure of $\mathcal{D}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.*

We have the following properties

Theorem 2.2.9. *Let $s \in \mathbb{R}$. The spaces H^s with the scalar product* (2.5) *are Hilbert. There holds*

$$S(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$$

and $S(\mathbb{R}^n)$ is dense H^s.

Theorem 2.2.10. *The space H^s, $s \in \mathbb{R}$ is reflexive and separable.*

There is also a duality result:

Theorem 2.2.11. *The dual of H^s is H^{-s} and the dual norm coincides with $|\ |_{-s}$.*

Finally, we introduce spaces

$$H^\infty = \bigcap_{s\in\mathbb{R}} H^s, \quad H^{-\infty} = \bigcup_{s\in\mathbb{R}} H^s.$$

The following inclusions hold

$$S \subset H^\infty \subset H^{-\infty} \subset S'.$$

Next we state and prove a theorem important for the Colombeau algebra and that we shall use in the sequel. Results of this type are given in [AF03].

Theorem 2.2.12. *The space $H^s(\mathbb{R}^n)$ is an algebra when $s > \frac{n}{2}$ and*

$$\|uv\|_{H^s} \leq c\|u\|_{H^s}\|v\|_{H^s}, \quad u, v \in H^s. \tag{2.6}$$

Proof. Let $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. There holds $\langle \xi \rangle^s \leq c(\langle \xi - \eta \rangle^s + \langle \eta \rangle^s)$ since

$$\begin{aligned}(1 + |\xi|^2)^p &\leq (1 + |\xi - \eta|^2 + |\eta|^2 + 2|\xi - \eta| \cdot |\eta|)^p \\ &\leq (1 + 2|\xi - \eta|^2 + 2|\eta|^2)^p \leq 2^p(1 + |\xi - \eta|^2 + 1 + |\eta|^2)^p \\ &\leq 2^p \cdot 2^p((1 + |\xi - \eta|^2)^p + (1 + |\eta|^2)^p),\end{aligned}$$

for any $p > 0$. This is similar as Peetre's inequality: $\langle \xi \rangle^s \leq 2^{|s|}\langle \xi - \eta \rangle^{|s|}\langle \eta \rangle^s, \; s \in \mathbb{R}$, see [Abe11]. Now

$$\|uv\|_{H^s} = |uv|_s = \|\langle \xi \rangle^s \widehat{(uv)}\|_2$$

and

$$\begin{aligned}\langle \xi \rangle^s |\widehat{(uv)}(\xi)| &\leq \langle \xi \rangle^s \int |\hat{u}(\xi - \eta)\hat{v}(\eta)| d\eta = \int \langle \xi \rangle^s |\hat{u}(\xi - \eta)\hat{v}(\eta)| d\eta \\ &\leq c \int \langle \xi - \eta \rangle^s |\hat{u}(\xi - \eta)\hat{v}(\eta)| d\eta + c \int \langle \eta \rangle^s |\hat{u}(\xi - \eta)\hat{v}(\eta)| d\eta \\ &= c|\langle \cdot \rangle^s \hat{u}| * |\hat{v}| + c|\hat{u}| * |\langle \cdot \rangle^s \hat{v}|.\end{aligned}$$

From Young's inequality it follows

$$\|\langle \xi \rangle^s \widehat{(uv)}\|_2 \leq c\|\langle \xi \rangle^s \hat{u}\|_2 \|\hat{v}\|_1 + c\|\hat{u}\|_1 \|\langle \xi \rangle^s \hat{v}\|_2.$$

Finally

$$\|\hat{u}\|_1 = \int \langle \xi \rangle^s \langle \xi \rangle^{-s} |\hat{u}| d\xi \lesssim \|\langle \xi \rangle^s \hat{u}\|_2 (\int \langle \xi \rangle^{-2s} d\xi)^{\frac{1}{2}}$$

and the result follows, since $\langle \xi \rangle^{-2s}$ is integrable for $s > \frac{n}{2}$. □

2.3 Vector valued functions

We will use spaces involving time. Let I be an interval in $\mathbb{R}$ and X a Banach space. By $C(I,X)$ we denote the space of continuous functions $u : I \to X$, that is, for all $t_0 \in I$,

$$\lim_{t \to t_0} \|u(t) - u(t_0)\|_X = 0.$$

Also, $C^m(I,X)$ is the space of functions $u : I \to X$ whose derivatives (in t) of order j belong to $C(I,X)$ for all $0 \leq j \leq m$. Finally, $C_0^m(I,X)$ are functions $u \in C^m(I,X)$ with compact support in I.

We introduce the definition of a measurable function, as in [CBH+98].

Definition 2.3.1. *A function $u : I \to X$ is measurable if there exists a set $E \subset I$ of measure zero and a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_0(I,X)$ such that $u_n(t) \to u(t)$ as $n \to \infty$ for all $t \in I \setminus E$.*

We also define integrability.

Definition 2.3.2. *A measurable function $u : I \to X$ is integrable if there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_0(I,X)$ such that*

$$\int_I \|u_n(t) - u(t)\|_X dt \to 0, \quad n \to \infty.$$

Now by $L^p(I,X)$ we denote the space of measurable functions $u : I \to X$, such that

$$\int_I \|u(t)\|_X^p dt < \infty \quad \text{for } 1 \leq p < \infty$$

or

$$\operatorname*{ess\,sup}_{t\in I} \|u(t)\|_X < \infty \quad \text{for } p = \infty.$$

The space $W^{m,p}(I,X)$ is the Banach spaces of (classes of) measurable functions $u : I \to X$, such that $\frac{\partial^j u}{\partial t^j} \in L^p(I,X)$ for every $0 \leq j \leq m$. This space is equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{j=1}^{m} \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^p}.$$

We will also often observe an integral of type $\int_I u(t)dt$, where $u(t) \in X$. In our work, this can be interpreted as the usual Lebesgue integral

$$U(x) = \int_I u(t,x)dt,$$

for fixed values of x. Equivalently it can be observed as a Bochner integral, defined analogously with approximation by vector–valued step functions. In particular, if X is the space of real numbers, then Bochner integrable functions are Lebesgue integrable functions ([CBH+98], [Mik78]). We present the following analogue to theorem 2.4.6 in the sequel.

Theorem 2.3.3 (Bochner). *Let $u : I \to X$ be a measurable function. Then u is integrable if and only if $\|u\|_X$ is integrable. Moreover,*

$$\left\|\int_I u\ dt\right\|_X \leq \int_I \|u\|_X.$$

It is also useful to define the derivative of a vector–valued function. We state the definitions of the Frechét derivative and Gâteaux derivative, see [Aub11].

Definition 2.3.4. *Let $U \subset I$ be an open subset and $t_0 \in U$. The map $f : I \to X$ is said to be Frechét differentiable at t_0 if there exists $A \in \mathcal{L}(I, X)$ such that*

$$\lim_{t \to t_0} \frac{\|f(t) - f(t_0) - A(t - t_0)\|}{|t - t_0|} = 0.$$

The map A is the Frechét derivative of f at t_0 (usually denoted by $Df(t_0)$).

Definition 2.3.5. *Let $f : I \to X$. If the limit*

$$A_t(v) = \lim_{s \to 0} \frac{f(t + sv) - f(t)}{s}$$

exists for each $v \in I$ and the map $v \mapsto A_t(v)$ is a continuous linear map, then we say f is Gâteaux differentiable at t and A_t is called the Gâteaux derivative of f at t.

If a function is Frechét differentiable at t, then it is Gâteaux differentiable at t and the two derivatives coincide. The converse, however, does not hold in general.

2.4 Important inequalities

Besides the Sobolev inequality (2.3) we will use several important inequalities which we list in this section. The first two are very well–known.

Theorem 2.4.1 (Hölder). *Let* $1 \leq p \leq \infty$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $1/p + 1/q = 1$. *Then* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{2.7}$$

Theorem 2.4.2 (Young). *Let* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *i* $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. *Then the convolution* $f * g \in L^r(\mathbb{R}^d)$ *and*

$$\|f * g\|_r \leq \|f\|_p\|g\|_q. \tag{2.8}$$

We state two versions of the Gronwall inequality. The first is as in [Dra03].

Theorem 2.4.3 (Gronwall's inequality). *Let* $A(t)$ *be continuous and nonnegative on* $[0, T]$ *and satisfy*

$$A(t) \leq E(t) + \int_0^t r(s)A(s)ds, \quad 0 \leq t \leq T,$$

where $r(t)$ *is a nonnegative integrable function on* $[0, T]$ *with* $E(t)$ *bounded on* $[0, T]$. *Then*

$$A(t) \leq |E(t)| \exp\left(\int_0^t r(s)ds\right), \quad 0 \leq t \leq T.$$

The second inequality is a variant of the theorem appearing in [EK09] and we prove this version.

Theorem 2.4.4 (Gronwall's inequality). *Let* $A : [0, \infty) \to \mathbb{R}$ *be a measurable function that is bounded on bounded intervals,* $E \geq 0$ *and* $r(t)$ *a nonnegative integrable function on* $[0, t]$ *for any* $t \in [0, \infty)$. *Let*

$$0 \leq A(t) \leq E + \int_0^t r(s)A(s)ds, \quad t \geq 0, \tag{2.9}$$

then

$$A(t) \leq E \exp\left(\int_0^t r(s)ds\right), \quad t \geq 0.$$

Proof. We show first by induction that

$$A(t) \leq E \cdot \sum_{k=0}^{n} \frac{R(t)^k}{k!} + R_n(t), \tag{2.10}$$

holds for any n, where

$$R(t) = \int_0^t r(s)ds$$

and

$$R_n(t) = \int_0^t \frac{R(s)^n}{n!} r(s)A(s)ds.$$

The case $n = 0$ is the inequality (2.9). Let (2.10) hold and let us show that it holds for $n+1$. By (2.10) and (2.9)

$$\begin{aligned}
A(t) &\leq E \cdot \sum_{k=0}^{n} \frac{R(t)^k}{k!} + \int_0^t \frac{R(s)^n}{n!} r(s) \left(E + \int_0^s r(s_1)A(s_1)ds_1 \right) ds \\
&= E \cdot \sum_{k=0}^{n} \frac{R(t)^k}{k!} + E\frac{R(t)^{n+1}}{(n+1)!} + \int_0^t \int_0^s \frac{R(s)^n}{n!} r(s)r(s_1)A(s_1)ds_1 ds \\
&= E \cdot \sum_{k=0}^{n+1} \frac{R(t)^k}{k!} + \int_0^t \int_0^{s_1} \frac{R(s)^n}{n!} r(s)r(s_1)A(s_1)ds ds_1 \\
&= E \cdot \sum_{k=0}^{n+1} \frac{R(t)^k}{k!} + \int_0^t r(s_1)A(s_1) \int_0^{s_1} \frac{R(s)^n}{n!} r(s)ds\, ds_1 \\
&= E \cdot \sum_{k=0}^{n+1} \frac{R(t)^k}{k!} + \int_0^t r(s_1)A(s_1)\frac{R(s_1)^{n+1}}{(n+1)!} ds_1 = E \cdot \sum_{k=0}^{n+1} \frac{R(t)^k}{k!} + R_{n+1}(t).
\end{aligned}$$

Here we used $R'(t) = r(t)$ and Fubini–Tonelli theorem, the function

$$\frac{R(s)^n}{n!} r(s)r(s_1)A(s_1)$$

being measurable and nonnegative. Now for the remainder R_n there holds

$$R_n(t) \leq (\sup_{[0,t]} A(t))\frac{R(t)^{n+1}}{(n+1)!} \to 0, \quad n \to \infty,$$

since $A(t)$ is bounded, $r(s)$ is integrable and all quantities are nonnegative, so $R_n(t) \to 0$ as $n \to \infty$ for any t in $[0, \infty)$. Since (2.10) holds for any $n \in \mathbb{N}$ it follows it holds in the limiting case, too, hence

$$A(t) \leq E \exp(R(t)), \quad t \in [0, \infty),$$

which completes the proof. □

We now state the Gagliardo–Nirenberg inequality as in [Caz03].

Theorem 2.4.5 (Gagliardo-Nirenberg). *let $1 \leq p, q, r \leq \infty$ and let j, m be two integers such that $0 \leq j < m$. If*

$$\frac{1}{p} = \frac{j}{n} + b\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-b}{q},$$

for some $b \in [j/m, 1]$ ($b < 1$ if $r > 1$ and $m - j - \frac{n}{r} = 0$), then there exists $C = C(n, m, j, q, r)$ so that

$$\sum_{|\alpha|=j} \|D^\alpha u(t)\|_p \leq C\Big(\sum_{|\alpha|=m} \|D^\alpha u(t)\|_r\Big)^b \|u(t)\|_q^{1-b} \quad \forall u \in \mathcal{D}(\mathbb{R}^n) \tag{2.11}$$

Theorem 2.4.6 (Minkowski). *Let $S_1 \subset \mathbb{R}^m$, $S_2 \subset \mathbb{R}^n$ and $F : S_1 \times S_2 \to \mathbb{R}$ is measurable. For $1 \leq p < \infty$ there holds*

$$(\int_{S_2} |\int_{S_1} F(x,y)dx|^p dy)^{\frac{1}{p}} \leq \int_{S_1} (\int_{S_2} |F(x,y)|^p dy)^{\frac{1}{p}} dx. \tag{2.12}$$

Theorem 2.4.7 (Riesz–Thorin convexity theorem [Hör90]). *If T is a linear map from $L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ to $L^{q_1}(\mathbb{R}^n) \cap L^{q_2}(\mathbb{R}^n)$ such that*

$$\|Tf\|_{q_j} \leq M_j \|f\|_{p_j} \quad j = 1, 2,$$

and if $1/p = t/p_1 + (1-t)/p_2$, $1/q = t/q_1 + (1-t)/q_2$ for some $t \in (0, 1)$, then

$$\|Tf\|_q \leq M_1^t M_2^{1-t} \|f\|_p, \quad f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n).$$

2.5 Colombeau algebra

We now present the definition of a H^2 - based Colombeau algebra. Different types of these algebras are described for example in [Gro+13] and in original works [Col00]. Also see [BO92] for $L^p - L^q$ - based algebras.

The product of two distributions is not defined, only the product of a smooth function and a distribution. If we try to extend this operation we will not be able to conserve the associative property, as this example shows:

$$0 = (\delta(x) \cdot x) \cdot vp\frac{1}{x} \neq \delta(x) \cdot (x \cdot vp\frac{1}{x}) = \delta(x),$$

where $vp\frac{1}{x}$ denotes the Cauchy principal value of $\frac{1}{x}$. These and other problems were the motivation for defining an associative, commutative algebra containing the space of distributions. Specifically, desirable properties for an algebra $(\mathcal{A}(\Omega), +, \cdot)$, for an open set Ω, are the following

(i) $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$ and $f(x) \equiv 1$ is the unity in $\mathcal{A}(\Omega)$.

(ii) There exist differential operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$, $i = 1, ..., n$ that are linear and satisfy the Leibniz rule.

(iii) $\partial_i|_{\mathcal{D}'}$ is the usual partial derivative, $i = 1, ..., n$.

(iv) The restriction $\cdot|_{C^\infty \times C^\infty}$ coincides with the pointwise product of functions.

The following (special) Colombeau algebra satisfies these conditions and is defined as follows (see [Gro+13]). Let $\Omega \subset \mathbb{R}^n$ open and

$$
\begin{aligned}
\mathcal{E}^s(\Omega) &:= (C^\infty(\Omega))^{(0,1]} \\
\mathcal{E}^s_M(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega \; \forall \alpha \in \mathbb{N}_0^n \; \exists N \in \mathbb{N} \text{ with} \\
&\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}), \; \varepsilon \to 0\} \\
\mathcal{N}^s(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega \; \forall \alpha \in \mathbb{N}_0^n \; \forall m \in \mathbb{N} \text{ with} \\
&\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{m}), \; \varepsilon \to 0\}.
\end{aligned}
$$

Elements of $\mathcal{E}^s_M(\Omega)$ and $\mathcal{N}^s(\Omega)$ are called moderate resp. negligible functions. The special Colombeau algebra is the quotient space

$$\mathcal{G}^s(\Omega) := \mathcal{E}^s_M(\Omega)/\mathcal{N}^s(\Omega).$$

If $u \in \mathcal{D}'(\Omega)$, then the embedding $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}^s(\Omega)$ is given by

$$u \mapsto [(u * \rho_\varepsilon)_\varepsilon],$$

where $\rho \in S(\mathbb{R}^n)$ is a mollifier such that

$$\int \rho(x) dx = 1, \tag{2.13}$$

$$\int x^\alpha \rho(x) = 0, \quad \forall |\alpha| \geq 1 \tag{2.14}$$

and $\rho_\varepsilon = \varepsilon^{-n} \rho(\frac{x}{\varepsilon})$. This type of mollifier assures that (iv) holds. There is no mollifier in $\mathcal{D}(\mathbb{R}^n)$ which satisfies both (2.13) and (2.14). On the other hand, $\rho \in S(\mathbb{R}^n)$ can be constructed by taking the inverse Fourier transform of a function from $S(\mathbb{R}^n)$ which equals 1 in a neighborhood of zero.

The H^2–based algebra we use is as in [NPR03]. One more paper using similar spaces is [NOP05]. Let $\mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$ (respectively $\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^n))$, $T>0$ denote the vector space of nets $(u_\varepsilon)_\varepsilon$ of functions

$$u_\varepsilon \in C([0,T),H^2(\mathbb{R}^n))\cap C^1([0,T),L^2(\mathbb{R}^n)),\ \varepsilon\in(0,1),$$

with the property that there exists $N\in\mathbb{N}$ (respectively, for every $M\in\mathbb{N}$) such that

$$\max\{\sup_{t\in[0,T)}\|u_\varepsilon(t)\|_{H^2},\ \sup_{t\in[0,T)}\|\partial_t u_\varepsilon(t)\|_{L^2}\}=\mathcal{O}(\varepsilon^{-N}),\ \varepsilon\to 0$$

$$\Big(\text{respectively}$$

$$\max\{\sup_{t\in[0,T)}\|u_\varepsilon(t)\|_{H^2},\ \sup_{t\in[0,T)}\|\partial_t u_\varepsilon(t)\|_{L^2}\}=\mathcal{O}(\varepsilon^{M}),\ \varepsilon\to 0\Big).$$

The quotient space

$$\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^n)=\mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^n)/\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$$

is a Colombeau type vector space. For $n\le 3$ this is a multiplicative algebra, since $H^2(\mathbb{R}^n)$ itself is an algebra for $n\le 3$.

The space $\mathcal{G}_{H^2}(\mathbb{R}^n)$ is defined in a similar way:

$$\begin{aligned}
\mathcal{E}^2(\mathbb{R}^n) &:= (H^2(\mathbb{R}^n))^{(0,1]}\\
\mathcal{E}_{H^2}(\mathbb{R}^n) &:= \{(u_\varepsilon)_\varepsilon\in\mathcal{E}^2(\mathbb{R}^n)\mid \exists N\in\mathbb{N}\ \|u_\varepsilon(x)\|_{H^2}=\mathcal{O}(\varepsilon^{-N}),\ \varepsilon\to 0\}\\
\mathcal{N}_{H^2}(\mathbb{R}^n) &:= \{(u_\varepsilon)_\varepsilon\in\mathcal{E}^2(\mathbb{R}^n)\mid \forall m\in\mathbb{N}\ \|u_\varepsilon(x)\|_{H^2}=\mathcal{O}(\varepsilon^{m}),\ \varepsilon\to 0\},\\
\mathcal{G}_{H^2}(\mathbb{R}^n) &:= \mathcal{E}_{H^2}(\mathbb{R}^n)/\mathcal{N}_{H^2}(\mathbb{R}^n).
\end{aligned}$$

This space is also an algebra in the case $n\le 3$.

The basic operations of addition, multiplication and differentiation are done component–wise, that is

$$u+v=[(u_\varepsilon+v_\varepsilon)_\varepsilon],\quad u\cdot v=[(u_\varepsilon\cdot v_\varepsilon)_\varepsilon],\quad \partial^\alpha u=[(\partial^\alpha u_\varepsilon)_\varepsilon].$$

We define differentiation on this algebra, although it is not a closed operation. If $u\in\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$, then $\partial^\alpha u$ is represented by $\partial^\alpha u_\varepsilon$ which has moderate growth in $L^2(\mathbb{R}^n)$ and giving rise to an element of a quotient vector space $\mathcal{G}_{C,L^2}([0,T)\times\mathbb{R}^n)$, defined analogously as $\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$ - with the difference that representatives have bounded growth only in L^2–norm, for any $t \in [0,T)$. We will see that the equations (1.1) – (1.3) have sense in $\mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^n)$. Also it is easily seen that $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n) \subset \mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^n)$.

We also mention the space $\mathcal{G}_{\infty,\infty}(\mathbb{R}^n)$ defined as follows

$$
\begin{aligned}
\mathcal{E}(\mathbb{R}^n) &:= (C^\infty(\mathbb{R}^n))^{(0,1]} \\
\mathcal{E}_{\infty,\infty}(\mathbb{R}^n) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0^n \ \exists N \in \mathbb{N} \ \|\partial^\alpha u_\varepsilon(x)\|_\infty = \mathcal{O}(\varepsilon^{-N}), \ \varepsilon \to 0\} \\
\mathcal{N}_{\infty,\infty}(\mathbb{R}^n) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0^n \ \forall m \in \mathbb{N} \ \|\partial^\alpha u_\varepsilon(x)\|_\infty = \mathcal{O}(\varepsilon^{m}), \ \varepsilon \to 0\}, \\
\mathcal{G}_{\infty,\infty}(\mathbb{R}^n) &:= \mathcal{E}_{\infty,\infty}(\mathbb{R}^n)/\mathcal{N}_{\infty,\infty}(\mathbb{R}^n).
\end{aligned}
$$

This is a special case of the $L^p - L^q$–based algebras defined in [BO92]. We can embed the delta function in this space by a convolution with a mollifier as before, and actually, $\delta * \rho_\varepsilon = \rho_\varepsilon$ so that ρ_ε itself is a representative of the delta function. Also in this way, $W^{-\infty,\infty}(\mathbb{R}^n)$ is embedded in $\mathcal{G}_{\infty,\infty}(\mathbb{R}^n)$ and $W^{\infty,\infty}(\mathbb{R}^n)$ is a subalgebra of $\mathcal{G}_{\infty,\infty}(\mathbb{R}^n)$, which was shown in [BO92].

We will prove that in this algebra, one more representative of the delta function is given by a *strict delta net*, defined as follows. We follow the approach given in [Gro+13].

Definition 2.5.1. *A strict delta net is a family of functions $\phi_\varepsilon \in \mathcal{E}_{\infty,\infty}$ which satisfies*

i) $\operatorname{supp}(\phi_\varepsilon) \to \{0\}, \ \varepsilon \to 0,$

ii) $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \phi_\varepsilon(x)dx = 1$

iii) $\int |\phi_\varepsilon(x)|dx$ *is bounded uniformly in* ε.

A strict delta net can be defined using ρ_ε as $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$, where χ is a cut–off function and ρ_ε is as before. Specifically, $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = 1, \ |x| \leq 1$ and $\chi(x) = 0, \ |x| \geq 2$.

In the sequel we will use the following estimates for ρ_ε and ϕ_ε. Since $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, we have

$$
\begin{aligned}
\|\partial^\alpha \rho_\varepsilon\|_{L^p}^p &= \int_{\mathbb{R}^n} \varepsilon^{-np} |\partial^\alpha(\rho(\frac{x}{\varepsilon}))|^p dx = \int_{\mathbb{R}^n} \varepsilon^{-np} |\frac{1}{\varepsilon^{|\alpha|}}(\partial^\alpha \rho)(\frac{x}{\varepsilon})|^p dx \\
&= \int_{\mathbb{R}^n} \varepsilon^{-np+n-|\alpha|p} |\partial^\alpha \rho(t)|^p dt = c\varepsilon^{n(1-p)-|\alpha|p} \sim \varepsilon^{-N},
\end{aligned}
$$

for some $N \in \mathbb{N}$, $1 < p < \infty$ and any multi - index α. Moreover, $\|\rho_\varepsilon\|_\infty = \varepsilon^{-n} \max |\rho(\frac{x}{\varepsilon})| = c\varepsilon^{-n}$, for any $\varepsilon > 0$. We also use mollifiers of type $\rho_{h_\varepsilon} = h_\varepsilon^n \rho(x h_\varepsilon)$, where $h_\varepsilon \to \infty$, $\varepsilon \to 0$, for example $h_\varepsilon = \ln \varepsilon^{-1}$, and these mollifiers admit completely analogous estimates as above.

Since the derivatives $\partial^\alpha(\chi(\frac{x}{\sqrt{\varepsilon}}))$ are bounded by

$$\sup_{x\in\mathbb{R}^n} |\varepsilon^{-|\alpha|/2}(\partial^\alpha\chi)(\frac{x}{\sqrt{\varepsilon}})| \lesssim \varepsilon^{-|\alpha|/2},$$

it is not hard to see that $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$ admits analogous estimates as ρ_ε in the L^p–norm. Now we prove the following theorem.

Theorem 2.5.2. *There exists a strict delta net ϕ_ε such that the difference $\rho_\varepsilon - \phi_\varepsilon$ belongs to $\mathcal{N}_{\infty,\infty}(\mathbb{R}^n)$ and both ρ_ε and ϕ_ε are representatives for the embedded delta function $[(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{\infty,\infty}(\mathbb{R}^n)$.*

Proof. For ϕ_ε we choose specifically $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$ as above. This defines a strict delta net as in Definition 2.5.1. The difference $\rho_\varepsilon - \phi_\varepsilon = 0$ for $|x| \le \sqrt{\varepsilon}$. Further

$$\begin{aligned}\|\rho_\varepsilon - \phi_\varepsilon\|_\infty = \|\rho_\varepsilon(x)(1-\chi(\frac{x}{\sqrt{\varepsilon}}))\|_\infty &\le \sup_{x>\sqrt{\varepsilon}} |\varepsilon^{-n}\rho(\frac{x}{\varepsilon})| \\ &\le C_q\varepsilon^{-n}\sup_{x>\sqrt{\varepsilon}}(1+|\frac{x}{\varepsilon}|)^{-q} \le C_q\varepsilon^{q/2-n}.\end{aligned}$$

Since $\rho \in S(\mathbb{R}^n)$ this estimate holds for any $q > 0$ so we have $\mathcal{N}_{\infty,\infty}(\mathbb{R}^n)$ estimates of order zero. Taking a derivative of arbitrary order of $\rho_\varepsilon - \phi_\varepsilon$ we will need to bound terms involving $\partial^\beta(\rho_\varepsilon(x)) \cdot \partial^\alpha((1-\chi)(\frac{x}{\sqrt{\varepsilon}}))$, which again vanishes for $x \le \sqrt{\varepsilon}$. We can repeat a similar analysis as before, but now

$$\partial^\alpha((1-\chi)(\frac{x}{\sqrt{\varepsilon}})) = \varepsilon^{\frac{-|\alpha|}{2}}\partial^\alpha(1-\chi)(\frac{x}{\varepsilon}) \lesssim \varepsilon^{-\frac{|\alpha|}{2}}.$$

It follows

$$\begin{aligned}\sup_{x>\sqrt{\varepsilon}} |\partial^\beta(\rho_\varepsilon(x)) \cdot \partial^\alpha((1-\chi)(\frac{x}{\sqrt{\varepsilon}}))| &\le \sup_{x>\sqrt{\varepsilon}} |\varepsilon^{-n-|\beta|}(\partial^\beta\rho)(\frac{x}{\varepsilon})| \cdot \varepsilon^{-\frac{|\alpha|}{2}} \\ &\le C_q\varepsilon^{-(n+|\beta|+|\alpha|/2)}\varepsilon^{q/2}\end{aligned}$$

for any $q > 0$. In this way we can obtain necessary estimates of arbitrary order, and conclude

$$\rho_\varepsilon - \phi_\varepsilon \in \mathcal{N}_{\infty,\infty}(\mathbb{R}^n).$$

□

We can embed functions in the space $\mathcal{G}_{H^2}(\mathbb{R}^n)$ by convolution with a mollifier ρ_ε, too. We will discuss only the embedding of the delta function and prove some more general properties when embedding the space $\mathcal{G}_{C^1,H^2}([0,,T) \times \mathbb{R}^n)$.

Theorem 2.5.3. *There exists a strict delta net ϕ_ε such that the difference $\rho_\varepsilon - \phi_\varepsilon$ belongs to $\mathcal{N}_{H^2}(\mathbb{R}^n)$ and both ρ_ε and ϕ_ε are representatives for the embedded delta function $[(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^n)$.*

Proof. The proof will be similar to the proof of Theorem 2.5.2. Let $\phi_\varepsilon(x) = \chi_\varepsilon(x)\rho_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$. Then

$$\begin{aligned}
\|\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon\|_2^2 &= \int_{\mathbb{R}^n} \rho_\varepsilon^2(x)(1-\chi(\frac{x}{\sqrt{\varepsilon}}))^2 dx \leq \int_{|x|>\sqrt{\varepsilon}} \rho_\varepsilon^2(x)dx \\
&\leq \int_{|x|>\sqrt{\varepsilon}} \varepsilon^{-n}(1+|\frac{x}{\varepsilon}|)^{-2q}dx = \int_{|x|>\sqrt{\varepsilon}} \varepsilon^{-n}(1+\frac{|x|}{\varepsilon})^{-2q+n+1-(n+1)}dx \\
&\leq \varepsilon^{-n} \sup_{x>\sqrt{\varepsilon}} (1+\frac{|x|}{\varepsilon})^{-2q+n+1} \int_{|x|>\sqrt{\varepsilon}} (1+\frac{|x|}{\varepsilon})^{-(n+1)}dx \\
&\leq \varepsilon^{-n}\varepsilon^{q-(n+1)/2}\varepsilon^n \int_{|y|>1/\sqrt{\varepsilon}} \frac{1}{(1+|y|)^{n+1}}dy \\
&\leq \varepsilon^{q-(n+1)/2} \int_{y\in\mathbb{R}^n} \frac{1}{(1+|y|)^{n+1}}dy.
\end{aligned}$$

The above integral is finite and independent of ε. So for arbitrary M we can choose $q = M + \frac{n+1}{2} \Leftrightarrow -2q+m+1 < 0$ so that $q > \frac{n+1}{2}$ so that the above estimates hold and

$$\|\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon\|_2^2 < \varepsilon^M, \quad \varepsilon \leq \varepsilon_1 < 1.$$

From the proof of Theorem 2.5.2, we see that the derivatives of $\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon$ can be bounded similarly in the L^2–norm. □

Let us now prove that we can embed some functions in $\mathcal{G}_{H^2}(\mathbb{R}^n)$ using a strict delta net.

Theorem 2.5.4. *Let $f \in H^2(\mathbb{R}^n)$. Then $f * \rho_\varepsilon - f * \phi_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^n)$, where ϕ_ε is a strict delta net defined by $\phi_\varepsilon = \chi_\varepsilon\rho_\varepsilon$, $\chi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})$ and χ is a cut–off function as before.*

Proof. Young's inequality implies

$$\|f * (\rho_\varepsilon - \phi_\varepsilon)\|_2 \lesssim \|f\|_2 \|(1-\chi_\varepsilon)\rho_\varepsilon\|_1$$

We can bound $\|(1-\chi_\varepsilon)\rho_\varepsilon\|_1$ by ε^M for any $M \in \mathbb{N}$, $\varepsilon \to 0$, in the same way as in the proof of Theorem 2.5.3. Also, $\partial^\alpha(f * (\rho_\varepsilon - \phi_\varepsilon)) = (\partial^\alpha f) * (\rho_\varepsilon - \phi_\varepsilon)$ and the proof follows. $\square$

We give two theorems explaining the product of elements from different algebras.

Theorem 2.5.5. *Let $u \in \mathcal{G}_{\infty,\infty}(\mathbb{R}^n)$ and $v \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$. Then, $u \cdot v \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$.*

Proof. Let $u_\varepsilon \in \mathcal{E}_{\infty,\infty}(\mathbb{R}^n)$ and $v_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$. We have

$$\|u_\varepsilon v_\varepsilon(t)\|_2 \lesssim \|u_\varepsilon\|_\infty \|v(t)\|_2 \lesssim \varepsilon^{-N} \quad \varepsilon \to 0,$$

for any $t \in [0,T)$. A similar situation holds for derivatives $\partial^\alpha(u_\varepsilon v_\varepsilon)$, $|\alpha| \leq 2$, since in this case we have terms of form $\partial^\beta u_\varepsilon \partial^\gamma v_\varepsilon$ which can be bounded as above. In the same way, product of $n^1_\varepsilon \in \mathcal{N}^{(}_{\infty,\infty}\mathbb{R}^n)$ and $n^2_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ is negligible in $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ and also $u_\varepsilon \cdot n^2_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$, $v_\varepsilon \cdot n^1_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$. Taking another representative of u and v, $u_\varepsilon + n^1_\varepsilon$ and $v_\varepsilon + n^2_\varepsilon$ it follows

$$(u_\varepsilon + n^1_\varepsilon)(v_\varepsilon + n^2_\varepsilon) = u_\varepsilon \cdot v_\varepsilon + u_\varepsilon \cdot n^2_\varepsilon + v_\varepsilon \cdot n^1_\varepsilon + n^1_\varepsilon \cdot n^2_\varepsilon = u_\varepsilon \cdot v_\varepsilon + n^3_\varepsilon,$$

where $n^3_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ so the product is well-defined. $\square$

Theorem 2.5.6. *Let $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ and ρ_ε is the representative of δ in $\mathcal{G}_{H^2}(\mathbb{R}^n)$. Then $u \cdot [(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$.*

Proof. The proof is similar to the proof of the previous theorem, since any derivative of ρ_ε is bounded also in the norm $\|\cdot\|_\infty$. Moreover, terms $\|n^1_\varepsilon u_\varepsilon\|_{H^2}$ and $\|n^1_\varepsilon n^2_\varepsilon\|_{H^2}$ are negligible, due to Theorem 2.2.12. $\square$

Next, since the initial condition is a function depending on x only, we define a restriction of an element $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$.

Definition 2.5.7. *Let $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ with a representative $u_\varepsilon \in \mathcal{E}_{C^1,H^2}$. Since $u_\varepsilon \in C([0,T), H^2(\mathbb{R}^3))$, the function $u_\varepsilon(\cdot,0)$ is in $\mathcal{E}_{H^2}$. Also, if $u_\varepsilon \in \mathcal{N}_{C^1,H^2}$, then $u_\varepsilon(\cdot,0)$ is in $\mathcal{N}_{H^2}$. We define the restriction of u to $\{0\} \times \mathbb{R}^n$ as the class $[u_\varepsilon(\cdot,0)]_\varepsilon \in \mathcal{G}_{H^2}$.*

Also relevant to our equations is the following definition.

Definition 2.5.8. *We say that* $a \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ *is of* $(\ln)^j$*–type,* $j \in (0,1]$ *if it has a representative* $a_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^n)$ *such that*

$$\|a_\varepsilon\|_2 = \mathcal{O}(\ln^j \varepsilon^{-1}), \quad \varepsilon \to 0.$$

Note that a function $a \in H^\infty(\mathbb{R}^n)$ is itself a representative in $\mathcal{G}_{H^2}(\mathbb{R}^n)$ (which can be proved as in the proof of Theorem 2.5.9 which is given in the sequel) and this is an example of a function that is of $(\ln)^j$–type for any $j \in (0,1]$. The reason for this is that its L^2–norm is a constant independent of ε. Further, if $a \in L^2(\mathbb{R}^n)$ and we embed it by $a \mapsto [(a * \rho_\varepsilon)_\varepsilon]$, then $\|a * \rho_\varepsilon\|_2 \lesssim \|a\|_2\|\rho_\varepsilon\|_1 = \|a\|_2$ and $[(a * \rho_\varepsilon)_\varepsilon]$ is also of $\ln^j$–type for any $j \in (0,1]$.

Finally, we discuss embedding functions in the space $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$.

Theorem 2.5.9. *Define the function* $\iota : W^{1,\infty}([0,T), L^2(\mathbb{R}^n)) \to \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$, $n \leq 3$ *by*

$$\iota(u) = [(u_\varepsilon)_\varepsilon]$$

where

$$u_\varepsilon(x,t) = \int_{\mathbb{R}^n} u(y,t)\rho_\varepsilon(x-y)dy \quad \text{for any } t \in [0,T). \tag{2.15}$$

(i) *This function is a linear injection. Restriction of the derivative* ∂^α*, for any* $\alpha \in \mathbb{N}^n$*, from* $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ *to* $W^{1,\infty}([0,T), L^2(\mathbb{R}^n))$ *is the usual distributional derivative.*

(ii) *The same embedding turns* $C^1([0,T), H^\infty(\mathbb{R}^n))$ *into a subalgebra of* $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$.

Proof. (i) For fixed values of t, (2.15) it is the usual convolution with a mollifier. Then, for any $|\alpha| \leq 2$ and every $t \in [0,T)$

$$\|\partial^\alpha(u * \rho_\varepsilon)\|_{L^2} = \|u * \partial^\alpha \rho_\varepsilon\|_{L^2} \leq \|u\|_{L^2}\|\partial^\alpha \rho_\varepsilon\|_{L^1} \sim \varepsilon^{-N}$$

for some $N \in \mathbb{N}$. Here we used Young's inequality (2.8). Also

$$\|\partial_t(u_\varepsilon(x,t))\|_{L^2} = \|\int_{\mathbb{R}^n} \partial_t(u(y,t))\rho_\varepsilon(x-y)dy\|_{L^2} = \|\partial_t u * \rho_\varepsilon\|_{L^2}, \text{ for every t} \in [0,\text{T}),$$

which is bounded (by a constant) again due to Young's inequality. So u_ε gives rise to an element $[u_\varepsilon] \in \mathcal{G}_{C^1,H^2}$. Moreover, for fixed values of t, we know that

$\|u * \phi_\varepsilon - u\|_{L^2} \to 0$. The embedding $u \hookrightarrow [u_\varepsilon]$ is thus an injection as a consequence of uniqueness of limit in L^2. Specifically, if $v_\varepsilon \in [u_\varepsilon]$, then

$$v = \lim_{\varepsilon \to 0} v_\varepsilon = \lim_{\varepsilon \to 0}(u_\varepsilon + n_\varepsilon) = u,$$

for every $t \in [0, T)$. We conclude that

$$W^{1,\infty}([0,T), L^2(\mathbb{R}^n)) \hookrightarrow \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n).$$

For partial derivatives in x there holds $\partial_x^\alpha(u * \rho_\varepsilon) = \partial_x^\alpha u * \rho_\varepsilon$ for any t, so $\iota(\partial_x^\alpha u) = \partial_x^\alpha [u * \rho_\varepsilon]$ and the derivative in $\mathcal{G}_{C^1,H^2}$ coincides with the (distributional) derivative in $W^{1,\infty}([0,T), L^2(\mathbb{R}^n))$. The same holds for the derivative in t since $\partial_t^\alpha u_\varepsilon = \partial_t^\alpha u * \rho_\varepsilon$ for any t.

(ii) We need to show that $u_\varepsilon - u \in \mathcal{N}_{C^1,H^2}$, where u_ε is given by (2.15) and $u \in C^1([0,T), H^\infty)$. The reason for this is the following. If we observed a constant embedding $u \mapsto [u]$, then $[u \cdot v] = [u_\varepsilon \cdot v_\varepsilon]$ is automatically satisfied. On the other hand, we need to use convolution to be able to embed other functions, too. So if u and u_ε given by (2.15) represent the same class, then

$$[(u \cdot v)_\varepsilon] = [u_\varepsilon \cdot v_\varepsilon].$$

We continue as in as in [BO92]. For fixed values of t there holds

$$\|u_\varepsilon - u\|_2^2 = \|u * \rho_\varepsilon - u\|_2^2 = \int | \int (u(x - \varepsilon y) - u(x))\rho(y)dy|^2 dx$$

We can apply the Taylor's formula to u up to the order of m. Since $\int y^\alpha \rho(y)dy = 0$ for $|\alpha| \le m$ (by (2.14)) we obtain

$$\begin{aligned}
\|u_\varepsilon - u\|_2^2 &= \int | \sum_{|\alpha|=m+1} \int \frac{(-\varepsilon y)^{\alpha})}{m!} \int_0^1 (1-\sigma)^m \partial^\alpha u(x - \sigma\varepsilon y) d\sigma \rho(y) dy|^2 dx \\
&\le C(m,q) \max_{|\alpha|=m+1} \int \left| \int \frac{(-\varepsilon y)^{\alpha})}{m!} \rho(y) \int_0^1 (1-\sigma)^m \partial^\alpha u(x - \sigma\varepsilon y) d\sigma dy \right|^2 dx \\
&\le C(m,q) \max_{|\alpha|=m+1} \int \int \left| \frac{(\varepsilon y)^{\alpha})}{m!} \rho(y) \int_0^1 (1-\sigma)^m \partial^\alpha u(x - \sigma\varepsilon y) d\sigma \right|^2 dx dy \\
&\le \frac{\varepsilon^{m+1}}{m!} C(m,q) \max_{|\alpha|=m+1} \int |y^\alpha \rho(y)| \int \int_0^1 |\partial^\alpha u(y - \sigma\varepsilon y)|^2 d\sigma dx dy \\
&\le c\varepsilon^{m+1} \int |y|^{m+1} |\rho(y)| dy \max_{|\alpha|=m+1} \|\partial^\alpha u\|_2.
\end{aligned}$$

So for any $m \in \mathbb{N}$ and sufficiently small ε we have

$$\|u_\varepsilon - u\|_2 \le c\varepsilon^m.$$

The same holds for $\partial_x^\alpha u$, $|\alpha| \le 2$. Finally, for any t

$$\|\partial_t(u - u_\varepsilon)\|_2 = \|\partial_t u - \partial_t u * \rho_\varepsilon\|_2 \le \varepsilon^M, \ \forall M \in \mathbb{N}, \varepsilon \to 0,$$

as above. $\square$

2.5.1 Notion of a solution

In this section let us observe the following Schrödinger equation:

$$\begin{aligned} iu_t + \triangle u + g(u) &= 0, \\ u(0) &= a. \end{aligned} \tag{2.16}$$

Definition 2.5.10. *We say that $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ is a solution of* (2.16) *if for an initial condition a and its representative $a_\varepsilon = a * \rho_\varepsilon$, there exists a representative $u_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ such that*

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) &= M_\varepsilon, \\ u_\varepsilon(0) &= a_\varepsilon + n_\varepsilon, \end{aligned} \tag{2.17}$$

for some $n_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^n)$, where $\sup_{t\in[0,T)} \|M_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^M)$, for any $M \in \mathbb{N}$.

If the above statement holds for some u_ε, then it holds for all representatives of the class $u = [u_\varepsilon]$: we show this for the linear part, and leave the analysis of $g(u)$ for Chapter 4 and Chapter 5. Let $v_\varepsilon = u_\varepsilon + N_\varepsilon$, $N_\varepsilon \in \mathcal{N}_{C^1,H^2}$, then

$$i(v_\varepsilon)_t + \triangle v_\varepsilon = i(u_\varepsilon)_t + \triangle u_\varepsilon + i(N_\varepsilon)_t + \triangle N_\varepsilon = M_\varepsilon + i(N_\varepsilon)_t + \triangle N_\varepsilon,$$

where $\|M_\varepsilon\|_{L^2} \sim \varepsilon^M$, for any $t \in [0,T)$. Now since $N_\varepsilon \in \mathcal{N}_{C^1,H^2}$, it follows $\|i(N_\varepsilon)_t + \triangle N_\varepsilon\|_{L^2} \sim \varepsilon^M$ for any $t \in [0,T)$. Also,

$$v_\varepsilon(0) = u_\varepsilon(0) + N_\varepsilon(0) = a_\varepsilon + n_\varepsilon + N_\varepsilon(0) = a_\varepsilon + N_\varepsilon^1$$

where $N_\varepsilon^1 \in \mathcal{N}_{H^2}$.

We also always start by solving precisely

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) &= 0, \\ u_\varepsilon(0) &= a_\varepsilon, \end{aligned}$$

$a_\varepsilon = a * \rho_\varepsilon$, since it follows from the previous analysis that $[u_\varepsilon]$ is indeed a solution.

Definition 2.5.11. *We say that a solution of* (2.16) *is unique if for any two solutions* $u, v \in \mathcal{G}_{C^1,H^2}$ *there holds* $\sup_{t\in[0,T)} \|u_\varepsilon - v_\varepsilon\|_{L^2} = O(\varepsilon^M)$, *for any* $M \in \mathbb{N}$.

These definitions justify the use of spaces based on nets $u_\varepsilon \in C([0,T), H^2(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n))$, $\varepsilon \in (0,1)$, for $n \leq 3$, which is natural for the equation in question.

2.5.2 Compatibility

We will see in Section 3 that for $a \in H^2(\mathbb{R}^n)$, $n \leq 3$, there is a unique solution $u \in C([0,T), H^2(\mathbb{R}^n))$ of the cubic equation (1.1). The space $H^2(\mathbb{R}^n)$ is embedded in the Colombeau algebra $\mathcal{G}_{H^2}(\mathbb{R}^n)$, which can again be proved as in Theorem 2.5.9. If there is a unique solution of (1.1) in $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$, then there is a representative u_ε that solves

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon &= u_\varepsilon |u_\varepsilon|^2, \\ u_\varepsilon(0) &= a * \rho_\varepsilon \end{aligned}$$

for $a \in H^2$ (as mentioned in the previous section, we always show that there is a solution to the equation without negligible functions, so the above claim will be justified). Ideally, classes $[(u_\varepsilon)_\varepsilon]$ and $[(u * \rho_\varepsilon)_\varepsilon]$ will coincide. But we are usually able to prove a slightly weaker version of this equality of classes, given by the following definition.

Definition 2.5.12. *We say that* $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ *is associated with a distribution* $v(t) \in \mathcal{D}'(\mathbb{R}^n)$ *for any* $t \in [0,T)$ *if there is a representative* u_ε *of* u *such that* $u_\varepsilon \to v$ *in* $\mathcal{D}'(\mathbb{R}^n)$ *for any* $t \in [0,T)$ *as* $\varepsilon \to 0$. *We denote association by* $u \approx v$.

Note that in the case $a \in C^1([0,T), H^\infty)$, then a represents itself and the same holds for the corresponding solution $u \in C^1([0,T), H^\infty)$ so in this case we automatically have compatibility between the two solutions.

We are usually able to prove $\|u - u_\varepsilon\|_{L^2} \to 0,\ \varepsilon \to 0$, for every $t \in [0,T)$ (the "L^2–association") from which it follows $[(u_\varepsilon)_\varepsilon] \approx u$. This motivates the following definition.

Definition 2.5.13. *We say that there is compatibility between a classical (Sobolev) solution and the Colombeau solution of*

$$
\begin{aligned}
iu_t + \triangle u + g(u) &= 0 \\
u(0) &= a
\end{aligned}
$$

if $\sup_{[0,T)} \|u_\varepsilon - u\|_{L^2} \to 0$ *as* $\varepsilon \to 0$, *where* $u_\varepsilon \in \mathcal{E}_{C^1,H^2}$ *is a solution of*

$$
\begin{aligned}
i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) &= 0 \\
u_\varepsilon(0) &= a * \rho_\varepsilon.
\end{aligned}
$$

Looking outside the context of equivalence classes, the tools we derive - primarily estimates - can be of use for discussing different types of convergences. For example, there is no well- posedness theory for (1.2), but analyzing the net of solutions can give insight in that direction.

Uniqueness in the Colombeau algebra also differs from the usual notion. It is possible that different representatives $u_\varepsilon + n_\varepsilon$ solve the regularized equation, but in the limiting case, they all converge to the same limit - if they do converge, that is if there is compatibility.

Generally, there are several papers showing instability or non-uniqueness in some distributional spaces, e.g. [CCT03], [Chr05], [HW82]. This indicates uniqueness is a potential problem when observing singular solutions and is one more reason to stress the importance of compatibility.

3 Semilinear Schödinger equation

In this section we describe the theory of a general semilinear Schrödinger equation, namely

$$
\begin{aligned}
iu_t + \triangle u + g(u) &= 0, \\
u(0) &= a,
\end{aligned}
\tag{3.1}
$$

based on [Caz03]. The regularized equations we consider are of type (3.1), so the theory we present in this chapter serves as a starting point for later results. Also, the tools used in the classical theory are useful for our analysis in the Colombeau algebra, too. The space dimension in this chapter is arbitrary $n \in \mathbb{N}$ unless stated otherwise.

3.1 The evolution operator

We start with some properties of the Laplacian operator $A = \triangle$. It is well - known that $A : C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a densely defined symmetric operator. Namely

$$(Au, v)_2 = (u, Av)_2, \quad \forall u, v \in C_0^\infty(\mathbb{R}^n),$$

from which it follows that $(Au, u)_2$ is real. On the other hand, if we observe $A : H^2 \subset L^2 \to L^2$, we obtain a bounded operator:

$$\|Au\|_{L^2} < \|u\|_{H^2}.$$

The scalar product $(,)_2$ is continuous on $L^2 \times L^2$, but also on $H^2 \times H^2$ since $u_n \to u$ in H^2 implies $u_n \to u$ in L^2 and this further implies $(u_n, v_n)_2 \to (u, v)_2$ in $\mathbb{C}$. Further since C_0^∞ is dense in H^2 we have

$$\begin{aligned}(Au, v)_2 &= (A \lim u_n, \lim v_n)_2 = (\lim Au_n, \lim v_n)_2 = \lim(Au_n, v_n)_2 \\ &= \lim(u_n, Av_n)_2 = (u, Av)_2, \quad \forall u, v \in H^2.\end{aligned}$$

So for $u \in H^2$ we have that (Au, u) is real. This fact is used in deriving energy equalities.

We now focus on some approximation properties of **dissipative** operators, following [CBH+98]. All of the following statements are proved in [CBH+98]. An unbounded operator $A : D(A) \subset X \to X$ is dissipative if

$$\|u - \lambda Au\| \geq \|u\|,$$

for all $u \in D(A)$ and all $\lambda > 0$. An unbounded operator is m**–dissipative** if it is dissipative and for all $\lambda > 0$ and all $f \in X$ there exists $u \in D(A)$ such that

$$u - \lambda Au = f. \tag{3.2}$$

From these definitions, u is the unique solution of (3.2), and in addition $\|u\| \leq \|f\|$. Let $J_\lambda = (I - \lambda A)^{-1}$ so that $u = J_\lambda f$ is the solution of (3.2). Finally, let $A_\lambda = AJ_\lambda = \frac{J_\lambda - I}{\lambda}$. If $\overline{D(A)} = X$ and A is m–dissipative, then $A_\lambda u \to Au$ as $\lambda \to 0$ for all $u \in D(A)$. We state a theorem relevant for our setting

Theorem 3.1.1. *If X is a Hilbert space and A is densely defined self adjoint operator in in X such that $A \leq 0$ ($(Au, u) \leq 0$ for all $u \in D(A)$), then A is m–dissipative. If A is skew–adjoint ($A^* = -A$), then A and $-A$ are m–dissipative.*

Now let us state different ways of defining the Schrödinger operator $i\triangle u$. Let $Y = L^2(\mathbb{R}^n)$ and B be a linear operator in Y such that

$$\begin{aligned}D(B) &= \{u \in H^1(\mathbb{R}^n),\ \triangle u \in Y\}; \\ Bu &= i\triangle u, \quad \forall u \in D(B).\end{aligned} \tag{3.3}$$

This operator B is skew–adjoint, B and $-B$ are m–dissipative operators with dense domains. Further, let $X = H^{-1}(\mathbb{R}^n)$ and given $u \in X$ let $\varphi_u \in H^1$ be the solution of $-\triangle\varphi_u + \varphi_u = u$ in X. Then X can be equipped with the scalar product

$$(u, v)_{-1} = (\varphi_u, \varphi_v)_{H^1} = \int_{\mathbb{R}^n} (\nabla\varphi_u \cdot \overline{\nabla\varphi_v} + \varphi_u\overline{\varphi_v})dx.$$

Now an operator A on X defined in the following way

$$\begin{aligned} D(A) &= H^1; \\ Au &= i\triangle u, \quad \forall u \in D(A), \end{aligned} \tag{3.4}$$

enjoys the same properties as B: A is skew-adjoint and A and $-A$ are dissipative with dense domains.

We now aim to connect this with the notion of a propagator (the evolution operator) $\mathcal{T}(t)$. First we state some definitions regarding the exponential operator e^A. Let X be a Banach space and $A \in \mathcal{L}(X)$. By e^A we denote the sum of the series $\sum_{n\geq 0} \frac{1}{n!}A^n$. The series is convergent in the norm of $\mathcal{L}(X)$ and if A and B commute, then $e^{A+B} = e^A e^B$. Further, for a fixed operator A, the function $t \mapsto e^{tA}$ belongs to $C^\infty(\mathbb{R}, \mathcal{L}(X))$ and there holds

$$\frac{d}{dt}e^{tA} = A \cdot e^{tA}$$

for all $t \in \mathbb{R}$. Moreover, the following result holds

Theorem 3.1.2. *Let $A \in \mathcal{L}(X)$. For all $T > 0$ and all $x \in X$, there exists a unique solution $u \in C^1([0,T], X)$ of the problem:*

$$\begin{aligned} u'(t) &= Au(t), \\ u(0) &= x. \end{aligned}$$

The solution is given by $u(t) = e^{tA}x$, for all $t \in [0,T]$.

Let A be an m–dissipative operator on X - a Banach space and J_λ, A_λ be as before. Set $\mathcal{T}_\lambda(t) = e^{tA_\lambda}$, $t \geq 0$.

Theorem 3.1.3. *For all $x \in X$ the sequence $u_\lambda(t) = \mathcal{T}_\lambda(t)x$ converges uniformly on bounded intervals of $[0,T]$ to a function $u \in C((0,\infty), X)$ as $\lambda \to 0$. We set $\mathcal{T}(t)x = u(t)$ for all $x \in X$ and $t \geq 0$. Then,*

$$\begin{aligned} &\mathcal{T}(t) \in \mathcal{L}(X) \text{ and } \|\mathcal{T}(t)\| \leq 1, \quad \forall t \geq 0, \\ &\mathcal{T}(0) = I, \\ &\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(t), \quad \forall s, t \geq 0. \end{aligned}$$

In addition, for all $x \in D(A)$, $u(t) = \mathcal{T}(t)x$ is the unique solution of the problem

$$u'(t) = Au(t)$$
$$u(0) = x$$

and $u \in C([0,\infty), D(A)) \cap C^1([0,\infty), X)$. Finally, $\mathcal{T}(t)Ax = A\mathcal{T}(t)x$, $\forall x \in D(A)$, $t \geq 0$.

We can now also discuss the notion of a one–parameter family $(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(X)$. This family is called a **contraction semigroup** if

- $\|\mathcal{T}(t)\| \leq 1, \quad \forall t \geq 0,$
- $\mathcal{T}(0) = I,$
- $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(t), \quad \forall s,t \geq 0.$
- for all $x \in X$, the function $t \mapsto \mathcal{T}(t)x$ belongs to $C([0,\infty), X)$.

The **generator** of $(\mathcal{T}(t))_{t\geq 0}$ is the linear operator A defined by

$$D(A) = \{x \in X;\ \frac{\mathcal{T}(t)x - x}{h} \text{ has a limit in } X \text{ as } h \to 0\},$$

and
$$Ax = \lim_{h\to 0} \frac{\mathcal{T}(t)x - x}{h}.$$

We now paraphrase the Hille–Yosida–Phillips theorem ([CBH+98, Theorem 3.4.4.]).

Theorem 3.1.4. *If $(\mathcal{T}(t))_{t\geq 0}$ is a contraction semigroup, then its generator A is m–dissipative and $D(A)$ is dense in X. Conversely, if A is an m–dissipative with dense domain and $(\mathcal{T}(t))_{t\geq 0}$ is the semigroup corresponding to A given by Theorem 3.1.3, then its generator is exactly A.*

The family $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is called an **isometry group** in X if

- $\|\mathcal{T}(t)x\| = \|x\|, \quad \forall t \in \mathbb{R}, \forall x \in X,$
- $\mathcal{T}(0) = I,$
- $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(t), \quad \forall s,t \geq 0.$
- for all $x \in X$, the function $t \mapsto \mathcal{T}(t)x$ belongs to $C(\mathbb{R}, X)$.

The following theorem holds

Theorem 3.1.5. *Let A be an m–dissipative operator with dense domain, and let $(\mathcal{T}(t))_{t\geq 0}$ be the contraction semigroup generated by A. Then $(\mathcal{T}(t))_{t\geq 0}$ is the restriction to $\mathbb{R}_+$ of an isometry group if and only if $-A$ is m–dissipative.*

To summarize, let us apply this theory to the Schrödinger operators (3.3) and (3.4). Let $(\mathcal{S}(t))_{t\in\mathbb{R}}$ and $(\mathcal{T}(t))_{t\in\mathbb{R}}$ be the isometry groups generated by B and A defined by (3.3) and (3.4). Keeping the same notation, there holds

$$\mathcal{S}(t)\varphi = \mathcal{T}(t)\varphi, \quad \forall t \in \mathbb{R}, \quad \forall \varphi \in Y.$$

The following theorem holds

Theorem 3.1.6. *Let $\varphi \in H^1$ and let $u(t) = \mathcal{T}(t)\varphi$. Then u is the unique solution to the problem*

$$\begin{aligned} iu_t + \triangle u &= 0, \\ u(0) &= \varphi \end{aligned}$$

and $u \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. If $\triangle\varphi \in L^2$, then $u \in C^1(\mathbb{R}, L^2)$ and $\triangle u \in C(\mathbb{R}, L^2)$.

All of the above holds in the case of a general domain $\Omega \subset \mathbb{R}^n$, but when we have specifically $\mathbb{R}^n$, then we can derive additional properties and explicitly express $(\mathcal{T}(t))_{t\in\mathbb{R}}$ in Fourier variables. The following theorem holds.

Theorem 3.1.7. *Let $p \in [2, \infty]$, $1/p + 1/p' = 1$ and $t > 0$. Then $\mathcal{T}(t)$ can be extended to an operator belonging to $\mathcal{L}(L^{p'}, L^p)$ and*

$$\|\mathcal{T}(t)\varphi\|_p \leq (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{p})}\|\varphi\|_{p'}, \quad \text{for all } \phi \in L^{p'} \tag{3.5}$$

Proof. Take the equation

$$iu_t + \triangle u = 0, \quad u(0,x) = \varphi(x), \tag{3.6}$$

for φ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Then,

$$\begin{aligned} \widehat{u(t)}_t(\xi) &= -i|\xi|^2\widehat{u(t)}(\xi) \quad \text{and} \\ \widehat{u(t)}(\xi) &= e^{-it|\xi|^2}\widehat{\varphi}(\xi). \end{aligned}$$

It follows that

$$u(t) = \int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\cdot\xi} \widehat{\varphi}(\xi) d\xi$$

is a solution and because of the previous theorem there holds $u(t) = \mathcal{T}(t)\varphi(x)$. We can also write $u(t) = K(t) * \varphi$ where

$$K(t) = \mathcal{F}^{-1}(e^{-i|\xi|^2 t}(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{i|x|^2}{4t}}.$$

It follows

$$\|\mathcal{T}(t)\varphi\|_\infty \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \|\varphi\|_1,$$

for all $t \neq 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Thus, one can extend $\mathcal{T}(t)$ to an operator in $\mathcal{L}(L^1, L^\infty)$, such that the above inequality holds for $\varphi \in L^1$. Similarly, $\mathcal{T}(t) \in \mathcal{L}(L^2, L^2)$ and is unitary. The general case follows from the Riesz–Thorin convexity theorem 2.4.7. $\square$

The operator $\mathcal{T}(t)$ is unitary on $L^2(\mathbb{R}^n)$, but also on $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. It is a Fourier multiplier and as such, commutes with other Fourier multipliers, including constant coefficient differential operators.

Finally, we wish to generalize Theorem 3.1.6 to the nonlinear case. The following holds ([CBH+98, Section 4.3.]).

Theorem 3.1.8 (Duhamel's formula). *Let $F : X \to X$ be a Lipschitz continuous function on bounded subsets of X. If u is a solution of the problem*

$$\begin{cases} u \in C([0,T], D(A)) \cap C^1([0,T], X); & (3.7) \\ u'(t) = Au(t) + F(u(t)) \quad \forall t \in [0,T]; & (3.8) \\ u(0) = x, & (3.9) \end{cases}$$

then

$$u(t) = \mathcal{T}(t)x + \int_0^t \mathcal{T}(t-s)F(u(s))ds, \quad \forall t \in [0,T]. \tag{3.10}$$

Conversely, if u satisfies (3.10) *then* (3.7) – (3.9) *hold.*

Let us now define the type of solution relevant for our setting, now following [Caz03]. We consider distributive solutions of (3.1).

Definition 3.1.9. *Let $g \in C(H^1, H^{-1})$, $a \in H^1$ and I is an interval such that $0 \in I$.*

(i) A weak H^1 solution u of (3.1) *is a function*

$$u \in L^\infty(I, H^1) \cap W^{1,\infty}(I, H^{-1})$$

such that $iu_t + \triangle u + g(u) = 0$ in H^{-1} for a.a. $t \in I$ and $u(0) = a$.

(ii) A strong H^1 solution u of (3.1) *is a function*

$$u \in C(I, H^1) \cap C^1(I, H^{-1})$$

such that $iu_t + \triangle u + g(u) = 0$ in H^{-1} for all $t \in I$ and $u(0) = a$.

In the following we will deal with strong solutions. Note that

$$iu_t + \triangle u + g(u) = 0 \text{ } in \text{ } H^{-1} \Leftrightarrow \langle \varphi, iu_t + \triangle u + g(u) \rangle = 0, \ \forall \varphi \in H^1,$$

so when $u \in H^2$ and $g(u), u_t \in L^2$ the expression $iu_t + \triangle u + g(u)$ is in L^2 and the above becomes

$$\int_{\mathbb{R}^n} (iu_t + \triangle u + g(u))\varphi = 0, \quad \forall \varphi \in H^1 \text{ } and \text{ } \forall t \in I.$$

It follows $iu_t + \triangle u + g(u) = 0$ for almost all $x \in \mathbb{R}^n$. So when we have a H^2 solution in the sense of Definition 3.1.9, then we have that (3.1) holds point-wise on $\mathbb{R}^n$.

Remark 3.1.10. *The boundary condition $u(t) \to 0$ as $|x| \to \infty$ is usually a part of defining a solution to the problem* (3.1)*, but in our relevant case $u \in H^2$ it holds since it holds for all H^2 functions, see* (2.3).

We can state the Duhamel's formula specified for our setting and as in [Caz03].

Theorem 3.1.11 (Duhamel's formula). *Let I be an interval such that $0 \in I$, let $g \in C(H^1, H^{-1})$ and $a \in H^1$. If g is bounded on bounded sets and $u \in L^\infty(I, H^1)$, then u is a weak H^1 solution of* (3.1) *on I if and only if*

$$u(t) = \mathcal{T}(t)a + i\int_0^t \mathcal{T}(t-s)g(u(s))ds \text{ for all } t \in I, \quad for \text{ } a.a. \text{ } t \in I. \tag{3.11}$$

A function $u \in C(I, H^1)$ is a strong H^1 solution of (3.1) *on I if and only if it satisfies* (3.11) *for all $t \in I$.*

Note that

$$\mathcal{T}(t-s)f(s) = \int e^{-i(t-s)|\xi|^2 + ix\xi}\widehat{f(s,x)}(\xi)d\xi$$

and the Fourier transform $\widehat{f(s,x)}$ is in the x variable and s denotes the time variable.

3.2 Strichartz estimates and uniqueness

We start by introducing the model nonlinearity $g(u)$.

Definition 3.2.1. *Let* $g(u) = -(Vu + u|u|^2 + (w * |u|^2)u)$ *for* $w \in W^{2,p}$*,* $p > 2$ *and* w *is even,* $V \in C_0^\infty(\mathbb{R}^n)$*.*

Definition 3.2.2. *We say that a pair* (q,r) *is admissible if*

$$\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r}) \quad \text{and} \tag{3.12}$$

$$2 \leq r \leq \frac{2n}{n-2} \; (2 \leq r \leq \infty \text{ if } n = 1, \; 2 \leq r < \infty \text{ if } n = 2). \tag{3.13}$$

Note that if (q,r) is admissible, then $2 \leq q \leq \infty$. Also, $(\infty, 2)$ is always admissible; $(2, \frac{2n}{n-2})$ is admissible for $n \leq 3$.

When dealing with the whole space $\mathbb{R}^n$, Strichartz estimates are a very useful tool. We now present them, as in [Caz03, Theorem 2.3.3.]

Theorem 3.2.3 (Strichartz's estimates)**.** *The following properties hold*

- *For every* $\varphi \in L^2(\mathbb{R}^n)$*, the function* $t \to \mathcal{T}(t)\varphi$ *belongs to*

  $$L^q(\mathbb{R}, L^r(\mathbb{R}^n)) \cap C(\mathbb{R}, L^2(\mathbb{R}^n))$$

  for every admissible pair (q,r)*. Furthermore, there exists a constant* C *such that*

  $$\|\mathcal{T}(\cdot)\varphi\|_{L^q(\mathbb{R},L^r)} \leq C\|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\mathbb{R}^n).$$

- *Let* I *be an interval in* $\mathbb{R}$ *(bounded or not),* $J = \overline{I}$ *and* $t_0 \in J$*. If* (γ, ρ) *is an admissible pair and* $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^n))$*, then for any admissible pair* (q,r)*, the function*

  $$t \to \Phi_f(t) = \int_{t_0}^{t} \mathcal{T}(t-s) f(s) ds \quad \text{for } t \in I$$

belongs to $L^q(I, L^r(\mathbb{R}^n)) \cap C(J, L^2(\mathbb{R}^n))$. Moreover, there exists a constant C independent of I such that

$$\|\Phi_f\|_{L^q(I,L^r)} \leq C\|f\|_{L^{\gamma'}(I,L^{\rho'})} \quad \forall f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^n)).$$

Lemma 3.2.4. *Let $I \ni 0$ be an interval. Let $1 \leq a_j < s_j \leq \infty$ and $\phi_j \in L^{s_j}(I)$, for $1 \leq j \leq k$. If there exists a constant $C \geq 0$ such that*

$$\sum_{j=1}^{k} \|\phi_j\|_{L^{s_j}(J)} \leq C \sum_{j=1}^{k} \|\phi_j\|_{L^{a_j}(J)}$$

for every interval J such that $0 \in J \subset I$, then $\phi_1 = \cdots = \phi_k = 0$ a.e. on I.

Theorem 3.2.5. *Let g be as in Definition 3.2.1. If $a \in H^1$ and u_1, u_2 are two weak H^1 solutions of* (3.1) *on some interval $I \ni 0$, then $u_1 = u_2$.*

Proof. Let $u, v \in L^\infty(I, H^1) \cap W^{1,\infty}(I, H^{-1})$ be two solutions of (3.1) and let us assume that I is a bounded interval. By (3.11)

$$u(t) - v(t) = i \int_0^t \mathcal{T}(t-s)\,(g(u(s)) - g(v(s)))\,ds \quad \text{for a.a. } t \in I.$$

Let us denote $W(t) = u(t) - v(t)$ and $f(t) = g(u) - g(v)$ so we can write

$$W(t) = i \int_0^t \mathcal{T}(t-s) f(s) ds \quad \text{for a.a. } t \in I.$$

Specifically, f is a sum of terms $f_j(t) = g_j(u) - g_j(v)$ where $g_1(u) = Vu$, $g_2(u) = (w * |u|^2)u$ and $g_3(u) = u|u|^2$. Now we will show that

$$\|f_j\|_{L^{\gamma'_j}(I,L^{\rho'_j})} \leq C\|W\|_{L^{\gamma'_j}(I,L^{r_j})} \tag{3.14}$$

for some admissible pairs (q_j, r_j) and (γ_j, ρ_j). It is not difficult to see that

$$\|f_j\|_2 \leq C\|W\|_2$$

for $j=1$. this also holds for $j=2$ since

$$\begin{aligned}
&\|(w*|u|^2)u-(w*|v|^2)v\|_2 = \|(w*|u|^2)(u-v)+v\cdot w*(|u|^2-|v|^2)\|_2 \\
&\lesssim \|w\|_\infty \|u^2\|_1 \|u-v\|_2 + \|v\cdot w*(|u|-|v|)(|u|+|v|)\|_2 \\
&\lesssim T\|w\|_\infty \|u\|_2^2 \|u-v\|_2 + \|v\|_2 \|w\|_\infty \|(|u|-|v|)(|u|+|v|)\|_2 \\
&\lesssim \|u-v\|_2(\|w\|_\infty(\|u\|_2^2+\|v\|_2(\|u\|_2+\|v\|_2))) \\
&= c(\|w\|_\infty, \|a\|_2)\|W\|_2.
\end{aligned}$$

Here Young's inequality (2.8) was used, for $r=2=p$, $q=1$ and $|\,|x|-|y|\,|\le|x-y|$. Applying the $L^{\gamma_j'}$–norm in t we obtain (3.14) for $r_j=r_j'=\rho_j=\rho_j'=2$, $q_j=\gamma_j=\infty$ and $\gamma_j'=q_j'=1$.
Regarding the cubic term, observe that

$$\begin{aligned}
|u|u|^2-v|v|^2| &= |u(|u|-|v|)(|u|+|v|)+|v|^2(u-v)| \\
&\le |u||u-v|(|u|+|v|)+|v|^2|u-v| = |u-v|(|u|+|v|)^2.
\end{aligned}$$

Using Hölder inequality we obtain

$$\|f_3\|_{\frac{4}{3}} \le \|u-v\|_4(\|u\|_4+\|v\|_4)^2$$

The norms $\|u\|_4$ and $\|v\|_4$ are bounded by the H^1 norm: using (2.11) inequality for $j=0,\ p=4,\ r=q=2,\ m=1$

$$\|u\|_4 \lesssim (\sum_{|\alpha|=1}\|\partial^\alpha u\|_2)^b\|u\|_2^{1-b} \le \|u\|_{H^1}^b\|u\|_{H^1}^{1-b} = \|u\|_{H^1}$$

where $b=\frac{1}{2}$ for $n=2$ and $b=\frac{3}{4}$ for $n=3$. In return $\|u\|_{H^1}$ is bounded (in particular by $c(\|a\|_{H^1})$ which we will see in the following section). So $H^1\hookrightarrow L^4$ and $|I|<\infty$ imply that (3.14) holds for $j=3$ also, and here $\rho_j'=r_j'=\frac{4}{3},\ \rho_j=r_j=4,\ q_j=\gamma_j=\frac{8}{n},\ q_j'=\gamma_j'=\frac{8}{8-n}$. Now we apply Strichartz estimates to $W(t)=\sum W_j$. First, there holds

$$\begin{aligned}
\|W_j\|_{L^{s_l}(I,L^{r_l})} &\le C\|f_j\|_{L^{\gamma_j'}(I,L^{\rho_j'})} \quad \text{and} \\
\sum_l \|W_j\|_{L^{s_l}(I,L^{r_l})} &\le C\|f_j\|_{L^{\gamma_j'}(I,L^{\rho_j'})}
\end{aligned}$$

for any admissible pair (s_l,r_l). Then

$$\sum_{j=1}^3\|W\|_{L^{s_j}(I,L^{r_j})} \le \sum_{j,l}\|W_l\|_{L^{s_j}(I,L^{r_j})} \le C\sum_{j=1}^3\|f_j\|_{L^{\gamma_j'}(I,L^{\rho_j'})} \le C\sum_{j=1}^3\|W_j\|_{L^{\gamma_j'}(I,L^{r_j})}$$

for $1 \leq \gamma_j' < s_j \leq \infty$, since $s_j = \frac{8}{n}$, $\gamma_j' = \frac{8}{8-n}$ for $j = 3$ and $s_j = \infty$, $\gamma_j' = 1$ for $j = 1, 2$. Now denoting

$$\phi_j(t) = \|W\|_{L^{r_j}}$$

the result follows from Lemma 3.2.4.

□

3.3 Well - posedness in Sobolev spaces

In this section we describe well - posedness of (3.1) in Sobolev spaces, mainly in H^2 but also in the energy space H^1 and then H^3. This theory is used in sections 4 and 5.

By local well–posedness we mean the following.

Definition 3.3.1. *We say that the initial value problem is locally well–posed in H^m, $m \in \mathbb{N}$, if the following properties hold:*

- *A solution of* (3.1) *is unique in H^m.*
- *For every $a \in H^m$, there exists a strong H^m solution of* (3.1) *defined on a maximal interval $(-T_{min}, T_{max})$ (a "maximal" solution) with $T_{max} = T_{max}(a) \in (0, \infty]$ and $T_{min} = T_{min}(a) \in (0, \infty]$.*
- *There is blowup alternative: if $T_{max} < \infty$, then $\lim_{t \to T_{max}} \|u(t)\|_{H^m} = +\infty$ (respectively, if $T_{min} < \infty$, then $\lim_{t \to -T_{min}} \|u(t)\|_{H^m} = +\infty$).*

Remark 3.3.2. *In our work, we do not discuss continuous dependence of the solution on initial data, so we do not include this notion in the definition of well–posedness.*

Definition 3.3.3. *If there is local well–posedness and additionally $T_{max} = T_{min} = \infty$, then we say that there is global well–posedness.*

Theorem 3.3.4. *Let g be as in Definition 3.2.1. For every $a \in H^2$ there exist $T_{min}, T_{max} > 0$ and a unique, maximal solution $u \in C((-T_{min}, T_{max}), H^2) \cap C^1((-T_{min}, T_{max})), L^2)$ of* (3.1). *Furthermore, the blowup alternative holds.*

Proof. Given $M, T > 0$ to be chosen later and $I = (-T, T)$, observe the space

$$\begin{aligned} E =&\{u \in L^\infty(I, H^1) \cap W^{1,\infty}(I, L^2) \cap W^{1,q}(I, L^r); \\ & u(0) = a, \quad \|u\|_{L^\infty(I,H^1)} + \|u\|_{W^{1,\infty}(I,L^2)} + \|u\|_{W^{1,q}(I,L^r)} \le M\}, \end{aligned}$$

where $(q, r) = (\frac{8}{n}, 4)$ is admissible. This is a complete metric space, where the metric is defined by

$$d(u, v) = \|u - v\|_{L^\infty(I,H^1)} + \|u - v\|_{L^q(I,L^r)}.$$

Let

$$\Phi(u)(t) = \mathcal{T}(t)a + \mathcal{G}(u)(t),$$

where

$$\mathcal{G}(u)(t) = i \int_0^t \mathcal{T}(t - s)g(u(s))ds.$$

Denote by $g_1(u) = Vu$, $g_2(u) = (w * |u|^2)u$, $g_3(u) = u|u|^2$. Like in the proof of Theorem 3.2.5

$$\begin{aligned} \|g_1(u) - g_1(v)\|_2 &\le \|V\|_\infty \|u - v\|_2 \\ \|g_2(u) - g_2(v)\|_2 &\le c(\|w\|_\infty, \|u\|_2, \|v\|_2)\|u - v\|_2 \\ \|g_3(u) - g_3(v)\|_{\frac{4}{3}} &\le c(\|u\|_4, \|v\|_4)\|u - v\|_4. \end{aligned}$$

Using the same notation as before, $\rho_1' = \rho_2' = 2 = r_1 = r_2 = \rho_1 = \rho_2$; $\rho_3' = \frac{4}{3}$, $r_3 = \rho_3 = 4$. Further, if $\|u\|_{H^1} \le M$, then

$$\|\mathcal{G}(u)(t)\|_2 \le TK(M).$$

We now estimate $\frac{\partial}{\partial t} g_j(u)$ in the following way

$$\begin{aligned} \left\| \frac{\partial}{\partial t}(Vu) \right\|_2 &\le \|V\|_\infty \|u_t\|_2 \\ \left\| \frac{\partial}{\partial t}((w * |u|^2)u \right\|_2 &\lesssim \|w\|_\infty \|u\|_2^2 \|u_t\|_2^2 \\ \left\| \frac{\partial}{\partial t}(u|u|^2) \right\|_{\frac{4}{3}} &\lesssim \||u|^2 u_t\|_{\frac{4}{3}} \lesssim \|u\|_4^2 \|u_t\|_4. \end{aligned}$$

Let $v(t) = \mathcal{G}(u)(t)$. Then v satisfies

$$iv_t + \triangle v + g(u) = 0, \quad v(0) = 0.$$

It follows

$$\|\triangle v\|_2 \lesssim \|v_t\|_2 + \|g(u)\|_2.$$

Further,

$$\begin{aligned}
v_t(t) &= i\frac{\partial}{\partial t}\int_0^t \mathcal{T}(t-s)g(u(s))ds = i\frac{\partial}{\partial t}\int_0^t \mathcal{T}(s)g(u(t-s))ds \\
&= i\mathcal{T}(t)g(u(0))\cdot 1 - i\mathcal{T}(0)g(u(t))\cdot 0 + i\int_0^t \mathcal{T}(s)\frac{\partial}{\partial t}(g(u(t-s))ds \\
&= i\mathcal{T}(t)g(u(0)) + \int_0^t \mathcal{T}(t-s)\frac{\partial}{\partial t}(g(u(s))ds.
\end{aligned}$$

By Strichartz estimates

$$\|\int_0^t \mathcal{T}(t-s)f_t(s)ds\|_{L^\infty(I,L^2)} \leq c\|f_t\|_{L^{\gamma'}(I,L^{\rho'})},$$

where (γ,ρ) is an admissible pair. In our case above

$$\begin{aligned}
\|v_t\|_{L^\infty(I,L^2)} &\lesssim \|g(u(0))\|_2 + \|\int_0^t \mathcal{T}(t-s)\frac{\partial}{\partial t}(g(u(s)))ds\|_{L^\infty(I,L^2)} \\
&\lesssim \|g(u(0))\|_2 + \sum_{j=1}^{3}\|\frac{\partial}{\partial t}(g_j(u(t)))\|_{L^{\gamma_j'}(I,L^{\rho_j'})}
\end{aligned}$$

and $(\gamma_j',\rho_j') = (\infty,2)$ for $j=1,2$ (and here we just apply the fact that $\mathcal{T}$ is unitary on L^2), $(\gamma_3',\rho_3') = (\frac{8}{8-n},\frac{4}{3})$ so that $(\gamma,\rho) = (\frac{8}{n},4)$ is admissible. Combining all the estimates

$$\begin{aligned}
&\|\mathcal{G}(u)(t)\|_{L^\infty(I,H^2)} \lesssim \|v\|_{L^\infty(I,L^2)} + \|\triangle v\|_{L^\infty(I,L^2)} \lesssim T(K(M) + \|g(a)\|_{L^\infty(I,L^2)} \\
&+ \|g(u)\|_{L^\infty(I,L^2)} + \|V\|_\infty\|u_t\|_{L^\infty(I,L^2)} + \|w\|_\infty\|u\|^2_{L^\infty(I,L^2)}\|u_t\|^2_{L^\infty(I,L^2)}) \\
&+ \|\|u\|_4^2\|u_t\|_4\|_{L^{\frac{8}{8-n}}(I)}.
\end{aligned} \tag{3.15}$$

Note that here we used that the upper bound for $\|v\|_{H^2}$ is essentially $\|v\|_2 + \|\triangle v\|_2$. This is due to

$$\sum_{|\alpha|\leq 2}\|\partial^\alpha v\|_2 = \sum_{|\alpha|\leq 2}\|\xi^\alpha\hat{v}\|_2 \leq \sum_{|\alpha|\leq 2}\|(1+\xi^2)^{\alpha/2}\hat{v}\|_2 \leq c\|(1+|\xi|^2)\hat{v}\|_2 = c(\|v\|_2+\|\triangle v\|_2).$$

On the other hand

$$\begin{aligned}
\Big(\int_{-T}^{T}(\|u\|_4^2\|u_t\|_4)^{\frac{8}{8-n}}dt\Big)^{\frac{8-n}{8}} &\lesssim \Big(\int \|u_t\|_4^{\frac{8}{8-n}\cdot\frac{8-n}{n}}dt\Big)^{\frac{n}{8}}\Big(\int \|u\|_4^{2\cdot\frac{8}{8-n}\cdot\frac{8-n}{8-2n}}dt\Big)^{\frac{8-2n}{8}} \\
&\lesssim \|u_t\|_{L^{\frac{8}{n}}(I,L^4)}\|u\|^2_{L^p(I,H^1)} \\
&\lesssim T\|u_t\|_{L^{\frac{8}{n}}(I,L^4)}\|u\|^2_{L^\infty(I,H^1)},
\end{aligned}$$

where $p=\frac{8}{8-2n}$ and $(\frac{8}{n},4)$ is admissible. Returning to (3.15)

$$\begin{aligned}
\|\Phi(u)(t)\|_{H^2} &\le T(\|a\|_{H^2}+c_1\|a\|_{H^1}+c_2M+c_3M^4+c_4M^3) \\
&\le c_0T(\|a\|_{H^2}+\|a\|_{H^1}+M+M^4+M^3),
\end{aligned}$$

with c_0 independent of M and T. Now choosing $M=\|a\|_{H^2}$ and T sufficiently small we obtain

$$\|\Phi\|_{L^\infty(I,H^2)}\le M. \tag{3.16}$$

Repeating some of the arguments

$$\Big\|\frac{\partial}{\partial t}\Phi\Big\|_{L^\infty(I,L^2)}\lesssim Tc_0(\|a\|_{H^1}+M+M^4+M^3)$$

and by Strichartz estimates, similarly as before

$$\Big\|\frac{\partial}{\partial t}\Phi\Big\|_{L^q(I,L^r)}\lesssim \|g(a)\|_2+\sum_{j=1}^{3}\Big\|\frac{\partial}{\partial t}g_j(u))\Big\|_{L^{\gamma'_j}(I,L^{\rho'_j})}$$

for (γ_j,ρ_j) admissible. Specifically, $(\gamma_j,\rho_j)=(\infty,2)$ and $(\gamma'_j,\rho'_j)=(1,2)$ for $j=1,2$ so that

$$\begin{aligned}
\|\frac{\partial}{\partial t}(Vu)\|_{L^1(I,L^2)} &\lesssim T\|Vu_t\|_{L^\infty(I,L^2)} \\
\|\frac{\partial}{\partial t}((w*|u|^2)u)\|_{L^1(I,L^2)} &\lesssim T\|u\|^2_{L^\infty(I,L^2)}\|u_t\|^2_{L^\infty(I,L^2)}
\end{aligned}$$

and $(\gamma'_3,\rho'_3)=(\frac{8}{8-n},\frac{4}{3})$ as before. Finally, choosing T possibly smaller,

$$\|\Phi(t)\|_{L^\infty(I,H^1)}+\|\Phi(t)\|_{W^{1,\infty}(I,L^2)}+\|\Phi(t)\|_{W^{1,q}(I,L^r)}\le M$$

so that $\Phi:E\to E$. A similar, though simpler argument shows that Φ is a contraction on E, so it has a unique fixed point on E. Moreover, by (3.16), $u\in L^\infty(I,H^2)$. It can be shown that $u\in C(I,H^2)\cap C^1(I,L^2)$. By Theorem 3.2.5 uniqueness on the whole space H^2 follows. We can now define a unique maximal solution (for

$T_{max} = \sup\{T\}$). Since I depended on $\|a\|_{H^2}$, the blowup alternative can be shown by contradiction. $\square$

We present also local well – posedness in the energy space H^1 and a theorem which lays ground for the proof of global well – posedness. This theorem is a simplified version of [Caz03, Theorem 3.3.5.] and [Caz03, Theorem 3.3.9.].

Theorem 3.3.5. *Let g be as in Definition 3.2.1. For every $M > 0$ there exists $T(M) > 0$ with the following property: For every $a \in H^1$ such that $\|a\|_{H^1} \leq M$ there exists a weak H^1 solution u of* (3.1) *on $I = (-T(M), T(M))$. In addition,*

$$\|u\|_{L^\infty(I,H^1)} \leq 2M, \tag{3.17}$$

$$\|u(t)\|_2 = \|a\|_2 \tag{3.18}$$

$$H(u(t)) \leq H(a). \tag{3.19}$$

If the solution is unique, then the solution is maximal, the blowup alternative holds and $H(u(t)) = H(a)$ for all $t \in (-T_{min}, T_{max})$.

The notion of **higher regularity** is also important for this subject. For example, if the initial data is $a \in H^1$ there is a unique solution $u \in C((-T_{min}, T_{max}), H^1)$. If we further assume that $a \in H^2$, we know there is a maximal solution $u \in C((-T^1_{min}, T^1_{max}), H^2)$, but do the two solutions coincide? Since a H^2 solution is also a H^1 solution and from uniqueness it follows that they surely coincide on the smaller of the two intervals $(-T_{min}, T_{max})$, $(-T^1_{min}, T^1_{max})$, also $(-T^1_{min}, T^1_{max}) \subset (-T_{min}, T_{max})$. So the question becomes: is $T_{max} = T^1_{max}$ and $T_{min} = T^1_{min}$? In the case relevant for our analysis the answer is affirmative. The following theorem is a consequence of [Caz03, Theorem 5.3.1.] and [Caz03, Remark 5.3.3.].

Theorem 3.3.6. *Let g be as in Definition 3.2.1 and $a \in H^1$. Let $u \in C((-T_{min}, T_{max}), H^1)$ be the maximal solution of* (3.1). *If $a \in H^2$ it follows that $u \in C((-T_{min}, T_{max}), H^2)$.*

For the cubic case, higher regularity holds in H^3 also. The following theorem is a consequence of [Caz03, Theorem 4.10.1.], [Caz03, Remark 4.10.3.] and [Caz03, Theorem 5.4.1].

Theorem 3.3.7. *Let $n \in \{2, 3\}$, g as in Definition 3.2.1 and $a \in H^2(\mathbb{R}^n)$. Let $u \in C((-T_{min}, T_{max}), H^2(\mathbb{R}^n))$ be the maximal solution of* (3.1). *If $a \in H^3(\mathbb{R}^n)$, then $u \in C((-T_{min}, T_{max}), H^3(\mathbb{R}^n))$.*

Proof. The crux of this theorem is to prove well – posedness in H^3 which is done in [Caz03, Theorem 4.10.1.] for the case $g(u) = u|u|^2$. So we present a proof with slight modifications because of the additional term $Vu + (w * |u|^2)u$. Higher regularity then follows in the same way as in the proof of [Caz03, Theorem 5.4.1].

Given $M, T > 0$ to be chosen later, let $I = (-T, T)$ and

$$E = \{u \in L^\infty(I, H^3) : \ \|u\|_{L^\infty(I,H^3)} \leq M\}.$$

We define distance as

$$d(u,v) = \|u - v\|_{L^\infty(I,L^2)},$$

and with it E is a complete metric space. Consider now

$$\Phi(u)(t) = \mathcal{T}(t)a + i\int_0^t \mathcal{T}(t-s)g(u(s))ds,$$

with $u \in E$ and $t \in I$. We derive the following inequalities

$$\begin{aligned}\|\Phi(u)(t)\|_{H^3} &\leq \|a\|_{H^3} + T\|g(u)\|_{H^3} \\ &\leq \|a\|_{H^3} + T(C(M)M + C_1M + C_2M^3)\end{aligned}$$

where $C_1 = C_1(\|V\|_\infty, \|\partial^\alpha V\|_\infty, \|\partial^\beta V\|_\infty)$ and $C(M)$ is as in [Caz03, Theorem 4.10.1.]. The third constant comes from $\|(w * |u|^2)u\|_{H^3}$ and note that in order to bound this term, it is enough to observe that

$$\|(\partial^\alpha w * |u|^2)\partial^\beta u\|_2 \leq \|\partial^\alpha w * |u|^2\|_\infty \|\partial^\beta u\|_2 \leq c\|w\|_\infty \|\partial^\alpha(|u|^2)\|_1 \|\partial^\beta u\|_2$$

for $|\alpha| \leq 2, |\beta| \leq 2$. Now $\partial^\alpha(|u|^2)$ is at most a sum of terms $\partial^\beta u \partial^\alpha u$ which in the L^1 norm is bounded by $\|u\|_{H^2}^2$ (using Hölder inequality). To conclude, $C_2 = C_2(\|w\|_\infty)$.

Also,

$$\begin{aligned}\|(w * |u|^2)u - (w * |v|^2)v\|_2 &\leq \|(w * |u|^2)(u - v) + v \cdot w * (|u|^2 - |v|^2)\|_2 \\ &\lesssim \|w\|_\infty \|u^2\|_1 \|u - v\|_2 + \|v \cdot w * (|u| - |v|)(|u| + |v|)\|_2 \\ &\lesssim \|w\|_\infty \|u\|_2^2 \|u - v\|_2 + \|v\|_2 \|w\|_\infty \|(|u| - |v|)(|u| + |v|)\|_2 \\ &\leq c\|u - v\|_2(\|w\|_\infty(\|u\|_2^2 + \|v\|_2(\|u\|_2 + \|v\|_2))) \\ &\leq 3c_1M^2.\end{aligned}$$

It follows

$$
\begin{aligned}
\|\Phi(u)(t) - \Phi(v)(t)\|_{L^2} &\leq T(C(M) + C_3 + 3c_1M^2)\|u - v\|_{L^\infty(I,L^2)} \\
&\leq T(C(M) + C_1 + C_4M^2)\|u - v\|_{L^\infty(I,L^2)}
\end{aligned}
$$

where $C_3 = \|V\|_\infty$. So if $M = 2\|a\|_{H^3}$ and $T(C(M) + C_1 + C_4M^2) \leq \frac{1}{2}$, then Φ is a strict contraction on E. Uniqueness and other properties follow in a similar manner. □

We now derive conservation of energy and charge for the H^2 solution and each of the equations (1.1) – (1.3). Note that (3.18) and (3.19) hold for the weaker case $u \in H^1$, but we only prove them in the simpler case $u \in H^2$, since then the equality in (3.1) has sense in L^2. To prove Theorem 3.3.5, approximate solutions are needed.

Theorem 3.3.8. *Let $a \in H^2$. For the cubic equation* (1.1) *there holds*

$$
\|u(t)\|_{L^2} = \|a\|_{L^2}, \tag{3.20}
$$

$$
H(u(t)) = H(a), \tag{3.21}
$$

where $H(u(t)) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^n} |u|^4 dx$ denotes the Hamiltonian.

Proof. From Theorem 3.3.4, we know that there is a solution $u \in C((-T_{min}, T_{max}), H^2) \cap C^1((-T_{min}, T_{max})), L^2)$, so the equation (1.1) has sense in L^2 so we can multiply by $\overline{u}$ and integrate over $\mathbb{R}^n$ - in other words take the scalar product by u in L^2:

$$
(iu_t, u)_2 + (\triangle u, u)_2 - (u|u|^2, u)_2 = 0.
$$

Now if we take the imaginary part, as we have seen before, because of symmetry of $\triangle$ we have $\mathrm{Im}(\triangle u, u)_2 = 0$. Also, $(u|u|^2, u)_2 = \int |u|^4 \in \mathbb{R}$, and we obtain:

$$
\begin{aligned}
\frac{1}{2i}((iu_t, u)_2 - \overline{(iu_t, u)_2}) &= 0 \\
\frac{1}{2i} \cdot i((u_t, u)_2 + (u, u_t)_2) &= 0 \\
\frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L^2} &= 0,
\end{aligned}
$$

so (3.20) holds. Further, taking the scalar product of (1.1) with $u_t \in L^2$ and taking the real part we obtain

$$\begin{aligned}
&\frac{1}{2}(i(u_t, u_t)_2 - i(u_t, u_t)_2) + \frac{1}{2}((\triangle u, u_t)_2 + (u_t, \triangle u)_2) \\
&-\frac{1}{2}((u|u|^2, u_t)_2 + (u_t, u|u|^2)_2) = 0, \\
&\frac{1}{2}\Big(\int(\overline{u}_t \triangle u + u_t \triangle \overline{u}) - \int |u|^2(u\overline{u}_t + \overline{u}u_t)\Big) = 0.
\end{aligned}$$

Now, $|u|^2 \partial_t(|u|^2) = \frac{1}{2}\partial_t |u|^4$ and the above is equivalent to

$$\int(\overline{u}_t \triangle u + u_t \triangle \overline{u}) - \frac{1}{2}\frac{\partial}{\partial t}\int |u|^4 = 0. \tag{3.22}$$

In order to obtain conservation of energy, we have to apply some density arguments. Let now $u, u_t \in C_0^\infty$. There holds

$$\begin{aligned}
\frac{\partial}{\partial t}\int |\nabla u|^2 &= \frac{\partial}{\partial t}\int(|u_{x_1}|^2 + \ldots + |u_{x_n}|^2) \\
&= \int u_{x_1 t}\overline{u}_{x_1} + u_{x_1}\overline{u_{x_1 t}} + \ldots + u_{x_n t}\overline{u}_{x_n} + u_{x_n}\overline{u_{x_n t}} \\
&= -\int(u_t \overline{u_{x_1 x_1}} + \overline{u}_t u_{x_1 x_1} + \ldots + u_t \overline{u_{x_n x_n}} + \overline{u}_t u_{x_n x_n}) \\
&= -\int(u_t \triangle \overline{u} + \overline{u}_t \triangle u).
\end{aligned}$$

Here we used integration by parts. Further, this formula is equivalent to

$$\int |\nabla u|^2 = \int |\nabla u(0)|^2 - \int_0^t \int(u_t \triangle \overline{u} + \overline{u}_t \triangle u). \tag{3.23}$$

Let now $u \in H^2$, $u_t \in L^2$ and $u_n \in C_0^\infty$ such that $(u_n)_t \in C_0^\infty$ and $u_n \to u$ in H^2, $(u_n)_t \to u_t$ in L^2 for any $t \in (-T_{min}, T_{max})$. Such a sequence is for example $u_n(t,x) = (\rho_n(x) * u(t,x))\xi_n(x)$ for a mollifier ρ_n and a cut–off function ξ_n, see [Bre10].

Then $\int |\nabla u|^2 = \lim_{n\to\infty}\int |\nabla u_n|^2$ for any t and also $\int(u_t \triangle \overline{u} + \overline{u}_t \triangle u) = \lim\limits_{n\to\infty}\int((u_n)_t \triangle \overline{u_n} + \overline{u_{nt}} \triangle u_n)$ since

$$\begin{aligned}
\int |(u_n)_t \triangle \overline{u}_n - u_t \triangle \overline{u}| &= \int |\triangle \overline{u}_n((u_n)_t - u_t) + u_t(\triangle \overline{u}_n - \triangle \overline{u})| \\
&\le \|\triangle \overline{u}_n\|_2 \|(u_n)_t - u_t\|_2 + \|u_t\|_2 \|\triangle \overline{u}_n - \triangle \overline{u}\|_2 \to 0,\ n \to \infty.
\end{aligned}$$

Since (3.23) holds for u_n it holds also for u and finally it is equivalent to

$$\int(u_t \triangle \overline{u} + \overline{u}_t \triangle u) = -\frac{\partial}{\partial t}\int |\nabla u|^2.$$

Returning to (3.22) we obtain

$$\frac{\partial}{\partial t}\left(\int |\nabla u|^2 + \frac{1}{2}\int |u|^4\right) = 0$$

and (3.21) holds. □

Theorem 3.3.9. *Let $a \in H^2$, $V \in C_0^\infty(\mathbb{R}^n)$ be a real valued function and let*

$$\begin{aligned} iu_t + \triangle u &= u|u|^2 + Vu, \\ u(0) &= a. \end{aligned}$$

There holds

$$\|u\|_{L^2} = \|a\|_{L^2}, \tag{3.24}$$

$$H(u(t)) = H(a), \tag{3.25}$$

where $H(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^n} |u|^4 + \frac{1}{2}\int V|u|^2$.

Proof. The proof is analogous to the proof of Theorem 3.3.8, noting that $V|u|^2$ is real and $V\frac{\partial}{\partial t}|u|^2 = \frac{\partial}{\partial t}(V|u|^2)$. □

Theorem 3.3.10. *Let $a \in H^2$, $w \in W^{2,p}$, $p > 2$ real valued and even and $V \in C_0^\infty$. Let*

$$\begin{aligned} iu_t + \triangle u &= (w * |u|^2)u + Vu, \\ u(0) &= a. \end{aligned}$$

There holds

$$\|u\|_{L^2} = \|a\|_{L^2}, \tag{3.26}$$

$$H(u(t)) = H(a), \tag{3.27}$$

where $H(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{2}\int V|u|^2 + \frac{1}{4}\int (w * |u|^2)|u|^2$.

Proof. Conservation of charge (3.26) holds in the same way as before, due to the fact that $(w * |u|^2)|u|^2$ is real. Let us take the L^2 scalar product with u_t, then the real part and observe just the term with the Hartree nonlinearity (others are as before)

$$\frac{1}{2}\int_{\mathbb{R}^3} (w * |u|^2)(u\overline{u}_t + \overline{u}u_t)dx = \frac{1}{2}\int_{\mathbb{R}^3} (w * |u|^2)\frac{\partial}{\partial t}|u|^2 dx$$

On the other hand

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^3}(w*|u|^2)|u|^2dx = \int_{\mathbb{R}^3}((w*|u|^2)\frac{\partial}{\partial t}|u|^2 + (w*\frac{\partial}{\partial t}|u|^2)|u|^2)dx \\
&= -\int_{\mathbb{R}^3}((w*|u|^2)\frac{\partial}{\partial t}|u|^2 + (w*\frac{\partial}{\partial t}|u|^2)|u|^2)dx \\
&\Rightarrow \int_{\mathbb{R}^3}((w*|u|^2)\frac{\partial}{\partial t}|u|^2 + (w*\frac{\partial}{\partial t}|u|^2)|u|^2)dx = 0 \\
&\Rightarrow \int_{\mathbb{R}^3}(w*|u|^2)\frac{\partial}{\partial t}|u|^2 = -(w*\frac{\partial}{\partial t}|u|^2)|u|^2dx = (w*\frac{\partial}{\partial t}|u|^2)|u|^2dx \\
&\Rightarrow \int_{\mathbb{R}^3}(w*|u|^2)\frac{\partial}{\partial t}|u|^2 = \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3}(w*|u|^2)|u|^2dx.
\end{aligned}
$$

We used the fact that w is even to exchange the minus sign. Finally,

$$
\frac{1}{2}\int_{\mathbb{R}^3}(w*|u|^2)(u\overline{u}_t + \overline{u}u_t)dx = \frac{1}{4}\frac{\partial}{\partial t}\int_{\mathbb{R}^3}(w*|u|^2)|u|^2dx
$$

and (3.27) follows. □

Conservation of charge and energy is used to prove global well-posedness. We use ideas from [Caz03, Theorem 3.4.1.].

Theorem 3.3.11. *Let g be as in Definition 3.2.1 and $a \in H^1$. There exists a global H^1 solution of* (3.1) *on* $\mathbb{R}$ *satisfying* (3.18) *and* (3.19).

Proof. Due to Theorem 3.3.5 there is a local solution. From conservation of energy, the following bounds are derived

$$
\begin{aligned}
\int|\nabla u|^2 = \sum_{|\alpha|=1}\|\partial^\alpha u\|_2^2 &\leq 2H(a) \\
&= \int|\nabla a|^2 + \int V|a|^2 + \frac{1}{2}\int(w*|a|^2)|a|^2 + \frac{1}{2}\int|a|^4 \\
&\leq \sum_{|\alpha|=1}\|\partial^\alpha a\|_2^2 + \|V\|_\infty\|a\|_2^2 + \frac{1}{2}\|w\|_\infty\|a\|_2^2\|a\|_2^2 + \frac{1}{2}\|a\|_4^4.
\end{aligned}
$$

Using (2.11) inequality for $j=0,\ p=4,\ r=q=2,\ m=1$

$$
\|a\|_4 \leq (\sum_{|\alpha|=1}\|\partial^\alpha a\|_2)^b\|a\|_2^{1-b} \leq \|a\|_{H^1}^b\|a\|_{H^1}^{1-b} = \|a\|_{H^1}
$$

where $b = \frac{1}{2}$ for $n = 2$ and $b = \frac{3}{4}$ for $n = 3$. So there holds

$$\begin{aligned}
\|u(t)\|_{H^1} &= \sqrt{\|u\|_2^2 + \sum_{|\alpha|=1} \|\partial^\alpha u\|_2^2} \\
&\leq (\|a\|_2^2 + \sum_{|\alpha|=1} \|\partial^\alpha a\|_2^2 + \|V\|_\infty \|a\|_2^2 + \frac{1}{2}\|w\|_\infty \|a\|_2^4 + \frac{1}{2}\|a\|_{H^1}^4)^{\frac{1}{2}} \\
&\leq ((1 + \|V\|_\infty)\|a\|_{H^1}^2 + \frac{1}{2}(\|w\|_\infty + 1)\|a\|_{H^1}^4)^{\frac{1}{2}} \\
&= \sqrt{M_1\|a\|_{H^1}^2 + M_2\|a\|_{H^1}^4}
\end{aligned}$$

Let $M = \sqrt{M_1\|a\|_{H^1}^2 + M_2\|a\|_{H^1}^4}$, we see that $\|a\|_{H^1} \leq M$. From Theorem 3.3.5, there is a H^1 solution on $[0, T(M)]$ such that $T(M)$ is the same for any initial condition whose H^2 norm is bounded with M. Based on Theorem 3.3.5 conservation of energy also holds and so $\|u(t)\|_{H^1} \leq M < \infty$ on $[0, T(M)]$. Setting $\tilde{a} = u(T(M)) \in H^2$ we see that again, there exist a H^1 solution $\tilde{u}$ (with initial value $\tilde{a}$ on $[0, T(M)]$) which again satisfies energy conservation. We define a function

$$u(t) = \begin{cases} u(t), & t \in [0, T(M)], \\ \tilde{u}(t - T(M)), & t \in [T(M), 2T(M)]. \end{cases}$$

This function is a solution on $[0, 2T(M)]$ and it is unique (due to Strichartz conditions). Moreover,

$$\begin{aligned}
\|u(t)\|_2 = \|\tilde{u}(t - T(M))\|_2 = \|\tilde{a}\|_2 = \|u(T(M)\|_2 = \|a\|_2, \\
H(u(t)) = H(\tilde{u}(t - T(M))) = H(\tilde{a}) = H(u(T(M)) = H(a),
\end{aligned}$$

for $t \in [T(M), 2T(M)]$. So newly defined u satisfies conservation of energy and charge and also $\|u(2T(M))\|_{H^1} \leq M$. Therefore this argument can be repeated so that we obtain a solution on $[0, \infty)$ such that conservation of energy holds for any $t \geq 0$. Similar arguments holds for $t \leq 0$ and we additionally conclude that $\|u\|_{H^1} < \infty$ for any $t \in \mathbb{R}$. $\square$

Theorem 3.3.12. *Let g be as in Definition 3.2.1. If $a \in H^2$, then there is a global solution $u \in C(\mathbb{R}, H^2)$ which satisfies conservation of charge and energy.*

Proof. The proof follows from Theorem 3.3.6. In other words conditions for higher regularity hold and the H^1 and the H^2 solutions coincide on all $\mathbb{R}$. $\square$

Also as a consequence of Theorem 3.3.7 we have the following theorem.

Theorem 3.3.13. *Let g be as in Definition 3.2.1. If $a \in H^3$ there is a global solution $u \in C(\mathbb{R}, H^3)$ which satisfies conservation of charge and energy.*

4 Well - posedness of the cubic equations in the Colombeau algebra

In this chapter we present original results related to equations (1.1) and (1.2). The term "well–posedness" in the title is used now for existence of a unique solution in $\mathcal{G}_{H^2,C^1}$. For (1.1) we are able to show compatibility with the Sobolev H^2 solution, too.

4.1 The delta potential

Consider first the equation with the delta potential

$$\begin{aligned} iu_t + \triangle u &= u|u|^2 + \delta u, \\ u(0,x) &= a(x), \ a \in \mathcal{G}_{H^2}(\mathbb{R}^3). \end{aligned} \tag{4.1}$$

This equation is a model for Bose–Einstein condensates (BEC) and $\delta(x)$ describes a localized external potential applied to the condensate. A lot of research is directed to understand the interaction between its soliton solution and the delta–like impurity. A soliton is a solitary wave (wave packet) solution, traveling unchanged in shape with constant velocity and occurs due to cancellation of dispersive and nonlinear effects. Here we turn to examining existence and uniqueness of a solution in the Colombeau algebra. The question of compatibility in the sense of Definition 2.5.13 remains open. We do not have a candidate for any kind of classical solution of (4.1).

The estimates we derive in this section are applicable to the cubic equation (1.1), too, so this is the reason we start with the equation with the potential. A representative of δ is chosen such that the regularized version of (4.1) is

$$\begin{aligned} i\partial_t u_\varepsilon + \triangle u_\varepsilon &= u_\varepsilon|u_\varepsilon|^2 + \phi_{h_\varepsilon} u_\varepsilon, \\ u_\varepsilon(0,x) &= a_\varepsilon(x), \end{aligned} \tag{4.2}$$

where $\phi_{h_\varepsilon}(x)$ is a strict delta net as in Section 2.5. Later on, one will see that we have to take $h_\varepsilon \sim (\ln \varepsilon^{-1})^{5/19}$. Let $\varepsilon > 0$. We have seen that conservation of charge (3.24) and energy (3.25) hold, where now

$$H(a_\varepsilon) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla a_\varepsilon|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |a_\varepsilon|^4 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{h_\varepsilon} |a_\varepsilon|^2 dx.$$

It follows

$$H(u_\varepsilon(t)) = H(a_\varepsilon) \geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx$$

and

$$\begin{aligned}
\|u_\varepsilon(t)\|_{H^1} &= \sqrt{\|u_\varepsilon\|_2^2 + \sum_{|\alpha|=1} \|\partial^\alpha u_\varepsilon\|_2^2} \\
&\leq (\|a_\varepsilon\|_2^2 + \sum_{|\alpha|=1} \|\partial^\alpha a_\varepsilon\|_2^2 + \|\phi_{h_\varepsilon}\|_\infty \|a_\varepsilon\|_2^2 + \frac{1}{2}\|a_\varepsilon\|_{H^1}^4)^{\frac{1}{2}} \\
&\leq ((1 + \|\phi_{h_\varepsilon}\|_\infty)\|a\|_{H^1}^2 + \frac{1}{2}\|a\|_{H^1}^4)^{\frac{1}{2}} \\
&\leq \sqrt{(1 + ch_\varepsilon^n)\|a\|_{H^1}^2 + M_2\|a\|_{H^1}^4}
\end{aligned} \tag{4.3}$$

We have used inequality (2.11) with $j = 0$, $m = 1$, $a = \frac{3}{4}$, $p = 4$, and $r = q = 2$.

Let us now show that the Definition 2.5.10 is independent of the representative when $g(u_\varepsilon) = u_\varepsilon |u_\varepsilon|^2 + \phi_{h_\varepsilon} u$. Let $v_\varepsilon = u_\varepsilon + N_\varepsilon$, $N_\varepsilon \in \mathcal{N}_{C^1,H^2}$. We have seen

$$i(v_\varepsilon)_t + \triangle v_\varepsilon + v_\varepsilon |v_\varepsilon|^2 = M_\varepsilon + f(u_\varepsilon, N_\varepsilon),$$

where $\|M_\varepsilon\|_2 \sim \varepsilon^M$ for any $t \in [0, T)$. Now for $f(u_\varepsilon, N_\varepsilon)$ we have:

$$\begin{aligned}
f(u_\varepsilon, N_\varepsilon) &= u_\varepsilon |N_\varepsilon|^2 + N_\varepsilon |u_\varepsilon|^2 + N_\varepsilon |N_\varepsilon|^2 + (u_\varepsilon + N_\varepsilon)(\overline{u}_\varepsilon N_\varepsilon + u_\varepsilon \overline{N}_\varepsilon) + \phi_{h_\varepsilon} N_\varepsilon, \text{ and} \\
\|f\|_2 &\lesssim \|N_\varepsilon\|_\infty^2 \|u_\varepsilon\|_2 + \|u_\varepsilon\|_\infty^2 \|N_\varepsilon\|_2 + \|N_\varepsilon\|_\infty^2 \|N_\varepsilon\|_2 + \|\phi_{h_\varepsilon}\|_\infty \|N_\varepsilon\|_2.
\end{aligned}$$

Noting that $\|N_\varepsilon\|_\infty \leq \|N_\varepsilon\|_{H^2}$ (Sobolev embedding (2.3)) we have that each term above is bounded by ε^M for any $t \in [0, T)$.

Let us now state the main theorem of this section.

Theorem 4.1.1. *Let $a \in \mathcal{G}_{H^2}$ such that there exists a representative a_ε which satisfies the following:*

$$\|a_\varepsilon\|_{H^3} = \mathcal{O}(\varepsilon^{-N}), \quad \text{and} \quad \|a_\varepsilon\|_{H^1} = \mathcal{O}(h_\varepsilon) \quad \textit{for some } N \in \mathbb{N},\ \varepsilon \to 0 \tag{4.4}$$

where $h_\varepsilon \sim (\ln \varepsilon^{-1})^{\frac{5}{11}}$*. Then for any* $T > 0$ *there exists a generalized solution* $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ *of* (4.1).

Remark 4.1.2. *For simplicity, we bound the norm of the initial condition with the same* h_ε *used to regularize the delta function.*

Proof. For each $\varepsilon \in (0,1)$ there exists a unique global solution $u_\varepsilon \in C(\mathbb{R}, H^3)$. This is a consequence of Theorem 3.3.13. From conservation of charge $\|u_\varepsilon(t)\|_2 = \|a_\varepsilon\|_2 \sim \varepsilon^{-N}$ for any $t \in [0,T)$. Also (4.3) holds which implies $\|u(t)\|_{H^1} \sim \varepsilon^{-N}$ for some $N \in \mathbb{N}$. It remains to obtain estimates for second order derivatives.

We first apply a second order derivative in x to the nonlinear part

$$\begin{aligned}\partial^\alpha(u^2\overline{u} + \phi_{h_\varepsilon} u) &= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} ((\partial^\beta u_\varepsilon^2)(\partial^{\alpha-\beta}\overline{u}) + (\partial^\beta \phi_{h_\varepsilon})(\partial^{\alpha-\beta} u_\varepsilon)) \\ &\lesssim u_\varepsilon^2 \partial^\alpha \overline{u_\varepsilon} + \sum_{|\beta|=1} \partial^\beta (u_\varepsilon^2) \partial^{\alpha-\beta} \overline{u_\varepsilon} + \partial^\alpha (u_\varepsilon)^2 \overline{u_\varepsilon} \\ &\quad + \phi_{h_\varepsilon} \partial^\alpha u_\varepsilon + \sum_{|\beta|=1} \partial^\beta \phi_{h_\varepsilon} \partial^{\alpha-\beta} u_\varepsilon + \partial^\alpha \phi_{h_\varepsilon} u_\varepsilon.\end{aligned}$$

where $|\alpha| = 2$. Note that

$$\partial^\alpha u_\varepsilon^2 = 2u_\varepsilon \partial^\alpha u_\varepsilon + \sum_{|\beta|=1} \partial^\beta u_\varepsilon \partial^{\alpha-\beta} u_\varepsilon.$$

In order to bound $\|\partial^\alpha(u_\varepsilon|u_\varepsilon|^2 + \phi_{h_\varepsilon} u_\varepsilon)\|_2$ we essentially need to bound

$$\|u_\varepsilon^2 \partial^\alpha u_\varepsilon + u_\varepsilon \partial^\beta u_\varepsilon \partial^\gamma u_\varepsilon + \phi_{h_\varepsilon} \partial^\alpha u_\varepsilon + \partial^\beta \phi_{h_\varepsilon} \partial^\gamma u_\varepsilon + \partial^\alpha \phi_{h_\varepsilon} u_\varepsilon\|_2, \tag{4.5}$$

where $|\gamma| = |\beta| = 1$. The idea here will be to go from the L^2–norm to $L^{\frac{10}{3}}$–norm using Hölder and Gagliardo–Nirenberg inequalities. Then, by (3.5) we go to the $L^{\frac{10}{7}}$–norm, which we bound (essentially) with $\|a_\varepsilon\|_{H^3}$ using Gronwall's inequality (Theorem 2.4.4).

The following estimates hold for any $t \in [0,T)$. Differentiating Duhamel's formula (3.11) twice and using (3.5) and Minkowski integral inequality (2.12)

$$\|\partial^\alpha u_\varepsilon\|_2 \le \|\partial^\alpha a_\varepsilon\|_2 + c\int_0^t \|\partial^\alpha \Big(u_\varepsilon(s)|u_\varepsilon(s)|^2 + \phi_{h_\varepsilon} u_\varepsilon(s)\Big)\|_2 ds. \tag{4.6}$$

where $|\alpha| = 2$. We can estimate each of the terms in (4.5) in the following way. There holds

$$
\begin{aligned}
\|u_\varepsilon^2 \partial^\alpha u_\varepsilon\|_2 =& \Big(\int |u_\varepsilon|^4 |\partial^\alpha u_\varepsilon|^2\Big)^{\frac{1}{2}} \\
\lesssim& \|u_\varepsilon\|_{10}^2 \|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \\
\lesssim& \Big(\sum_{|\beta|=1} \|\partial^\beta u_\varepsilon\|_{\frac{10}{3}}\Big)^{\frac{3}{2}} \|u(t)\|_2^{\frac{1}{2}} \|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}.
\end{aligned}
$$

Here we used the Hölder inequality (2.7) for $p = \frac{5}{2},\ q = \frac{5}{3}$ and the Gagliardo–Nirenberg inequality (2.11) for $j = 0,\ m = 1,\ p = 10,\ r = \frac{10}{3},\ q = 2,\ b = \frac{3}{4}$. Further,

$$
\begin{aligned}
\|u_\varepsilon(\partial^\beta u_\varepsilon)(\partial^\gamma u_\varepsilon)\|_2 \lesssim& \|u_\varepsilon\|_6 \|\partial^\beta u_\varepsilon \partial^\gamma u_\varepsilon\|_3 \lesssim \|u_\varepsilon\|_6 \|\partial^\beta u_\varepsilon\|_6 \|\partial^\gamma u_\varepsilon\|_6 \\
\lesssim& \Big(\sum_{|\alpha|=1} \|\partial^\alpha u_\varepsilon(t)\|_2\Big)\Big(\sum_{|\alpha|=2} \|\partial^\alpha u_\varepsilon(t)\|_{\frac{10}{3}}\Big)^{\frac{20}{13}} \|u_\varepsilon(t)\|_2^{\frac{6}{13}}.
\end{aligned}
$$

In the first line, Hölder inequality was used for $p = 3,\ q = \frac{3}{2}$ first and then for $p = q = 2$. In the second line, Gagliardo–Nirenberg inequality was used for $\|u_\varepsilon\|_6$ first, where $j = 0,\ m = 1,\ r = q = 2,\ b = 1$ and then for $\|\partial^\beta u\|_6$ and $\|\partial^\gamma u_\varepsilon\|_6$ with $j = 1$, $m = 2$, $p = 6$, $r = \frac{10}{3}$, $q = 2$, $b = \frac{10}{13}$. Finally,

$$
\begin{aligned}
&\|u_\varepsilon \partial^\alpha \phi_{h_\varepsilon}\|_2 \le \|\partial^\alpha \phi_{h_\varepsilon}\|_\infty \|u_\varepsilon\|_2, \quad \|\partial^\beta \phi_{h_\varepsilon} \partial^\gamma u_\varepsilon\|_2 \le \|\partial^\beta \phi_{h_\varepsilon}\|_\infty \|\partial^\gamma u_\varepsilon\|_2, \\
&\|\phi_{h_\varepsilon} \partial^\alpha u_\varepsilon\|_2 \le \|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \|\phi_{h_\varepsilon}\|_5.
\end{aligned}
$$

In the last line, Hölder inequality was used for $p = \frac{5}{3},\ q = \frac{5}{2}$. The norms $\|\partial^\alpha \phi_{h_\varepsilon}\|_p$, $p \in \{\infty, 5\},\ |\alpha| \le 2$ are controlled by h_ε^m for some m. It remains to obtain bounds for $\|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}}$ and $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}},\ |\gamma| = 1,\ |\alpha| = 2$. Again we use Duhamel's formula (3.11), estimate (3.5) for $p = \frac{10}{3},\ p' = \frac{10}{7}$ and the fact that $\mathcal{T}(t)$ commutes with ∂^γ

$$
\begin{aligned}
&\|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}} \le \|\partial^\gamma(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \\
&+ c\int_0^t \frac{1}{(t-s)^{\frac{3}{5}}} \|\partial^\gamma \left(u_\varepsilon(s)|u_\varepsilon(s)|^2 + \phi_{h_\varepsilon} u_\varepsilon(s)\right)\|_{\frac{10}{7}} ds, \\
&\|\partial^\gamma(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \le \Big(\sum_{|\alpha|=2} \|\partial^\alpha(\mathcal{T}(t)a_\varepsilon)\|_2\Big)^{\frac{4}{5}} \|\mathcal{T}(t)a_\varepsilon\|_2^{\frac{1}{5}} \le \|a_\varepsilon\|_{H^2}
\end{aligned}
$$

where the Gagliardo-Nirenberg inequality (2.11) was used, $j = 1,\ m = 2,\ p = 10/3,\ q = r = 2,\ b = 4/5$. Applying the Hölder inequality and (2.11) again with $j = 0$, $m = 1$, $p = 5$, $r = q = 2$, $b = \frac{9}{10}$, we derive the following inequalities

$$
\begin{aligned}
\|\partial^\gamma \phi_{h_\varepsilon} u_\varepsilon\|_{\frac{10}{7}} \leq& \|a_\varepsilon\|_2 \|\partial^\gamma \phi_{h_\varepsilon}\|_5, \\
\|\phi_{h_\varepsilon} \partial^\gamma u_\varepsilon\|_{\frac{10}{7}} \leq& \|\partial^\gamma u_\varepsilon\|_2 \|\phi_{h_\varepsilon}\|_5 \leq (2H(a_\varepsilon))^{\frac{1}{2}} \|\phi_{h_\varepsilon}\|_5, \text{ and} \\
\|\partial^\gamma u_\varepsilon |u_\varepsilon|^2\|_{\frac{10}{7}} \leq& \|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}} \|u_\varepsilon\|_5^2 \\
\leq& \|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}} \Big(\sum_{|\alpha|=1} \|\partial^\alpha u_\varepsilon\|_2 \Big)^{\frac{1}{5}} \|u_\varepsilon\|_2^{\frac{9}{5}} \\
\leq& \|\partial^\gamma u_\varepsilon(t)\|_{\frac{10}{3}} H(a_\varepsilon)^{\frac{1}{10}} \|a_\varepsilon\|_2^{\frac{9}{5}}.
\end{aligned}
$$

Gronwall's inequality implies

$$
\|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}} \leq c_1(a_\varepsilon, \phi_{h_\varepsilon}) \cdot \exp(c_2(a_\varepsilon, \phi_{h_\varepsilon})) \tag{4.7}
$$

where

$$
c_1(a_\varepsilon, \phi_{h_\varepsilon}) = \|a_\varepsilon\|_{H^2} + T^{\frac{2}{5}} (\|a_\varepsilon\|_2 \|\partial^\gamma \phi_{h_\varepsilon}\|_5 + H(a_\varepsilon)^{\frac{1}{2}} \|\phi_{h_\varepsilon}\|_5)
$$

and

$$
c_2(a_\varepsilon, \phi_{h_\varepsilon}) = T^{\frac{2}{5}} H(a_\varepsilon)^{\frac{1}{10}} \|a_\varepsilon\|_2^{\frac{9}{5}}.
$$

Let $f_\varepsilon = c_1(a_\varepsilon, \phi_{h_\varepsilon}) \cdot \exp(c_2(a_\varepsilon, \phi_{h_\varepsilon}))$. Recall that

$$
\begin{aligned}
H(a_\varepsilon) &\lesssim (1 + \|\phi_{h_\varepsilon}\|_\infty) \|a_\varepsilon\|_{H^1}^2 + \|a\|_{H^1}^4 \\
&\sim (1 + h_\varepsilon^n) h_\varepsilon^2 + h_\varepsilon^4 \lesssim h_\varepsilon^4.
\end{aligned}
$$

It follows

$$
c_2(a_\varepsilon, \phi_{h_\varepsilon}) \lesssim h_\varepsilon^{\frac{2}{5}} \cdot h_\varepsilon^{\frac{9}{5}} = h_\varepsilon^{\frac{11}{5}}
$$

and

$$
c_1(a_\varepsilon, \phi_{h_\varepsilon}) \lesssim h_\varepsilon^m, \text{ for some } m \in \mathbb{N}.
$$

Now

$$
\|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}} \lesssim h_\varepsilon^m \cdot (\exp(h_\varepsilon^{\frac{11}{5}}))^{T^{2/5}} \lesssim (\ln \varepsilon^{-1})^p (\exp((\ln \varepsilon^{-1}))^{T^{2/5}} \lesssim \varepsilon^{-N},
$$

since $h_\varepsilon^{\frac{11}{5}} = (\ln \varepsilon^{-1})^{\frac{5}{11} \cdot \frac{11}{5}}$. Finally,

$$
\sup_{[0,T)} \|\partial^\gamma u_\varepsilon(t)\|_{\frac{10}{3}} \leq c\varepsilon^{-N}, \quad \varepsilon \to 0, \text{ for some } N.
$$

Similarly

$$\begin{aligned}
&\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \leq \|\partial^\alpha(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \\
&+ c\int_0^t \frac{1}{(t-s)^{\frac{3}{5}}}\|\partial^\alpha(u_\varepsilon(s)|u_\varepsilon(s)|^2 + \phi_{h_\varepsilon}u_\varepsilon(s))\|_{\frac{10}{7}} ds \text{ and,} \\
&\|\partial^\alpha(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \leq \Big(\sum_{|\alpha|=3}\|\partial^\alpha(\mathcal{T}(t)a_\varepsilon)\|_2\Big)^{\frac{13}{15}}\|\mathcal{T}(t)a_\varepsilon\|_2^{\frac{2}{15}} \leq \|a_\varepsilon\|_{H^3}.
\end{aligned}$$

Now we have

$$\partial^\alpha(g(u_\varepsilon)) \lesssim (|u_\varepsilon|^2+\phi_{h_\varepsilon})\partial^\alpha u_\varepsilon + \sum_{|\beta|=1}\partial^\beta(|u_\varepsilon|^2+\phi_{h_\varepsilon})\partial^{\alpha-\beta}u_\varepsilon + (\partial^\alpha|u_\varepsilon|^2+\partial^\alpha\phi_{h_\varepsilon})u_\varepsilon$$

and we need to bound the following terms

$$\|\partial^\alpha u\cdot|u|^2\|_{\frac{10}{7}} \leq \|\partial^\alpha u\|_{\frac{10}{3}}\|u_\varepsilon\|_5^2 \leq \|\partial^\alpha u\|_{\frac{10}{3}}(\sum_{|\alpha|=1}\|\partial^\alpha u\|_2)^{\frac{2}{10}}\|u_\varepsilon\|_2^{\frac{9}{5}}$$

where we used Hölder and Gagliardo–Nirenberg inequality as before. Then

$$\begin{aligned}
\|\phi_{h_\varepsilon}\partial^\alpha u_\varepsilon\|_{\frac{10}{7}} &\leq \|\partial^\alpha u\|_{\frac{10}{3}}\|\phi_{h_\varepsilon}\|_{\frac{5}{2}}, \\
\|\partial^\beta\phi_{h_\varepsilon}\partial^\gamma u\|_{\frac{10}{7}} &\leq \|\partial^\beta\phi_{h_\varepsilon}\|_5\|\partial^\gamma u_\varepsilon\|_2, \\
\|\partial^\alpha\phi_{h_\varepsilon}u_\varepsilon\|_{\frac{10}{7}} &\leq \|\partial^\alpha\phi_{h_\varepsilon}\|_5\|u_\varepsilon\|_2, \\
\|\partial^\gamma u_\varepsilon\partial^\beta u_\varepsilon u_\varepsilon\|_{\frac{10}{7}} &\leq \|u_\varepsilon\partial^\gamma u_\varepsilon\|_{\frac{5}{2}}\|\partial^\beta u_\varepsilon\|_{\frac{10}{3}} \leq \|u_\varepsilon\|_{10}^2\|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}}\|\partial^\beta u_\varepsilon\|_{\frac{10}{3}} \\
&\leq (\sum_{|\alpha|=1}\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}})^{\frac{3}{4}\cdot 2}\|u_\varepsilon\|_2^{\frac{1}{4}\cdot 2}\|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}}\|\partial^\beta u_\varepsilon\|_{\frac{10}{3}} \\
&\leq (\sum_{|\alpha|=1}\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}})^{\frac{7}{2}}\|u_\varepsilon\|_2^{\frac{1}{2}},
\end{aligned}$$

where for the last term we used Gagliardo–Nirenberg inequality (2.11) with $p = 10,\ j=0,\ m=1,\ r=\frac{10}{3},\ q=2,\ b=\frac{3}{4}$. Finally,

$$\begin{aligned}
&\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \leq \|a_\varepsilon\|_{H^3} + \int_0^t \frac{1}{(t-s)^{\frac{3}{5}}}\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} c_3(a_\varepsilon,\phi_{h_\varepsilon})ds + c_4(a_\varepsilon,\phi_{h_\varepsilon}) \text{ and} \\
&\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \leq (\|a_\varepsilon\|_{H^3}+c_4(a_\varepsilon,\phi_{h_\varepsilon}))\exp(c_3(a_\varepsilon,\phi_{h_\varepsilon})\cdot T^{\frac{2}{5}}), \qquad (4.8)
\end{aligned}$$

where

$$c_3(a_\varepsilon,\phi_{h_\varepsilon}) = H(a_\varepsilon)^{\frac{1}{10}}\|a_\varepsilon\|_2^{\frac{9}{5}} + \|\phi_{h_\varepsilon}\|_{\frac{5}{2}}$$

and

$$c_4(a_\varepsilon, \phi_{h_\varepsilon}) = H(a_\varepsilon)^{\frac{1}{2}}\|\partial^\beta \phi_{h_\varepsilon}\|_5 + \|a_\varepsilon\|_2\|\partial^\alpha \phi_{h_\varepsilon}\|_5 + \|a_\varepsilon\|_2^{\frac{1}{2}} f_\varepsilon^{\frac{7}{2}}.$$

Denote by g_ε the expression on the right hand side of (4.8). It follows that $g_\varepsilon \le c\varepsilon^{-N}$ for $\varepsilon \to 0$ and for some N, since

$$\|\phi_{h_\varepsilon}\|_{\frac{5}{2}} \lesssim h_\varepsilon^{\frac{9}{5}} \sim ((\ln \varepsilon^{-1})^{\frac{5}{11}})^{\frac{9}{5}} \le \ln \varepsilon^{-1},\ \varepsilon \to 0$$

and we have again $H(a_\varepsilon)^{\frac{1}{10}}\|a_\varepsilon\|_2^{\frac{9}{5}} \lesssim h_\varepsilon^{\frac{11}{5}} = \ln \varepsilon^{-1}$. Also $c_4 \lesssim h_\varepsilon^m$ for some $m \in \mathbb{N}$, so we conclude

$$\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \lesssim (\varepsilon^{-N} + h_\varepsilon^m)\varepsilon^{-N_1}. \tag{4.9}$$

Note that $\|\partial^\beta u_\varepsilon\|_{\frac{10}{3}}$ and $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}$ are bounded on $[0,T)$ (an assumption needed for Gronwall's inequality), since the Gagliardo–Nirenberg inequality implies

$$\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \le \Big(\sum_{|\alpha|=3}\|\partial^\alpha u_\varepsilon\|_2\Big)^{\frac{13}{15}}\|u_\varepsilon\|_2^{\frac{2}{15}} < \infty \text{ for each } t \in [0,T).$$

The H^3–norm of the solution is bounded on bounded intervals in t because the global well–posedness holds. One can bound $\|\partial^\beta u_\varepsilon\|_{\frac{10}{3}}$ similarly. Returning to (4.6) we see that

$$\begin{aligned}\sup_{[0,T)}\|\partial^\alpha u_\varepsilon\|_2 \le & \|a_\varepsilon\|_{H^2} + g_\varepsilon f_\varepsilon^{\frac{3}{2}}\|a_\varepsilon\|_2^{\frac{1}{2}} + H(a_\varepsilon)^{\frac{1}{2}} g_\varepsilon^{\frac{20}{13}}\|a_\varepsilon\|_2^{\frac{6}{13}} \\ & + \|\partial^\alpha \phi_{h_\varepsilon}\|_\infty\|a_\varepsilon\|_2 + H(a_\varepsilon)^{\frac{1}{2}}\|\partial^\beta\phi_{h_\varepsilon}\|_\infty + g_\varepsilon\|\phi_{h_\varepsilon}\|_5, \quad (4.10)\\ \sup_{[0,T)}\|\partial^\alpha u_\varepsilon(t)\|_2 = & \mathcal{O}(\varepsilon^{-N}),\ \text{for some } N. \quad (4.11)\end{aligned}$$

Returning to expressions g_ε and f_ε, we see that the above estimate is exponential in $\|a\|_{H^1}$ and $\|\phi_{h_\varepsilon}\|_{\frac{5}{2}}$ (raised to a power), and the other quantities are $\|a_\varepsilon\|_{H^m}$, $m \le 3$ and $\|\partial^\alpha \phi_{h_\varepsilon}\|_p$, $\alpha \le 2$, and some $p \ge 1$; these quantities are multiplied and raised to certain fractional powers.

Moderateness of $\sup_{[0,T)}\|\partial_t u_\varepsilon(t)\|_2$ follows easily from (4.2), since from the Gagliardo-Nirenberg inequality it follows that

$$\||u_\varepsilon(t)|^2 u_\varepsilon(t)\|_2 \le \|\nabla u_\varepsilon(t)\|_2^3.$$

Moreover,

$$u_\varepsilon \in C([0,T), H^2(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n)),\ \varepsilon \in (0,1),$$

which completes the proof. $\square$

We are able to show uniqueness for a special class of solutions.

Definition 4.1.3. *Let $u, v \in \mathcal{G}_{C^1,H^2}$ be any two classes such that for each class there exists a representative solving*

$$\begin{aligned} i\partial_t u_\varepsilon + \triangle u_\varepsilon &= u_\varepsilon |u_\varepsilon|^2 + \phi_{h_\varepsilon} u_\varepsilon + N_\varepsilon, \\ u_\varepsilon(0,x) &= a_\varepsilon(x) + n_\varepsilon(x) \end{aligned} \tag{4.12}$$

where $N_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$ and $n_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^n)$ (similarly for v). If it follows that $\sup_{[0,T)} \|u_\varepsilon - v_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$ for any $M \in \mathbb{N}$, we say that the solution is unique.

Note that from the existence proof we know that at least one u exists with such a property ($N_\varepsilon = n_\varepsilon = 0$).

Theorem 4.1.4. *If $h_\varepsilon \sim \ln^s \ln^q \varepsilon^{-1}$, where $s = \frac{7}{25}$, $q = \frac{1}{500}$ and $a \in \mathcal{G}_{H^3(\mathbb{R}^3)}$, $\|a_\varepsilon\|_{H^3} \sim h_\varepsilon$, the solution is unique in the above sense.*

Proof. Let u, v be as above and $w_\varepsilon = u_\varepsilon - v_\varepsilon$. Then w_ε solves

$$\begin{aligned} i(w_\varepsilon)_t + \triangle w_\varepsilon =& u_\varepsilon |u_\varepsilon|^2 - (u_\varepsilon - w_\varepsilon)(|u_\varepsilon|^2 - u_\varepsilon \overline{w_\varepsilon} - w_\varepsilon \overline{u}_\varepsilon + |w_\varepsilon|^2) + \phi_{h_\varepsilon} w_\varepsilon + N_\varepsilon \\ w_\varepsilon(0,x) =& n_\varepsilon(x), \end{aligned} \tag{4.13}$$

where $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{H^3}(\mathbb{R}^n)$, $(N_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$. The first equation is simplified to

$$\begin{aligned} & i(w_\varepsilon)_t + \triangle w_\varepsilon - |u_\varepsilon|^2 u_\varepsilon + (u_\varepsilon - w_\varepsilon)(|u_\varepsilon|^2 - u_\varepsilon \overline{w_\varepsilon} - \overline{u_\varepsilon} w_\varepsilon + |w_\varepsilon|^2) + N_\varepsilon - \phi_{h_\varepsilon} = 0, \\ & i(w_\varepsilon)_t + \triangle w_\varepsilon = u_\varepsilon^2 \overline{w_\varepsilon} + 2w_\varepsilon |u_\varepsilon|^2 - 2u_\varepsilon |w_\varepsilon|^2 - w_\varepsilon^2 \overline{u_\varepsilon} + w_\varepsilon |w_\varepsilon|^2 + \phi_{h_\varepsilon} - N_\varepsilon. \end{aligned}$$

Multiplying by $\overline{w}_\varepsilon$, integrating on $\mathbb{R}^3$ and taking the imaginary part we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |w_\varepsilon|^2 dx =& \operatorname{Im} \int_{\mathbb{R}^n} \left(2\mathrm{Re}(u_\varepsilon \overline{w_\varepsilon}) u_\varepsilon \overline{w_\varepsilon} - |w_\varepsilon|^2 u_\varepsilon \overline{w_\varepsilon} - N_\varepsilon \overline{w_\varepsilon} \right) dx \\ \leq & \int_{\mathbb{R}^n} \left(2|u_\varepsilon w_\varepsilon|^2 + |u_\varepsilon||w_\varepsilon|^3 + |N_\varepsilon w_\varepsilon| \right) dx. \end{aligned} \tag{4.14}$$

Furthermore, for arbitrary $M \in \mathbb{N}$

$$\begin{aligned}\sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \le & \|n_\varepsilon(x)\|_2^2 + \sup_{[0,T)} \Big(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty\Big) \int_0^T \|w_\varepsilon(t)\|_2^2 d\tau \\ & + \sup_{[0,T)} \|w_\varepsilon(t)\|_2 \|N_\varepsilon(t)\|_2, \\ \sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \le & \varepsilon^M \exp\big(\sup_{[0,T)} \Big(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty\Big)\big). \end{aligned} \tag{4.15}$$

The following estimates are needed for completing the proof. In order to bound $\|u_\varepsilon\|_\infty$ we aim to bound $\|u_\varepsilon\|_{H^2}$ by a function of the initial condition, since then we can control $\|u_\varepsilon\|_\infty$ by $\sqrt{\ln \varepsilon^{-1}}$ (otherwise, we can only control it by ε^{-N}). For that we repeat the procedure of the existence proof. First we derive estimates for L^2 and H^1 norm of u_ε the solution of (4.12) (we do not have classical conservation of charge and energy, nevertheless, we use similar arguments). Multiplying (4.12) by $\overline{u}_\varepsilon$, integrating over $\mathbb{R}^n$ and taking the real part we obtain

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|u_\varepsilon\|_2^2 &= \operatorname{Im} \int N_\varepsilon \overline{u}_\varepsilon dx \\ \frac{1}{2}\|u_\varepsilon\|_2^2 &= \|a_\varepsilon + n_\varepsilon\|_2 + \operatorname{Im} \int_0^t \int N_\varepsilon \overline{u}_\varepsilon dx ds \\ &\le \|a_\varepsilon + n_\varepsilon\|_2 + \int_0^t \|N_\varepsilon\|_2 \|u_\varepsilon\|_2 \\ &\lesssim h_\varepsilon + \varepsilon^M + \varepsilon^M \cdot \varepsilon^{-N} \lesssim h_\varepsilon, \end{aligned}$$

since $u \in \mathcal{G}_{C^1,H^2}$. Further, multiplying (4.12) by $\overline{u}_t$, integrating over $\mathbb{R}^n$ and taking the real part

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\frac{1}{2}\int |\nabla u_\varepsilon|^2 dx + \frac{1}{4}\int |u_\varepsilon|^4 dx + \frac{1}{2}\int \phi_{h_\varepsilon}|u_\varepsilon|^2 dx\Big) \le \int |N_\varepsilon||u_\varepsilon| dx, \\ &H(u(t)) \le H(a_\varepsilon + n_\varepsilon) + \|N_\varepsilon\|_2 \|u_\varepsilon\|_2 \\ &\lesssim H(a_\varepsilon + n_\varepsilon) + \varepsilon^M \quad \text{and} \\ &\sum_{|\gamma|=1} \|\partial^\gamma u_\varepsilon\|_2 = \sqrt{\int |\nabla u_\varepsilon|^2 dx} \le \sqrt{H(u(t))} \\ &\lesssim ((1 + \|\phi_{h_\varepsilon}\|_\infty)\|a_\varepsilon + n_\varepsilon\|_{H^1}^2 + \|a_\varepsilon + n_\varepsilon\|_{H^1}^4 + \varepsilon^M)^{\frac{1}{2}} \\ &\lesssim (h_\varepsilon^n h_\varepsilon^2 + h_\varepsilon^4)^{\frac{1}{2}} \lesssim h_\varepsilon^{\frac{5}{2}}. \end{aligned}$$

Using the same procedure as in the existence proof we obtain $\|\partial^\gamma u_\varepsilon\|_{\frac{10}{3}} \lesssim f_\varepsilon^1$ where

$$f_\varepsilon^1 \sim (\|a_\varepsilon + n_\varepsilon\|_{H^2} + \|u_\varepsilon\|_2 \|\partial^\gamma \phi_{h_\varepsilon}\|_5 + \|\partial^\gamma u_\varepsilon\|_2 \|\phi_{h_\varepsilon}\|_5) \exp((\sum_{|\alpha|=1} \|\partial^\alpha u_\varepsilon\|_2)^{\frac{1}{5}} \|u_\varepsilon\|_2)$$

$$\lesssim (h_\varepsilon + \varepsilon^M + h_\varepsilon h_\varepsilon^{(5+12)/5} + h_\varepsilon^{\frac{5}{2}} h_\varepsilon^{12/5}) \exp(h_\varepsilon^{\frac{1}{2}} h_\varepsilon)$$
$$= (h_\varepsilon + \varepsilon^M + h_\varepsilon^{22/5} + h_\varepsilon^{49/10}) \exp(h_\varepsilon^{3/2}) \lesssim h_\varepsilon^{49/10} \exp(h_\varepsilon^{3/2}).$$

Further, $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}} \lesssim g_\varepsilon^1$ where

$$g_\varepsilon^1 \sim (\|a_\varepsilon + n_\varepsilon\|_{H^3} + H(u_\varepsilon(t))^{\frac{1}{2}} \|\partial^\beta \phi_{h_\varepsilon}\|_5 + \|u_\varepsilon\|_2 \|\partial^\gamma \phi_{h_\varepsilon}\|_5 + \|u_\varepsilon\|_2^{\frac{1}{2}} (f_\varepsilon^1)^{\frac{7}{2}}) \cdot$$
$$\exp(H(u_\varepsilon(t))^{\frac{1}{10}} \|u_\varepsilon\|_2^{\frac{9}{5}} + \|\phi_{h_\varepsilon}\|_{\frac{5}{2}}))$$
$$\lesssim (h_\varepsilon + h_\varepsilon^{\frac{5}{2}} h_\varepsilon^{\frac{17}{5}} + h_\varepsilon h_\varepsilon^{\frac{17}{5}} + h_\varepsilon^{\frac{1}{2}} h_\varepsilon^{7 \cdot 49/20} (\exp h^{\frac{3}{2}})^{7/2}) \exp(h_\varepsilon^{\frac{1}{2}} h_\varepsilon^{\frac{9}{5}} + h_\varepsilon^{\frac{9}{5}})$$
$$\lesssim (h_\varepsilon^{17/2} + h_\varepsilon^{353/20} (\exp h_\varepsilon^{3/2})^{7/2}) \exp(h_\varepsilon^{9/5}).$$

Also, estimating $\|\partial^\alpha u_\varepsilon\|_2$ as in the existence proof

$$\sup_{[0,T)} \|\partial^\alpha u_\varepsilon\|_2 \le \|a_\varepsilon + n_\varepsilon\|_{H^2} + g_\varepsilon^1 (f_\varepsilon^1)^{\frac{3}{2}} \|u_\varepsilon\|_2^{\frac{1}{2}} + H(u_\varepsilon)^{\frac{1}{2}} (g_\varepsilon^1)^{\frac{20}{13}} \|u_\varepsilon\|_2^{\frac{6}{13}}$$
$$+ \|\partial^\alpha \phi_{h_\varepsilon}\|_\infty \|u_\varepsilon\|_2 + H(u_\varepsilon)^{\frac{1}{2}} \|\partial^\beta \phi_{h_\varepsilon}\|_\infty + g_\varepsilon^1 \|\phi_{h_\varepsilon}\|_5 + T \sup_{[0,T)} \|\partial^\alpha N_\varepsilon\|_2, \tag{4.16}$$
$$\lesssim h_\varepsilon + g_\varepsilon^1 (f_\varepsilon^1)^{\frac{3}{2}} h_\varepsilon^{\frac{1}{2}} + h_\varepsilon^{\frac{5}{2}} (g_\varepsilon^1)^{\frac{20}{13}} h_\varepsilon^{\frac{6}{13}} + h_\varepsilon^3 h_\varepsilon + h_\varepsilon^{\frac{5}{2}} h_\varepsilon^4 + g_\varepsilon^1 h_\varepsilon^{\frac{12}{5}} + \varepsilon^M$$
$$\lesssim h_\varepsilon + h_\varepsilon^{\frac{3}{2}} g_\varepsilon^1 (f_\varepsilon^1)^{\frac{3}{2}} + h_\varepsilon^{\frac{77}{26}} (g_\varepsilon^1)^{\frac{20}{13}} + h_\varepsilon^4 + h_\varepsilon^{\frac{13}{2}} + h_\varepsilon^{\frac{12}{5}} g_\varepsilon^1. \tag{4.17}$$

We can now use the Sobolev embedding $\|u_\varepsilon(t)\|_\infty^2 \lesssim \|u_\varepsilon(t)\|_{H^2}^2$. Choosing $h_\varepsilon \sim \ln^s \ln^q \varepsilon^{-1}$ where $s = \frac{7}{25}$ and $q = \frac{1}{500}$ and using the fact that $\ln^s \ln^q \varepsilon^{-1} \le \ln^q \varepsilon^{-1}$, $\varepsilon \to 0$ for $s \le 1$, each term in (4.17) can be estimated by $\sqrt{\ln \varepsilon^{-1}}$. Thus,

$$\|u_\varepsilon(t)\|_{H^2} \sim \sqrt{\ln \varepsilon^{-1}}. \tag{4.18}$$

Returning to (4.15), it follows that for any $M \in \mathbb{N}$ $\|w_\varepsilon(t)\|_2^2 \le \varepsilon^M$, which completes the proof. □

4.2 The cubic Schrödinger equation

Now we study the cubic equation without potential

$$\begin{aligned} iu_t + \triangle u &= u|u|^2, \\ u(0) &= a \end{aligned} \tag{4.19}$$

in two and three space dimensions. There are many physical phenomena that are connected with (4.19). In dimension three it represents dynamics of the interacting Bose gas. Other applications are related to small amplitude gravity waves and dynamics of quantum plasma. The equation also describes propagation of short optical pulses in optical fibers, see [GKY90]. Its soliton solutions are referred to as dark solitons, the expression coming from optics.

We list several estimates known for this equation which are useful for our analysis and then focus on well - posedness in the Colombeau algebra.

From theorems 3.3.4 and 3.3.11 we see there is local and global well - posedness in H^2. Also, conservation of charge (3.20) and energy (3.21) holds.

In one dimension, for any $s \geq 0$ the norm $\|u(t)\|_{H^s}$ is uniformly bounded w.r.t. to $t \in \mathbb{R}$. In two and three dimensions $u(t) \in H^s$ holds for every t and there exists $T = T(\|a\|_{H^s})$ such that

$$\|u(t)\|_{H^s} \leq C\|a\|_{H^s}, \quad t \in [0, T].$$

In [Bou98], it was shown that in 3D there is scattering and a uniform bound

$$\|u(t)\|_{H^s} \leq C \exp(\|a\|_{H^s}), \quad \text{for all } t \geq 0, \quad s \geq 1. \tag{4.20}$$

This paper was an extension of results form [GV85] and [LS78], based on the Morawetz' inequality.

In [Col+01] (inequality (3.25)), it was shown by a similar argument that global in time solutions in 2D also satisfy a uniform bound

$$\|u(t)\|_{H^s} \leq c\|a\|_{H^s}, \quad \text{for all } t \geq 0, \quad s \geq 1. \tag{4.21}$$

We use Bourgain's estimate (4.20) for the existence proof. But estimates from Section 4.1 are needed for the uniqueness proof. From this section we can conclude that if u is a solution of (4.19) and $a \in H^3$, then it satisfies the following bound

$$\|u\|_{H^2} \leq p_k\left(\|a\|_{H^1}, \|a\|_{H^2}, \|a\|_{H^3}, \exp(c\|a\|_{H^1})\right),$$

where p_k is a function of fractional power k. This follows from relations (4.7), (4.8) and (4.18) with (dropping the subscript ε)

$$\begin{aligned} f &= \|a\|_{H^2} \exp(c\|a\|_{H^1}), \\ g &= (\|a\|_{H^3} + \|a\|_2 f^{\frac{7}{2}}) \exp(c_1\|a\|_{H^1}). \end{aligned}$$

4.2.1 Existence and uniqueness

From the previous section it follows that Definition 2.5.10 is independent of the representative. The main theorem of this section is the following.

Theorem 4.2.1. *Let $n \in \{2,3\}$, $T > 0$, $a \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ such that there exists a representative a_ε which satisfies the following:*

$$\|a_\varepsilon\|_{H^2} \leq h_\varepsilon \tag{4.22}$$

where $h_\varepsilon \sim \varepsilon^{-N}$ for $n = 2$ and $h_\varepsilon \sim N \ln \varepsilon^{-1}$ for $n = 3$, for some $N \in \mathbb{N}$. Then there exists a solution $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ of (4.19). If, additionally $\|a_\varepsilon\|_{H^3} \sim \ln^s \ln^q \varepsilon^{-1}$, where $s = \frac{5}{7}$, $q = \frac{1}{24}$, the solution is unique in the sense of Definition 4.1.3.

Proof. Existence. Let us take the equation (4.19) written in the form of representatives

$$\begin{aligned} i\partial_t u_\varepsilon + \triangle u_\varepsilon - |u_\varepsilon|^2 u_\varepsilon = 0 \\ u_\varepsilon(x,0) = a_\varepsilon(x) \end{aligned} \tag{4.23}$$

As we have seen before, there exists a unique solution $u_\varepsilon \in C^([0,T], H^2(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n))$ for every $T > 0$ and ε. Estimates (4.20) and (4.21) together with assumption (4.22) imply

$$\sup_{t\geq 0} \|\partial^\alpha u_\varepsilon(t)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}),\ \varepsilon \to 0,$$

for $|\alpha| \leq 2$. Again boundedness of $\|\partial_t u_\varepsilon(t)\|_2$ follows easily from (4.23). We can conclude that u, represented by the net of functions $(u_\varepsilon)_\varepsilon$ belongs to the space $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ that solves the problem (4.19) in the sense of Definition 2.5.10.

Uniqueness. Let $u, v \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$, $n \in \{2,3\}$ be two solutions of (4.19) with representatives u_ε and v_ε satisfying

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon = |u_\varepsilon|^2 u_\varepsilon + \phi_{h_\varepsilon} u_\varepsilon + N_\varepsilon, \\ u_\varepsilon(0) = a_\varepsilon + n_\varepsilon, \end{aligned} \tag{4.24}$$

for $N_\varepsilon \in \mathcal{N}_{C^1,H^2},\ n_\varepsilon \in \mathcal{N}_{H^2}$.

Let $w_\varepsilon = u_\varepsilon - v_\varepsilon$. Then w_ε solves:

$$\begin{aligned} i(w_\varepsilon)_t + \triangle w_\varepsilon - (|u_\varepsilon|^2 u_\varepsilon - |u_\varepsilon - w_\varepsilon|^2 (u_\varepsilon - w_\varepsilon)) + N_\varepsilon = 0, \\ w_\varepsilon(x,0) = n_\varepsilon(x), \end{aligned} \tag{4.25}$$

where $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{H^3}(\mathbb{R}^n),\ (N_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$. The first equation is simplified to

$$\begin{aligned} &i(w_\varepsilon)_t + \triangle w_\varepsilon - |u_\varepsilon|^2 u_\varepsilon + (u_\varepsilon - w_\varepsilon)(|u_\varepsilon|^2 - u_\varepsilon \overline{w_\varepsilon} - \overline{u_\varepsilon} w_\varepsilon + |w_\varepsilon|^2) + N_\varepsilon = 0, \\ &i(w_\varepsilon)_t + \triangle w_\varepsilon = u_\varepsilon^2 \overline{w_\varepsilon} + 2w_\varepsilon |u_\varepsilon|^2 - 2u_\varepsilon |w_\varepsilon|^2 - w_\varepsilon^2 \overline{u_\varepsilon} + w_\varepsilon |w_\varepsilon|^2 - N_\varepsilon \end{aligned}$$

If we multiply (4.25) by $\overline{w_\varepsilon}$, integrate over $\mathbb{R}^n$ and take the imaginary part

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |w_\varepsilon|^2 dx =\ &\operatorname{Im} \int_{\mathbb{R}^n} \left(2\operatorname{Re}(u_\varepsilon \overline{w_\varepsilon}) u_\varepsilon \overline{w_\varepsilon} - |w_\varepsilon|^2 u_\varepsilon \overline{w_\varepsilon} - N_\varepsilon \overline{w_\varepsilon} \right) dx \\ \leq &\int_{\mathbb{R}^n} \left(2|u_\varepsilon w_\varepsilon|^2 + |u_\varepsilon||w_\varepsilon|^3 + |N_\varepsilon w_\varepsilon| \right) dx. \end{aligned} \tag{4.26}$$

Integration with respect to t gives

$$\begin{aligned} \|w_\varepsilon(t)\|_2^2 \leq &\|n_\varepsilon\|_2^2 + \int_0^t \Big(2\|u_\varepsilon(t)\|_\infty^2 \|w_\varepsilon(t)\|_2^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_2^2 \\ &+ \|N_\varepsilon\|_2 \|w_\varepsilon(t)\|_2 \Big) d\tau \\ \sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \leq &\|n_\varepsilon\|_2^2 + 2 \sup_{[0,T)} \left(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty \right) \int_0^T \|w_\varepsilon(t)\|_2^2 d\tau \\ &+ \sup_{[0,T)} \|w_\varepsilon(t)\|_2 \|N_\varepsilon\|_2, \\ \sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \leq &\varepsilon^M \exp(\sup_{[0,T)} \left(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty \right)), \end{aligned} \tag{4.27}$$

for arbitrary $M \in \mathbb{N}$. The Sobolev inequality $\|u_\varepsilon(t)\|_\infty \le \|u_\varepsilon(t)\|_{H^2}$ holds. But, estimates (4.20) and (4.21) can not be directly used bellow, since equation (4.24) is not homogeneous. These bounds are derived in Theorem 4.1.4 (relation (4.16)) and the difference now is that the terms with ϕ_{h_ε} are missing. Condition for h_ε can now be relaxed to $h_\varepsilon \sim \ln^s \ln^q \varepsilon^{-1}$, where $s = \frac{5}{7}$, $q = \frac{1}{24}$ which implies $\|u_\varepsilon(t)\|_{H^2} \sim \sqrt{\ln \varepsilon^{-1}}$.

Applying Gronwall's inequality (2.4.3) to (4.27) we obtain

$$\sup_{0 \le t \le T} \|w_\varepsilon(t)\|_2 = O(\varepsilon^M), \quad \varepsilon \to 0, \quad \text{for any } M \in \mathbb{N}, \tag{4.28}$$

implying that the solution is unique in the sense of Definition 4.1.3. □

4.2.2 Compatibility with the classical solution

We now prove that there is compatibility between the Sobolev H^2 solution and the Colombeau solution of (4.19) in the sense of Definition 2.5.13. Let ϕ_ε be a mollifier as defined in Section 2.5. The following holds

Theorem 4.2.2. *Let u be the classical H^2 solution of the cubic Schrödinger equation in $n \in \{2, 3\}$ dimensions:*

$$\begin{aligned} iu_t + \triangle u - |u|^2 u = 0 \quad &\text{on} \quad \mathbb{R}^n \times (0, \infty) \\ u(0) = a, & \end{aligned}$$

for $a \in H^3(\mathbb{R}^n)$. Let $T > 0$. The solution u_ε to the equation (4.23) with initial data $a_\varepsilon = a * \phi_\varepsilon$ converges to u in the $L^2(\mathbb{R}^n)$ norm for every $t < T$.

Proof. Since

$$\|\partial_x^\alpha (a * \phi_\varepsilon)\|_2 = \|\partial_x^\alpha a * \phi_\varepsilon\|_2 \le \|\partial_x^\alpha a\|_2 \|\phi_\varepsilon\|_1 = \|\partial_x^\alpha a\|_2$$

for $|\alpha| \le 3$, uniformly with respect to ε, we obtain condition (4.22). It follows that the regularized initial data give rise to a unique solution in the space $\mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^n)$.

Let $v_\varepsilon = u - u_\varepsilon$. Then $u \in H^2$ implies that $\|u(t)\|_\infty$ is finite, and $u_\varepsilon \in H^2$ for each $\varepsilon > 0$ gives, based on (4.20),

$$\begin{aligned}\|v_\varepsilon(t)\|_\infty \le \|u(t)\|_\infty + \|u_\varepsilon(t)\|_\infty &\le c_1 + \|u_\varepsilon(t)\|_{H^2} \\ &\le c_1 + \exp(\|a * \phi_\varepsilon\|_{H^2}) \le c_1 + c_2,\end{aligned}$$

Also,

$$\|\partial_x^\gamma v_\varepsilon(t)\|_2 \le \|\partial_x^\gamma u(t)\|_2 + \|\partial_x^\gamma u_\varepsilon(t)\|_2 \le c, \quad |\gamma| \le 2$$

Further, v_ε satisfies

$$\begin{aligned} i\partial_t v_\varepsilon + \triangle v_\varepsilon - (|u|^2 u - |u - v_\varepsilon|^2(u - v_\varepsilon)) = 0,& \\ v_\varepsilon(x,0) = a(x) - a * \phi_\varepsilon(x).&\end{aligned}$$

Like in the uniqueness proof, one can see that

$$\|v_\varepsilon(t)\|_2^2 \le \|a - a * \phi_\varepsilon\|_2^2 \ \exp((\|u(t)\|_\infty^2 + \|u(t)\|_\infty \|v_\varepsilon(t)\|_\infty)T).$$

Therefore,

$$\|v_\varepsilon(t)\|_2^2 \le C\|a - a * \phi_\varepsilon\|_2^2 \to 0, \quad \varepsilon \to 0.$$

This completes the proof. □

The Hartree equation

5

We observe now the Hartree equation with a delta potential:

$$\begin{aligned} iu_t + \triangle u - (w * |u|^2)u &= \delta u, \\ u(0) &= a. \end{aligned} \tag{5.1}$$

We will study this equation in the Colombeau setting and then try to connect the theory related to a different formulation of (5.1), namely

$$\begin{aligned} iu_t + \triangle_\alpha u = (w * |u|^2)u, \\ u(0) = a \end{aligned} \tag{5.2}$$

Here, $-\triangle u + \delta u$ is understood as a singular perturbation of the negative Laplacian. Let us describe shortly the related theory as in [GM18] and [MOS18].

5.1 Singular Laplacian and well - posedness in the singular Sobolev space

Observe a one–parameter family of operators $\triangle_\alpha$, $\alpha \in (-\infty, \infty]$, defined by

$$\begin{aligned} \mathcal{D}(-\triangle_\alpha) &= \{\psi \in L^2(\mathbb{R}^3) |\ \psi = \phi_\lambda + \frac{\phi_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} G_\lambda,\ \phi_\lambda \in H^2(\mathbb{R}^3), \\ (-\triangle_\alpha + \lambda)\psi &= (-\triangle + \lambda)\phi_\lambda, \end{aligned}$$

where $\lambda > 0$ is an arbitrarily fixed constant and

$$G_\lambda(x) := \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}$$

is the Green's function for the Laplacian, that is, the distributional solution to $(-\triangle + \lambda)G_\lambda = \delta$ in $\mathcal{D}'(\mathbb{R}^3)$. Note that $G_\lambda \in L^2(\mathbb{R}^3)$. The operator $\triangle_\alpha$ induces the

Schrödinger propagator $t \mapsto e^{it\triangle_\alpha}$, analogous to the usual propagator. The space H^2_α is exactly $\mathcal{D}(-\triangle_\alpha)$ with the norm

$$\|\psi\|_{H^2_\alpha} = \|(I - \triangle_\alpha)\psi\|_2.$$

For arbitrary $\psi = \phi_\lambda + \frac{\phi_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} G_\lambda \in H^2_\alpha$ there holds

$$\|\psi\|_{H^2_\alpha} \approx \|\phi_\lambda\|_{H^2}.$$

A function u is a solution of (5.2) if $u \in C(I, H^2_\alpha(\mathbb{R}^3))$ for some interval $I \subset \mathbb{R}$ with $0 \in I$ and the Duhamel's formula

$$u(t) = e^{it\triangle_\alpha} a - i \int_0^t e^{i(t-s)\triangle_\alpha}(w * |u(s)|^2)u(s)ds \tag{5.3}$$

holds. Local and global well–posedness in H^2_α is defined in the same way as for H^2 spaces.

5.2 Higher regularity

We are interested in connecting the Colombeau solution of (5.1) and the singular Sobolev solution of (5.2). In that purpose, we prove the following theorem.

Theorem 5.2.1. *Let $w \in W^{2,p}(\mathbb{R}^3),\ p > 2$ and w is even. The Cauchy problem* (5.2) *is locally well-posed in the space*

$$V = \{u \in H^2(\mathbb{R}^3), u \text{ is odd}\} \subset H^2(\mathbb{R}^3) \cap H^2_\alpha(\mathbb{R}^3)$$

and there is also global well – posedness.

Remark 5.2.2. *This theorem is already known, only we present a different proof. Odd functions are L^2–orthogonal to spherically symmetric functions, and on such a space the operator $\triangle_\alpha$ is the same as $\triangle$, see [MOS18].*

Proof. The proof is based on methods from [Caz03], similar to the ones in [MOS18], but taking a different form in the usual Sobolev space.

Note that V is closed: if $u_n \in V$ converges to u in the H^2–norm, then it converges also in the L^∞–norm and for almost all x we have

$$u(-x) = \lim_{n\to\infty} u_n(-x) = -\lim_{n\to\infty} u_n(x) = -u(x).$$

Hence u is odd and $u \in H^2(\mathbb{R}^3)$ implying that $u \in V$. As a closed subset of a complete metric space $H^2(\mathbb{R}^3)$, V is itself complete. We will now use the fixed point theorem on the space

$$
\begin{aligned}
V_M &= \{u \in L^\infty([-T,T],V) : \sup_{t\in[-T,T]} \|u(t)\|_{H^2} \le M\},\\
d(u,v) &= \|u-v\|_{L_t^\infty,L_x^2},
\end{aligned}
$$

where T and M will be determined later. Note that on the intersection of spaces $H^2(\mathbb{R}^3)$ and $H^2_\alpha(\mathbb{R}^3)$, the norms $\|\cdot\|_{H^2}$ and $\|\cdot\|_{H^2_\alpha}$ are equivalent and the characterization of this space is that $u \in H^2(\mathbb{R}^3)$ and $u(0) = 0$. The operator $-\triangle_\alpha$ acts as $-\triangle$ on the space of $H^2(\mathbb{R}^3)$ functions which vanish at zero.

From Duhamel's formula we have:

$$\|\Phi(u)\|_{H^2} \le \|e^{it\triangle_\alpha}a\|_{H^2} + T\|e^{it\triangle_\alpha}(w*|u|^2)u\|_{H^2}$$

Since $\triangle_\alpha a = \triangle a$ for $a \in H^2 \cap H^2_\alpha$, we have $e^{it\triangle_\alpha}a = e^{it\triangle}a$. Also, we will see that $(w*|u|^2)u \in H^2$ and for $u \in V_M$ there holds $u(0) = 0$, so $((w*|u|^2)u)(0) = 0)$ and $e^{it\triangle_\alpha}(w*|u|^2)u = e^{it\triangle}(w*|u|^2)u$. It follows that

$$
\begin{aligned}
\|\Phi(u)\|_{H^2} &\le \|e^{it\triangle}a\|_{H^2} + T\|e^{it\triangle}(w*|u|^2)u\|_{H^2}\\
&\le \|a\|_{H^2} + T\|(w*|u|^2)u\|_{H^2} \le \|a\|_{H^2} + C_1T\|w\|_\infty\|u\|^3_{H^2}
\end{aligned}
\tag{5.4}
$$

The term $(w*|u|^2)u$ is in H^2 for $u \in V_M$ due to following inequalities:

$$
\begin{aligned}
\|\partial_x^\alpha((w*|u|^2)u)\|_2 &\lesssim \|w*(\partial_x^\alpha|u|^2)\|_\infty\|u\|_2 + 2\|w*(\partial_x^\beta|u|^2)\|_\infty\|\partial_x^\beta u\|_2\\
&\quad + \|w*|u|^2\|_\infty\|\partial^\beta u\|_2\\
&\lesssim \|w\|_\infty\|u\|^3_{H^2} + 2\|w\|_\infty\|u\|^3_{H^2} + \|w\|_\infty\|u\|^3_{H^2} = c\|w\|_\infty\|u\|^3_{H^2},
\end{aligned}
\tag{5.5}
$$

where $|\alpha| = 2$ and $|\beta| = 1$. Note that $W^{2,p} \subset L^\infty$, for $p > 2$. To prove that Φ is a contraction observe

$$\|\Phi(u)-\Phi(v)\|_2 \le C_2T\|u-v\|_2\big(\|w\|_\infty(\|u\|_2^2 + \|v\|_2(\|u\|_2 + \|v\|_2))\big),$$

where estimates are derived as in proof of Theorem 3.3.7. Choosing $M = 2\|a\|_{H^2}$ and $T = \frac{1}{4}\big(\max\{C_1,C_2\}M^2\|w\|_\infty\big)^{-1}$ we have first from (5.4)

$$\|\Phi(u)\|_{H^2} \le \frac{M}{2} + C_1 \cdot \frac{1}{4}\frac{1}{M^2C_1\|w\|_\infty} \cdot \|w\|_\infty M^3 < M$$

for $\|u\|_{H^3} \leq M$, since $\frac{1}{\max\{C_1,C_2\}} \leq \frac{1}{C_1}$. Further,

$$\|\Phi(u)-\Phi(v)\|_2 \leq C_2 \cdot \frac{1}{4C_2M^2\|w\|_\infty}\|u-v\|_2 \cdot \|w\|_\infty 3M^2 = \frac{3}{4}\|u-v\|_2,$$

so that Φ is a contraction. To conclude that $\Phi : V_M \to V_M$ we need to show that $\Phi(u)$ is odd. Returning to Duhamel's formula, we have that

$$\Phi(u) = e^{it\triangle}a - i\int_0^t e^{i(t-s)\triangle}(w * |u|^2)u \, ds. \tag{5.6}$$

Firstly, $e^{it\triangle}a$ is odd:

$$(e^{it\triangle}a)(-x) = \int_{\mathbb{R}^3} e^{-it|\xi|^2}e^{x\xi}\hat{a}(\xi)d\xi = \int_{\mathbb{R}^3} e^{-it|\xi|^2}e^{ix\xi_1}\hat{a}(-\xi_1)d\xi_1 = -(e^{it\triangle}a)(x).$$

We used a substitution $\xi = -\xi_1$ and the fact that $\hat{a}$ is also odd:

$$\hat{a}(-\xi) = \int_{\mathbb{R}^3} e^{ix\xi}a(x)dx = \int_{\mathbb{R}^3} e^{-iy\xi}a(-y)dy = -\hat{a}(\xi),$$

since a is odd. Similarly,

$$e^{i(t-s)\triangle}(w * |u|^2)u = \int_{\mathbb{R}^3} e^{ix\xi}e^{-i(t-s)|\xi|^2}((w * \widehat{|u|^2})u)(\xi)d\xi.$$

Now, we prove that $(w * \widehat{|u|^2})u$ is odd. By the convolution theorem

$$((w * \widehat{|u|^2})u) = \widehat{w * |u|}^2 * \hat{u}.$$

As before, we see that $\hat{u}$ is odd, since u is odd. On the other hand, $\widehat{w * |u|}^2$ is even:

$$\widehat{w * |u|}^2 = \hat{w} \cdot \widehat{|u|}^2$$

and this is a product of two even functions since $|u|^2$ is even. Finally, convolution of an odd and an even function is odd:

$$\begin{aligned}(f * g)(-x) &= \int_{\mathbb{R}^3} f(-x-y)g(y)dy = \int_{\mathbb{R}^3} f(-x+s)g(-s)ds \\ &= -\int_{\mathbb{R}^3} f(x-s)g(s)ds = -(f * g)(x),\end{aligned}$$

where f is odd and g is even. Since both terms in (5.6) are odd, we conclude that Φ is odd.

Uniqueness on the whole space V follows from Theorem 3.2.5 in the following manner. Suppose there is another solution v of (5.2). Since $\triangle_\alpha v = \triangle v$, for $v \in H^2(\mathbb{R}^3) \cap H^2_\alpha(\mathbb{R}^3)$ (and also for u), both u and v are also H^1 solutions of

$$\begin{aligned} iu_t + \triangle u &= (w * |u|^2)u, \\ u(0,x) &= a(x), \end{aligned}$$

and conditions of Theorem 3.2.5 are fulfilled. It follows that $u = v$.

As in Theorem 3.3.4, we can extend such a solution over a maximal time interval for which the blow-up alternative holds. Also, Theorem 3.3.11 holds and there is global well–posedness. $\square$

5.3 Maximal time interval

Now we know that if $a \in H^2_\alpha$, then there is a unique solution in H^2_α on a maximal interval (T_*, T^*). If additionally, $a \in V$ there is a unique solution in V on a maximal interval (T_1, T_2). Since a solution in V is a $H^2_\alpha(\mathbb{R}^3)$ solution, we see that $(T_1, T_2) \subset (T_*, T^*)$. But furthermore, since on the intersection of spaces, there holds:

$$\|u(t)\|_{H^2} \approx \|u(t)\|_{H^2_\alpha} \to \infty, \text{ for } t \to T^*,$$

it follows that $T_2 = T^*$ and similarly, $T_1 = T_*$.

(If we assume that $T_2 < T^*$ it would follow

$$\|u(t)\|_{H^2_\alpha} \approx \|u(t)\|_{H^2} \to \infty, \ t \to T_2,$$

which is a contradiction with the blowup of $\|u(t)\|_{H^2_\alpha}$ at T^*.)

5.4 Hartree equation in the Colombeau algebra

Return now to the original equation (5.1). Let us again confirm that if u is a solution in the sense of Definition 2.5.10, that is if (2.17) holds for some u_ε, then it holds for all representatives of the class $u = [u_\varepsilon]$: let $v_\varepsilon = u_\varepsilon + N_\varepsilon$, $N_\varepsilon \in \mathcal{N}_{C^1,H^2}$, then

$$
\begin{aligned}
& i(v_\varepsilon)_t + \triangle v_\varepsilon - (w * |v_\varepsilon|^2)v_\varepsilon - \phi_\varepsilon v_\varepsilon = i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon - \phi_\varepsilon u_\varepsilon \\
& + i(N_\varepsilon)_t + \triangle N_\varepsilon - \phi_\varepsilon N_\varepsilon - ((w * |u_\varepsilon|^2)N_\varepsilon + (w * (|N_\varepsilon|^2 + u_\varepsilon \overline{N}_\varepsilon + N_\varepsilon \overline{u}_\varepsilon))(u_\varepsilon + N_\varepsilon)) \\
& = M_\varepsilon + i(N_\varepsilon)_t + \triangle N_\varepsilon - \phi_\varepsilon N_\varepsilon - g(w, u_\varepsilon, N_\varepsilon),
\end{aligned}
$$

where $\|M_\varepsilon\|_{L^2} \sim \varepsilon^M$, for any $t \in [0,T)$. Now since $N_\varepsilon \in \mathcal{N}_{C^1,H^2}$, it follows $\|i(N_\varepsilon)_t + \triangle N_\varepsilon\|_{L^2} \sim \varepsilon^M$ for any $t \in [0,T)$. Furthermore,

$$\|\phi_\varepsilon N_\varepsilon\|_{L^2} \leq \|\phi_\varepsilon\|_\infty \|N_\varepsilon\|_{L^2} \leq \|\phi_\varepsilon\|_{H^2} \|N_\varepsilon\|_{L^2} \sim \varepsilon^M.$$

For $g(w, u_\varepsilon, N_\varepsilon)$ we have the following bounds

$$
\begin{aligned}
& \|(w * |u_\varepsilon|^2)N_\varepsilon\|_{L^2} \leq \|w\|_\infty \|u_\varepsilon\|_{L^2}^2 \|N_\varepsilon\|_{L^2} \sim \varepsilon^M, \\
& \|(w * |N_\varepsilon|^2)u_\varepsilon\|_{L^2} \leq \|w\|_\infty \|N_\varepsilon\|_{L^2}^2 \|u_\varepsilon\|_{L^2} \sim \varepsilon^M, \\
& \|(w * u_\varepsilon \overline{N}_\varepsilon)u_\varepsilon\|_{L^2} \leq \|w\|_\infty \|N_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^2} \sim \varepsilon^M,
\end{aligned}
$$

and completely analogously for the remaining terms. Finally,

$$v_\varepsilon(0) = u_\varepsilon(0) + N_\varepsilon(0) = a_\varepsilon + n_\varepsilon + N_\varepsilon(0) = a_\varepsilon + N_\varepsilon^1$$

where $N_\varepsilon^1 \in \mathcal{N}_{H^2}$.

5.4.1 Existence and uniqueness

We consider regularized version of (5.1):

$$
\begin{aligned}
i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon &= \phi_{h_\varepsilon} u_\varepsilon & (5.7) \\
u_\varepsilon(0,x) &= a_\varepsilon(x), & (5.8)
\end{aligned}
$$

where $h_\varepsilon > 0$ will be determined later and $w \in W^{2,p}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ (due to the Sobolev embedding) and $a_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^3)$. We have seen in Section 3.3 that conservation of energy and charge holds, that is

$$\|u_\varepsilon(t)\|_2 = \|a_\varepsilon\|_2$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx + \frac{1}{4}\int_{\mathbb{R}^3}(w*|u_\varepsilon|^2)|u_\varepsilon|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{h_\varepsilon}|u_\varepsilon|^2dx = H(a_\varepsilon),$$

where

$$H(a_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla a_\varepsilon|^2dx + \frac{1}{4}\int_{\mathbb{R}^3}(w*|a_\varepsilon|^2)|a_\varepsilon|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{h_\varepsilon}|a_\varepsilon|^2dx.$$

Also, using Young's inequality

$$\|u_\varepsilon(t)\|_{H^1} \le \|a_\varepsilon\|_{L^2} + \sqrt{2H(a_\varepsilon)} \le \|a_\varepsilon\|_{L^2} + \sqrt{c\|a_\varepsilon\|^2_{H^2} + \frac{1}{2}\|w\|_\infty\|a_\varepsilon\|^4_2}.$$

It follows that

$$\|u_\varepsilon(t)\|_{H^1} \le \|a_\varepsilon\|_{L^2} + c_1\|a_\varepsilon\|_{H^2}\sqrt{1 + c_2\|a_\varepsilon\|^2_{H^2}}. \tag{5.9}$$

Moreover,

$$H(a_\varepsilon) \le C(\|a_\varepsilon\|^2_{H^2} + \|a_\varepsilon\|^4_{H^2})$$

and

$$\sum_{|\alpha|=1}\|\partial^\alpha_x u_\varepsilon(t)\|_2 \le c\sqrt{H(a_\varepsilon)}.$$

The following theorem holds.

Theorem 5.4.1. *Let $a \in \mathcal{G}_{H^2}$ be of $\ln^{\frac{1}{3}}$–type. Then for any $T > 0$ there exists a unique solution $u \in \mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$ of* (5.1).

Proof. We know that for each $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in C([0,T),H^2(\mathbb{R}^3)) \cap C([0,T),L^2)$, for any $T > 0$. We need to prove that $\sup_{0\le t<T}\|u_\varepsilon(t)\|_{H^2} = \mathcal{O}(\varepsilon^{-N})$ and $\sup_{0\le t<T}\|\partial_t u_\varepsilon(t)\|_{L^2} = \mathcal{O}(\varepsilon^{-N})$ for some $N \in \mathbb{N}$. We know that $\|u_\varepsilon(t)\|_2 = \|a_\varepsilon\|_2 \le Ch_\varepsilon$. Then using that $h_\varepsilon \sim (ln\varepsilon^{-1})^{\frac{1}{3}}$ it follows $\|u_\varepsilon(t)\|_2 \le C\varepsilon^{-N_1}$ for any $N_1 \in \mathbb{N}$.

Using derived estimates we have that $H(a_\varepsilon) \le c(ln\varepsilon^{-1})^{\frac{4}{3}}$. Therefore $\sum_{|\alpha|=1}\|\partial^\alpha_x u_\varepsilon(t)\|_2 \le c(ln\varepsilon^{-1})^{\frac{2}{3}} \le \varepsilon^{-N_2}$ for any $N_2 \in \mathbb{N}$.

Next we differentiate Duhamel's formula twice in x and for $|\alpha| = 2$ it follows that for any $t \in [0, T)$

$$
\begin{aligned}
\|\partial_x^\alpha u_\varepsilon(t)\|_2 \le \|\partial_x^\alpha a_\varepsilon\|_2 & \\
&+ \sum_{\beta\le\alpha} c_{\alpha\beta} \sum_{\gamma\le\beta} c_{\beta\gamma} \|w\|_\infty \int_0^t \|\partial^{\beta-\gamma} u_\varepsilon(s)\|_2 \|\partial^\gamma u_\varepsilon(s)\|_2 \|\partial^{\alpha-\beta} u_\varepsilon(s)\|_2 ds \\
&+ \int_0^t \|\partial_x^\alpha(\phi_{h_\varepsilon} u_\varepsilon(s))\|_{L^2} ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\partial^\alpha u_\varepsilon\|_2 \le \|\partial_x^\alpha a_\varepsilon\|_2 &+ \|w\|_\infty \int_0^t \|u_\varepsilon\|_2^2 \|\partial^\alpha u_\varepsilon\|_2 ds \\
&+ \sum_{|\beta|=1} c_{\alpha\beta} \sum_{\gamma\le\beta} c_{\beta\gamma} \|w\|_\infty \int_0^t \|\partial^{\beta-\gamma} u_\varepsilon\|_2 \|\partial^\gamma u_\varepsilon\|_2 \|\partial^{\alpha-\beta} u_\varepsilon\|_2 \\
&+ \sum_{\gamma\le\alpha} c_{\alpha\gamma} \|w\|_\infty \int_0^t \|\partial^{\alpha-\gamma} u_\varepsilon\|_2 \|\partial^\gamma u_\varepsilon\|_2 \|u_\varepsilon\|_2 ds + \int_0^t \|\partial_x^\alpha(\phi_{h_\varepsilon} u_\varepsilon)\|_{L^2} ds \\
&\le E(t) + 3\|w\|_\infty \int_0^t \|a_\varepsilon\|_2^2 \|\partial^\alpha u_\varepsilon\|_2 ds + \|\phi_{h_\varepsilon}\|_\infty \int_0^t \|\partial^\alpha u_\varepsilon\|_2 ds,
\end{aligned}
$$

where

$$
\begin{aligned}
E(t) = \|\partial^\alpha a_\varepsilon\|_2 &+ c \sum_{|\beta|=1} c_{\alpha\beta} \|w\|_\infty \int_0^t \|a_\varepsilon\|_2 \|\partial^\beta u_\varepsilon\|_2 \|\partial^{\alpha-\beta} u_\varepsilon\|_2 ds \\
&+ \|\partial^\alpha \phi_{h_\varepsilon}\|_\infty \int_0^t \|u_\varepsilon\|_2 ds + c_1 \sum_{|\beta|=1} \|\partial^{\alpha-\beta} \phi_{h_\varepsilon}\|_\infty \int_0^t \|\partial^\beta u_\varepsilon(s)\|_2 ds.
\end{aligned}
$$

Applying Gronwall's inequality we obtain

$$
\|\partial^\alpha u_\varepsilon\|_2 \le |E(t)| e^{T(3\|w\|_\infty \|a_\varepsilon\|_2^2 + \|\phi_{h_\varepsilon}\|_\infty)} = c(\varepsilon)|E(t)|.
$$

Assumptions of the theorem imply that $c(\varepsilon) = e^{T(3\|w\|_\infty \|a_\varepsilon\|_2^2 + \|\phi_{h_\varepsilon}\|_\infty)} \le c\varepsilon^{-N_3}$ for some $N_3 \in \mathbb{N}$. Next we use estimates (5.9) to derive bounds for $|E(t)|$, that is we obtain

$$
|E(t)| \le \|\partial^\alpha a_\varepsilon\|_2 + CT\|a_\varepsilon\|_2(\|a_\varepsilon\|_2 + c_1\|a_\varepsilon\|_{H^2}\sqrt{1 + c_2\|a_\varepsilon\|_{H^2}^2}) + cTh_\varepsilon^6 + cTh_\varepsilon^4 \varepsilon^{-N_2}.
$$

Hence there exists $N \in \mathbb{N}$ such that $\|\partial_x^\alpha u_\varepsilon(x, t)\|_2 \le c\varepsilon^{-N}$.

It remains to estimate $\|(u_\varepsilon)_t\|_2$, but this follows directly from the equation and all the estimates that we derived. Therefore $u_\varepsilon \in \mathcal{E}_{C^1,H^2}$. Since also

$$\sup_{t\in[0,T)} \|i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon - \phi_{h_\varepsilon} u_\varepsilon\|_{L^2} = O(\varepsilon^M), \quad \forall M \in \mathbb{N}$$

we proved that there exist a solution of (5.1).

Uniqueness. Suppose that there is another solution $v \in \mathcal{G}_{C^1,H^2}$ and $V_\varepsilon = u_\varepsilon - v_\varepsilon$. Then h_ε satisfies the following equation, using $v_\varepsilon = u_\varepsilon - V_\varepsilon$:

$$\begin{aligned} i(V_\varepsilon)_t + \triangle V_\varepsilon &= u_\varepsilon(w * (|V_\varepsilon|^2 - u_\varepsilon \overline{V_\varepsilon} - V_\varepsilon \overline{u_\varepsilon})) + V_\varepsilon(w * |u_\varepsilon - V_\varepsilon|^2) + \phi_\varepsilon V_\varepsilon + N_\varepsilon \\ V_\varepsilon(0) &= n_\varepsilon \end{aligned} \tag{5.10}$$

Multiplying by $\overline{V_\varepsilon}$, integrating over $\mathbb{R}^3$ and taking the imaginary part we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|V_\varepsilon\|_2^2 &= \operatorname{Im}\left(\int u_\varepsilon \overline{V_\varepsilon}(w * (|V_\varepsilon|^2 - u_\varepsilon \overline{V_\varepsilon} - V_\varepsilon \overline{u_\varepsilon})) + N_\varepsilon \overline{V_\varepsilon}\right) \\ &\lesssim \|w * |V_\varepsilon|^2\|_\infty \|V_\varepsilon\|_2 \|V_\varepsilon\|_2 + \|w * |V_\varepsilon u_\varepsilon|\|_\infty \|u_\varepsilon\|_2 \|V_\varepsilon\|_2 + \|N_\varepsilon\|_2 \|V_\varepsilon\|_2 \\ &\le \|w\|_\infty \left(\|V_\varepsilon\|_2^2 \|u_\varepsilon\|_2 \|V_\varepsilon\|_2 + \|V_\varepsilon\|_2^2 \|u_\varepsilon\|_2^2\right) + \|N_\varepsilon\|_2 \|V_\varepsilon\|_2, \end{aligned}$$

that is

$$\frac{1}{2}\frac{d}{dt}\|V_\varepsilon\|_2^2 \le c\|V_\varepsilon\|_2^2(\|u_\varepsilon\|_2\|V_\varepsilon\|_2 + \|u_\varepsilon\|_2^2) + \|N_\varepsilon\|_2\|V_\varepsilon\|_2 \tag{5.11}$$

Here we used Young's inequality for $r = \infty,\ p = 1,\ q = \infty$. We know $\|u_\varepsilon\|_2 = \|a_\varepsilon\|_2 \le ch_\varepsilon \le c(\ln \varepsilon^{-1})^{\frac{1}{2}}$, $\varepsilon \to 0$. On the other hand, v_ε satisfies

$$\begin{aligned} i(v_\varepsilon)_t + \triangle v_\varepsilon = (w * |v_\varepsilon|^2)v_\varepsilon + \phi_\varepsilon v_\varepsilon + N_\varepsilon, \\ v_\varepsilon(0) = a_\varepsilon + n_\varepsilon, \end{aligned}$$

where $\|N_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^M)$, for any $t \in [0,T)$ and $n_\varepsilon \in \mathcal{N}_{H^2}$. Multiplying by $\overline{v}_\varepsilon$, integrating on $\mathbb{R}^3$ and taking the imaginary part we obtain:

$$\frac{d}{dt}\|v_\varepsilon\|_2^2 \le \|N_\varepsilon\|_2\|v_\varepsilon\|_2.$$

Integrating in t we obtain

$$\begin{aligned}
\|v_\varepsilon\|^2 &\le \|a_\varepsilon + n_\varepsilon\|_2 + \int_0^t \|N_\varepsilon\|_2 \|v(s)\|_2 ds \\
&\le \|a_\varepsilon\|_2 + \|n_\varepsilon\|_2 + T \sup_{t\in[0,T)} \|v_\varepsilon(t)\|_2 \|N_\varepsilon(t)\|_2 \le h_\varepsilon + \varepsilon^M \le 2h_\varepsilon \\
&\Rightarrow \|v_\varepsilon\|_2 \le c\sqrt{h_\varepsilon} \le c(\ln \varepsilon^{-1})^{\frac{1}{2}}, \ \varepsilon \to 0,
\end{aligned}$$

since $\|v_\varepsilon\|_2 \sim \varepsilon^{-N}$ (because v is a Colombeau solution) and $\|v_\varepsilon\|_2 \|N_\varepsilon\|_2 = \varepsilon^{-N} \cdot \varepsilon^{M_1} \sim \varepsilon^M$, for any $M \in \mathbb{N}$.
Integrating (5.11) in t and using Gronwall inequality we obtain

$$\begin{aligned}
\|V_\varepsilon\|_2^2 &\le (\|n_\varepsilon\|_2^2 + T \sup_{0\le t\le T} \|N_\varepsilon\|_2 \|V_\varepsilon\|_2) + \int_0^t c \ln \varepsilon^{-1} \|V_\varepsilon(\tau)\|_2^2 d\tau \\
\|V_\varepsilon\|_2^2 &\le (\|n_\varepsilon\|_2^2 + T \sup_{0\le t\le T} \|N_\varepsilon\|_2 \|V_\varepsilon\|_2) \exp(T \ln \varepsilon^{-1})
\end{aligned}$$

from which it follows

$$\sup_{0\le t\le T} \|V_\varepsilon\|_2 = O(\varepsilon^M), \ \varepsilon \to 0, \ \text{for any } M \ \in \mathbb{N} \tag{5.12}$$

which completes the proof. □

5.4.2 Compatibility

Given the Cauchy problem (5.2) for $a \in V = \{u \in H^2(\mathbb{R}^3), u \text{ is odd}\}$, we know from Section 5.2 that there is a unique solution $u \in V$. Since $H^2(\mathbb{R}^3) \hookrightarrow \mathcal{G}_{H^2}(\mathbb{R}^3)$, for such an initial condition there is a unique solution of (5.1) in $\mathcal{G}_{C^1,H^2}$, also (this is proved in detail in the sequel). This means there is a representative u_ε such that

$$\begin{aligned}
i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon &= \phi_{h_\varepsilon} u_\varepsilon \\
u_\varepsilon(0) &= a_\varepsilon,
\end{aligned} \tag{5.13}$$

for some regularization a_ε of a. We now focus on proving that

$$\sup_{[0,T)} \|u_\varepsilon(t) - u(t)\|_2 \to 0.$$

This is not exactly compatibility in the sense of Definition 2.5.13, but both (5.13) and (5.2) are equations used to solve (5.1) in a way, so it makes sense to examine this type of convergence.

In this section, we choose an even mollifier ϕ_ε. Note that $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\varepsilon^{-3}\rho(\frac{x}{\varepsilon})$, where we can choose an even function χ with the desired properties and also an even function ρ. This is because ρ can be constructed as an inverse Fourier transform of an even function $\hat{\rho}$, which is then also even. In Section 2.5 we have seen that $\hat{\rho}$ should be 1 in a neighborhood of zero and in $S(\mathbb{R}^3)$, so it can be chosen to be even.

Recall that we can embed a to $\mathcal{G}_{H^2}$ by $a \mapsto [(a * \phi_\varepsilon)_\varepsilon]$ (Theorem 2.5.4). Now we assert that $a_\varepsilon = a * \phi_\varepsilon$ satisfies the appropriate growth conditions:

$$\|\partial_x^\alpha (a * \phi_\varepsilon)\|_2 = \|\partial_x a * \phi_\varepsilon\|_2 \leq \|\partial_x^\alpha a\|_2 \|\phi_\varepsilon\|_1 \leq \|\partial_x^\alpha a\|_2, \tag{5.14}$$

which is a constant independent from ε since $a \in H^2$. We conclude that the conditions of Theorem 5.4.1 are satisfied and u_ε gives rise to a unique solution in $\mathcal{G}_{C^1,H^2}$.

We now show that u_ε is odd for each ε and thus $u_\varepsilon(0) = 0$ for each ε. Firstly, $a_\varepsilon = a * \phi_\varepsilon$ is odd, as a convolution of an odd and an even function. Further, we can repeat the fixed point argument from Section 5.2 on the space

$$W = \{u_\varepsilon \in H^2(\mathbb{R}^3) \ : \ u_\varepsilon(-x) = -u_\varepsilon(x)\},$$

for every ε. The key difference being in bounding the term $\|\phi_{h_\varepsilon} u_\varepsilon\|_{H^2}$:

$$\|\partial_x^\alpha \phi_{h_\varepsilon} \partial_x^\alpha u_\varepsilon\|_{L^2} \leq \|\partial_x^\alpha \phi_{h_\varepsilon}\|_\infty \|\partial_x^\alpha u_\varepsilon\|_{L^2}$$

for $\alpha \leq 2$. Norms $\|\partial_x^\alpha \phi_{h_\varepsilon}\|_\infty$ are bounded by $\|\partial_x^\alpha \phi_{h_\varepsilon}\|_{H^m}$ (Sobolev embedding) for some m and this is finite for each ε since ϕ_{h_ε} is smooth and compactly supported. Also, since ϕ_{h_ε} is even, $\phi_{h_\varepsilon} u_\varepsilon$ is odd for u_ε odd, so the proof can be analogously conducted.

Therefore we can formulate the following theorem.

Theorem 5.4.2. *Let $a \in V$ and let u be the (fractional) Sobolev solution $u \in V$ of* (5.2)*. Let $[(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ be the Colombeau solution of* (5.1)*. Then* $\sup_{[0,T)} \|u_\varepsilon(t) - u(t)\|_2 \to 0$.

Proof. First, note that problem (5.2) for $a \in H^2(\mathbb{R}^3) \cap H^2_\alpha(\mathbb{R}^3)$ is equivalent to

$$\begin{aligned} iu_t + \triangle u - (w * |u|^2)u &= 0 \\ u(0,x) &= a(x) \end{aligned}$$

and $u(0,t) = 0 = u_\varepsilon(0,t)$. Let $V_\varepsilon = u_\varepsilon - u$. Then V_ε satisfies:

$$i(V_\varepsilon)_t + \triangle V_\varepsilon = (w * |u_\varepsilon|^2)u_\varepsilon - (w * |u|^2)u + \phi_{h_\varepsilon} u_\varepsilon$$

Like in the uniqueness proof,

$$\|V_\varepsilon\|_2^2 \leq \|a - a * \phi_\varepsilon\|_2^2 + c\int_0^t \|V_\varepsilon\|_2^2(\|u_\varepsilon\|_2\|V_\varepsilon\|_2 + \|u_\varepsilon\|_2^2)ds + \int_0^t \|V_\varepsilon\phi_{h_\varepsilon}u_\varepsilon\|_1 ds \tag{5.15}$$

Both u and u_ε satisfy conservation of charge, so $\|u_\varepsilon\|_2 = \|a_\varepsilon\|_2$ and $\|V_\varepsilon\|_2 \leq \|a\|_2 + \|a_\varepsilon\|_2 \leq c$ independently of ε as we showed before. It remains to obtain bounds for $\|V_\varepsilon\phi_\varepsilon u_\varepsilon\|_1$. We will show that

$$\int_{\mathbb{R}^3} |\phi_{h_\varepsilon}(x)(V_\varepsilon u_\varepsilon)(s,x)|dx \to 0, \quad \varepsilon \to 0 \quad \text{for any } s \in [0,T), \tag{5.16}$$

using the Lebesgue dominated convergence theorem. Then it will follow

$$\int_0^t \|\phi_{h_\varepsilon}V_\varepsilon(s)u_\varepsilon(s)\|_1 ds \to 0, \quad \varepsilon \to 0,$$

again using the dominated convergence theorem, but in t. The expression $\|\phi_{h_\varepsilon}V_\varepsilon(s)u_\varepsilon(s)\|_1$ converges to zero pointwise in t and we will see later it is bounded by a constant for ε small enough and a constant is integrable on $[0,t]$, $t \leq T$.

Observe that

$$\begin{aligned}\int\limits_{\mathbb{R}^3} |h_\varepsilon^3\rho(xh_\varepsilon)\chi(x\sqrt{h_\varepsilon})V_\varepsilon(x)u_\varepsilon(x)|dx &= \int\limits_{|\kappa|\leq 2\sqrt{h_\varepsilon}} |\rho(\kappa)\chi(\frac{\kappa}{\sqrt{h_\varepsilon}})V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})|\, d\kappa \\ &\leq \int\limits_{|\kappa|\leq 2\sqrt{h_\varepsilon}} |\rho(\kappa)V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})|\, d\kappa\end{aligned}$$

Now we focus on proving that $|\rho(\kappa)V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})|$ converges pointwise to zero, for any $t \in [0,T)$. For this we need equicontinuity of u_ε in zero, so first we shall prove the following (recall that $u_\varepsilon(0) = 0$ for each ε):

$$\forall\delta > 0\ \exists\delta_1 > 0\ \forall\varepsilon > 0 :\ |x| < \delta_1 \Rightarrow |u_\varepsilon(x)| < \delta. \tag{5.17}$$

We argue by contradiction, suppose

$$\exists\delta > 0\ \forall\delta_1 > 0\ \exists\varepsilon_0 > 0 :\ |x| < \delta_1 \wedge |u_{\varepsilon_0}(x)| \geq \delta.$$

For such δ and ε_0, define $\mu(\xi) = u_{\varepsilon_0}((1-\xi)x)$, $\xi \in [0,1]$. Note here that for each ε the solution u_ε is actually in $H^3 \subset C^1$ since $a_\varepsilon = a * \phi_\varepsilon \in H^3$ for each ε:

$$\|\partial^\alpha(a * \phi_\varepsilon)\|_2 = \|a * \partial^\alpha \phi_\varepsilon\|_2 \leq \|a\|_2 \|\partial^\alpha \phi_\varepsilon\|_1 < \infty \quad \forall \varepsilon > 0.$$

Since

$$|\mu(0) - \mu(1)| = |u_{\varepsilon_0}| \geq \delta,$$

by the mean–value theorem there exists $\xi_0 \in (0,1)$ such that $|\mu'(\xi_0, x)| \geq \delta$, for all $|x| < \delta_1$ (δ_1 will be determined later). Also

$$\int_{|x|<\delta_1} |\mu'(\xi_0, x)| dx \geq \int_{|x|<\delta_1} \delta = (2\delta_1)^3 \delta.$$

On the other hand,

$$|\mu'(\xi)| = |(\nabla u_{\varepsilon_0})((1-\xi)x) \cdot (-x)| \leq |(\nabla u_{\varepsilon_0})((1-\xi)x)| \cdot |x|.$$

It follows

$$\int_{|x|<\delta_1} |\mu'(\xi, x)| dx \leq \|\nabla u_{\varepsilon_0}\|_2 \left(\int_{|x|<\delta_1} |x|^2 dx \right)^{1/2} < M \cdot \sqrt{\frac{4\pi\delta_1^5}{5}}, \quad \text{for each } \xi \in [0,1].$$

Here we used that $\|\nabla u_\varepsilon\|_2$ is bounded by a constant M independent of ε which follows from (5.9) and (5.14) - it is bounded by $\|a_\varepsilon\|_{H^2}$ and this in return is bounded by a constant. Choosing $\delta_1 = \frac{\pi M^2}{90\delta^2}$ we obtain

$$\int_{|x|<\delta_1} |\mu'(\xi, x)| dx < C \; \forall \xi \in [0,1] \wedge \exists \xi_0 \in (0,1) : \int_{|x|<\delta_1} |\mu'(\xi_0, x)| dx \geq C$$

which is a contradiction.

We can now use equicontinuity in zero (5.17) to prove pointwise convergence of $|V_\varepsilon(\frac{\kappa}{h_\varepsilon}) u_\varepsilon(\frac{\kappa}{h_\varepsilon})|$ to zero. Let $\delta > 0$. There exists δ_1 such that

$$\left|\frac{\kappa}{h_\varepsilon}\right| \leq \delta_1 \Rightarrow \left|u_\varepsilon(\frac{\kappa}{h_\varepsilon})\right| < \frac{\delta}{2} \text{ for any } \varepsilon > 0.$$

Since $\left|\frac{\kappa}{h_\varepsilon}\right| \leq \left|\frac{2\sqrt{h_\varepsilon}}{h_\varepsilon}\right| = \left|\frac{2}{\sqrt{h_\varepsilon}}\right| \to 0, \; \varepsilon \to 0$, inequality $\left|\frac{\kappa}{h_\varepsilon}\right| \leq \delta_1$ holds for small enough ε. Further,

$$|V_\varepsilon(\frac{\kappa}{h_\varepsilon})| = |u_\varepsilon(\frac{\kappa}{h_\varepsilon}) - u(\frac{\kappa}{h_\varepsilon})| < \frac{\delta}{2} + \frac{\delta}{2},$$

because for $\delta/2$ and small enough ε, there holds $u(\frac{\kappa}{h_\varepsilon}) < \delta/2$ because of continuity of $u \in H^2(\mathbb{R}^3) \subset C(\mathbb{R}^3)$ and $u(0) = 0$. Finally, for $\varepsilon \le \varepsilon_1$ and any κ there holds

$$|V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})| < \delta. \tag{5.18}$$

We conclude that $|\rho(\kappa)V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})|$ converges to zero pointwise. From (5.18) it also follows that

$$|\rho(\kappa)V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})| \le c|\rho(\kappa)| \in L^1,$$

for $\varepsilon \le \varepsilon_1$, and any $c = \delta$. By this, the conditions of the dominated convergence theorem are satisfied and

$$\int\limits_{|\kappa|\le 2\sqrt{h_\varepsilon}} |\rho(\kappa)V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})|\, d\kappa \to 0, \quad \varepsilon \to 0,$$

which implies (5.16).

Recall now that we need also for $\|\phi_{h_\varepsilon}V_\varepsilon(s)u_\varepsilon(s)\|_1$ to be bounded in s by a constant independent of ε and it is since

$$\|\phi_{h_\varepsilon}V_\varepsilon(s)u_\varepsilon(s)\|_1 \le \int\limits_{|\kappa|\le 2\sqrt{h_\varepsilon}} |\rho(\kappa)V_\varepsilon(\frac{\kappa}{h_\varepsilon})u_\varepsilon(\frac{\kappa}{h_\varepsilon})|\, d\kappa \le \int_{\mathbb{R}^3} |\rho(\kappa)| \cdot \delta \le c, \quad \varepsilon \le \varepsilon_1.$$

Returning to (5.15), we have

$$\|u_\varepsilon - u\|_2^2 \le \|a - a * \phi_\varepsilon\|_2^2 + C\int_0^t \|u_\varepsilon - u\|_2^2 ds + \int_0^t \|\phi_{h_\varepsilon}V_\varepsilon(s)u_\varepsilon(s)\|_1\, ds.$$

Applying Gronwall's theorem 2.4.3 we obtain

$$\|u_\varepsilon - u\|_2^2 \le (\|a - a * \phi_\varepsilon\|_2^2 + \int_0^T \|\phi_{h_\varepsilon}V_\varepsilon(s)u_\varepsilon(s)\|_1\, ds) \cdot \exp(CT) \to 0, \quad \varepsilon \to 0,$$

for any $t \in [0, T)$, which completes the proof. □

6 Conclusion

This work is focused on three initial value problems. Existence and uniqueness in the H^2–based Colombeau algebra was shown. For the cubic equation (1.1), compatibility with the Sobolev H^2 solution was shown. For the equation (1.2), question of the convergence of the net of solutions remains open. Here, the solution of the regularized equation gives rise to a solution in the Colombeau algebra. But, the question of a more "classical" solution candidate is unanswered. For the Hartree equation, we show that the net of solutions of the regularized equation converges to the solution of the fractional equation (5.2).

We based our analysis on well–posedness results in Sobolev spaces, developed by many authors and described in [Caz03]. We also used the more recent theory of well–posedness in singular Sobolev spaces developed in [MOS18].

Important part of the thesis are estimates. For the cubic equations we derive an estimate which is exponential in $\|a\|_{H^1}$ and of fractional power of $\|a\|_{H^3}$.

The tools developed can be used for further analysis. For example, convergence of the net of solutions can be examined in different spaces; variations of these equations can be observed.

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Biography

Nevena Dugandžija was born on 7th September, 1988 in Foča, Bosnia and Herzegovina. In 2007, she enrolled in the Bachelor program of financial mathematics at the Faculty of Sciences in Novi Sad. She completed the four year program in 2011 with an average grade of 9.29 and continued to attend the Master studies of techno–mathematics at the same faculty. In 2012, she did a 6–month internship at Robert Bosch GmbH in Stuttgart, Germany, in the field of digital image processing. In 2013, she completed the master program and enrolled in the doctoral studies at the Department of Mathematics and Informatics, Faculty of Sciences, Novi Sad. She passed all the exams of the program with an average grade of 9.92. From 2014 to 2018 she was employed as a research associate at the same department. From 2018 to present she is at the position of teaching assistant. She is the coauthor of 1 paper and was a presenter on several scientific conferences.

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Rezultati u disertaciji su sadržani u radovima [DN19].

U disertaciji posmatramo tri tipa Šredingerovih jednačina. Za sva tri tipa pokazano je da postoji jedinstveno rešenje u H^2 tipu Kolombo algebre. Neki prostori distribucija utopljeni su u ovaj tip

algebre i na taj način se daje smisao nelinearnim operacijama u jednačini. Izvedene su ocene za H^2 normu rešenja regularizovanih jednačina.

Kompatibilnost sa klasičnim rešenjem pokazana je za kubnu jednačinu bez potencijala, kao i za Hartrijevu jednačinu. U slučaju kubne jednačine, kompatibilnost se odnosi na konvergenciju mreže rešenja ka Soboljevljevom H^2 rešenju. Za Hartrijevu jednačinu pokazujemo konvergenciju mreže rešenja ka rešenju u prostoru $\tilde{H}^2_\alpha$, frakcionom (singularnom) Soboljevljevom prostoru.

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Član:	dr Marko Nedeljkov, redovni profesor, Prirodno-matematički fakultet, Univerzitet u Novom Sadu, mentor
Član:	dr Ivana Vojnović, docent, Prirodno-matematički fakultet, Univerzitet u Novom Sadu
Član:	dr Alesandro Mikelanđeli, spoljni saradnik; istraživač, Visoka škola naprednih studija, Trst, Italija; Institut za primenjenu matematiku, Bon, Nemačka

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AB

We observed three types of Schrödinger equations. For all types, we proved that there exists a unique solution in a H^2–based Colombeau algebra. Some spaces of distributions are embedded in this algebra and in that way the nonlinear operations in the equations are given meaning. We derived estimates for the H^2–norm of solutions of the regularized equations.

Compatibility with a classical solution was shown for the cubic equation without potential and for the Hartree equation. For the cubic equation, this means that the net of solutions converges to a Sobolev H^2 solution. For the Hartree equation, we proved convergence of a net of solutions to a solution in the space $\tilde{H}^2_\alpha$, the fractional (singular) Sobolev space.

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President:	Dr Danijela Rajter–Ćirić, full professor, Faculty of Sciences, University of Novi Sad
Member:	Dr Marko Nedeljkov, full professor, Faculty of Sciences, University of Novi Sad,
Member:	Dr Ivana Vojnović, assistant professor, Faculty of Sciences, University of Novi Sad,
Member:	Dr Alessandro Michelangeli, external collaborator; experienced researcher, Scuola Internazionale Superiore di Studi Avanzanti (SISSA), Trieste, Italy; Institut für Angewandte Mathematik, Bonn, Germany

Овај Образац чини саставни део докторске дисертације, односно докторског уметничког пројекта који се брани на Универзитету у Новом Саду. Попуњен Образац укоричити иза текста докторске дисертације, односно докторског уметничког пројекта.

План третмана података

Назив пројекта/истраживања

Нелинеарна Шредингерова једначина са сингуларитетима

Назив институције/институција у оквиру којих се спроводи истраживање

а) Природно – математички факултет, Универзитет у Новом Саду

б)

в)

Назив програма у оквиру ког се реализује истраживање

-

1. Опис података

1.1 Врста студије

Укратко описати тип студије у оквиру које се подаци прикупљају

Пошто је истраживање искључиво теоријског карактера, није вршено никакво прикупљање података. Из тог разлога се остатак обрасца не односи на њега, те је подразумевани одговор у свакој рубрици: није вршено прикупљање података.

1.2 Врсте података

а) квантитативни

б) квалитативни

1.3. Начин прикупљања података

а) анкете, упитници, тестови

б) клиничке процене, медицински записи, електронски здравствени записи в) генотипови: навести врсту ________________________________

г) административни подаци: навести врсту _______________________

д) узорци ткива: навести врсту_________________________________

ђ) снимци, фотографије: навести врсту_____________________________

е) текст, навести врсту _______________________________________

ж) мапа, навести врсту _______________________________________

з) остало: описати ____

1.3 Формат података, употребљене скале, количина података

1.3.1 Употребљени софтвер и формат датотеке:

a) Excel фајл, датотека __________________

b) SPSS фајл, датотека __________________

c) PDF фајл, датотека ___________________

d) Текст фајл, датотека __________________

e) JPG фајл, датотека ___________________

f) Остало, датотека ____________________

1.3.2. Број записа (код квантитативних података)

а) број варијабли ____

б) број мерења (испитаника, процена, снимака и сл.) ______________________

1.3.3. Поновљена мерења

а) да

б) не

Уколико је одговор да, одговорити на следећа питања:

а) временски размак између поновљених мера је ______________________________

б) варијабле које се више пута мере односе се на ________________________________

в) нове верзије фајлова који садрже поновљена мерења су именоване као ____________

Напомене: ____

Да ли формати и софтвер омогућавају дељење и дугорочну валидност података?

а) Да

б) Не

Ако је одговор не, образложити ____

____

2. Прикупљање података

2.1 Методологија за прикупљање/генерисање података

2.1.1. У оквиру ког истраживачког нацрта су подаци прикупљени?

а) експеримент, навести тип ____

б) корелационо истраживање, навести тип _____________________________________

ц) анализа текста, навести тип ____

д) остало, навести шта ____

2.1.2 Навести врсте мерних инструмената или стандарде података специфичних за одређену научну дисциплину (ако постоје).

____

____

2.2 Квалитет података и стандарди

2.2.1. Третман недостајућих података

а) Да ли матрица садржи недостајуће податке? Да Не

Ако је одговор да, одговорити на следећа питања:

а) Колики је број недостајућих података? _________________________

б) Да ли се кориснику матрице препоручује замена недостајућих података? Да Не

в) Ако је одговор да, навести сугестије за третман замене недостајућих података

____

2.2.2. На који начин је контролисан квалитет података? Описати

____

____

2.2.3. На који начин је извршена контрола уноса података у матрицу?

____

____

3. Третман података и пратећа документација

3.1. Третман и чување података

3.1.1. Подаци ће бити депоновани у ____________________________________ *репозиторијум.*

3.1.2. URL адреса ____

3.1.3. DOI ____

3.1.4. Да ли ће подаци бити у отвореном приступу?

а) Да

б) Да, али после ембарга који ће трајати до ____________________________________

в) Не

Ако је одговор не, навести разлог ____

3.1.5. Подаци неће бити депоновани у репозиторијум, али ће бити чувани.

Образложење

____

____

3.2 Метаподаци и документација података

3.2.1. Који стандард за метаподатке ће бити примењен? ___________________________________

3.2.1. Навести метаподатке на основу којих су подаци депоновани у репозиторијум.

____

____

Ако је потребно, навести методе које се користе за преузимање података, аналитичке и процедуралне информације, њихово кодирање, детаљне описе варијабли, записа итд.

____

____

____

____

3.3 Стратегија и стандарди за чување података

3.3.1. До ког периода ће подаци бити чувани у репозиторијуму? ________________________

3.3.2. Да ли ће подаци бити депоновани под шифром? Да Не

3.3.3. Да ли ће шифра бити доступна одређеном кругу истраживача? Да Не

3.3.4. Да ли се подаци морају уклонити из отвореног приступа после извесног времена?

Да Не

Образложити

____

____

4. Безбедност података и заштита поверљивих информација

Овај одељак МОРА бити попуњен ако ваши подаци укључују личне податке који се односе на учеснике у истраживању. За друга истраживања треба такође размотрити заштиту и сигурност података.

4.1 Формални стандарди за сигурност информација/података

Истраживачи који спроводе испитивања с људима морају да се придржавају Закона о заштити података о личности *(https://www.paragraf.rs/propisi/zakon_o_zastiti_podataka_o_licnosti.html)* и одговарајућег институционалног кодекса о академском интегритету.

4.1.2. Да ли је истраживање одобрено од стране етичке комисије? Да Не

Ако је одговор Да, навести датум и назив етичке комисије која је одобрила истраживање

____

4.1.2. Да ли подаци укључују личне податке учесника у истраживању? Да Не

Ако је одговор да, наведите на који начин сте осигурали поверљивост и сигурност информација везаних за испитанике:

а) Подаци нису у отвореном приступу

б) Подаци су анонимизирани

ц) Остало, навести шта

____

____

5. Доступност података

5.1. Подаци ће бити

а) јавно доступни

б) доступни само уском кругу истраживача у одређеној научној области

ц) затворени

Ако су подаци доступни само уском кругу истраживача, навести под којим условима могу да их користе:

____

____

Ако су подаци доступни само уском кругу истраживача, навести на који начин могу приступити подацима:

____

____

5.4. Навести лиценцу под којом ће прикупљени подаци бити архивирани.

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6. Улоге и одговорност

6.1. Навести име и презиме и мејл адресу власника (аутора) података

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6.2. Навести име и презиме и мејл адресу особе која одржава матрицу с подацима

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6.3. Навести име и презиме и мејл адресу особе која омогућује приступ подацима другим истраживачима

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