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Convolution and Localization Operators in Ultradistribution Spaces

-doctoral dissertation-

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Dedicated to my loving parents

Contents

Preface

The convolution of distributions was widely researched by many authors. Starting with Schwartz, who gave a definition for convolution of distributions, many other authors addressed the problem and gave alternative definitions of convolution and proved that they are equivalent with the Schwartz's one. The convolution of ultradistributions was already addressed in the Beurling case two decades ago. In fact, the analogous definitions that appear in the distributional setting, with appropriate changes, also apply in Beurling ultradistribution. Of course, the proof for their equivalence is more difficult because of the topological properties of the spaces under consideration. In this work, we study the convolution of Roumieu ultradistributions. Besides the analogous form of the Schwartz's definition, we give several other and proof their equivalence. Because of the topological properties of the corresponding spaces, they are not complete analogues to the definitions in the distributional setting. Furthermore, the proof of their equivalence is different than in the Beurling case. In fact, we will make a detour and study ε tensor products of specific locally convex spaces in order to prove the desired equivalence. Beside its theoretical importance, we will need this result in the last chapter.

The second main line of discourse is devoted to the study of localization operators on ultradistribution spaces, or rather a specific subclass whose elements are called Anti-Wick operators. We will be mainly interested in their connection to the Weyl quantization for symbols belonging to specific global symbol classes of Shubin type. The functional frame in which we will study this connection will be the spaces of tempered ultradistribution of Beurling and Roumieu type. By considering the convolution with the gaussian kernel, we will extend the definition of Anti-Wick quantization (Anti-Wick operators) for symbols that are not necessarily tempered ultradistributions.

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Chapter 0

Introduction

The aim of this work is to study the relationship between specific type of localization operators called Anti-Wick operators and a certain class of pseudodifferential operators in ultradistributional setting. The Anti-Wick operators first appear in a paper of Berezin [1], and later, in a paper of Daubechies [14], by the name of localization operators. In the latter paper they were proposed as a mathematical tool to localize a signal on the time-frequency plane. Anti-Wick operators were extensively studied during the years by many authors, primarily in the setting of Schwartz distributions. In Nicola and Rodino [36] and Shubin [53] there is systematic approach to the theory of Anti-Wick operators in distributional setting (see also the references therein). We also encourage the reader to see two recent papers on localization operators by Cordero and Gröchenig [11], [12]. Anti-Wick operators appear in approximation of pseudodifferential operators; see Cordoba and Fefferman [13], Folland [18], Tataru [55]. They can also be used in proving the Sharp Garding inequality (see [36]).

The aim of this work is twofold. The first goal is giving a relation between the Anti-Wick operators and the Weyl quantization of symbols in specific symbol classes. The second goal is enlarging the class of Anti-Wick operators.

The work is divided into five chapters.

In the first chapter we settle the basic notations that we will use. We give a brief survey on the theory of ultradistributions developed by Komatsu in [26], [27] and [28]. We also give definitions and basic facts for some subspaces of ultradistributions. Of special interest will be the spaces of tempered ultradistributions (which are a generalisation of the Gelfand-Shilov spaces) defined and studied by Pilipović in [40], [41], [42] (see also [33]) and other authors. Probably the best reference for the properties of this space is the book of Carmichael, Kamiński and Pilipović [10] with a systematic approach to the theory. These spaces were recently used by Pilipović and Teofanov in [44] and [45] in the theory of modulation spaces. Besides few technical results, we state and prove a very important kernel theorem for tempered ultradistributions in this chapter which will be of big importance for the rest of the work. We assume that the reader has deep knowledge in functional analysis and omit any background material on that subject in the introduction (Schaefer [49], Treves [56], Köthe [30], [31], are just a few good references).

We make a slight detour in the second chapter and study the Laplace transform on ultradistribution spaces. The two main theorems proven there characterise ultradistributions defined on the whole $\mathbb{R}^d$ through the estimates of their Laplace transforms. These results will be of particular importance for the last chapter.

The third chapter is devoted to the convolution of ultradistributions. We will be mainly interested in the Roumieu case. The convolution of Beurling ultradistributions was studied by Pilipović [41] and Kamiński, Kovačević and Pilipović [25]. Besides its theoretical importance, to motivate the study of convolution one doesn't need to look further then the most simple examples. For if $P(D) = \sum_{|\alpha|\leq n} c_\alpha D^\alpha$ is an ordinary partial differential operator then $P(D)u$ can be rewritten as $P(\delta) * u$, where $P(\delta)$ is the (ultra)distribution $\sum_{|\alpha|\leq n} c_\alpha D^\alpha \delta$. In ultradistributional setting one can consider infinite such sums with appropriate conditions on the coefficients c_α.

In the fourth chapter we define certain global symbol classes of Shubin type and study the resulting pseudodifferential operators which are of infinite order. They act continuously on the spaces of tempered ultradistributions and are constructed in such way that they give a well suited environment for studying Anti-Wick quantization, i.e. Anti-Wick operators with symbols in these classes. Many authors studied pseudodifferential operators of finite and infinite order that act continuously on Gevrey classes, constructed appropriate local symbol classes and developed corresponding calculi (see for example Matsuzawa [34], Hashimoto, Matsuzawa and Morimoto [22] for pseudodifferential operators of finite order and Zanghirati [59] for infinite order). For the global symbol classes and corresponding pseudodifferential operators of finite and infinite order we refer to Cappiello [3]-[6], Cappiello and Rodino [7], Cappiello, Gramchev and Rodino [8]. The symbol classes and the corresponding operators constructed in these papers are of (SG)-polyhomogeneous type in the setting of Gelfand-Shilov spaces and are employed, with great success, in the study of (SG)-hyperbolic Cauchy problems. It is important to note that in the analytic case, local symbol classes and corresponding pseudodifferential operators of infinite order were considered by Boutet de Monvel [2].

The fifth chapter is devoted to the Anti-Wick quantization. We first investigate its relation to the Weyl quantization when the symbols belong to the symbol classes constructed before. Then, by using the theory developed in the previous chapters, we enlarge the class of Anti-Wick operators. Probably the most interesting features of Anti-Wick operators are the positiveness, respectively the self-adjointness, of the operator when the symbol is positive, respectively real-valued. Also, when the corresponding symbol is in L^∞, the Anti-Wick operator can be extended as bounded operator on L^2 and its norm is not bigger then the L^∞ norm of the symbol.

Throughout this work, all the results that are borrowed have explicit reference next to them which refer to the paper or book were they can be found and are without a proof. All the results that are obtained by the author together with his advisor are without a reference and are presented with proofs. All of them can be found in [46], [43], [48] and [47].

Chapter 1

Preliminaries

1.1 Basic Facts and Notation

The sets of natural (including zero), integer, positive integer, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{C}$. For multi-indexes $\alpha, \beta \in \mathbb{N}^d$, we set

$$|\alpha| = \alpha_1 + \ldots \alpha_d; \quad \alpha! = \alpha_1! \cdot \ldots \cdot \alpha_d!; \quad \beta \leq \alpha \Leftrightarrow \beta_j \leq \alpha_j,\ \forall j = 1, \ldots, d;$$

$$\beta < \alpha \Leftrightarrow \beta \leq \alpha \text{ and } \beta \neq \alpha; \quad \text{for } \beta \leq \alpha,\ \binom{\alpha}{\beta} = \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}.$$

We use the symbols, for $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$,

$$\langle x \rangle = (1 + |x|^2)^{1/2}; \quad x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d};$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d} \text{ where } D_j^{\alpha_j} = i^{-1} \frac{\partial^{\alpha_j}}{\partial x_j{}^{\alpha_j}}; \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}.$$

If $z \in \mathbb{C}^d$, by z^2 we will denote $z_1^2 + ... + z_d^2$. Note that, if $x \in \mathbb{R}^d$, $x^2 = |x|^2$. For $x, y \in \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{N}^d$, the following equalities and inequalities hold

$$(x + y)^\alpha = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} x^{\alpha - \gamma} y^\gamma; \quad \alpha!\beta! \leq (\alpha + \beta)!;$$

$$(\alpha + \beta)! \leq 2^{|\alpha| + |\beta|} \alpha!\beta!; \quad |\alpha|! \leq d^{|\alpha|} \alpha!.$$

Also, for $n \in \mathbb{N}$, the number of all multi-indexes $\alpha \in \mathbb{N}^d$ such that $|\alpha| = n$ is $\binom{n + d - 1}{d - 1}$ and the number of all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq n$ is $\binom{n + d}{n}$.

For a measurable (Lebesgue measurable) subset K of $\mathbb{R}^d$ we will denote by $|K|$ the Lebesgue measure of K.

Let f be a function defined on the convex domain $U \subseteq \mathbb{R}^d$ that has continuous partial derivatives up to order $n + 1$ ($n \in \mathbb{N}$), in U. Then, we have the Taylor's formula

$$f(y) = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} \partial^\alpha f(x)(y - x)^\alpha$$

$$+\sum_{|\alpha|=n+1}\frac{n+1}{\alpha!}(y-x)^{\alpha}\int_0^1(1-t)^n\partial^{\alpha}f((1-t)x+ty)dt,\ \text{for } x,y\in U.$$

Let $P=\{\zeta\in\mathbb{C}^d|\,|w_1-\zeta_1|\leq r_1,...,|w_d-\zeta_d|\leq r_d\}$ is a polydisc in $\mathbb{C}^d$. If f is analytic on a neighbourhood of P then the Cauchy integral formula holds

$$\partial^{\alpha}f(z)=\frac{\alpha!}{(2\pi i)^d}\oint_{|w_1-\zeta_1|=r_1}...\oint_{|w_d-\zeta_d|=r_d}\frac{f(\zeta)}{(\zeta_1-z_1)^{\alpha_1+1}\cdot...\cdot(\zeta_d-z_d)^{\alpha_d+1}}d\zeta_1...d\zeta_d,$$

for $z\in\operatorname{int}P$ and $\alpha\in\mathbb{N}^d$. Let K be a $d+1$-real dimensional piecewise smooth surface with a boundary in $\mathbb{C}^d$ and let the boundary ∂K be a d-real dimensional piecewise smooth surface. If f is analytic on a neighbourhood of K, then we have the Cauchy-Poincaré theorem

$$\int_{\partial K}f(z)dz_1\wedge...\wedge dz_d=0.$$

1.2 Function Space. Ultradistributions

The space of all *locally integrable functions* on U, where U is an open subset of $\mathbb{R}^d$ will be denoted by $L^{1,loc}(U)$. It consists of all measurable functions $f:U\to\mathbb{C}$ such that $\int_K|f(x)|dx<\infty$, for every $K\subset\subset U$ (we will always use this notation for a compact subset of an open set). As standard, $L^p\left(\mathbb{R}^d\right)$, $1\leq p\leq\infty$, stands for the Banach space (from now on, abbreviated as (B) -space) of all measurable functions $f:\mathbb{R}^d\to\mathbb{C}$ such that

$$\|f\|_{L^p}=\left(\int_{\mathbb{R}^d}|f(x)|^p dx\right)^{1/p}<\infty\ \text{for}\, p<\infty;\ \|f\|_{L^{\infty}}=\operatorname{ess\,sup}|f|<\infty\ \text{for}\, p=\infty.$$

The inner product in $L^2\left(\mathbb{R}^d\right)$ will be denoted by $(\cdot,\cdot)$.

For an open subset U of $\mathbb{R}^d$, by $\mathcal{C}^{\infty}(U)$ will be denoted the space of all infinitely differentiable functions on U. We will often drop the notation U when $U=\mathbb{R}^d$. For the definition and the properties of the test spaces of infinitely differentiable functions and the corresponding spaces of distributions we refer the reader to [51] (see also [56], [21]).

If U is an open subset of $\mathbb{C}^d$, then by $\mathcal{O}(U)$ we denote the space of all analytic functions on U.

Following [26], we denote by M_p, $p\in\mathbb{N}$, a sequence of positive numbers such that $M_0=1$. We will impose the following condition on M_p:

$(M.1)$ (logarithmic convexity) $M_p^2\leq M_{p-1}M_{p+1}$, $p\in\mathbb{Z}_+$;

$(M.2)$ (stability under ultradifferential operators) $M_p\leq c_0H^p\min_{0\leq q\leq p}\{M_{p-q}M_q\}$, $p,q\in\mathbb{N}$, for some $c_0,H\geq 1$;

$(M.3)$ (strong non-quasi-analyticity) $\sum_{p=q+1}^{\infty}\frac{M_{p-1}}{M_p}\leq c_0q\frac{M_q}{M_{q+1}}$, $q\in\mathbb{Z}_+$;

although in some assertions we could assume the weaker ones:

$(M.2)'$ (stability under differential operators) $M_{p+1} \leq c_0 H^{p+1} M_p$, $p \in \mathbb{N}$, for some $c_0, H \geq 1$;

$(M.3)'$ (non-quasi-analyticity) $\displaystyle\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$.

For $s > 1$, the *Gevrey sequence* $M_p = p!^s$ satisfies $(M.1)$, $(M.2)$ and $(M.3)$.

For $\alpha \in \mathbb{N}^d$, M_α will mean $M_{|\alpha|}$, $|\alpha| = \alpha_1 + ... + \alpha_d$. Recall (see [26]), $m_p = M_p/M_{p-1}$, $p \in \mathbb{Z}_+$ and if M_p satisfies $(M.1)$ and $(M.3)'$, the *associated function* for the sequence M_p is defined by

$$M(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

It is non-negative, continuous, monotonically increasing function, which vanishes for sufficiently small $\rho > 0$ and increases more rapidly then $\ln \rho^p$ when ρ tends to infinity, for any $p \in \mathbb{Z}_+$. If M_p satisfies $(M.1)$ and $(M.3)'$ then for each $k \in \mathbb{N}$, $k^p p!/M_p \to 0$ when $p \to \infty$ (see [26]). We will often use the following proposition.

Proposition 1.2.1. *([26]) Let M_p satisfies $(M.1)$ and $(M.3)'$. M_p satisfies $(M.2)$ if and only if $2M(\rho) \leq M(H\rho) + \ln c_0$.*

Let $U \subseteq \mathbb{R}^d$ be an open set and $K \subset\subset U$. Then $\mathcal{E}^{\{M_p\},h}(K)$ is the space of all $\varphi \in \mathcal{C}^\infty(U)$ which satisfy $\displaystyle\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$ and $\mathcal{D}_K^{\{M_p\},h}$ is the space of all $\varphi \in \mathcal{C}^\infty\left(\mathbb{R}^d\right)$ with supports in K, which satisfy $p_{K,h}(\varphi) = \displaystyle\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$. One verifies that it is a (B) - space with the norm $p_{K,h}$. Define as locally convex spaces (from now on, abbreviated as l.c.s.)

$$\mathcal{E}^{(M_p)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h \to 0} \mathcal{E}^{\{M_p\},h}(K), \quad \mathcal{E}^{\{M_p\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \to \infty} \mathcal{E}^{\{M_p\},h}(K),$$

$$\mathcal{D}_K^{(M_p)} = \varprojlim_{h \to 0} \mathcal{D}_K^{\{M_p\},h}, \quad \mathcal{D}^{(M_p)}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}_K^{(M_p)},$$

$$\mathcal{D}_K^{\{M_p\}} = \varinjlim_{h \to \infty} \mathcal{D}_K^{\{M_p\},h}, \quad \mathcal{D}^{\{M_p\}}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}_K^{\{M_p\}}.$$

The elements of the space $\mathcal{E}^{(M_p)}(U)$, resp. $\mathcal{E}^{\{M_p\}}(U)$, are called *ultradifferentiable functions of Beurling*, resp. *of Roumieu type*, and the elements of the space $\mathcal{D}^{(M_p)}(U)$, resp. $\mathcal{D}^{\{M_p\}}(U)$ are called *ultradifferentiable functions with compact support of Beurling*, resp. *of Roumieu type*. If (M_p) satisfies $(M.1)$ and $(M.3)'$, non of these spaces are trivial; in the sequel, we will always assume the M_p satisfies this two conditions. They are complete, bornological, Montel spaces. Moreover, $\mathcal{E}^{(M_p)}(U)$ and $\mathcal{D}_K^{(M_p)}$ are (FS) - spaces; $\mathcal{D}_K^{\{M_p\}}$ and $\mathcal{D}^{\{M_p\}}(U)$ are (DFS) - spaces; $\mathcal{D}^{(M_p)}(U)$ is a (LFS) - space; $\mathcal{E}^{\{M_p\}}(U)$ is a $(DLFS)$ - space. If in addition M_p satisfies $(M.2)'$ then all of the above spaces are nuclear. The spaces of *ultradistributions* and *ultradistributions with compact support of Beurling*, resp. *Roumieu* *type* are defined as the strong duals of $\mathcal{D}^{(M_p)}(U)$ and $\mathcal{E}^{(M_p)}(U)$, resp. $\mathcal{D}^{\{M_p\}}(U)$ and $\mathcal{E}^{\{M_p\}}(U)$. These are complete, bornological, Montel spaces. Moreover, $\mathcal{D}'^{(M_p)}(U)$ is a $(DLFS)$ - space; $\mathcal{D}'^{\{M_p\}}(U)$ is a (FS) - space; $\mathcal{E}'^{(M_p)}(U)$ is a (DFS) - space; $\mathcal{E}'^{\{M_p\}}(U)$ is a (LFS) - space. If (M_p) satisfies $(M.2)'$ then they are all nuclear. For the properties of these spaces, we refer to [26], [27] and [28]. In the future we will not emphasise the set U when $U = \mathbb{R}^d$. Following [26], the common notation for the symbols (M_p) and $\{M_p\}$ will be $*$. $\mathcal{D}^*(U)$ is continuously and densely injected in $\mathcal{E}^*(U)$. Hence we have the continuous inclusion $\mathcal{E}'^*(U) \to \mathcal{D}'^*(U)$.

Theorem 1.2.1. *([26]) Let U be an open subset of $\mathbb{R}^d$. Then $\mathcal{E}^*(U)$ is topological algebra under the pointwise multiplication. $\mathcal{D}^*(U)$ is topological $\mathcal{E}^*(U)$-module in which the multiplication is hypocontinuous.*

If U' and U are two open subsets of $\mathbb{R}^d$, $U' \subseteq U$, then the inclusion $\mathcal{D}^*(U') \to \mathcal{D}^*(U)$ is continuous. Hence, its dual mapping $\rho^U_{U'} : \mathcal{D}'^*(U) \to \mathcal{D}'^*(U')$, is continuous. For $T \in \mathcal{D}'^*(U)$, $\rho^U_{U'}T$ is the restriction of T to U' and it will be denoted by T (if there is no confusion). Obviously, if U, U' and U'' are open subsets of $\mathbb{R}^d$ such that $U'' \subseteq U' \subseteq U$ then the restrictions obey the chain rule $\rho^U_{U''} = \rho^{U'}_{U''} \circ \rho^U_{U'}$.

Theorem 1.2.2. *([26]) The spaces $\mathcal{D}'^*(U)$, $U \subseteq \mathbb{R}^d$, with the restriction mappings $\rho^U_{U'}$ form a sheaf on $\mathbb{R}^d$ which is soft on any open set in $\mathbb{R}^d$. Namely they satisfy the following three properties:*

(i) Let $U = \bigcup U_j$ be an open covering of an open set U in $\mathbb{R}^d$. If $T \in \mathcal{D}'^*(U)$ and $\rho^U_{U_j}T = 0$ for all j then $T = 0$.

(ii) Let $U = \bigcup U_j$ be an open covering of an open set U in $\mathbb{R}^d$. If $T_j \in \mathcal{D}'^*(U_j)$ are compatible in the sense that $\rho^{U_j}_{U_j \cap U_k}T_j = \rho^{U_k}_{U_j \cap U_k}T_k$ for all $U_j \cap U_k \neq \emptyset$, then there is $T \in \mathcal{D}'^*(U)$ whose restriction to U_j is equal to T_j.

(iii) Let F be a relatively closed set in an open set U in $\mathbb{R}^d$. If $T \in \mathcal{D}'^*(U')$ on an open neighbourhood U' of F in U, then there is $S \in \mathcal{D}'^*(U)$ such that $\rho^{U'}_{U''}T = \rho^U_{U''}S$ on an open neighbourhood U'' of F in U.

For $\varphi \in \mathcal{E}^*(U)$ and $T \in \mathcal{D}'^*(U)$, φT defined by $\langle \varphi T, \psi \rangle = \langle T, \varphi\psi \rangle$ is well defined element of $\mathcal{D}'^*(U)$. Moreover, we have the following theorems.

Theorem 1.2.3. *([26]) The multiplication $(\varphi, T) \mapsto \varphi T$, $\mathcal{E}^*(U) \times \mathcal{D}'^*(U) \to \mathcal{D}'^*(U)$, is hypocontinuous bilinear mapping.*

Theorem 1.2.4. *([26]) Each $\varphi \in \mathcal{E}^*(U)$ induces a sheaf homomorphism $\varphi : \mathcal{D}'^* \to \mathcal{D}'^*$ over U under the multiplication. Namely for each pair of open subsets $U'' \subseteq U' \subseteq U$ we have $\rho^{U'}_{U''} \circ \varphi = \varphi \circ \rho^{U'}_{U''} : \mathcal{D}'^*(U') \to \mathcal{D}'^*(U'')$.*

Theorem 1.2.5. *([26]) The multiplication is a hypocontinuous bilinear mapping on $\mathcal{E}^*(U) \times \mathcal{E}'^*(U)$ into $\mathcal{E}'^*(U)$ and on $\mathcal{D}^*(U) \times \mathcal{D}'^*(U)$ into $\mathcal{E}'^*(U)$*

Let U, U_1 and U_2 are open subsets of $\mathbb{R}^d$ such that $U = U_1 - U_2 = \{x \in \mathbb{R}^d | \, x = x_1 - x_2, \, x_1 \in U_1, x_2 \in U_2\}$. Suppose that $T \in \mathcal{E}'^*(U_2)$ and $\varphi \in \mathcal{E}^*(U)$, or that $T \in \mathcal{D}'^*(U)$ and $\varphi \in \mathcal{D}^*(U_2)$, or that $T \in \mathcal{E}'^*(-U_2)$ and $\varphi \in \mathcal{D}^*(U_1)$. Define the convolution $T * \varphi$ by $T * \varphi(x) = \langle T(y), \varphi(x - y) \rangle$. If M_p satisfies $(M.1)$, $(M.2)$ and $(M.3)'$ we have the following theorems.

Theorem 1.2.6. *([26]) The convolution is a hypocontinuous bilinear mapping:*

$$\mathcal{E}'^*(U_2) \times \mathcal{E}^*(U) \to \mathcal{E}^*(U_1); \quad \mathcal{D}'^*(U) \times \mathcal{D}^*(U_2) \to \mathcal{E}^*(U_1);$$

$$\mathcal{E}'^*(-U_2) \times \mathcal{D}^*(U_1) \to \mathcal{D}^*(U).$$

Theorem 1.2.7. *([26]) The bilinear mapping $(S,T) \mapsto S * T$, $\langle S * T, \varphi\rangle = \left\langle T, \check{S} * \varphi\right\rangle$, (where $\check{S}(x) = S(-x)$) is well defined and hypocontinuous as a bilinear mapping:*

$$\mathcal{E}'^*(-U_2) \times \mathcal{E}'^*(U_1) \to \mathcal{E}'^*(U); \ \mathcal{D}'^*(-U) \times \mathcal{E}'^*(U_1) \to \mathcal{D}'^*(U_2);$$

$$\mathcal{E}'^*(U_2) \times \mathcal{D}'^*(U) \to \mathcal{D}'^*(U_1).$$

In Chapter 3 we will define convolution for more general pairings (S,T) of ultradistributions. A sequence of nonnegative ultradifferentiable functions $\mu_n \in \mathcal{D}^*\left(\mathbb{R}^d\right)$ will be called a *δ-sequence* if $\mu_n \to \delta$, when $n \to \infty$ in $\mathcal{E}'^*\left(\mathbb{R}^d\right)$, where δ is the Dirac's δ ultradistribution. Such a sequence can always be constructed. For example, we can take $\mu \in \mathcal{D}^*(U)$ where U is the open unit ball in $\mathbb{R}^d$ with centre at 0, to be such that $\mu \geq 0$ and $\int_{\mathbb{R}^d} \mu(x)dx = 1$ (such function exists by Lemma 5.1 of [26]). Then, define $\mu_n(x) = n^d\mu(nx)$, $n \in \mathbb{Z}_+$. One easily checks that μ_n converge to δ in $\mathcal{E}'^*\left(\mathbb{R}^d\right)$. By using appropriate δ-sequence and cut-off functions and theorems 1.2.7, 1.2.3 and 1.2.5 one easily proves that $\mathcal{D}^*(U)$ is dense in $\mathcal{D}'^*(U)$ and in $\mathcal{E}'^*(U)$, where U is an open subset of $\mathbb{R}^d$. Obviously $\mathcal{D}^*(U)$ is continuously injected in $\mathcal{D}'^*(U)$ and $\mathcal{E}'^*(U)$.

It is said that $P(\xi) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \xi^\alpha$, $\xi \in \mathbb{R}^d$, is an *ultrapolynomial of class* (M_p), resp. $\{M_p\}$, whenever the coefficients c_α satisfy the estimate $|c_\alpha| \leq CL^{|\alpha|}/M_\alpha$, $\alpha \in \mathbb{N}^d$, for some $L > 0$ and $C > 0$, resp. for every $L > 0$ there exists $C_L > 0$. The corresponding operator $P(D) = \sum_\alpha c_\alpha D^\alpha$ is an *ultradifferential operator of class* (M_p), resp. $\{M_p\}$ and if M_p satisfies $(M.2)$, they act continuously on $\mathcal{E}^{(M_p)}(U)$ and $\mathcal{D}^{(M_p)}(U)$, resp. $\mathcal{E}^{\{M_p\}}(U)$ and $\mathcal{D}^{\{M_p\}}(U)$ and the corresponding spaces of ultradistributions. Moreover, each ultradifferential operator $P(D)$ of class $*$ induces sheaf homomorphism $P(D) : \mathcal{D}'^* \to \mathcal{D}'^*$ (cf. [26]). If T is an ultradistribution and φ an ultradifferentiable function on appropriate open subsets of $\mathbb{R}^d$ such that the convolution $T * \varphi$ can be defined, as in theorem 1.2.6, then $P(D)(T * \varphi) = P(D)T * \varphi = T * P(D)\varphi$, where $P(D)$ is ultradifferential operator of class $*$. Similarly, if S and T are ultradistributions as in theorem 1.2.7, i.e. $S * T$ can be defined, then $P(D)(T * S) = P(D)T * S = T * P(D)S$.

We say that $f \in L^{1,loc}\left(\mathbb{R}^d\right)$ is of *ultrapolynomial growth of class* $*$ if there exists a ultrapolynomial of class $*$ and a constant $C > 0$ such that $|f(x)| \leq CP(x)$ a.e. If M_p satisfies $(M.2)$ and $(M.3)$, this is equivalent to the following:

there exist $m, C > 0$, resp. for every $m > 0$ there exists $C > 0$, such that $|f(x)| \leq Ce^{M(m|x|)}$ a.e.

Remark 1.2.1. Some authors use the term sub-exponential growth for ultrapolynomial growth, when working with Gelfand-Shilov spaces. However, this term means completely different thing in Komatsu's notions. We will restrict ourselves to only use the term ultrapolynomial growth.

We say that a subset K of $\mathbb{R}^d$ has the *cone property* if for each $x \in K$ there are a neighbourhood $U \cap K$ of x, a unit vector e in $\mathbb{R}^d$ and a positive number ε_0 such that $(U \cap K) + \varepsilon e$ is in the interior of K for any $0 < \varepsilon < \varepsilon_0$. Let U_1 and U_2 are open subsets of $\mathbb{R}^{d_1}_x$ and $\mathbb{R}^{d_2}_y$ respectively. Let K_1 and K_2 be compact subsets of U_1 and U_2 respectively, that satisfy the cone property. We have the following very important theorem.

Theorem 1.2.8. *([27]) Let M_p satisfies $(M.1)$, $(M.2)$ and $(M.3)'$. Then the bilinear mapping which assigns to each pair of functions $\varphi(x)$ on U_1 and $\psi(y)$ on U_2 the product $\varphi(x)\psi(y)$ on $U_1 \times U_2$ induces the following isomorphisms of locally convex spaces:*

$$\mathcal{E}^{(M_p)}(U_1)\hat{\otimes}\mathcal{E}^{(M_p)}(U_2) \cong \mathcal{E}^{(M_p)}(U_1 \times U_2); \quad \mathcal{E}^{\{M_p\}}(U_1)\hat{\otimes}\mathcal{E}^{\{M_p\}}(U_2) \cong \mathcal{E}^{\{M_p\}}(U_1 \times U_2);$$

$$\mathcal{D}^{(M_p)}_{K_1}\hat{\otimes}\mathcal{D}^{(M_p)}_{K_2} \cong \mathcal{D}^{(M_p)}_{K_1\times K_2}; \quad \mathcal{D}^{\{M_p\}}_{K_1}\hat{\otimes}\mathcal{D}^{\{M_p\}}_{K_2} \cong \mathcal{D}^{\{M_p\}}_{K_1\times K_2};$$

$$\mathcal{D}^{\{M_p\}}(U_1)\hat{\otimes}\mathcal{D}^{\{M_p\}}(U_2) \cong \mathcal{D}^{\{M_p\}}(U_1 \times U_2). \tag{1.1}$$

The completion of the tensor products in the above theorem are in the topology $\pi = \epsilon$ (all of the space in the above theorem are nuclear and hence the topologies π and ϵ coincide). Note that we don't have the corresponding isomorphism to (1.1) in the (M_p) case. In [28] it is proved that $\mathcal{D}^{(M_p)}(U_1)\hat{\otimes}_\iota\mathcal{D}^{(M_p)}(U_2) \cong \mathcal{D}^{(M_p)}(U_1 \times U_2)$ where ι stands for the inductive tensor product topology. In general it is stronger than the π topology even when the spaces are nuclear. But we will never use this fact (the only good references that the author knows about the inductive tensor product topology are Grothendieck [20] and Komatsu [28]). However, from this immediately follows that $\mathcal{D}^{(M_p)}(U_1) \otimes \mathcal{D}^{(M_p)}(U_2)$ is dense in $\mathcal{D}^{(M_p)}(U_1 \times U_2)$.

If E and F are two l.c.s. we will denote by $\mathcal{L}(E, F)$ the space of all continuous linear mappings from E into F and by $\mathcal{L}_b(E, F)$ this space equipped with the topology of bounded convergence. Denote by $\mathbf{B}^s(E, F)$ the space of all separately continuous bilinear functionals on $E \times F$. If E and F are barrelled then we can define on $\mathbf{B}^s(E, F)$ the topology of bibounded convergence, i.e. the topology of uniform convergence on the sets $A \times B$ where A and B are bounded subsets of E and F respectively, and denote it by $\mathbf{B}^s_b(E, F)$. The following is the kernel theorem for ultradistributions (we will sometimes refer to it as Komatsu kernel theorem).

Theorem 1.2.9. *([27]) Let M_p satisfies $(M.1)$, $(M.2)$ and $(M.3)'$. Let $*$ be either (M_p) or $\{M_p\}$. Then we have the canonical isomorphisms of locally convex spaces:*

$$\begin{aligned}\mathbf{B}^s_b\left(\mathcal{D}^*(U_1), \mathcal{D}^*(U_2)\right) &\cong \mathcal{L}_b\left(\mathcal{D}^*(U_1), \mathcal{D}'^*(U_2)\right) \cong \mathcal{L}_b\left(\mathcal{D}^*(U_2), \mathcal{D}'^*(U_1)\right)\\ &\cong \mathcal{D}'^*(U_1)\hat{\otimes}\mathcal{D}'^*(U_2) \cong \mathcal{D}'^*(U_1 \times U_2).\end{aligned}$$

The topology of the tensor product in the above theorem is $\pi = \epsilon$ (because $\mathcal{D}'^*(U_1)$ and $\mathcal{D}'^*(U_2)$ are nuclear these topologies coincide).

The theory of vector valued ultradifferentiable functions and vector valued ultradistributions is developed in [28]. We will only need results about vector valued ultradifferentiable functions in few occasions in chapter 3. Instead of listing them here we will give precise references when they are needed.

By $\mathfrak{R}$ is denoted the set of all positive sequences which monotonically increase to infinity. For $(r_p) \in \mathfrak{R}$, consider the sequence $N_0 = 1$, $N_p = M_p \prod_{j=1}^{p} r_j$, $p \in \mathbb{Z}_+$. One easily sees that this sequence satisfies $(M.1)$ and $(M.3)'$ if M_p does and its associated function will be denoted by $N_{r_p}(\rho)$, i.e. $N_{r_p}(\rho) = \sup_{p\in\mathbb{N}} \log_+ \dfrac{\rho^p}{M_p \prod_{j=1}^{p} r_j}$, $\rho > 0$. Note, for given (r_p) and every $k > 0$ there is $\rho_0 > 0$ such that $N_{r_p}(\rho) \leq M(k\rho)$, for $\rho > \rho_0$. In the next chapters we will need the following technical results.

Lemma 1.2.1. *Let $(k_p) \in \mathfrak{R}$. There exists $(k'_p) \in \mathfrak{R}$ such that $k'_p \leq k_p$ and*

$$\prod_{j=1}^{p+q} k'_j \leq 2^{p+q} \prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_j, \text{ for all } p, q \in \mathbb{Z}_+.$$

Proof. Define $k'_1 = k_1$ and inductively $k'_j = \min\left\{k_j, \dfrac{j}{j-1} k'_{j-1}\right\}$, for $j \geq 2$, $j \in \mathbb{N}$. Obviously $k'_j \leq k_j$ and one easily checks that (k'_j) is monotonically increasing. To prove that k'_j tends to infinity, suppose the contrary. Then, because (k'_j) is a monotonically increasing sequence of positive numbers, it follows that it is bounded by some $C > 0$. Because $(k_j) \in \mathfrak{R}$, there exists j_0, such that, for all $j \geq j_0$, $j \in \mathbb{N}$, $k_j \geq 2C$. So, for all $j \geq j_0 + 1$, $k'_j = \dfrac{j}{j-1} k'_{j-1}$. We get that $k'_j = \dfrac{j}{j_0} k'_{j_0} \to \infty$, when $j \to \infty$, which is a contradiction. Hence $(k'_j) \in \mathfrak{R}$. Note that, for all $p, j \in \mathbb{Z}_+$, we have $k'_{p+j} \leq \dfrac{p+j}{j} k'_j$. Hence $\prod_{j=1}^{p+q} k'_j = \prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_{p+j} \leq$ $\prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} \dfrac{p+j}{j} k'_j = \dfrac{(p+q)!}{p!q!} \prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_j \leq 2^{p+q} \prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_j$. □

Hence, for every $(k_p) \in \mathfrak{R}$, we can find $(k'_p) \in \mathfrak{R}$, as in lemma 1.2.1, such that $N_{k_p}(\rho) \leq N_{k'_p}(\rho)$, $\rho > 0$ and the sequence $N_0 = 1$, $N_p = M_p \prod_{j=1}^{p} k'_j$, $p \in \mathbb{Z}_+$, satisfies $(M.2)$ if M_p does.

Lemma 1.2.2. *let $g : [0,\infty) \to [0,\infty)$ be an increasing function that satisfies the following estimate:*

for every $L > 0$ there exists $C > 0$ such that $g(\rho) \leq M(L\rho) + \ln C$.

Then there exists subordinate function $\epsilon(\rho)$ such that $g(\rho) \leq M(\epsilon(\rho)) + \ln C'$, for some constant $C' > 1$.

For the definition of subordinate function see [26].

Proof. If $g(\rho)$ is bounded then the claim of the lemma is trivial (we can take C' large enough such that the inequality will hold for arbitrary subordinate function). Assume that g is not bounded. We can easily find continuous strictly increasing function $f : [0,\infty) \to [0,\infty)$ which majorizes g such that for every $L > 0$ there exists $C > 0$ such that $f(\rho) \leq M(L\rho) + \ln C$. Hence, there exists $\rho_1 > 0$ such that $f(\rho) > 0$ for $\rho \geq \rho_1$. There exists $\rho_0 > 0$ such that $M(\rho) = 0$ for $\rho \leq \rho_0$ and $M(\rho) > 0$ for $\rho > \rho_0$. Because $M(\rho)$ is continuous and strictly increasing on the interval $[\rho_0, \infty)$ and $\lim_{\rho\to\infty} M(\rho) = \infty$, M is bijection from $[\rho_0, \infty)$ to $[0, \infty)$ with continuous and strictly increasing inverse $M^{-1} : [0, \infty) \to [\rho_0, \infty)$. Define $\epsilon(\rho)$ on $[\rho_1, \infty)$ in the following way $\epsilon(\rho) = M^{-1}(f(\rho))$ and define it linearly on $[0, \rho_1)$ such that it will be continuous on $[0, \infty)$ and $\epsilon(0) = 0$. Then $\epsilon(\rho)$ is strictly increasing and continuous on $[0, \infty)$. Moreover, for $\rho \in [\rho_1, \infty)$, it satisfies $f(\rho) = M(\epsilon(\rho))$. Hence, there exists $C' > 1$ such that $f(\rho) \leq M(\epsilon(\rho)) + \ln C'$, for $\rho \geq 0$. It remains to prove that $\epsilon(\rho)/\rho \to 0$ when $\rho \to \infty$. Assume the contrary. Then, there exist $L > 0$ and a strictly increasing sequence ρ_j which tends to infinity when $j \to \infty$, such that $\epsilon(\rho_j) \geq 2L\rho_j$, i.e. $f(\rho_j) \geq M(2L\rho_j)$. For this L, by the condition for f, choose $C > 1$ such that $f(\rho) \leq M(L\rho) + \ln C$. Then we have $M(2L\rho_j) \leq M(L\rho_j) + \ln C$, which contradicts the fact that $e^{M(\rho)}$ increases faster then ρ^p for any p. One can obtain this contradiction by using equality (3.11) of [26]. □

For $(t_j) \in \mathfrak{R}$, denote by T_k the product $\prod_{j=1}^{k} t_j$ and $T_0 = 1$. For U open subset of $\mathbb{R}^d$, in [28] it is proven that the seminorms $p_{K,(t_j)}(\varphi) = \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in K} \frac{|D^\alpha\varphi(x)|}{T_\alpha M_\alpha}$, when K ranges over the compact subsets of U and (t_j) in $\mathfrak{R}$, give the topology of $\mathcal{E}^{\{M_p\}}(U)$. Also, for $K \subset\subset \mathbb{R}^d$, the topology of $\mathcal{D}^{\{M_p\}}_K$ is given by the seminorms $p_{K,(t_j)}$, when (t_j) ranges in $\mathfrak{R}$. From this it follows that $\mathcal{D}^{\{M_p\}}_K = \varprojlim_{(t_j)\in\mathfrak{R}} \mathcal{D}^{M_p}_{K,(t_j)}$, where $\mathcal{D}^{M_p}_{K,(t_j)}$ is the (B) - space of all $\mathcal{C}^\infty$ functions supported by K for which the norm $p_{K,(t_j)}$ is finite.

From now on, we always assume that M_p satisfies $(M.1)$, $(M.2)$ and $(M.3)$. We denote by $\tilde{\mathcal{S}}^{M_p,m}_2\left(\mathbb{R}^d\right)$, $m > 0$, the space of all smooth functions φ which satisfy

$$\sigma_{m,2}(\varphi) := \left(\sum_{\alpha,\beta\in\mathbb{N}^d} \int_{\mathbb{R}^d} \left|\frac{m^{|\alpha|+|\beta|}\langle x\rangle^{|\beta|} D^\alpha\varphi(x)}{M_\alpha M_\beta}\right|^2 dx\right)^{1/2} < \infty,$$

supplied with the topology induced by the norm $\sigma_{m,2}$. The elements of the space $\mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right) = \varprojlim_{m\to\infty} \tilde{\mathcal{S}}^{M_p,m}_2\left(\mathbb{R}^d\right)$, resp. $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right) = \varinjlim_{m\to 0} \tilde{\mathcal{S}}^{M_p,m}_2\left(\mathbb{R}^d\right)$, will be called *tempered ultradifferentiable function of Beurling*, resp. *of Roumieu type*. The strong dual of $\mathcal{S}^{(M_p)}$, resp. $\mathcal{S}^{\{M_p\}}$, is the space of *tempered ultradistributions of Beurling*, resp. *of Roumieu type*, in notation $\mathcal{S}'^{(M_p)}$, resp. $\mathcal{S}'^{\{M_p\}}$. All the good properties of $\mathcal{S}^*$ and its strong dual follow from the equivalence of the sequence of norms $\sigma_{m,2}$, $m > 0$, with each of the following sequences of norms (see [10], [40]):

(a) $\sigma_{m,p}$, $m > 0$; $p \in [1, \infty)$ is fixed;

(b) $\sigma_{m,\infty}$, $m > 0$, where $\sigma_{m,\infty}(\varphi) := \sup_{\alpha,\beta\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{m^{|\alpha|+|\beta|}\langle x\rangle^{|\beta|}\left|D^\alpha\varphi(x)\right|}{M_\alpha M_\beta}$;

(c) $\|\cdot\|_m$, $m>0$, where $\|\varphi\|_m := \sup_{\alpha\in\mathbb{N}^d} \frac{m^{|\alpha|}\left\|D^\alpha\varphi(\cdot)e^{M(m|\cdot|)}\right\|_{L^\infty}}{M_\alpha}$.

If we denote by $\tilde{\mathcal{S}}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$ the space of all infinitely differentiable functions on $\mathbb{R}^d$ for which the norm $\sigma_{m,\infty}$ is finite (obviously it is a (B) - space), then $\mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right) = \lim\limits_{\substack{\longleftarrow\\ m\to\infty}} \tilde{\mathcal{S}}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$ and $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right) = \lim\limits_{\substack{\longrightarrow\\ m\to 0}} \tilde{\mathcal{S}}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$. Also, for $m_2 > m_1$, the inclusion $\tilde{\mathcal{S}}_\infty^{M_p,m_2}\left(\mathbb{R}^d\right) \to \tilde{\mathcal{S}}_\infty^{M_p,m_1}\left(\mathbb{R}^d\right)$ is a compact mapping. Also, if we denote by $\mathcal{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$ the space of all infinitely differentiable functions on $\mathbb{R}^d$ for which the norm $\|\cdot\|_m$ is finite (obviously it is a (B) - space), then $\mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right) = \lim\limits_{\substack{\longleftarrow\\ m\to\infty}} \mathcal{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$ and $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right) = \lim\limits_{\substack{\longrightarrow\\ m\to 0}} \mathcal{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$. Moreover, for $m_2 > m_1$, the inclusion $\mathcal{S}_\infty^{M_p,m_2}\left(\mathbb{R}^d\right) \to \mathcal{S}_\infty^{M_p,m_1}\left(\mathbb{R}^d\right)$ is a compact mapping. So, $\mathcal{S}^*\left(\mathbb{R}^d\right)$ is a (FS) - space in the (M_p) case, resp. a (DFS) - space in the $\{M_p\}$ case and its (FS), resp. (DFS) structure, can be given by either of the above two ways. Hence they are complete, bornological, Montel spaces. Moreover, they are nuclear spaces. In [42] and [10] it is proved that $\mathcal{S}^{\{M_p\}} = \lim\limits_{\substack{\longleftarrow\\ (r_i),(s_j)\in\mathfrak{R}}} \tilde{\mathcal{S}}^{M_p}_{(r_p),(s_q)}$, where $\tilde{\mathcal{S}}^{M_p}_{(r_p),(s_q)} = \left\{\varphi\in\mathcal{C}^\infty\left(\mathbb{R}^d\right) \mid \gamma_{(r_p),(s_q)}(\varphi)<\infty\right\}$ and

$\gamma_{(r_p),(s_q)}(\varphi) = \sup_{\alpha,\beta\in\mathbb{N}^d} \frac{\left\|\langle x\rangle^{|\beta|}D^\alpha\varphi(x)\right\|_{L^2}}{\left(\prod_{p=1}^{|\alpha|}r_p\right)M_\alpha\left(\prod_{q=1}^{|\beta|}s_q\right)M_\beta}$. Also, $\mathcal{S}^{\{M_p\}} = \lim\limits_{\substack{\longleftarrow\\ (r_i),(s_j)\in\mathfrak{R}}} \mathcal{S}^{M_p}_{(r_p),(s_q)}$, where $\mathcal{S}^{M_p}_{(r_p),(s_q)} = \left\{\varphi\in\mathcal{C}^\infty\left(\mathbb{R}^d\right) \mid \|\varphi\|_{(r_p),(s_q)}<\infty\right\}$ and

$$\|\varphi\|_{(r_p),(s_q)} = \sup_{\alpha\in\mathbb{N}^d} \frac{\left\|D^\alpha\varphi(\cdot)e^{N_{s_p}(|\cdot|)}\right\|_{L^\infty}}{M_\alpha\prod_{p=1}^{|\alpha|}r_p}.$$

We have the continuous and dense inclusions $\mathcal{D}^*\to\mathcal{S}^*$ and $\mathcal{S}^*\to\mathcal{E}^*$. Hence the inclusions $\mathcal{S}'^*\to\mathcal{D}'^*$ and $\mathcal{E}'^*\to\mathcal{S}'^*$ are continuous. One easily proves that $\mathcal{E}'^*$ is dense in $\mathcal{S}'^*$. Hence $\mathcal{D}^*$ is continuously and densely injected in $\mathcal{S}'^*$. Moreover, ultradifferential operators of class * act continuously on $\mathcal{S}^*$ and $\mathcal{S}'^*$.

We will need the following kernel theorem for $\mathcal{S}'^*$. The (M_p) case was already considered in [33] (the authors used the characterisation of Fourier-Hermite coefficients of the elements of the space in the proof of the kernel theorem).

Proposition 1.2.2. *The following isomorphisms of locally convex spaces hold*

$$\begin{aligned}\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\hat{\otimes}\mathcal{S}^*\left(\mathbb{R}^{d_2}\right) &\cong \mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)\cong\mathcal{L}_b\left(\mathcal{S}'^*\left(\mathbb{R}^{d_1}\right),\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)\right),\\ \mathcal{S}'^*\left(\mathbb{R}^{d_1}\right)\hat{\otimes}\mathcal{S}'^*\left(\mathbb{R}^{d_2}\right) &\cong \mathcal{S}'^*\left(\mathbb{R}^{d_1+d_2}\right)\cong\mathcal{L}_b\left(\mathcal{S}^*\left(\mathbb{R}^{d_1}\right),\mathcal{S}'^*\left(\mathbb{R}^{d_2}\right)\right).\end{aligned}$$

Proof. Note that $\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\otimes\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)$ is dense in $\mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)$. This is true because of the continuous and dense inclusion $\mathcal{D}^*\left(\mathbb{R}^{d_1+d_2}\right)\to\mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)$ and because $\mathcal{D}^*\left(\mathbb{R}^{d_1}\right)\otimes\mathcal{D}^*\left(\mathbb{R}^{d_2}\right)$ is dense in $\mathcal{D}^*\left(\mathbb{R}^{d_1+d_2}\right)$ (see theorem 2.1 of [27]). We need to prove that $\mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)$ induces on $\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\otimes\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)$ the topology $\pi=\epsilon$ (the π and the ϵ topologies are the same because $\mathcal{S}^*$ is nuclear). Because the bilinear mapping $(\varphi,\psi)\mapsto\varphi\otimes\psi$, $\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\times\mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)\to\mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)$ is separately continuous it follows that it is continuous. This is true in the (M_p) case because $\mathcal{S}^{(M_p)}$ is (FS)-space (hence a F - space) and it is true in the $\{M_p\}$ case because

$\mathcal{S}^{\{M_p\}}$ is (DFS) - space (hence a barrelled (DF) - space). The continuity of this bilinear mapping proves that the inclusion $\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\otimes_\pi \mathcal{S}^*\left(\mathbb{R}^{d_2}\right)\rightarrow \mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)$ is continuous, hence the topology π is stronger than the induced one. Let A' and B' be equicontinuous subsets of $\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)$ and $\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)$, respectively. There exist $h>0$ and $C>0$ such that $\sup\limits_{T\in A'}|\langle T,\varphi\rangle|\leq C\|\varphi\|_h$ and $\sup\limits_{F\in B'}|\langle F,\psi\rangle|\leq C\|\psi\|_h$ in the (M_p) case, resp. there exist $(k_p),(k'_p)\in\mathfrak{R}$ and $C>0$ such that $\sup\limits_{T\in A'}|\langle T,\varphi\rangle|\leq C\|\varphi\|_{(k_p),(k'_p)}$ and $\sup\limits_{F\in B'}|\langle F,\psi\rangle|\leq C\|\psi\|_{(k_p),(k'_p)}$ in the $\{M_p\}$ case. We consider first the $\{M_p\}$ case. By lemma 1.2.1, without losing generality we can assume that $\prod_{j=1}^{p+q}k_j\leq 2^{p+q}\prod_{j=1}^{p}k_j\prod_{j=1}^{q}k_j$, $p\in\mathbb{Z}_+$ and the same for (k'_j). Put $r_j=k_j/(2H)$ and $r'_j=k'_j/(2H)$, $j\in\mathbb{Z}_+$. For all $T\in A'$ and $F\in B'$, we have

$$\begin{aligned}
|\langle T_x\otimes F_y,\chi(x,y)\rangle| &= |\langle F_y,\langle T_x,\chi(x,y)\rangle\rangle|\leq C\sup_{y,\beta}\frac{|\langle T_x,D^\beta_y\chi(x,y)\rangle|e^{N_{k'_p}(|y|)}}{M_\beta\prod_{j=0}^{|\beta|}k_j}\\
&\leq C^2\sup_{x,y,\alpha,\beta}\frac{\left|D^\alpha_xD^\beta_y\chi(x,y)\right|e^{N_{k'_p}(|x|)}e^{N_{k'_p}(|y|)}}{M_\alpha M_\beta\prod_{j=0}^{|\alpha|}k_j\prod_{j=0}^{|\beta|}k_j}\\
&\leq c_0^2C^2\sup_{x,y,\alpha,\beta}\frac{\left|D^\alpha_xD^\beta_y\chi(x,y)\right|e^{N_{r'_j}(|(x,y)|)}}{M_{\alpha+\beta}\prod_{j=0}^{|\alpha|+|\beta|}r_j}=c_0^2C^2\|\chi\|_{(r_p),(r'_p)},
\end{aligned}$$

where, in the third inequality we used proposition 1.2.1 for $N_{k'_p}(\lambda)$. Similarly, in the (M_p) case one obtains $\sup\limits_{T\in A',\,F\in B'}|\langle T_x\otimes F_y,\chi(x,y)\rangle|\leq c_0^2C^2\|\chi\|_{hH}$. Hence, the ϵ topology on $\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\otimes\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)$ is weaker than the induced one from $\mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)$. This gives the isomorphism $\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\hat{\otimes}\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)\cong\mathcal{S}^*\left(\mathbb{R}^{d_1+d_2}\right)$. Proposition 50.5 of [56] yields

$$\begin{aligned}
&\mathcal{S}^*\left(\mathbb{R}^{d_1}\right)\hat{\otimes}\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)\cong\mathcal{L}_b\left(\mathcal{S}'^*\left(\mathbb{R}^{d_1}\right),\mathcal{S}^*\left(\mathbb{R}^{d_2}\right)\right)\ \text{ and}\\
&\mathcal{S}'^*\left(\mathbb{R}^{d_1}\right)\hat{\otimes}\mathcal{S}'^*\left(\mathbb{R}^{d_2}\right)\cong\mathcal{L}_b\left(\mathcal{S}^*\left(\mathbb{R}^{d_1}\right),\mathcal{S}'^*\left(\mathbb{R}^{d_2}\right)\right)
\end{aligned}$$

($\mathcal{S}^*$ is a Montel space). Now, because $\mathcal{S}^{(M_p)}$ is (F) - space, theorem 9.9 of [49] gives the isomorphism $\mathcal{S}'^{(M_p)}\left(\mathbb{R}^{d_1}\right)\hat{\otimes}\mathcal{S}'^{(M_p)}\left(\mathbb{R}^{d_2}\right)\cong\mathcal{S}'^{(M_p)}\left(\mathbb{R}^{d_1+d_2}\right)$. In the $\{M_p\}$ case, $\mathcal{S}^{\{M_p\}}$ is (DFS) - space, i.e. the strong dual of the (FS) - space $\mathcal{S}'^{\{M_p\}}$, hence this theorem implies the same isomorphism in the $\{M_p\}$ case. □

Denote by $\mathcal{O}'^*_C$ the space of convolutors for $\mathcal{S}^*$, i.e. the space of all $T\in\mathcal{S}'^*$ for which the mapping $\varphi\mapsto T*\varphi$ is well defined and continuous mapping from $\mathcal{S}^*$ to itself. Denote by $\mathcal{O}^*_M$ the space of multipliers for $\mathcal{S}^*$, i.e. the space of all $\psi\in\mathcal{E}^*$ for which the mapping $\varphi\mapsto\psi\varphi$ is well defined and continuous mapping from $\mathcal{S}^*$ to itself. For the properties of these spaces we refer to [17].

As in [42], we define $\mathcal{D}^*_{L^\infty}\left(\mathbb{R}^d\right)$ by

$$\mathcal{D}^{(M_p)}_{L^\infty}\left(\mathbb{R}^d\right)=\varprojlim_{h\rightarrow\infty}\mathcal{D}^{M_p}_{L^\infty,h}\left(\mathbb{R}^d\right),\ \text{resp. }\mathcal{D}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^d\right)=\varinjlim_{h\rightarrow 0}\mathcal{D}^{M_p}_{L^\infty,h}\left(\mathbb{R}^d\right),$$

where $\mathcal{D}^{M_p}_{L^\infty,h}\left(\mathbb{R}^d\right)$ is the (B) - space of all $\varphi\in\mathcal{C}^\infty\left(\mathbb{R}^d\right)$ for which the norm $\sup\limits_{\alpha\in\mathbb{N}^d}\dfrac{h^{|\alpha|}\left\|D^\alpha\varphi\right\|_{L^\infty}}{M_\alpha}$ is finite. $\mathcal{D}^{(M_p)}_{L^\infty}$ is a (F) - space. Also, define $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^d\right)$

as the space of all $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$ functions such that, for every $(t_j)\in\mathfrak{R}$, the norm $\|\varphi\|_{(t_j)}=\sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d}\frac{|D^\alpha\varphi(x)|}{T_\alpha M_\alpha}$ is finite. The space $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ is complete l.c.s. because $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}=\varprojlim_{(t_j)\in\mathfrak{R}}\tilde{\mathcal{D}}^{M_p}_{L^\infty,(t_j)}$, where $\tilde{\mathcal{D}}^{M_p}_{L^\infty,(t_j)}$ is the (B) - space of all $\mathcal{C}^\infty$ functions ϕ for which the norm $\|\phi\|_{(t_j)}$ is finite. In [42] it is proved that $\mathcal{D}^{\{M_p\}}_{L^\infty}=\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ as sets and the former has a stronger topology than the latter. Denote by $\dot{\mathcal{B}}^{(M_p)}$, resp. $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ the completion of $\mathcal{D}^{(M_p)}$, resp. $\mathcal{D}^{\{M_p\}}$, in $\mathcal{D}^{(M_p)}_{L^\infty}$, resp. $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$. Then, $\dot{\mathcal{B}}^{(M_p)}$ is a (F) - space. Also $\mathcal{S}^{(M_p)}$, resp. $\mathcal{S}^{\{M_p\}}$, is continuously injected into $\dot{\mathcal{B}}^{(M_p)}$, resp. $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$. The strong dual of $\dot{\mathcal{B}}^{(M_p)}$, resp. $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$, will be denoted by $\mathcal{D}'^{(M_p)}_{L^1}$, resp. $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. They are continuously injected into $\mathcal{S}'^{(M_p)}$, resp. $\mathcal{S}'^{\{M_p\}}$, and hence into $\mathcal{D}'^{(M_p)}$, resp. $\mathcal{D}'^{\{M_p\}}$. Ultradifferential operators of class (M_p), resp. $\{M_p\}$, act continuously on $\dot{\mathcal{B}}^{(M_p)}$, resp. $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$, and on $\mathcal{D}'^{(M_p)}_{L^1}$, resp. $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. For the further properties of these spaces we refer to [42]. The following lemma characterises the elements of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$.

Lemma 1.2.3. *$\varphi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ if and only if $\varphi\in\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ and for every $\varepsilon>0$ and $(t_j)\in\mathfrak{R}$ there exists a compact set K such that* $\sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d\backslash K}\frac{|D^\alpha\varphi(x)|}{T_\alpha M_\alpha}<\varepsilon$.

Proof. Let E be the subspace of $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ defined by the conditions of the lemma. It is enough to prove that E is complete and that $\mathcal{D}^{\{M_p\}}$ is dense in E.

To prove that E is complete, it is enough to prove that it is closed. Let φ_ν be a net from E that converges to $\varphi\in\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$. Let $\varepsilon>0$ and $(t_j)\in\mathfrak{R}$ be fixed. Then there exists ν_0 such that, for all $\nu\geq\nu_0$, $\|\varphi-\varphi_\nu\|_{(t_j)}<\varepsilon/2$. Because $\varphi_{\nu_0}\in E$ (with $\varepsilon/2$ instead of ε) we have, for $x\in\mathbb{R}^d\backslash K$ and $\alpha\in\mathbb{N}^d$,

$$\frac{|D^\alpha\varphi(x)|}{T_\alpha M_\alpha}\leq\frac{|D^\alpha\varphi(x)-D^\alpha\varphi_{\nu_0}(x)|}{T_\alpha M_\alpha}+\frac{|D^\alpha\varphi_{\nu_0}(x)|}{T_\alpha M_\alpha}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

that is $\varphi\in E$.

The proof will be done if we prove that $\mathcal{D}^{\{M_p\}}$ is sequently dense in E. Let $\varphi\in E$. Take $\chi\in\mathcal{D}^{\{M_p\}}$ such that $\chi=1$ on the ball $K_{\mathbb{R}^d}(0,1)$ and $\chi=0$ out of $K_{\mathbb{R}^d}(0,2)$. Then $|D^\alpha\chi(x)|\leq C_1h^{|\alpha|}M_\alpha$ for some $h>0$ and $C_1>0$. For $n\in\mathbb{Z}_+$, put $\chi_n(x)=\chi(x/n)$ and $\varphi_n=\chi_n\varphi$. Then $\varphi_n\in\mathcal{D}^{\{M_p\}}$. Let $(t_j)\in\mathfrak{R}$. We have

$$\begin{aligned}
&\frac{|D^\alpha\varphi(x)-D^\alpha\varphi_n(x)|}{T_\alpha M_\alpha}\\
\leq&\quad\frac{|1-\chi(x/n)|\,|D^\alpha\varphi(x)|}{T_\alpha M_\alpha}+\sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}}\binom{\alpha}{\beta}\frac{\left|D^\beta\chi(x/n)\right|\left|D^{\alpha-\beta}\varphi(x)\right|}{n^{|\beta|}T_\alpha M_\alpha}\\
\leq&\quad\frac{|1-\chi(x/n)|\,|D^\alpha\varphi(x)|}{T_\alpha M_\alpha}+\frac{C_1\|\varphi\|_{(t_j/2)}}{n}\sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}}\binom{\alpha}{\beta}\frac{h^{|\beta|}T_{\alpha-\beta}}{2^{|\alpha|-|\beta|}T_\alpha}\\
\leq&\quad\varepsilon+\frac{C_1C_2\|\varphi\|_{(t_j/2)}}{n},\ n>n_0,
\end{aligned}$$

independently of x and α, for large enough n_0. This implies the assertion. $\square$

By the above lemma one easily check that if $\varphi \in \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ and $\psi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ then $\varphi\psi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}$. We have the following easy fact.

Lemma 1.2.4. *The bilinear mapping* $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty} \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}} \to \dot{\tilde{\mathcal{B}}}^{\{M_p\}}$, $(\varphi, \psi) \mapsto \varphi\psi$, *is continuous.*

1.3 Fourier Transform

For $f \in L^1\left(\mathbb{R}^d\right)$, its *Fourier transform* is defined by $(\mathcal{F}f)(\xi) = \displaystyle\int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx$, $\xi \in \mathbb{R}^d$. The Fourier transform is an isomorphism of $\mathcal{S}^*\left(\mathbb{R}^d\right)$ and it extends to isomorphism of $\mathcal{S}'^*\left(\mathbb{R}^d\right)$. Also, it is an isometry of $L^2\left(\mathbb{R}^d\right)$. Its inverse mapping $\mathcal{F}^{-1}$ is given by $\left(\mathcal{F}^{-1}f\right)(\xi) = \dfrac{1}{(2\pi)^d} \displaystyle\int_{\mathbb{R}^d} e^{ix\xi} f(x)dx$, when $f \in L^1\left(\mathbb{R}^d\right)$. For $\alpha \in \mathbb{N}^d$, the following identities are valid for elements of $\mathcal{S}^*$ or of $\mathcal{S}'^*$:

$$(\mathcal{F}\left(D^\alpha f\right))(\xi) = \xi^\alpha \left(\mathcal{F}f\right)(\xi); \quad \left(\mathcal{F}\left(x^\alpha f\right)\right)(\xi) = (-1)^{|\alpha|} D^\alpha (\mathcal{F}f)(\xi).$$

If $T \in \mathcal{S}'^*$ and $\varphi \in \mathcal{S}^*$ or $\varphi \in \mathcal{O}'^*_C$ then

$$\mathcal{F}(\varphi * T) = (\mathcal{F}\varphi) \cdot (\mathcal{F}T), \quad \mathcal{F}(\varphi T) = (2\pi)^{-d} (\mathcal{F}\varphi) * (\mathcal{F}T).$$

Chapter 2

Laplace Transform in Spaces of Ultradistributions

The Laplace transform of distributions was defined and studied by Schwartz, [51]. Later, Carmichael and Pilipović in [9] (see also [10]), considered the Laplace transform in Σ'_α of Beurling-Gevrey tempered ultradistributions and obtained some results concerning the so-called tempered convolution. In particular, they gave a characterisation of the space of Laplace transforms of elements from Σ'_α supported by an acute closed cone in $\mathbb{R}^d$. Komatsu has given a great contribution to the investigations of the Laplace transform in ultradistribution and hyperfunction spaces considering them over appropriate domains, see [29] and references therein (see also [60]). Michalik in [35] and Lee and Kim in [32] have adapted the space of ultradistribution and Fourier hyperfunctions to the definition of the Laplace transform, following ideas of Komatsu. Our approach is different. We develop the theory within the space of already constructed ultradistributions of Beurling and Roumieu type. The ideas in the proofs of the two main theorems of this chapter (theorem 2.1.1 and theorem 2.1.2) are similar to those in [57] in the case of Schwartz distributions. In these theorems are characterised ultradistributions defined on the whole $\mathbb{R}^d$ through the estimates of their Laplace transforms. These results will be needed in the last chapter.

2.1 Laplace Transform

For a set $B \subseteq \mathbb{R}^d$ denote by $\operatorname{ch} B$ the convex hull of B.

Theorem 2.1.1. *Let B be a connected open set in $\mathbb{R}^d_\xi$ and $T \in \mathcal{D}'^*(\mathbb{R}^d_x)$ be such that, for all $\xi \in B$, $e^{-x\xi}T(x) \in \mathcal{S}'^*(\mathbb{R}^d_x)$. Then the Fourier transform $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}T(x)\right)$ is an analytic function of $\zeta = \xi + i\eta$ for $\xi \in \operatorname{ch} B$, $\eta \in \mathbb{R}^d$. Furthermore, it satisfies the following estimates:*

for every $K \subset\subset \operatorname{ch} B$ there exist $k > 0$ and $C > 0$, resp. for every $k > 0$ there exists $C > 0$, such that

$$|\mathcal{F}_{x\to\eta}(e^{-x\xi}T(x))(\xi + i\eta)| \leq Ce^{M(k|\eta|)}, \forall \xi \in K, \forall \eta \in \mathbb{R}^d. \tag{2.1}$$

Proof. Let K be a fixed compact subset of $\operatorname{ch} B$. There exists $0<\varepsilon<1/4$ and $\xi^{(1)},...,\xi^{(l)}\in B$ such that the convex hull Π of the set $\{\xi^{(1)},...,\xi^{(l)}\}$ contains the closed 4ε neighbourhood of K (obviously $\Pi\subset\subset \operatorname{ch} B$). We shell prove that the set

$$\left\{S\in\mathcal{D}'^{*}|\, S(x)=T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}},\xi\in K\right\} \tag{2.2}$$

is bounded in $\mathcal{S}'^{*}$. Note that by the condition in the theorem $T(x)e^{-x\xi}\in\mathcal{S}'^{*}$ and $e^{\varepsilon\sqrt{1+|x|^2}}$ is the restriction on the real axis of the function $e^{\varepsilon\sqrt{1+z^2}}$ that is analytic and single valued on the strip $\mathbb{R}^d+i\{y\in\mathbb{R}^d|\,|y|<1/4\}$, and hence $e^{\varepsilon\sqrt{1+|x|^2}}$ is in $\mathcal{E}^{*}$. Note that

$$T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}}=\sum_{k=1}^{l}e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)T(x)e^{-x\xi^{(k)}}, \tag{2.3}$$

where $a(x,\xi)=e^{-x\xi}\left(\displaystyle\sum_{k=1}^{l}e^{-x\xi^{(k)}}\right)^{-1}$. The function $a(x,\xi)$ satisfies the following conditions:

i) $0<a(x,\xi)\leq 1$, $(x,\xi)\in\mathbb{R}^d\times\Pi$;

ii) $e^{\varepsilon'\sqrt{1+|x|^2}}a(x,\xi)\leq e^{\varepsilon'}$, $(x,\xi)\in\mathbb{R}^d\times K$, and $\forall\varepsilon'\leq 4\varepsilon$;

iii) $a(x,\xi)\in\mathcal{C}^{\infty}\left(\mathbb{R}^{2d}\right)$.

iii) it's obvious. To prove *i)*, take $\xi\in\Pi$. Then there exist $t_1,...,t_l\geq 0$ such that $\xi=\displaystyle\sum_{k=1}^{l}t_k\xi^{(k)}$ and $\displaystyle\sum_{k=1}^{l}t_k=1$. Then, by the weighted arithmetic mean-geometric mean inequality, we have

$$e^{-x\xi}=\prod_{k=1}^{l}e^{-xt_k\xi^{(k)}}\leq\sum_{k=1}^{l}t_ke^{-x\xi^{(k)}}\leq\sum_{k=1}^{l}e^{-x\xi^{(k)}},$$

from where it follows *i)*. For the prove of *ii)*, note that, for $(x,\xi)\in\mathbb{R}^d\times K$,

$$e^{\varepsilon'\sqrt{1+|x|^2}}a(x,\xi)\leq e^{\varepsilon'+\varepsilon'|x|}a(x,\xi)=e^{\varepsilon'}\max_{|t|\leq\varepsilon'}e^{-tx}a(x,\xi)=e^{\varepsilon'}\max_{|t|\leq\varepsilon'}a(x,\xi+t)\leq e^{\varepsilon'},$$

where the last inequality follows from *i)*.

Now we will estimate the derivatives of $a(x,\xi)$. Let $s=\max\limits_{\xi\in\Pi}|\xi|$. Then $a(z,\xi)$ is an analytic function of $z=x+iy$ on the strip $\mathbb{R}^d+i\{y\in\mathbb{R}^d|\,|y|s<\pi/4\}$, for every fixed $\xi\in\Pi$, because

$$\left|\sum_{k=1}^{l}e^{-z\xi^{(k)}}\right|^2=\left|\sum_{k=1}^{l}e^{-x\xi^{(k)}}e^{-iy\xi^{(k)}}\right|^2\geq\left(\sum_{k=1}^{l}e^{-x\xi^{(k)}}\cos y\xi^{(k)}\right)^2$$

$$\geq \left(\sum_{k=1}^{l} e^{-x\xi^{(k)}} \frac{\sqrt{2}}{2} \right)^2,$$

and hence

$$\left| \sum_{k=1}^{l} e^{-z\xi^{(k)}} \right| \geq \frac{\sqrt{2}}{2} \sum_{k=1}^{l} e^{-x\xi^{(k)}} > 0, \tag{2.4}$$

Take $0 < r < 1/\sqrt{d}$ so small such that $rs\sqrt{d} < \pi/4$. Then, from Cauchy integral formula, we have

$$|\partial_z^\alpha a(x,\xi)| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1 - x_1| \leq r, \ldots, |w_d - x_d| \leq r} \left| \frac{e^{-w\xi}}{\sum_{k=1}^{l} e^{-w\xi^{(k)}}} \right|.$$

If we use the inequality (2.4), we get (we put $w = u + iv$)

$$\begin{aligned}
\left| \frac{e^{-(u+iv)\xi}}{\sum_{k=1}^{l} e^{-(u+iv)\xi^{(k)}}} \right| &\leq \frac{\sqrt{2}e^{-u\xi}}{\sum_{k=1}^{l} e^{-u\xi^{(k)}}} = \frac{\sqrt{2}e^{-x\xi}e^{-(u-x)\xi}}{\sum_{k=1}^{l} e^{-x\xi^{(k)}}e^{-(u-x)\xi^{(k)}}} \\
&\leq \frac{\sqrt{2}e^{-x\xi}e^{|u-x||\xi|}}{\sum_{k=1}^{l} e^{-x\xi^{(k)}}e^{-|u-x|\left|\xi^{(k)}\right|}} \leq \frac{\sqrt{2}e^{-x\xi}e^{rs\sqrt{d}}}{\sum_{k=1}^{l} e^{-x\xi^{(k)}}e^{-rs\sqrt{d}}} \\
&= \sqrt{2}e^{2rs\sqrt{d}} a(x,\xi).
\end{aligned}$$

So, we obtain the estimate

$$|\partial_x^\alpha a(x,\xi)| \leq \sqrt{2}e^{2s} \frac{\alpha!}{r^{|\alpha|}} a(x,\xi). \tag{2.5}$$

Note that, by the previous estimate and the property $ii)$ of $a(x,\xi)$, it follows that $a(x,\xi) \in \mathcal{S}^*$ for every $\xi \in K$ and the set $\{a(x,\xi) | \, \xi \in K\}$ is a bounded set in $\mathcal{S}^*$. We will estimate the derivatives of $e^{\varepsilon\sqrt{1+|x|^2}}$. The function $e^{\varepsilon\sqrt{1+z^2}}$ is analytic on the strip $\mathbb{R}^d + i\{y \in \mathbb{R}^d | \, |y| < 1/4\}$, where we take the principal branch of the square root which is single valued and analytic on $\mathbb{C}\backslash(-\infty, 0]$. If we take $r < 1/(8d)$, from the Cauchy integral formula, we get the estimate

$$\left| \partial_z^\alpha e^{\varepsilon\sqrt{1+|x|^2}} \right| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1 - x_1| \leq r, \ldots, |w_d - x_d| \leq r} \left| e^{\varepsilon\sqrt{1+w^2}} \right|.$$

Put $w = u + iv$ and estimate as follows

$$\begin{aligned}
\left| e^{\varepsilon\sqrt{1+w^2}} \right| &= e^{\mathrm{Re}\left(\varepsilon\sqrt{1+w^2}\right)} \leq e^{\left|\varepsilon\sqrt{1+w^2}\right|} \leq e^{\varepsilon\sqrt[4]{(1+|u|^2-|v|^2)^2+4(uv)^2}} \\
&\leq e^{\varepsilon\sqrt{1+|u|^2-|v|^2+2|uv|}} \leq e^{\varepsilon\sqrt{1+2|u|^2}} \leq e^{\varepsilon\sqrt{1+4|u-x|^2+4|x|^2}} \\
&\leq e^{\varepsilon\sqrt{1+1+4|x|^2}} \leq e^{2\varepsilon\sqrt{1+|x|^2}}.
\end{aligned}$$

Hence

$$\left| \partial_x^\alpha e^{\varepsilon\sqrt{1+|x|^2}} \right| \leq \frac{\alpha!}{r^{|\alpha|}} e^{2\varepsilon\sqrt{1+|x|^2}}. \tag{2.6}$$

If we take r small enough we can make the previous estimates for the derivatives of $a(x,\xi)$ and $e^{\varepsilon\sqrt{1+|x|^2}}$ to hold for the same r. Now we obtain

$$\begin{aligned}\left|D_x^\alpha\left(e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)\right)\right| &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{(\alpha-\beta)!}{r^{|\alpha-\beta|}}e^{2\varepsilon\sqrt{1+|x|^2}}\cdot\sqrt{2}e^{2s}\frac{\beta!}{r^{|\beta|}}a(x,\xi)\\ &\leq \sqrt{2}e^{2s}\frac{\alpha!}{r^{|\alpha|}}2^{|\alpha|}e^{2\varepsilon\sqrt{1+|x|^2}}a(x,\xi).\end{aligned}$$

Using the property $ii)$ of the function $a(x,\xi)$, we get

$$\left|D_x^\alpha\left(e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)\right)\right| \leq \sqrt{2}e^{2s}\frac{\alpha!2^{|\alpha|}}{r^{|\alpha|}}e^{2\varepsilon\sqrt{1+|x|^2}}a(x,\xi) \leq \sqrt{2}e^{2s+2\varepsilon}\frac{\alpha!2^{|\alpha|}}{r^{|\alpha|}}, \quad (2.7)$$

for all $\xi\in K$. By this estimate and proposition 7 of [17] one has $e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)$ is a multiplier for $\mathcal{S}'^*$. Because of (2.3), (2.2) is a subset of $\mathcal{S}'^*$. Now to prove that (2.2) is bounded in $\mathcal{S}'^*$. We will give the prove only in the $\{M_p\}$ case, the (M_p) case is similar. Let $\psi\in\mathcal{S}^{\{M_p\}}$. There exists $h>0$ such that $\psi\in\tilde{\mathcal{S}}_\infty^{M_p,h}$. Note that

$$\left\langle e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)T(x)e^{-x\xi^{(k)}},\psi(x)\right\rangle = \left\langle T(x)e^{-x\xi^{(k)}}, e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)\psi(x)\right\rangle,$$

for all $k\in\{1,...,l\}$, for all $\xi\in K$. Choose $m\leq h/4$. By (2.7), we have

$$\begin{aligned}&\frac{m^{|\alpha|+|\beta|}\langle x\rangle^\beta\left|D^\alpha\left(e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)\psi(x)\right)\right|}{M_\alpha M_\beta}\\ &\leq m^{|\alpha|+|\beta|}\langle x\rangle^\beta\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{\sqrt{2}e^{2s+2\varepsilon}(\alpha-\gamma)!2^{|\alpha-\gamma|}|D^\gamma\psi(x)|}{r^{|\alpha-\gamma|}M_\alpha M_\beta}\\ &\leq C_1\sigma_{h,\infty}(\psi)\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{h^{|\alpha|+|\beta|}(\alpha-\gamma)!2^{|\alpha-\gamma|}}{4^{|\alpha|+|\beta|}r^{|\alpha-\gamma|}M_{\alpha-\gamma}h^{|\gamma|+|\beta|}}\\ &\leq C_1\sigma_{h,\infty}(\psi)\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{h^{|\alpha|-|\gamma|}(\alpha-\gamma)!}{2^{|\alpha|}r^{|\alpha-\gamma|}M_{\alpha-\gamma}} \leq C\sigma_{h,\infty}(\psi),\ \forall\xi\in K.\end{aligned}$$

Hence $e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)T(x)e^{-x\xi^{(k)}}$, $\xi\in K$, is bounded in $\mathcal{S}'^{\{M_p\}}$. By (2.3), the set (2.2) is bounded in $\mathcal{S}'^{\{M_p\}}$.

We will prove that $e^{-\varepsilon\sqrt{1+|x|^2}}\in\mathcal{S}^*$. In order to do that, we will estimate the derivatives of $e^{-\varepsilon\sqrt{1+|x|^2}}$ with the Cauchy integral formula (similarly as for $e^{\varepsilon\sqrt{1+|x|^2}}$). We obtain

$$\left|\partial_z^\alpha e^{-\varepsilon\sqrt{1+|x|^2}}\right| \leq \frac{\alpha!}{r^{|\alpha|}}\sup_{|w_1-x_1|\leq r,\dots,|w_d-x_d|\leq r}\left|e^{-\varepsilon\sqrt{1+w^2}}\right|,$$

where, $0<r<1/(8d)$. Let $w=u+iv$. Then, if we put

$$\rho=\sqrt{\left(1+|u|^2-|v|^2\right)^2+4(uv)^2},$$

$$\cos\theta = \frac{1+|u|^2-|v|^2}{\sqrt{\left(1+|u|^2-|v|^2\right)^2+4(uv)^2}},\ \sin\theta = \frac{2uv}{\sqrt{\left(1+|u|^2-|v|^2\right)^2+4(uv)^2}}$$

(where $\theta \in (-\pi,\pi)$), we have that $\theta \in (-\pi/2,\pi/2)$ (because $\cos\theta > 0$ and $\theta \in (-\pi,\pi)$) and

$$\begin{aligned}\operatorname{Re}\sqrt{1+|u|^2-|v|^2+2iuv} &= \operatorname{Re}\sqrt{\rho(\cos\theta+i\sin\theta)} = \operatorname{Re}\sqrt{\rho}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\\ &= \sqrt{\rho}\cos\frac{\theta}{2} \geq \frac{\sqrt{\rho}}{2},\end{aligned}$$

where the second equality holds because we take the principal branch of $\sqrt{z}$. Because $r < 1/(8d)$, we get

$$\begin{aligned}\left|e^{-\varepsilon\sqrt{1+w^2}}\right| &= e^{\operatorname{Re}\left(-\varepsilon\sqrt{1+w^2}\right)} \leq e^{-\frac{\varepsilon}{2}\sqrt[4]{(1+|u|^2-|v|^2)^2+4(uv)^2}} \leq e^{-\frac{\varepsilon}{2}\sqrt{1+|u|^2-|v|^2}}\\ &\leq e^{-\frac{\varepsilon}{2}\sqrt{1+\frac{|x|^2}{2}-|u-x|^2-|v|^2}} \leq e^{-\frac{\varepsilon}{4}\sqrt{1+|x|^2}}.\end{aligned}$$

Hence, we obtain

$$\left|\partial_x^\alpha e^{-\varepsilon\sqrt{1+|x|^2}}\right| \leq \frac{\alpha!}{r^{|\alpha|}}e^{-\frac{\varepsilon}{4}\sqrt{1+|x|^2}}. \tag{2.8}$$

From this, it easily follows that $e^{-\varepsilon\sqrt{1+|x|^2}} \in \mathcal{S}^*$. So $e^{-x\xi}T(x) \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right)$, for $\xi \in K$, because $e^{-x\xi}T(x) = T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}}e^{-\varepsilon\sqrt{1+|x|^2}}$ and we proved that $T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}} \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right)$, for $\xi \in K$.

Put $f(\xi+i\eta) = \mathcal{F}_{x\to\eta}(e^{-x\xi}T(x))$. We will prove that f is an analytic function on $\operatorname{ch}B + i\mathbb{R}^d$. Let U be an arbitrary bounded open subset of $\operatorname{ch}B$ such that $K = \overline{U} \subset\subset \operatorname{ch}B$. For $\psi \in \mathcal{S}^*$ and $\xi \in U$, we have

$$\begin{aligned}\langle f(\xi+i\eta),\psi(\eta)\rangle &= \left\langle \mathcal{F}_{x\to\eta}\left(e^{-x\xi}T(x)\right),\psi(\eta)\right\rangle = \left\langle e^{-x\xi}T(x),\mathcal{F}(\psi)(x)\right\rangle\\ &= \left\langle e^{-x\xi}T(x),\int_{\mathbb{R}^d} e^{-ix\eta}\psi(\eta)d\eta\right\rangle\\ &= \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x), e^{-\varepsilon\sqrt{1+|x|^2}}\int_{\mathbb{R}^d} e^{-ix\eta}\psi(\eta)d\eta\right\rangle\\ &= \left\langle\left(e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x)\right)\otimes 1_\eta, e^{-\varepsilon\sqrt{1+|x|^2}}e^{-ix\eta}\psi(\eta)\right\rangle\\ &= \int_{\mathbb{R}^d}\left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle\psi(\eta)d\eta.\end{aligned}$$

Hence

$$f(\xi+i\eta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle. \tag{2.9}$$

First we will prove that $f \in \mathcal{C}^\infty\left(U \times \mathbb{R}^d_\eta\right)$. We will prove the differentiability only in ξ_1 and in the $\{M_p\}$ case. The existence of the rest of the derivatives is proved in analogous way and the (M_p) case is treated similarly. Let $\xi^{(0)} =$

$\left(\xi_1^{(0)},...,\xi_d^{(0)}\right) = \left(\xi_1^{(0)},\xi'\right) \in U$, $\xi = \left(\xi_1^{(0)}+\xi_1,\xi_2^{(0)},...,\xi_d^{(0)}\right) = \left(\xi_1^{(0)}+\xi_1,\xi'\right)$, $x = (x_1,...,x_d) = (x_1,x')$. Let $0 < |\xi_1| < \delta < \varepsilon < 1$ such that the ball with radius δ and centre in $\xi^{(0)}$ is contained in U. Then, by using (2.3) and (2.9), we obtain

$$\frac{f(\xi+i\eta)-f(\xi^{(0)}+i\eta)}{\xi_1} - \left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-x_1)e^{-x\xi^{(0)}}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$$
$$= \sum_{k=1}^{l}\left\langle e^{-ix\eta}e^{-x\xi^{(k)}}T(x)e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi)-a\left(x,\xi^{(0)}\right)}{\xi_1}+x_1a\left(x,\xi^{(0)}\right)\right),\right.$$
$$\left. e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle.$$

It is enough to prove that, for every $\psi\in\mathcal{S}^{\{M_p\}}$,

$$e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi)-a\left(x,\xi^{(0)}\right)}{\xi_1}+x_1a\left(x,\xi^{(0)}\right)\right)\psi(x)\longrightarrow 0,$$

when $\xi_1\to 0$, in $\mathcal{S}^{\{M_p\}}$. First note that

$$e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi)-a\left(x,\xi^{(0)}\right)}{\xi_1}+x_1a\left(x,\xi^{(0)}\right)\right)$$
$$= e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right).$$

Now, we get

$$\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1 = \frac{1}{\xi_1}\sum_{n=1}^{\infty}\frac{(-1)^nx_1^n\xi_1^n}{n!}+x_1 = \sum_{n=2}^{\infty}\frac{(-1)^nx_1^n\xi_1^{n-1}}{n!}.$$

So, for $j\in\mathbb{N}$, $j\geq 2$ and $0<|\xi_1|<\delta<\varepsilon<1$, we have

$$\begin{aligned}\left|D_{x_1}^j\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right| &= \left|D_{x_1}^j\left(\sum_{n=2}^{\infty}\frac{(-1)^nx_1^n\xi_1^{n-1}}{n!}\right)\right|\\ &= \left|\sum_{n=j}^{\infty}\frac{(-1)^nn!x_1^{n-j}\xi_1^{n-1}}{(n-j)!n!}\right| \leq |\xi_1|\sum_{n=j}^{\infty}\frac{|x_1|^{n-j}|\xi_1|^{n-2}}{(n-j)!}\\ &\leq |\xi_1|\sum_{n=j}^{\infty}\frac{|x_1|^{n-j}|\xi_1|^{n-j}}{(n-j)!}\leq \delta e^{|x_1|\delta}.\end{aligned}$$

Using similar technic, we obtain the estimates

$$\left|D_{x_1}\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right|\leq\delta|x_1|e^{|x_1|\delta}\text{ and }\left|\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right|\leq\delta|x_1|^2e^{|x_1|\delta}.$$

So, in all cases, we have $\left|D^j_{x_1}\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right| \leq \delta\langle x_1\rangle^2 e^{|x_1|\delta}$. By using (2.7), we get (for simpler notation we write j for the d-tuple $(j,0,...,0)$)

$$\begin{aligned}
&\left|D^\alpha\left(e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\psi(x)\right)\right|\\
&\quad= \left|\sum_{\beta\leq\alpha}\sum_{j\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{j}D^{\beta-j}\left(e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\right)\right.\\
&\qquad\qquad \left.\cdot D^j\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)D^{\alpha-\beta}\psi(x)\right|\\
&\quad\leq \sum_{\beta\leq\alpha}\sum_{j\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{j}\sqrt{2}e^{2s}\frac{(\beta-j)!2^{|\beta-j|}}{r^{|\beta-j|}}e^{2\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\\
&\qquad\qquad \cdot\delta\langle x_1\rangle^2 e^{|x_1|\delta}|D^{\alpha-\beta}\psi(x)|\\
&\quad\leq C\delta\langle x_1\rangle^2\sum_{\beta\leq\alpha}\sum_{j\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{j}\left(\frac{2}{r}\right)^{|\beta-j|}(\beta-j)!|D^{\alpha-\beta}\psi(x)|,
\end{aligned}$$

where we used the inequality $e^{2\varepsilon\sqrt{1+|x|^2}}a(x,\xi^{(0)})e^{|x_1|\delta} \leq e^{3\varepsilon\sqrt{1+|x|^2}}a(x,\xi^{(0)}) \leq e^{3\varepsilon}$, which follows from the property $ii)$ of $a(x,\xi)$. Because $\psi\in\mathcal{S}^{\{M_p\}}$, there exists $m>0$ such that $\psi\in\tilde{\mathcal{S}}^{M_p,m}_\infty$. Choose h such that $h<m/4$, $h<1/4$ and $hH<m$. We get

$$\begin{aligned}
&\frac{h^{|\alpha|+|\beta|}\langle x\rangle^\beta\left|D^\alpha\left(e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\psi(x)\right)\right|}{M_\alpha M_\beta}\\
&\quad\leq C\delta\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}(\gamma-j)!\frac{\langle x_1\rangle^2\langle x\rangle^{|\beta|}h^{|\alpha|+|\beta|}|D^{\alpha-\gamma}\psi(x)|}{M_{\alpha-\gamma}M_{\gamma-j}M_jM_\beta}\\
&\quad\leq C_1\delta\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}(\gamma-j)!\frac{\langle x\rangle^{|\beta|+2}h^{|\alpha|+|\beta|}H^{|\beta|+2}|D^{\alpha-\gamma}\psi(x)|}{M_{\alpha-\gamma}M_{\gamma-j}M_jM_{\beta+2}}\\
&\quad\leq C_2\delta\sigma_{m,\infty}(\psi)\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}(\gamma-j)!\frac{h^{|\alpha|+|\beta|}H^{|\beta|}}{m^{|\alpha|-|\gamma|}m^{|\beta|+2}M_{\gamma-j}M_j}\\
&\quad\leq C_3\delta\sigma_{m,\infty}(\psi)\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}\left(\frac{h}{m}\right)^{|\alpha|-|\gamma|}\left(\frac{hH}{m}\right)^{|\beta|}\frac{h^{|\gamma|}(\gamma-j)!}{M_{\gamma-j}M_j}\\
&\quad\leq C_0\delta\sigma_{m,\infty}(\psi),
\end{aligned}$$

where we use $(M.2)$ and the fact $\frac{k^p p!}{M_p}\to 0$, when $p\to\infty$. Now, from this it follows that

$$e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi)-a\left(x,\xi^{(0)}\right)}{\xi_1}+x_1 a\left(x,\xi^{(0)}\right)\right)\psi(x)\longrightarrow 0,\ \xi_1\to 0$$

in $\mathcal{S}^{\{M_p\}}$ and by the above remarks, the differentiability of $f(\xi+i\eta)$ on $U\times\mathbb{R}^d_\eta$ follows. From the previous, we conclude that

$$\partial^\alpha_\xi f(\xi+i\eta)=\left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-x)^\alpha e^{-x\xi}T(x)e^{-ix\eta},e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$$

and similarly $\partial^\alpha_\eta f(\xi+i\eta)=\left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-ix)^\alpha e^{-x\xi}T(x)e^{-ix\eta},e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$. From this and the arbitrariness of U, the analyticity of $f(\xi+i\eta)$ follows because it satisfies the Cauchy-Riemann equations. So, for $\zeta=\xi+i\eta$, we get

$$f(\zeta)=\left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\zeta}T(x),e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle \tag{2.10}$$

and $\partial^\alpha_\zeta f(\zeta)=\left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-x)^\alpha e^{-x\zeta}T(x),e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$, for $\zeta\in U+i\mathbb{R}^d_\eta$, for each fixed U (ε depends on U).

Now we will prove the estimates (2.1) for $f(\xi+i\eta)$. Let $K\subset\subset \operatorname{ch}B$ be arbitrary but fixed. First we will consider the (M_p) case. We know that $\mathcal{S}^{(M_p)}$ is a (FS) - space and $\mathcal{S}^{(M_p)}=\varprojlim_{h\to\infty}\tilde{\mathcal{S}}^{M_p,h}_\infty$. If we denote the closure of $\mathcal{S}^{(M_p)}$ in $\tilde{\mathcal{S}}^{M_p,h}_\infty$ by $\tilde{\tilde{\mathcal{S}}}^{M_p,h}_\infty$ then $\mathcal{S}^{(M_p)}=\varprojlim_{h\to\infty}\tilde{\tilde{\mathcal{S}}}^{M_p,h}_\infty$ and the projective limit is reduced. Then $\mathcal{S}'^{(M_p)}=\varinjlim_{h\to\infty}\tilde{\tilde{\mathcal{S}}}'^{M_p,h}_\infty$ which is injective inductive limit with compact maps (because the projective limit is with compact maps). Because we proved that the set $\left\{S\in\mathcal{D}'^*\middle|\, S(x)=T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}},\xi\in K\right\}$ is bounded in $\mathcal{S}'^{(M_p)}$, it follows that there exists $h>0$ such that

$$\left\{S\in\mathcal{D}'^*\middle|\, S(x)=T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}},\xi\in K\right\}\subseteq\tilde{\tilde{\mathcal{S}}}'^{M_p,h}_\infty$$

and it's bounded there. By (2.8), we have the estimate

$$\begin{aligned}
&\frac{h^{|\alpha|+|\beta|}\langle x\rangle^\beta\left|D^\alpha_x\left(e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right)\right|}{M_\alpha M_\beta}\\
&\quad\le\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\frac{(2h)^{|\alpha|-|\gamma|}(2h)^{|\gamma|}h^{|\beta|}\langle x\rangle^\beta|\eta|^\gamma(\alpha-\gamma)!e^{-\frac{\varepsilon}{4}\sqrt{1+|x|^2}}}{2^{|\alpha|}r^{|\alpha-\gamma|}M_{\alpha-\gamma}M_\gamma M_\beta}\\
&\quad\le C_1\frac{1}{2^{|\alpha|}}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\left(\frac{2h}{r}\right)^{|\alpha|-|\gamma|}\frac{(\alpha-\gamma)!e^{M(h\langle x\rangle)}e^{M(2h|\eta|)}e^{-\frac{\varepsilon}{4}\langle x\rangle}}{M_{\alpha-\gamma}}\le C'e^{M(2h|\eta|)},
\end{aligned}$$

where we used that $e^{M(h\langle x\rangle)}e^{-\frac{\varepsilon}{4}\langle x\rangle}$ is bounded and $k^p p!/M_p\to 0$ when $p\to\infty$. Then, for $\xi\in K$ and $\eta\in\mathbb{R}^d$,

$$\begin{aligned}
|f(\xi+i\eta)| &= \left|\left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x),e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle\right|\\
&\le C\left\|e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right\|_{\tilde{\tilde{\mathcal{S}}}^{M_p,h}_\infty}\le\tilde{C}e^{M(2h|\eta|)}.
\end{aligned}$$

Now we will consider the $\{M_p\}$ case. $\mathcal{S}^{\{M_p\}}$ is a (DFS) - space and $\mathcal{S}^{\{M_p\}} = \varinjlim_{h\to 0} \tilde{\mathcal{S}}_\infty^{M_p,h}$, where the inductive limit is injective with compact maps. Let $h>0$ be fixed. For shorter notation, let

$$F = \left\{ S \in \mathcal{D}'^* \middle| \, S(x) = T(x) e^{-x\xi + \varepsilon\sqrt{1+|x|^2}}, \, \xi \in K \right\}$$

and denote by J the inclusion $\tilde{\mathcal{S}}_\infty^{M_p,h} \to \mathcal{S}^{\{M_p\}}$. Because we already proved that F is a bounded subset of $\mathcal{S}'^{\{M_p\}}$, its image under ${}^tJ$ (the transposed mapping of J) is a bounded subset of $\tilde{\mathcal{S}}_\infty'^{M_p,h}$. By the above calculations we see that $e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}$ is in $\tilde{\mathcal{S}}_\infty^{M_p,m}$, for every $m>0$. Hence, for $\xi \in K$ and $\eta \in \mathbb{R}^d$, we have

$$\begin{aligned} |f(\xi+i\eta)| &= \left| \left\langle e^{\varepsilon\sqrt{1+|x|^2}} e^{-x\xi} T(x), e^{-ix\eta} e^{-\varepsilon\sqrt{1+|x|^2}} \right\rangle \right| \\ &= \left| \left\langle {}^tJ \left(e^{\varepsilon\sqrt{1+|x|^2}} e^{-x\xi} T(x) \right), e^{-ix\eta} e^{-\varepsilon\sqrt{1+|x|^2}} \right\rangle \right| \\ &\leq C_h' \left\| e^{-ix\eta} e^{-\varepsilon\sqrt{1+|x|^2}} \right\|_{\tilde{\mathcal{S}}_\infty^{M_p,h}} \leq C_h e^{M(2h|\eta|)}, \end{aligned}$$

where we used the above estimate for $\dfrac{h^{|\alpha|+|\beta|}\langle x\rangle^\beta \left| D^\alpha \left(e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}} \right) \right|}{M_\alpha M_\beta}$. □

Remark 2.1.1. If, for $S \in \mathcal{D}'^*$, the conditions of the theorem are fulfilled, we call $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right)$ the *Laplace transform* of S and denote it by $\mathcal{L}(S)$. Moreover, by (2.10),

$$\mathcal{L}(S)(\zeta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}} e^{-x\zeta} S(x), e^{-\varepsilon\sqrt{1+|x|^2}} \right\rangle, \tag{2.11}$$

for $\zeta \in U + i\mathbb{R}^d_\eta$, where $\overline{U} \subset\subset \operatorname{ch} B$ and ε depends on U.

Note that, if for $S \in \mathcal{D}'^*$ the conditions of the theorem are fulfilled for $B=\mathbb{R}^d$, then the choice of ε can be made uniform for all $K \subset\subset \mathbb{R}^d$.

We will construct certain class of ultrapolynomials similar to those in [26], (see (10.9)' in [26]), which will have the added beneficence of not having zeroes in a strip containing the real axis.

Let $c>0$ be fixed. Let $k>0$, $l>0$ and $(k_p)\in\mathfrak{R}$, $(l_p)\in\mathfrak{R}$ be arbitrary but fixed. Choose $q\in\mathbb{Z}_+$ such that $\dfrac{c\sqrt{d}}{lm_p} < \dfrac{1}{2}$, for all $p\in\mathbb{N}$, $p\geq q$ in the (M_p) case and $\dfrac{c\sqrt{d}}{l_pm_p} < \dfrac{1}{2}$, for all $p\in\mathbb{N}$, $p\geq q$ in the $\{M_p\}$ case. Consider the entire functions

$$P_l(w) = \prod_{j=q}^{\infty}\left(1+\frac{w^2}{l^2m_j^2}\right), \; w\in\mathbb{C}^d \tag{2.12}$$

in the (M_p) case, resp.

$$P_{l_p}(w) = \prod_{j=q}^{\infty}\left(1+\frac{w^2}{l_j^2m_j^2}\right), \; w\in\mathbb{C}^d \tag{2.13}$$

in the $\{M_p\}$ case. It is easily verified that the entire function $P_l(w_1,0,...,0)$, respectively $P_{l_p}(w_1,0,...,0)$, of one variable satisfies the condition c) of proposition 4.6 of [26]. Hence, $P_l(w)$, resp. $P_{l_p}(w)$, satisfies the equivalent conditions a) and b) of proposition 4.5 of [26]. Hence, there exist $L>0$ and $C'>0$, resp. for every $L>0$ there exists $C'>0$, such that $|P_l(w)|\leq C'e^{M(L|w|)}$, resp. $|P_{l_p}(w)|\leq C'e^{M(L|w|)}$, for all $w\in\mathbb{C}^d$ and $P_l(D)$, resp. $P_{l_p}(D)$, are ultradifferential operators of (M_p), resp. $\{M_p\}$, type. It is easy to check that $P_l(w)$ and $P_{l_p}(w)$ don't have zeroes in $W=\mathbb{R}^d+i\{v\in\mathbb{R}^d|\,|v_j|\leq c,\, j=1,...,d\}$. For $w=u+iv\in W$, $|u|\geq 2c\sqrt{d}$, we have $\left|w^2\right|\geq\dfrac{|w|^2}{4}$ and $\left|1+\dfrac{w^2}{l_j^2m_j^2}\right|\geq 1$, for $j\geq q$. We estimate as follows

$$\begin{aligned}
|P_{l_p}(w)| &= \left|\prod_{j=q}^{\infty}\left(1+\frac{w^2}{l_j^2m_j^2}\right)\right| = \sup_p\prod_{j=q}^{p}\left|1+\frac{w^2}{l_j^2m_j^2}\right| \geq \sup_p\prod_{j=q}^{p}\frac{|w^2|}{l_j^2m_j^2}\\
&\geq \sup_p\prod_{j=q}^{p}\frac{|w|^2}{4l_j^2m_j^2} = \frac{\prod_{j=1}^{q-1}4l_j^2}{|w|^{2q-2}}\left(\sup_p\frac{|w|^pM_{q-1}}{M_p\prod_{j=1}^{p}2l_j}\right)^2\\
&= C_0'\left(\frac{M_{q-1}\prod_{j=1}^{q-1}k_j}{|w|^{q-1}}\right)^2 e^{2N_{2l_p}(|w|)} \geq C_0'\frac{e^{N_{2l_p}(|w|)}}{e^{2N_{k_p}(|w|)}},
\end{aligned}$$

where we put $C_0'=\displaystyle\prod_{j=1}^{q-1}\frac{4l_j^2}{k_j^2}$ and $l_p=l$ and $k_p=k$ in the (M_p) case. For $w\in W$, because $P_l(w)$, resp. $P_{l_p}(w)$, doesn't have zeroes in W, we get that there exist $C_0>0$ such that

$$|P_l(w)|\geq C_0\frac{e^{M(|w|/(2l))}}{e^{2M(|w|/k)}},\ \text{resp.}\ |P_{l_p}(w)|\geq C_0\frac{e^{N_{2l_p}(|w|)}}{e^{2N_{k_p}(|w|)}},\ w\in W. \tag{2.14}$$

Now, by using Cauchy integral formula, we can estimate the derivatives of $1/P_l(x)$, resp. $1/P_{l_p}(\xi)$. We will introduce some notations to make the calculations less cumbersome. For $r>0$, denote by $B_r(a)$ the polydisc with centre at a and radii r, i.e. $\{z\in\mathbb{C}^d|\,|z_j-a_j|<r,\, j=1,2,...,d\}$ and by $T_r(a)$ the corresponding polytorus $\{z\in\mathbb{C}^d|\,|z_j-a_j|=r,\, j=1,2,...,d\}$. We will do it for the $\{M_p\}$ case, for the (M_p) case it is similar. We already know that on W, $1/P_{l_p}(w)$ is analytic function (P_{l_p} doesn't have zeroes in W). Hence

$$\left|\partial_w^\alpha\frac{1}{P_{l_p}(x)}\right|\leq\frac{\alpha!}{r^{|\alpha|}}\cdot\left\|\frac{1}{P_{l_p}(z)}\right\|_{L^\infty(T_r(x))}\leq\frac{\alpha!}{C_0r^{|\alpha|}}\cdot\left\|\frac{e^{2N_{k_p}(|z|)}}{e^{N_{2l_p}(|z|)}}\right\|_{L^\infty(T_r(x))},$$

for arbitrary but fixed $r\leq c$ (so $\overline{B_r(x)}\subseteq W$). For $x\in\mathbb{R}^d\backslash B_{2r\sqrt{d}}(0)$, there exists $j\in\{1,...,d\}$ such that $|x_j|\geq 2r\sqrt{d}$. Then, on $T_r(x)$, $|z|\geq|x|-|z-x|=|x|-r\sqrt{d}\geq|x|/2$, i.e. $e^{N_{2l_p}(|z|)}\geq e^{N_{2l_p}(|x|/2)}=e^{N_{4l_p}(|x|)}$. Moreover, for such x, we have

$$e^{2N_{k_p}(|z|)}\leq e^{2N_{k_p}(|x|+r\sqrt{d})}\leq 4e^{2N_{k_p}(2r\sqrt{d})}e^{2N_{k_p}(2|x|)}=C_1e^{2N_{k_p}(2|x|)},$$

where in the last inequality we used that $e^{M(\lambda+\nu)} \leq 2e^{M(2\lambda)}e^{M(2\nu)}$, for $\lambda \geq 0$, $\nu \geq 0$. So, we obtain $\left|\partial_w^\alpha \frac{1}{P_{l_p}(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} \frac{e^{2N_{k_p}(2|x|)}}{e^{N_{4l_p}(|x|)}}$. For x in $B_{2r\sqrt{d}}(0)$, $\left\|e^{2N_{k_p}(|z|)}e^{-N_{2l_p}(|z|)}\right\|_{L^\infty(T_r(x))}$ is bounded, so we can conclude that the above inequality holds, possible with another constant C. Analogously, we can prove that, for the (M_p) case, $\left|\partial_w^\alpha \frac{1}{P_l(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} \frac{e^{2M(2|x|/k)}}{e^{M(|x|/(4l))}}$. This is important, because, if $k > 0$ is fixed, resp. $(k_p) \in \mathfrak{R}$ is fixed, then we can find $l > 0$, resp. $(l_p) \in \mathfrak{R}$, such that $e^{2M(2|x|/k)}e^{-M(|x|/(4l))} \leq C''e^{-M(|x|/k)}$, resp. $e^{2N_{k_p}(2|x|)}e^{-N_{4l_p}(|x|)} \leq C''e^{-N_{k_p}(|x|)}$, for some $C'' > 0$. This inequality trivially follows from proposition 1.2.1 in the (M_p) case. To prove the inequality in the $\{M_p\}$ case, first note that $e^{2N_{k_p}(2|x|)}e^{N_{k_p}(|x|)} \leq e^{3N_{k_p/2}(|x|)}$. By lemma 1.2.1, there exists $(k'_p) \in \mathfrak{R}$ such that $k'_p \leq k_p/2$ and $\prod_{j=1}^{p+q} k'_j \leq 2^{p+q} \prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_j$, for all $p, q \in \mathbb{Z}_+$. So $e^{3N_{k_p/2}(|x|)} \leq e^{3N_{k'_p}(|x|)}$. If we put $N_0 = 1$ and $N_p = M_p \prod_{j=1}^{p} k'_j$, for $p \in \mathbb{Z}_+$, then, by the properties of (k'_p), it follows that N_p satisfies $(M.1)$, $(M.2)$ and $(M.3)'$ where the constant H in $(M.2)$ for this sequence is equal to $2H$. Moreover, note that $N(\lambda) = N_{k'_p}(\lambda)$, for all $\lambda \geq 0$. We can now use proposition 1.2.1 for $N(|x|)$ (i.e. for $N_{k'_p}(|x|)$) and obtain $e^{3N_{k'_p}(|x|)} \leq c''e^{N_{k'_p}(4H^2|x|)} = c''e^{N_{k'_p/(4H^2)}(|x|)}$, for some $c'' > 0$. Now take l_p such that $4l_p = k'_p/(4H^2)$, $p \in \mathbb{Z}_+$ and the desired inequality follows. So, we obtain

$$\left|\partial_x^\alpha \frac{1}{P_l(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-M(|x|/k)}, \text{ resp. } \left|\partial_x^\alpha \frac{1}{P_{l_p}(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-N_{k_p}(|x|)},\ x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d,$$

where C depends on k and l, resp. (k_p) and (l_p), and M_p; $r \leq c$ arbitrary but fixed. Moreover, from the above observation and (2.14), we obtain

$$|P_l(w)| \geq \tilde{C}e^{M(|w|/k)}, \text{ resp. } |P_{l_p}(w)| \geq \tilde{C}e^{N_{k_p}(|w|)},\ w \in W, \tag{2.15}$$

for some $\tilde{C} > 0$.

We summarise the results obtained above in the following proposition.

Proposition 2.1.1. *Let $c > 0$ and $k > 0$, resp. $c > 0$ and $(k_p) \in \mathfrak{R}$ are arbitrary but fixed. Then there exist $l > 0$ and $q \in \mathbb{Z}_+$, resp. there exist $(l_p) \in \mathfrak{R}$ and $q \in \mathbb{Z}_+$ such that $P_l(z) = \prod_{j=q}^{\infty}\left(1 + \frac{z^2}{l^2m_j^2}\right)$, resp. $P_{l_p}(z) = \prod_{j=q}^{\infty}\left(1 + \frac{z^2}{l_j^2m_j^2}\right)$, is an entire function that doesn't have zeroes on the strip $W = \mathbb{R}^d + i\{y \in \mathbb{R}^d \,|\, |y_j| \leq c,\ j = 1, ..., d\}$. $P_l(x)$, resp. $P_{l_p}(x)$, is an ultrapolynomial of class *. Moreover $|P_l(z)| \geq \tilde{C}e^{M(|z|/k)}$, resp. $|P_{l_p}(z)| \geq \tilde{C}e^{N_{k_p}(|z|)}$, $z \in W$, for some $\tilde{C} > 0$ and $\left|\partial_x^\alpha \frac{1}{P_l(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-M(|x|/k)}$, resp. $\left|\partial_x^\alpha \frac{1}{P_{l_p}(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-N_{k_p}(|x|)}$, $x \in \mathbb{R}^d$, $\alpha \in \mathbb{N}^d$, where C depends on k and l, resp. (k_p) and (l_p), and M_p; $r \leq c$ arbitrary but fixed.*

Theorem 2.1.2. *Let B be a connected open set in $\mathbb{R}^d_\xi$ and f an analytic function on $B+i\mathbb{R}^d_\eta$. Let f satisfies the condition:*

for every compact subset K of B there exist $C>0$ and $k>0$, resp. for every $k>0$ there exists $C>0$, such that

$$|f(\xi+i\eta)|\leq Ce^{M(k|\eta|)},\ \forall\xi\in K,\forall\eta\in\mathbb{R}^d. \tag{2.16}$$

Then, there exists $S\in\mathcal{D}'^*(\mathbb{R}^d_x)$ such that $e^{-x\xi}S(x)\in\mathcal{S}'^*(\mathbb{R}^d_x)$, for all $\xi\in B$ and

$$\mathcal{L}(S)(\xi+i\eta)=\mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right)(\xi+i\eta)=f(\xi+i\eta),\ \xi\in B,\ \eta\in\mathbb{R}^d. \tag{2.17}$$

Proof. Because of (2.16), for every fixed $\xi\in B$, $f_\xi=f(\xi+i\eta)\in\mathcal{S}'^*(\mathbb{R}^d_\eta)$. Put $T_\xi(x)=\mathcal{F}^{-1}_{\eta\to x}\left(f_\xi(\eta)\right)(x)\in\mathcal{S}'^*(\mathbb{R}^d_x)$ and $S_\xi(x)=e^{x\xi}T_\xi(x)\in\mathcal{D}'^*(\mathbb{R}^d_x)$. We will show that S_ξ does not depend on $\xi\in B$. Let U be an arbitrary, but fixed, bounded connected open subset of B, such that $K=\overline{U}\subset\subset B$.

Let $c>2$ be such that $|\xi_j|\leq c/2$, for $\xi=(\xi_1,...,\xi_d)\in K$. In the (M_p) case, choose $s>0$ such that $\displaystyle\int_{\mathbb{R}^d}e^{M(k|\eta|)}e^{-M(\frac{s}{2}|\eta|)}d\eta<\infty$ and $e^{2M(k|\eta|)}\leq\tilde{c}e^{M(\frac{s}{2}|\eta|)}$, for some constant $\tilde{c}>0$. For the $\{M_p\}$ case, by the conditions in the theorem, for every $k>0$ there exists $C>0$, such that $\ln_+|f(\xi+i\eta)|\leq M(k|\eta|)+\ln C$ for all $\xi\in K$ and $\eta\in\mathbb{R}^d$. The same estimate holds for the nonnegative increasing function

$$g(\rho)=\sup_{|\eta|\leq\rho}\sup_{\xi\in K}\ln_+|f(\xi+i\eta)|.$$

If we use lemma 1.2.2 for this function we get that there exists subordinate function $\epsilon(\rho)$ and a constant $C>1$ such that $g(\rho)\leq M(\epsilon(\rho))+\ln C$. From this we have that $\ln_+|f(\xi+i\eta)|\leq g(|\eta|)\leq M(\epsilon(|\eta|))+\ln C$, i.e.

$$|f(\xi+i\eta)|\leq Ce^{M(\epsilon(|\eta|))},\ \forall\xi\in K,\forall\eta\in\mathbb{R}^d, \tag{2.18}$$

for some $C>1$. By lemma 3.12 of [26], there exists another sequence $\tilde{N}_p$, which satisfies $(M.1)$, such that $\tilde{N}(\rho)\geq M(\epsilon(\rho))$ and $k'_p=\tilde{n}_p/m_p\to\infty$ when $p\to\infty$. Take $(k_p)\in\mathfrak{R}$ such that $k_p\leq k'_p$, $p\in\mathbb{Z}_+$. Then

$$e^{N_{k_p}(\rho)}=\sup_p\frac{\rho^p}{M_p\prod_{j=1}^p k_j}\geq\sup_p\frac{\rho^p}{M_p\prod_{j=1}^p k'_j}=e^{\tilde{N}(\rho)}\geq e^{M(\epsilon(\rho))}.$$

Hence, from (2.18), it follows that $|f(\xi+i\eta)|\leq Ce^{N_{k_p}(|\eta|)}$, for all $\xi\in K$ and $\eta\in\mathbb{R}^d$. Choose $(s_p)\in\mathfrak{R}$ such that $\displaystyle\int_{\mathbb{R}^d}e^{N_{k_p}(|\eta|)}e^{-N_{2s_p}(|\eta|)}d\eta<\infty$ and $e^{2N_{k_p}(|\eta|)}\leq\tilde{c}e^{N_{2s_p}(|\eta|)}$, for some $\tilde{c}>0$.

Now, for the chosen c and s, resp. (s_p), by the discussion before the theorem, we can find $l>0$, resp. $(l_p)\in\mathfrak{R}$, and entire functions $P_l(w)$ as in (2.12), resp. $P_{l_p}(w)$ as in (2.13), such that they don't have zeroes in $W=\mathbb{R}^d+i\{v\in\mathbb{R}^d||v_j|\leq c,\ j=1,...,d\}$ and the following estimates hold

$$\left|\partial^\alpha_x\frac{1}{P_l(x)}\right|\leq C\cdot\frac{\alpha!}{r^{|\alpha|}}e^{-M(s|x|)},\text{ resp. }\left|\partial^\alpha_x\frac{1}{P_{l_p}(x)}\right|\leq C\cdot\frac{\alpha!}{r^{|\alpha|}}e^{-N_{s_p}(|x|)},\ x\in\mathbb{R}^d,\alpha\in\mathbb{N}^d,$$

where C depends on s and l, resp. (s_p) and (l_p), and M_p; $r \leq c$ is arbitrary but fixed. For shorter notation, we will denote $P_l(w)$ and $P_{l_p}(w)$ by $P(w)$ in both cases. Define the entire functions $P_\xi(w) = P(w - i\xi) = \prod_{j=q}^{\infty}\left(1 + \frac{(w-i\xi)^2}{l^2 m_j^2}\right)$ in the (M_p) case, resp. $P_\xi(w) = P(w - i\xi) = \prod_{j=q}^{\infty}\left(1 + \frac{(w-i\xi)^2}{l_j^2 m_j^2}\right)$ in the $\{M_p\}$ case. As we noted in the construction of the entire functions $P(w)$ (the discussion before the theorem), $P(w)$ satisfies the equivalent conditions a) and b) of proposition 4.5 of [26]. Hence, there exist $L > 0$ and $C' > 0$, resp. for every $L > 0$ there exists $C' > 0$, such that $|P(w)| \leq C' e^{M(L|w|)}$, $w \in \mathbb{C}^d$ and $P(D)$ are ultradifferential operators of (M_p), resp. $\{M_p\}$, type. So, we obtain

$$|P_\xi(w)| = |P(w - i\xi)| \leq C' e^{M(L|w-i\xi|)} \leq C'' e^{M(2L|w|)},\ w \in \mathbb{C}^d,$$

because $\xi = (\xi_1, ..., \xi_d)$ is such that $|\xi_j| \leq c/2$, for $j = 1, ..., d$. Hence, by proposition 4.5 of [26], $P_\xi(D)$ is an ultradifferential operator of class (M_p), resp. of class $\{M_p\}$, for every $\xi = (\xi_1, ..., \xi_d)$ such that $|\xi_j| \leq c/2$, $j = 1, ..., d$. Moreover, by the properties of $P(w)$, it follows that $P_\xi(w)$ is an entire function that doesn't have zeroes in $\mathbb{R}^d + i\{v \in \mathbb{R}^d | \, |v_j| \leq c/2,\ j = 1, ..., d\}$ for all $\xi \in K$. So, by using the Cauchy integral formula to estimate the derivatives, one obtains that $P_\xi(\eta)$ and $1/P_\xi(\eta)$ are multipliers for $\mathcal{S}'^*(\mathbb{R}^d_\eta)$. Also, by (2.15), we have $|P_\xi(\eta)| = |P(\eta - i\xi)| \geq \tilde{C} e^{M(s|\eta - i\xi|)} \geq \tilde{C}' e^{M(\frac{s}{2}|\eta|)}$, for all $\xi \in K$ and $\eta \in \mathbb{R}^d$ in the (M_p) case and similarly, $|P_\xi(\eta)| = |P(\eta - i\xi)| \geq \tilde{C} e^{N_{s_p}(|\eta - i\xi|)} \geq \tilde{C}' e^{N_{2s_p}(|\eta|)}$, for all $\xi \in K$ and $\eta \in \mathbb{R}^d$, in the $\{M_p\}$ case. For $\xi \in B$, put $f_\xi(\eta) = f(\xi + i\eta)$. Then $f_\xi(\eta)/P_\xi(\eta) \in L^1\left(\mathbb{R}^d_\eta\right) \cap \mathcal{E}^*\left(\mathbb{R}^d_\eta\right)$, for all $\xi \in K$. Observe that

$$\begin{aligned} e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(f_\xi(\eta)\right)(x) &= e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)P_\xi(\eta)}{P_\xi(\eta)}\right)(x) \\ &= e^{x\xi}P_\xi(D_x)\left(\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x)\right), \end{aligned}$$

i.e.

$$S_\xi(x) = e^{x\xi}P_\xi(D_x)\left(\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x)\right). \tag{2.19}$$

Let $P(w) = \sum_\alpha c_\alpha w^\alpha$. For simpler notation, put $R(\eta) = f_\xi(\eta)/P_\xi(\eta)$ and calculate as follows

$$\begin{aligned} P(D_x)\left(e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}(R)(x)\right) &= \sum_\alpha c_\alpha \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-i\xi)^\beta e^{x\xi}D_x^{\alpha-\beta}\mathcal{F}^{-1}_{\eta\to x}(R)(x) \\ &= e^{x\xi}\sum_\alpha c_\alpha \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-i\xi)^\beta D_x^{\alpha-\beta}\mathcal{F}^{-1}_{\eta\to x}(R)(x). \end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{\alpha} c_\alpha \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (-i\xi)^\beta D_x^{\alpha-\beta} \mathcal{F}^{-1}_{\eta\to x}(R)(x) \\
&\qquad = \mathcal{F}^{-1}_{\eta\to x}\left(\sum_{\alpha} c_\alpha \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (-i\xi)^\beta \eta^{\alpha-\beta} R(\eta)\right)(x) \\
&\qquad = \mathcal{F}^{-1}_{\eta\to x}\left(\sum_{\alpha} c_\alpha (\eta - i\xi)^\alpha R(\eta)\right)(x) = \mathcal{F}^{-1}_{\eta\to x}\left(P(\eta - i\xi) R(\eta)\right)(x) \\
&\qquad = \mathcal{F}^{-1}_{\eta\to x}\left(P_\xi(\eta) R(\eta)\right)(x) = P_\xi(D_x)\mathcal{F}^{-1}_{\eta\to x}(R)(x).
\end{aligned}$$

From this and (2.19), we get $S_\xi(x) = P(D_x)\left(e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x)\right)$. Now, for $w = \eta - i\xi$, we have

$$\begin{aligned}
e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{f(\xi + i\eta)e^{(\xi+i\eta)x}}{P(\eta - i\xi)} d\eta \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d - i\xi} \frac{f(iw)e^{iwx}}{P(w)} dw.
\end{aligned}$$

The function $\frac{f(iw)e^{iwx}}{P(w)}$ is analytic for $iw \in U + i\mathbb{R}^d$, i.e. $w \in \mathbb{R}^d - iU$ (because $P(w)$ is analytic in the last set and doesn't have zeroes there). Using the growth estimates for f and P, from the theorem of Cauchy-Poincaré, it follows that the last integral doesn't depend on $\xi \in U$. From this and the arbitrariness of U it follows that $S_\xi(x)$ doesn't depend on $\xi \in B$. We will denote this by $S(x)$. Now, by the observations in the beginning, it follows that $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right) = f_\xi$ as ultradistributions in η for every fixed $\xi \in B$. By theorem 2.1.1, it follows that $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right)$ is analytic function for $\zeta = \xi + i\eta \in B + i\mathbb{R}^d$, hence the equality (2.17) holds pointwise. □

Remark 2.1.2. If f is an analytic function on $O = B + i\mathbb{R}^d_\eta$ and satisfies the conditions of the previous theorem then, by this theorem and theorem 2.1.1, it follows that f is analytic on $\operatorname{ch} B + i\mathbb{R}^d_\eta$ and satisfies the estimates (2.1) for every $K \subset\subset \operatorname{ch} B$.

Chapter 3

Convolution of Ultradistributions

Existence of convolution of distributions was considered by Schwartz [50], [51] and later by many authors in various directions. In [50], it is proved that if $S, T \in \mathcal{D}'\left(\mathbb{R}^d\right)$ are two distributions such that $(S \otimes T)\varphi^{\Delta} \in \mathcal{D}'_{L^1}\left(\mathbb{R}^{2d}\right)$, for every $\varphi \in \mathcal{D}$, then the convolution $S * T$ can always be defined as an element of $\mathcal{D}'\left(\mathbb{R}^d\right)$. Later on, Shiraishi in [52] proved that this condition is equivalent to the condition that for every $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{S}\right) T \in \mathcal{D}'_{L^1}\left(\mathbb{R}^d\right)$. Many authors gave alternative definitions of convolution of two distributions which were shown to be equivalent to the definition given by Schwartz (see, for example [15], [16], [23], [24], [37]-[39], [52], [54]). We refer also to an interesting recent paper related to the existence of the convolution [37]. In the case of ultradistributions, the existence of convolution of two Beurling ultradistributions was studied in [41] where the convolution is defined in analogous form to that of Schwartz. In the first section of this chapter we will briefly present the theory for the existence of convolution of Beurling ultradistributions before we move to the main part of this chapter (for the systematic approach in the Beurling case we refer to [41] and [50]). In the second section we will prove several very important facts about the ε tensor product of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$ with a complete l.c.s. that are key components in the proof of the main result in the third section. The third section is devoted to the existence of the convolution of Roumieu ultradistributions. The main theorem there gives the equivalence of several definitions of convolution, among which are the ones that corresponds to the Schwartz's definition and Shiraishi's one.

3.1 Convolution of Beurling Ultradistributions

All the results that we give here are from [25] and [41]. We will mention only the important facts that will be needed for future references.

The key component in the Beurling case is the fact that $\mathcal{B}^{(M_p)}$ is a (F) - space. In fact, it is proved that the bidual of $\mathcal{B}^{(M_p)}$ is isomorphic to $\mathcal{D}^{(M_p)}_{L^\infty}$, i.e. $\left(\mathcal{D}'^{(M_p)}_{L^1}\right)'_b$ and $\mathcal{D}^{(M_p)}_{L^\infty}$ are isomorphic l.c.s. Equip $\mathcal{D}^{(M_p)}_{L^\infty}$ with the topology of compact convergence (from the duality $\left\langle \mathcal{D}'^{(M_p)}_{L^1}, \mathcal{D}^{(M_p)}_{L^\infty}\right\rangle$) and denote it by $\mathcal{D}^{(M_p)}_{L^\infty,c}$. One actually proves that $\mathcal{B}^{(M_p)}$ is distinguished (F) - space and hence $\mathcal{D}'^{(M_p)}_{L^1}$ is barrelled and bornological (note that is also complete as the strong dual of a (F) - space). The topology of compact convergence on $\mathcal{D}_{L^\infty}^{(M_p)}$ is the same as the topology of compact convex circled convergence (from the duality $\left\langle \mathcal{D}_{L^1}'^{(M_p)}, \mathcal{D}_{L^\infty}^{(M_p)} \right\rangle$). That is the reason for the index c. The inclusions $\mathcal{D}_{L^\infty,c}^{(M_p)} \to \mathcal{E}^{(M_p)}$ and $\mathcal{D}_{L^\infty}^{(M_p)} \to \mathcal{D}_{L^\infty,c}^{(M_p)}$ are continuous. One proves that the bounded sets of $\mathcal{D}_{L^\infty}^{(M_p)}$ and of $\mathcal{D}_{L^\infty,c}^{(M_p)}$ are the same. Moreover, the induced topology by $\mathcal{D}_{L^\infty,c}^{(M_p)}$ on a bounded subset of $\mathcal{D}_{L^\infty,c}^{(M_p)}$ is the same as the induced one by $\mathcal{E}^{(M_p)}$. Also, $\mathcal{D}^{(M_p)}$ is dense in $\mathcal{D}_{L^\infty,c}^{(M_p)}$ and the dual of $\mathcal{D}_{L^\infty,c}^{(M_p)}$ is algebraically isomorphic to $\mathcal{D}_{L^1}'^{(M_p)}$. Because of this, if $S, T \in \mathcal{D}'^{(M_p)}\left(\mathbb{R}^d\right)$ are such that, for every $\varphi \in \mathcal{D}^{(M_p)}\left(\mathbb{R}^d\right)$, $(S \otimes T)\varphi^\Delta \in \mathcal{D}_{L^\infty}^{(M_p)}\left(\mathbb{R}^{2d}\right)$, then the convolution of S and T can be defined by

$$\langle S * T, \varphi \rangle = {}_{\mathcal{D}_{L^1}'^{(M_p)}}\left\langle (S \otimes T)\varphi^\Delta, 1 \right\rangle_{\mathcal{D}_{L^\infty,c}^{(M_p)}},$$

where 1 is the constant function which is always equal to 1. Moreover, one actually proves that the mapping $\varphi \mapsto (S \otimes T)\varphi^\Delta$, $\mathcal{D}^{(M_p)}\left(\mathbb{R}^d\right) \to \mathcal{D}_{L^1}'^{(M_p)}\left(\mathbb{R}^{2d}\right)$, is continuous and hence, $S * T$ is well defined ultradistribution. By the properties of the topology of $\mathcal{D}_{L^\infty,c}^{(M_p)}$, if ψ_n is a bounded sequence in $\mathcal{D}_{L^\infty}^{(M_p)}$ which converges to the constant function 1 in $\mathcal{E}^{(M_p)}$ then it converges to 1 also in $\mathcal{D}_{L^\infty,c}^{(M_p)}$ and, for $G \in \mathcal{D}_{L^1}'^{(M_p)}$, $\langle G, \psi_n \rangle \to \langle G, 1 \rangle$, when $n \to \infty$. This, in particular, is satisfied if the sequence ψ_n is defined by $\psi_n(x) = \psi(x/n)$, $n \in \mathbb{Z}_+$, for $\psi \in \mathcal{D}^{(M_p)}$ such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ when $|x| \leq 1$ and $\psi(x) = 0$ when $|x| > 2$.

In the case of Beurling ultradistributions, in [25], the equivalence of this definition and the analogous form of the Shiraishi's definition, as well as few other definitions, was proved. For future references, we will give the theorem here (for its proof, we refer to [25]).

Theorem 3.1.1. *([25]) Let $S, T \in \mathcal{D}'^{(M_p)}\left(\mathbb{R}^d\right)$. The following statements are equivalent:*

i) the convolution of S and T exists;

ii) for all $\varphi \in \mathcal{D}^{(M_p)}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{S}\right) T \in \mathcal{D}_{L^1}'^{(M_p)}\left(\mathbb{R}^d\right)$ and the convolution of S and T is given by $\langle S * T, \varphi \rangle = \left\langle \left(\varphi * \check{S}\right) T, 1 \right\rangle$;

iii) for all $\varphi \in \mathcal{D}^{(M_p)}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{T}\right) S \in \mathcal{D}_{L^1}'^{(M_p)}\left(\mathbb{R}^d\right)$ and the convolution of S and T is given by $\langle S * T, \varphi \rangle = \left\langle \left(\varphi * \check{T}\right) S, 1 \right\rangle$;

iv) for all $\varphi, \psi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{S}\right)(\psi * T) \in L^1\left(\mathbb{R}^d\right)$.

3.2 On the ε Tensor Products with $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$

Let E be a l.c.s. and A a subset of E. A point $e \in E$ is said to be a *sequential limit point* of A if there is a sequence in A which converges to e in E. The set of all sequential limit points of A is called the *sequential limit set* of A. A is said to be *sequentially closed* if it coincides with its sequential limit set. It is easy to verify that intersection of sequentially closed sets is always sequentially closed. Hence there is the smallest sequentially closed set that contains A which we call the *sequential closure* of A. The sequential limit set of A is obviously a subset of the sequential closure of A, but the latter can be strictly larger then the former.

Let E and F be l.c.s. and $\mathcal{L}_c(E,F)$ denote the space of continuous linear mappings from E into F with the topology of uniform convergence on convex circled compact subsets of E. E'_c denotes the dual of E equipped with the topology of uniform convergence on convex circled compact subsets of E. As in Komatsu [28] and Schwartz [50], we define the ε *tensor product* of E and F, denoted by $E\varepsilon F$, as the space of all bilinear functionals on $E'_c \times F'_c$ which are hypocontinuous with respect to the equicontinuous subsets of E' and F'. It is equipped with the topology of uniform convergence on products of equicontinuous subsets of E' and F'. Moreover, the following isomorphisms hold:

$$E\varepsilon F \cong \mathcal{L}_\epsilon\left(E'_c, F\right) \cong \mathcal{L}_\epsilon\left(F'_c, E\right), \tag{3.1}$$

where $\mathcal{L}_\epsilon\left(E'_c, F\right)$ is the space of all continuous linear mappings from E'_c to F equipped with the ϵ topology of uniform convergence on equicontinuous subsets of E', similarly for $\mathcal{L}_\epsilon\left(F'_c, E\right)$. It is proved in [50] that if both E and F are complete then $E\varepsilon F$ is complete. The tensor product $E \otimes F$ is injected in $E\varepsilon F$ under $(e\otimes f)(e',f') = \langle e, e'\rangle\langle f, f'\rangle$. The induced topology on $E\otimes F$ is the ϵ topology and we have the topological imbedding $E\otimes_\epsilon F \hookrightarrow E\varepsilon F$.

We recall the following definitions (c.f. Komatsu [28] and Schwartz [50]).

Definition 3.2.1. The l.c.s. E is said to have the *sequential approximation property* (resp. the *weak sequential approximation property*) if the identity mapping $\mathrm{Id} : E \to E$ is in the sequential limit set (resp. the sequential closure) of $E' \otimes E$ in $\mathcal{L}_c(E,E)$.

The l.c.s. E is said to have the *weak approximation property* if the identity mapping $\mathrm{Id} : E \to E$ is in the closure of $E'\otimes E$ in $\mathcal{L}_c(E,E)$.

Remark 3.2.1. The reader should not confuse the weak approximation property with the *approximation property* defined by Grothendieck [20]. The latter means that the identity mapping $\mathrm{Id} : E \to E$ is in the closure of $E'\otimes E$ in $\mathcal{L}_p(E,E)$, where the index p stands for the topology of precompact convergence. In fact Grothendieck gives the definition of the approximation property by requiring $E'\otimes E$ to be dense in $\mathcal{L}_p(E,E)$. But this can be shown that it is equivalent to the previous definition (see [49]). In general, if E has the approximation property then it has the weak approximation property. Obviously, if E is quasi-complete then the weak approximation property and the approximation property are the same thing. We refer to [31] and [49] for further properties on the approximation property.

We also need the next proposition ([28], proposition 1.4., p. 659).

Proposition 3.2.1. *([28]) If E and F are complete l.c.s. and if either E or F has the weak approximation property then $E\varepsilon F$ is isomorphic to $E\hat{\otimes}_\epsilon F$.*

For $K \subset\subset \mathbb{R}^d$, we denote by $\mathcal{C}_0(K)$ the (B) - space of all continuous functions supported by K endowed with $\|\cdot\|_{L^\infty}$ norm. Throughout this chapter, we will often denote by $K_{\mathbb{R}^d}(x,t)$ the close ball in $\mathbb{R}^d$ with radius $t > 0$ and centre at x.

Lemma 3.2.1. *Let K_1 and K_2 be two compact subsets of $\mathbb{R}^d$ such that $K_1 \subset\subset \operatorname{int}K_2$. Then there exists a sequence S_n in $(\mathcal{C}_0(K_1))' \otimes \mathcal{C}_0(K_2)$ such that $S_n \to \mathrm{Id}$, as $n \to \infty$, in $\mathcal{L}_c(\mathcal{C}_0(K_1), \mathcal{C}_0(K_2))$.*

Proof. For every $n \in \mathbb{Z}_+$, choose a finite open covering $\{U_{1,n}, ..., U_{k_n,n}\}$ of K_1 of open sets each with diameter less than $1/n$ such that $\bar{U}_{j,n} \subseteq \operatorname{int}K_2$, $j = 1, ..., k_n$. Let $\chi_{j,n}$, $j = 1, ..., k_n$, be a continuous partition of unity subordinated to $\{U_{1,n}, ..., U_{k_n,n}\}$. For every $j \in \{1, ..., k_n\}$, choose a point $x_{j,n} \in \operatorname{supp}\chi_{j,n} \cap K_1$. Define

$$S_n = \sum_{j=1}^{k_n} \delta(\cdot - x_{j,n}) \otimes \chi_{j,n} \in (\mathcal{C}_0(K_1))' \otimes \mathcal{C}_0(K_2).$$

Let $V = \left\{\varphi \in \mathcal{C}_0(K_2) \middle| \|\varphi\|_{L^\infty} \leq \varepsilon\right\}$ and B a compact convex circled subset of $\mathcal{C}_0(K_1)$. Let

$$\mathcal{M}(B, V) = \left\{T \in \mathcal{L}(\mathcal{C}_0(K_1), \mathcal{C}_0(K_2)) \,\middle|\, T(B) \subseteq V\right\}.$$

By the Arzela - Ascoli theorem, for the chosen ε there exists $\eta > 0$ such that for all $x, y \in K_1$ such that $|x - y| < \eta$, $|\varphi(x) - \varphi(y)| \leq \varepsilon$ for all $\varphi \in B$. Let $n_0 \in \mathbb{N}$ is so large such that $1/n_0 < \eta$. Then, for $n \geq n_0$ and $x \in K_1$, we have

$$\begin{aligned} |S_n(\varphi)(x) - \varphi(x)| &= \left|\sum_{j=1}^{k_n} \varphi(x_{j,n}) \chi_{j,n}(x) - \sum_{j=1}^{k_n} \varphi(x)\chi_{j,n}(x)\right| \\ &\leq \sum_{j=1}^{k_n} |\varphi(x_{j,n}) - \varphi(x)| \chi_{j,n}(x) \leq \varepsilon, \end{aligned}$$

for all $\varphi \in B$. Note that, for $x \in K_2 \backslash K_1$, $\varphi(x) = 0$ and

$$|S_n(\varphi)(x)| \leq \sum_{j=1}^{k_n} |\varphi(x_{j,n})| \chi_{j,n}(x) \leq \varepsilon.$$

So, $S_n - \mathrm{Id} \in \mathcal{M}(B, V)$ for $n \geq n_0$. □

Lemma 3.2.2. *B is a precompact subset of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}(\mathbb{R}^d)$ if and only if B is bounded in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}(\mathbb{R}^d)$ and for every $\varepsilon > 0$ and $(t_j) \in \mathfrak{R}$, there exists $K \subset\subset \mathbb{R}^d$ such that* $\sup\limits_{\varphi \in B} \sup\limits_{\substack{\alpha \in \mathbb{N}^d \\ x \in \mathbb{R}^d \backslash K}} \dfrac{|D^\alpha \varphi(x)|}{T_\alpha M_\alpha} \leq \varepsilon.$

Proof. $\Rightarrow$. Let $\varepsilon > 0$ and $(t_j) \in \mathfrak{R}$ and $V = \left\{\varphi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}} \middle| \|\varphi\|_{(t_j)} \leq \varepsilon/2\right\}$. There exist $\varphi_1, ..., \varphi_n \in B$ such that for each $\varphi \in B$ there exists $j \in \{1, ..., n\}$ such that $\varphi \in \varphi_j + V$. Let $K \subset\subset \mathbb{R}^d$ such that $|D^\alpha \varphi_j(x)|/(T_\alpha M_\alpha) \leq \varepsilon/2$ for all $x \in \mathbb{R}^d \backslash K$, $\alpha \in \mathbb{N}^d$, $j \in \{1, ..., n\}$. Let $\varphi \in B$. There exists $j \in \{1, ..., n\}$ such that $\|\varphi - \varphi_j\|_{(t_j)} \leq \varepsilon/2$. The proof follows from

$$\frac{|D^\alpha \varphi(x)|}{T_\alpha M_\alpha} \leq \frac{|D^\alpha (\varphi(x) - \varphi_j(x))|}{T_\alpha M_\alpha} + \frac{|D^\alpha \varphi_j(x)|}{T_\alpha M_\alpha} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ x \in \mathbb{R}^d \backslash K, \alpha \in \mathbb{N}^d.$$

$\Leftarrow$. Let $V = \left\{\varphi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \,\middle|\, \|\varphi\|_{(t_j)} \leq \varepsilon\right\}$. Since B is bounded in the Montel space $\mathcal{E}^{\{M_p\}}\left(\mathbb{R}^d\right)$, it is precompact in $\mathcal{E}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Thus, there exists a finite subset $B_0 = \{\varphi_1, ..., \varphi_n\}$ of B such that, for every $\varphi \in B$, there exists $j \in \{1, ..., n\}$ such that $p_{K,(t_j)}(\varphi - \varphi_j) \leq \varepsilon$, where K is the compact set for which the assumption in the lemma holds for the chosen ε and (t_j). If $\varphi \in B$ is fixed, take such $\varphi_j \in B_0$. Then, $\dfrac{|D^\alpha \varphi(x) - D^\alpha \varphi_j(x)|}{T_\alpha M_\alpha} \leq \varepsilon$, for all $x \in K, \alpha \in \mathbb{N}^d$. Also, by the assumption,

$$\frac{|D^\alpha \varphi(x) - D^\alpha \varphi_j(x)|}{T_\alpha M_\alpha} \leq \frac{|D^\alpha \varphi(x)|}{T_\alpha M_\alpha} + \frac{|D^\alpha \varphi_j(x)|}{T_\alpha M_\alpha} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all $x \in \mathbb{R}^d \backslash K$, $\alpha \in \mathbb{N}^d$. So, the proof follows. □

Proposition 3.2.2. $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$ *has the weak sequential approximation property.*

Proof. Let $K_n = K_{\mathbb{R}^d}(0, 2^{n-1})$, $n \geq 1$. Let $\theta \in \mathcal{D}^{\{M_p\}}_{K_1}$ is such that $\theta = 1$ on $K_{\mathbb{R}^d}(0, 1/2)$. Define $\theta_n(x) = \theta(x/2^n)$, $n \in \mathbb{Z}_+$. Then $\theta_n \in \mathcal{D}^{\{M_p\}}_{K_{n+1}}$ and $\theta_n = 1$ on K_n. Let $T_n \in \mathcal{L}\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right), \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$, defined by $T_n(\varphi) = \theta_n \varphi$. Let $\mu \in \mathcal{D}^{\{M_p\}}_{K_1}$, $\mu \geq 0$, is such that $\displaystyle\int_{\mathbb{R}^d} \mu(x)dx = 1$ and define a δ-sequence $\mu_m = m^d \mu(m\cdot)$, $m \in \mathbb{Z}_+$. For each fixed $n \in \mathbb{Z}_+$, by lemma 3.2.1, we find

$$S_{k,n} = \sum_{l=1}^{j_{k,n}} \delta\left(\cdot - x_{l,k,n}\right) \otimes \chi_{l,k,n} \in (\mathcal{C}_0(K_{n+1}))' \otimes \mathcal{C}_0(K_{n+2})$$

such that $S_{k,n} \to \mathrm{Id}$, when $k \to \infty$, in $\mathcal{L}_c\left(\mathcal{C}_0(K_{n+1}), \mathcal{C}_0(K_{n+2})\right)$, where $\chi_{l,k,n}$ are continuous function with values in $[0,1]$ that have compact support in int K_{n+2} and $x_{l,k,n}$ are points in supp $\chi_{l,k,n} \cap K_{n+1}$. Moreover the support of $\chi_{l,k,n}$ has diameter less then $1/k$ and $\displaystyle\sum_{l=1}^{j_{k,n}} \chi_{l,k,n}(x) \leq 1$ on K_{n+2} and $\displaystyle\sum_{l=1}^{j_{k,n}} \chi_{l,k,n}(x) = 1$ on K_{n+1}. Define, for $k, m, n \in \mathbb{Z}_+$,

$$T_{k,m,n} = \sum_{l=1}^{j_{k,n}} \theta_n \delta\left(\cdot - x_{l,k,n}\right) \otimes \left(\mu_m * \chi_{l,k,n}\right) \text{ and } T_{m,n} : \varphi \mapsto T_{m,n}(\varphi) = \mu_m * (\theta_n \varphi).$$

First we will prove that for each fixed $m, n \in \mathbb{Z}_+$, $T_{k,m,n} \to T_{m,n}$ in the space $\mathcal{L}_c\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right), \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$, when $k \to \infty$. Let

$$V = \left\{\varphi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \,\middle|\, \|\varphi\|_{(t_j)} \leq \varepsilon\right\},$$

B a convex circled compact subset of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$ and

$$\mathcal{M}(B,V)=\left\{T\in\mathcal{L}\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right),\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)\Big|\,T(B)\subseteq V\right\}$$

(a neighbourhood of zero in $\mathcal{L}_c\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right),\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$). Let $\varphi\in B$. Then, for $\alpha\in\mathbb{N}^d$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}
&\frac{|D^\alpha T_{k,m,n}(\varphi)(x)-D^\alpha T_{m,n}(\varphi)(x)|}{T_\alpha M_\alpha}\\
&\quad= \frac{1}{T_\alpha M_\alpha}\left|(D^\alpha\mu_m)*\left(\sum_{l=1}^{j_{k,n}}\theta_n\left(x_{l,k,n}\right)\varphi\left(x_{l,k,n}\right)\chi_{l,k,n}-\theta_n\varphi\right)(x)\right|\\
&\quad\leq m^d\|\mu\|_{(t_j/m)}\int_{K_{n+2}}\sum_{l=1}^{j_{k,n}}\left|\theta_n\left(x_{l,k,n}\right)\varphi\left(x_{l,k,n}\right)-\theta_n(y)\varphi(y)\right|\chi_{l,k,n}(y)dy.
\end{aligned}$$

Because the mapping $\varphi\mapsto\theta_n\varphi$, $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\to\mathcal{C}_0(K_{n+1})$ is continuous, it maps the compact set B in a compact set in $\mathcal{C}_0(K_{n+1})$, which we denote by B_1. By the Arzela - Ascoli theorem, for the chosen ε there exists $\eta>0$ such that for all $x,y\in K_{n+1}$ such that

$$|x-y|<\eta\Rightarrow|\theta_n(x)\varphi(x)-\theta_n(y)\varphi(y)|\leq\frac{\varepsilon}{m^d\|\mu\|_{(t_j/m)}|K_{n+2}|},\ \varphi\in B.$$

If we take k_0 large enough such that $1/k_0<\eta$, then, for all $k\geq k_0$,

$$\frac{|D^\alpha T_{k,m,n}(\varphi)(x)-D^\alpha T_{m,n}(\varphi)(x)|}{T_\alpha M_\alpha}\leq\varepsilon,\ x\in\mathbb{R}^d,\alpha\in\mathbb{N}^d,\varphi\in B.$$

That is $T_{k,m,n}-T_{m,n}\in\mathcal{M}(B,V)$ for all $k\geq k_0$. Now we prove that, for each fixed $n\in\mathbb{Z}_+$, $T_{m,n}\to T_n$, when $m\to\infty$, in $\mathcal{L}_c\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right),\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$. We use the notation as above. Because of lemma 1.2.1, without losing generality, we can assume that (t_j) is such that $T_{p+q}\leq 2^{p+q}T_pT_q$, for all $p,q\in\mathbb{N}$. Then, for $\varphi\in B,\alpha\in\mathbb{N}^d,x\in\mathbb{R}^d$,

$$\frac{|D^\alpha T_{m,n}(\varphi)(x)-D^\alpha T_n(\varphi)(x)|}{T_\alpha M_\alpha}\leq\int_{\mathbb{R}^d}\mu_m(y)\frac{|D^\alpha(\theta_n\varphi)(x-y)-D^\alpha(\theta_n\varphi)(x)|}{T_\alpha M_\alpha}dy.$$

Let $t'_1=t_1/(4H)$ and $t'_p=t_{p-1}/(2H)$, for $p\in\mathbb{N}$, $p\geq 2$. Then $(t'_j)\in\mathfrak{R}$. For the moment, denote $\theta_n\varphi$ by φ_n. By the mean value theorem, we have

$$\begin{aligned}
|D^\alpha\varphi_n(x-y)-D^\alpha\varphi_n(x)| &\leq 2\sqrt{d}\|\varphi_n\|_{(t'_j)}T'_{|\alpha|+1}M_{|\alpha|+1}|y|\\
&\leq c_0t_1M_1\sqrt{d}\|\varphi_n\|_{(t'_j)}T_\alpha M_\alpha|y|.
\end{aligned}$$

Note that $\|\varphi_n\|_{(t'_j)}\leq\|\theta\|_{(t'_j/2)}\|\varphi\|_{(t'_j/2)}$. So, by the definition of μ_m, we obtain

$$\frac{|D^\alpha T_{m,n}(\varphi)(x)-D^\alpha T_n(\varphi)(x)|}{T_\alpha M_\alpha}\leq\frac{c_0t_1M_1\sqrt{d}\|\theta\|_{(t'_j/2)}\|\varphi\|_{(t'_j/2)}}{m}.$$

There exists $C > 0$ such that $\sup_{\varphi\in B}\|\varphi\|_{(t'_j/2)} \leq C$. If we take large enough m_0, such that $1/m_0 \leq \varepsilon/\left(c_0 C t_1 M_1\sqrt{d}\|\theta\|_{(t'_j/2)}\right)$, then, for all $m \geq m_0$, $T_{m,n} - T_n \in \mathcal{M}(B,V)$.

Now, we prove that $T_n \to \mathrm{Id}$ in $\mathcal{L}_c\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right), \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$. Let B, V and $\mathcal{M}(B,V)$ be the same as above. There exists $C > 0$ such that $\|\varphi\|_{(t_j/2)} \leq C$, for all $\varphi \in B$. Moreover, by lemma 3.2.2, for the chosen ε and (t_j), there exists $K \subset\subset \mathbb{R}^d$ such that $\dfrac{|D^\alpha\varphi(x)|}{T_\alpha M_\alpha} \leq \dfrac{\varepsilon}{2\left(1+\|\theta\|_{L^\infty}\right)}$ for all $\alpha \in \mathbb{N}^d$, $x \in \mathbb{R}^d\backslash K$ and $\varphi \in B$. There exists n_0 such that $K \subset\subset \mathrm{int}K_{n_0}$ and $C\|\theta\|_{(t_j/2)}/2^{n_0} \leq \varepsilon/2$. So, for $n \geq n_0$, we have

$$\begin{aligned}
&\frac{|D^\alpha T_n(\varphi)(x) - D^\alpha\varphi(x)|}{T_\alpha M_\alpha}\\
&\quad\leq \left|1-\theta(x/2^n)\right|\frac{|D^\alpha\varphi(x)|}{T_\alpha M_\alpha} + \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}}\binom{\alpha}{\beta}\frac{\left|D^\beta\theta(x/2^n)\right|\left|D^{\alpha-\beta}\varphi(x)\right|}{2^{n|\beta|}T_\alpha M_\alpha}\\
&\quad\leq \frac{\varepsilon}{2} + \frac{\|\theta\|_{(t_j/2)}\|\varphi\|_{(t_j/2)}}{2^n} \leq \varepsilon,
\end{aligned}$$

that is $T_n - \mathrm{Id} \in \mathcal{M}(B,V)$, for all $n \geq n_0$. Thus, Id belongs to the sequential closure of $\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\right)'\left(\mathbb{R}^d\right) \otimes \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$. □

If E is a complete l.c.s., by proposition 3.2.2, proposition 3.2.1 and (3.1), we have the following isomorphisms of l.c.s.

$$\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\varepsilon E \cong \mathcal{L}_\epsilon\left(\left(\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\right)'_c\left(\mathbb{R}^d\right), E\right) \cong \mathcal{L}_\epsilon\left(E'_c, \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right) \cong \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\hat{\otimes}_\epsilon E. \tag{3.2}$$

Let E be a complete l.c.s. Define the space $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$ as the space of all smooth $E-$valued functions φ on $\mathbb{R}^d$ so that

i) for each continuous seminorm q of E and $(t_j) \in \mathfrak{R}$ there exists $C > 0$ such that $q_{(t_j)}(\varphi) = \sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d} q\left(\dfrac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\right) \leq C$,

ii) for every $\varepsilon > 0$, $(t_j) \in \mathfrak{R}$ and q a continuous seminorm on E, there exists $K \subset\subset \mathbb{R}^d$ such that $q\left(\dfrac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\right) \leq \varepsilon$, for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d\backslash K$.

We equip $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$ with the locally convex topology generated by the seminorms $q_{(t_j)}$, where q are continuous seminorms on E and $(t_j) \in \mathfrak{R}$. This is Hausdorff topology and hence, $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$ is a l.c.s.

Proposition 3.2.3. *$\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$ and $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\varepsilon E$ are isomorphic l.c.s.*

Proof. By (3.2), it is enough to prove that $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d;E\right)\cong\mathcal{L}_\epsilon\left(E'_c,\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$. Let $\varphi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d;E\right)$, $e'\in E'$ and $\tilde{\varphi}_{e'}(x)=\langle e',\varphi(x)\rangle, x\in\mathbb{R}^d$. Clearly, $\tilde{\varphi}_{e'}$ is smooth and $D^\alpha\tilde{\varphi}_{e'}=\langle e',D^\alpha\varphi\rangle$. Let $(t_j)\in\mathfrak{R}$ and $\varepsilon>0$. Then

$$\frac{|D^\alpha\tilde{\varphi}_{e'}(x)|}{T_\alpha M_\alpha}=\left|\left\langle e',\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\right\rangle\right|\leq C_1 q\left(\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\right)\leq C_1 q_{(t_j)}(\varphi),\alpha\in\mathbb{N}^d,x\in\mathbb{R}^d,$$

and there exists $K\subset\subset\mathbb{R}^d$ such that $q\left(D^\alpha\varphi(x)/(T_\alpha M_\alpha)\right)\leq\varepsilon/C_1$, for all $\alpha\in\mathbb{N}^d$ and $x\in\mathbb{R}^d\backslash K$. Similarly as above, one obtains that $\left|D^\alpha\tilde{\varphi}_{e'}(x)\right|/(T_\alpha M_\alpha)\leq\varepsilon$ for all $\alpha\in\mathbb{N}^d$ and $x\in\mathbb{R}^d\backslash K$, i.e. $\tilde{\varphi}_{e'}\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Let $\varphi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d;E\right)$. Consider the mapping $T_\varphi:E'\to\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $e'\mapsto T_\varphi(e')=\tilde{\varphi}_{e'}$. We prove that $T_\varphi\in\mathcal{L}\left(E'_c,\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$.

Let $A=\left\{\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\Big| x\in\mathbb{R}^d,\,\alpha\in\mathbb{N}^d\right\}$. We will prove that A is precompact in E. Let $U=\{e\in E|\,q_1(e)\leq r,...,q_n(e)\leq r\}$ be a neighbourhood of zero in E. For the chosen r, (t_j) and $q_1,...,q_n$, there exists $K\subset\subset\mathbb{R}^d$ such that $q_l\left(D^\alpha\varphi(x)/(T_\alpha M_\alpha)\right)\leq r/2$, for all $\alpha\in\mathbb{N}^d$, $x\in\mathbb{R}^d\backslash K$ and $l=1,...,n$. Moreover, there exists $C>0$ such that $q_{l,(t_j/2)}(\varphi)\leq C$, for all $l=1,...,n$. Take $s\in\mathbb{Z}_+$ such that $1/2^s\leq r/(2C)$. Then, if $|\alpha|\geq s$, we have $q_l\left(D^\alpha\varphi(x)/(T_\alpha M_\alpha)\right)\leq r/2$ for all $x\in\mathbb{R}^d$. The set $A'=\left\{D^\alpha\varphi(x)/(T_\alpha M_\alpha)\big| x\in K,\,|\alpha|<s\right\}$ is obviously compact in E. So, there exists a finite subset B'_0 of A' such that $A'\subseteq B'_0+U$. Take $x_1\in K$, $x_2\in\mathbb{R}^d\backslash K$ and let $\beta\in\mathbb{N}^d$ be a fixed d-tuple such that $|\beta|>s$. Consider the set $B_0=B'_0\bigcup\left\{D^\beta\varphi(x_1)/(T_\beta M_\beta),\varphi(x_2)\right\}\subseteq A$. If $|\alpha|<s$ and $x\in K$, $D^\alpha\varphi(x)/(T_\alpha M_\alpha)\in B_0+U$. If $|\alpha|\geq s$ and $x\in K$, we have

$$q_l\left(\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}-\frac{D^\beta\varphi(x_1)}{T_\beta M_\beta}\right)\leq q_l\left(\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\right)+q_l\left(\frac{D^\beta\varphi(x_1)}{T_\beta M_\beta}\right)\leq r,\,l=1,...,n.$$

Also, if $x\in\mathbb{R}^d\backslash K$ and $\alpha\in\mathbb{N}^d$, we have

$$q_l\left(\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}-\varphi(x_2)\right)\leq q_l\left(\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\right)+q_l\left(\varphi(x_2)\right)\leq r,\,l=1,...,n.$$

We obtain that $A\subseteq B_0+U$. Thus, A is precompact.

Let $V=\left\{\psi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\big|\,\|\psi\|_{(t_j)}\leq\varepsilon\right\}$ be a neighbourhood of zero in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$. Because A is precompact and E is complete l.c.s., $\tilde{A}$ - the closed convex circled hull of A is compact. Let $W=\left(1/\varepsilon\tilde{A}\right)^\circ$ ($^\circ$ means the polar). Let $e'\in W$. Then

$$\frac{|D^\alpha T_\varphi(e')(x)|}{T_\alpha M_\alpha}=\left|\left\langle e',\frac{D^\alpha\varphi(x)}{T_\alpha M_\alpha}\right\rangle\right|\leq\varepsilon,\,\alpha\in\mathbb{N}^d,x\in\mathbb{R}^d,$$

and the continuity of T_φ follows.

The topology of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d;E\right)$ is the one induced by $\mathcal{L}_\epsilon\left(E'_c,\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$ when we consider it as a subspace of the latter by the injection $\varphi\mapsto T_\varphi$. To prove this, let

$\mathcal{M}(B,V)$ be a neighbourhood of zero in $\mathcal{L}_\epsilon\left(E'_c, \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$, where V is as above and B is an equicontinuous subset of E'. Let $U = \{e \in E|\, q_1(e) \leq r, ..., q_n(e) \leq r\}$ be a neighbourhood of zero in E such that $|\langle e', e\rangle| \leq \varepsilon$, when $e \in U$ and $e' \in B$. Let

$$W = \left\{\varphi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right) \middle|\, q_{1,(t_j)}(\varphi) \leq r, ..., q_{n,(t_j)}(\varphi) \leq r\right\}.$$

Then, for $\varphi \in W$, $\dfrac{D^\alpha \varphi(x)}{T_\alpha M_\alpha} \in U$ for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. Hence, for $e' \in B$, $|D^\alpha T_\varphi(e')(x)|\,/(T_\alpha M_\alpha) \leq \varepsilon$, $\alpha \in \mathbb{N}^d, x \in \mathbb{R}^d$, i.e. $T_\varphi \in \mathcal{M}(B,V)$, for all $\varphi \in W$. Conversely, let W be a neighbourhood of zero in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$ given as above. Consider U as above and $B = U^\circ$. If $\varphi \in W$ and $e' \in B$, then $\|T_\varphi(e')\|_{(t_j)} \leq 1$. Let $V = \left\{\psi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right) \middle|\, \|\psi\|_{(t_j)} \leq 1\right\}$ and $\tilde{G} = \mathcal{M}(B,V) \cap \left\{T_\varphi \middle|\, \varphi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)\right\}$. Let $T_\varphi \in \tilde{G}$. Then, for all $e' \in B$, $T_\varphi(e') \in V$, i.e. $\|T_\varphi(e')\|_{(t_j)} \leq 1$. So, we have

$$\left|\left\langle e', \frac{D^\alpha \varphi(x)}{T_\alpha M_\alpha}\right\rangle\right| = \frac{|D^\alpha T_\varphi(e')(x)|}{T_\alpha M_\alpha} \leq 1,\ \alpha \in \mathbb{N}^d, x \in \mathbb{R}^d, e' \in B.$$

We obtain that, for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, $\dfrac{D^\alpha \varphi(x)}{T_\alpha M_\alpha} \in B^\circ = U^{\circ\circ} = U$. But this means that $\varphi \in W$. Hence, we proved that $\varphi \mapsto T_\varphi$ is a topological imbedding of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$ into $\mathcal{L}_\epsilon\left(E'_c, \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$. It remains to prove that this mapping is a surjection. By theorem 1.12 of [28], $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \varepsilon E \cong \mathcal{L}_\epsilon\left(E'_c, \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$ is identified with the space of all $f \in \mathcal{C}\left(\mathbb{R}^d; E\right)$ such that:

i) for any $e' \in E'$, the function $\langle e', f(\cdot)\rangle$ is in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$;

ii) for every equicontinuous set A' in E', the set $\{\langle e', f(\cdot)\rangle|\, e' \in A'\}$ is relatively compact in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$.

Every such f generates an operator $L' \in \mathcal{L}\left(E'_c, \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$ by $L'(e')(\cdot) = \tilde{f}_{e'} = \langle e', f(\cdot)\rangle$, which gives the algebraic isomorphism between the space of all $f \in \mathcal{C}\left(\mathbb{R}^d; E\right)$ which satisfy the above conditions and $\mathcal{L}\left(E'_c, \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)$. We will prove that every such f belongs to $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$ and obtain the desired surjectivity. So, let $f \in \mathcal{C}\left(\mathbb{R}^d; E\right)$ be a function that satisfies the conditions *i)* and *ii)*. By the above conditions, $\tilde{f}_{e'} \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \subseteq \mathcal{E}^{\{M_p\}}\left(\mathbb{R}^d\right)$, so, by theorem 3.10 of [28], we get that $f \in \mathcal{E}^{\{M_p\}}\left(\mathbb{R}^d; E\right)$. Hence f is smooth E-valued and from the quoted theorem it follows that $D^\alpha \tilde{f}_{e'}(x) = \langle e', D^\alpha f(x)\rangle$. Let $(t_j) \in \mathfrak{R}$. Then

$$\left|\left\langle e', \frac{D^\alpha f(x)}{T_\alpha M_\alpha}\right\rangle\right| = \frac{\left|D^\alpha \tilde{f}_{e'}(x)\right|}{T_\alpha M_\alpha} \leq \left\|\tilde{f}_{e'}\right\|_{(t_j)}.$$

Hence, the set $\left\{\dfrac{D^\alpha f(x)}{T_\alpha M_\alpha}\Big|\, \alpha\in\mathbb{N}^d,\ x\in\mathbb{R}^d\right\}$ is weakly bounded, hence it is bounded in E. Let q be a continuous seminorm in E and $U=\{e\in E|\, q(e)\leq\varepsilon\}$. There exists $C>0$ such that $q\left(D^\alpha f(x)/(T_\alpha M_\alpha)\right)\leq C$, for all $\alpha\in\mathbb{N}^d$ and $x\in\mathbb{R}^d$. Since $A'=W^\circ$ is equicontinuous set in E', $\left\{\tilde{f}_{e'}|\, e'\in A'\right\}$ is relatively compact in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$. By lemma 3.2.2, for the chosen (t_j), there exists $K\subset\subset\mathbb{R}^d$ such that $\left|D^\alpha\tilde{f}_{e'}(x)\right|/(T_\alpha M_\alpha)\leq 1$, for all $\alpha\in\mathbb{N}^d$, $x\in\mathbb{R}^d\backslash K$ and $e'\in A'$. We obtain that, for $\alpha\in\mathbb{N}^d$ and $x\in\mathbb{R}^d\backslash K$, $\dfrac{D^\alpha f(x)}{T_\alpha M_\alpha}\in A'^\circ=U^{\circ\circ}=U$. But then, $q\left(D^\alpha f(x)/(T_\alpha M_\alpha)\right)\leq\varepsilon$, for all $\alpha\in\mathbb{N}^d$ and $x\in\mathbb{R}^d\backslash K$. We obtain that $f\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d;E\right)$. □

By this proposition, if we take $E=\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^m\right)$, we get

$$\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d;\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^m\right)\right)\cong\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\varepsilon\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^m\right)\cong\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\hat{\otimes}_\epsilon\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^m\right). \tag{3.3}$$

Proposition 3.2.4. $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1+d_2}\right)\cong\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1}\right)\hat{\otimes}_\epsilon\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$.

Proof. By (3.3), it is enough to prove $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1+d_2}\right)\cong\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1};\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)\right)$. Let $f\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1};\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)\right)$. For each fixed $x\in\mathbb{R}^{d_1}$ put $\varphi_x=f(x)\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$. For every $(x,y)\in\mathbb{R}^{d_1+d_2}$ define the scalar valued function $\varphi(x,y)=\varphi_x(y)$, $y\in\mathbb{R}^{d_2}$. Put $\theta_{j,x}=\partial_{x_j}f(x)\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$. Let $\left(x^{(0)},y^{(0)}\right)$ be a fixed point in $\mathbb{R}^{d_1+d_2}$, $\varepsilon>0$ and $(t_j)\in\mathfrak{R}$. Then, because f is infinitely differentiable $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$-valued function, there exists $\eta>0$ such that, for all $x\in\mathbb{R}^{d_1}$ with $\left|x-x^{(0)}\right|<\eta$, we have

$$\left\|\left|x-x^{(0)}\right|^{-1}\left(f(x)-f\left(x^{(0)}\right)-\sum_{j=1}^{d_1}\theta_{j,x^{(0)}}\left(x_j-x_j^{(0)}\right)\right)\right\|_{(t_j)}\leq\varepsilon,$$

Hence, $\left|x-x^{(0)}\right|^{-1}\left|\varphi_x(y)-\varphi_{x^{(0)}}(y)-\displaystyle\sum_{j=1}^{d_1}\theta_{j,x^{(0)}}(y)\left(x_j-x_j^{(0)}\right)\right|\leq\varepsilon$, for all $y\in\mathbb{R}^{d_2}$ and $\left|x-x^{(0)}\right|<\eta$. Since $\varphi_{x^{(0)}}$ is smooth, we have

$$\begin{aligned}
&\left|\varphi_x(y)-\varphi_{x^{(0)}}\left(y^{(0)}\right)-\sum_{j=1}^{d_1}\theta_{j,x^{(0)}}\left(y^{(0)}\right)\left(x_j-x_j^{(0)}\right)-\sum_{j=1}^{d_2}\partial_{y_j}\varphi_{x^{(0)}}\left(y^{(0)}\right)\left(y_j-y_j^{(0)}\right)\right|\\
&\quad\leq\ \left|\varphi_x(y)-\varphi_{x^{(0)}}(y)-\sum_{j=1}^{d_1}\theta_{j,x^{(0)}}(y)\left(x_j-x_j^{(0)}\right)\right|\\
&\qquad+\left|\varphi_{x^{(0)}}(y)-\varphi_{x^{(0)}}\left(y^{(0)}\right)-\sum_{j=1}^{d_2}\partial_{y_j}\varphi_{x^{(0)}}\left(y^{(0)}\right)\left(y_j-y_j^{(0)}\right)\right|
\end{aligned}$$

$$+\sum_{j=1}^{d_1}\left|\theta_{j,x^{(0)}}(y)-\theta_{j,x^{(0)}}\left(y^{(0)}\right)\right|\left|x_j-x_j^{(0)}\right|$$
$$\leq \varepsilon\left|x-x^{(0)}\right|+\varepsilon\left|y-y^{(0)}\right|+\varepsilon\left|x-x^{(0)}\right|\leq 2\varepsilon\left(\left|x-x^{(0)}\right|+\left|y-y^{(0)}\right|\right),$$

for all $(x,y)\in\mathbb{R}^{d_1+d_2}$ such that $\left|x-x^{(0)}\right|<\eta$ and $\left|y-y^{(0)}\right|<\eta$, for some small enough $\eta>0$ (in the last inequality we used the continuity of the functions $\theta_{j,x^{(0)}}$). By the arbitrariness of $\left(x^{(0)},y^{(0)}\right)$, we obtain that $\varphi(x,y)$ is differentiable and $\partial_{x_j}\varphi(x,y)=\theta_{j,x}(y)$, i.e. $\partial_{x_j}\varphi(x,\cdot)=\partial_{x_j}f(x)$, $j=1,...,d_1$, and $\partial_{y_j}\varphi(x,y)=\partial_{y_j}\varphi_x(y)$, $j=1,...,d_2$. Similarly, one easily proves that φ is a $\mathcal{C}^\infty$ function and $D_x^\alpha D_y^\beta\varphi(x,y)=D_y^\beta\left(D_x^\alpha f(x)\right)(y)$ for all $\alpha\in\mathbb{N}^{d_1}$, $\beta\in\mathbb{N}^{d_2}$ and $(x,y)\in\mathbb{R}^{d_1+d_2}$.

Let $(t_j)\in\mathfrak{R}$ and $\alpha\in\mathbb{N}^{d_1}$, $\beta\in\mathbb{N}^{d_2}$ and $(x,y)\in\mathbb{R}^{d_1+d_2}$. Then

$$\frac{\left|D_x^\alpha D_y^\beta\varphi(x,y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}\leq\left\|\frac{D^\alpha f(x)}{T_\alpha M_\alpha}\right\|_{(t_j)}\leq\sup_{\alpha\in\mathbb{N}^{d_1}}\sup_{x\in\mathbb{R}^{d_1}}\left\|\frac{D^\alpha f(x)}{T_\alpha M_\alpha}\right\|_{(t_j)},$$

which is a seminorm in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1};\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)\right)$. Moreover, if $\varepsilon>0$, then there exists $K_1\subset\subset\mathbb{R}^{d_1}$ such that $\sup_{\alpha\in\mathbb{N}^{d_1}}\sup_{x\in\mathbb{R}^{d_1}\backslash K_1}\left\|\frac{D^\alpha f(x)}{T_\alpha M_\alpha}\right\|_{(t_j)}\leq\varepsilon$. In the proof of proposition 3.2.3 we proved that $A=\left\{\frac{D^\alpha f(x)}{T_\alpha M_\alpha}\middle|\,\alpha\in\mathbb{N}^{d_1},\,x\in\mathbb{R}^{d_1}\right\}$ is a precompact subset of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$. So, by lemma 3.2.2, for the chosen (t_j) and ε, there exists $K_2\subset\subset\mathbb{R}^{d_2}$such that

$$\frac{\left|D_y^\beta\left(D_x^\alpha f(x)\right)(y)\right|}{T_\alpha T_\beta M_\alpha M_\beta}\leq\varepsilon,\alpha\in\mathbb{N}^{d_1},\beta\in\mathbb{N}^{d_2},x\in\mathbb{R}^{d_1},y\in\mathbb{R}^{d_2}\backslash K_2.$$

Then

$$\frac{\left|D_x^\alpha D_y^\beta\varphi(x,y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}\leq\varepsilon,(x,y)\in\mathbb{R}^{d_1+d_2}\backslash K,K=K_1\times K_2,\alpha\in\mathbb{N}^{d_1},\beta\in\mathbb{N}^{d_2}.$$

Hence, we obtained that $\varphi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1+d_2}\right)$ and that the injection

$$f\mapsto\varphi,\ \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1};\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)\right)\to\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1+d_2}\right)$$

is continuous.

Now, let $\varphi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1+d_2}\right)$. Let f be the mapping $x\mapsto\varphi(x,\cdot)$, $\mathbb{R}^{d_1}\to\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$. Obviously, it is well defined. Note that $\partial_{x_j}\varphi(x,\cdot)\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$, $j=1,...,d_1$. Let $x^{(0)}\in\mathbb{R}^{d_1}$ and $(t_j)\in\mathfrak{R}$. Because of lemma 1.2.1, without losing generality, we can assume that (t_j) is such that $T_{n+k}\leq 2^{n+k}T_nT_k$, for all $n,k\in\mathbb{N}$. Then, by Taylor expanding $D^\beta\varphi(x,y)$ in $x^{(0)}$ up to order 1, we obtain

$$\left|x-x^{(0)}\right|^{-1}\left|D_y^\beta\varphi(x,y)-D_y^\beta\varphi\left(x^{(0)},y\right)-\sum_{j=1}^{d_1}\partial_{x_j}D_y^\beta\varphi\left(x^{(0)},y\right)\left(x_j-x_j^{(0)}\right)\right|$$
$$=\left|x-x^{(0)}\right|^{-1}\left|\sum_{|\gamma|=2}\frac{2}{\gamma!}\left(x-x^{(0)}\right)^\gamma\int_0^1(1-t)\partial_x^\gamma D_y^\beta\varphi\left((1-t)x^{(0)}+tx,y\right)dt\right|$$

$$\leq \; 2^{d_1+1}\left|x-x^{(0)}\right| \|\varphi\|_{(t_j/(2H))}T_{\beta+2}M_{\beta+2}(2H)^{-|\beta|-2}$$
$$\leq \; 2^{d_1+1}C\left|x-x^{(0)}\right| \|\varphi\|_{(t_j/(2H))}T_{\beta}M_{\beta},$$

where, in the last inequality, we used that $T_{2+|\beta|}M_{2+|\beta|}\leq c_0T_2M_2(2H)^{2+|\beta|}T_\beta M_\beta$ and put $C=c_0T_2M_2$ which depends only on (M_p) and (t_j). Because this holds for all $y\in\mathbb{R}^{d_2}$ and $\beta\in\mathbb{N}^{d_2}$, we get that f is differentiable as an $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$-valued function at $x^{(0)}$. Because $x^{(0)}$ was arbitrary, it follows that f is differentiable at all points and similarly, we can prove that f is infinitely differentiable as a $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)$-valued function. Moreover, $D^\alpha f(x)=D^\alpha_x\varphi(x,\cdot)$. Let $(t_j),(\tilde{t}_j)\in\mathfrak{R}$. By lemma 1.2.1, we can choose $(t'_j)\in\mathfrak{R}$ such that $t'_j\leq t_j$, $t'_j\leq\tilde{t}_j$ and $T'_{j+k}\leq 2^{j+k}T'_jT'_k$, for all $j,k\in\mathbb{N}$. Because

$$\frac{\left|D^\alpha_xD^\beta_y\varphi(x,y)\right|}{T_\alpha\tilde{T}_\beta M_\alpha M_\beta}\leq\frac{c_0(2H)^{|\alpha|+|\beta|}\left|D^\alpha_xD^\beta_y\varphi(x,y)\right|}{T'_{\alpha+\beta}M_{\alpha+\beta}}\leq c_0\|\varphi\|_{(t'_j/(2H))},$$

for all $\alpha\in\mathbb{N}^{d_1}$, $\beta\in\mathbb{N}^{d_2}$ and $(x,y)\in\mathbb{R}^{d_1+d_2}$, we get

$$\sup_{\alpha\in\mathbb{N}^{d_1}}\sup_{x\in\mathbb{R}^{d_1}}\left\|\frac{D^\alpha f(x)}{T_\alpha M_\alpha}\right\|_{(\tilde{t}_j)}\leq c_0\|\varphi\|_{(t'_j/(2H))}.$$

Let $(t_j),(\tilde{t}_j)\in\mathfrak{R}$, $\varepsilon>0$ be fixed and choose $(t'_j)\in\mathfrak{R}$ as above. Denote $t''_j=t'_j/(2H)$. Then there exists $K\subset\subset\mathbb{R}^{d_1+d_2}$ such that

$$\frac{\left|D^\alpha_xD^\beta_y\varphi(x,y)\right|}{T''_{\alpha+\beta}M_{\alpha+\beta}}\leq\frac{\varepsilon}{c_0},\alpha\in\mathbb{N}^{d_1},\beta\in\mathbb{N}^{d_2},(x,y)\in\mathbb{R}^{d_1+d_2}\backslash K.$$

Let K_1 be the projection of K on $\mathbb{R}^{d_1}$. Then K_1 is a compact subset of $\mathbb{R}^{d_1}$ and if $x\in\mathbb{R}^{d_1}\backslash K_1$ is fixed, by the above estimates, we have that $\left\|\frac{D^\alpha f(x)}{T_\alpha M_\alpha}\right\|_{(\tilde{t}_j)}\leq\varepsilon$, for all $\alpha\in\mathbb{N}^{d_1}$. Because $x\in\mathbb{R}^{d_1}\backslash K_1$ is arbitrary, we have $f\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1};\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)\right)$. From the above estimates, it follows that the mapping $\varphi\mapsto f$, $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1+d_2}\right)\to\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1};\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)\right)$, which is obviously injection, is continuous. Observe that the composition in both directions of the two mappings defined above is the identity mapping. So $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1+d_2}\right)\cong\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_1};\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d_2}\right)\right)$. □

3.3 Existence of Convolution of Two Roumieu Ultradistributions

We follow in this section the ideas for the convolution of Schwartz distributions but since in our case the topological properties are more delicate, the proofs are adequately more complicate.

We define an alternative l.c. topology on $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ such that its dual is algebraically isomorphic to $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ (c.f. [39] for the case of Schwartz distributions). Let

$g \in \mathcal{C}_0\left(\mathbb{R}^d\right)$ (the space of all continuous functions that vanish at infinity) and $(t_j) \in \mathfrak{R}$. The seminorms

$$p_{g,(t_j)}(\varphi) = \sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d} \frac{|g(x)D^\alpha\varphi(x)|}{T_\alpha M_\alpha},\ \varphi \in \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$$

generate l.c. topology on $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ and this space with this topology is denoted by $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$. Note that the inclusions $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty} \to \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$ and $\mathcal{D}^{\{M_p\}} \to \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty} \to \mathcal{E}^{\{M_p\}}$ are continuous.

Lemma 3.3.1. *Let $P(D) = \sum_\alpha c_\alpha D^\alpha$ be an ultradifferential operator of class $\{M_p\}$. Then $P(D)$ is a continuous mapping from $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$ to $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$.*

Proof. We know that c_α are constants such that for every $L > 0$ there exists $C > 0$ such that $\sup_\alpha |c_\alpha| M_\alpha / L^{|\alpha|} \leq C$. So, by lemma 3.4 of [28], there exists $(r_j) \in \mathfrak{R}$ and $C_1 > 0$ such that $\sup_\alpha |c_\alpha| R_\alpha M_\alpha \leq C_1$. Let $g \in \mathcal{C}_0$ and $(t_j) \in \mathfrak{R}$. Take $(s'_j) \in \mathfrak{R}$ such that $s'_j \leq r_j$ and $s'_j \leq t_j$ ($S_k \leq T_k, S_k \leq R_k$). By lemma 1.2.1, there exists $(s_j) \in \mathfrak{R}$ such that $s_j \leq s'_j$ and $S_{j+k} \leq 2^{j+k} S_j S_k$, for all $j, k \in \mathbb{N}$. Then, for $\varphi \in \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$, we have

$$\begin{aligned}
&\frac{|g(x)D^\alpha\left(P(D)\varphi(x)\right)|}{T_\alpha M_\alpha}\\
&\leq \sum_\beta \frac{|c_\beta|\left|g(x)D^{\alpha+\beta}\varphi(x)\right|}{T_\alpha M_\alpha} \leq C_1 p_{g,(s_j/(4H))}(\varphi) \sum_\beta \frac{S_{\alpha+\beta} M_{\alpha+\beta}}{(4H)^{|\alpha|+|\beta|} T_\alpha R_\beta M_\alpha M_\beta}\\
&\leq c_0 C_1 p_{g,(s_j/(4H))}(\varphi) \sum_\beta \frac{S_\alpha S_\beta}{2^{|\alpha|+|\beta|} T_\alpha R_\beta} \leq C_2 p_{g,(s_j/(4H))}(\varphi), \alpha \in \mathbb{N}^d, x \in \mathbb{R}^d.
\end{aligned}$$

Note that we can perform the same calculations as above without g. This implies that $P(D)$ is a continuous mapping from $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ into $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$. □

Denote by $\left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'$ the strong dual of $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$.

Lemma 3.3.2. *$\mathcal{D}^{\{M_p\}}$ is sequentially dense in $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$. In particular, the inclusion $\left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)' \to \mathcal{D}'^{\{M_p\}}$ is continuous.*

Proof. Let $\varphi \in \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$. Let $\chi \in \mathcal{D}^{\{M_p\}}$ be such that $\chi = 1$ on $\{x \in \mathbb{R}^d \,|\, |x| \leq 1\}$ and $\chi = 0$ on $\{x \in \mathbb{R}^d \,|\, |x| > 2\}$. For $n \in \mathbb{Z}_+$, denote by $\chi_n(x) = \chi(x/n)$ and $\varphi_n(x) = \chi_n(x)\varphi(x)$. Then, obviously $\varphi_n \in \mathcal{D}^{\{M_p\}}$. There exist $h > 0$ and $C_1 > 0$ such that $|D^\alpha \chi(x)| \leq C_1 h^{|\alpha|} M_\alpha$. For $g \in \mathcal{C}_0$ and $(t_j) \in \mathfrak{R}$, we have

$$\begin{aligned}
&\frac{|g(x)D^\alpha\varphi(x) - g(x)D^\alpha\varphi_n(x)|}{T_\alpha M_\alpha}\\
&\leq |1-\chi(x/n)||g(x)| \frac{|D^\alpha\varphi(x)|}{T_\alpha M_\alpha} + \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \binom{\alpha}{\beta} \frac{|g(x)|\left|D^\beta\chi(x/n) D^{\alpha-\beta}\varphi(x)\right|}{n^{|\beta|} T_\alpha M_\alpha}
\end{aligned}$$

$$\leq \ |1-\chi(x/n)||g(x)|\|\varphi\|_{(t_j)} + C_1 \frac{\|g\|_{L^\infty}\|\varphi\|_{(t_j/2)}}{n} \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \binom{\alpha}{\beta} \frac{h^{|\beta|}}{2^{|\alpha|-|\beta|}T_\beta}$$
$$\leq \ |1-\chi(x/n)||g(x)|\|\varphi\|_{(t_j)} + c_1 C_1 \frac{\|g\|_{L^\infty}\|\varphi\|_{(t_j/2)}}{n},$$

where c_1 is a constant such that $2^{|\beta|}h^{|\beta|}/T_\beta \leq c_1$ for all $\beta \in \mathbb{N}^d$. Because $g \in \mathcal{C}_0$, the last tends to zero when $n \to \infty$, uniformly for $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$. □

Lemma 3.3.3. *The bilinear mapping $(\varphi, \psi) \mapsto \varphi\psi$, $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty} \times \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty} \to \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$ is continuous.*

Proof. Let $g \in \mathcal{C}_0$ and $(t_j) \in \mathfrak{R}$. Obviously, $\tilde{g}(x) = \sqrt{|g(x)|} \in \mathcal{C}_0$. Let $\varphi, \psi \in \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$. Then

$$\begin{aligned} \frac{|g(x)D^\alpha(\varphi(x)\psi(x))|}{2^\alpha T_\alpha M_\alpha} &\leq \frac{1}{2^\alpha}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{|g(x)|\left|D^\beta\varphi(x)\right|\left|D^{\alpha-\beta}\psi(x)\right|}{T_\alpha M_\alpha} \\ &\leq C p_{\tilde{g},(t_j/2)}(\varphi) p_{\tilde{g},(t_j/2)}(\psi),\ x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d. \end{aligned}$$

□

Proposition 3.3.1. *The sets $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ and $\left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'$ are equal and the inclusion $\left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)' \to \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ is continuous.*

Proof. Since, $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ is continuously and densely injected in $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$, it follows that the injection $\left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)' \to \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ is continuous. Let $T \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. Then, by theorem 1 of [42], there exist an ultradifferential operator $P(D)$, of class $\{M_p\}$ and $F_1, F_2 \in L^1$ such that $T = P(D)F_1 + F_2$. Let $\varphi \in \mathcal{D}^{\{M_p\}}$. Then

$$|\langle P(D)F_1, \varphi\rangle| = |\langle F_1, P(-D)\varphi\rangle| = \left|\int_{\mathbb{R}^d} F_1(x)P(-D)\varphi(x)dx\right|.$$

Because $F_1 \in L^1 \subseteq \mathcal{M}^1$ (integrable measures), by proposition 1.2.1. of [39], there exists $g_1 \in \mathcal{C}_0$ such that $\left|\int_{\mathbb{R}^d} F_1(x)f(x)dx\right| \leq \|fg_1\|_{L^\infty}$, for all $f \in \mathcal{BC}$ ($\mathcal{BC}$ is the space of continuous bounded functions on $\mathbb{R}^d$). Let $(t_j) \in \mathfrak{R}$. We obtain, by lemma 3.3.1, that for some $\tilde{g}_1 \in \mathcal{C}_0$, $(t'_j) \in \mathfrak{R}$ and $C_1 > 0$,

$$|\langle P(D)F_1, \varphi\rangle| \leq \|g_1 P(-D)\varphi\|_{L^\infty} \leq p_{g_1,(t_j)}(P(-D)\varphi) \leq C_1 p_{\tilde{g}_1,(t'_j)}(\varphi).$$

Similarly, there exist $\tilde{g}_2 \in \mathcal{C}_0$, $(t''_j) \in \mathfrak{R}$ and $C_2 > 0$ such that $|\langle F_2, \varphi\rangle| \leq C_2 p_{\tilde{g}_2,(t''_j)}(\varphi)$, for all $\varphi \in \mathcal{D}^{\{M_p\}}$. By lemma 3.3.2, $T \in \left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'$. □

Lemma 3.3.4. *Let $S, T \in \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^d\right)$ are such that, for every $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $(S \otimes T)\varphi^\Delta \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$. Then $F : \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right) \to \left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^{2d}\right)$ defined by $F(\varphi) = (S \otimes T)\varphi^\Delta$ is linear and continuous.*

Proof. By proposition 3.3.1, $(S\otimes T)\varphi^{\Delta} \in \left(\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^{2d}\right)$ for every $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Because $\mathcal{D}^{\{M_p\}}$ is bornological, it is enough to prove that F maps bounded sets into bounded sets. Let B be a bounded set in $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Then, there exist $K \subset\subset \mathbb{R}^d$ and $h > 0$ such that $B \subseteq \mathcal{D}^{\{M_p\},h}_K$ and B is bounded there. It is obvious that, without losing generality, we can assume that $K = K_{\mathbb{R}^d}(0,q)$, for some $q > 0$. Take $\chi \in \mathcal{D}^{\{M_p\}}$ such that $\chi = 1$ on K and 0 outside some bounded neighbourhood of K. Then, for $\varphi \in B$ and $\psi \in \tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^{2d}\right)$, we have

$$\left\langle (S\otimes T)\varphi^{\Delta}, \psi\right\rangle = \left\langle (S\otimes T)\chi^{\Delta}\varphi^{\Delta}, \psi\right\rangle = \left\langle (S\otimes T)\chi^{\Delta}, \varphi^{\Delta}\psi\right\rangle,$$

where, in the last equality, we used that $(S\otimes T)\chi^{\Delta} \in \left(\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^{2d}\right)$ and $\varphi^{\Delta} \in \tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^{2d}\right)$ when $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Let $\psi \in B_1$ for some bounded set B_1 in $\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^{2d}\right)$. Let $g \in \mathcal{C}_0\left(\mathbb{R}^{2d}\right)$ and $(t_j) \in \mathfrak{R}$. Then, for $\varphi \in B$ and $\psi \in B_1$, we have

$$\begin{aligned}
&\frac{\left|g(x,y)D^{\alpha}_x D^{\beta}_y\left(\varphi^{\Delta}(x,y)\psi(x,y)\right)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}\\
&\quad\leq \sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{|g(x,y)|\left|D^{\gamma+\delta}\varphi(x+y)\right|\left|D^{\alpha-\gamma}_x D^{\beta-\delta}_y\psi(x,y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}\\
&\quad\leq p_{K,h}(\varphi)p_{g,(t_j/2)}(\psi)\sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{(2h)^{|\gamma|+|\delta|}}{2^{|\alpha|+|\beta|}T_{\gamma+\delta}} \leq Cp_{K,h}(\varphi)p_{g,(t_j/2)}(\psi).
\end{aligned}$$

Since $p_{K,h}(\varphi)$ and $p_{g,(t_j/2)}(\psi)$ are bounded when $\varphi \in B$ and $\psi \in B_1$, we obtain that the set $\left\{\theta \in \tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^{2d}\right) \,\middle|\, \theta = \varphi^{\Delta}\psi,\ \varphi \in B,\ \psi \in B_1\right\}$ is bounded in $\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^{2d}\right)$. This implies that $\left\langle (S\otimes T)\varphi^{\Delta}, \psi\right\rangle = \left\langle (S\otimes T)\chi^{\Delta}, \varphi^{\Delta}\psi\right\rangle$ is bounded, for $\varphi \in B$ and $\psi \in B_1$. Thus, $F(\varphi) = (S\otimes T)\varphi^{\Delta}$, $\varphi \in B$ is bounded. □

Definition 3.3.1. Let $S, T \in \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^d\right)$ be such that for every $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $(S\otimes T)\varphi^{\Delta} \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$. Define the convolution of S and T, $S*T \in \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^d\right)$, by

$$\langle S*T, \varphi\rangle = {}_{\left(\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'}\left\langle (S\otimes T)\varphi^{\Delta}, 1\right\rangle_{\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}};\ (1 \in \tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}).$$

Because of lemma 3.3.4, the mapping

$$\varphi \mapsto {}_{\left(\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'}\left\langle (S\otimes T)\varphi^{\Delta}, 1\right\rangle_{\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}},\ \mathcal{D}^{\{M_p\}} \to \mathbb{C}$$

is continuous. More precisely $\varphi \mapsto (S\otimes T)\varphi^{\Delta}$, $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right) \to \left(\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^{2d}\right)$ is continuous by lemma 3.3.4. Also, the identity mapping $\left(\tilde{\dot{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^{2d}\right) \to$

$\left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'_\sigma\left(\mathbb{R}^{2d}\right)$ is continuous (the latter is the dual of $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$ with the weak* topology) and because $1 \in \tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$, it is continuous functional on $\left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'_\sigma\left(\mathbb{R}^{2d}\right)$. Hence $S * T$ is well defined ultradistribution.

For every $a > 0$, define the space $\dot{\mathcal{B}}^{\{M_p\}}_a = \left\{\varphi \in \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right) \,\middle|\, \operatorname{supp}\varphi \subseteq \Delta_a\right\}$, where $\Delta_a = \left\{(x,y) \in \mathbb{R}^{2d} \middle| \, |x+y| \leq a\right\}$. With the seminorms $\|\varphi\|_{(t_j)}$ (now over $\mathbb{R}^{2d}$), $\dot{\mathcal{B}}^{\{M_p\}}_a$ becomes a l.c.s. Define the space $\dot{\mathcal{B}}^{\{M_p\}}_\Delta = \varinjlim_{a\to\infty} \dot{\mathcal{B}}^{\{M_p\}}_a$, where the inductive limit is strict; $\dot{\mathcal{B}}^{\{M_p\}}_\Delta$ is a l.c.s. because we have a continuous inclusion $\dot{\mathcal{B}}^{\{M_p\}}_\Delta \to \mathcal{E}^{\{M_p\}}$.

Lemma 3.3.5. *Let $a > 0$. Then $\mathcal{D}^{\{M_p\}}_{\Delta_a}\left(\mathbb{R}^{2d}\right) = \left\{\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right) \,|\, \operatorname{supp}\varphi \subseteq \Delta_a\right\}$ is sequentially dense in $\dot{\mathcal{B}}^{\{M_p\}}_a$.*

Proof. Let $\varphi \in \dot{\mathcal{B}}^{\{M_p\}}_a$. Take $\chi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ such that $\chi(x,y) = 1$ on $K_{\mathbb{R}^{2d}}(0,1)$ and $\chi(x,y) = 0$ out of $K_{\mathbb{R}^{2d}}(0,2)$. For $n \in \mathbb{Z}_+$, put $\chi_n(x,y) = \chi(x/n, y/n)$. Then $\varphi_n = \chi_n\varphi \in \mathcal{D}^{\{M_p\}}_{\Delta_a}\left(\mathbb{R}^{2d}\right)$ for all $n \in \mathbb{Z}_+$. Let $(t_j) \in \mathfrak{R}$. We have

$$\begin{aligned}
&\frac{\left|D^\alpha_x D^\beta_y \varphi(x,y) - D^\alpha_x D^\beta_y \varphi_n(x,y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}\\
&\quad\leq |1-\chi(x/n,y/n)| \frac{\left|D^\alpha_x D^\beta_y \varphi(x,y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}\\
&\qquad + \sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta\\ \gamma+\delta\neq 0}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} \frac{\left|D^\gamma_x D^\delta_y \chi(x/n,y/n)\right| \left|D^{\alpha-\gamma}_x D^{\beta-\delta}_y \varphi(x,y)\right|}{n^{|\gamma|+|\delta|}T_{\alpha+\beta}M_{\alpha+\beta}}\\
&\quad\leq |1-\chi(x/n,y/n)| \frac{\left|D^\alpha_x D^\beta_y \varphi(x,y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}} + \frac{C_1C_2\|\varphi\|_{(t_j/2)}}{n}
\end{aligned}$$

By lemma 1.2.3 and by the way we chose χ, it follows that the above two terms tend to zero uniformly in $(x,y) \in \mathbb{R}^{2d}$ and $\alpha, \beta \in \mathbb{N}^d$ when $n \to \infty$. □

Because $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right) = \bigcup_{a\in\mathbb{R}_+} \mathcal{D}^{\{M_p\}}_{\Delta_a}\left(\mathbb{R}^{2d}\right)$, by lemma 3.3.5, it follows that $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ is dense in $\dot{\mathcal{B}}^{\{M_p\}}_\Delta$. Moreover, one easily checks that the inclusions $\dot{\mathcal{B}}^{\{M_p\}}_\Delta \to \mathcal{E}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ and $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right) \to \dot{\mathcal{B}}^{\{M_p\}}_\Delta$ are continuous, hence, the inclusion $\left(\dot{\mathcal{B}}^{\{M_p\}}_\Delta\right)' \to \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ is continuous ($\left(\dot{\mathcal{B}}^{\{M_p\}}_\Delta\right)'$ is the strong dual of $\dot{\mathcal{B}}^{\{M_p\}}_\Delta$).

Theorem 3.3.1. *Let $S, T \in \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^d\right)$. The following statements are equivalent:*

i) the convolution of S and T exists;

ii) $S \otimes T \in \left(\dot{\mathcal{B}}^{\{M_p\}}_\Delta\right)'$;

iii) *for all* $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{S}\right)T \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^d\right)$ *and for every compact subset* K *of* $\mathbb{R}^d$, $(\varphi,\chi) \mapsto \left\langle\left(\varphi * \check{S}\right)T,\chi\right\rangle$, $\mathcal{D}^{\{M_p\}}_K \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \to \mathbb{C}$, *is a continuous bilinear mapping;*

iv) *for all* $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{T}\right)S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^d\right)$ *and for every compact subset* K *of* $\mathbb{R}^d$, $(\varphi,\chi) \mapsto \left\langle\left(\varphi * \check{T}\right)S,\chi\right\rangle$, $\mathcal{D}^{\{M_p\}}_K \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \to \mathbb{C}$, *is a continuous bilinear mapping;*

v) *for all* $\varphi,\psi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{S}\right)(\psi * T) \in L^1\left(\mathbb{R}^d\right)$.

Proof. *i)* $\Rightarrow$ *ii)*. Let $a>0$. Choose $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ such that $\varphi = 1$ on $K_{\mathbb{R}^d}(0,a)$ and $\varphi = 0$ on the complement of some bounded neighbourhood of this set. Then, there exist $(t_j) \in \mathfrak{R}$ and $C>0$ such that $\left|\left\langle(S\otimes T)\varphi^{\Delta},\psi\right\rangle\right| \leq C\|\psi\|_{(t_j)}$ for all $\psi \in \mathcal{D}^{\{M_p\}}_{\Delta_a}\left(\mathbb{R}^{2d}\right) \subseteq \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$. Since $\left\langle(S\otimes T)\varphi^{\Delta},\psi\right\rangle = \left\langle S\otimes T,\varphi^{\Delta}\psi\right\rangle = \langle S\otimes T,\psi\rangle$, it follows that $|\langle S\otimes T,\psi\rangle| \leq C\|\psi\|_{(t_j)}$ for all $\psi \in \mathcal{D}^{\{M_p\}}_{\Delta_a}\left(\mathbb{R}^{2d}\right)$. By lemma 3.3.5, it follows that $S\otimes T$ is a continuous linear mapping from $\dot{\mathcal{B}}^{\{M_p\}}_a$ to $\mathbb{C}$. Hence $S\otimes T \in \left(\dot{\mathcal{B}}^{\{M_p\}}_{\Delta}\right)'$.

ii) $\Rightarrow$ *i)*. Let $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ with support in $K_{\mathbb{R}^d}(0,a)$ for some $a>0$. Then, for that a, there exist $(t_j) \in \mathfrak{R}$ and $C>0$ such that $|\langle S\otimes T,\psi\rangle| \leq C\|\psi\|_{(t_j)}$ for all $\psi \in \dot{\mathcal{B}}^{\{M_p\}}_a$. Let $\psi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$. Then $\varphi^{\Delta}\psi \in \mathcal{D}^{\{M_p\}}_{\Delta_a} \subseteq \dot{\mathcal{B}}^{\{M_p\}}_a$ and by lemma 1.2.4

$$\left|\left\langle(S\otimes T)\varphi^{\Delta},\psi\right\rangle\right| = \left|\left\langle S\otimes T,\varphi^{\Delta}\psi\right\rangle\right| \leq C\left\|\varphi^{\Delta}\psi\right\|_{(t_j)} \leq \tilde{C}\left\|\psi\right\|_{(t'_j)},$$

for some $(t'_j) \in \mathfrak{R}$ and $\tilde{C}>0$ that depend on φ and (t_j). Thus, $(S\otimes T)\varphi^{\Delta} \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$.

i) and *ii)* $\Rightarrow$ *iii)*. Let F and K_1 be compact subsets of $\mathbb{R}^d$. Take K to be a compact set in $\mathbb{R}^d$ such that $F \subset\subset \operatorname{int} K$ and let $\varphi \in \mathcal{D}^{\{M_p\}}_{K_1}$, $\psi \in \mathcal{D}^{\{M_p\}}_K$ and $\chi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Then

$$\left\langle\left(\left(\varphi * \check{S}\right)T\right) * \psi,\chi\right\rangle = \left\langle S\otimes T,\varphi(x+y)\left(\check{\psi} * \chi\right)(y)\right\rangle.$$

There exists $a>0$ such that $\operatorname{supp}\varphi^{\Delta}(x,y)\left(\check{\psi} * \chi\right)(y) \subseteq \Delta_a$, for all $\varphi \in \mathcal{D}^{\{M_p\}}_{K_1}$, $\psi \in \mathcal{D}^{\{M_p\}}_K$ and $\chi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Then, for that a, there exist $(t_j) \in \mathfrak{R}$ and $C_1>0$ such that $|\langle S\otimes T,\theta\rangle| \leq C_1\|\theta\|_{(t_j)}$ for all $\theta \in \dot{\mathcal{B}}^{\{M_p\}}_a$. So we obtain

$$\left|\left\langle\left(\left(\varphi * \check{S}\right)T\right) * \psi,\chi\right\rangle\right| = C_1\left\|\varphi^{\Delta}(x,y)\left(\check{\psi} * \chi\right)(y)\right\|_{(t_j)}.$$

We have

$$\begin{aligned}
\frac{\left|D^{\alpha}_x D^{\beta}_y\left(\varphi^{\Delta}(x,y)\left(\check{\psi} * \chi\right)(y)\right)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}} &\leq \sum_{\delta\leq\beta}\binom{\beta}{\delta}\frac{\left|D^{\alpha+\beta-\delta}\varphi(x+y)\right|\left|D^{\delta}\left(\check{\psi} * \chi\right)(y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}\\
&\leq \|\varphi\|_{(t_j/2)}\sum_{\delta\leq\beta}\binom{\beta}{\delta}\frac{\left|D^{\delta}\left(\check{\psi} * \chi\right)(y)\right|}{2^{|\alpha|+|\beta|-|\delta|}T_{\delta}M_{\delta}}\\
&\leq |K|\|\varphi\|_{(t_j/2)}\|\psi\|_{(t_j/2)}\|\chi\|_{L^{\infty}}.
\end{aligned}$$

Hence $\left|\left\langle\left(\left(\varphi*\check{S}\right)T\right)*\psi,\chi\right\rangle\right|\leq C_1|K|\|\varphi\|_{(t_j/2)}\|\psi\|_{(t_j/2)}\|\chi\|_{L^\infty}$, for all $\varphi\in\mathcal{D}^{\{M_p\}}_{K_1}$, $\psi\in\mathcal{D}^{\{M_p\}}_K$ and $\chi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Thus, $\left(\left(\varphi*\check{S}\right)T\right)*\psi\in\mathcal{M}^1$. Since $\left(\left(\varphi*\check{S}\right)T\right)*\psi\in\mathcal{E}^{\{M_p\}}\left(\mathbb{R}^d\right)$, it follows that $\left(\left(\varphi*\check{S}\right)T\right)*\psi\in L^1$. Let $\varphi\in\mathcal{D}^{\{M_p\}}_{K_1}$ be fixed. Then the mapping $\psi\mapsto\left(\left(\varphi*\check{S}\right)T\right)*\psi$, $\mathcal{D}^{\{M_p\}}_K\to\mathcal{D}'^{\{M_p\}}$ is continuous and has a closed graph. Since $\left(\left(\varphi*\check{S}\right)T\right)*\psi\in L^1$, this above mapping from $\mathcal{D}^{\{M_p\}}_K$ to L^1, has a closed graph as well and so, it is continuous. (L^1 is a (B) - space and $\mathcal{D}^{\{M_p\}}_K$ is a (DFS) - space.) Hence, there exist $(r_j)\in\mathfrak{R}$ and $C_1>0$ such that

$$\left\|\left(\left(\varphi*\check{S}\right)T\right)*\psi\right\|_{L^1}\leq C_1\|\psi\|_{K,(r_j)}. \tag{3.4}$$

By lemma 1.2.1, we can assume, without losing generality, that (r_j) is such that $R_{j+k}\leq 2^{j+k}R_jR_k$, for all $j,k\in\mathbb{N}$. Let $r'_j=r_j/(2H)$ and $\theta\in\mathcal{D}^{M_p}_{F,(r'_j)}$. Then, there exist $\psi_n\in\mathcal{D}^{\{M_p\}}_K$, $n\in\mathbb{Z}_+$ such that $\psi_n\to\theta$ in $\mathcal{D}^{M_p}_{K,(r_j)}$. The mapping $\theta\mapsto\left(\left(\varphi*\check{S}\right)T\right)*\theta$, $\mathcal{D}^{M_p}_{K,(r_j)}\to\mathcal{D}'^{\{M_p\}}$ is continuous. So, if $\psi_n\in\mathcal{D}^{\{M_p\}}_K$ tends to $\theta\in\mathcal{D}^{M_p}_{F,(r'_j)}$ in the topology of $\mathcal{D}^{M_p}_{K,(r_j)}$ then $\left(\left(\varphi*\check{S}\right)T\right)*\psi_n\to\left(\left(\varphi*\check{S}\right)T\right)*\theta$ in $\mathcal{D}'^{\{M_p\}}$. By (3.4), we have $\left\|\left(\left(\varphi*\check{S}\right)T\right)*\psi_n\right\|_{L^1}\leq C_1\|\psi_n\|_{K,(r_j)}$. So, $\left(\left(\varphi*\check{S}\right)T\right)*\psi_n$ is a Cauchy sequence in L^1, hence it must be convergent and it must converge to $\left(\left(\varphi*\check{S}\right)T\right)*\theta$, because it converge to that ultradistribution in $\mathcal{D}'^{\{M_p\}}$. Consequently, $\left(\left(\varphi*\check{S}\right)T\right)*\theta\in L^1$ for all $\theta\in\mathcal{D}^{M_p}_{F,(r'_j)}$ and if we let $n\to\infty$ in the last inequality, we get $\left\|\left(\left(\varphi*\check{S}\right)T\right)*\theta\right\|_{L^1}\leq C_1\|\theta\|_{K,(r_j)}$, for all $\theta\in\mathcal{D}^{M_p}_{F,(r'_j)}$. By corollary 1 of [42], it follows that $\left(\varphi*\check{S}\right)T\in\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. Now, we prove that the mapping $(\varphi,\chi)\mapsto\left\langle\left(\varphi*\check{S}\right)T,\chi\right\rangle$, $\mathcal{D}^{\{M_p\}}_K\left(\mathbb{R}^d\right)\times\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\to\mathbb{C}$, is continuous, for every compact set K. There exists $a>0$ such that $K\subset\subset_{\mathbb{R}^d}(0,a)$. Take $\theta\in\mathcal{D}^{\{M_p\}}$ such that $\theta=1$ on $K_{\mathbb{R}^d}(0,a)$ and $\theta=0$ on the complement of some bounded neighbourhood of this ball. Then $\varphi^\Delta\theta^\Delta=\varphi^\Delta$ for all $\varphi\in\mathcal{D}^{\{M_p\}}_K$. Let $\varphi\in\mathcal{D}^{\{M_p\}}_K$ and $\chi,\psi_n\in\mathcal{D}^{\{M_p\}}$, $n\in\mathbb{Z}_+$, such that $\psi_n\to\delta$, when n tends to infinity, in $\mathcal{E}'^{\{M_p\}}$. Then

$$\begin{aligned}\left\langle\left(\varphi*\check{S}\right)T,\chi\right\rangle &= \lim_{n\to\infty}\left\langle\left(\left(\varphi*\check{S}\right)T\right)*\psi_n,\chi\right\rangle\\ &= \lim_{n\to\infty}\left\langle S\otimes T,\varphi^\Delta(x,y)\left(\check{\psi}_n*\chi\right)(y)\right\rangle=\left\langle S\otimes T,\varphi^\Delta(x,y)\chi(y)\right\rangle\\ &= \left\langle S\otimes T,\varphi^\Delta(x,y)\theta^\Delta(x,y)\chi(y)\right\rangle=\left\langle (S\otimes T)\varphi^\Delta,1_x\otimes\chi(y)\right\rangle,\end{aligned}$$

where the last tow terms are in the sense of the duality $\left\langle\dot{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty},\left(\dot{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\right\rangle$. Now, let $\chi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$ and take $\psi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ such that $\psi=1$ on $K_{\mathbb{R}^d}(0,1)$ and $\psi=0$ out of $K_{\mathbb{R}^d}(0,2)$. Put $\psi_n(x)=\psi(x/n)$, $n\in\mathbb{Z}_+$, and $\chi_n(x)=\psi_n(x)\chi(x)$. Then, one easily checks that $1_x\otimes\chi_n(y)\to 1_x\otimes\chi(y)$ in $\dot{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^{2d}\right)$, $n\to\infty$. Because $\left(\varphi*\check{S}\right)T\in\left(\dot{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^d\right)=\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^d\right)$ and $(S\otimes T)\varphi^\Delta\in\left(\dot{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^{2d}\right)=\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$ (c.f. proposition 3.3.1), we have

$$\left\langle\left(\varphi*\check{S}\right)T,\chi\right\rangle = \lim_{n\to\infty}\left\langle\left(\varphi*\check{S}\right)T,\chi_n\right\rangle=\lim_{n\to\infty}\left\langle(S\otimes T)\varphi^\Delta,1_x\otimes\chi_n(y)\right\rangle$$

$$= \left\langle (S\otimes T)\varphi^{\Delta}, 1_x\otimes\chi(y)\right\rangle, \varphi\in\mathcal{D}_K^{\{M_p\}}, \chi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right).$$

Also $(S\otimes T)\theta^{\Delta}\in\left(\tilde{\tilde{\mathcal{D}}}_{L^\infty}^{\{M_p\}}\right)'\left(\mathbb{R}^{2d}\right)$ and by the construction of θ, $(S\otimes T)\theta^{\Delta}\varphi^{\Delta}=(S\otimes T)\varphi^{\Delta}$. Hence

$$\left\langle\left(\varphi*\check{S}\right)T,\chi\right\rangle=\left\langle(S\otimes T)\theta^{\Delta},\varphi^{\Delta}(x,y)\chi(y)\right\rangle, \varphi\in\mathcal{D}_K^{\{M_p\}}, \chi\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}(\mathbb{R}^d). \quad (3.5)$$

Since the bilinear mapping

$$(\varphi(x),\chi(y))\mapsto\varphi^{\Delta}(x,y)\chi(y), \mathcal{D}_K^{\{M_p\}}\times\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\to\tilde{\tilde{\mathcal{D}}}_{L^\infty}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$$

is continuous and $(S\otimes T)\theta^{\Delta}\in\left(\tilde{\tilde{\mathcal{D}}}_{L^\infty}^{\{M_p\}}\right)'\left(\mathbb{R}^{2d}\right)$, it follows that the bilinear mapping

$$(\varphi(x),\chi(y))\mapsto\left\langle(S\otimes T)\theta^{\Delta},\varphi^{\Delta}(x,y)\chi(y)\right\rangle, \mathcal{D}_K^{\{M_p\}}\times\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\to\mathbb{C}$$

is continuous. Hence, by (3.5), we obtain the desired continuity.

i) and *ii)* $\Rightarrow$ *iv)* The proof is analogous to *ii)* $\Rightarrow$ *iii)*.

ii) $\Rightarrow$ *v)*. Let $K\subset\subset\mathbb{R}^d$ and let $\varphi,\psi\in\mathcal{D}_K^{\{M_p\}}$, $\chi\in\mathcal{D}^{\{M_p\}}$. Then

$$\begin{aligned}\left\langle\left(\varphi*\check{S}\right)(\psi*T),\chi\right\rangle &= \langle\langle S(x),\varphi(x+t)\rangle\langle T(y),\psi(t-y)\rangle,\chi(t)\rangle\\ &= \langle((S\otimes T)(x,y))\otimes 1_t,\varphi(x+t)\psi(t-y)\chi(t)\rangle\\ &= \left\langle(S\otimes T)(x,y),\int_{\mathbb{R}^d}\varphi(x+t)\psi(t-y)\chi(t)dt\right\rangle.\end{aligned}$$

Let $\theta(x,y)=\int_{\mathbb{R}^d}\varphi(x+t)\psi(t-y)\chi(t)dt$. Let $a>0$ be such that $K\subset\subset K_{\mathbb{R}^d}(0,a)$. We will prove that $\operatorname{supp}\theta\subseteq\Delta_{2a}$. Let (x,y) be such that $|x+y|>2a$. Then we have

$$2a<|x+y|\leq|x+t|+|t-y|, \forall t\in\mathbb{R}^d.$$

Let $t_0\in\mathbb{R}^d$ be fixed. Then $|x+t_0|>a$ or $|t_0-y|>a$. If $|x+t_0|>a$, then $\varphi(x+t_0)=0$ and if $|t_0-y|>a$, then $\psi(t_0-y)=0$. In any case $\varphi(x+t_0)\psi(t_0-y)=0$ and this holds for arbitrary $t_0\in\mathbb{R}^d$. So, we obtain that $\operatorname{supp}\theta\subseteq\Delta_{2a}$. Now, because $\varphi,\psi\in\mathcal{D}_K^{\{M_p\}}$, there exist $h_1,h_2,C_1,C_2>0$ such that $|D^{\alpha}\varphi(x)|\leq C_1h_1^{|\alpha|}M_{\alpha}$ and $|D^{\alpha}\psi(x)|\leq C_2h_2^{|\alpha|}M_{\alpha}$. Let $(t_j)\in\mathfrak{R}$. We have

$$\begin{aligned}\frac{\left|D_x^{\alpha}D_y^{\beta}\theta(x,y)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}} &\leq \int_{\mathbb{R}^d}\frac{|D^{\alpha}\varphi(x+t)|\left|D^{\beta}\psi(t-y)\right||\chi(t)|}{T_{\alpha+\beta}M_{\alpha+\beta}}dt\\ &\leq \|\chi\|_{L^\infty}\int_K\frac{|D^{\alpha}\varphi(x+y+t)|\left|D^{\beta}\psi(t)\right|}{T_{\alpha+\beta}M_{\alpha+\beta}}dt\leq C_1C_2C_3|K|\|\chi\|_{L^\infty}.\end{aligned}$$

It follows that the mapping $\chi\mapsto\int_{\mathbb{R}^d}\varphi(x+t)\psi(t-y)\chi(t)dt$, $\mathcal{C}_0\left(\mathbb{R}^d\right)\to\dot{\mathcal{B}}_{2a}^{\{M_p\}}$ is continuous, i.e. this mapping is continuous as a mapping from $\mathcal{C}_0\left(\mathbb{R}^d\right)$ to $\dot{\mathcal{B}}_{\Delta}^{\{M_p\}}$. But, $S\otimes T\in\left(\dot{\mathcal{B}}_{\Delta}^{\{M_p\}}\right)'$, so the mapping

$$\chi\mapsto\left\langle(S\otimes T)(x,y),\int_{\mathbb{R}^d}\varphi(x+t)\psi(t-y)\chi(t)dt\right\rangle, \mathcal{C}_0\left(\mathbb{R}^d\right)\to\mathbb{C},$$

is continuous. Since $\left(\varphi * \check{S}\right)(\psi * T) \in \mathcal{M}^1$ and it belongs to $\mathcal{E}^{\{M_p\}}$, it follows $\left(\varphi * \check{S}\right)(\psi * T) \in L^1$.

iii) $\Rightarrow$ *i)*. Let $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ and let $K \subset\subset \mathbb{R}^d$ such that $\operatorname{supp}\varphi \subset\subset \operatorname{int}K$. By the assumption, the bilinear mapping $G : \mathcal{D}^{\{M_p\}}_K \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \to \mathbb{C}$, $G(\psi, \chi) = \left\langle \left((\psi\varphi) * \check{S}\right) T, \chi\right\rangle$, is continuous. Hence G extends to a linear continuous mapping, $\hat{G}$, on the completion of the tensor product $\mathcal{D}^{\{M_p\}}_K \hat{\otimes} \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$ ($\mathcal{D}^{\{M_p\}}_K$ is nuclear and the π topology coincides with the ϵ topology). Let $\theta \in \mathcal{D}^{\{M_p\}}_K$ be a function such that $\theta = 1$ on $\operatorname{supp}\varphi$. Then, the mapping $F : \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \to \mathcal{D}^{\{M_p\}}_K$, $F(\chi) = \theta\chi$ is continuous. So, the mapping

$$F \otimes_\epsilon \mathrm{Id} : \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \otimes_\epsilon \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \to \mathcal{D}^{\{M_p\}}_K \otimes_\epsilon \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$$

is continuous and by proposition 3.2.4, we have the continuous extension

$$F \hat{\otimes}_\epsilon \mathrm{Id} : \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right) \to \mathcal{D}^{\{M_p\}}_K \hat{\otimes}_\epsilon \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right).$$

Thus, we have the continuous mapping

$$\tilde{G} : \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right) \xrightarrow{F\hat{\otimes}_\epsilon Id} \mathcal{D}^{\{M_p\}}_K \hat{\otimes}_\epsilon \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \xrightarrow{\hat{G}} \mathbb{C},$$

i.e. $\tilde{G} \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$. For $\psi, \chi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$,

$$\begin{aligned} \tilde{G}(\psi \otimes \chi) &= \hat{G}\left(F(\psi) \otimes \chi\right) = G(\theta\psi, \chi) = \left\langle \left((\theta\psi\varphi) * \check{S}\right) T, \chi\right\rangle \\ &= \langle (S \otimes T)\varphi^\Delta, \psi(x+y)\chi(y)\rangle. \end{aligned}$$

Let Θ be the linear transformation $\Theta(x, y) = (x + y, y)$ and denote by $\tilde{\Theta}$ the linear operator $\tilde{\Theta} f(x', y') = f \circ \Theta(x, y) = f(x + y, y)$. It yields an isomorphism of $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ and of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$, hence, the transposed mapping ${}^t\tilde{\Theta}$ is an isomorphism of $\mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ and of $\tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$. Thus

$$\tilde{G}(\psi \otimes \chi) = \left\langle (S \otimes T)\varphi^\Delta, \tilde{\Theta}(\psi \otimes \chi)\right\rangle = \left\langle {}^t\tilde{\Theta}\left((S \otimes T)\varphi^\Delta\right), \psi \otimes \chi\right\rangle.$$

Because $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right) \otimes \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ is dense in $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$, $\tilde{G} = {}^t\tilde{\Theta}\left((S \otimes T)\varphi^\Delta\right)$ in $\mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$. $\tilde{G} \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$, so ${}^t\tilde{\Theta}\left((S \otimes T)\varphi^\Delta\right) \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$, hence $(S \otimes T)\varphi^\Delta \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^{2d}\right)$.

iv) $\Rightarrow$ *i)* The proof is analogous to the previous one.

v) $\Rightarrow$ *i)*. Let K and K_1 be compact subsets of $\mathbb{R}^d$ such that $K_1 \subset\subset \operatorname{int}K$ and both satisfy the cone property. Observe the mapping $G : \mathcal{D}^{\{M_p\}}_K \times \mathcal{D}^{\{M_p\}}_K \to \mathcal{M}^1$, $G(\varphi, \psi) = \left(\varphi * \check{S}\right)(\psi * T)$. Note that the mapping $\varphi \mapsto \left(\varphi * \check{S}\right)(\psi * T)$ is continuous from $\mathcal{D}^{\{M_p\}}_K$ to $\mathcal{D}'^{\{M_p\}}$ and hence, it has a closed graph. Because $\mathcal{M}^1$ is a (B) - space and $\mathcal{D}^{\{M_p\}}_K$ is a (DFS) - space, from the closed graph theorem, it follows that G is separately continuous in φ and similarly in ψ. $\mathcal{D}^{\{M_p\}}_K$ is a (DFS) - space, hence G is continuous. It can be extended to a continuous mapping, $\hat{G}$, on the completion of the tensor product $\mathcal{D}^{\{M_p\}}_K \hat{\otimes} \mathcal{D}^{\{M_p\}}_K$. Since $\mathcal{D}^{\{M_p\}}_K \hat{\otimes} \mathcal{D}^{\{M_p\}}_K \cong$

$\mathcal{D}^{\{M_p\}}_{K\times K}$ (theorem 1.2.8), the mapping $\mathcal{D}^{\{M_p\}}_{K\times K}\times\mathcal{C}_0\left(\mathbb{R}^d\right)\rightarrow\mathbb{C}$, $(f,\theta)\mapsto\langle\hat{G}(f),\theta\rangle$, is continuous because it is the composition of the mappings

$$\mathcal{D}^{\{M_p\}}_{K\times K}\times\mathcal{C}_0\left(\mathbb{R}^d\right)\xrightarrow{\hat{G}\times Id}\mathcal{M}^1\left(\mathbb{R}^d\right)\times\mathcal{C}_0\left(\mathbb{R}^d\right)\xrightarrow{\langle\cdot,\cdot\rangle}\mathbb{C},$$

where the last mapping is the duality of $\mathcal{C}_0$ and $\mathcal{M}^1$. Hence, the mapping $\mathcal{D}^{\{M_p\}}_{K\times K}\times\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)\rightarrow\mathbb{C}$, $(f,\chi)\mapsto\langle\hat{G}(f),\chi\rangle$, is continuous. So, this mapping can be extended to $\tilde{G}$ on the completion of the tensor product $\mathcal{D}^{\{M_p\}}_{K\times K}\hat{\otimes}\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$. Take $\theta\in\mathcal{D}^{\{M_p\}}_{K}$ such that $\theta=1$ on K_1 and put $\theta_1(x)=\theta(x)$ and $\theta_2(y)=\theta(y)$. Because $\psi\mapsto\theta_1\theta_2\psi$, $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)\rightarrow\mathcal{D}^{\{M_p\}}_{K\times K}$, is continuous, the mapping $\psi\otimes\varphi\mapsto\theta_1\theta_2\psi\otimes\varphi$, $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)\otimes_\epsilon\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d}\right)\rightarrow\mathcal{D}^{\{M_p\}}_{K\times K}\otimes_\epsilon\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{d}\right)$ is continuous and it extends to a continuous mapping V on the completion of these spaces. By proposition 3.2.4, the composition $\tilde{G}\circ V$ is continuous from $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{3d}\right)$ to $\mathbb{C}$. That means that there exist $(t_j)\in\mathfrak{R}$ and $C_1>0$ such that $\left|\tilde{G}\circ V(f)\right|\leq C_1\|f\|_{(t_j)}$, for all $f\in\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{3d}\right)$. Let $\varphi,\psi,\chi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, then

$$\begin{aligned}\tilde{G}\circ V(\varphi\otimes\psi\otimes\chi) &= \tilde{G}(\theta_1\varphi\otimes\theta_2\psi\otimes\chi)=\left\langle\left((\theta_1\varphi)*\check{S}\right)((\theta_2\psi)*T),\chi\right\rangle\\ &= \langle(S(x)\otimes T(y))\otimes 1_t,\theta_1(x+t)\varphi(x+t)\theta_2(t-y)\psi(t-y)\chi(t)\rangle.\end{aligned}$$

By nuclearity and theorem 1.2.8, we have continuous dense inclusions

$$\left(\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)\otimes\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)\otimes\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)\rightarrow\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{3d}\right).$$

So, for $\tilde{\varphi}\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{3d}\right)$, there exists a net $\tilde{\varphi}_\nu\in\left(\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)\otimes\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)\right)\otimes\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ such that $\tilde{\varphi}_\nu\rightarrow\tilde{\varphi}$ in $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{3d}\right)$. But then the convergence holds in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{3d}\right)$ and, for $\tilde{\varphi}\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{3d}\right)$,

$$\tilde{G}\circ V(\tilde{\varphi})=\langle(S(x)\otimes T(y))\otimes 1_t,\theta_1(x+t)\theta_2(t-y)\tilde{\varphi}(x+t,t-y,t)\rangle.$$

Let $\varphi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $K_1=K_{\mathbb{R}^d}(0,a)$, where $a>0$ is such that $\operatorname{supp}\varphi\subset\subset\operatorname{int}K_1$. Let $K=K_{\mathbb{R}^d}(0,a+2)$ and $K'=K_{\mathbb{R}^d}(0,a+1)$. Choose $\theta\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ to be equal to 1 on K' and has a support in $\operatorname{int}K$. Take $\mu\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ with support in the open unit ball and $\displaystyle\int_{\mathbb{R}^d}\mu(x)dx=1$. Let $\chi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ be arbitrary and consider the function $f(x,y,t)=\varphi(x-y)\chi(x-t,t-y)\mu(x)$. Obviously $f\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{3d}\right)$ and

$$\begin{aligned}&\tilde{G}\circ V(f)\\ &\quad= \langle(S(x)\otimes T(y))\otimes 1_t,\theta_1(x+t)\theta_2(t-y)f(x+t,t-y,t)\rangle\\ &\quad= \langle(S(x)\otimes T(y))\otimes 1_t,\theta_1(x+t)\theta_2(t-y)\varphi(x+y)\chi(x,y)\mu(x+t)\rangle.\end{aligned}$$

By construction $\theta_1(x+t)\mu(x+t)=\mu(x+t)$, for all $x,t\in\mathbb{R}^d$. Let $x,y,t\in\mathbb{R}^d$ are such that $\varphi(x+y)\mu(x+t)\neq 0$. Then $|x+y|<a$ and $|x+t|<1$. So, $|t-y|\leq|x+y|+|x+t|<a+1$, hence $\theta_2(t-y)=1$. We have

$$\tilde{G}\circ V(f) = \langle(S(x)\otimes T(y))\otimes 1_t,\varphi(x+y)\chi(x,y)\mu(x+t)\rangle$$

$$
\begin{aligned}
&= \left\langle S(x)\otimes T(y), \varphi(x+y)\chi(x,y)\int_{\mathbb{R}^d}\mu(x+t)dt\right\rangle \\
&= \left\langle (S(x)\otimes T(y))\varphi^{\Delta}(x,y), \chi(x,y)\right\rangle .
\end{aligned}
$$

We estimate the derivatives of f as follows

$$
\begin{aligned}
&\frac{\left|D_x^{\alpha}D_y^{\beta}D_t^{\gamma}f(x,y,t)\right|}{T_{\alpha+\beta+\gamma}M_{\alpha+\beta+\gamma}} \\
&\leq \frac{1}{T_{\alpha+\beta+\gamma}M_{\alpha+\beta+\gamma}}\sum_{\beta'\leq\beta}\sum_{\gamma'\leq\gamma}\sum_{\alpha'\leq\alpha}\sum_{\alpha''\leq\alpha'}\binom{\beta}{\beta'}\binom{\gamma}{\gamma'}\binom{\alpha}{\alpha'}\binom{\alpha'}{\alpha''} \\
&\quad \cdot\left|D^{\alpha-\alpha'+\beta-\beta'}\varphi(x-y)\right|\left|D_x^{\alpha''+\gamma'}D_y^{\beta'+\gamma-\gamma'}\chi(x-t,t-y)\right|\left|D^{\alpha'-\alpha''}\mu(x)\right| \\
&\leq \|\varphi\|_{(t_j/4)}\|\mu\|_{(t_j/4)}\|\chi\|_{(t_j/4)}\sum_{\beta'\leq\beta}\sum_{\gamma'\leq\gamma}\sum_{\alpha'\leq\alpha}\sum_{\alpha''\leq\alpha'}\binom{\beta}{\beta'}\binom{\gamma}{\gamma'}\binom{\alpha}{\alpha'}\binom{\alpha'}{\alpha''}4^{-|\alpha|-|\beta|-|\gamma|} \\
&\leq \|\varphi\|_{(t_j/4)}\|\mu\|_{(t_j/4)}\|\chi\|_{(t_j/4)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left|\left\langle (S(x)\otimes T(y))\varphi^{\Delta}(x,y), \chi(x,y)\right\rangle\right| &= \left|\tilde{G}\circ V(f)\right| \leq C_1\|f\|_{(t_j)} \\
&\leq C_1\|\varphi\|_{(t_j/4)}\|\mu\|_{(t_j/4)}\|\chi\|_{(t_j/4)}.
\end{aligned}
$$

for $\chi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$. Since $\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ is dense in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$, the proof follows. □

Remark 3.3.1. Let $\chi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$ is equal to 1 on the $K_{\mathbb{R}^d}(0,1)$ and have a support in $K_{\mathbb{R}^d}(0,2)$. Put $\chi_n(x)=\chi(x/n)$, $n\in\mathbb{Z}_+$. If for S and T the equivalent conditions of the above theorem hold and $\varphi\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, then, similarly as in the proof of *ii)* $\Rightarrow$ *iii)*, we can prove that $\left\langle\left(\varphi*\check{S}\right)T,\chi_n\right\rangle=\left\langle(S\otimes T)\varphi^{\Delta},1_x\otimes\chi_n(y)\right\rangle$. But then, by construction, $\chi_n\to 1$ in $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^d\right)$ and $1_x\otimes\chi_n(y)\to 1_{x,y}$ in $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\left(\mathbb{R}^{2d}\right)$. Hence $\langle S*T,\varphi\rangle=\left\langle(S\otimes T)\varphi^{\Delta},1\right\rangle=\left\langle\left(\varphi*\check{S}\right)T,1\right\rangle$. Similarly, $\langle S*T,\varphi\rangle=\left\langle\left(\varphi*\check{T}\right)S,1\right\rangle$.

Chapter 4

Pseudodifferential Operators of Infinite Order in Spaces of Tempered Ultradistributions

Pseudodifferential operators that act continuously on Gevrey classes were vastly studied during the years. A lot of local symbol classes that give rise to such operators (both of finite and infinite order) were constructed by many authors. Also, global symbol classes and corresponding operators (of finite and infinite order), as well as their symbolic calculus were developed in [3], [4], [5], [6], [7], [8] (see also [36]). The functional frame in which those were studied are the Gelfand - Shilov spaces of Roumieu type. The symbol classes developed there are well suited for studying polyhomogeneous operators. In this chapter we develop a global calculus for some classes of pseudodifferential operators of infinite order. The symbol classes and the corresponding pseudodifferential operators that we will develop here are of Shubin type. The functional frame in which the considered symbol classes and the corresponding pseudodifferential operators will be studied is going to be Komatsu ultradistributions, more precisely the spaces of tempered ultradistributions of Beurling and Roumieu type. Our symbol classes are similar to those in [5] and [6], but the weights that control the growth of the derivatives of the symbols are constructed in such way that they give well suited environment for studying Anti-Wick and Weyl operators on the space of tempered ultradistributions. In this chapter, we develop calculus for our symbol classes.

In the first section of this chapter, we give the definition of the symbol classes as well as their basic topological properties. We study pseudodifferential operators $\mathrm{Op}_\tau(a)$, arising from τ-quantization of symbols that belong to these symbol classes. We prove a theorem that gives the hypocontinuity of the mapping $(a, u) \mapsto \mathrm{Op}_\tau(a)u$, for u in the test space.

We start the second section with the definition of the spaces of asymptotic expansion. We state and prove results concerning change of quantization, composition of operators and asymptotic expansion of the symbol of the transposed operator.

4.1 Definition and Basic Properties of the Symbol Classes

Let $a \in \mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. For $\tau \in \mathbb{R}$, consider the ultradistribution

$$K_\tau(x,y) = \mathcal{F}^{-1}_{\xi\rightarrow x-y}(a)((1-\tau)x+\tau y,\xi) \in \mathcal{S}'^*\left(\mathbb{R}^{2d}\right). \tag{4.1}$$

Let $\mathrm{Op}_\tau(a)$ be the operator from $\mathcal{S}^*$ to $\mathcal{S}'^*$ corresponding to the kernel $K_\tau(x,y)$, i.e.

$$\langle \mathrm{Op}_\tau(a)u, v\rangle = \langle K_\tau, v\otimes u\rangle,\ u,v\in\mathcal{S}^*\left(\mathbb{R}^d\right). \tag{4.2}$$

a will be called the τ*-symbol* of the *pseudodifferential operator* $\mathrm{Op}_\tau(a)$ and $\mathrm{Op}_\tau(a)$ will be referred as the τ*-quantization* of a. When $\tau=0$, we will denote $\mathrm{Op}_0(a)$ by $a(x,D)$ and this is called *standard*, or *left*, *quantization*. For $\tau=1$ one obtains the so-called *right quantization*. The case $\tau=1/2$ is particularly interesting and yields the *Weyl quantization* and it will be denoted by a^w. We will return to further study the relationship between the Weyl and another, very important, quantization in the next chapter.

When $a\in\mathcal{S}^*\left(\mathbb{R}^{2d}\right)$,

$$\mathrm{Op}_\tau(a)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}a((1-\tau)x+\tau y,\xi)u(y)dyd\xi, \tag{4.3}$$

where the integral is absolutely convergent.

Proposition 4.1.1. *The correspondence $a\mapsto K_\tau$ is an isomorphism of $\mathcal{S}^*\left(\mathbb{R}^{2d}\right)$, of $\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$ and of $L^2\left(\mathbb{R}^{2d}\right)$. The inverse map is given by*

$$a(x,\xi) = \mathcal{F}_{y\rightarrow\xi}K_\tau(x+\tau y, x-(1-\tau)y).$$

Proof. The partial Fourier transform and the composition with the change of variable $\Xi(x,y)=((1-\tau)x+\tau y, x-y)$ are isomorphisms of $\mathcal{S}^*\left(\mathbb{R}^{2d}\right)$, of $\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$ and of $L^2\left(\mathbb{R}^{2d}\right)$. The last part is just an easy computation. □

Operators with symbols in $\mathcal{S}^*$ correspond to kernels in $\mathcal{S}^*$ and by proposition 1.2.2, those extend to continuous operators from $\mathcal{S}'^*$ to $\mathcal{S}^*$. We will call these **-regularizing operators*.

Now we will define the announced global symbol classes. Let A_p and B_p be positive sequences that satisfy $(M.1)$, $(M.3)'$ and $A_0=1$ and $B_0=1$. Moreover, let $A_p\subset M_p$ and $B_p\subset M_p$ i.e. there exist $c_0>0$ and $L>0$ such that $A_p\leq c_0L^pM_p$ and $B_p\leq c_0L^pM_p$, for all $p\in\mathbb{N}$ (it is obvious that without losing generality we can assume that this c_0 is the same with c_0 from the conditions $(M.2)$ and $(M.3)$ for M_p). For $0<\rho\leq 1$, define $\Gamma^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h,m\right)$ as the space of all $a\in\mathcal{C}^\infty\left(\mathbb{R}^{2d}\right)$ for which the following norm is finite

$$\|a\|_{h,m,\Gamma} = \sup_{\alpha,\beta}\sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}e^{-M(m|\xi|)}e^{-M(m|x|)}}{h^{|\alpha|+|\beta|}A_\alpha B_\beta}.$$

It is easily verified that it is a (B) - space. Define

$$\begin{aligned}
\Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};m\right) &= \lim_{\substack{\longleftarrow\\ h\to 0}} \Gamma^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h,m\right),\\
\Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right) &= \lim_{\substack{\longrightarrow\\ m\to \infty}} \Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};m\right),\\
\Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h\right) &= \lim_{\substack{\longleftarrow\\ m\to 0}} \Gamma^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h,m\right),\\
\Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right) &= \lim_{\substack{\longrightarrow\\ h\to \infty}} \Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h\right).
\end{aligned}$$

Remark 4.1.1. $\Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};m\right)$ and $\Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h\right)$ are (F) - spaces. Obviously, the inclusions $\Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};m\right)\to \mathcal{S}'^{(M_p)}\left(\mathbb{R}^{2d}\right)$ and $\Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h\right)\to \mathcal{S}'^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ are continuous, hence $\Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and $\Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ are Hausdorff l.c.s. Moreover, as inductive limits of barrelled and bornological l.c.s., they are barrelled and bornological.

Remark 4.1.2. By proposition 7 of [17] it follows that every element of $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ is a multiplier for $\mathcal{S}'^{*}\left(\mathbb{R}^{2d}\right)$.

Remark 4.1.3. Examples of nontrivial elements of $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ are given by every ultrapolynomial of class $*$.

Proposition 4.1.2. *For every $a\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ there exists a sequence χ_j, $j\in\mathbb{Z}_+$, in $\mathcal{D}^*\left(\mathbb{R}^{2d}\right)$ such that $\chi_j\to a$ in $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$.*

Proof. Let $\varphi(x)\in\mathcal{D}^{(B_p)}\left(\mathbb{R}^{d}\right)$ and $\psi(\xi)\in\mathcal{D}^{(A_p)}\left(\mathbb{R}^{d}\right)$, in the (M_p) case, resp. $\varphi(x)\in\mathcal{D}^{\{B_p\}}\left(\mathbb{R}^{d}\right)$ and $\psi(\xi)\in\mathcal{D}^{\{A_p\}}\left(\mathbb{R}^{d}\right)$ in the $\{M_p\}$ case, are such that $0\leq\varphi,\psi\leq 1$, $\varphi(x)=1$ when $|x|\leq 1/4$, $\psi(\xi)=1$ when $|\xi|\leq 1/4$ and $\varphi(x)=0$ when $|x|\geq 1/2$, $\psi(\xi)=0$ when $|\xi|\geq 1/2$ (such functions exist because A_p and B_p satisfy $(M.3)'$). Put $\chi(x,\xi)=\varphi(x)\psi(\xi)$, $\chi_n(x,\xi)=\chi(x/n,\xi/n)$ for $n\in\mathbb{Z}_+$. It easily checked that $\chi,\chi_n\in\mathcal{D}^{(M_p)}\left(\mathbb{R}^{2d}\right)$, resp. $\chi,\chi_n\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$. Let $a\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$. Then, one easily proves that $a_n(x,\xi)=\chi_n(x,\xi)a(x,\xi)$ is an element of $\mathcal{D}^*\left(\mathbb{R}^{2d}\right)$. We will prove that $a_n\to a$ in $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$.

The (M_p) case. It is enough to prove that there exists $m>0$ such that for every $h>0$, $a_n\to a$ in $\Gamma^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h,m\right)$. Take m such that $a\in\Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};m\right)$. Then, obviously, $a_n,a\in\Gamma^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};h,m''\right)$ for all $m''\geq m$ and all $h>0$. Let $h>0$ be fixed. By proposition 1.2.1 we have $e^{2M(m|x|)}\leq c_0e^{M(mH|x|)}$ and $e^{2M(m|\xi|)}\leq c_0e^{M(mH|\xi|)}$. For simplicity in notation we will put $m'=mH$. Choose $h_1>0$ such that $4h_1<h$. From the way we chose χ, there exists $C_0>0$ such that $\left|D^{\alpha}_{\xi}D^{\beta}_{x}\chi(x,\xi)\right|\leq C_0h_1^{|\alpha|+|\beta|}A_\alpha B_\beta$, for all $\alpha,\beta\in\mathbb{N}^d$. We estimate as follows

$$\frac{\left|D^{\alpha}_{\xi}D^{\beta}_{x}a(x,\xi)-D^{\alpha}_{\xi}D^{\beta}_{x}(\chi_n(x,\xi)a(x,\xi))\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}e^{-M(m'|\xi|)}e^{-M(m'|x|)}}{h^{|\alpha|+|\beta|}A_\alpha B_\beta}$$

$$\leq\ \frac{(1-\chi_n(x,\xi))\left|D^{\alpha}_{\xi}D^{\beta}_{x}a(x,\xi)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}e^{-M(m'|\xi|)}e^{-M(m'|x|)}}{h^{|\alpha|+|\beta|}A_\alpha B_\beta}$$

$$+\sum_{\substack{\gamma\leq\alpha,\,\delta\leq\beta\\ \gamma+\delta\neq 0}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{\left|D_\xi^{\alpha-\gamma}D_x^{\beta-\delta}a(x,\xi)\right|\left|D_\xi^\gamma D_x^\delta\chi(x/n,\xi/n)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}{n^{|\gamma|+|\delta|}h^{|\alpha|+|\beta|}A_\alpha B_\beta e^{M(m'|\xi|)}e^{M(m'|x|)}}$$
$$\leq\ C'(1-\chi_n(x,\xi))\|a\|_{h,m}e^{-M(m|\xi|)}e^{-M(m|x|)}+S,$$

where S is the sum in the previous inequality. First, observe that

$$C'(1-\chi_n(x,\xi))\|a\|_{h,m}e^{-M(m|\xi|)}e^{-M(m|x|)}\leq C'\|a\|_{h,m}e^{-M(mn/4)}\to 0,$$

when $n\to\infty$. It's left to estimate S:

$$\frac{1}{2^{|\alpha|+|\beta|}}\sum_{\substack{\gamma\leq\alpha,\,\delta\leq\beta\\ \gamma+\delta\neq 0}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{\left|D_\xi^{\alpha-\gamma}D_x^{\beta-\delta}a(x,\xi)\right|\left|D_\xi^\gamma D_x^\delta\chi(x/n,\xi/n)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}{n^{|\gamma|+|\delta|}(h/2)^{|\alpha|+|\beta|}A_\alpha B_\beta e^{M(m'|\xi|)}e^{M(m'|x|)}}$$

$$\leq\ C_1\frac{\|a\|_{h/2,m}}{2^{|\alpha|+|\beta|}}\sum_{\substack{\gamma\leq\alpha,\,\delta\leq\beta\\ \gamma+\delta\neq 0}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{\left|D_\xi^\gamma D_x^\delta\chi(x/n,\xi/n)\right|\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|}}{n^{|\gamma|+|\delta|}(h/2)^{|\gamma|+|\delta|}A_\gamma B_\delta e^{M(m|\xi|)}e^{M(m|x|)}}$$

$$\leq\ C\|a\|_{h/2,m}\frac{1}{2^{|\alpha|+|\beta|}}\sum_{\substack{\gamma\leq\alpha,\,\delta\leq\beta\\ \gamma+\delta\neq 0}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\left(1+\frac{n}{\sqrt{2}}\right)^{|\gamma|+|\delta|}\frac{h_1^{|\gamma|+|\delta|}e^{-M(mn/4)}}{n^{|\gamma|+|\delta|}(h/2)^{|\gamma|+|\delta|}}$$

$$\leq\ C\|a\|_{h/2,m}\frac{e^{-M(mn/4)}}{2^{|\alpha|+|\beta|}}\sum_{\substack{\gamma\leq\alpha,\,\delta\leq\beta\\ \gamma+\delta\neq 0}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\left(\frac{4h_1}{h}\right)^{|\gamma|+|\delta|}\leq C\|a\|_{h/2,m}e^{-M(mn/4)}$$

and the last term obviously converges to zero when $n\to\infty$. Hence, we prove that, for every $h>0$, $a_n\to a$ in $\Gamma_{A_p,B_p}^{M_p,\infty}\left(\mathbb{R}^{2d};h,m'\right)$, from what the claim follows.

The $\{M_p\}$ case. It is enough to prove that there exists $h>0$ such that for every $m>0$, $a_n\to a$ in $\Gamma_{A_p,B_p,\rho}^{M_p,\infty}\left(\mathbb{R}^{2d};h,m\right)$. From the way we chose χ, there exist $C_0>0$ and $h_1>1$ such that $\left|D_\xi^\alpha D_x^\beta\chi(x,\xi)\right|\leq C_0h_1^{|\alpha|+|\beta|}A_\alpha B_\beta$, for all $\alpha,\beta\in\mathbb{N}^d$. Let $h>0$ be such that $a\in\Gamma_{A_p,B_p,\rho}^{\{M_p\},\infty}\left(\mathbb{R}^{2d};h/(2h_1)\right)$. It is clear that, without losing generality, we can assume that $h>4h_1$. Obviously, $a\in\Gamma_{A_p,B_p}^{\{M_p\},\infty}\left(\mathbb{R}^{2d};h\right)$. Let $m>0$ be arbitrary but fixed. Similarly as in the (M_p) case, we have that $e^{2M(m|x|/H)}\leq c_0e^{M(m|x|)}$ and $e^{2M(m|\xi|/H)}\leq c_0e^{M(m|\xi|)}$. For simpler notation, put $m'=m/H$. Similarly as above, we estimate

$$\frac{\left|D_\xi^\alpha D_x^\beta a(x,\xi)-D_\xi^\alpha D_x^\beta(\chi_n(x,\xi)a(x,\xi))\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}e^{-M(m|\xi|)}e^{-M(m|x|)}}{h^{|\alpha|+|\beta|}A_\alpha B_\beta}$$
$$\leq\ C'(1-\chi_n(x,\xi))\|a\|_{h,m'}e^{-M(m'|\xi|)}e^{-M(m'|x|)}+C\|a\|_{h/2,m'}e^{-M(m'n/4)}$$
$$\leq\ C'\|a\|_{h,m'}e^{-M(m'n/4)}+C\|a\|_{h/2,m'}e^{-M(m'n/4)},$$

which obviously tends to zero when $n\to\infty$. □

Theorem 4.1.1. *Let $a\in\Gamma_{A_p,B_p,\rho}^{*,\infty}\left(\mathbb{R}^{2d}\right)$. Then the integral (4.3) is well defined as an iterated integral. The ultradistribution $\operatorname{Op}_\tau(a)u$, $u\in\mathcal{S}^*$, coincides with the function defined by that iterated integral.*

Proof. The (M_p) case. Because $a \in \Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, there exists $m > 0$ such that, for every $h > 0$ there exists $C_1 > 0$ such that

$$\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \leq C_1 \frac{h^{|\alpha|+|\beta|} A_\alpha B_\beta e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}},\ \forall \alpha,\beta \in \mathbb{N}^d,\ \forall (x,\xi) \in \mathbb{R}^{2d}.$$

Hence, $b(x,\xi) = \displaystyle\int_{\mathbb{R}^d} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy$ is well defined and $b \in \mathcal{C}^\infty\left(\mathbb{R}^{2d}\right)$. Choose $m_0 > 0$ large enough such that, for all $m' \geq m_0$,

$$\int_{\mathbb{R}^d} e^{M(2m|\tau y|)} e^{-M(m'|y|)} dy < \infty.$$

Because $u \in \mathcal{S}^{(M_p)}$, for such m' we get $\displaystyle\sup_{\alpha\in\mathbb{N}^d} \frac{m'^{|\alpha|}\left\|D^\alpha u(y) e^{M(m'|y|)}\right\|_{L^\infty}}{M_\alpha} < \infty$. One obtains

$$\begin{aligned}
&|\xi^\alpha b(x,\xi)| \\
&= \left|\int_{\mathbb{R}^d} e^{i(x-y)\xi} D^\alpha_y \left(a((1-\tau)x + \tau y, \xi) u(y)\right) dy\right| \\
&\leq \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} |\tau|^{|\gamma|} \left|D^\gamma_x a((1-\tau)x+\tau y,\xi)\right| \left|D^{\alpha-\gamma} u(y)\right| dy \\
&\leq C \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} |\tau|^{|\gamma|} h^{|\gamma|} B_\gamma e^{M(m|\xi|)} e^{M(m|(1-\tau)x+\tau y|)} \frac{M_{\alpha-\gamma} e^{-M(m'|y|)}}{m'^{|\alpha|-|\gamma|}} dy \\
&\leq C' \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} (|\tau| h L)^{|\gamma|} M_\gamma e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)} e^{M(2m|\tau y|)} \cdot \frac{M_{\alpha-\gamma} e^{-M(m'|y|)}}{m'^{|\alpha|-|\gamma|}} dy \\
&\leq C'' e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)} M_\alpha \left(|\tau| h L + \frac{1}{m'}\right)^{|\alpha|},
\end{aligned}$$

where we used $B_p \subset M_p$. For $l > 0$ consider $P_l(\xi)$. By proposition 2.1.1, we can choose l such that $|P_l(\xi)| \geq c'' e^{M(r|\xi|)}$ where $r > 0$ is chosen such that $\displaystyle\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$ and $P_l(\xi)$ is never zero. Also, if we represent $P_l(\xi) = \sum_\alpha c_\alpha \xi^\alpha$, there exists $L' > 0$ and $C' > 0$ such that $|c_\alpha| \leq C' L'^{|\alpha|}/M_\alpha$. Choose $h > 0$ so small and $m' \geq m_0$ so large such that $\left(|\tau| h L + \dfrac{1}{m'}\right) L' < \dfrac{1}{4}$. Then, we have

$$\begin{aligned}
|P_l(\xi) b(x,\xi)| &\leq \sum_\alpha |c_\alpha| \left|\xi^\alpha b(x,\xi)\right| \\
&\leq C'' e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)} \sum_\alpha |c_\alpha| M_\alpha \left(|\tau| h L + \frac{1}{m'}\right)^{|\alpha|} \\
&\leq C_0 e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)}.
\end{aligned}$$

Hence $\int_{\mathbb{R}^d} |b(x,\xi)|d\xi$ is finite for every x, i.e. (4.3) is well defined as iterated integral. From this estimate also follows that $b(x,\xi)v(x) \in L^1(\mathbb{R}^{2d})$, for any $v \in \mathcal{S}^{(M_p)}$.

Let us consider the $\{M_p\}$ case. Because $a \in \Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$, there exists $h>0$ such that, for every $m>0$ there exists $C_1>0$ such that

$$\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \leq C_1 \frac{h^{|\alpha|+|\beta|}A_\alpha B_\beta e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}, \forall \alpha,\beta\in\mathbb{N}^d, \forall(x,\xi)\in\mathbb{R}^{2d}.$$

Hence, $b(x,\xi) = \int_{\mathbb{R}^d} e^{i(x-y)\xi}a((1-\tau)x+\tau y,\xi)u(y)dy$ is well defined and $b \in \mathcal{C}^\infty(\mathbb{R}^{2d})$. Put

$$g(\lambda) = \sup_{|(x,\xi)|\leq\lambda}\sup_{\alpha,\beta}\ln_+ \frac{\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}{h^{|\alpha|+|\beta|}A_\alpha B_\beta}.$$

g is an increasing function and by proposition 1.2.1, it satisfies the condition of lemma 1.2.2. Hence, there exists subordinate function $\epsilon(\lambda)$ and a constant $C'>1$ such that $g(\lambda)\leq M(\epsilon(\lambda))+\ln C'$. We get that

$$\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \leq C' \frac{h^{|\alpha|+|\beta|}A_\alpha B_\beta e^{M(\epsilon(|(x,\xi)|))}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}, \forall \alpha,\beta\in\mathbb{N}^d,\ \forall(x,\xi)\in\mathbb{R}^{2d}.$$

By lemma 3.12 of [26], there exist another sequence $\tilde{N}_p$, which satisfies $(M.1)$, such that $\tilde{N}(\lambda)\geq M(\epsilon(\lambda))$ and $k'_p = \tilde{n}_p/m_p \to \infty$ when $p\to\infty$. There exist $(k''_p)\in\mathfrak{R}$ such that $k''_p \leq k'_p$, for $p\in\mathbb{Z}_+$. Then

$$e^{N_{k''_p}(\lambda)} = \sup_p \frac{\lambda^p}{M_p\prod_{j=1}^p k''_j} \geq \sup_p \frac{\lambda^p}{M_p\prod_{j=1}^p k'_j} = e^{\tilde{N}(\lambda)} \geq e^{M(\epsilon(\lambda))}.$$

From this, we obtain the estimate

$$\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \leq C \frac{h^{|\alpha|+|\beta|}A_\alpha B_\beta e^{N_{k_p}(|\xi|)}e^{N_{k_p}(|x|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}, \forall \alpha,\beta\in\mathbb{N}^d, \forall(x,\xi)\in\mathbb{R}^{2d},$$

where we choose $(k_p)\in\mathfrak{R}$ such that $e^{N_{k''_p}(|(x,\xi)|)} \leq c'e^{N_{k_p}(|\xi|)}e^{N_{k_p}(|x|)}$, for some $c'>0$. Because $u\in\mathcal{S}^{\{M_p\}}$, there exists $h_1>0$ such that for every $(s_p)\in\mathfrak{R}$, $\sup_\alpha \frac{h_1^{|\alpha|}\left\|e^{N_{s_p}(|x|)}D^\alpha u(x)\right\|_{L^\infty}}{M_\alpha} < \infty$. Choose $(s_p)\in\mathfrak{R}$, such that

$$\int_{\mathbb{R}^d} e^{N_{k_p/2}(|\tau y|)}e^{-N_{s_p}(|y|)}dy < \infty.$$

Then, we have

$$\begin{aligned} &|\xi^\alpha b(x,\xi)| \\ &\quad = \left|\int_{\mathbb{R}^d} e^{i(x-y)\xi} D^\alpha_y \left(a((1-\tau)x+\tau y,\xi)u(y)\right)dy\right| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^d}|\tau|^{|\gamma|}\left|D_x^\gamma a((1-\tau)x+\tau y,\xi)\right|\left|D^{\alpha-\gamma}u(y)\right|dy\\
&\leq C'\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^d}|\tau|^{|\gamma|}h^{|\gamma|}B_\gamma e^{N_{k_p}(|\xi|)}e^{N_{k_p}(|(1-\tau)x+\tau y|)}\frac{e^{-N_{s_p}(|y|)}M_{\alpha-\gamma}}{h_1^{|\alpha|-|\gamma|}}dy\\
&\leq C''\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^d}\frac{(|\tau|hL)^{|\gamma|}M_\gamma e^{N_{k_p}(|\xi|)}e^{N_{k_p/2}(|(1-\tau)x|)}e^{N_{k_p/2}(|\tau y|)}M_{\alpha-\gamma}}{h_1^{|\alpha|-|\gamma|}e^{N_{s_p}(|y|)}}dy\\
&\leq C\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{(|\tau|hL)^{|\gamma|}e^{N_{k_p}(|\xi|)}e^{N_{k_p/2}(|(1-\tau)x|)}M_\alpha}{h_1^{|\alpha|-|\gamma|}}\\
&= Ce^{N_{k_p}(|\xi|)}e^{N_{k_p/2}(|(1-\tau)x|)}M_\alpha\left(|\tau|hL+\frac{1}{h_1}\right)^{|\alpha|},
\end{aligned}$$

where we used $B_p\subset M_p$. For $(l_p)\in\mathfrak{R}$ consider $P_{l_p}(\xi)$. By proposition 2.1.1 we can choose $(l_p)\in\mathfrak{R}$ such that $|P_{l_p}(\xi)|\geq c''e^{N_{r_p}(|\xi|)}$ where $(r_p)\in\mathfrak{R}$ is such that $\int_{\mathbb{R}^d}e^{N_{k_p}(|\xi|)}e^{-N_{r_p}(|\xi|)}d\xi<\infty$ and $P_{l_p}(\xi)$ is never zero. Also, if we represent $P_{l_p}(\xi)=\sum_\alpha c_\alpha\xi^\alpha$, then for any $L'>0$ there exists $C'>0$ such that $|c_\alpha|\leq C'L'^{|\alpha|}/M_\alpha$. Choose $L'>0$ such that, $\left(|\tau|hL+\frac{1}{h_1}\right)L'<\frac{1}{4}$. By the above estimate, we have

$$\begin{aligned}
\left|P_{l_p}(\xi)b(x,\xi)\right| &\leq \sum_\alpha|c_\alpha|\left|\xi^\alpha b(x,\xi)\right|\\
&\leq Ce^{N_{k_p}(|\xi|)}e^{N_{k_p/2}(|(1-\tau)x|)}\sum_\alpha|c_\alpha|M_\alpha\left(|\tau|hL+\frac{1}{h_1}\right)^{|\alpha|}\\
&\leq C_0e^{N_{k_p}(|\xi|)}e^{N_{k_p/2}(|(1-\tau)x|)}.
\end{aligned}$$

Hence $\int_{\mathbb{R}^d}|b(x,\xi)|d\xi$ is finite for every x, i.e. (4.3) is well defined as iterated integral. From this estimate also follows that $b(x,\xi)v(x)\in L^1\left(\mathbb{R}^{2d}\right)$, for any $v\in\mathcal{S}^{\{M_p\}}$.

Hence, in both cases we get that $\int_{\mathbb{R}^d}|b(x,\xi)|d\xi$ is finite for every x, i.e. (4.3) is well defined as iterated integral, and $b(x,\xi)v(x)\in L^1\left(\mathbb{R}^{2d}\right)$, for any $v\in\mathcal{S}^*$. We will temporary denote $F(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}b(x,\xi)d\xi$. From the above estimates it is obvious that $F\in\mathcal{S}'^*$. By Fubini's theorem, we have

$$\langle F,v\rangle=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i(x-y)\xi}a((1-\tau)x+\tau y,\xi)u(y)v(x)dydxd\xi.$$

By the growth condition of a, it is obvious that the integral

$$\int_{\mathbb{R}^{2d}}e^{i(x-y)\xi}a((1-\tau)x+\tau y,\xi)u(y)v(x)dydx$$

converges. If we put the change of variable $\Xi(x,y) = ((1-\tau)x+\tau y, x-y)$ in the last term of the above equality we obtain $\langle F, v\rangle = \left\langle a, \mathcal{F}_2^{-1}\left((v\otimes u)\circ \Xi^{-1}\right)\right\rangle = \langle \mathrm{Op}_\tau(a)u, v\rangle$, which completes the proof of the theorem. □

We will define more general classes of operators and symbols.

Definition 4.1.1. Denote by $\Pi^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h,m\right)$ the (B) - space of all $a \in \mathcal{C}^\infty\left(\mathbb{R}^{3d}\right)$ with the norm

$$\|a\|_{h,m,\Pi} = \sup_{\alpha,\beta,\gamma\in\mathbb{N}^d}\sup_{(x,y,\xi)\in\mathbb{R}^{3d}} \frac{\left|D^\alpha_\xi D^\beta_x D^\gamma_y a(x,y,\xi)\right| \langle (x,y,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|}\langle x-y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}A_\alpha B_{\beta+\gamma}} \cdot e^{-M(m|\xi|)}e^{-M(m|x|)}e^{-M(m|y|)}.$$

Define

$$\begin{aligned}
\Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};m\right) &= \varprojlim_{h\to 0} \Pi^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h,m\right),\\
\Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right) &= \varinjlim_{m\to \infty} \Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};m\right),\\
\Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h\right) &= \varprojlim_{m\to 0} \Pi^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h,m\right),\\
\Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right) &= \varinjlim_{h\to \infty} \Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h\right).
\end{aligned}$$

$\Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};m\right)$ and $\Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h\right)$ are (F) - spaces. Similarly as for the spaces $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, one proves that $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$ are barrelled and bornological l.c.s.

One easily sees that, for $a \in \Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$, the function $b(x,\xi) = a(x,x,\xi)$ belongs to $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$. Moreover, if $p \in \Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and $\tau \in \mathbb{R}$, then $a(x,y,\xi) = p((1-\tau)x+\tau y,\xi)$ belongs to $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$.

Remark 4.1.4. The Γ and Π classes defined here are appropriate generalisation (for symbols of infinite order) in ultradistributional setting of the corresponding classes in the setting of Schwartz distributions (see [53] for the corresponding Γ and Π symbol classes and calculus in the setting of Schwartz distributions).

Lemma 4.1.1. *Let $h>0$ be fixed. For every bounded set B in $\Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h\right)$, there exist $C>0$ and $(k_p)\in\mathfrak{R}$ such that, for all $a\in B$,*

$$\sup_{\alpha,\beta,\gamma\in\mathbb{N}^d}\sup_{(x,y,\xi)\in\mathbb{R}^{3d}} \frac{\left|D^\alpha_\xi D^\beta_x D^\gamma_y a(x,y,\xi)\right| \langle (x,y,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|}\langle x-y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}A_\alpha B_{\beta+\gamma}} \cdot e^{-N_{k_p}(|\xi|)}e^{-N_{k_p}(|x|)}e^{-N_{k_p}(|y|)} \leq C.$$

Proof. Because B is bounded, for every $m>0$ there exists a constant $C_m>0$ (which depends on m) such that, for every $a\in B$, $\|a\|_{h,m,\Pi}\leq C_m$, i.e.

$$\frac{\left|D^\alpha_\xi D^\beta_x D^\gamma_y a(x,y,\xi)\right| \langle (x,y,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|}\langle x-y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}A_\alpha B_{\beta+\gamma}} \leq C_m e^{M(m|\xi|)}e^{M(m|x|)}e^{M(m|y|)},$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$ and all $\alpha, \beta, \gamma \in \mathbb{N}^d$. Without losing generality, we can take $C_m \geq 1$. Put

$$g_a(x, y, \xi) = \sup_{\alpha,\beta,\gamma} \ln_+ \left(\frac{\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)\right| \langle (x, y, \xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x - y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \right).$$

Then, by proposition 1.2.1, we have

$$\begin{aligned} g_a(x, y, \xi) &\leq M(m|\xi|) + M(m|x|) + M(m|y|) + \ln C_m \\ &\leq 3M(m|(x, y, \xi)|) + \ln C_m \leq M(mH^2|(x, y, \xi)|) + \ln(c_0^2 C_m). \end{aligned}$$

Now, define $\tilde{g}_a(\lambda) = \sup_{|(x,y,\xi)|\leq\lambda} g_a(x, y, \xi)$. Then $\tilde{g}_a(\lambda) \leq M(mH^2\lambda) + \ln(c_0^2 C_m)$, for $\lambda \geq 0$ and $a \in B$. Then, if we put $\tilde{g}(\lambda) = \sup_{a\in B} \tilde{g}_a(\lambda)$, we have $\tilde{g}(\lambda) \leq M(mH^2\lambda) + \ln(c_0^2 C_m)$, for $\lambda \geq 0$. $\tilde{g}_a(\lambda)$ is an increasing function of λ for every $a \in B$, hence $\tilde{g}(\lambda)$ is an increasing function of λ. So $\tilde{g}$ satisfies the conditions in lemma 1.2.2. Hence, there exist subordinate function $\epsilon(\lambda)$ and a constant $C' > 1$ such that $\tilde{g}(\lambda) \leq M(\epsilon(\lambda)) + \ln C'$. We get that

$$\begin{aligned} \ln_+ \left(\frac{\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)\right| \langle (x, y, \xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x - y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \right) &\leq \tilde{g}\left(|(x, y, \xi)|\right) \\ &\leq M\left(\epsilon\left(|(x, y, \xi)|\right)\right) + \ln C', \end{aligned}$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$, $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $a \in B$, i.e.

$$\frac{\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)\right| \langle (x, y, \xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x - y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \leq C' e^{M(\epsilon(|(x,y,\xi)|))},$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$, $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $a \in B$. By lemma 3.12 of [26], there exists another sequence $\tilde{N}_p$, which satisfies $(M.1)$, such that $\tilde{N}(\lambda) \geq M(\epsilon(\lambda))$ and $k'_p = \tilde{n}_p/m_p \to \infty$ when $p \to \infty$. There exists $(k''_p) \in \mathfrak{R}$ such that $k''_p \leq k'_p$, for $p \in \mathbb{Z}_+$. Then

$$e^{N_{k''_p}(\lambda)} = \sup_p \frac{\lambda^p}{M_p \prod_{j=1}^p k''_j} \geq \sup_p \frac{\lambda^p}{M_p \prod_{j=1}^p k'_j} = \sup_p \frac{\lambda^p \tilde{N}_0}{\tilde{N}_p} = e^{\tilde{N}(\lambda)} \geq e^{M(\epsilon(\lambda))}.$$

From this, we obtain the estimate

$$\frac{\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)\right| \langle (x, y, \xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x - y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \leq C e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)},$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$, $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $a \in B$, where we choose $(k_p) \in \mathfrak{R}$ such that $e^{N_{k''_p}(|(x,y,\xi)|)} \leq c' e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}$, for some constant $c' > 0$. □

Lemma 4.1.2. *Let $a \in \Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$. For $\delta > 0$ and $u, \chi \in \mathcal{S}^*\left(\mathbb{R}^d\right)$, such that $\chi(0) = 1$, define*

$$I_{\chi,\delta}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(x, y, \xi)\chi(\delta\xi)u(y)dyd\xi.$$

*Then $I_{\chi,\delta}(x)$ has a limit when $\delta \to 0^+$ and the limit doesn't depend on χ. Moreover, the limit function is continuous and has ultrapolynomial growth of class *.*

Proof. The (M_p) case. Let $a \in \Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}; m\right)$. For $l > 0$ consider $P_l(\xi)$. By proposition 2.1.1, $P_l(\xi)$ is never zero and we can choose $P_l(\xi)$ such that, $|P_l(\xi)| \geq c_1 e^{M(r|\xi|)}$, where $r > 0$ is such that $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$. Also, we have the estimate $\left| D^\alpha_\xi \frac{1}{P_l(\xi)} \right| \leq c_1' \frac{\alpha!}{d_1^{|\alpha|}}$, for some $c_1' > 0$ and $d_1 > 0$. On the other hand if we represent $P_l(\xi) = \sum_\alpha c_\alpha \xi^\alpha$ then there exist $L_0 > 0$ and $C_0 > 0$ such that $|c_\alpha| \leq C_0 L_0^{|\alpha|}/M_\alpha$. Observe that

$$e^{i(x-y)\xi} = \frac{1}{P_l(y-x)} P_l(-D_\xi) \left(\frac{1}{P_l(\xi)} P_l(-D_y) e^{i(x-y)\xi} \right).$$

Then we have

$$\begin{aligned} &I_{\chi,\delta}(x) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{e^{i(x-y)\xi}}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi)\left(a(x,y,\xi)\chi(\delta\xi)u(y)\right) \right) dy d\xi. \quad (4.4) \end{aligned}$$

Because $u, \chi \in \mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right)$, for every $s > 0$

$$\sup_{\alpha\in\mathbb{N}^d} \frac{s^{|\alpha|} \left\| e^{M(s|\xi|)} D^\alpha \chi(\xi) \right\|_{L^\infty}}{M_\alpha} < \infty, \quad \sup_{\alpha\in\mathbb{N}^d} \frac{s^{|\alpha|} \left\| e^{M(s|y|)} D^\alpha u(y) \right\|_{L^\infty}}{M_\alpha} < \infty.$$

Now, we estimate as follows

$$\begin{aligned} &\left| P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi)\left(a(x,y,\xi)\chi(\delta\xi)u(y)\right) \right) \right| \\ &\leq \sum_{\alpha,\gamma} |c_\alpha|\, |c_\gamma| \sum_{\substack{\alpha'\leq\alpha \\ \gamma'\leq\gamma}} \sum_{\gamma''\leq\gamma'} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \\ &\quad \cdot \left| D^{\alpha'}_\xi D^{\gamma''}_y a(x,y,\xi) \right| \left| D^{\gamma'-\gamma''}_y \frac{1}{P_l(y-x)} \right| \delta^{|\alpha|-|\alpha'|} \left| D^{\alpha-\alpha'}_\xi \chi(\delta\xi) \right| \left| D^{\gamma-\gamma'}_y u(y) \right| \\ &\leq C' \sum_{\alpha,\gamma} |c_\alpha|\, |c_\gamma| \sum_{\substack{\alpha'\leq\alpha \\ \gamma'\leq\gamma}} \sum_{\gamma''\leq\gamma'} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \frac{(\gamma'-\gamma'')!}{d_1^{|\gamma'|-|\gamma''|}} \\ &\quad \cdot \frac{h^{|\alpha'|+|\gamma''|} \langle x-y \rangle^{\rho|\alpha'|+\rho|\gamma''|} A_{\alpha'} B_{\gamma''} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{\langle (x,y,\xi) \rangle^{\rho|\alpha'|+\rho|\gamma''|}} \\ &\quad \cdot \delta^{|\alpha|-|\alpha'|} \frac{M_{\alpha-\alpha'} M_{\gamma-\gamma'} e^{-M(s\delta|\xi|)}}{s^{|\alpha|-|\alpha'|+|\gamma|-|\gamma'|} e^{M(s|y|)}} \\ &\leq C'' \sum_{\alpha,\gamma} \frac{L_0^{|\alpha|+|\gamma|}}{M_\alpha M_\gamma} \sum_{\substack{\alpha'\leq\alpha \\ \gamma'\leq\gamma}} \sum_{\gamma''\leq\gamma'} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \frac{(\gamma'-\gamma'')!(4L_0)^{|\gamma'|-|\gamma''|}}{d_1^{|\gamma'|-|\gamma''|}(4L_0)^{|\gamma'|-|\gamma''|}} \\ &\quad \cdot \frac{(2Lh)^{|\alpha'|+|\gamma'|} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{(2Lh)^{|\gamma'|-|\gamma''|} M_{\gamma'-\gamma''}} \delta^{|\alpha|-|\alpha'|} \frac{M_\alpha M_\gamma}{s^{|\alpha|-|\alpha'|+|\gamma|-|\gamma'|} e^{M(s|y|)}} \end{aligned}$$

$$
\begin{aligned}
&\le C_1 \frac{e^{M(m|\xi|)}e^{M(m|x|)}e^{M(m|y|)}}{e^{M(s|y|)}} \\
&\quad \cdot \sum_{\alpha,\gamma} \left(\frac{\delta L_0}{s}\right)^{|\alpha|} \left(\frac{L_0}{s}\right)^{|\gamma|} \sum_{\substack{\alpha'\le\alpha\\ \gamma'\le\gamma}} \binom{\alpha}{\alpha'}\binom{\gamma}{\gamma'} \left(\frac{2sLh}{\delta}\right)^{|\alpha'|} (2sLh)^{|\gamma'|} \\
&\quad \cdot \sum_{\gamma''\le\gamma'} \binom{\gamma'}{\gamma''} \frac{1}{(8L_0Lh)^{|\gamma'|-|\gamma''|}} \\
&= C_1 \frac{e^{M(m|\xi|)}e^{M(m|x|)}e^{M(m|y|)}}{e^{M(s|y|)}} \sum_{\alpha,\gamma} \left(\frac{\delta L_0}{s} + 2LL_0h\right)^{|\alpha|} \left(\frac{L_0}{s} + 2L_0Lh + \frac{1}{4}\right)^{|\gamma|}.
\end{aligned}
$$

Choose h such that $LL_0h < 1/8$ and then choose s such that the above sum converge for $\delta = 1$ and denote its value by C_2 (then, obviously, for $0 < \delta < 1$ the sum is not greater than C_2). Moreover, choose s large enough, such that $\int_{\mathbb{R}^d} e^{M(m|y|)}e^{-M(s|y|)}dy < \infty$. Hence

$$
\begin{aligned}
|I_{\chi,\delta}(x)| &\le \frac{C_1C_2}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{1}{|P_l(\xi)|} \frac{e^{M(m|\xi|)}e^{M(m|x|)}e^{M(m|y|)}}{e^{M(s|y|)}} dyd\xi \\
&\le Ce^{M(m|x|)} \int_{\mathbb{R}^d} \frac{e^{M(m|\xi|)}}{e^{M(r|\xi|)}} d\xi \cdot \int_{\mathbb{R}^d} \frac{e^{M(m|y|)}}{e^{M(s|y|)}} dy,
\end{aligned}
$$

which is finite for every x. Note that $a(x,y,\xi)\chi(\delta\xi)u(y) \to a(x,y,\xi)u(y)$ in $\mathcal{E}^{(M_p)}\left(\mathbb{R}^{2d}_{y,\xi}\right)$ for each fixed x when $\delta \to 0^+$, so

$$
\frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) \left(a(x,y,\xi)\chi(\delta\xi)u(y)\right)\right)
$$

tends to $\frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x,y,\xi)u(y)\right)$ in $\mathcal{E}^{(M_p)}\left(\mathbb{R}^{2d}_{y,\xi}\right)$ for each fixed x, when $\delta \to 0^+$. If we take the limit in (4.4) as $\delta \to 0^+$, from dominated convergence, it follows that

$$
\lim_{\delta\to 0^+} I_{\chi,\delta}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{e^{i(x-y)\xi}}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) \left(a(x,y,\xi)u(y)\right)\right) dyd\xi.
$$

Moreover, by similar estimates as above, one proves that the function in the last integral can be dominated by $Ce^{M(m|x|)} \frac{e^{M(m|\xi|)}}{e^{M(r|\xi|)}} \cdot \frac{e^{M(m|y|)}}{e^{M(s|y|)}}$. Thus, $\lim_{\delta\to 0^+} I_{\chi,\delta}(x)$ is a continuous function with (M_p) - ultrapolynomial growth. Note that the choice of P_l does not depend on χ and u, only on m such that $a \in \Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};m\right)$. Hence, one can choose the same P_l for all $a \in \Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};m\right)$. From this, the claim in the lemma follows.

The $\{M_p\}$ case. Let $a \in \Pi^{\{M_p\},\infty}_{A_p,B_p}\left(\mathbb{R}^{3d};h\right)$. By lemma 4.1.1 there exists $(k_p) \in \mathfrak{R}$ such that

$$\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x,y,\xi)\right|$$
$$\leq C_0 \frac{h^{|\alpha|+|\beta|+|\gamma|}\langle x-y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma} e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{\langle (x,y,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}, \tag{4.5}$$

for all $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $(x,y,\xi) \in \mathbb{R}^{3d}$. For $(l_p) \in \mathfrak{R}$ consider $P_{l_p}(\xi)$. By proposition 2.1.1, we can choose $P_{l_p}(\xi)$ such that, $\left|P_{l_p}(\xi)\right| \geq c'' e^{N_{r_p}(|\xi|)}$, where $(r_p) \in \mathfrak{R}$ is such that $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p}(|\xi|)} d\xi < \infty$. On the other hand, if we represent $P_{l_p}(\xi) = \sum_\alpha c_\alpha \xi^\alpha$, then for every $L' > 0$ there exists $C' > 0$ such that $|c_\alpha| \leq C' L'^{|\alpha|}/M_\alpha$. Also, we have the same estimates, as in the (M_p) case, for the derivatives of $1/P_{l_p}(\xi)$, i.e $\left|D_\xi^\alpha \frac{1}{P_{l_p}(\xi)}\right| \leq c_1' \frac{\alpha!}{d_1^{|\alpha|}}$, for some $c_1' > 0$ and $d_1 > 0$. Because $u, \chi \in \mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)$, there exists $s > 0$, such that

$$\sup_{\alpha\in\mathbb{N}^d} \frac{s^{|\alpha|}\left\|e^{M(s|\xi|)} D^\alpha \chi(\xi)\right\|_{L^\infty}}{M_\alpha} < \infty, \quad \sup_{\alpha\in\mathbb{N}^d} \frac{s^{|\alpha|}\left\|e^{M(s|y|)} D^\alpha u(y)\right\|_{L^\infty}}{M_\alpha} < \infty.$$

(We can choose s to be the same for u and χ). Similarly as for the (M_p) case, one obtains (4.4), but with P_{l_p} in place of P_l and obtains the estimate

$$\left|P_{l_p}(D_y)\left(\frac{1}{P_{l_p}(y-x)} P_{l_p}(D_\xi)\left(a(x,y,\xi)\chi(\delta\xi)u(y)\right)\right)\right|$$
$$\leq C_1 \frac{e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{e^{M(s|y|)}} \sum_{\alpha,\gamma}\left(\frac{\delta L'}{s} + 2LL'h\right)^{|\alpha|}\left(\frac{L'}{s} + 2L'Lh + \frac{1}{4}\right)^{|\gamma|}.$$

Choose L', small enough, such that the above sum converges for $\delta = 1$ and denote its value by C_2. Similarly as above, we obtain the estimate

$$|I_{\chi,\delta}(x)| \ \leq\ C e^{N_{k_p}(|x|)} \int_{\mathbb{R}^d} e^{-N_{r_p}(|\xi|)} e^{N_{k_p}(|\xi|)} d\xi \cdot \int_{\mathbb{R}^d} e^{N_{k_p}(|y|)} e^{-M(s|y|)} dy.$$

The first integral converges by the choice of (r_p) and the convergence of the second can be easily proven. By similar arguments as in the (M_p) case and dominated convergence, the claim of the lemma follows. Note that the choice of P_{l_p} does not depend on u and χ, only on a. □

By the lemma, $\lim_{\delta\to 0^+} I_{\chi,\delta}(x)$ is in $\mathcal{S}'^*\left(\mathbb{R}^d\right)$. For $a \in \Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$ define the operator $A : \mathcal{S}^*\left(\mathbb{R}^d\right) \to \mathcal{S}'^*\left(\mathbb{R}^d\right)$, $Au(x) = \lim_{\delta\to 0^+} I_{\chi,\delta}(x)$. By the proof of the above lemma we obtain that

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{e^{i(x-y)\xi}}{P_l(\xi)} P_l(D_y)\left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x,y,\xi) u(y)\right) dy d\xi,$$

for the (M_p) case, respectively

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{e^{i(x-y)\xi}}{P_{l_p}(\xi)} P_{l_p}(D_y)\left(\frac{1}{P_{l_p}(y-x)} P_{l_p}(D_\xi) a(x,y,\xi) u(y)\right) dy d\xi,$$

for the $\{M_p\}$ case and moreover, the choice of P_l in the (M_p) case, respectively P_{l_p} in the $\{M_p\}$ case does not depend on $u \in \mathcal{S}^*$. If $P_{l'}$, resp. $P_{l'_p}$, is another operator such that $|P_{l'}(\xi)| \geq c_1 e^{M(r|\xi|)}$, resp. $\left|P_{l'_p}(\xi)\right| \geq c'' e^{N_{r_p}(|\xi|)}$, where $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$, resp. $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p}(|\xi|)} d\xi < \infty$, then $Au(x)$ can be given in the above form with $P_{l'}$ in place of P_l, resp. $P_{l'_p}$ in place of P_{l_p}. To prove the continuity of A, put

$$K(x,y,\xi) = e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x,y,\xi) u(y) \right)$$

in the (M_p) case, resp., the same but with P_{l_p} in place of P_l, in the $\{M_p\}$ case. For $v \in \mathcal{S}^*$,

$$\langle Au(x), v(x) \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} K(x,y,\xi) v(x) dy d\xi dx.$$

Let $v \in \mathcal{S}^*$ be fixed. If $u \in B$, where B is a bounded set in $\mathcal{S}^*$, similarly as in the proof of the above lemma, one can prove that $\langle Au(x), v(x) \rangle$ is bounded when $u \in B$. Hence the set $A(B)$ is simply bounded in $\mathcal{S}'^*$, consequently it is strongly bounded. Because $\mathcal{S}^*$ is bornological and A maps bounded sets into bounded sets it must be continuous.

Theorem 4.1.2. *The mapping* $(a,u) \mapsto Au$, $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right) \times \mathcal{S}^*\left(\mathbb{R}^d\right) \to \mathcal{S}^*\left(\mathbb{R}^d\right)$, *is hypocontinuous.*

Proof. Because $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$ and $\mathcal{S}^*\left(\mathbb{R}^d\right)$ are barrelled it is enough to prove that the mapping is separately continuous. We will consider first the (M_p) case. It is enough to prove that, for every $m > 0$, the mapping $\Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}; m\right) \times \mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right) \to \mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right)$ is separately continuous. We will prove that it is continuous i.e. that for every $s > 0$, there exists a constant $C > 0$ and $h > 0$, $t > 0$ such that $\|Au\|_s \leq C\|a\|_{h,m,\Pi}\|u\|_t$, where $\|\phi\|_s = \sup_\alpha \frac{s^{|\alpha|}\left\|D^\alpha\phi(\cdot)e^{M(s|\cdot|)}\right\|_{L^\infty}}{M_\alpha}$ are the seminorms in $\mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right)$. Let $s > 0$. Obviously, without losing generality, we can assume that $s \geq 1$. Choose $P_l(\xi)$ as in the proof of the above lemma and represent Au in the form

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{e^{i(x-y)\xi}}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x,y,\xi) u(y) \right) dy d\xi.$$

In the proof of the above lemma we proved that P_l can be chosen the same for all $a \in \Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}; m\right)$ (it depends only on m). By proposition 2.1.1 $P_l(\xi)$ is never zero and we can choose l small enough such that $\left|D^\alpha_\xi \frac{1}{P_l(\xi)}\right| \leq c'_1 \frac{\alpha!}{d_1^{|\alpha|}} e^{-M(r|\xi|)}$, for some $c'_1 > 0$ and $d_1 > 0$, where $r > 0$ is such that $\int_{\mathbb{R}^d} \frac{e^{M(m|\xi|)}e^{M(2s|\xi|)}}{e^{M(r|\xi|)}} d\xi$ converges and $e^{M(\frac{r}{2}|x|)} \geq \tilde{C} e^{M(s|x|)} e^{M(m|x|)}$. On the other hand, if we represent

$P_l(\xi) = \sum_\alpha c_\alpha \xi^\alpha$, there exist $L_0 \geq 1$ and $C_0 > 0$ such that $|c_\alpha| \leq C_0 L_0^{|\alpha|}/M_\alpha$. Then, for $a \in \Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}; m\right)$ and $u \in \mathcal{S}^{(M_p)}$, we have

$$\left| D_x^\beta \left(e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x,y,\xi) u(y) \right) \right) \right|$$

$$\begin{aligned}
\leq\ & \sum_{\beta' \leq \beta} \sum_{\alpha,\gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} |c_\alpha| \, |c_\gamma| \, \frac{|\xi|^{|\beta|-|\beta'|}}{|P_l(\xi)|} \\
& \cdot \left| D_y^{\gamma'-\gamma''} D_x^{\beta'-\beta''} \left(\frac{1}{P_l(y-x)} \right) \right| \left| D_\xi^\alpha D_x^{\beta''} D_y^{\gamma''} a(x,y,\xi) \right| \left| D_y^{\gamma-\gamma'} u(y) \right| \\
\leq\ & C_1 \|a\|_{h,m,\Pi} \|u\|_t \sum_{\beta' \leq \beta} \sum_{\alpha,\gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \frac{L_0^{|\alpha|+|\gamma|}}{M_\alpha M_\gamma} \\
& \cdot \frac{|\xi|^{|\beta|-|\beta'|}}{e^{M(r|\xi|)}} \cdot \frac{(\beta'-\beta''+\gamma'-\gamma'')! e^{-M(r|x-y|)}}{d_1^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}} \cdot \frac{M_{\gamma-\gamma'} e^{-M(t|y|)}}{t^{|\gamma|-|\gamma'|}} \\
& \cdot \frac{h^{|\alpha|+|\beta''|+|\gamma''|} \langle x-y \rangle^{\rho|\alpha|+\rho|\beta''|+\rho|\gamma''|} A_\alpha B_{\beta''+\gamma''} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{\langle (x,y,\xi) \rangle^{\rho|\alpha|+\rho|\beta''|+\rho|\gamma''|}} \\
\leq\ & C_2 \|a\|_{h,m,\Pi} \|u\|_t \sum_{\beta' \leq \beta} \sum_{\alpha,\gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} L_0^{|\alpha|+|\gamma|} \\
& \cdot \frac{|\xi|^{|\beta|-|\beta'|}}{e^{M(r|\xi|)}} \cdot \frac{(\beta'-\beta''+\gamma'-\gamma'')! e^{-M(r|x-y|)}}{d_1^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}} \cdot \frac{e^{-M(t|y|)}}{t^{|\gamma|-|\gamma'|} M_{\gamma'-\gamma''}} \\
& \cdot (2hL)^{|\alpha|+|\beta''|+|\gamma''|} H^{|\beta''|+|\gamma''|} M_{\beta''} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)} \\
\leq\ & C_2 \|a\|_{h,m,\Pi} \|u\|_t \frac{M_\beta e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(r|\xi|)} e^{M(t|y|)} e^{M(r|x-y|)}} \\
& \cdot \sum_{\beta' \leq \beta} \sum_{\alpha,\gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} L_0^{|\alpha|+|\gamma|} \frac{(2s|\xi|)^{|\beta|-|\beta'|}}{M_{\beta-\beta'}} \\
& \cdot \frac{(\beta'-\beta''+\gamma'-\gamma'')!}{d_1^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}} \cdot \frac{(2hL)^{|\alpha|+|\beta''|+|\gamma''|} H^{|\beta''|+|\gamma''|}}{(2s)^{|\beta|-|\beta'|} t^{|\gamma|-|\gamma'|} M_{\beta'-\beta''} M_{\gamma'-\gamma''}} \\
\leq\ & C_3 \|a\|_{h,m,\Pi} \|u\|_t \frac{M_\beta e^{M(m|\xi|)} e^{M(2s|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(r|\xi|)} e^{M(t|y|)} e^{M(r|x-y|)}} \\
& \cdot \sum_{\beta' \leq \beta} \sum_{\alpha,\gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} (2hLL_0)^{|\alpha|} L_0^{|\gamma|} \\
& \cdot \frac{(\beta'-\beta''+\gamma'-\gamma'')! (4sHL_0)^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}}{d_1^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|} M_{\beta'-\beta''+\gamma'-\gamma''} (4sHL_0)^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}} \\
& \cdot \frac{(2hL)^{|\beta''|+|\gamma''|} H^{|\beta'|+|\gamma'|}}{t^{|\gamma|-|\gamma'|} (2s)^{|\beta|-|\beta'|}} \\
\leq\ & C_4 \|a\|_{h,m,\Pi} \|u\|_t \frac{M_\beta e^{M(m|\xi|)} e^{M(2s|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(r|\xi|)} e^{M(t|y|)} e^{M(r|x-y|)}}
\end{aligned}$$

$$\begin{aligned}
&\cdot\sum_{\beta'\leq\beta}\sum_{\alpha,\gamma}\sum_{\gamma'\leq\gamma}\sum_{\gamma''\leq\gamma'}\sum_{\beta''\leq\beta'}\binom{\beta}{\beta'}\binom{\beta'}{\beta''}\binom{\gamma}{\gamma'}\binom{\gamma'}{\gamma''}(2hLL_0)^{|\alpha|}L_0^{|\gamma|}\\
&\cdot\frac{(2hL)^{|\beta''|+|\gamma''|}H^{|\beta'|+|\gamma'|}}{(4sHL_0)^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}t^{|\gamma|-|\gamma'|}(2s)^{|\beta|-|\beta'|}}\\
= \;& C_4\|a\|_{h,m,\Pi}\|u\|_t\frac{M_\beta e^{M(m|\xi|)}e^{M(2s|\xi|)}e^{M(m|x|)}e^{M(m|y|)}}{e^{M(r|\xi|)}e^{M(t|y|)}e^{M(r|x-y|)}}\\
&\cdot\left(\frac{1}{2s}+2hLH+\frac{1}{4sL_0}\right)^{|\beta|}\sum_{\alpha,\gamma}(2hLL_0)^{|\alpha|}\left(\frac{L_0}{t}+2hLL_0H+\frac{1}{4s}\right)^{|\gamma|}.
\end{aligned}$$

Note that $e^{M(\frac{r}{2}|x|)}\leq C_6e^{M(r|x-y|)}e^{M(r|y|)}$. For the chosen r we choose t such that the integral $\int_{\mathbb{R}^d}\frac{e^{M(m|y|)}e^{M(r|y|)}}{e^{M(t|y|)}}dy$ converges and moreover, we take h small enough and t large enough such that the above sum converges. Moreover, choose h small enough such that $\frac{1}{2s}+2hLH+\frac{1}{4sL_0}\leq\frac{1}{s}$. Then for the derivatives of Au we obtain

$$\left|D_x^\beta Au(x)\right|\leq C\|a\|_{h,m,\Pi}\|u\|_te^{-M(s|x|)}\frac{M_\beta}{s^{|\beta|}},$$

which is the desired estimate.

Now we will consider the $\{M_p\}$ case. Note that it is enough to prove that, for every $h>0$, $\Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h\right)\times\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)\rightarrow\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)$ is separately continuous. Because $\Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h\right)$ and $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)$ are bornological it is enough to prove that this mapping maps products of bounded sets into bounded sets in $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Let B_1 and B_2 be bounded sets in $\Pi^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h\right)$, respectively in $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)$. Then, by lemma 4.1.1, there exist $\tilde{C}_1>0$ and $(k_p)\in\mathfrak{R}$ such that

$$\frac{\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x,y,\xi)\right|\langle(x,y,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}e^{-N_{k_p}(|\xi|)}e^{-N_{k_p}(|x|)}e^{-N_{k_p}(|y|)}}{h^{|\alpha|+|\beta|+|\gamma|}\langle x-y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}A_\alpha B_{\beta+\gamma}}\leq\tilde{C}_1,\quad(4.6)$$

for all $a\in B_1$, $(x,y,\xi)\in\mathbb{R}^{3d}$ and $\alpha,\beta,\gamma\in\mathbb{N}^d$. We know that $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)=\varinjlim_{s\rightarrow 0}\mathcal{S}^{M_p,s}_\infty\left(\mathbb{R}^d\right)$, where $\mathcal{S}^{M_p,s}_\infty\left(\mathbb{R}^d\right)$ is the (B) - space with the norm

$$\|\phi\|_s=\sup_\alpha\frac{s^{|\alpha|}\left\|D^\alpha\phi(\cdot)e^{M(s|\cdot|)}\right\|_{L^\infty}}{M_\alpha}$$

and $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)$ is a (DFS) - space generated by this inductive limit (the linking mappings are compact inclusions, see Chapter 1). So, there exists $t>0$ such that $B_2\subseteq\mathcal{S}^{M_p,t}_\infty\left(\mathbb{R}^d\right)$ and it is bounded there. Hence, there exists $\tilde{C}_2>0$ such that $\|u\|_t\leq\tilde{C}_2$, for all $u\in B_2$. On the other hand, we know that $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right)=\varprojlim_{(s_p),(s'_p)\in\mathfrak{R}}\mathcal{S}^{M_p}_{(s_p),(s'_p)}\left(\mathbb{R}^d\right)$, where $\mathcal{S}^{M_p}_{(s_p),(s'_p)}\left(\mathbb{R}^d\right)$ is the (B) - space with the norm $\|\phi\|_{(s_p),(s'_p)}=\sup_\alpha\frac{\left\|D^\alpha\phi(\cdot)e^{N_{s'_p}(|\cdot|)}\right\|_{L^\infty}}{M_\alpha\prod_{j=1}^{|\alpha|}s_j}$. Hence, it is enough to prove that, for arbitrary $(s_p),(s'_p)\in\mathfrak{R}$, $\|Au\|_{(s_p),(s'_p)}$ is bounded for all $a\in B_1$ and $u\in B_2$. So, let $(s_p),(s'_p)\in\mathfrak{R}$ be fixed. Represent Au as

$$Au(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}}\frac{e^{i(x-y)\xi}}{P_{l_p}(\xi)}P_{l_p}(D_y)\left(\frac{1}{P_{l_p}(y-x)}P_{l_p}(D_\xi)a(x,y,\xi)u(y)\right)dyd\xi.$$

In the proof of lemma 4.1.2 we proved that the choice of P_{l_p} depends only on (k_p) such that (4.6) holds. But (k_p) is the same for all $a\in B_1$ hence we can choose P_{l_p} the same for all $a\in B_1$. By proposition 2.1.1, $P_{l_p}(\xi)$ is never zero and we can choose $(l_p)\in\mathfrak{R}$ such that, $\left|D^\alpha_\xi\frac{1}{P_{l_p}(\xi)}\right|\leq c'_1\frac{\alpha!}{d_1^{|\alpha|}}e^{-N_{r_p}(|\xi|)}$, for some $c'_1>0$ and $d_1>0$, where $(r_p)\in\mathfrak{R}$ is chosen such that $\int_{\mathbb{R}^d}\frac{e^{N_{k_p}(|\xi|)}e^{N_{s_p}(|\xi|)}}{e^{N_{r_p}(|\xi|)}}d\xi$ converges and $e^{N_{2r_p}(|x|)}\geq\tilde{C}e^{N_{s'_p}(|x|)}e^{N_{k_p}(|x|)}$ (see also the remarks after the proof of lemma 4.1.2). On the other hand, if we represent $P_{l_p}(\xi)=\sum_\alpha c_\alpha\xi^\alpha$, then for every $L'>0$ there exists $C'>0$ such that $|c_\alpha|\leq C'L'^{|\alpha|}/M_\alpha$. For $a\in B_1$ and $u\in B_2$, similarly as in the (M_p) case, one obtains the estimate

$$\begin{aligned}
&\left|D^\beta_x\left(e^{i(x-y)\xi}\frac{1}{P_{l_p}(\xi)}P_{l_p}(D_y)\left(\frac{1}{P_{l_p}(y-x)}P_{l_p}(D_\xi)a(x,y,\xi)u(y)\right)\right)\right|\\
&\leq C_2\tilde{C}_1\tilde{C}_2\frac{M_\beta e^{N_{k_p}(|\xi|)}e^{N_{k_p}(|x|)}e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)}e^{M(t|y|)}e^{N_{r_p}(|x-y|)}}\\
&\quad\cdot\sum_{\beta'\leq\beta}\sum_{\alpha,\gamma}\sum_{\gamma'\leq\gamma}\sum_{\gamma''\leq\gamma'}\sum_{\beta''\leq\beta'}\binom{\beta}{\beta'}\binom{\beta'}{\beta''}\binom{\gamma}{\gamma'}\binom{\gamma'}{\gamma''}\frac{L'^{|\alpha|+|\gamma|}|\xi|^{|\beta|-|\beta'|}}{M_{\beta-\beta'}\prod_{j=1}^{|\beta|-|\beta'|}s_j}\\
&\quad\cdot\prod_{j=1}^{|\beta|-|\beta'|}s_j\cdot\frac{(\beta'-\beta''+\gamma'-\gamma'')!}{d_1^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}}\cdot\frac{(2hL)^{|\alpha|+|\beta''|+|\gamma''|}H^{|\beta''|+|\gamma''|}}{t^{|\gamma|-|\gamma'|}M_{\beta'-\beta''}M_{\gamma'-\gamma''}}\\
&\leq C_3\tilde{C}_1\tilde{C}_2\frac{M_\beta e^{N_{k_p}(|\xi|)}e^{N_{s_p}(|\xi|)}e^{N_{k_p}(|x|)}e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)}e^{M(t|y|)}e^{N_{r_p}(|x-y|)}}\\
&\quad\cdot\sum_{\beta'\leq\beta}\sum_{\alpha,\gamma}\sum_{\gamma'\leq\gamma}\sum_{\gamma''\leq\gamma'}\sum_{\beta''\leq\beta'}\binom{\beta}{\beta'}\binom{\beta'}{\beta''}\binom{\gamma}{\gamma'}\binom{\gamma'}{\gamma''}(2hLL')^{|\alpha|}L'^{|\gamma|}\\
&\quad\cdot\frac{(\beta'-\beta''+\gamma'-\gamma'')!}{d_1^{|\beta'|-|\beta''|+|\gamma'|-|\gamma''|}M_{\beta'-\beta''+\gamma'-\gamma''}}\cdot\frac{(2hL)^{|\beta''|+|\gamma''|}H^{|\beta'|+|\gamma'|}}{t^{|\gamma|-|\gamma'|}}\cdot\prod_{j=1}^{|\beta|-|\beta'|}s_j\\
&\leq C_4\tilde{C}_1\tilde{C}_2\frac{M_\beta e^{N_{k_p}(|\xi|)}e^{N_{s_p}(|\xi|)}e^{N_{k_p}(|x|)}e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)}e^{M(t|y|)}e^{N_{r_p}(|x-y|)}}\\
&\quad\cdot\sum_{\beta'\leq\beta}\sum_{\alpha,\gamma}\sum_{\gamma'\leq\gamma}\sum_{\gamma''\leq\gamma'}\sum_{\beta''\leq\beta'}\binom{\beta}{\beta'}\binom{\beta'}{\beta''}\binom{\gamma}{\gamma'}\binom{\gamma'}{\gamma''}(2hLL')^{|\alpha|}L'^{|\gamma|}\\
&\quad\cdot\frac{(2hL)^{|\beta''|+|\gamma''|}H^{|\beta'|+|\gamma'|}}{t^{|\gamma|-|\gamma'|}}\cdot\prod_{j=1}^{|\beta|-|\beta'|}s_j\\
&\leq C_5\tilde{C}_1\tilde{C}_2\frac{M_\beta e^{N_{k_p}(|\xi|)}e^{N_{s_p}(|\xi|)}e^{N_{k_p}(|x|)}e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)}e^{M(t|y|)}e^{N_{r_p}(|x-y|)}}
\end{aligned}$$

$$\cdot 2^{|\beta|}\prod_{j=1}^{|\beta|}s_j\sum_{\alpha,\gamma}(2hLL')^{|\alpha|}\left(\frac{L'}{t}+2hLL'H+L'H\right)^{|\gamma|},$$

where, in the last inequality, we used that $\dfrac{\lambda^p}{\prod_{j=1}^{p}s_j}\rightarrow 0$, when $p\rightarrow\infty$, for any fixed $\lambda>0$ (i.e. it is bounded for all $p\in\mathbb{Z}_+$). (This follows from the fact that $(s_p)\in\mathfrak{R}$.) Note that $e^{N_{2r_p}(|x|)}\leq C_6e^{N_{r_p}(|x-y|)}e^{N_{r_p}(|y|)}$. Also, the integral $\displaystyle\int_{\mathbb{R}^d}\frac{e^{N_{k_p}(|y|)}e^{N_{r_p}(|y|)}}{e^{M(t|y|)}}dy$ converges (this easily follows from the fact that $e^{N_{k_p}(|y|)}\leq c''e^{M(t'|y|)}$ for every $t'>0$, where the constant c'' depends on t'; similarly for $e^{N_{r_p}(|y|)}$). Take L' such that the sum converges. Then, for the derivatives of Au, we obtain $\left|D^\beta_x Au(x)\right|\leq C\tilde{C}_1\tilde{C}_2e^{-N_{s'_p}(|x|)}M_\beta\prod_{j=1}^{|\beta|}(2s_j)$, i.e. $\|Au\|_{(2s_p),(s'_p)}\leq C\tilde{C}_1\tilde{C}_2$, for all $a\in B_1$ and $u\in B_2$. □

Let $\tau\in\mathbb{R}$ be fixed. The inclusion $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)\rightarrow\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$, $b\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, $b\mapsto a$, where $a(x,y,\xi)=b((1-\tau)x+\tau y,\xi)$, is continuous. Moreover, if $u,\phi\in\mathcal{S}^*\left(\mathbb{R}^d\right)$ such that $\phi(0)=1$, by theorem 4.1.1, we have

$$\begin{aligned}\operatorname{Op}_\tau(b)u(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i(x-y)\xi}b((1-\tau)x+\tau y,\xi)u(y)dyd\xi\\ &= \lim_{\delta\rightarrow 0^+}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}}e^{i(x-y)\xi}b((1-\tau)x+\tau y,\xi)\phi(\delta\xi)u(y)dyd\xi.\end{aligned}$$

Hence, the operator $\operatorname{Op}_\tau(b)$ coincides with the operator B corresponding to b when we observe $b((1-\tau)x+\tau y,\xi)$ as an element of $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$. We get that the mapping $(b,u)\mapsto\operatorname{Op}_\tau(b)u$, $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)\times\mathcal{S}^*\left(\mathbb{R}^d\right)\rightarrow\mathcal{S}^*\left(\mathbb{R}^d\right)$, is hypocontinuous. For $b\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, denote its kernel by $K(x,y)$. If we consider the transposed of the operator $\operatorname{Op}_\tau(b)$ then its kernel is $K(y,x)$. On the other hand, by (4.1), $K(y,x)=\mathcal{F}^{-1}_{\xi\rightarrow x-y}(b)(\tau x+(1-\tau)y,-\xi)$. Hence ${}^t\operatorname{Op}_\tau(b(x,\xi))=\operatorname{Op}_{1-\tau}(b(x,-\xi))$ i.e. ${}^t\operatorname{Op}_\tau(b)$ is pseudo-differential operator and by the above it is a continuous mapping from $\mathcal{S}^*\left(\mathbb{R}^d\right)$ to $\mathcal{S}^*\left(\mathbb{R}^d\right)$. Using this we can extend $\operatorname{Op}_\tau(b)$ to a continuous operator from $\mathcal{S}'^*\left(\mathbb{R}^d\right)$ to $\mathcal{S}'^*\left(\mathbb{R}^d\right)$ by $\langle\operatorname{Op}_\tau(b)u,v\rangle=\langle u,{}^t\operatorname{Op}_\tau(b)v\rangle$, $u\in\mathcal{S}'^*\left(\mathbb{R}^d\right)$, $v\in\mathcal{S}^*\left(\mathbb{R}^d\right)$.

For $b\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, one can also consider the formal adjoint of $\operatorname{Op}_\tau(b)$, in notation $\operatorname{Op}_\tau(b)^*$, defined by $\langle\operatorname{Op}_\tau(b)^*u,\overline{v}\rangle=\langle\overline{u},\operatorname{Op}_\tau(b)v\rangle$, $u,v\in\mathcal{S}^*\left(\mathbb{R}^d\right)$. Similarly as for the transposed operator, one proves that the kernel of $\operatorname{Op}_\tau(b)^*$ is $\overline{K(y,x)}$, where $K(x,y)$ is the kernel of $\operatorname{Op}_\tau(b)$, and by (4.1)

$$\overline{K(y,x)}=\mathcal{F}^{-1}_{\xi\rightarrow x-y}\left(\overline{b}\right)(\tau x+(1-\tau)y,\xi).$$

Hence $\operatorname{Op}_\tau(b)^*=\operatorname{Op}_{1-\tau}\left(\overline{b}\right)$ and it is continuous mapping from $\mathcal{S}^*\left(\mathbb{R}^d\right)$ into $\mathcal{S}^*\left(\mathbb{R}^d\right)$.

Observe that, even for general $b\in\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$ we can perform the same calculation for the kernels of ${}^t\operatorname{Op}_\tau(b)$ and $\operatorname{Op}_\tau(b)^*$ and obtain

$${}^t\operatorname{Op}_\tau(b(x,\xi))=\operatorname{Op}_{1-\tau}(b(x,-\xi))\text{ and }\operatorname{Op}_\tau(b)^*=\operatorname{Op}_{1-\tau}\left(\overline{b}\right),$$

as continuous operators from $\mathcal{S}^*\left(\mathbb{R}^d\right)$ into $\mathcal{S}'^*\left(\mathbb{R}^d\right)$. An interesting consequence of this is that if $b \in \mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$ is real-valued, i.e. $b = \overline{b}$, then $(b^w)^* = b^w$.

We need the following technical lemmas.

Lemma 4.1.3. *Let M_p be a sequence which satisfies $(M.1)$, $(M.2)$ and $(M.3)$ and m a positive real. Then, for all $n \in \mathbb{Z}_+$, $M(mm_n) \leq 2(c_0 m + 2)n \ln H + \ln c_0$, where c_0 is the constant form the conditions $(M.2)$ and $(M.3)$. If $(t_p) \in \mathfrak{R}$ then, $N_{t_p}(mm_n) \leq n \ln H + \ln c$ for all $n \in \mathbb{Z}_+$, where the constant c depends only on M_p, (t_p) and m, but not on n.*

Proof. By $(M.3)$, for all $p \geq n+1$, $p \in \mathbb{N}$, we have

$$\frac{1}{m_{n+1}} + \frac{1}{m_{n+2}} + \dots + \frac{1}{m_p} \leq c_0 \frac{n}{m_{n+1}} \leq c_0 \frac{n}{m_n}.$$

If we multiply the above inequality with m_p and use the fact that the sequence m_n is monotonically increasing, we obtain $p - n \leq c_0 \frac{n m_p}{m_n}$, i.e. $\frac{mm_n}{m_p} \leq c_0 \frac{mn}{p-n}$. Hence, for $p \geq [c_0 m]n + 2n \geq n+1$, we obtain that $mm_n \leq m_p$. Denote by k the term $[c_0 m] + 2$. $M(\rho)$ is monotonically increasing, so $M(mm_n) \leq M(m_{kn})$. For $p \geq kn$, we have

$$\frac{m_{kn}^{p+1}}{M_{p+1}} = \frac{m_{kn}^p}{M_p} \cdot \frac{m_{kn}}{m_{p+1}} \leq \frac{m_{kn}^p}{M_p}.$$

Hence $M(m_{kn}) = \sup_p \ln_+ \frac{m_{kn}^p}{M_p} = \sup_{p \leq kn} \ln_+ \frac{m_{kn}^p}{M_p}$. For $p \leq kn$, $p \in \mathbb{N}$, we have

$$\frac{m_{kn}^p}{M_p} \leq \frac{m_{kn+1} \cdot m_{kn+2} \cdot \dots \cdot m_{kn+p}}{M_p} = \frac{M_{kn+p}}{M_p M_{kn}} \leq c_0 H^{kn+p} \leq c_0 H^{2kn},$$

where, in the second inequality, we used $(M.2)$. We obtained

$$M(mm_n) \leq M(m_{kn}) = \sup_{p \leq kn} \ln_+ \frac{m_{kn}^p}{M_p} \leq 2kn \ln H + \ln c_0 \leq 2(c_0 m + 2)n \ln H + \ln c_0,$$

which completes the proof for the first part. For the second part, denote by T_p the product $\prod_{j=1}^p t_j$. Observe that, for $p \in \mathbb{Z}_+$, we have

$$\frac{m^p m_n^p}{T_p M_p} \leq \frac{m^p m_{n+1} \cdot m_{n+2} \cdot \dots \cdot m_{n+p}}{T_p M_p} = \frac{m^p M_{n+p}}{T_p M_p M_n} \leq c_0 H^n \frac{(mH)^p}{T_p} \leq c H^n,$$

where, in the last inequality, we used the fact that (t_p) monotonically increases to infinity. Obviously c does not depend on p or n, only on m, (t_p) and M_p. From this we obtain $N_{t_p}(mm_n) \leq n \ln H + \ln c$, which completes the second part of the lemma. □

Lemma 4.1.4. *Let M_p be a sequence which satisfies $(M.1)$ and $(M.3)'$ and $R > 1+\dfrac{1}{M_1}$ be arbitrary. There exist a sequence $\psi_n \in \mathcal{D}^*\left(\mathbb{R}^d\right)$, $n \in \mathbb{N}$, such that*

$$\sum_{n=0}^{\infty}\psi_n = 1,\ \operatorname{supp}\psi_0 \subseteq \left\{\xi\in\mathbb{R}^d |\, \langle\xi\rangle < 3RM_1\right\},$$
$$\operatorname{supp}\psi_n \subseteq \left\{\xi\in\mathbb{R}^d |\, 2Rm_n < \langle\xi\rangle < 3Rm_{n+1}\right\},$$

for $n \in \mathbb{Z}_+$ and for every $h > 0$ there exists $C > 0$, resp. there exist $h > 0$ and $C > 0$ such that

$$|D^\alpha\psi_0(\xi)| \leq C\left(\frac{h}{RM_1}\right)^{|\alpha|} M_\alpha,\ \text{and}\ |D^\alpha\psi_n(\xi)| \leq C\left(\frac{h}{Rm_n}\right)^{|\alpha|} M_\alpha,\ \forall n\in\mathbb{Z}_+,$$

for all $\xi\in\mathbb{R}^d$ and $\alpha\in\mathbb{N}^d$.

Proof. Let $\phi\in\mathcal{D}^*$ such that $0\leq\phi\leq 1$, $\phi(\xi)=1$, for $\langle\xi\rangle<\sqrt{6}$, $\phi(\xi)=0$, for $\langle\xi\rangle>3$. Put

$$\psi_0(\xi) = \phi\left(\frac{\xi}{RM_1}\right),\ \psi_n(\xi) = \phi\left(\frac{\xi}{Rm_{n+1}}\right) - \phi\left(\frac{\xi}{Rm_n}\right).$$

It is easy to check that ψ_n, $n\in\mathbb{N}$, satisfy the claim in the lemma. □

Let $\rho_0 = \inf\{\rho\in\mathbb{R}_+ |\, A_p \subset M_p^\rho\}$. Obviously $0 < \rho_0 \leq 1$. In general, the infimum can not be reached.

Counterexample. Let $r_1 = 1$ and $r_p = p^{1-1/(2\sqrt{\ln p})}$ for $p\in\mathbb{N}$, $p\geq 2$. The sequences r_p and $p^{1/(2\sqrt{\ln p})}$ are monotonically increasing. Put $R_p = \prod_{j=1}^{p} r_p$. Take $M_0 = 1$, $M_p = p!^2R_p$ and $A_p = p!^2$. Then, obviously, A_p satisfies $(M.1)$, $(M.2)$ and $(M.3)$. One easily checks that M_p satisfies $(M.1)$, $(M.2)$ and $(M.3)$. It is clear that $A_p\subset M_p$. Note that $A_p \not\subset M_p^{2/3}$. In the contrary, there will exist $C>0$ and $L>0$ such that $p!^2 \leq CL^p p!^{4/3}R_p^{2/3}$, i.e. $\dfrac{p!}{L^{3p/2}R_p}\leq C_1$, for all $p\in\mathbb{Z}_+$, where we put $C_1 = C^{3/2}$. This is impossible, because this means that $\sum_{j=1}^{p}\ln\dfrac{j}{L^{3/2}r_j}$ is bounded from above for all $p\in\mathbb{Z}_+$, but

$$\lim_{j\to\infty}\ln\frac{j}{L^{3/2}r_j} = \lim_{j\to\infty}\ln\frac{j^{1/(2\sqrt{\ln j})}}{L^{3/2}} = \lim_{j\to\infty}\left(\frac{\sqrt{\ln j}}{2} - \frac{3}{2}\ln L\right) = \infty.$$

On the other hand, note that for $\lambda > 2/3$, $A_p\subset M_p^\lambda$. This is true because

$$\frac{p!^2}{p!^{2\lambda}R_p^\lambda} = \frac{p!^{2(1-\lambda)}}{R_p^\lambda} = \prod_{j=2}^{p}\frac{j^{2(1-\lambda)}}{j^{\lambda-\lambda/(2\sqrt{\ln j})}} = \prod_{j=2}^{p}\frac{j^{\lambda/(2\sqrt{\ln j})}}{j^{3\lambda-2}}$$

and the last term converges to zero when $p\to\infty$ (note that $3\lambda-2>0$ when $\lambda>2/3$). From now on we will assume that ρ is such that $\rho_0\leq\rho\leq 1$ if the infimum can be reached, otherwise $\rho_0 < \rho \leq 1$.

For $0 < r < 1$, define the set $\Omega_r = \{(x, y) \in \mathbb{R}^{2d} | \, |x - y| > r\langle x\rangle\}$.

Lemma 4.1.5. *Let $0 < r < 1$. There exists $\theta \in \mathcal{E}^*\left(\mathbb{R}^{2d}\right)$ such that $0 \leq \theta \leq 1$, $\theta = 0$ on $\mathbb{R}^{2d} \backslash \Omega_{r/4}$, $\theta = 1$ on $\Omega_{3r/4}$ and for every $h > 0$ there exists $C > 0$, resp. there exist $h > 0$ and $C > 0$, such that $\left|D_x^\beta D_y^\gamma \theta(x, y)\right| \leq C h^{|\beta|+|\gamma|} M_{\beta+\gamma}$, for all $(x, y) \in \mathbb{R}^{2d}$, $\alpha, \beta \in \mathbb{N}^d$.*

Proof. Let $f(x, y) = 1$ on $\Omega_{r/2}$ and $f(x, y) = 0$ on $\mathbb{R}^{2d} \backslash \Omega_{r/2}$. Let $\mu \in \mathcal{D}^*\left(\mathbb{R}^{2d}\right)$ is such that $\mu \geq 0$ with support in the closed ball with centre at the origin and radius $r/16$ and $\int_{\mathbb{R}^{2d}} \mu(x, y) dx dy = 1$. Put $\theta = f * \mu$. Then, one easily checks that θ satisfies the conditions in the lemma. □

Proposition 4.1.3. *Let $a \in \Pi^{*,\infty}_{A_p, B_p, \rho}\left(\mathbb{R}^{3d}\right)$ and A be the operator corresponding to a as defined above. The kernel K of this operator is an element of $\mathcal{C}^\infty(\Omega_r)$ for every $0 < r < 1$ and for every such Ω_r and every $h > 0$, resp. there exists $h > 0$, such that*

$$\sup_{\beta, \gamma \in \mathbb{N}^d} \sup_{(x, y) \in \overline{\Omega_r}} \frac{h^{\beta+\gamma}\left|D_x^\beta D_y^\gamma K(x, y)\right| e^{M(h|(x, y)|)}}{M_{\beta+\gamma}} < \infty. \tag{4.7}$$

*Moreover, if there exists r, $0 < r < 1$, such that $a(x, y, \xi) = 0$ for $(x, y, \xi) \in \left(\mathbb{R}^{2d} \backslash \Omega_r\right) \times \mathbb{R}^d$ then $K \in \mathcal{S}^*\left(\mathbb{R}^{2d}\right)$, i.e. A is *-regularizing.*

Proof. Let $\psi_n \in \mathcal{D}^*\left(\mathbb{R}^d\right)$ be as in lemma 4.1.4, where R will be chosen later. Then, note that the sum $\sum_{n=0}^{\infty} 1_y \otimes \psi_n(\xi)$ converges to $1_{y,\xi}$ in $\mathcal{E}^*\left(\mathbb{R}^{2d}_{y,\xi}\right)$ (with 1_y we denote the function of variable y that is identically equal to 1, similarly $1_{y,\xi}$ is the function of variables (y, ξ) that is identically equal to 1). Because $a(x, y, \xi)$ is an element of $\mathcal{E}^*\left(\mathbb{R}^{2d}_{y,\xi}\right)$, for every fixed x, we get

$$a(x, y, \xi) = a(x, y, \xi) \sum_{n=0}^{\infty} \psi_n(\xi) = \sum_{n=0}^{\infty} \left(\psi_n(\xi) a(x, y, \xi)\right),$$

in $\mathcal{E}^*\left(\mathbb{R}^{2d}_{y,\xi}\right)$. Let $u \in \mathcal{S}^*\left(\mathbb{R}^d\right)$. Because $1/P_l(y-x)$ and $1/P_l(\xi)$, resp. $1/P_{l_p}(y-x)$ and $1/P_{l_p}(\xi)$ are elements of $\mathcal{E}^*\left(\mathbb{R}^{2d}_{y,\xi}\right)$, for $* = (M_p)$, resp. $* = \{M_p\}$, for fixed x, we get

$$\begin{aligned} &\frac{1}{P_l(\xi)} P_l(D_y)\left(\frac{1}{P_l(y-x)} P_l(D_\xi)\left(a(x, y, \xi) u(y)\right)\right) \\ &\qquad = \sum_{n=0}^{\infty} \frac{1}{P_l(\xi)} P_l(D_y)\left(\frac{1}{P_l(y-x)} P_l(D_\xi)\left(a(x, y, \xi) \psi_n(\xi) u(y)\right)\right), \end{aligned}$$

in the (M_p) case and with P_{l_p} in place of P_l in the $\{M_p\}$ case, in $\mathcal{E}^*\left(\mathbb{R}^{2d}_{y,\xi}\right)$. If we choose l small enough such that $|P_l(\xi)| \geq c_1 e^{M(r|\xi|)} \geq c_1' e^{2M(r'|\xi|)}$, where $r' > 0$ is such that $\int_{\mathbb{R}^d} e^{M(m|\xi|)}e^{-M(r'|\xi|)}d\xi < \infty$, by the properties of ψ_n similarly as in the proof of lemma 4.1.2, we obtain

$$\left|\frac{1}{P_l(\xi)}P_l(D_y)\left(\frac{1}{P_l(y-x)}P_l(D_\xi)\left(a(x,y,\xi)\psi_n(\xi)u(y)\right)\right)\right| \leq C\frac{e^{M(m|x|)}e^{M(m|\xi|)}}{e^{M(r'|\xi|)}e^{M(r'Rm_n)}}\cdot\frac{e^{M(m|y|)}}{e^{M(s|y|)}}$$

in the (M_p) case, where m is such that $a \in \Pi^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};m\right)$ and s is from the $\mathcal{S}^{(M_p)}$ - seminorms of u. Respectively, if we choose $(l_p) \in \mathfrak{R}$ small enough such that

$$\left|P_{l_p}(\xi)\right| \geq c'' e^{N_{r_p}(|\xi|)} \geq c''_1 e^{2N_{r'_p}(|\xi|)},$$

where $(r'_p) \in \mathfrak{R}$ is such that $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)}e^{-N_{r'_p}(|\xi|)}d\xi < \infty$, we get

$$\left|\frac{1}{P_{l_p}(\xi)}P_{l_p}(D_y)\left(\frac{1}{P_{l_p}(y-x)}P_{l_p}(D_\xi)\left(a(x,y,\xi)\psi_n(\xi)u(y)\right)\right)\right| \leq C\frac{e^{N_{k_p}(|x|)}e^{N_{k_p}(|\xi|)}}{e^{N_{r'_p}(|\xi|)}e^{N_{r'_p}(Rm_n)}}\cdot\frac{e^{N_{k_p}(|y|)}}{e^{M(s|y|)}}$$

in the $\{M_p\}$ case, where (k_p) is such that (4.5) holds for a and s depends on u. Hence, by dominated convergence,

$$\begin{aligned} &Au(x)\\ &= \frac{1}{(2\pi)^d}\sum_{n=0}^{\infty}\int_{\mathbb{R}^{2d}}\frac{e^{i(x-y)\xi}}{P_l(\xi)}P_l(D_y)\left(\frac{1}{P_l(y-x)}P_l(D_\xi)\left(a(x,y,\xi)\psi_n(\xi)u(y)\right)\right)dyd\xi\\ &= \frac{1}{(2\pi)^d}\sum_{n=0}^{\infty}\int_{\mathbb{R}^{2d}}e^{i(x-y)\xi}a(x,y,\xi)\psi_n(\xi)u(y)dyd\xi, \end{aligned}$$

in the (M_p) case, resp. the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case and the convergence is uniform for x in compact subsets of $\mathbb{R}^d$ and in $\mathcal{S}'^*\left(\mathbb{R}^d\right)$. For simpler notation, put $a_n(x,y,\xi) = a(x,y,\xi)\psi_n(\xi)$ and A_n for the associated operator to a_n. Then, we get $Au(x) = \sum_{n=0}^{\infty} A_n u(x)$, where the convergence is uniform for x in compact subsets of $\mathbb{R}^d$ and in $\mathcal{S}'^*\left(\mathbb{R}^d\right)$. So $\sum_{k=0}^{n} A_k \to A$, when $n\to\infty$, in $\mathcal{L}_\sigma\left(\mathcal{S}^*\left(\mathbb{R}^d\right),\mathcal{S}'^*\left(\mathbb{R}^d\right)\right)$. $\mathcal{S}^*$ is barrelled, so, by the Banach - Steinhaus theorem (see [49], theorem 4.6), it follows that $\sum_{k=0}^{n} A_k \to A$, when $n\to\infty$, in the topology of precompact convergence. But $\mathcal{S}^*$ is Montel space, so the convergence holds in $\mathcal{L}_b\left(\mathcal{S}^*\left(\mathbb{R}^d\right),\mathcal{S}'^*\left(\mathbb{R}^d\right)\right)$ (the topology of bounded convergence). Hence, if we denote by $K(x,y)$ the kernel of A and by K_n the kernel of A_n, by proposition 1.2.2, we get $K=\sum_{n=0}^{\infty} K_n$, where the convergence holds in $\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. Now, observe that

$$K_n(x,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x,y,\xi)\psi_n(\xi)d\xi$$

and K_n is a $\mathcal{C}^\infty$ function. Take R such that $Rm_1\geq 1$. Later on we will impose more conditions on R. Let $r\in(0,1)$ be fixed. First, we will observe the (M_p) case. There exists $m>0$ such that $a\in \Pi^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d};h,m\right)$, for all $h>0$. Let m' be arbitrary but fixed positive real number. We want to prove (4.7) for this m'. Obviously, without losing generality, we can assume that $m'\geq 1$. Let $(x,y)\in\Omega_r$ be arbitrary but fixed. Let $q\in\{1,...,d\}$ be such that $|x_q-y_q|\geq|x_j-y_j|$, for all $j\in\{1,...,d\}$. Then $|x_q-y_q|>\frac{r}{d}\langle x\rangle$. We calculate

$$\begin{aligned}
&D^\beta_x D^\gamma_y K_n(x,y)\\
&= \frac{1}{(2\pi)^d}\sum_{\substack{\beta'+\beta''=\beta\\ \gamma'+\gamma''=\gamma}}\sum_{k=0}^{n}\sum_{\substack{k'+k''=k\\ k''\leq\beta''_q+\gamma''_q}}\binom{\beta}{\beta'}\binom{\gamma}{\gamma'}\binom{n}{k}\binom{k}{k'}\\
&\quad\cdot\frac{(\beta''+\gamma'')!}{(\beta''+\gamma''-e_qk'')!}\frac{(-1)^{|\gamma''|+n}}{(x_q-y_q)^n i^{k''}}\\
&\quad\cdot\int_{\mathbb{R}^d} e^{i(x-y)\xi}\frac{1}{P_l(y-x)}P_l(D_\xi)\left(\xi^{\beta''+\gamma''-e_qk''}D^{k'}_{\xi_q}D^{\beta'}_x D^{\gamma'}_y a(x,y,\xi)D^{n-k}_{\xi_q}\psi_n(\xi)\right)d\xi.
\end{aligned}$$

On Ω_r we have the following inequality

$$|(x,y)|\leq|x|+|y|\leq\langle x\rangle+|x-y|+|x|\leq 2\langle x\rangle+|x-y|\leq\left(\frac{2}{r}+1\right)|x-y|. \tag{4.8}$$

Hence, by using proposition 1.2.1, we can find $m''>0$ such that $e^{M(m''|x-y|)}\geq c''e^{M(m|x|)}e^{M(m|y|)}e^{M(m'|(x,y)|)}$ on Ω_r. Take $l'\geq m''$. Then we have

$$e^{M(l'|\xi|)}\geq c'''e^{M(m''|\xi|)}. \tag{4.9}$$

By proposition 2.1.1, we can find small enough $l>0$ such that $|P_l(\xi)|\geq c''e^{M(l'|\xi|)}$. On the other, hand if we represent $P_l(D)$ as $\sum_\alpha c_\alpha D^\alpha$, then there exist $C'_1>0$ and $L_0>0$ such that $|c_\alpha|\leq C'_1L_0^{|\alpha|}/M_\alpha$. We will estimate the part in the integral for $n\in\mathbb{Z}_+$ as follows

$$\begin{aligned}
&\left|\frac{1}{P_l(y-x)}P_l(D_\xi)\left(\xi^{\beta''+\gamma''-e_qk''}D^{k'}_{\xi_q}D^{\beta'}_x D^{\gamma'}_y a(x,y,\xi)D^{n-k}_{\xi_q}\psi_n(\xi)\right)\right|\\
&\leq \frac{1}{|P_l(y-x)|}\sum_\alpha|c_\alpha|\sum_{\alpha'\leq\alpha}\sum_{\substack{\alpha''+\alpha'''=\alpha'\\ \alpha'''\leq\beta''+\gamma''-e_qk''}}\binom{\alpha}{\alpha'}\binom{\alpha'}{\alpha''}\frac{(\beta''+\gamma''-e_qk'')!}{(\beta''+\gamma''-e_qk''-\alpha''')!}
\end{aligned}$$

$$\cdot|\xi|^{|\beta''+\gamma''-e_qk''|-|\alpha'''|}\left|D_\xi^{\alpha''+e_qk'}D_x^{\beta'}D_y^{\gamma'}a(x,y,\xi)\right|\left|D_\xi^{\alpha-\alpha'+e_q(n-k)}\psi_n(\xi)\right|$$
$$\leq C_1e^{-M(l'|x-y|)}\sum_\alpha|c_\alpha|\sum_{\alpha'\leq\alpha}\sum_{\substack{\alpha''+\alpha'''=\alpha'\\ \alpha'''\leq\beta''+\gamma''-e_qk''}}\binom{\alpha}{\alpha'}\binom{\alpha'}{\alpha''}$$
$$\cdot\frac{(\beta''+\gamma''-e_qk'')!}{(\beta''+\gamma''-e_qk''-\alpha''')!}\cdot|\xi|^{|\beta''+\gamma''-e_qk''|-|\alpha'''|}\cdot\frac{h_1^{|\alpha|-|\alpha'|+n-k}M_{\alpha-\alpha'+n-k}}{(Rm_n)^{|\alpha|-|\alpha'|+n-k}}$$
$$\cdot\frac{h^{|\alpha''|+|\beta'|+|\gamma'|+k'}\langle x-y\rangle^{\rho|\alpha''|+\rho k'+\rho|\beta'|+\rho|\gamma'|}A_{\alpha''+k'}B_{\beta'+\gamma'}}{\langle(x,y,\xi)\rangle^{\rho|\alpha''|+\rho k'+\rho|\beta'|+\rho|\gamma'|}}\cdot e^{M(m|\xi|)}e^{M(m|x|)}e^{M(m|y|)},$$

on the support of ψ_n. Note that $\langle x-y\rangle\leq 2(1+|x|^2+|y|^2)^{1/2}\leq 2\langle(x,y,\xi)\rangle$. Hence

$$\langle x-y\rangle^{\rho|\alpha''|+\rho|\beta'|+\rho|\gamma'|}\leq 2^{\rho|\alpha''|+\rho|\beta'|+\rho|\gamma'|}\langle(x,y,\xi)\rangle^{\rho|\alpha''|+\rho|\beta'|+\rho|\gamma'|}.$$

Also, $(\beta''+\gamma''-e_qk'')!\leq 2^{|\beta''+\gamma''-e_qk''|}(\beta''+\gamma''-e_qk''-\alpha''')!\alpha'''!$. Moreover $B_{\beta'+\gamma'}\leq c_0'L^{|\beta'|+|\gamma'|}M_{\beta'+\gamma'}$ and $A_{\alpha''+k'}\leq c_0'L^{|\alpha''|+k'}M^\rho_{\alpha''+k'}$. Let

$$T_n=\left\{\xi\in\mathbb{R}^d|\,2Rm_n\leq\langle\xi\rangle\leq 3Rm_{n+1}\right\}.$$

By construction, $\operatorname{supp}\psi_n\subseteq T_n$. Note that, on T_n,

$$\frac{|\xi|^{|\beta''+\gamma''-e_qk''|-|\alpha'''|}}{\langle(x,y,\xi)\rangle^{\rho k'}}\leq\frac{\langle\xi\rangle^{|\beta''+\gamma''-e_qk''|-|\alpha'''|}}{\langle\xi\rangle^{\rho k'}}\leq\frac{(3Rm_{n+1})^{|\beta''|+|\gamma''|}}{(3Rm_{n+1})^{|\alpha'''|+k''}(2Rm_n)^{\rho k'}}.$$

Because m_n is monotonically increasing, $m_n^{n-k}\geq m_n\cdot m_{n-1}\cdot\ldots\cdot m_{k+1}=M_n/M_k\geq M_{n-k}$ and similarly, $m_n^{k'}\geq M_{k'}$ and $m_n^{k''}\geq M_{k''}$. Moreover, there exists $\tilde{c}>0$ such that $M_p^\rho\leq\tilde{c}M_p$. We use this to estimate the above integral. By Fatou's lemma we have $\int_{\mathbb{R}^d}|\sum\ldots|\,d\xi\leq\sum\int_{\mathbb{R}^d}|\ldots|\,d\xi$. Considering the parts that are depended on α, α', α'' and α''', after using the above inequalities, one obtains

$$e^{M(3mRm_{n+1})}\sum_\alpha\sum_{\alpha'\leq\alpha}\sum_{\substack{\alpha''+\alpha'''=\alpha'\\ \alpha'''\leq\beta''+\gamma''-e_qk''}}\binom{\alpha}{\alpha'}\binom{\alpha'}{\alpha''}\frac{L_0^{|\alpha|-|\alpha'''|}L_0^{|\alpha'''|}}{M_\alpha}$$
$$\cdot\frac{\alpha'''!(2h)^{|\alpha''|}L^{|\alpha''|+k'}M^\rho_{\alpha''+k'}h_1^{|\alpha|-|\alpha'|}M_{\alpha-\alpha'+n-k}}{(3Rm_{n+1})^{|\alpha'''|+k''}(2Rm_n)^{\rho k'}(Rm_n)^{|\alpha|-|\alpha'|+n-k}}\cdot|T_n|$$
$$\leq C_2e^{M(3mRm_{n+1})}\sum_\alpha\sum_{\alpha'\leq\alpha}\sum_{\substack{\alpha''+\alpha'''=\alpha'\\ \alpha'''\leq\beta''+\gamma''-e_qk''}}\binom{\alpha}{\alpha'}\binom{\alpha'}{\alpha''}\frac{L_0^{|\alpha|-|\alpha'''|}}{M_\alpha}$$
$$\cdot\frac{(2h)^{|\alpha''|}L^{|\alpha|+n}h_1^{|\alpha|-|\alpha'|}H^{|\alpha|+n}M_{\alpha'''}M_{\alpha''}M^\rho_{k'}M_{\alpha-\alpha'}M_{n-k}}{(Rm_n)^{|\alpha'''|+k''}(Rm_n)^{\rho k'}(Rm_n)^{|\alpha|-|\alpha'|+n-k}}\cdot|T_n|$$
$$\leq\frac{C_2|T_n|e^{M(3mRm_{n+1})}}{R^{\rho n}}\sum_\alpha\sum_{\alpha'\leq\alpha}\sum_{\substack{\alpha''+\alpha'''=\alpha'\\ \alpha'''\leq\beta''+\gamma''-e_qk''}}\binom{\alpha}{\alpha'}\binom{\alpha'}{\alpha''}$$
$$\cdot\frac{(HL)^{|\alpha|+n}(2L_0h)^{|\alpha''|}(L_0h_1)^{|\alpha|-|\alpha'|}}{(RM_1)^{|\alpha|-|\alpha''|}m_n^{k''}}$$

$$\leq \frac{C_2|T_n|(HL)^n e^{M(3mRm_{n+1})}}{R^{\rho n}M_{k''}} \sum_{\alpha} \frac{(HL)^{|\alpha|}}{(RM_1)^{|\alpha|}} \sum_{\alpha'\leq\alpha} \binom{\alpha}{\alpha'} \cdot (1+2L_0 hRM_1)^{|\alpha'|} (L_0 h_1)^{|\alpha|-|\alpha'|}$$
$$= \frac{C_2|T_n|(HL)^n e^{M(3mRm_{n+1})}}{R^{\rho n}M_{k''}} \sum_{\alpha} \left(\frac{HL}{RM_1} + 2hHLL_0 + \frac{h_1 HLL_0}{RM_1}\right)^{|\alpha|}.$$

Take R such that $\frac{HL}{RM_1}+\frac{1}{8}+\frac{1}{RM_1}\leq\frac{1}{2}$ and take h and h_1 small enough such that $2hHLL_0 \leq 1/8$ and $h_1 HLL_0 \leq 1$. Then, the sum will be uniformly convergent for all h and h_1 for which the previous inequalities hold. The choice of R depends only on A_p, B_p and M_p (and not on L_0, hence not on the operator P_l). Also, the choice of h and h_1 depend on A_p, B_p, M_p and the operator P_l, but not on R. Before we continue, note that, from the way we choose q, we have the following inequality

$$1+|x-y|^2 \leq \langle x\rangle^2 + d|x_q-y_q|^2 \leq \frac{d^2}{r^2}|x_q-y_q|^2 + d|x_q-y_q|^2 \leq \left(\frac{d}{r}+d\right)^2 |x_q-y_q|^2.$$

For shorter notation, put $r_1 = \frac{d}{r}+d$. So, we obtain $\langle x-y\rangle \leq r_1|x_q-y_q|$. Now, for the estimate of $\left|D_x^\beta D_y^\gamma K_n(x,y)\right|$, by using (4.9), we obtain

$$\left|D_x^\beta D_y^\gamma K_n(x,y)\right|$$
$$\leq C_3 \sum_{\substack{\beta'+\beta''=\beta\\ \gamma'+\gamma''=\gamma}} \sum_{k=0}^{n} \sum_{\substack{k'+k''=k\\ k''\leq\beta''_q+\gamma''_q}} \binom{\beta}{\beta'}\binom{\gamma}{\gamma'}\binom{n}{k}\binom{k}{k'} \frac{(\beta''+\gamma'')!}{(\beta''+\gamma''-e_q k'')!} \frac{\langle x-y\rangle^{\rho k'}}{|x_q-y_q|^n}$$
$$\cdot 2^{|\beta''+\gamma''-e_q k''|}(3Rm_{n+1})^{|\beta''|+|\gamma''|} h^{|\beta'|+|\gamma'|+k'} 2^{|\beta'|+|\gamma'|} L^{|\beta'|+|\gamma'|} M_{\beta'+\gamma'} h_1^{n-k}$$
$$\cdot e^{-M(l'|y-x|)} e^{M(m|x|)} e^{M(m|y|)} \frac{|T_n|(HL)^n e^{M(3mRm_{n+1})}}{R^{\rho n}M_{k''}}$$
$$\leq C_3 r_1^n \sum_{\substack{\beta'+\beta''=\beta\\ \gamma'+\gamma''=\gamma}} \sum_{k=0}^{n} \sum_{k'+k''=k} \binom{\beta}{\beta'}\binom{\gamma}{\gamma'}\binom{n}{k}\binom{k}{k'} \frac{4^{|\beta''|+|\gamma''|}k''!}{M_{k''}} \cdot \frac{h_1^{k''}}{h_1^{k''}}$$
$$\cdot \frac{(3m'R^2 m_{n+1})^{|\beta''|+|\gamma''|}}{2^{k''}(m'R)^{|\beta''|+|\gamma''|}M_{\beta''+\gamma''}} h^{|\beta'|+|\gamma'|+k'}(2L)^{|\beta'|+|\gamma'|} h_1^{n-k} M_{\beta+\gamma}$$
$$\cdot e^{-M(m'|(x,y)|)} \frac{|T_n|(HL)^n e^{M(3mRm_{n+1})}}{R^{\rho n}}.$$

Note that $\frac{(3m'R^2 m_{n+1})^{|\beta''|+|\gamma''|}}{M_{\beta''+\gamma''}} \leq e^{M(3m'R^2 m_{n+1})}$. Also, by using $(M.2)$, we obtain

$$|T_n| = \omega_d\left(\left(9R^2 m_{n+1}^2-1\right)^{d/2} - \left(4R^2 m_n^2 - 1\right)^{d/2}\right) \leq \omega_d (3Rm_{n+1})^d$$

$$\leq \omega_d (3c_0 R M_1)^d H^{(n+1)d}$$

(ω_d is the volume of the d-dimensional unit ball). By proposition 1.2.1

$$e^{M(3mRm_{n+1})} e^{M(3m'R^2 m_{n+1})} \leq c_0 e^{M(3Hm'R^2 m_{n+1})},$$

where we take $R \geq m$ (which depends only on a). We obtain

$$\begin{aligned} \left|D_x^\beta D_y^\gamma K_n(x,y)\right| &\leq C_4 (3c_0 R M_1 H)^d \frac{M_{\beta+\gamma} (HL)^n H^{nd} e^{M(3Hm'R^2 m_{n+1})} r_1^n}{e^{M(m'|(x,y)|)} R^{\rho n}} \\ &\cdot \left(\frac{4}{m'R} + 2hL\right)^{|\beta|+|\gamma|} \left(h_1 + h + \frac{h_1}{2}\right)^n. \end{aligned}$$

By lemma 4.1.3 we have

$$e^{M(3Hm'R^2 m_{n+1})} \leq c_0 H^{2(3c_0 Hm'R^2+2)(n+1)} = c_0 H^{2(3c_0 Hm'R^2+2)} \left(H^{2(3c_0 Hm'R^2+2)}\right)^n.$$

Take $R^\rho > H^{d+1} L r_1$ and $R \geq 8$. For the fixed m' in the beginning of the proof, choose h small enough such that $2hL \leq 1/(2m')$. Then $\dfrac{4}{m'R} + 2hL \leq \dfrac{1}{m'}$. For the chosen R, choose h and h_1 smaller then the chosen before such that $H^{2(3c_0 Hm'R^2+2)} \left(h_1 + h + \dfrac{h_1}{2}\right) \leq 1$. (Note that the choice of R and hence the choice of ψ_n, $n \in \mathbb{N}$, depends only on A_p, B_p, M_p and a, but not on the operator P_l or m'.) Then $\sum_{n=1}^{\infty} \left|D_x^\beta D_y^\gamma K_n(x,y)\right|$ will converge and we have the following estimate

$$\sum_{n=1}^{\infty} \left|D_x^\beta D_y^\gamma K_n(x,y)\right| \leq C \frac{M_{\beta+\gamma}}{e^{M(m'|(x,y)|)} m'^{|\beta|+|\gamma|}}.$$

For $\left|D_x^\beta D_y^\gamma K_0(x,y)\right|$, by similar procedure, we obtain the same estimate. Hence (4.7) holds and the proof for the (M_p) case is complete.

The $\{M_p\}$ case. We will prove that for every $(t_p), (t'_p) \in \mathfrak{R}$,

$$\sup_{\beta,\gamma \in \mathbb{N}^d} \sup_{(x,y)\in\overline{\Omega_r}} \frac{\left|D_x^\beta D_y^\gamma K(x,y)\right| e^{N_{t_p}(|(x,y)|)}}{T'_{\beta+\gamma} M_{\beta+\gamma}} < \infty, \tag{4.10}$$

for every fixed $0 < r < 1$, where $T'_{\beta+\gamma} = \prod_{j=1}^{|\beta|+|\gamma|} t'_j$ and $T'_0 = 1$. From this, the claim in the lemma follows. To prove this, fix $0 < r < 1$ and take $\theta \in \mathcal{E}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ as in lemma 4.1.5. Define $\tilde{K} = K\theta$. Then $\tilde{K}$ is $\mathcal{C}^\infty$ function and for every $(t_p), (t'_p) \in \mathfrak{R}$, $\sup_{\beta,\gamma\in\mathbb{N}^d} \sup_{(x,y)\in\mathbb{R}^{2d}} \dfrac{\left|D_x^\beta D_y^\gamma \tilde{K}(x,y)\right| e^{N_{t_p}(|(x,y)|)}}{T'_{\beta+\gamma} M_{\beta+\gamma}} < \infty$. Hence $\tilde{K} \in \mathcal{S}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$. So, there exists $h > 0$ such that

$$\sup_{\beta,\gamma\in\mathbb{N}^d} \sup_{(x,y)\in\mathbb{R}^{2d}} \frac{h^{|\beta|+|\gamma|} \left|D_x^\beta D_y^\gamma \tilde{K}(x,y)\right| e^{M(h|(x,y)|)}}{M_{\beta+\gamma}} < \infty.$$

But, $\tilde{K}(x,y) = K(x,y)$ on $\Omega_{3r/4}$ and the desired estimate follows. Now, to prove (4.10). Let $a \in \Pi^{\{M_p\},\infty}_{A_p,B_p}(\mathbb{R}^{3d})$. Then there exists $h>0$ such that $a \in \Pi^{M_p,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{3d};h,m)$, for all $m>0$. By lemma 4.1.1, there exist $(k_p)\in\mathfrak{R}$ and $c_0'>0$ such that

$$\left|D^\alpha_\xi D^\beta_x D^\gamma_y a(x,y,\xi)\right| \leq c_0' \frac{h^{|\alpha|+|\beta|+|\gamma|}\langle x-y\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma} e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{\langle (x,y,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}},$$

for all $\alpha,\beta,\gamma\in\mathbb{N}^d$ and $(x,y,\xi)\in\mathbb{R}^{3d}$. Let $(t_p),(t_p')\in\mathfrak{R}$ be fixed. For $(l_p)\in\mathfrak{R}$ consider $P_{l_p}(\xi)$. By proposition 2.1.1, we can choose $P_{l_p}(\xi)$ such that, $\left|P_{l_p}(\xi)\right| \geq c'' e^{N_{l_p'}(|\xi|)}$ where $(l_p')\in\mathfrak{R}$ is such that $e^{N_{l_p'}(|x-y|)} \geq c_1 e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)} e^{N_{t_p}(|(x,y)|)}$ on Ω_r. This is possible because of (4.8). On the other hand, if we represent $P_{l_p}(\xi) = \sum_\alpha c_\alpha \xi^\alpha$ then for every $L'>0$ there exists $C'>0$ such that $|c_\alpha| \leq C' L'^{|\alpha|}/M_\alpha$. By the same calculations, one obtains the same form for $D^\beta_x D^\gamma_y K_n(x,y)$ as in the (M_p) case, but with P_{l_p} in place of P_l. The prove continues in the same way as above. We will point out only the notable differences. The first difference is in the estimate of the part that is depended on α, α', α'' and α''' (for $n\in\mathbb{Z}_+$) and the integral over $\mathbb{R}^d_\xi$, where in the $\{M_p\}$ case one obtains the estimate

$$\frac{C_2|T_n|(HL)^n e^{N_{k_p}(3Rm_{n+1})}}{R^{\rho n} M_{k''}} \sum_\alpha \left(\frac{HL}{RM_1} + 2hHLL' + \frac{h_1 HLL'}{RM_1}\right)^{|\alpha|}.$$

The convergence of this sum follows from the fact that we can take R arbitrary large and L' arbitrary small. Moving on to the estimate of $\left|D^\beta_x D^\gamma_y K_n(x,y)\right|$, in similar fashion, one obtains the following

$$\begin{aligned}
&\left|D^\beta_x D^\gamma_y K_n(x,y)\right| \\
&\quad\leq C_3 r_1^n \sum_{\substack{\beta'+\beta''=\beta\\ \gamma'+\gamma''=\gamma}} \sum_{k=0}^{n} \sum_{k'+k''=k} \binom{\beta}{\beta'}\binom{\gamma}{\gamma'}\binom{n}{k}\binom{k}{k'} \frac{12^{|\beta''|+|\gamma''|}k''!}{M_{k''}} \frac{R^{|\beta''|+|\gamma''|}}{2^{k''}} \\
&\qquad \cdot \frac{m_{n+1}^{|\beta''|+|\gamma''|}}{M_{\beta''+\gamma''}} h^{|\beta'|+|\gamma'|+k'} (2L)^{|\beta'|+|\gamma'|} h_1^{n-k} M_{\beta+\gamma} e^{-N_{t_p}(|(x,y)|)} \\
&\qquad \cdot \frac{|T_n|(HL)^n e^{N_{k_p}(3Rm_{n+1})}}{R^{\rho n}}.
\end{aligned}$$

By using the increasingness of m_p and $(M.2)$, we get

$$\frac{m_{n+1}^{|\beta''|+|\gamma''|}}{M_{\beta''+\gamma''}} \leq \frac{m_{n+2}\cdot m_{n+3}\cdot\ldots\cdot m_{n+1+|\beta''|+|\gamma''|}}{M_{\beta''+\gamma''}} = \frac{M_{n+1+\beta''+\gamma''}}{M_{\beta''+\gamma''}M_{n+1}} \leq c_0 H^{n+1+|\beta''|+|\gamma''|}.$$

We obtain the estimate:

$$\begin{aligned}
&\left|D^\beta_x D^\gamma_y K_n(x,y)\right| \\
&\quad\leq C_4 \frac{M_{\beta+\gamma}|T_n|(H^2L)^n e^{N_{k_p}(3Rm_{n+1})} r_1^n}{e^{N_{t_p}(|(x,y)|)} R^{\rho n}} (12RH+2hL)^{|\beta|+|\gamma|} \left(h_1+h+\frac{1}{2}\right)^n.
\end{aligned}$$

By lemma 4.1.3 $e^{N_{k_p}(3Rm_{n+1})} \leq cH^{n+1}$, where c depends only on (k_p), R and M_p (does not depend on n). Now, if we use the same estimate for $|T_n|$ as in the (M_p) case, if we take large enough R, the sum $\sum_{n=1}^{\infty} \left|D_x^\beta D_y^\gamma K_n(x,y)\right|$ will converge and we obtain

$$\sum_{n=1}^{\infty} \left|D_x^\beta D_y^\gamma K(x,y)\right| \leq C \frac{M_{\beta+\gamma}}{e^{N_{t_p}(|(x,y)|)}} (12RH + 2hL)^{|\beta|+|\gamma|}.$$

One obtains similar estimates for $\left|D_x^\beta D_y^\gamma K_0(x,y)\right|$. Hence we obtain (4.10) and the proof for the $\{M_p\}$ case is complete. It remains to prove the fact that if there exists r, $0 < r < 1$, such that $a(x,y,\xi) = 0$ for $(x,y,\xi) \in \left(\mathbb{R}^{2d} \backslash \Omega_r\right) \times \mathbb{R}^d$ then $K \in \mathcal{S}^*\left(\mathbb{R}^{2d}\right)$. But this trivially follows from the proved growth condition of $D_x^\beta D_y^\gamma K(x,y)$ and the fact that for $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$, $K_n(x,y) = 0$ for all $n \in \mathbb{N}$, hence, $K = 0$ on $\mathbb{R}^{2d} \backslash \Omega_r$. □

4.2 Symbolic Calculus

Let $\rho_1 = \inf\{\rho \in \mathbb{R}_+ |\, A_p \subset M_p^\rho\}$ and $\rho_2 = \inf\{\rho \in \mathbb{R}_+ |\, B_p \subset M_p^\rho\}$ and put $\rho_0 = \max\{\rho_1, \rho_2\}$. Then $0 < \rho_0 \leq 1$ and for every ρ such that $\rho_0 \leq \rho \leq 1$, if the larger infimum can be reached, or, otherwise $\rho_0 < \rho \leq 1$, $A_p \subset M_p^\rho$ and $B_p \subset M_p^\rho$. So, for every such ρ, there exists $c_0' > 0$ and $L > 0$ (which depend on ρ) such that, $A_p \leq c_0' L^p M_p^\rho$, $B_p \leq c_0' L^p M_p^\rho$. Moreover, because M_p tends to infinity, there exists $\tilde{c} > 0$ such that $M_p^\rho \leq \tilde{c} M_p$, for all such ρ. From now on we suppose that $\rho_0 \leq \rho \leq 1$, if the larger infimum can be reached, or otherwise $\rho_0 < \rho \leq 1$.

For $t > 0$, put $Q_t = \left\{(x,\xi) \in \mathbb{R}^{2d} |\, \langle x \rangle < t, \langle \xi \rangle < t\right\}$ and $Q_t^c = \mathbb{R}^{2d} \backslash Q_t$. Denote by $FS_{A_p,B_p,\rho}^{M_p,\infty}\left(\mathbb{R}^{2d}; B, h, m\right)$ the vector space of all formal series $\sum_{j=0}^{\infty} a_j(x,\xi)$ such that $a_j \in \mathcal{C}^\infty\left(\operatorname{int} Q_{Bm_j}^c\right)$, $D_\xi^\alpha D_x^\beta a_j(x,\xi)$ can be extended to continuous function on $Q_{Bm_j}^c$ for all $\alpha, \beta \in \mathbb{N}^d$ and

$$\sup_{j \in \mathbb{N}} \sup_{\alpha,\beta} \sup_{(x,\xi) \in Q_{Bm_j}^c} \frac{\left|D_\xi^\alpha D_x^\beta a_j(x,\xi)\right| \langle (x,\xi) \rangle^{\rho|\alpha|+\rho|\beta|+2j\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha|+|\beta|+2j} A_\alpha B_\beta A_j B_j} < \infty.$$

In the above, we use the convention $m_0 = 0$ and hence $Q_{Bm_0}^c = \mathbb{R}^{2d}$. It is easy to check that $FS_{A_p,B_p,\rho}^{M_p,\infty}\left(\mathbb{R}^{2d}; B, h, m\right)$ is a (B) - space. Define

$$\begin{aligned}
FS_{A_p,B_p,\rho}^{(M_p),\infty}\left(\mathbb{R}^{2d}; B, m\right) &= \varprojlim_{h \to 0} FS_{A_p,B_p,\rho}^{M_p,\infty}\left(\mathbb{R}^{2d}; B, h, m\right), \\
FS_{A_p,B_p,\rho}^{(M_p),\infty}\left(\mathbb{R}^{2d}\right) &= \varinjlim_{B,m \to \infty} FS_{A_p,B_p,\rho}^{(M_p),\infty}\left(\mathbb{R}^{2d}; B, m\right), \\
FS_{A_p,B_p,\rho}^{\{M_p\},\infty}\left(\mathbb{R}^{2d}; B, h\right) &= \varprojlim_{m \to 0} FS_{A_p,B_p,\rho}^{M_p,\infty}\left(\mathbb{R}^{2d}; B, h, m\right),
\end{aligned}$$

$$FS^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right) = \varinjlim_{B,h\to\infty} FS^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};B,h\right).$$

Then, $FS^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};B,m\right)$ and $FS^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};B,h\right)$ are (F) - spaces. Note that the inclusion mappings $FS^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};B,m\right) \to \prod_{j=0}^{\infty}\mathcal{E}^{(M_p)}\left(\operatorname{int} Q^c_{Bm_j}\right)$ and $FS^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};B,h\right) \to \prod_{j=0}^{\infty}\mathcal{E}^{\{M_p\}}\left(\operatorname{int} Q^c_{Bm_j}\right)$, $\sum_{j=0}^{\infty} a_j \mapsto (a_0,a_1,a_2,...)$, are continuous, so $FS^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and $FS^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ are Hausdorff l.c.s. Moreover, as inductive limits of barrelled and bornological spaces they are barrelled and bornological. Note, also, that the inclusions $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right) \to FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, defined as $a \mapsto \sum_{j\in\mathbb{N}} a_j$, where $a_0 = a$ and $a_j = 0$, $j \geq 1$, is continuous.

Definition 4.2.1. Two sums, $\sum_{j\in\mathbb{N}} a_j, \sum_{j\in\mathbb{N}} b_j \in FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, are said to be equivalent, in notation $\sum_{j\in\mathbb{N}} a_j \sim \sum_{j\in\mathbb{N}} b_j$, if there exist $m>0$ and $B>0$, resp. there exist $h>0$ and $B>0$, such that for every $h>0$, resp. for every $m>0$,

$$\sup_{N\in\mathbb{Z}_+}\sup_{\alpha,\beta}\sup_{(x,\xi)\in Q^c_{Bm_N}} \frac{\left|D^\alpha_\xi D^\beta_x \sum_{j<N}\left(a_j(x,\xi)-b_j(x,\xi)\right)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2N\rho}}{h^{|\alpha|+|\beta|+2N}A_\alpha B_\beta A_N B_N}\cdot$$
$$\cdot e^{-M(m|\xi|)}e^{-M(m|x|)} < \infty.$$

From now on, we assume that A_p and B_p satisfy $(M.2)$. Without losing generality we can assume that the constants c_0 and H from the condition $(M.2)$ for A_p and B_p are the same as the corresponding constants for M_p.

Theorem 4.2.1. *Let $a \in \Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ be such that $a \sim 0$. Then, for every $\tau\in\mathbb{R}$, $\operatorname{Op}_\tau(a)$ is *-regularizing.*

Proof. First we will prove the following lemma.

Lemma 4.2.1. *Let $0<l\leq 1$ and $B>1$. There exists $C>0$ depending on B, l and M_p and $\tilde{m}>0$ depending only on B and M_p and not on l such that*

$$\inf\left\{\frac{M_n}{l^n\rho^n}\middle| n\in\mathbb{Z}_+,\ \rho\geq Bm_n\right\} \leq Ce^{-M(l\tilde{m}\rho)},\ \textit{for all } \rho\geq BM_1.$$

Proof. For shorter notation put

$$f(\rho) = \inf\left\{\frac{M_n}{l^n\rho^n}\middle| n\in\mathbb{Z}_+,\ \rho\geq Bm_n\right\}$$

and $T_{\rho,0} = \{n\in\mathbb{Z}_+ |\, \rho\geq Bm_n\}$, $T_{\rho,1} = \{n\in\mathbb{Z}_+ |\, \rho< Bm_n\}$. Obviously $T_{\rho,0}\cup T_{\rho,1} = \mathbb{Z}_+$ and they are not empty. For $n\in\mathbb{Z}_+$, denote by $\mathbb{Z}_{+,n}$ the set $\{1,...,n\}$. By the properties of m_n, there exists $k\in\mathbb{Z}_+$ (which depends on ρ) such that $T_{\rho,0} =$

$\{1, 2, ..., k\}$. In the proof of lemma 4.1.3, we proved that, for $s \in \mathbb{Z}_+$, $\dfrac{m_{k+s+1}}{m_{k+1}} \geq \dfrac{s}{c_0(k+1)}$. Take $s = 2k([c_0]+1)$, and for shorter notation, put $t = 2[c_0]+2$. Then $m_{k+kt+1} > m_{k+1}$. For $q \in \mathbb{Z}_+$, we get $Bm_{k+kt+q} \geq Bm_{k+kt+1} > Bm_{k+1} \geq l\rho$. Then, for $q \in \mathbb{Z}_+$, we have

$$\frac{B^{k+kt+q}M_{k+kt+q}}{l^{k+kt+q}\rho^{k+kt+q}} = \frac{B^{k+kt+q-1}M_{k+kt+q-1}}{l^{k+kt+q-1}\rho^{k+kt+q-1}} \cdot \frac{Bm_{k+kt+q}}{l\rho} > \frac{B^{k+kt+q-1}M_{k+kt+q-1}}{l^{k+kt+q-1}\rho^{k+kt+q-1}}.$$

So, we obtain

$$e^{-M(l\rho/B)} = \inf_{n\in\mathbb{N}} \frac{B^n M_n}{l^n \rho^n} = \inf_{n\in\mathbb{Z}_{+,k+kt}} \frac{B^n M_n}{l^n \rho^n}, \tag{4.11}$$

for $\rho > BM_1/l$ (the infimum can not be obtained for $n = 0$). Now, let $0 \leq q \leq t$, $q \in \mathbb{N}$ and $n \in T_{\rho,0}$. One has

$$\frac{B^{n+qk}M_{n+qk}}{l^{n+qk}\rho^{n+qk}} \geq \frac{B^n M_n}{l^n \rho^n}\left(\frac{B^k M_k}{l^k \rho^k}\right)^q \geq f(\rho)^{q+1} \geq f(\rho)^{t+1},$$

where the last inequality holds because $f(\rho) \leq 1$ when $\rho > BM_1/l$. Hence, by (4.11), $e^{-M(l\rho/B)} \geq f(\rho)^{t+1}$, for $\rho > BM_1/l$. Repeated use of proposition 1.2.1 yields

$$(t+1)M\left(\frac{l\rho}{BH^{t+1}}\right) \leq 2^{t+1}M\left(\frac{l\rho}{BH^{t+1}}\right) \leq M\left(\frac{l\rho}{B}\right) + \ln c',$$

i.e. $f(\rho) \leq e^{-\frac{1}{t+1}M(l\rho/B)} \leq Ce^{-M(l\tilde{m}\rho)}$, $\forall \rho > BM_1/l$, where we put $\tilde{m} = 1/(BH^{t+1})$, which depends only on B and the sequence M_p (recall that $t = 2[c_0]+2$). For $BM_1 \leq \rho \leq BM_1/l$, $f(\rho)$ is bounded so the same inequality holds, possibly with another C. □

We continue the proof of the theorem. It is enough to prove that $a \in \mathcal{S}^*$, because then the claim will follow from proposition 4.1.1. Because $a \sim 0$, in the (M_p) case, there exist $m > 0$ and $B > 0$, such that for every $h > 0$ there exists $C > 0$, resp. in the $\{M_p\}$ case, there exist $h > 0$ and $B > 0$, such that for every $m > 0$ there exists $C > 0$, such that

$$\begin{aligned}\left|D_\xi^\alpha D_x^\beta a(x,\xi)\right| &\leq C\frac{h^{|\alpha|+|\beta|+2N}A_\alpha B_\beta A_N B_N e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho N}\langle(x,\xi)\rangle^{\rho|\beta|+\rho N}}\\ &\leq C_1\frac{h^{|\alpha|+|\beta|}A_\alpha B_\beta e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}} \cdot \frac{(hL)^{2N}M_N^{2\rho}}{\langle(x,\xi)\rangle^{2N\rho}},\end{aligned}$$

for all $N \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{N}^d$, $(x,\xi) \in Q^c_{Bm_N}$. It is obvious that without losing generality we can assume that $B > 1$. In the (M_p) case let $m' > 0$ be arbitrary but fixed. Let $(x,\xi) \in Q^c_{Bm_1}$. Then, there exists $N \in \mathbb{Z}_+$ such that $(x,\xi) \in Q_{Bm_{N+1}} \backslash Q_{Bm_N}$. We estimate as follows

$$\begin{aligned}
&\frac{m'^{|\alpha|+|\beta|}\left|D_\xi^\alpha D_x^\beta a(x,\xi)\right| e^{M(m'|(x,\xi)|)}}{M_{\alpha+\beta}}\\
&\qquad\leq\; C_1\frac{(m'h)^{|\alpha|+|\beta|}A_\alpha B_\beta e^{M(m|\xi|)}e^{M(m|x|)}e^{M(m'|(x,\xi)|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}M_{\alpha+\beta}}\cdot\frac{(hL)^{2N}M_N^{2\rho}}{\langle(x,\xi)\rangle^{2N\rho}}\\
&\qquad\leq\; C_2\frac{(m'hL)^{|\alpha|+|\beta|}M_{\alpha+\beta}^{\rho}e^{2M(mBm_{N+1})}e^{M(2m'Bm_{N+1})}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}M_{\alpha+\beta}}\cdot\frac{(hL)^{2N}M_N^{2\rho}}{(Bm_N)^{2N\rho}}\\
&\qquad\leq\; C_3(m'hL)^{|\alpha|+|\beta|}(hL)^{2N}e^{2M(mBm_{N+1})}e^{M(2m'Bm_{N+1})},
\end{aligned}$$

where, in the last inequality, we used $m_N^N\geq M_N$. By lemma 4.1.3, we have

$$\begin{aligned}
&e^{2M(mBm_{N+1})}e^{M(2m'Bm_{N+1})}\\
&\qquad\leq\; c_0^3H^{4(c_0mB+2)(N+1)}H^{2(2c_0m'B+2)(N+1)}\\
&\qquad=\; c_0^3H^{4(c_0mB+2)}H^{2(2c_0m'B+2)}\left(H^{4(c_0mB+2)}H^{2(2c_0m'B+2)}\right)^N.
\end{aligned}$$

Take h small enough such that $m'hL\leq 1$ and $h^2L^2H^{4(c_0mB+2)}H^{2(2c_0m'B+2)}\leq 1$. We get

$$\frac{m'^{|\alpha|+|\beta|}\left|D_\xi^\alpha D_x^\beta a(x,\xi)\right| e^{M(m'|(x,\xi)|)}}{M_{\alpha+\beta}}\leq C$$

for all $\alpha,\beta\in\mathbb{N}^d$ and $(x,\xi)\in Q^c_{Bm_1}$. For $(x,\xi)\in Q_{Bm_1}$ the same estimate will hold, possibly for another $C>0$, because $a\in\Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)\subseteq\mathcal{E}^{(M_p)}\left(\mathbb{R}^{2d}\right)$ and Q_{Bm_1} is bounded.

In the $\{M_p\}$ case, by the above observations, we have

$$\begin{aligned}
\left|D_\xi^\alpha D_x^\beta a(x,\xi)\right| \;&\leq\; C_1\frac{h^{|\alpha|+|\beta|}A_\alpha B_\beta e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}\\
&\qquad\cdot\left(\inf\left\{\frac{\left(h^{1/\rho}L^{1/\rho}\right)^N M_N}{\langle(x,\xi)\rangle^N}\Big| N\in\mathbb{Z}_+,\,(x,\xi)\in Q^c_{Bm_N}\right\}\right)^{2\rho}.
\end{aligned}$$

and it is obvious that without losing generality we can assume that $h\geq 1$ and $L\geq 1$ (L is the constant from $A_p\subset M_p^\rho$ and $B_p\subset M_p^\rho$). Now, note that

$$\begin{aligned}
&\inf\left\{\frac{\left(h^{1/\rho}L^{1/\rho}\right)^N M_N}{\langle(x,\xi)\rangle^N}\Big| N\in\mathbb{Z}_+,\,(x,\xi)\in Q^c_{Bm_N}\right\}\\
&\leq\inf\left\{\frac{\left(h^{1/\rho}L^{1/\rho}\right)^N M_N}{\langle(x,\xi)\rangle^N}\Big| N\in\mathbb{Z}_+,\,\langle(x,\xi)\rangle\geq 2Bm_N\right\}\leq C'e^{-M\left(\tilde{m}\langle(x,\xi)\rangle/(hL)^{1/\rho}\right)},
\end{aligned}$$

for all $\langle(x,\xi)\rangle\geq 2BM_1$, where in the last inequality we used the above lemma with $l=(hL)^{-1/\rho}\leq 1$. Proposition 1.2.1 yields $e^{M(m|\xi|)}e^{M(m|x|)}\leq c_0e^{M(mH|(x,\xi)|)}$. Because $A_p\subset M_p^\rho$ and $B_p\subset M_p^\rho$, we have

$$\left|D_\xi^\alpha D_x^\beta a(x,\xi)\right|\leq C_2(L^2h)^{|\alpha|+|\beta|}M_{\alpha+\beta}e^{M(mH|(x,\xi)|)}e^{-M\left(|(x,\xi)|\tilde{m}/(hL)^{1/\rho}\right)},$$

for all $\alpha,\beta\in\mathbb{N}^d$ and $\langle(x,\xi)\rangle\geq 2BM_1$. For $\langle(x,\xi)\rangle\leq 2BM_1$ the same estimate will hold, possibly for another $C>0$ and $\tilde{h}>0$ instead of L^2h, because $a\in\Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)\subseteq\mathcal{E}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$ and the set $\left\{(x,\xi)\in\mathbb{R}^{2d}|\,\langle(x,\xi)\rangle\leq 2BM_1\right\}$ is bounded. m can be arbitrary small, so if we take m small enough we have $e^{M(mH|(x,\xi)|)}e^{-M\left(|(x,\xi)|\tilde{m}/(hL)^{\lambda/\rho}\right)}\leq C_3e^{-M(m'|(x,\xi)|)}$ for some, small enough, $m'>0$, which completes the proof in the $\{M_p\}$ case. □

Theorem 4.2.2. *Let* $\sum_{j\in\mathbb{N}}a_j\in FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ *be given. Than, there exists* $a\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, *such that* $a\sim\sum_{j\in\mathbb{N}}a_j$.

Proof. Define $\varphi(x)\in\mathcal{D}^{(B_p)}\left(\mathbb{R}^d\right)$ and $\psi(\xi)\in\mathcal{D}^{(A_p)}\left(\mathbb{R}^d\right)$, in the (M_p) case, resp. $\varphi(x)\in\mathcal{D}^{\{B_p\}}\left(\mathbb{R}^d\right)$ and $\psi(\xi)\in\mathcal{D}^{\{A_p\}}\left(\mathbb{R}^d\right)$ in the $\{M_p\}$ case, such that $0\leq\varphi,\psi\leq 1$, $\varphi(x)=1$ when $\langle x\rangle\leq 2$, $\psi(\xi)=1$ when $\langle\xi\rangle\leq 2$ and $\varphi(x)=0$ when $\langle x\rangle\geq 3$, $\psi(\xi)=0$ when $\langle\xi\rangle\geq 3$. Put $\chi(x,\xi)=\varphi(x)\psi(\xi)$, $\chi_n(x,\xi)=\chi\left(\frac{x}{Rm_n},\frac{\xi}{Rm_n}\right)$ for $n\in\mathbb{Z}_+$ and $R>0$ and put $\chi_0(x,\xi)=0$. It is easily checked that $\chi,\chi_n\in\mathcal{D}^{(M_p)}\left(\mathbb{R}^{2d}\right)$, resp. $\chi,\chi_n\in\mathcal{D}^{\{M_p\}}\left(\mathbb{R}^{2d}\right)$.

The (M_p) case. Let $m,B>0$ are such that $\sum_j a_j\in FS^{M_p,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d};B,h,m\right)$ for all $h>0$. For $R\geq 2B$, $a(x,\xi)=\sum_j\left(1-\chi_j(x,\xi)\right)a_j(x,\xi)$ is a well defined $\mathcal{C}^\infty\left(\mathbb{R}^{2d}\right)$ function. We will prove that for sufficiently large R, $a\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and $a\sim\sum_j a_j(x,\xi)$ which will complete the proof in the (M_p) case. For $0<h<1$, using the fact that $1-\chi_j(x,\xi)=0$ for $(x,\xi)\in Q_{Rm_j}$, we have the estimates

$$\frac{\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}e^{-M(m|\xi|)}e^{-M(m|x|)}}{(8h)^{|\alpha|+|\beta|}A_\alpha B_\beta}$$

$$\leq \sum_{j\in\mathbb{N}}\sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\left|D^{\alpha-\gamma}_\xi D^{\beta-\delta}_x a_j(x,\xi)\right|e^{-M(m|\xi|)}e^{-M(m|x|)}$$
$$\cdot\frac{\left|D^\gamma_\xi D^\delta_x\left(1-\chi_j(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|}}{(8h)^{|\alpha|+|\beta|}A_\alpha B_\beta}$$

$$\leq C_0\sum_{j\in\mathbb{N}}\sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{h^{|\alpha|-|\gamma|+|\beta|-|\delta|+2j}A_{\alpha-\gamma}B_{\beta-\delta}A_jB_j}{(8h)^{|\alpha|+|\beta|}A_\alpha B_\beta}$$
$$\cdot\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|-2\rho j}\left|D^\gamma_\xi D^\delta_x\left(1-\chi_j(x,\xi)\right)\right|$$

$$\leq C_0\sum_{j\in\mathbb{N}}\frac{1}{8^{|\alpha|+|\beta|}}h^{2j}L^{2j}M_j^{2\rho}\left|1-\chi_j(x,\xi)\right|\langle(x,\xi)\rangle^{-2\rho j}$$
$$+C_0\sum_{j\in\mathbb{N}}\frac{1}{8^{|\alpha|+|\beta|}}\sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\ (\delta,\gamma)\neq(0,0)}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}$$
$$\cdot\frac{h^{2j}L^{2j}M_j^{2\rho}\left|D^\gamma_\xi D^\delta_x\left(1-\chi_j(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|-2\rho j}}{h^{|\gamma|+|\delta|}A_\gamma B_\delta}$$

$$= \quad S_1 + S_2,$$

where S_1 and S_2 are the first and the second sum, correspondingly. To estimate S_1 note that, on the support of $1-\chi_j$ the inequality $\langle (x,\xi)\rangle \geq Rm_j$ holds. One obtains

$$S_1 \leq C_0 \sum_{j\in\mathbb{N}} \frac{(hL)^{2j} M_j^{2\rho}}{R^{2\rho j} m_j^{2\rho j}} \leq C_0 \sum_{j\in\mathbb{N}} \frac{(hL)^{2j}}{R^{2\rho j}} < \infty,$$

for large enough R (in the second inequality we use the fact that $m_j^j \geq M_j$). For the estimate of S_2, note that $D_\xi^\gamma D_x^\delta (1-\chi_j(x,\xi)) = 0$ when $(x,\xi)\in Q_{3Rm_j}^c$, because $(\delta,\gamma)\neq(0,0)$ and $\chi_j(x,\xi)=0$ on $Q_{3Rm_j}^c$. So, for $(x,\xi)\in Q_{3Rm_j}$, we have that $\langle(x,\xi)\rangle \leq \langle x\rangle + \langle\xi\rangle \leq 6Rm_j$. Moreover, from the construction of χ, we have that for the chosen h, there exists $C_1>0$ such that $\left|D_\xi^\alpha D_x^\beta \chi(x,\xi)\right| \leq C_1 h^{|\alpha|+|\beta|} A_\alpha B_\beta$. By using $m_j^j \geq M_j$, one obtains

$$\begin{aligned}
S_2 \quad &\leq \quad C_2 \sum_{j\in\mathbb{N}} \frac{1}{8^{|\alpha|+|\beta|}} \sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} \frac{(hL)^{2j} 6^{\rho|\gamma|+\rho|\delta|} M_j^{2\rho} (Rm_j)^{\rho|\gamma|+\rho|\delta|}}{R^{2\rho j} m_j^{2\rho j} (Rm_j)^{|\gamma|+|\delta|}} \\
&\leq \quad C_3 \sum_{j\in\mathbb{N}} \frac{(hL)^{2j}}{R^{2\rho j}},
\end{aligned}$$

which is convergent for large enough R. Hence $a \in \Gamma_{A_p,B_p,\rho}^{M_p,\infty}\left(\mathbb{R}^{2d}; 8h, m\right)$ for all $0<h<1$, from what we obtain $a\in\Gamma_{A_p,B_p,\rho}^{(M_p),\infty}\left(\mathbb{R}^{2d}\right)$. Now, to prove that $a \sim \sum_{j\in\mathbb{N}} a_j(x,\xi)$. Note that, for $(x,\xi)\in Q_{3Rm_N}^c$, $a - \sum_{j<N} a_j = \sum_{j\geq N} (1-\chi_j)\, a_j$. This easily follows from the definition of χ_j and the fact that m_n is monotonically increasing.

$$\begin{aligned}
&\frac{\left|D_\xi^\alpha D_x^\beta \sum_{j\geq N} (1-\chi_j(x,\xi))\, a_j(x,\xi)\right| \langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha|+|\beta|+2N} A_\alpha B_\beta A_N B_N} \\
&\leq \sum_{j\geq N} \frac{(1-\chi_j(x,\xi)) \left|D_\xi^\alpha D_x^\beta a_j(x,\xi)\right| \langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha|+|\beta|+2N} A_\alpha B_\beta A_N B_N} \\
&\quad + \sum_{j\geq N} \sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} \left|D_\xi^{\alpha-\gamma} D_x^{\beta-\delta} a_j(x,\xi)\right| e^{-M(m|\xi|)} e^{-M(m|x|)} \\
&\qquad \cdot \frac{\left|D_\xi^\gamma D_x^\delta (1-\chi_j(x,\xi))\right| \langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N}}{(8(1+H)h)^{|\alpha|+|\beta|+2N} A_\alpha B_\beta A_N B_N} \\
&\leq C_0 \sum_{j\geq N} \frac{(1-\chi_j(x,\xi))\, h^{2j-2N} A_j B_j}{(1+H)^{2N} \langle(x,\xi)\rangle^{2\rho j - 2\rho N} A_N B_N} \\
&\quad + C_0 \sum_{j\geq N} \frac{1}{8^{|\alpha|+|\beta|}} \sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}} \binom{\alpha}{\gamma}\binom{\beta}{\delta}
\end{aligned}$$

$$\cdot \frac{h^{2j-2N}\left|D_\xi^\gamma D_x^\delta\left(1-\chi_j(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|}A_jB_j}{(1+H)^{2N}h^{|\gamma|+|\delta|}\langle(x,\xi)\rangle^{2\rho j-2\rho N}A_\gamma B_\delta A_N B_N}$$

$$= S_1 + S_2,$$

where S_1 and S_2 are the first and the second sum, correspondingly. To estimate S_1, observe that on the support of $1-\chi_j$ the inequality $\langle(x,\xi)\rangle \geq Rm_j$ holds. Using the monotone increasingness of m_n and $(M.2)$ for A_p and B_p, one obtains

$$\begin{aligned} S_1 &\leq C_0' \sum_{j\geq N} \frac{h^{2j-2N}H^{2j}A_{j-N}B_{j-N}}{(1+H)^{2N}R^{2\rho j-2\rho N}m_j^{2\rho j-2\rho N}} \leq C_4 \sum_{j\geq N} \frac{h^{2j-2N}H^{2j}L^{2j-2N}M_{j-N}^{2\rho}}{(1+H)^{2N}R^{2\rho j-2\rho N}m_{j-N}^{2\rho j-2\rho N}} \\ &= C_4 \frac{H^{2N}}{(1+H)^{2N}} \sum_{j=0}^{\infty}\left(\frac{hHL}{R^\rho}\right)^{2j} \leq C_4 \sum_{j=0}^{\infty}\left(\frac{hHL}{R^\rho}\right)^{2j} < \infty, \end{aligned}$$

uniformly, for $N \in \mathbb{Z}_+$, for large enough R. For S_2, note that $D_\xi^\gamma D_x^\delta\left(1-\chi_j(x,\xi)\right) = 0$ when $(x,\xi) \in Q_{3Rm_j}^c$, because $(\delta,\gamma) \neq (0,0)$ and $\chi_j(x,\xi) = 0$ on $Q_{3Rm_j}^c$. Moreover, from the construction of χ, we have that for the chosen h, there exists $C_1 > 0$ such that $\left|D_\xi^\alpha D_x^\beta \chi(x,\xi)\right| \leq C_1 h^{|\alpha|+|\beta|}A_\alpha B_\beta$. Now

$$\begin{aligned} S_2 &\leq C_5 \sum_{j\geq N} \frac{1}{8^{|\alpha|+|\beta|}} \sum_{\substack{\gamma\leq\alpha,\delta\leq\beta \\ (\delta,\gamma)\neq(0,0)}} \binom{\alpha}{\gamma}\binom{\beta}{\delta} \frac{h^{2j-2N}6^{|\gamma|+|\delta|}H^{2j}A_{j-N}B_{j-N}}{(1+H)^{2N}R^{2\rho j-2\rho N}m_j^{2\rho j-2\rho N}} \\ &\leq C_6 \sum_{j\geq N} \frac{h^{2j-2N}H^{2j}A_{j-N}B_{j-N}}{(1+H)^{2N}R^{2\rho j-2\rho N}m_j^{2\rho j-2\rho N}}, \end{aligned}$$

which we already proved that is bounded uniformly for $N \in \mathbb{Z}_+$. Hence, we obtained

$$\sup_{N\in\mathbb{Z}_+}\sup_{\alpha,\beta}\sup_{(x,\xi)\in Q_{3Rm_N}^c}\left|D_\xi^\alpha D_x^\beta \sum_{j\geq N}\left(1-\chi_j(x,\xi)\right)a_j(x,\xi)\right| \cdot \frac{\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N}e^{-M(m|\xi|)}e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha|+|\beta|+2N}A_\alpha B_\beta A_N B_N} < \infty,$$

for arbitrary $h > 0$, i.e. $a \sim \sum_{j\in\mathbb{N}} a_j(x,\xi)$. For the $\{M_p\}$ case, let $h, B > 0$ are such that $a \in FS_{A_p,B_p,\rho}^{M_p,\infty}\left(\mathbb{R}^{2d}; B, h, m\right)$ for all $m > 0$. Then, for $R \geq 2B$ we define $a(x,\xi) = \sum_{j\in\mathbb{N}}\left(1-\chi_j(x,\xi)\right)a_j(x,\xi)$ and similarly as above, one proves that, for sufficiently large R, a satisfies the claim in the theorem. □

Now we will prove theorems for change of quantization and composition of operators. Note that, unlike in [5] and [6], we do not impose additional conditions on A_p and B_p in the composition theorem.

Theorem 4.2.3. *Let $\tau,\tau_1\in\mathbb{R}$ and $a\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$. Then, there exists $b\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and *-regularizing operator T such that $\operatorname{Op}_{\tau_1}(a)=\operatorname{Op}_{\tau}(b)+T$. Moreover,*

$$b(x,\xi)\sim\sum_{\beta}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}\partial^{\beta}_{\xi}D^{\beta}_{x}a(x,\xi),\ \textit{in } FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right).$$

Proof. Put $p_j(x,\xi)=\sum_{|\beta|=j}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}\partial^{\beta}_{\xi}D^{\beta}_{x}a(x,\xi)$. One easily verifies that $\sum_j p_j\in FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$. Take the sequence $\chi_j(x,\xi)$, $j\in\mathbb{N}$, constructed in the proof of theorem 4.2.2, such that $b=\sum_j(1-\chi_j)p_j$ is an element of $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and $b\sim\sum_j p_j$. By the observations after theorem 4.1.2, the operators $\operatorname{Op}_{\tau_1}(a)$ and $\operatorname{Op}_{\tau}(b)$ coincide with the operators A and B corresponding to a and b when we observe $a((1-\tau_1)x+\tau_1 y,\xi)$ and $b((1-\tau)x+\tau y,\xi)$ as elements of $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$. It is clear that it is enough to prove that the kernel of $A-B$ is in $\mathcal{S}^*\left(\mathbb{R}^{2d}\right)$. To prove that, write

$$\begin{aligned}&a((1-\tau_1)x+\tau_1 y,\xi)-b((1-\tau)x+\tau y,\xi)\\&\quad=(\chi_0 a)((1-\tau_1)x+\tau_1 y,\xi)+\sum_{n=0}^{\infty}((\chi_{n+1}-\chi_n)((1-\tau)x+\tau y,\xi))\\&\qquad\cdot\left(a((1-\tau_1)x+\tau_1 y,\xi)-\sum_{j=0}^{n}p_j((1-\tau)x+\tau y,\xi)\right).\end{aligned}$$

By construction $\chi_0=0$, so $\chi_0 a=0$. Note that the above sum is locally finite and it converges in $\mathcal{E}^*\left(\mathbb{R}^{3d}\right)$. Denote by A_n the operator corresponding to

$$\begin{aligned}a_n(x,y,\xi)&=(\chi_{n+1}-\chi_n)\left((1-\tau)x+\tau y,\xi\right)\\&\quad\cdot\left(a((1-\tau_1)x+\tau_1 y,\xi)-\sum_{j=0}^{n}p_j((1-\tau)x+\tau y,\xi)\right)\end{aligned}$$

considered as an element of $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$. For $u\in\mathcal{S}^*\left(\mathbb{R}^d\right)$, we obtain

$$\begin{aligned}&Au(x)-Bu(x)\\&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}}\frac{e^{i(x-y)\xi}}{P_l(\xi)}P_l(D_y)\left(\frac{1}{P_l(y-x)}P_l(D_\xi)\left(\sum_{n=0}^{\infty}a_n(x,y,\xi)u(y)\right)\right)dyd\xi,\end{aligned}$$

in the (M_p) case and the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case. Note that, because of the convergence of the sum in $\mathcal{E}^*\left(\mathbb{R}^{3d}\right)$, we can interchange the sum with the ultradifferential operators and with $1/P_l(y-x)$ and $1/P_l(\xi)$, resp. with $1/P_{l_p}(y-x)$ and $1/P_{l_p}(\xi)$. For $v\in\mathcal{S}^*\left(\mathbb{R}^d\right)$, by the way we define p_j and using the fact about the support of χ_n, with similar technic as in the proof of lemma 4.1.2, one proves that

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}^{3d}}\left|\frac{1}{P_l(\xi)}P_l(D_y)\left(\frac{1}{P_l(y-x)}P_l(D_\xi)\left(a_n(x,y,\xi)u(y)\right)\right)v(x)\right|dyd\xi dx<\infty,$$

for sufficiently small l and sufficiently large R (from the definition of χ_n) in the (M_p) case, resp. the same but with P_{l_p} in place of P_l for sufficiently small $(l_p) \in \mathfrak{R}$ and sufficiently large R (from the definition of χ_n) in the $\{M_p\}$ case. Hence, from monotone and dominated convergence it follows that

$$\begin{aligned}
&\langle Au - Bu, v\rangle \\
&\quad= \frac{1}{(2\pi)^d}\sum_{n=0}^{\infty}\int_{\mathbb{R}^{3d}} e^{i(x-y)\xi}\frac{1}{P_l(\xi)} \\
&\qquad\qquad \cdot P_l(D_y)\left(\frac{1}{P_l(y-x)}P_l(D_\xi)\left(a_n(x,y,\xi)u(y)\right)\right)v(x)dyd\xi dx \\
&\quad= \frac{1}{(2\pi)^d}\sum_{n=0}^{\infty}\int_{\mathbb{R}^{3d}} e^{i(x-y)\xi}a_n(x,y,\xi)u(y)v(x)dyd\xi dx = \sum_{n=0}^{\infty}\langle A_n u, v\rangle
\end{aligned}$$

in the (M_p) case, resp. the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case. Hence, $\sum_{k=0}^{n} A_k u \to Au - Bu$, when $n \to \infty$ in $\mathcal{S}'^*\left(\mathbb{R}^d\right)$ for every fixed $u \in \mathcal{S}^*\left(\mathbb{R}^d\right)$. But then, because $\mathcal{S}^*$ is barrelled, by the Banach - Steinhaus theorem (see [49], theorem 4.6), $\sum_{k=0}^{n} A_k \to Au - Bu$, when $n \to \infty$ in the topology of precompact convergence in $\mathcal{L}\left(\mathcal{S}^*\left(\mathbb{R}^d\right), \mathcal{S}'^*\left(\mathbb{R}^d\right)\right)$. $\mathcal{S}^*$ is Montel, hence the convergence holds in $\mathcal{L}_b\left(\mathcal{S}^*\left(\mathbb{R}^d\right), \mathcal{S}'^*\left(\mathbb{R}^d\right)\right)$. If we denote by K and K_n, $n \in \mathbb{N}$, the kernels of the operators $A - B$ and A_n, $n \in \mathbb{N}$ correspondingly, then, by proposition 1.2.2, it follows that $K = \sum_{n=0}^{\infty} K_n$, where the convergence is in $\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. Let $r = 1/(8(1+|\tau|+|\tau_1|))$. Take $\theta \in \mathcal{E}^*\left(\mathbb{R}^{2d}\right)$ as in lemma 4.1.5 and put $\tilde{\theta} = 1-\theta$. θ and $\tilde{\theta}$ are obviously multipliers for $\mathcal{S}'^*$. By proposition 4.1.3 and the properties of θ, $\theta K \in \mathcal{S}^*\left(\mathbb{R}^{2d}\right)$. It is enough to prove that $\tilde{\theta}K \in \mathcal{S}^*\left(\mathbb{R}^{2d}\right)$. Note that $\tilde{\theta}K = \sum_n \tilde{\theta}K_n$. Our goal is to prove that $\sum_n \tilde{\theta}K_n \in \mathcal{S}^*$. Observe that

$$\begin{aligned}
K_n(x,y) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi}\left(\chi_{n+1}-\chi_n\right)((1-\tau)x+\tau y,\xi) \\
&\qquad \cdot\left(a((1-\tau_1)x+\tau_1 y,\xi) - \sum_{j=0}^{n} p_j((1-\tau)x+\tau y,\xi)\right)d\xi,
\end{aligned}$$

for all $n \in \mathbb{N}$. Put $\left\{\begin{array}{l} x' = (1-\tau)x+\tau y, \\ y' = x-y, \end{array}\right.$ from what we obtain

$$\left\{\begin{array}{l} x = x'+\tau y', \\ y = x'-(1-\tau)y'. \end{array}\right.$$

Hence $a((1-\tau_1)x+\tau_1 y,\xi) = a(x'+(\tau-\tau_1)y',\xi)$. If we Taylor expand the right hand side in $y'=0$, we get

$$a((1-\tau_1)x+\tau_1 y,\xi) = \sum_{|\beta|\leq n}\frac{1}{\beta!}(\tau-\tau_1)^{|\beta|}\partial_x^\beta a(x',\xi)(x-y)^\beta + W_{n+1}(x,y,\xi),$$

where W_{n+1} is the reminder of the expansion:

$$W_{n+1}(x,y,\xi)$$
$$= (n+1)\sum_{|\beta|=n+1}\frac{1}{\beta!}(x-y)^\beta(\tau-\tau_1)^{|\beta|}\int_0^1(1-t)^n\partial_x^\beta a(x'+t(\tau-\tau_1)y',\xi)dt.$$

If we insert the above expression for a in the expression for K_n we obtain

$$\begin{aligned}
K_n(x,y)& \\
=\ & \frac{1}{(2\pi)^d}\sum_{|\beta|\le n}\frac{(\tau-\tau_1)^{|\beta|}}{\beta!}\int_{\mathbb{R}^d}e^{i(x-y)\xi}(-D_\xi)^\beta\left(\left(\chi_{n+1}-\chi_n\right)(x',\xi)\partial_x^\beta a(x',\xi)\right)d\xi\\
&+\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{i(x-y)\xi}\left(\chi_{n+1}-\chi_n\right)(x',\xi)W_{n+1}(x,y,\xi)d\xi\\
&-\frac{1}{(2\pi)^d}\sum_{j=0}^n\int_{\mathbb{R}^d}e^{i(x-y)\xi}\left(\chi_{n+1}-\chi_n\right)(x',\xi)p_j((1-\tau)x+\tau y,\xi)d\xi\\
=\ & S_{1,n}(x,y)+S_{2,n}(x,y)-S_{3,n}(x,y).
\end{aligned}$$

Our goal is to prove that each of the sums $\sum_n\tilde{\theta}(S_{1,n}-S_{3,n})$ and $\sum_n\tilde{\theta}S_{2,n}$, is $\mathcal{S}^*$ function. Because of the way we defined p_j, one obtains

$$\begin{aligned}
S_{1,n}(x,y)-S_{3,n}(x,y) &= \frac{1}{(2\pi)^d}\sum_{0\ne|\beta|\le n}\sum_{0\ne\delta\le\beta}\binom{\beta}{\delta}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}\\
&\cdot\int_{\mathbb{R}^d}e^{i(x-y)\xi}\left(D_\xi^\delta\left(\chi_{n+1}-\chi_n\right)\right)(x',\xi)D_\xi^{\beta-\delta}\partial_x^\beta a(x',\xi)d\xi.
\end{aligned}$$

Put

$$\begin{aligned}
\tilde{S}_{\beta,n}(x,y) &= \frac{1}{(2\pi)^d}\sum_{0\ne\delta\le\beta}\binom{\beta}{\delta}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}\\
&\cdot\int_{\mathbb{R}^d}e^{i(x-y)\xi}\left(D_\xi^\delta\left(\chi_{n+1}-\chi_n\right)\right)(x',\xi)D_\xi^{\beta-\delta}\partial_x^\beta a(x',\xi)d\xi.
\end{aligned}$$

Obviously $\tilde{S}_{\beta,n}\in\mathcal{E}^*\left(\mathbb{R}^{2d}\right)\cap\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. Let $w\in\mathcal{S}^*\left(\mathbb{R}^{2d}\right)$. Note that

$$\begin{aligned}
\langle\tilde{S}_{\beta,n},w\rangle &= \frac{1}{(2\pi)^d}\sum_{0\ne\delta\le\beta}\binom{\beta}{\delta}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}\int_{\mathbb{R}^{3d}}\frac{1}{P_l(\xi)}e^{i(x-y)\xi}\\
&\cdot P_l(D_y)\left(\left(D_\xi^\delta\left(\chi_{n+1}-\chi_n\right)\right)(x',\xi)D_\xi^{\beta-\delta}\partial_x^\beta a(x',\xi)w(x,y)\right)d\xi dxdy,
\end{aligned}$$

in the (M_p) case, where $l>0$ will be chosen later, resp. the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case, where $(l_p)\in\mathfrak{R}$ will be chosen later. We will consider first the (M_p) case. Then there exists $m>0$ such that $a\in\Gamma_{A_p,B_p,\rho}^{(M_p),\infty}\left(\mathbb{R}^{2d};m\right)$. Chose l such that $|P_l(\xi)|\ge c'e^{4M(m|\xi|)}$ (cf. proposition 2.1.1). On the other hand

$P_l(\xi)=\sum_\alpha c_\alpha \xi^\alpha$ and there exist $C_0>0$ and $L_0>0$ such that $|c_\alpha|\leq C_0 L_0^{|\alpha|}/M_\alpha$. Note that, when $(\chi_{n+1}-\chi_n)(x',\xi)\neq 0$, $\langle (x',\xi)\rangle \geq Rm_n$. Using this, one easily obtains that

$$\sum_{n=1}^{\infty}\sum_{0\neq|\beta|\leq n}\sum_{0\neq\delta\leq\beta}\binom{\beta}{\delta}\frac{(|\tau_1|+|\tau|)^{|\beta|}}{\beta!}\int_{\mathbb{R}^{3d}}\left|\frac{e^{i(x-y)\xi}}{P_l(\xi)}\right.$$

$$\left.\cdot P_l(D_y)\left(\left(D_\xi^\delta(\chi_{n+1}-\chi_n)\right)(x',\xi)D_\xi^{\beta-\delta}\partial_x^\beta a(x',\xi)w(x,y)\right)\right|d\xi dx dy<\infty,$$

for sufficiently large R (from the definition of χ_n). In the $\{M_p\}$ case, by lemma 4.1.1 there exists $(k_p)\in\mathfrak{R}$ such that the estimate in that lemma holds (we can regard $a((1-\tau)x+\tau y,\xi)$ as an element of $\Pi_{A_p,B_p,\rho}^{\{M_p\},\infty}\left(\mathbb{R}^{3d};h\right)$). Take $(l_p)\in\mathfrak{R}$ such that $\left|P_{l_p}(\xi)\right|\geq c'e^{4N_{k_p}(|\xi|)}$. One obtains the same estimate as above but with P_{l_p} in place of P_l, for sufficiently large R (from the definition of χ_n). From this we obtain that $\displaystyle\sum_{n=1}^{\infty}(S_{1,n}-S_{3,n})=\sum_{n=1}^{\infty}\sum_{0\neq|\beta|\leq n}\tilde{S}_{\beta,n}$ converges in $\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. Denote its limit by $\tilde{S}(x,y)$. Moreover, from the above, we can change the order of summation and integration. The local finiteness of $\sum_n(\chi_{n+1}-\chi_n)$ implies

$$\sum_{n\geq|\beta|}D_\xi^\delta(\chi_{n+1}(x',\xi)-\chi_n(x',\xi))=D_\xi^\delta(1-\chi_{|\beta|}(x',\xi))=-D_\xi^\delta\chi_{|\beta|}(x',\xi),$$

where the last equality follows from the fact that $\delta\neq 0$. In the (M_p) case, we obtain

$$\sum_{n=1}^{\infty}\sum_{0\neq|\beta|\leq n}\langle\tilde{S}_{\beta,n},w\rangle$$

$$\begin{aligned}
&= -\frac{1}{(2\pi)^d}\sum_{|\beta|=1}^{\infty}\sum_{0\neq\delta\leq\beta}\binom{\beta}{\delta}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}\\
&\quad\cdot\int_{\mathbb{R}^{3d}}\frac{1}{P_l(\xi)}e^{i(x-y)\xi}P_l(D_y)\left(D_\xi^\delta\chi_{|\beta|}(x',\xi)D_\xi^{\beta-\delta}\partial_x^\beta a(x',\xi)w(x,y)\right)d\xi dx dy\\
&= -\frac{1}{(2\pi)^d}\sum_{|\beta|=1}^{\infty}\sum_{0\neq\delta\leq\beta}\binom{\beta}{\delta}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}\int_{\mathbb{R}^{2d}}I_{\beta,\delta}(x,y)w(x,y)dx dy,
\end{aligned}$$

where we put $\displaystyle I_{\beta,\delta}(x,y)=\int_{\mathbb{R}^d}e^{i(x-y)\xi}D_\xi^\delta\chi_{|\beta|}(x',\xi)D_\xi^{\beta-\delta}\partial_x^\beta a(x',\xi)d\xi$. Similarly, in the $\{M_p\}$ case we obtain the same equality. Hence

$$-\frac{1}{(2\pi)^d}\sum_{|\beta|=1}^{\infty}\sum_{0\neq\delta\leq\beta}\binom{\beta}{\delta}\frac{1}{\beta!}(\tau_1-\tau)^{|\beta|}I_{\beta,\delta}(x,y)$$

converges to $\tilde{S}(x,y)$ in $\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. Now we will prove that $\tilde{\theta}\tilde{S}$ is $\mathcal{S}^*$ function. Denote

$$T_n=\left\{(x,\xi)\in\mathbb{R}^{2d}\,|\,|x|\leq 3Rm_n \text{ and } |\xi|\leq 3Rm_n\right\}\qquad(4.12)$$

and put $T_{\xi,n}$ to be the projection of T_n on $\mathbb{R}^d_\xi$. By construction $\operatorname{supp}\chi_{|\beta|} \subseteq T_{|\beta|}$. So, for the derivatives of $I_{\beta,\delta}(x,y)$ when $(x,y) \in \mathbb{R}^{2d}\backslash\Omega_r \supseteq \operatorname{supp}\tilde{\theta}$, we have

$$\begin{aligned}
&\left|D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x,y)\right| \\
&\quad\leq \sum_{\substack{\alpha\leq\beta' \\ \nu\leq\gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha \\ \nu'+\nu''=\nu}} \binom{\gamma'}{\nu}\binom{\beta'}{\alpha}\binom{\alpha}{\alpha'}\binom{\nu}{\nu'}(1+|\tau|)^{|\alpha'|+|\nu'|}(1+|\tau|)^{|\beta'|-|\alpha|+|\gamma'|-|\nu|} \\
&\qquad\cdot\int_{T_{\xi,|\beta|}} |\xi|^{|\alpha''|+|\nu''|}\left|D_\xi^\delta D_x^{\alpha'+\nu'}\chi_{|\beta|}(x',\xi)\right|\left|D_\xi^{\beta-\delta}D_x^{\beta+\gamma'-\nu+\beta'-\alpha}a(x',\xi)\right| d\xi \\
&\quad\leq C_1 \sum_{\substack{\alpha\leq\beta' \\ \nu\leq\gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha \\ \nu'+\nu''=\nu}} \binom{\gamma'}{\nu}\binom{\beta'}{\alpha}\binom{\alpha}{\alpha'}\binom{\nu}{\nu'}(1+|\tau|)^{|\beta'|+|\gamma'|-|\alpha''|-|\nu''|} \\
&\qquad\cdot\int_{T_{\xi,|\beta|}} |\xi|^{|\alpha''|+|\nu''|}\frac{h_1^{|\delta|+|\alpha'|+|\nu'|}A_\delta B_{\alpha'+\nu'}}{(Rm_{|\beta|})^{|\delta|+|\alpha'|+|\nu'|}} \\
&\qquad\cdot\frac{h^{|2\beta-\delta+\beta'+\gamma'-\alpha-\nu|}A_{\beta-\delta}B_{\beta+\beta'+\gamma'-\alpha-\nu}e^{M(m|\xi|)}e^{M(m|x'|)}}{\langle(x',\xi)\rangle^{\rho|2\beta-\delta+\beta'+\gamma'-\alpha-\nu|}}d\xi.
\end{aligned}$$

Because $\delta \neq 0$, $D_\xi^\delta D_x^{\alpha'+\nu'}\chi_{|\beta|}(x',\xi) = 0$ when $\chi_{|\beta|}(x',\xi) = 1$, hence when $|x'| \leq Rm_{|\beta|}$ and $|\xi| \leq Rm_{|\beta|}$. So, when $D_\xi^\delta D_x^{\alpha'+\nu'}\chi_{|\beta|}(x',\xi) \neq 0$ we have $\langle(x',\xi)\rangle \geq Rm_{|\beta|}$. We obtain

$$\left(Rm_{|\beta|}\right)^{|\delta|+|\alpha'|+|\nu'|}\langle(x',\xi)\rangle^{\rho|2\beta-\delta+\beta'+\gamma'-\alpha-\nu|} \geq \left(Rm_{|\beta|}\right)^{\rho|2\beta+\beta'+\gamma'-\alpha''-\nu''|}.$$

By assumption, there exists $c, L \geq 1$ such that $A_p \leq cL^pM_p^\rho$ and $B_p \leq cL^pM_p^\rho$. Hence

$$\begin{aligned}
&\frac{A_\delta B_{\alpha'+\nu'}A_{\beta-\delta}B_{\beta+\beta'+\gamma'-\alpha-\nu}}{\left(Rm_{|\beta|}\right)^{\rho|2\beta+\beta'+\gamma'-\alpha''-\nu''|}} \\
&\leq \frac{A_\beta B_{\beta+\beta'+\gamma'-\alpha''-\nu''}}{\left(Rm_{|\beta|}\right)^{\rho|2\beta+\beta'+\gamma'-\alpha''-\nu''|}} \leq \frac{c^2L^{|2\beta+\beta'+\gamma'-\alpha''-\nu''|}M^\rho_{2\beta+\beta'+\gamma'-\alpha''-\nu''}}{\left(Rm_{|\beta|}\right)^{2\rho|\beta|}\left(Rm_{|\beta|}\right)^{\rho|\beta'+\gamma'-\alpha''-\nu''|}} \\
&\leq \frac{C''(LH^2)^{|2\beta+\beta'+\gamma'-\alpha''-\nu''|}M_\beta^{2\rho}M_{\beta'+\gamma'}}{R^{2\rho|\beta|}m_{|\beta|}^{2\rho|\beta|}\left(RM_1\right)^{\rho|\beta'+\gamma'-\alpha''-\nu''|}M_{\alpha''+\nu''}} \leq \frac{C''(LH^2)^{|2\beta+\beta'+\gamma'-\alpha''-\nu''|}M_{\beta'+\gamma'}}{R^{2\rho|\beta|}\left(RM_1\right)^{\rho|\beta'+\gamma'-\alpha''-\nu''|}M_{\alpha''+\nu''}},
\end{aligned}$$

where, in the last inequality, we used that $m_n^n \geq M_n$. Also, note that when $(x,y) \in \mathbb{R}^{2d}\backslash\Omega_r$ and $\chi_{|\beta|}((1-\tau)x+\tau y,\xi) \neq 0$, we have the inequalities

$$\begin{aligned}
|x'| &= |(1-\tau)x+\tau y| \leq 3Rm_{|\beta|}, \\
|x|^2+|y|^2 &\leq 2|x|^2+|x-y|^2+2|x||x-y| \\
&\leq 2|x|^2+r^2\langle x\rangle^2+2r|x|\langle x\rangle \leq (2+r)^2\langle x\rangle^2, \\
1+|(1-\tau)x+\tau y|^2 &\geq 1+|x|^2+|\tau|^2|x-y|^2-2|\tau||x||x-y| \\
&\geq \langle x\rangle^2-\frac{\langle x\rangle^2}{4} \geq \frac{\langle x\rangle^2}{4},
\end{aligned}$$

(remember, $r = 1/(8(1+|\tau|+|\tau_1|)))$. Put $s = 2+r$ for shorter notation. Combining these inequalities we get $|(x,y)| \leq 2s\langle x'\rangle \leq 8sRm_{|\beta|}$. Using this and proposition 1.2.1, for arbitrary $m' > 0$, we obtain

$$e^{M(m|x'|)} \leq e^{M(3mRm_{|\beta|})}e^{M(8sm'Rm_{|\beta|})}e^{-M(m'|(x,y)|)} \leq c_0 e^{M\left(8s(m+m')HRm_{|\beta|}\right)}e^{-M(m'|(x,y)|)},$$

when $(x,y) \in \mathbb{R}^{2d}\backslash\Omega_r$ and $\chi_{|\beta|}((1-\tau)x+\tau y,\xi) \neq 0$. Using these inequalities in the estimate for $D_x^{\beta'}D_y^{\gamma'}I_{\beta,\delta}(x,y)$, for $(x,y) \in \mathbb{R}^{2d}\backslash\Omega_r$, we get

$$\begin{aligned}
&\left|D_x^{\beta'}D_y^{\gamma'}I_{\beta,\delta}(x,y)\right| \\
&\leq C_3 \sum_{\substack{\alpha\leq\beta'\\ \nu\leq\gamma'}}\sum_{\substack{\alpha'+\alpha''=\alpha\\ \nu'+\nu''=\nu}} \binom{\gamma'}{\nu}\binom{\beta'}{\alpha}\binom{\alpha}{\alpha'}\binom{\nu}{\nu'}(1+|\tau|)^{|\beta'|+|\gamma'|-|\alpha''|-|\nu''|}M_{\beta'+\gamma'} \\
&\quad \cdot \int_{T_{\xi,|\beta|}} |\xi|^{|\alpha''|+|\nu''|}\frac{(LH^2)^{|2\beta+\beta'+\gamma'-\alpha''-\nu''|}h_1^{|\delta|+|\alpha'|+|\nu'|}h^{|2\beta-\delta+\beta'+\gamma'-\alpha-\nu|}}{R^{2\rho|\beta|}\,(RM_1)^{\rho|\beta'+\gamma'-\alpha''-\nu''|}\,M_{\alpha''+\nu''}} \\
&\quad\quad \cdot e^{M(m|\xi|)}e^{M\left(8s(m+m')HRm_{|\beta|}\right)}e^{-M(m'|(x,y)|)}d\xi \\
&\leq C_4\frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)}}\sum_{\substack{\alpha\leq\beta'\\ \nu\leq\gamma'}}\sum_{\substack{\alpha'+\alpha''=\alpha\\ \nu'+\nu''=\nu}} \binom{\gamma'}{\nu}\binom{\beta'}{\alpha}\binom{\alpha}{\alpha'}\binom{\nu}{\nu'} \\
&\quad \cdot\frac{(LH^2)^{2|\beta|}h_1^{|\delta|+|\alpha'|+|\nu'|}h^{|2\beta-\delta+\beta'+\gamma'-\alpha-\nu|}e^{M\left(8s(m+m')HRm_{|\beta|}\right)}}{(m'R)^{|\alpha''|+|\nu''|}R^{2\rho|\beta|}} \\
&\quad \cdot\int_{T_{\xi,|\beta|}} e^{2M((m+m')R|\xi|)}d\xi,
\end{aligned}$$

where, in the last inequality, we used that

$$\frac{(1+|\tau|)^{|\beta'|+|\gamma'|-|\alpha''|-|\nu''|}(LH^2)^{|\beta'|+|\gamma'|-|\alpha''|-|\nu''|}}{(RM_1)^{\rho|\beta'+\gamma'-\alpha''-\nu''|}} \leq 1,$$

for large enough R. Moreover, on $T_{\xi,|\beta|}$, by proposition 1.2.1, we have $2M((m+m')R|\xi|) \leq M(3(m+m')HR^2m_{|\beta|}) + \ln c_0$. Lemma 4.1.3 implies

$$\begin{aligned}
&e^{M\left(3(m+m')HR^2m_{|\beta|}\right)}e^{M\left(8s(m+m')HRm_{|\beta|}\right)} \\
&\quad\leq c_0^2H^{2(3c_0(m+m')HR^2+2)|\beta|}H^{2(8c_0s(m+m')HR+2)|\beta|} \leq c_0^2H^{4\left(8c_0s(m+m')HR^2+2\right)|\beta|}.
\end{aligned}$$

Similarly as in the proof for proposition 4.1.3, we have $\left|T_{\xi,|\beta|}\right| \leq C_5R^dH^{d|\beta|}$, for some $C_5 > 0$. For the (M_p) case, m is fixed. It is clear that, without losing generality, we can assume that $m \geq 1$. Choose R such that $R \geq 4$ and $R^{2\rho} \geq 2(1+|\tau|+|\tau_1|)L^2H^{d+4}$. For arbitrary but fixed $m' > 0$, choose h such that $hH^{4\left(8c_0s(m+m')HR^2+2\right)} \leq 1$ and $2h \leq 1/(4m')$. Moreover, choose h_1 such that $h_1 \leq h$. Then we obtain

$$\left|D_x^{\beta'}D_y^{\gamma'}I_{\beta,\delta}(x,y)\right| \leq C_6R^d\frac{M_{\beta'+\gamma'}h^{|\beta|}}{e^{M(m'|(x,y)|)}}\left(\frac{L^2H^{d+4}}{R^{2\rho}}\right)^{|\beta|}\left(2h+\frac{1}{m'R}\right)^{|\beta'|+|\gamma'|}$$

$$\leq \quad C_6 R^d \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)}} \cdot \frac{1}{(2(1+|\tau|+|\tau_1|))^{|\beta|}} \cdot \frac{1}{(2m')^{|\beta'|+|\gamma'|}},$$

when $(x, y) \in \mathbb{R}^{2d} \backslash \Omega_r$. Note that the choice of R (and hence of χ_n, $n \in \mathbb{N}$) depends only on A_p, B_p, M_p, τ, τ_1 and a, but not on m'. By the definition of $\tilde{\theta}$ it follows that there exists $C' > 0$ such that

$$\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \leq C' R^d \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)}} \cdot \frac{1}{(2(1+|\tau|+|\tau_1|))^{|\beta|}} \cdot \frac{1}{m'^{|\beta'|+|\gamma'|}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. Hence

$$\sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (|\tau_1| + |\tau|)^{|\beta|} \left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \leq C \frac{M_{\beta'+\gamma'}}{m'^{|\beta'|+|\gamma'|} e^{M(m'|(x,y)|)}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. From the arbitrariness of m' it follows that $\tilde{\theta}\tilde{S} \in \mathcal{S}^{(M_p)}$. Now we consider the $\{M_p\}$ case. Then h and h_1 are fixed. Choose R such that $R^{2\rho} \geq 2(1+|\tau|+|\tau_1|)(h+h_1)hL^2H^{d+16}$ and then choose m and m' such that $8c_0 s(m+m')HR^2 \leq 1$. Then $H^{4\left(8c_0 s(m+m')HR^2+2\right)|\beta|} \leq H^{12|\beta|}$. Then we have

$$\left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x,y) \right|$$

$$\begin{aligned} &\leq \quad C_6 R^d \frac{M_{\beta'+\gamma'} \left(hL^2H^{d+16}\right)^{|\beta|} h_1^{|\delta|} h^{|\beta-\delta|}}{e^{M(m'|(x,y)|)} R^{2\rho|\beta|}} \left(h + h_1 + \frac{1}{m'R} \right)^{|\beta'|+|\gamma'|} \\ &\leq \quad C_6 R^d \frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)} (2(1+|\tau|+|\tau_1|))^{|\beta|}} \left(h + h_1 + \frac{1}{m'R} \right)^{|\beta'|+|\gamma'|}, \end{aligned}$$

when $(x, y) \in \mathbb{R}^{2d} \backslash \Omega_r$. By the definition of $\tilde{\theta}$ it follows that there exist $C' > 0$ and $\tilde{h} > 0$ such that

$$\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \leq C' R^d \frac{M_{\beta'+\gamma'} \tilde{h}^{|\beta'|+|\gamma'|}}{e^{M(m'|(x,y)|)} (2(1+|\tau|+|\tau_1|))^{|\beta|}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. Hence

$$\sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (|\tau_1| + |\tau|)^{|\beta|} \left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \leq C \frac{M_{\beta'+\gamma'} \tilde{h}^{|\beta'|+|\gamma'|}}{e^{M(m'|(x,y)|)}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$; i.e. $\tilde{\theta}\tilde{S} \in \mathcal{S}^{\{M_p\}}$.

It remains to prove that $\sum_{n=0}^{\infty} \tilde{\theta}(x,y) S_{2,n}(x,y) \in \mathcal{S}^*$. Note that

$$S_{2,n}(x,y)$$

$$= \quad \frac{n+1}{(2\pi)^d} \sum_{|\beta|=n+1} \sum_{\delta \leq \beta} \binom{\beta}{\delta} \frac{(-1)^{|\beta|}}{\beta!} (\tau - \tau_1)^{|\beta|} \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_\xi^\delta \left(\chi_{n+1} - \chi_n \right)(x', \xi)$$

$$\cdot \int_0^1 (1-t)^n D_\xi^{\beta-\delta} \partial_x^\beta a(x' + t(\tau - \tau_1)y', \xi) dt d\xi.$$

For brevity in notation, put

$$\begin{aligned} \tilde{I}_{\beta,\delta,n}(x,y) &= \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_\xi^\delta \left(\chi_{n+1} - \chi_n\right)(x', \xi) \\ &\qquad \cdot \int_0^1 (1-t)^n D_\xi^{\beta-\delta} \partial_x^\beta a(x' + t(\tau-\tau_1)y', \xi) dt d\xi. \end{aligned}$$

We will estimate $\left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta,\delta,n}(x,y) \right|$ when $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r \supseteq \operatorname{supp} \tilde{\theta}$.

$$\begin{aligned} &\left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta,\delta,n}(x,y) \right| \\ &\leq \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha \\ \nu'+\nu''=\nu}} \binom{\beta'}{\alpha} \binom{\gamma'}{\nu} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} (2(1+|\tau|+|\tau_1|))^{|\beta'|-|\alpha''|+|\gamma'|-|\nu''|} \\ &\quad \cdot \int_{\mathbb{R}^d} |\xi|^{|\alpha''|+|\nu''|} \left| D_\xi^\delta D_x^{\alpha'+\nu'} \left(\chi_{n+1} - \chi_n\right)(x', \xi) \right| \\ &\quad \cdot \int_0^1 (1-t)^n \left| D_\xi^{\beta-\delta} D_x^{\beta+\beta'-\alpha+\gamma'-\nu} a(x' + t(\tau-\tau_1)y', \xi) \right| dt d\xi \\ &\leq C_1 \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha \\ \nu'+\nu''=\nu}} \binom{\beta'}{\alpha} \binom{\gamma'}{\nu} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} (2(1+|\tau|+|\tau_1|))^{|\beta'|-|\alpha''|+|\gamma'|-|\nu''|} \\ &\quad \cdot \int_{T_{\xi,n+1}} |\xi|^{|\alpha''|+|\nu''|} \frac{h_1^{|\delta|+|\alpha'|+|\nu'|} A_\delta B_{\alpha'+\nu'}}{(Rm_n)^{|\alpha'|+|\nu'|+|\delta|}} \int_0^1 (1-t)^n \\ &\quad \cdot \frac{h^{2|\beta|-|\delta|+|\beta'|-|\alpha|+|\gamma'|-|\nu|} A_{\beta-\delta} B_{\beta+\beta'-\alpha+\gamma'-\nu} e^{M(m|\xi|)} e^{M(m|x'+t(\tau-\tau_1)y'|)}}{\langle (x' + t(\tau-\tau_1)y', \xi) \rangle^{\rho(2|\beta|-|\delta|+|\beta'|-|\alpha|+|\gamma'|-|\nu|)}} dt d\xi. \end{aligned}$$

Above, we already proved that on $\mathbb{R}^{2d} \backslash \Omega_r$, $\langle x \rangle \leq 2\langle x' \rangle$. Using this, by similar technic as there, one easily proves that $\langle (x' + t(\tau - \tau_1)y', \xi) \rangle \geq Rm_n$ when $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$ and $\chi_{n+1}(x', \xi) - \chi(x', \xi) \neq 0$. Also, for such x, y and ξ we have $|x' + t(\tau - \tau_1)y'| \leq |x'| + (|\tau| + |\tau_1|)|y'| \leq \langle x' \rangle + 2r(|\tau| + |\tau_1|)\langle x' \rangle \leq 8Rm_{n+1}$ and $|\xi| \leq 3Rm_{n+1}$. We obtain

$$\begin{aligned} &\left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta,\delta,n}(x,y) \right| \\ &\leq \frac{C_1}{n+1} \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha \\ \nu'+\nu''=\nu}} \binom{\beta'}{\alpha} \binom{\gamma'}{\nu} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} (2(1+|\tau|+|\tau_1|))^{|\beta'|-|\alpha''|+|\gamma'|-|\nu''|} \\ &\quad \cdot \frac{h_1^{|\delta|+|\alpha'|+|\nu'|} h^{2|\beta|-|\delta|+|\beta'|-|\alpha|+|\gamma'|-|\nu|} A_\beta B_{\beta+\beta'-\alpha''+\gamma'-\nu''} e^{M(3mRm_{n+1})} e^{M(8mRm_{n+1})}}{(Rm_n)^{\rho(2|\beta|+|\beta'|-|\alpha''|+|\gamma'|-|\nu''|)}} \\ &\qquad \cdot \int_{T_{\xi,n+1}} |\xi|^{|\alpha''|+|\nu''|} d\xi \end{aligned}$$

Because $A_p \subset M_p^\rho$, $B_p \subset M_p^\rho$ and $m_n^n \geq M_n$, we have

$$\begin{aligned}\frac{A_\beta B_{\beta+\beta'-\alpha''+\gamma'-\nu''}}{(Rm_n)^{\rho(2|\beta|+|\beta'|-|\alpha''|+|\gamma'|-|\nu''|)}} &\leq C' \frac{(HL)^{2n+2+|\beta'|-|\alpha''|+|\gamma'|-|\nu''|} M^\rho_{2n+2} M^\rho_{\beta'-\alpha''+\gamma'-\nu''}}{(Rm_n)^{\rho(2n+2)}(Rm_n)^{\rho(|\beta'|-|\alpha''|+|\gamma'|-|\nu''|)}}\\ &\leq C'' \frac{(H^3L)^{2n+2+|\beta'|-|\alpha''|+|\gamma'|-|\nu''|} M_n^{2\rho} M_{\beta'-\alpha''+\gamma'-\nu''}}{(Rm_n)^{\rho(2n+2)}(RM_1)^{\rho(|\beta'|-|\alpha''|+|\gamma'|-|\nu''|)}}\\ &\leq C''' \frac{(H^3L)^{2n+2+|\beta'|-|\alpha''|+|\gamma'|-|\nu''|} M_{\beta'+\gamma'}}{R^{\rho(2n+2)}(RM_1)^{\rho(|\beta'|-|\alpha''|+|\gamma'|-|\nu''|)} M_{\alpha''+\nu''}}.\end{aligned}$$

By proposition 1.2.1, $e^{M(3mRm_{n+1})}e^{M(8mRm_{n+1})} \leq c_0 e^{M(8mHRm_{n+1})}$. If we insert these inequalities in the above estimate, we have

$$\begin{aligned}\left|D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta,\delta,n}(x,y)\right| \leq\ & \frac{C_2 M_{\beta'+\gamma'}}{n+1} \sum_{\substack{\alpha\leq\beta'\\ \nu\leq\gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha\\ \nu'+\nu''=\nu}} \binom{\beta'}{\alpha}\binom{\gamma'}{\nu}\binom{\alpha}{\alpha'}\binom{\nu}{\nu'}\\ &\cdot \frac{h_1^{|\delta|+|\alpha'|+|\nu'|} h^{2n+2-|\delta|+|\beta'|-|\alpha|+|\gamma'|-|\nu|} e^{M(8mHRm_{n+1})}}{R^{\rho(n+1)} M_{\alpha''+\nu''}}\\ &\cdot \int_{T_{\xi,n+1}} |\xi|^{|\alpha''|+|\nu''|} d\xi,\end{aligned}$$

for large enough R such that $(RM_1)^\rho \geq 2H^3L(1+|\tau|+|\tau_1|)$ and $R^\rho \geq (H^3L)^2$. For $m' > 0$,

$$\int_{T_{\xi,n+1}} \frac{|\xi|^{|\alpha''|+|\nu''|}}{M_{\alpha''+\nu''}} d\xi \leq \frac{e^{M(3m'R^2m_{n+1})}}{(m'R)^{|\alpha''|+|\nu''|}} |T_{\xi,n+1}| \leq C_3 R^d \frac{H^{d(n+1)} e^{M(3m'R^2m_{n+1})}}{(m'R)^{|\alpha''|+|\nu''|}}.$$

Also, similarly as in the first part of the proof, $e^{M(m'|(x,y)|)} \leq e^{M(8sm'Rm_{n+1})}$ when $(x,y) \in \mathbb{R}^{2d}\backslash\Omega_r$ and $(\chi_{n+1}-\chi_n)(x',\xi) \neq 0$, where we put $s = 2+r$. Proposition 1.2.1 and lemma 4.1.3 yield

$$\begin{aligned}&e^{M(8mHRm_{n+1})}e^{M(3m'R^2m_{n+1})}\\ &\leq c_0 e^{M(8(m+m')H^2R^2m_{n+1})} e^{M(8sm'Rm_{n+1})} e^{-M(m'|(x,y)|)}\\ &\leq c_0^3 H^{2(8c_0(m+m')H^2R^2+2)(n+1)} H^{2(8c_0sm'R+2)(n+1)} e^{-M(m'|(x,y)|)}\\ &\leq c_0^3 H^{4(8c_0s(m+m')H^2R^2+2)(n+1)} e^{-M(m'|(x,y)|)}.\end{aligned}$$

In the (M_p) case, m is fixed. Choose R such that $R^\rho \geq 4(1+|\tau|+|\tau_1|)H^d$. Let m' be arbitrary but fixed. For the chosen R, choose h such that $hH^{4(8c_0s(m+m')H^2R^2+2)} \leq 1$ and $8m'h \leq 1$. Moreover choose $h_1 \leq h$. Note that the choice of R (and hence of χ_n, $n\in\mathbb{N}$) depends only on A_p, B_p, M_p, τ, τ_1 and a, but not on m'. We have

$$\begin{aligned}&\left|D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta,\delta,n}(x,y)\right|\\ &\leq C \frac{M_{\beta'+\gamma'}}{(n+1)e^{M(m'|(x,y)|)}} \sum_{\substack{\alpha\leq\beta'\\ \nu\leq\gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha\\ \nu'+\nu''=\nu}} \binom{\beta'}{\alpha}\binom{\gamma'}{\nu}\binom{\alpha}{\alpha'}\binom{\nu}{\nu'}\end{aligned}$$

$$\cdot \frac{h^{n+1} h^{|\beta'|-|\alpha''|+|\gamma'|-|\nu''|}}{(4(1+|\tau|+|\tau_1|))^{n+1}(m'R)^{|\alpha''|+|\nu''|}}$$

$$\leq C \frac{h^{n+1} M_{\beta'+\gamma'}}{(4(1+|\tau|+|\tau_1|))^{n+1}(n+1)e^{M(m'|(x,y)|)}} \left(2h + \frac{1}{m'R}\right)^{|\beta'|+|\gamma'|}$$

$$\leq C \frac{h^{n+1} M_{\beta'+\gamma'}}{(4(1+|\tau|+|\tau_1|))^{n+1}(n+1)e^{M(m'|(x,y)|)}(2m')^{|\beta'|+|\gamma'|}},$$

for $\beta', \gamma' \in \mathbb{N}^d$ and $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$. Hence $\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) \tilde{I}_{\beta,\delta,n}(x,y) \right) \right|$ satisfy the same estimate for all $(x,y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$, possibly with another constant C. From this and the arbitrariness of m', one easily obtains that $\sum_n \tilde{\theta} S_{2,n} \in \mathcal{S}^{(M_p)}$. In the $\{M_p\}$ case h and h_1 are fixed. Choose R such that $R^\rho \geq 4(1+|\tau|+|\tau_1|)(h+h_1)hH^{d+12}$. For the chosen R choose m and m' such that $8c_0 s(m+m')H^2R^2 \leq 1$. Then $H^{4(8c_0s(m+m')H^2R^2+2)(n+1)} \leq H^{12(n+1)}$. We obtain

$$\left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta,\delta,n}(x,y) \right|$$

$$\leq C \frac{M_{\beta'+\gamma'}}{(n+1)e^{M(m'|(x,y)|)}} \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha'+\alpha''=\alpha \\ \nu'+\nu''=\nu}} \binom{\beta'}{\alpha} \binom{\gamma'}{\nu} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'}$$

$$\cdot \frac{h_1^{|\alpha'|+|\nu'|} h^{|\beta'|-|\alpha|+|\gamma'|-|\nu|}}{(4(1+|\tau|+|\tau_1|))^{n+1}(m'R)^{|\alpha''|+|\nu''|}}$$

$$\leq C \frac{M_{\beta'+\gamma'}}{(n+1)(4(1+|\tau|+|\tau_1|))^{n+1}e^{M(m'|(x,y)|)}} \left(h + h_1 + \frac{1}{m'R}\right)^{|\beta'|+|\gamma'|},$$

for $\beta', \gamma' \in \mathbb{N}^d$ and $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$. Hence, there exist $\tilde{h} > 0$ and $C > 0$ such that

$$\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) \tilde{I}_{\beta,\delta,n}(x,y) \right) \right| \leq C \frac{M_{\beta'+\gamma'}}{(n+1)(4(1+|\tau|+|\tau_1|))^{n+1}e^{M(m'|(x,y)|)}} \tilde{h}^{|\beta'|+|\gamma'|}$$

for all $(x,y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. Now one easily obtains that $\sum_n \tilde{\theta} S_{2,n} \in \mathcal{S}^{\{M_p\}}$. We already pointed out that from this it follows that $K \in \mathcal{S}^*$, which completes the proof. □

Theorem 4.2.4. *Let $\tau \in \mathbb{R}$ and $a \in \Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$.*
*i) The transposed operator, ${}^t\mathrm{Op}_\tau(a)$, is still a pseudo-differential operator and it is equal to $\mathrm{Op}_{1-\tau}(a(x,-\xi))$. Moreover, there exist $b \in \Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and *-regularizing operator T such that ${}^t\mathrm{Op}_\tau(a) = \mathrm{Op}_\tau(b) + T$ and*

$$b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(1-2\tau)^{|\alpha|}(-\partial_\xi)^\alpha D_x^\alpha a(x,-\xi) \text{ in } FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right).$$

*ii) The formal adjoint $\mathrm{Op}_\tau(a)^*$, is still a pseudo-differential operator and it is equal to $\mathrm{Op}_{1-\tau}(\overline{a})$. Moreover, there exist $b_1 \in \Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and *-regularizing operator T_1 such that $\mathrm{Op}_\tau(a)^* = \mathrm{Op}_\tau(b_1) + T_1$ and*

$$b_1(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(1-2\tau)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha \overline{a}(x,\xi) \text{ in } FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right).$$

Proof. By the observation after theorem 4.1.2, ${}^t\mathrm{Op}_\tau(a(x,\xi)) = \mathrm{Op}_{1-\tau}(a(x,-\xi))$ and $\mathrm{Op}_\tau(a)^* = \mathrm{Op}_{1-\tau}(\overline{a})$. The rest follows from theorem 4.2.3. □

Theorem 4.2.5. *Let $a, b \in \Gamma^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$. There exist $f \in \Gamma^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$ and *-regularizing operator T such that $a(x,D)b(x,D) = f(x,D) + T$ and f has the asymptotic expansion*

$$f(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi) \text{ in } FS^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d}). \tag{4.13}$$

Proof. By the above theorem ${}^t b(x,D) = b_1(x,D) + T'$ where T' is *-regularizing operator and $b_1 \in \Gamma^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$ with asymptotic expansion

$$b_1(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(-\partial_\xi)^\alpha D_x^\alpha b(x,-\xi). \tag{4.14}$$

Again, by the above theorem, ${}^t b_1(x,D) = \mathrm{Op}_1(b_1(x,-\xi))$ and

$$b(x,D) = {}^t\mathrm{Op}_1(b(x,-\xi)) = {}^t\left({}^t b(x,D)\right).$$

Put $b_2(x,\xi) = b_1(x,-\xi)$. Then we have

$$b(x,D) = {}^t\left({}^t b(x,D)\right) = {}^t b_1(x,D) + {}^tT' = \mathrm{Op}_1(b_2) + {}^tT'.$$

We have $a(x,D)b(x,D) = a(x,D)\mathrm{Op}_1(b_2) + T_1$, where we put $T_1 = a(x,D){}^tT'$, which is *-regularizing. Because $\mathcal{F}(\mathrm{Op}_1(b_2)u)(\xi) = \int_{\mathbb{R}^d} e^{-iy\xi} b_2(y,\xi)u(y)dy$ and $\mathrm{Op}_1(b_2)u \in \mathcal{S}^*$,

$$a(x,D)\mathrm{Op}_1(b_2)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x,\xi)b_2(y,\xi)u(y)dyd\xi$$

and this is well defined as iterated integral by theorem 4.1.1. Observe that $\tilde{a}(x,y,\xi) = a(x,\xi)b_2(y,\xi)$ is an element of $\Pi^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{3d})$. To prove that one only has to use the inequalities $2\langle(x,\xi)\rangle\langle x-y\rangle \geq \langle(x,y,\xi)\rangle$ and $2\langle(y,\xi)\rangle\langle x-y\rangle \geq \langle(x,y,\xi)\rangle$ in the estimates for the derivatives of $\tilde{a}$. The operator $\tilde{A}$ corresponding to this $\tilde{a}$ is the same as $a(x,D)\mathrm{Op}_1(b_2)$. Let

$$p_j(x,\xi) = \sum_{|\beta|=j}\frac{1}{\beta!}\partial_\xi^\beta\left(a(x,\xi)D_x^\beta b_2(x,\xi)\right).$$

Obviously $\sum_j p_j \in FS^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$. Let $\chi_j(x,\xi)$, $j \in \mathbb{N}$, be the sequence constructed in the proof of theorem 4.2.2, such that $f = \sum_j(1-\chi_j)p_j$ is an element of $\Gamma^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$ and $f \sim \sum_j p_j$. By the observations after theorem 4.1.2, the operator $f(x,D)$ coincide with the operator F corresponding to f when we observe $f(x,\xi)$ as elements of $\Pi^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{3d})$. We will prove that the kernel of $\tilde{A} - F$ is in $\mathcal{S}^*(\mathbb{R}^{2d})$, i.e. $\tilde{A} - F$ is *-regularizing. Similarly as in the proof of theorem 4.2.3, $\tilde{a}(x,y,\xi) - f(x,\xi) = \sum_{n=0}^{\infty}\tilde{a}_n(x,y,\xi)$ where we put

$$\tilde{a}_n(x,y,\xi) = (\chi_{n+1}(x,\xi) - \chi_n(x,\xi))\left(\tilde{a}(x,y,\xi) - \sum_{j=0}^{n} p_j(x,\xi)\right),$$

which is obviously an element of $\Pi^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{3d}\right)$. Denote by $\tilde{A}_n$ its corresponding operator. Similarly as in the proof of theorem 4.2.3, we have $K(x,y) = \sum_n K_n(x,y)$, where K is the kernel of $\tilde{A}-F$, K_n is the kernel of $\tilde{A}_n$ and the convergence holds in $\mathcal{S}'^*$. Observe that

$$K_n(x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi}\left(\chi_{n+1}-\chi_n\right)(x,\xi)\left(a(x,\xi)b_2(y,\xi) - \sum_{j=0}^{n} p_j(x,\xi)\right)d\xi,$$

for all $n\in\mathbb{N}$. Let $r=1/8$. Take $\theta\in\mathcal{E}^*\left(\mathbb{R}^{2d}\right)$ as in lemma 4.1.5 and put $\tilde{\theta}=1-\theta$. θ and $\tilde{\theta}$ are obviously multipliers for $\mathcal{S}'^*$. By proposition 4.1.3 and the properties of θ, $\theta K\in\mathcal{S}^*\left(\mathbb{R}^{2d}\right)$. It is enough to prove that $\tilde{\theta}K\in\mathcal{S}^*\left(\mathbb{R}^{2d}\right)$. Note that $\tilde{\theta}K=\sum_n\tilde{\theta}K_n$. Our goal is to prove that $\sum_n\tilde{\theta}K_n\in\mathcal{S}^*$. Taylor expand $b_2(y,\xi)$ in the first variable to obtain

$$b_2(y,\xi) = \sum_{|\beta|\leq n}\frac{1}{\beta!}(y-x)^\beta\partial_x^\beta b_2(x,\xi) + W_{n+1}(x,y,\xi),$$

where W_{n+1} is the remainder of the expansion:

$$W_{n+1}(x,y,\xi) = (n+1)\sum_{|\beta|=n+1}\frac{1}{\beta!}(y-x)^\beta\int_0^1(1-t)^n\partial_x^\beta b_2(x+t(y-x),\xi)dt.$$

If we insert this in the expression for K_n, keeping in mind the definition of p_j, we have $K_n(x,y)=S_{1,n}(x,y)+S_{2,n}(x,y)$ where we put

$$\begin{aligned} S_{1,n}(x,y) &= \frac{1}{(2\pi)^d}\sum_{0\neq|\beta|\leq n}\sum_{0\neq\delta\leq\beta}\binom{\beta}{\delta}\frac{1}{\beta!}\\ &\quad\cdot\int_{\mathbb{R}^d} e^{i(x-y)\xi}D_\xi^\delta\left(\chi_{n+1}-\chi_n\right)(x,\xi)D_\xi^{\beta-\delta}\left(a(x,\xi)\partial_x^\beta b_2(x,\xi)\right)d\xi\\ S_{2,n}(x,y) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi}\left(\chi_{n+1}-\chi_n\right)(x,\xi)a(x,\xi)W_{n+1}(x,y,\xi)d\xi. \end{aligned}$$

Our goal is to prove that $\sum_n\tilde{\theta}S_{1,n}$ and $\sum_n\tilde{\theta}S_{2,n}$ are $\mathcal{S}^*$ functions. Similarly as in the proof of theorem 4.2.3, $\sum_n S_{1,n}$ converges in $\mathcal{S}'^*$ to $\tilde{S}$ and

$$\tilde{S} = -\frac{1}{(2\pi)^d}\sum_{|\beta|=1}^{\infty}\sum_{0\neq\delta\leq\beta}\binom{\beta}{\delta}\frac{1}{\beta!}I_{\beta,\delta},$$

where the convergence is in $\mathcal{S}'^*$, where we put

$$I_{\beta,\delta}(x,y) = \int_{\mathbb{R}^d} e^{i(x-y)\xi}D_\xi^\delta\chi_{|\beta|}(x,\xi)D_\xi^{\beta-\delta}\left(a(x,\xi)\partial_x^\beta b_2(x,\xi)\right)d\xi.$$

To prove that $-\frac{1}{(2\pi)^d}\sum_{|\beta|=1}^{\infty}\sum_{0\neq\delta\leq\beta}\binom{\beta}{\delta}\frac{1}{\beta!}\tilde{\theta}I_{\beta,\delta}$ is in $\mathcal{S}^*$ we have to estimate the derivatives of $I_{\beta,\delta}$ when $(x,y)\in\mathbb{R}^{2d}\backslash\Omega_r\supseteq\operatorname{supp}\tilde{\theta}$. Note that, we can choose m such that $a, b_2 \in \Gamma^{(M_p),\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}; m\right)$ in the (M_p) case, resp. we can choose h such that $a, b_2 \in \Gamma^{\{M_p\},\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}; h\right)$ in the $\{M_p\}$ case. Let T_n be as in (4.12) and put $T_{\xi,n}$ to be the projection of T_n on $\mathbb{R}^d_\xi$. By the way we constructed χ_n, it follows that $\operatorname{supp}\chi_{|\beta|} \subseteq T_{|\beta|}$.

$$\begin{aligned}
&\left|D^{\beta'}_x D^{\gamma'}_y I_{\beta,\delta}(x,y)\right|\\
&\quad\leq \sum_{\kappa\leq\beta-\delta}\sum_{\alpha\leq\beta'}\sum_{\alpha'+\alpha''=\alpha}\sum_{\alpha'''\leq\beta'-\alpha}\binom{\beta-\delta}{\kappa}\binom{\beta'}{\alpha}\binom{\alpha}{\alpha'}\binom{\beta'-\alpha}{\alpha'''}\int_{T_{\xi,|\beta|}}|\xi|^{|\alpha''|+|\gamma'|}\\
&\qquad\cdot\left|D^{\delta}_\xi D^{\alpha'}_x\chi_{|\beta|}(x,\xi)\right|\left|D^{\beta-\delta-\kappa}_\xi D^{\beta'-\alpha-\alpha'''}_x a(x,\xi) D^{\kappa}_\xi D^{\beta+\alpha'''}_x b_2(x,\xi)\right|d\xi\\
&\quad\leq C_1\sum_{\kappa\leq\beta-\delta}\sum_{\alpha\leq\beta'}\sum_{\alpha'+\alpha''=\alpha}\binom{\beta-\delta}{\kappa}\binom{\beta'}{\alpha}\binom{\alpha}{\alpha'}2^{|\beta'|-|\alpha|}\int_{T_{\xi,|\beta|}}|\xi|^{|\alpha''|+|\gamma'|}\\
&\qquad\cdot\frac{h_1^{|\delta|+|\alpha'|}A_\delta B_{\alpha'}h^{|2\beta-\delta+\beta'-\alpha|}A_{\beta-\delta}B_{\beta+\beta'-\alpha}e^{2M(m|\xi|)}e^{2M(m|x|)}}{(Rm_{|\beta|})^{|\delta|+|\alpha'|}\langle(x,\xi)\rangle^{\rho|2\beta-\delta+\beta'-\alpha|}}d\xi.
\end{aligned}$$

Because $\delta\neq 0$, $D^{\delta}_\xi D^{\alpha'}_x\chi_{|\beta|}(x,\xi)=0$ when $\chi_{|\beta|}(x,\xi)=1$, hence when $|x|\leq Rm_{|\beta|}$ and $|\xi|\leq Rm_{|\beta|}$. So, when $D^{\delta}_\xi D^{\alpha'}_x\chi_{|\beta|}(x,\xi)\neq 0$ we have $\langle(x,\xi)\rangle\geq Rm_{|\beta|}$. We obtain

$$R^{|\delta|+|\alpha'|}m_{|\beta|}^{|\delta|+|\alpha'|}\langle(x,\xi)\rangle^{\rho|2\beta-\delta+\beta'-\alpha|}\geq\left(Rm_{|\beta|}\right)^{\rho|2\beta+\beta'-\alpha''|}.$$

By assumption $A_p\subset M_p^\rho$, $B_p\subset M_p^\rho$ i.e. there exist $c\geq 1$ and $L\geq 1$ such that $A_p\leq cL^pM_p^\rho$ and $B_p\leq cL^pM_p^\rho$. Observe that

$$\begin{aligned}
&\frac{A_\delta B_{\alpha'}A_{\beta-\delta}B_{\beta+\beta'-\alpha}}{\left(Rm_{|\beta|}\right)^{\rho|2\beta+\beta'-\alpha''|}}\\
&\quad\leq\frac{A_\beta B_{\beta+\beta'-\alpha''}}{\left(Rm_{|\beta|}\right)^{\rho|2\beta+\beta'-\alpha''|}}\leq\frac{c^2L^{|2\beta+\beta'-\alpha''|}M^\rho_{2\beta+\beta'-\alpha''}}{\left(Rm_{|\beta|}\right)^{2\rho|\beta|}\left(Rm_{|\beta|}\right)^{\rho|\beta'-\alpha''|}}\\
&\quad\leq\frac{C'(LH)^{|2\beta+\beta'-\alpha''|}M^\rho_{2\beta}M_{\beta'-\alpha''}}{R^{2\rho|\beta|}m_{|\beta|}^{2\rho|\beta|}\left(Rm_{|\beta|}\right)^{\rho|\beta'-\alpha''|}}\leq\frac{C''(LH^2)^{|2\beta+\beta'-\alpha''|}M^{2\rho}_{\beta}M_{\beta'+\gamma'}}{R^{2\rho|\beta|}m_{|\beta|}^{2\rho|\beta|}\left(RM_1\right)^{\rho|\beta'-\alpha''|}M_{\alpha''+\gamma'}}\\
&\quad\leq\frac{C''(LH^2)^{|2\beta+\beta'-\alpha''|}M_{\beta'+\gamma'}}{R^{2\rho|\beta|}\left(RM_1\right)^{\rho|\beta'-\alpha''|}M_{\alpha''+\gamma'}},
\end{aligned}$$

where, in the last inequality, we used that $m_n^n\geq M_n$. Also, note that when $(x,y)\in\mathbb{R}^{2d}\backslash\Omega_r$ and $\chi_{|\beta|}(x,\xi)\neq 0$, we have the following inequalities

$$\begin{aligned}
|x|&\leq 3Rm_{|\beta|},\\
|x|^2+|y|^2&\leq 2|x|^2+|x-y|^2+2|x||x-y|\leq 2|x|^2+r^2\langle x\rangle^2+2r|x|\langle x\rangle\\
&\leq (2+r)^2\langle x\rangle^2,
\end{aligned}$$

Put $s=2+r$ for shorter notation. Combining these inequalities we get $|(x,y)|\leq 4sRm_{|\beta|}$. Using this and proposition 1.2.1, for arbitrary $m'>0$, we obtain

$$e^{2M(m|x|)}\leq c_0e^{M(3mHRm_{|\beta|})}e^{M(m'|(x,y)|)}e^{-M(m'|(x,y)|)}$$

$$\begin{aligned} &\leq c_0 e^{M(3mHRm_{|\beta|})} e^{M(4sm'Rm_{|\beta|})} e^{-M(m'|(x,y)|)} \\ &\leq c_0^2 e^{M\left(4s(m+m')H^2Rm_{|\beta|}\right)} e^{-M(m'|(x,y)|)}, \end{aligned}$$

when $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$ and $\chi_{|\beta|}(x,\xi) \neq 0$. Using these inequalities in the estimate for $D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x,y)$, for $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$, we get

$$\begin{aligned} &\left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x,y) \right| \\ &\leq C_3 \sum_{\kappa \leq \beta-\delta} \sum_{\alpha \leq \beta'} \sum_{\alpha'+\alpha''=\alpha} \binom{\beta-\delta}{\kappa} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} 2^{|\beta'|-|\alpha|} \\ &\quad \cdot \int_{T_{\xi,|\beta|}} |\xi|^{|\alpha''|+|\gamma'|} \frac{(LH^2)^{|2\beta+\beta'-\alpha''|} h_1^{|\delta|+|\alpha'|} h^{|2\beta-\delta+\beta'-\alpha|} M_{\beta'+\gamma'}}{R^{2\rho|\beta|} (RM_1)^{\rho|\beta'-\alpha''|} M_{\alpha''+\gamma'}} \\ &\quad \cdot e^{M(mH|\xi|)} e^{M\left(4s(m+m')H^2Rm_{|\beta|}\right)} e^{-M(m'|(x,y)|)} d\xi \\ &\leq C_4 \frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)}} \sum_{\kappa \leq \beta-\delta} \sum_{\alpha \leq \beta'} \sum_{\alpha'+\alpha''=\alpha} \binom{\beta-\delta}{\kappa} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \\ &\quad \cdot \int_{T_{\xi,|\beta|}} \frac{(LH^2)^{2|\beta|} h_1^{|\delta|+|\alpha'|} h^{|2\beta-\delta+\beta'-\alpha|} e^{2M((m+m')HR|\xi|)} e^{M\left(4s(m+m')H^2Rm_{|\beta|}\right)}}{(m'R)^{|\alpha''|+|\gamma'|} R^{2\rho|\beta|}} d\xi, \end{aligned}$$

where, in the last inequality, we used that

$$\frac{2^{|\beta'|-|\alpha|} (LH^2)^{|\beta'|-|\alpha''|}}{(RM_1)^{\rho|\beta'-\alpha''|}} \leq 1,$$

for large enough R. Moreover, on $T_{\xi,|\beta|}$, using proposition 1.2.1, we have $2M((m+m')HR|\xi|) \leq M(3(m+m')H^2R^2m_{|\beta|}) + \ln c_0$. Now, by lemma 4.1.3, we obtain

$$\begin{aligned} &e^{M\left(3(m+m')H^2R^2m_{|\beta|}\right)} e^{M\left(4s(m+m')H^2Rm_{|\beta|}\right)} \\ &\leq c_0^2 H^{2(3c_0(m+m')H^2R^2+2)|\beta|} H^{2\left(4c_0s(m+m')H^2R+2\right)|\beta|} \leq c_0^2 H^{4\left(4c_0s(m+m')H^2R^2+2\right)|\beta|}. \end{aligned}$$

Similarly as in the proof for proposition 4.1.3, we have $\left|T_{\xi,|\beta|}\right| \leq C_5 R^d H^{d|\beta|}$, for some $C_5 > 0$. For the (M_p) case, m is fixed. It is clear that, without losing generality, we can assume that $m \geq 1$. Choose R such that $R \geq 4$ and $R^{2\rho} \geq 8L^2H^{d+4}$. For arbitrary but fixed $m' > 0$, choose h such that $hH^{4\left(4c_0s(m+m')H^2R^2+2\right)} \leq 1$ and $2h \leq 1/(4m')$. Moreover, choose h_1 such that $h_1 \leq h$. Then we obtain

$$\begin{aligned} \left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x,y) \right| &\leq C_6 \frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)}} \sum_{\kappa \leq \beta-\delta} \sum_{\alpha \leq \beta'} \sum_{\alpha'+\alpha''=\alpha} \binom{\beta-\delta}{\kappa} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \\ &\quad \cdot \frac{(L^2H^4)^{|\beta|} h^{|\alpha'|} h^{|\beta|} h^{|\beta'|-|\alpha|}}{(m'R)^{|\alpha''|+|\gamma'|} R^{2\rho|\beta|}} \left|T_{\xi,|\beta|}\right| \\ &\leq C_7 R^d \frac{M_{\beta'+\gamma'} 2^{|\beta|} h^{|\beta|}}{e^{M(m'|(x,y)|)}} \left(\frac{L^2H^{d+4}}{R^{2\rho}}\right)^{|\beta|} \left(2h + \frac{1}{m'R}\right)^{|\beta'|+|\gamma'|} \\ &\leq C_7 R^d \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)} 4^{|\beta|}} \cdot \frac{1}{(2m')^{|\beta'|+|\gamma'|}}, \end{aligned}$$

when $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$. Note that the choice of R (and hence of χ_n, $n \in \mathbb{N}$) depends only on A_p, B_p, M_p, a and b_2 but not on m'. By the definition of $\tilde{\theta}$ it follows that there exists $C' > 0$ such that

$$\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \leq C' R^d \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)} 4^{|\beta|}} \cdot \frac{1}{m'^{|\beta'|+|\gamma'|}},$$

for all $(x,y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. Hence

$$\begin{aligned}
&\sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} \left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \\
&\quad \leq \; C' R^d \sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta! 4^{|\beta|}} \cdot \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)}} \cdot \frac{1}{m'^{|\beta'|+|\gamma'|}} \leq C \frac{M_{\beta'+\gamma'}}{m'^{|\beta'|+|\gamma'|} e^{M(m'|(x,y)|)}},
\end{aligned}$$

for all $(x,y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. From the arbitrariness of m' it follows that $\tilde{\theta}\tilde{S} \in \mathcal{S}^{(M_p)}$. Now we consider the $\{M_p\}$ case. Then h and h_1 are fixed. Choose R such that $R^{2\rho} \geq 8(h+h_1)hL^2H^{d+16}$ and then choose m and m' such that $4c_0 s(m+m')H^2R^2 \leq 1$. Then $H^{4\left(4c_0 s(m+m')H^2R^2+2\right)|\beta|} \leq H^{12|\beta|}$. Then we have

$$\begin{aligned}
&\left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x,y) \right| \\
&\quad \leq \; C_6 \frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)}} \sum_{\kappa \leq \beta-\delta} \sum_{\alpha \leq \beta'} \sum_{\alpha'+\alpha''=\alpha} \binom{\beta-\delta}{\kappa} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \\
&\qquad \cdot \frac{(hL^2H^4)^{|\beta|} h_1^{|\delta|} h_1^{|\alpha'|} h^{|\beta-\delta|} h^{|\beta'|-|\alpha|} H^{12|\beta|}}{(m'R)^{|\alpha''|+|\gamma'|} R^{2\rho|\beta|}} \left| T_{\xi,|\beta|} \right| \\
&\quad \leq \; C_7 R^d \frac{M_{\beta'+\gamma'} 2^{|\beta|} \left(hL^2H^{d+16}\right)^{|\beta|} h_1^{|\delta|} h^{|\beta-\delta|}}{e^{M(m'|(x,y)|)} R^{2\rho|\beta|}} \left(h + h_1 + \frac{1}{m'R} \right)^{|\beta'|+|\gamma'|} \\
&\quad \leq \; C_7 R^d \frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)} 4^{|\beta|}} \left(h + h_1 + \frac{1}{m'R} \right)^{|\beta'|+|\gamma'|},
\end{aligned}$$

when $(x,y) \in \mathbb{R}^{2d} \backslash \Omega_r$. By the definition of $\tilde{\theta}$ it follows that there exist $C' > 0$ and $\tilde{h} > 0$ such that

$$\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \leq C' R^d \frac{M_{\beta'+\gamma'} \tilde{h}^{|\beta'|+|\gamma'|}}{e^{M(m'|(x,y)|)} 4^{|\beta|}},$$

for all $(x,y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. Hence

$$\sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} \left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x,y) I_{\beta,\delta}(x,y) \right) \right| \leq C \frac{M_{\beta'+\gamma'} \tilde{h}^{|\beta'|+|\gamma'|}}{e^{M(m'|(x,y)|)}},$$

for all $(x,y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$, from what we obtain $\tilde{\theta}\tilde{S} \in \mathcal{S}^{\{M_p\}}$.

Next, we will prove that $\sum_n \tilde{\theta}(x,y) S_{2,n}(x,y) \in \mathcal{S}^*$. Note that

$$
\begin{aligned}
&S_{2,n}(x,y)\\
&\quad = \frac{n+1}{(2\pi)^d}\sum_{|\beta|=n+1}\sum_{\delta\le\beta}\sum_{\kappa\le\beta-\delta}\binom{\beta}{\delta}\binom{\beta-\delta}{\kappa}\frac{1}{\beta!}\int_{\mathbb{R}^d} e^{i(x-y)\xi}D_\xi^\delta\left(\chi_{n+1}-\chi_n\right)(x,\xi)\\
&\qquad\qquad \cdot D_\xi^\kappa a(x,\xi)\int_0^1(1-t)^n D_\xi^{\beta-\delta-\kappa}\partial_x^\beta b_2(x+t(y-x),\xi)dtd\xi.
\end{aligned}
$$

For brevity in notation, put

$$
\begin{aligned}
\tilde{I}_{\beta,\delta,n}(x,y) &= \sum_{\kappa\le\beta-\delta}\binom{\beta-\delta}{\kappa}\int_{\mathbb{R}^d} e^{i(x-y)\xi}D_\xi^\delta\left(\chi_{n+1}-\chi_n\right)(x,\xi)D_\xi^\kappa a(x,\xi)\\
&\qquad\cdot\int_0^1(1-t)^n D_\xi^{\beta-\delta-\kappa}\partial_x^\beta b_2(x+t(y-x),\xi)dtd\xi.
\end{aligned}
$$

We will estimate $\left|D_x^{\beta'}D_y^{\gamma'}\tilde{I}_{\beta,\delta,n}(x,y)\right|$ when $(x,y)\in\mathbb{R}^{2d}\backslash\Omega_r\supseteq\operatorname{supp}\tilde{\theta}$.

$$
\begin{aligned}
&\left|D_x^{\beta'}D_y^{\gamma'}\tilde{I}_{\beta,\delta,n}(x,y)\right|\\
&\quad\le C_1\sum_{\substack{\alpha\le\beta'\\ \nu\le\gamma'}}\sum_{\alpha'+\alpha''=\alpha}\sum_{\kappa\le\beta-\delta}\sum_{\alpha'''\le\beta'-\alpha}\binom{\beta'}{\alpha}\binom{\gamma'}{\nu}\binom{\alpha}{\alpha'}\binom{\beta-\delta}{\kappa}\binom{\beta'-\alpha}{\alpha'''}\\
&\qquad\cdot\int_{T_{\xi,n+1}}|\xi|^{|\alpha''|+|\nu|}\frac{h_1^{|\delta|+|\alpha'|}A_\delta B_{\alpha'}}{(Rm_n)^{|\alpha'|+|\delta|}}\int_0^1(1-t)^n\\
&\qquad\cdot\frac{h^{2|\beta|-|\delta|+|\beta'|-|\alpha|+|\gamma'|-|\nu|}A_{\beta-\delta}B_{\beta+\beta'-\alpha+\gamma'-\nu}e^{2M(m|\xi|)}e^{2M(m(|x|+|y|))}}{\langle(x,\xi)\rangle^{\rho(|\beta'|-|\alpha|-|\alpha'''|+|\kappa|)}\langle(x+t(y-x),\xi)\rangle^{\rho(2|\beta|-|\delta|-|\kappa|+|\alpha'''|+|\gamma'|-|\nu|)}}dtd\xi.
\end{aligned}
$$

When $(\chi_{n+1}-\chi_n)(x,\xi)\neq 0$ and $(x,y)\in\mathbb{R}^{2d}\backslash\Omega_r$, the inequalities $\langle(x,\xi)\rangle\ge Rm_m$ and $\langle(x+t(y-x),\xi)\rangle\ge Rm_m$ hold. Also $|x|+|y|\le 2|x|+|x-y|\le s\langle x\rangle\le 4sRm_{n+1}$, where we put $s=2+r$. Hence

$$
\begin{aligned}
&\left|D_x^{\beta'}D_y^{\gamma'}\tilde{I}_{\beta,\delta,n}(x,y)\right|\\
&\quad\le\frac{C_2}{n+1}\sum_{\substack{\alpha\le\beta'\\ \nu\le\gamma'}}\sum_{\alpha'+\alpha''=\alpha}\binom{\beta'}{\alpha}\binom{\gamma'}{\nu}\binom{\alpha}{\alpha'}2^{|\beta|-|\delta|+|\beta'|-|\alpha|}\int_{T_{\xi,n+1}}|\xi|^{|\alpha''|+|\nu|}d\xi\\
&\qquad\cdot\frac{h_1^{|\delta|+|\alpha'|}h^{2|\beta|-|\delta|+|\beta'|-|\alpha|+|\gamma'|-|\nu|}A_\beta B_{\beta+\beta'-\alpha''+\gamma'-\nu}}{(Rm_n)^{\rho(2|\beta|+|\beta'|-|\alpha''|+|\gamma'|-|\nu|)}}\cdot e^{M(3mHRm_{n+1})}e^{M(4smHRm_{n+1})}.
\end{aligned}
$$

Because $A_p\subset M_p^\rho$, $B_p\subset M_p^\rho$ and $m_n^n\ge M_n$, we have

$$
\begin{aligned}
\frac{A_\beta B_{\beta+\beta'-\alpha''+\gamma'-\nu}}{(Rm_n)^{\rho(2|\beta|+|\beta'|-|\alpha''|+|\gamma'|-|\nu|)}} &\le C'\frac{(HL)^{2n+2+|\beta'|-|\alpha''|+|\gamma'|-|\nu|}M_{2n+2}^\rho M_{\beta'-\alpha''+\gamma'-\nu}^\rho}{(Rm_n)^{\rho(2n+2)}(Rm_n)^{\rho(|\beta'|-|\alpha''|+|\gamma'|-|\nu|)}}\\
&\le C''\frac{(H^3L)^{2n+2+|\beta'|-|\alpha''|+|\gamma'|-|\nu|}M_n^{2\rho}M_{\beta'-\alpha''+\gamma'-\nu}}{(Rm_n)^{\rho(2n+2)}(RM_1)^{\rho(|\beta'|-|\alpha''|+|\gamma'|-|\nu|)}}
\end{aligned}
$$

$$\leq \quad C''' \frac{(H^3L)^{2n+2+|\beta'|-|\alpha''|+|\gamma'|-|\nu|}M_{\beta'+\gamma'}}{R^{\rho(2n+2)}(RM_1)^{\rho(|\beta'|-|\alpha''|+|\gamma'|-|\nu|)}M_{\alpha''+\nu}}.$$

By proposition 1.2.1, $e^{M(3mHRm_{n+1})}e^{M(4smHRm_{n+1})} \leq c_0 e^{M(4smH^2Rm_{n+1})}$. If we insert these inequalities in the above estimate, we have

$$\begin{aligned}\left|D_x^{\beta'}D_y^{\gamma'}\tilde{I}_{\beta,\delta,n}(x,y)\right| \quad &\leq \quad \frac{C_3 M_{\beta'+\gamma'}}{n+1}\sum_{\substack{\alpha\leq\beta'\\ \nu\leq\gamma'}}\sum_{\alpha'+\alpha''=\alpha}\binom{\beta'}{\alpha}\binom{\gamma'}{\nu}\binom{\alpha}{\alpha'}\\ &\quad\cdot\frac{h_1^{|\delta|+|\alpha'|}h^{2n+2-|\delta|+|\beta'|-|\alpha|+|\gamma'|-|\nu|}e^{M(4smH^2Rm_{n+1})}}{R^{\rho(n+1)}M_{\alpha''+\nu}}\\ &\quad\cdot\int_{T_{\xi,n+1}}|\xi|^{|\alpha''|+|\nu|}d\xi,\end{aligned}$$

for large enough R such that $(RM_1)^\rho \geq 2H^3L$ and $R^\rho \geq 2(H^3L)^2$. For $m' > 0$,

$$\int_{T_{\xi,n+1}}\frac{|\xi|^{|\alpha''|+|\nu|}}{M_{\alpha''+\nu}}d\xi \leq \frac{e^{M(3m'R^2m_{n+1})}}{(m'R)^{|\alpha''|+|\nu|}}|T_{\xi,n+1}| \leq C_4R^d\frac{H^{d(n+1)}e^{M(3m'R^2m_{n+1})}}{(m'R)^{|\alpha''|+|\nu|}}.$$

Also, $e^{M(m'|(x,y)|)} \leq e^{M(m'(|x|+|y|))} \leq e^{M(4sm'Rm_{n+1})}$ when $(\chi_{n+1}-\chi_n)(x,\xi) \neq 0$ and $(x,y)\in\mathbb{R}^{2d}\backslash\Omega_r$. Proposition 1.2.1 and lemma 4.1.3 yield

$$\begin{aligned}&e^{M(4smH^2Rm_{n+1})}e^{M(3m'R^2m_{n+1})}\\ &\qquad\leq \quad c_0e^{M(4s(m+m')H^3R^2m_{n+1})}e^{M(4sm'Rm_{n+1})}e^{-M(m'|(x,y)|)}\\ &\qquad\leq \quad c_0^3H^{2(4c_0s(m+m')H^3R^2+2)(n+1)}H^{2(4c_0sm'R+2)(n+1)}e^{-M(m'|(x,y)|)}\\ &\qquad\leq \quad c_0^3H^{4(4c_0s(m+m')H^3R^2+2)(n+1)}e^{-M(m'|(x,y)|)}.\end{aligned}$$

In the (M_p) case, m is fixed. Choose R such that $R^\rho \geq 4H^d$. Let m' be arbitrary but fixed. For the chosen R, choose h such that $hH^{4(4c_0s(m+m')H^3R^2+2)} \leq 1$ and $8m'h \leq 1$. Moreover choose $h_1 \leq h$. Note that the choice of R (and hence of χ_n, $n\in\mathbb{N}$) depends only on A_p, B_p, M_p, a and b_2 but not on m'. We have

$$\begin{aligned}\left|D_x^{\beta'}D_y^{\gamma'}\tilde{I}_{\beta,\delta,n}(x,y)\right| \quad &\leq \quad C\frac{M_{\beta'+\gamma'}}{(n+1)e^{M(m'|(x,y)|)}}\sum_{\substack{\alpha\leq\beta'\\ \nu\leq\gamma'}}\sum_{\alpha'+\alpha''=\alpha}\binom{\beta'}{\alpha}\binom{\gamma'}{\nu}\binom{\alpha}{\alpha'}\\ &\qquad\cdot\frac{h^{n+1}h^{|\beta'|-|\alpha''|+|\gamma'|-|\nu|}}{4^{n+1}(m'R)^{|\alpha''|+|\nu|}}\\ &\leq \quad C\frac{h^{n+1}M_{\beta'+\gamma'}}{4^{n+1}(n+1)e^{M(m'|(x,y)|)}}\left(2h+\frac{1}{m'R}\right)^{|\beta'|+|\gamma'|}\\ &\leq \quad C\frac{h^{n+1}M_{\beta'+\gamma'}}{4^{n+1}(n+1)e^{M(m'|(x,y)|)}(2m')^{|\beta'|+|\gamma'|}},\end{aligned}$$

for $\beta',\gamma'\in\mathbb{N}^d$ and $(x,y)\in\mathbb{R}^{2d}\backslash\Omega_r$. Hence $\left|D_x^{\beta'}D_y^{\gamma'}\left(\tilde{\theta}(x,y)\tilde{I}_{\beta,\delta,n}(x,y)\right)\right|$ satisfy the same estimate for all $(x,y)\in\mathbb{R}^{2d}$ and $\beta',\gamma'\in\mathbb{N}^d$, possibly with another constant

C. From this and the arbitrariness of m', one easily obtains that $\sum_n \tilde{\theta} S_{2,n} \in \mathcal{S}^{(M_p)}$. In the $\{M_p\}$ case h and h_1 are fixed. Choose R such that $R^\rho \geq 4(h+h_1)hH^{d+12}$. For the chosen R choose m and m' such that $4c_0 s(m+m')H^3R^2 \leq 1$. Then $H^{4(4c_0s(m+m')H^3R^2+2)(n+1)} \leq H^{12(n+1)}$. We obtain

$$\begin{aligned} \left|D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta,\delta,n}(x,y)\right| &\leq C\frac{M_{\beta'+\gamma'}}{(n+1)e^{M(m'|(x,y)|)}} \sum_{\substack{\alpha\leq\beta'\\ \nu\leq\gamma'}} \sum_{\alpha'+\alpha''=\alpha} \binom{\beta'}{\alpha}\binom{\gamma'}{\nu}\binom{\alpha}{\alpha'} \\ &\qquad \cdot \frac{h_1^{|\alpha'|}h^{|\beta'|-|\alpha|+|\gamma'|-|\nu|}}{4^{n+1}(m'R)^{|\alpha''|+|\nu|}} \\ &\leq C\frac{M_{\beta'+\gamma'}}{(n+1)4^{n+1}e^{M(m'|(x,y)|)}}\left(h+h_1+\frac{1}{m'R}\right)^{|\beta'|+|\gamma'|}, \end{aligned}$$

for $\beta',\gamma' \in \mathbb{N}^d$ and $(x,y)\in\mathbb{R}^{2d}\backslash\Omega_r$. Hence, there exist $\tilde{h}>0$ and $C>0$ such that

$$\left|D_x^{\beta'} D_y^{\gamma'}\left(\tilde{\theta}(x,y)\tilde{I}_{\beta,\delta,n}(x,y)\right)\right| \leq C\frac{M_{\beta'+\gamma'}}{(n+1)4^{n+1}e^{M(m'|(x,y)|)}}\tilde{h}^{|\beta'|+|\gamma'|}$$

for all $(x,y)\in\mathbb{R}^{2d}$ and $\beta',\gamma'\in\mathbb{N}^d$. Now one easily obtains that $\sum_n \tilde{\theta}S_{2,n} \in \mathcal{S}^{\{M_p\}}$. Hence, we proved that $a(x,D)b(x,D) = a(x,D)\mathrm{Op}_1(b_2)+T_1 = f(x,D)+T_2$, where T_2 is *-regularizing operator. It remains to prove (4.13). Obviously, it is enough to prove that $\sum_\beta \frac{1}{\beta!}\partial_\xi^\beta\left(a(x,\xi)D_x^\beta b_2(x,\xi)\right) \sim \sum_\beta \frac{1}{\beta!}\partial_\xi^\beta a(x,\xi)D_x^\beta b(x,\xi)$. For $N\in\mathbb{Z}_+$ we have

$$\begin{aligned} &\sum_{j=0}^{N-1}\sum_{|\beta|=j}\frac{1}{\beta!}\partial_\xi^\beta\left(a\cdot D_x^\beta b_2\right) \\ &\quad= \sum_{j=0}^{N-1}\sum_{|\alpha+\gamma|=j}\frac{1}{\alpha!\gamma!}\partial_\xi^\gamma a\cdot\left(\partial_\xi^\alpha D_x^{\alpha+\gamma}b_2 - \sum_{s=0}^{N-j-1}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\delta!}\partial_\xi^{\alpha+\delta}D_x^{\alpha+\gamma+\delta}b\right) \\ &\qquad+\sum_{j=0}^{N-1}\sum_{s=0}^{N-j-1}\sum_{|\alpha+\gamma|=j}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\alpha!\gamma!\delta!}\partial_\xi^\gamma a\cdot\partial_\xi^{\alpha+\delta}D_x^{\alpha+\gamma+\delta}b. \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{j=0}^{N-1}\sum_{s=0}^{N-j-1}\sum_{|\alpha+\gamma|=j}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\alpha!\gamma!\delta!}\partial_\xi^\gamma a\cdot\partial_\xi^{\alpha+\delta}D_x^{\alpha+\gamma+\delta}b \\ &\quad= \sum_{j=0}^{N-1}\sum_{s=0}^{N-j-1}\sum_{k=0}^{j}\sum_{\substack{|\alpha|=k\\ |\gamma|=j-k}}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\alpha!\gamma!\delta!}\partial_\xi^\gamma a\cdot\partial_\xi^{\alpha+\delta}D_x^{\alpha+\gamma+\delta}b \\ &\quad= \sum_{s=0}^{N-1}\sum_{j=0}^{N-s-1}\sum_{k=0}^{j}\sum_{\substack{|\alpha|=k\\ |\gamma|=j-k}}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\alpha!\gamma!\delta!}\partial_\xi^\gamma a\cdot\partial_\xi^{\alpha+\delta}D_x^{\alpha+\gamma+\delta}b \end{aligned}$$

$$= \sum_{s=0}^{N-1}\sum_{k=0}^{N-s-1}\sum_{j=k}^{N-s-1}\sum_{\substack{|\alpha|=k\\|\gamma|=j-k}}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\alpha!\gamma!\delta!}\partial_\xi^\gamma a\cdot\partial_\xi^{\alpha+\delta}D_x^{\alpha+\gamma+\delta}b$$

$$= \sum_{t=0}^{N-1}\sum_{s+k=t}\sum_{j=k}^{N-s-1}\sum_{|\gamma|=j-k}\sum_{\substack{|\alpha|=k\\|\delta|=s}}\frac{(-1)^{|\delta|}}{\alpha!\gamma!\delta!}\partial_\xi^\gamma a\cdot\partial_\xi^{\alpha+\delta}D_x^{\alpha+\gamma+\delta}b$$

$$= \sum_{t=0}^{N-1}\sum_{s+k=t}\sum_{j=k}^{N-s-1}\sum_{|\gamma|=j-k}\sum_{|\beta|=t}\sum_{\alpha+\delta=\beta}\frac{(-1)^{|\delta|}}{\alpha!\gamma!\delta!}\partial_\xi^\gamma a\cdot\partial_\xi^{\beta}D_x^{\beta+\gamma}b$$

$$= \sum_{j=0}^{N-1}\sum_{|\gamma|=j}\frac{1}{\gamma!}\partial_\xi^\gamma a\cdot D_x^\gamma b.$$

Hence, we have to estimate the derivatives of

$$\sum_{j=0}^{N-1}\sum_{|\alpha+\gamma|=j}\frac{1}{\alpha!\gamma!}\partial_\xi^\gamma a\cdot\partial_\xi^\alpha D_x^{\alpha+\gamma}\left(b_2-\sum_{s=0}^{N-j-1}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\delta!}\partial_\xi^\delta D_x^\delta b\right).$$

By construction $b(x,\xi)\sim\sum_{j=0}^{\infty}\sum_{|\delta|=j}\frac{(-1)^{|\delta|}}{\delta!}\partial_\xi^\delta D_x^\delta b(x,\xi)$. So, for $(x,\xi)\in Q^c_{Bm_N}$, we have

$$\Bigg|D_\xi^{\alpha'}D_x^{\beta'}\sum_{j=0}^{N-1}\sum_{|\alpha+\gamma|=j}\frac{1}{\alpha!\gamma!}\partial_\xi^\gamma a(x,\xi)$$
$$\cdot\partial_\xi^\alpha D_x^{\alpha+\gamma}\left(b_2(x,\xi)-\sum_{s=0}^{N-j-1}\sum_{|\delta|=s}\frac{(-1)^{|\delta|}}{\delta!}\partial_\xi^\delta D_x^\delta b(x,\xi)\right)\Bigg|$$
$$\leq C_1\sum_{j=0}^{N-1}\sum_{|\alpha+\gamma|=j}\sum_{\substack{\alpha''\leq\alpha'\\\beta''\leq\beta'}}\binom{\alpha'}{\alpha''}\binom{\beta'}{\beta''}$$
$$\cdot\frac{h^{|\alpha'|+|\beta'|+2N}A_{|\alpha'|+j}B_{|\beta'|+j}A_{N-j}B_{N-j}e^{2M(m|\xi|)}e^{2M(m|x|)}}{\alpha!\gamma!\langle(x,\xi)\rangle^{\rho(|\alpha'|+|\beta'|+2N)}}$$
$$\leq C\frac{(4Hh)^{|\alpha'|+|\beta'|+2N}A_{\alpha'}B_{\beta'}A_NB_Ne^{M(mH|\xi|)}e^{M(mH|x|)}}{\langle(x,\xi)\rangle^{\rho(|\alpha'|+|\beta'|+2N)}},$$

which gives the desired asymptotic expansion. □

For the next corollary we need the following technical lemma.

Lemma 4.2.2. *Let $a,b\in\Gamma^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$ are such that $a\sim\sum_j a_j$ and $b\sim\sum_j b_j$. Then $ab\sim\sum_{j=0}^{\infty}\sum_{s+k=j}a_sb_k$ and*

$$\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi)\sim\underbrace{0+\ldots+0}_{|\alpha|}+\sum_{j=|\alpha|}^{\infty}\sum_{s+k+|\alpha|=j}\partial_\xi^\alpha a_s(x,\xi)\partial_x^\alpha b_k(x,\xi) \qquad (4.15)$$

in $FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$*, for each* $\alpha\in\mathbb{N}^d$*. Moreover, there exist* $B>0$ *and* $m>0$ *such that, for every* $h>0$*, there exists* $C>0$*; resp. there exist* $B>0$ *and* $h>0$ *such that, for every* $m>0$*, there exists* $C>0$*; such that*

$$\sup_{\alpha}\sup_{N>|\alpha|}\sup_{\gamma,\delta}\sup_{(x,\xi)\in Q^c_{Bm_N}}\left|D^{\gamma}_{\xi}D^{\delta}_{x}\left(\partial^{\alpha}_{\xi}a(x,\xi)\partial^{\alpha}_{x}b(x,\xi)\right.\right.$$

$$\left.\left.-\sum_{j=|\alpha|}^{N-1}\sum_{s+k+|\alpha|=j}\partial^{\alpha}_{\xi}a_s(x,\xi)\partial^{\alpha}_{x}b_k(x,\xi)\right)\right|\frac{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}e^{-M(m|\xi|)}e^{-M(m|x|)}}{h^{|\gamma|+|\delta|+2N}A_\gamma B_\delta A_N B_N}\leq C.$$

Proof. By the conditions in the lemma, there exist $B>0$ and $m>0$ such that, for every $h>0$, there exists $\tilde{C}>0$; resp. there exist $B>0$ and $h>0$ such that, for every $m>0$, there exists $\tilde{C}>0$; such that

$$\sup_{j\in\mathbb{N}}\sup_{\gamma,\delta}\sup_{(x,\xi)\in Q^c_{Bm_j}}\frac{\left|D^{\gamma}_{\xi}D^{\delta}_{x}a_j(x,\xi)\right|\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2j\rho}e^{-M(m|\xi|)}e^{-M(m|x|)}}{h^{|\gamma|+|\delta|+2j}A_\gamma B_\delta A_j B_j}\leq\tilde{C},$$

$$\sup_{N\in\mathbb{Z}_+}\sup_{\gamma,\delta}\sup_{(x,\xi)\in Q^c_{Bm_N}}\frac{\left|D^{\gamma}_{\xi}D^{\delta}_{x}\left(a(x,\xi)-\sum_{j<N}a_j(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}}{h^{|\gamma|+|\delta|+2N}A_\gamma B_\delta A_N B_N}\cdot$$

$$\cdot e^{-M(m|\xi|)}e^{-M(m|x|)}\leq\tilde{C}$$

and the same estimate for $D^{\gamma}_{\xi}D^{\delta}_{x}b_j$ and $D^{\gamma}_{\xi}D^{\delta}_{x}\left(b-\sum_{j<N}b_j\right)$. One easily checks that $\underbrace{0+\dots+0}_{|\alpha|}+\sum_{j=|\alpha|}^{\infty}\sum_{s+k+|\alpha|=j}\partial^{\alpha}_{\xi}a_s\partial^{\alpha}_{x}b_k\in FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$, for each fixed $\alpha\in\mathbb{N}^d$. For $N>|\alpha|$ and $(x,\xi)\in Q^c_{Bm_N}$, observe that

$$\begin{aligned}\partial^{\alpha}_{\xi}a\cdot\partial^{\alpha}_{x}b\;&=\;\partial^{\alpha}_{\xi}a\cdot\left(\partial^{\alpha}_{x}b-\sum_{k=0}^{N-|\alpha|-1}\partial^{\alpha}_{x}b_k\right)\\&\quad+\sum_{k=0}^{N-|\alpha|-1}\left(\partial^{\alpha}_{\xi}a-\sum_{s=0}^{N-|\alpha|-k-1}\partial^{\alpha}_{\xi}a_s\right)\cdot\partial^{\alpha}_{x}b_k+\sum_{j=|\alpha|}^{N-1}\sum_{s+k=j-|\alpha|}\partial^{\alpha}_{\xi}a_s\partial^{\alpha}_{x}b_k.\end{aligned}$$

We have the estimates

$$\left|D^{\gamma}_{\xi}D^{\delta}_{x}\left(\left(\partial^{\alpha}_{\xi}a(x,\xi)-\sum_{s=0}^{N-|\alpha|-k-1}\partial^{\alpha}_{\xi}a_s(x,\xi)\right)\partial^{\alpha}_{x}b_k(x,\xi)\right)\right|$$

$$\leq\sum_{\substack{\gamma'\leq\gamma\\ \delta'\leq\delta}}\binom{\gamma}{\gamma'}\binom{\delta}{\delta'}\left|D^{\alpha+\gamma'}_{\xi}D^{\delta'}_{x}\left(a(x,\xi)-\sum_{s=0}^{N-|\alpha|-k-1}a_s(x,\xi)\right)\right|$$

$$\cdot\left|D^{\gamma-\gamma'}_{\xi}D^{\delta-\delta'+\alpha}_{x}b_k(x,\xi)\right|$$

$$\leq \tilde{C}^2 \sum_{\substack{\gamma'\leq\gamma\\ \delta'\leq\delta}} \binom{\gamma}{\gamma'}\binom{\delta}{\delta'}$$
$$\cdot\frac{h^{|\gamma'|+|\delta'|+2N-|\alpha|-2k}A_{\alpha+\gamma'}B_{\delta'}A_{N-|\alpha|-k}B_{N-|\alpha|-k}e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma'|+\rho|\delta'|+2N\rho-\rho|\alpha|-2\rho k}}$$
$$\cdot\frac{h^{|\gamma|-|\gamma'|+|\delta|-|\delta'|+|\alpha|+2k}A_{\gamma-\gamma'}B_{\delta-\delta'+\alpha}A_kB_ke^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|-\rho|\gamma'|+\rho|\delta|-\rho|\delta'|+\rho|\alpha|+2k\rho}}$$
$$\leq c_0^2\tilde{C}^2 \sum_{\substack{\gamma'\leq\gamma\\ \delta'\leq\delta}} \binom{\gamma}{\gamma'}\binom{\delta}{\delta'}\frac{h^{|\gamma|+|\delta|+2N}A_{\alpha+\gamma}B_{\alpha+\delta}A_{N-|\alpha|}B_{N-|\alpha|}e^{M(mH|\xi|)}e^{M(mH|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}}$$
$$\leq c_0^4\tilde{C}^2\frac{(2hH)^{|\gamma|+|\delta|+2N}A_\gamma B_\delta A_NB_Ne^{M(mH|\xi|)}e^{M(mH|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}},$$

for all $(x,\xi)\in Q^c_{Bm_N}$, $\gamma,\delta\in\mathbb{N}^d$ and the estimates are uniform for α, N and k. Hence

$$\left|D^\gamma_\xi D^\delta_x\sum_{k=0}^{N-|\alpha|-1}\left(\partial^\alpha_\xi a(x,\xi)-\sum_{s=0}^{N-|\alpha|-k-1}\partial^\alpha_\xi a_s(x,\xi)\right)\cdot\partial^\alpha_x b_k(x,\xi)\right|$$
$$\leq c_0^4\tilde{C}^2\frac{(4hH)^{|\gamma|+|\delta|+2N}A_\gamma B_\delta A_NB_Ne^{M(mH|\xi|)}e^{M(mH|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}},$$

for all $(x,\xi)\in Q^c_{Bm_N}$, $\gamma,\delta\in\mathbb{N}^d$ and the estimates are uniform for α and N, $N>|\alpha|$. Analogously, one easily obtains similar estimates for the derivatives of $\partial^\alpha_\xi a\cdot\left(\partial^\alpha_x b-\sum_{k=0}^{N-|\alpha|-1}\partial^\alpha_x b_k\right)$. Now we can estimate the derivatives of

$$\partial^\alpha_\xi a\cdot\partial^\alpha_x b-\sum_{j=|\alpha|}^{N-1}\sum_{s+k=j-|\alpha|}\partial^\alpha_\xi a_s\partial^\alpha_x b_k$$

and obtain the inequality in the lemma. Moreover, for fixed $\alpha\in\mathbb{N}^d$, to prove (4.15) it only remains to consider the case when $N\leq|\alpha|$ (we already consider the case when $N>|\alpha|$ above). But then $\sum_{j=|\alpha|}^{N-1}\sum_{s+k=j-|\alpha|}\partial^\alpha_\xi a_s\partial^\alpha_x b_k$ is empty and we only have to estimate the derivatives of $\partial^\alpha_\xi a\cdot\partial^\alpha_x b$ which is easy and we omit it ($a,b\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and α is fixed). □

Corollary 4.2.1. *Let $a,b\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ with asymptotic expansions $a\sim\sum_j a_j$ and $b\sim\sum_j b_j$. Then there exist $f\in\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$ and *-regularizing operator T such that $a(x,D)b(x,D)=f(x,D)+T$ and f has the following asymptotic expansion*

$$f(x,\xi)\sim\sum_{j=0}^\infty\sum_{s+k+l=j}\sum_{|\alpha|=l}\frac{1}{\alpha!}\partial^\alpha_\xi a_s(x,\xi)D^\alpha_x b_k(x,\xi)\ \text{in}\ FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right).$$

Proof. It is easy to check that the above formal sum is an element of the space $FS^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$. By theorem 4.2.5, we only have to prove that

$$\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)D_x^\alpha b(x,\xi)\sim\sum_{j=0}^{\infty}\sum_{s+k+l=j}\sum_{|\alpha|=l}\frac{1}{\alpha!}\partial_\xi^\alpha a_s(x,\xi)D_x^\alpha b_k(x,\xi).$$

For $N\in\mathbb{Z}_+$ and $(x,\xi)\in Q^c_{Bm_N}$, we have

$$\sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha a\cdot D_x^\alpha b-\sum_{j=0}^{N-1}\sum_{s+k+l=j}\sum_{|\alpha|=l}\frac{1}{\alpha!}\partial_\xi^\alpha a_s\cdot D_x^\alpha b_k$$

$$\begin{aligned}
&= \sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha a\cdot D_x^\alpha b-\sum_{j=0}^{N-1}\sum_{l=0}^{j}\sum_{s+k=j-l}\sum_{|\alpha|=l}\frac{1}{\alpha!}\partial_\xi^\alpha a_s\cdot D_x^\alpha b_k\\
&= \sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha a\cdot D_x^\alpha b-\sum_{l=0}^{N-1}\sum_{j=l}^{N-1}\sum_{s+k=j-l}\sum_{|\alpha|=l}\frac{1}{\alpha!}\partial_\xi^\alpha a_s\cdot D_x^\alpha b_k\\
&= \sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}\left(\partial_\xi^\alpha a\cdot D_x^\alpha b-\sum_{l=j}^{N-1}\sum_{s+k=l-j}\partial_\xi^\alpha a_s\cdot D_x^\alpha b_k\right).
\end{aligned}$$

By lemma 4.2.2, the derivatives of $\partial_\xi^\alpha a\cdot D_x^\alpha b-\sum_{l=j}^{N-1}\sum_{s+k=l-j}\partial_\xi^\alpha a_s\cdot D_x^\alpha b_k$ can be uniformly estimated, as in the lemma, for all α, N and $(x,\xi)\in Q^c_{Bm_N}$, such that $|\alpha|<N$, from what the desired equivalence follows. □

Chapter 5

Anti-Wick and Weyl Quantization on Ultradistribution Spaces

The Anti-Wick and the Weyl quantization of global symbols, as well as their connection, in the case of Schwartz distributions was vastly studied during the years (see for example [36] and [53] for a systematic approach to the theory). The importance in studying the Anti-Wick quantization lies in the facts that real valued symbols give rise to formally self-adjoint operators and positive symbols give rise to positive operators. On the other hand the Weyl quantization is important because it is closely connected with the Wigner transform and also, the Weyl quantization of real valued symbol is formally self-adjoint operator.

The results that we give here are related to the global symbol classes defined and studied in the previous chapter, which corresponding operators act continuously on the space of tempered ultradistributions of Beurling, resp. Roumieu type.

For a symbol a which is an element of the space of tempered (ultra)distributions, its Anti-Wick quantization is equal to the Weyl quantization of a symbol b that is given as the convolution of a and the gaussian kernel $e^{-|\cdot|^2}$. The purpose of this chapter is twofold. In the first section, after giving the definition and the basic facts about the short-time Fourier transform, we define Anti-Wick quantization. We extend results from [36] (see also [53]) to ultradistributions. More precisely, we give the connection between the Anti-Wick and Weyl quantization for symbols belonging to the symbol classes introduced before. The last two sections are devoted to finding the largest subspace of ultradistributions for which the convolution with $e^{s|\cdot|^2}$, $s \in \mathbb{R}\backslash\{0\}$, exists. The answer to this question in the case of Schwartz distributions was already given in [58]. This gives a way to extend the definition of Anti-Wick operators with symbols that are not necessarily tempered ultradistributions. In particular, we prove theorem 5.3.1, which gives such class of symbols.

5.1 Anti-Wick Quantization

For $\varphi \in \mathcal{S}^*(\mathbb{R}^d)$, $\varphi \neq 0$, and $u \in \mathcal{S}'^*(\mathbb{R}^d)$ we define the *short-time Fourier transform* of u with window φ by $V_\varphi u(y,\eta) = \mathcal{F}_{t\to\eta}\left(u(t)\overline{\varphi(t-y)}\right)$. Then V_φ acts continuously $\mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^{2d})$, $\mathcal{S}'^*(\mathbb{R}^d) \to \mathcal{S}'^*(\mathbb{R}^{2d})$ and $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ (for the properties of the short-time Fourier transform in connection with spaces of tempered ultradistributions, we refer to [19]). If $\varphi_1, \varphi_2 \in \mathcal{S}^*(\mathbb{R}^d)$ are non-zero and $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$ we define the *localization operator* $A_a^{\varphi_1,\varphi_2}$ by $\langle A_a^{\varphi_1,\varphi_2} u, \overline{v}\rangle = \left\langle a, V_{\varphi_1} u \overline{V_{\varphi_2} v}\right\rangle$, $u, v \in \mathcal{S}^*(\mathbb{R}^d)$. It is continuous operator from $\mathcal{S}^*(\mathbb{R}^d)$ to $\mathcal{S}'^*(\mathbb{R}^d)$. We will be particularly interested in the case when $\varphi_1(x) = \varphi_2(x) = \mathcal{G}_0(x) = \pi^{-d/4} e^{-\frac{1}{2}|x|^2}$. Obviously $\|\mathcal{G}_0\|_{L^2} = 1$. We will also use the notation $\mathcal{G}_{y,\eta}(x) = \pi^{-d/4} e^{ix\eta} e^{-\frac{1}{2}|x-y|^2}$. In this case we will denote the short-time Fourier transform just by V. Hence, for $u \in \mathcal{S}'^*$, Vu is the tempered ultradistribution in $\mathbb{R}^{2d}$ given by $Vu(y,\eta) = \mathcal{F}_{t\to\eta}(u(t)\mathcal{G}_0(t-y))$. We summarise the above results about the continuity of V in the following proposition.

Proposition 5.1.1. *The short-time Fourier transform acts continuously*

$$\mathcal{S}'^*(\mathbb{R}^d) \to \mathcal{S}'^*(\mathbb{R}^{2d}),\ \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^{2d}) \text{ and } L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}).$$

Moreover, $\|Vu\|_{L^2(\mathbb{R}^{2d})} = (2\pi)^{d/2}\|u\|_{L^2(\mathbb{R}^d)}$, *for* $u \in L^2(\mathbb{R}^d)$.

The adjoint map of V, $V^* : \mathcal{S}^*(\mathbb{R}^{2d}) \to \mathcal{S}^*(\mathbb{R}^d)$,

$$V^*F(t) = (2\pi)^d \int_{\mathbb{R}^d} \mathcal{F}^{-1}_{\eta\to t}(F(y,\eta))\,\mathcal{G}_0(t-y)dy,\ F \in \mathcal{S}^*(\mathbb{R}^{2d})$$

extends to a well defined and continuous map $\mathcal{S}'^*(\mathbb{R}^{2d}) \to \mathcal{S}'^*(\mathbb{R}^d)$ and $L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d)$ and $V^*V = (2\pi)^d\mathrm{Id}$. Now we can define Anti-Wick operators.

Definition 5.1.1. Let $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$. We define the *Anti-Wick operator* with symbol a as the map $A_a : \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}'^*(\mathbb{R}^d)$ given by $A_a u = (2\pi)^{-d} V^*(aVu)$, $u \in \mathcal{S}^*(\mathbb{R}^d)$. A_a will also be called the *Anti-Wick quantization* of a.

Observe that, if a is a multiplier for $\mathcal{S}^*(\mathbb{R}^{2d})$ (for example an element of $\Gamma^{*,\infty}_{A_p,B_p,\rho}(\mathbb{R}^{2d})$), then A_a maps $\mathcal{S}^*(\mathbb{R}^d)$ continuously into itself. Also, note that the above formula is equivalent to

$$\langle A_a u, \overline{v}\rangle = (2\pi)^{-d}\langle a, Vu\overline{Vv}\rangle, \quad u, v \in \mathcal{S}^*(\mathbb{R}^d), \tag{5.1}$$

hence A_a is precisely the localization operator $A_a^{\varphi_1,\varphi_2}$ when $\varphi_1(x) = \varphi_2(x) = \mathcal{G}_0(x)$. One easily proves the following proposition.

Proposition 5.1.2. *a) Let* $a_n \in \mathcal{S}'^*(\mathbb{R}^{2d})$ *be a sequence that converges to* a *in* $\mathcal{S}'^*(\mathbb{R}^{2d})$, *then* $A_{a_n}u \to A_a u$, *for every* $u \in \mathcal{S}^*(\mathbb{R}^d)$.
b) Let $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$ *be real valued. Then* A_a *is formally self-adjoint.*

If a is locally integrable function of *-ultrapolynomial growth (for example, if it is an element of $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$), then, by (5.1), we can represent the action of A_a as

$$A_a u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} a(y,\eta)\,(u,\mathcal{G}_{y,\eta})\,\mathcal{G}_{y,\eta}(x)dyd\eta, \quad u\in\mathcal{S}^*\left(\mathbb{R}^d\right).$$

The proof of the following proposition is the same as in the case of distribution and it will be omitted (see for example [36]).

Proposition 5.1.3. *Let $a\in\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. Then $A_a = b^w$ where $b\in\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$ is given by*

$$b(x,\xi) = \pi^{-d}\left(a(\cdot,\cdot)*e^{-|\cdot|^2-|\cdot|^2}\right)(x,\xi). \tag{5.2}$$

From now on we assume that $A_p = B_p$. Our goal is to represent the Anti-Wick operator A_a, for $a\in\Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ as a pseudo-differential operator b^w for some $b\in\Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. First, note that $|\eta|^{2k}\leq k!e^{|\eta|^2}$, for all $k\in\mathbb{N}$ and

$$\langle\eta\rangle^k \leq 2^k\sqrt{k!}e^{|\eta|^2/2}. \tag{5.3}$$

Theorem 5.1.1. *Let $a\in\Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Then there exists $\tilde{b}\in\Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ and *-regularizing operator T such that $A_a = \tilde{b}^w + T$. Moreover, $\tilde{b}$ has an asymptotic expansion $\sum_j p_j$ in $FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$, where $p_0 = a(x,\xi)$ and*

$$p_j(x,\xi) = \sum_{2j-1\leq|\alpha+\beta|\leq 2j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial^\alpha_\xi\partial^\beta_x a(x,\xi),\ j\in\mathbb{Z}_+,$$

where $c_{\alpha,\beta} = \dfrac{1}{\pi^d}\displaystyle\int_{\mathbb{R}^{2d}}\eta^\alpha y^\beta e^{-|y|^2-|\eta|^2}dyd\eta.$

Proof. First we will prove that $\sum_j p_j\in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Note that $c_{\alpha,\beta}=0$ if $|\alpha+\beta|$ is odd. Hence $p_j(x,\xi) = \displaystyle\sum_{|\alpha+\beta|=2j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial^\alpha_\xi\partial^\beta_x a(x,\xi)$. If we use the fact $|\eta|^k\leq \sqrt{k!}e^{|\eta|^2/2}$ we have $|c_{\alpha,\beta}|\leq c'\sqrt{|\alpha|!|\beta|!}$, where we put $c' = \dfrac{1}{\pi^d}\displaystyle\int_{\mathbb{R}^{2d}}e^{-|y|^2/2-|\eta|^2/2}dyd\eta$. For the derivatives of p_j we have

$$\begin{aligned}\left|D^\gamma_\xi D^\delta_x p_j(x,\xi)\right| &\leq C_1'\sum_{|\alpha+\beta|=2j}\frac{|c_{\alpha,\beta}|}{\alpha!\beta!}\cdot\frac{h^{|\gamma|+|\delta|+2j}A_{\alpha+\gamma}A_{\beta+\delta}e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho j}}\\ &\leq C_1\sum_{|\alpha+\beta|=2j}\frac{d^{2j}}{\sqrt{|\alpha|!|\beta|!}}\cdot\frac{(hH)^{|\gamma|+|\delta|+2j}A_\gamma A_\delta A_{2j}e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho j}}\\ &\leq C_2 2^{2j+2d-1}\frac{(hH)^{|\gamma|+|\delta|+2j}(dH)^{2j}A_\gamma A_\delta A_j A_j e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho j}},\end{aligned}$$

i.e., we obtain $\dfrac{\left|D^\gamma_\xi D^\delta_x p_j(x,\xi)\right|\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho j}e^{-M(m|\xi|)}e^{-M(m|x|)}}{(2dhH^2)^{|\gamma|+|\delta|+2j}A_\gamma A_\delta A_j A_j}\leq C$, for all $(x,\xi)\in\mathbb{R}^{2d}$, $\gamma,\delta\in\mathbb{N}$, $j\in\mathbb{N}$. Hence $\sum_j p_j\in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Take χ_j as in the proof of theorem 4.2.2 and define $\tilde{b} = \sum_j (1-\chi_j)p_j$. Then $\tilde{b} \in \Gamma^{*,\infty}_{A_p,A_p,\rho}(\mathbb{R}^{2d})$ and $\tilde{b} \sim \sum_j p_j$. It is enough to prove that $b - \tilde{b} \in \mathcal{S}^*$, for b defined as in (5.2). We have

$$b(x,\xi) - \tilde{b}(x,\xi) = \chi_0(x,\xi)b(x,\xi) + \sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x,\xi) \left(b(x,\xi) - \sum_{j=0}^{n} p_j(x,\xi) \right).$$

By definition, $\chi_0 = 0$. We Taylor expand a and we obtain

$$a(y,\eta) = \sum_{|\alpha|+|\beta|\leq 2n+1} \frac{1}{\alpha!\beta!} \partial^\alpha_\xi \partial^\beta_x a(x,\xi)(\eta-\xi)^\alpha (y-x)^\beta + r_{2n+2}(x,y,\xi,\eta),$$

where r_{2n+2} is the reminder

$$\begin{aligned} r_{2n+2}(x,y,\xi,\eta) &= (2n+2) \sum_{|\alpha+\beta|=2n+2} \frac{1}{\alpha!\beta!} (\eta-\xi)^\alpha (y-x)^\beta \\ &\quad \cdot \int_0^1 (1-t)^{2n+1} \partial^\alpha_\xi \partial^\beta_x a\,(x+t(y-x), \xi+t(\eta-\xi))\, dt. \end{aligned}$$

If we put this in the expression for $b - \tilde{b}$, keeping in mind the way we defined p_j, we obtain

$$b(x,\xi) - \tilde{b}(x,\xi) = \frac{1}{\pi^d} \sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x,\xi) \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha!\beta!} I_{\alpha,\beta}(x,\xi),$$

where we put

$$I_{\alpha,\beta}(x,\xi) = \int_0^1 \int_{\mathbb{R}^{2d}} \eta^\alpha y^\beta (1-t)^{2n+1} \partial^\alpha_\xi \partial^\beta_x a\,(x+ty, \xi+t\eta)\, e^{-|y|^2-|\eta|^2} dy d\eta dt.$$

We will estimate the derivatives of $I_{\alpha,\beta}$.

$$\begin{aligned} &\left|\partial^\gamma_\xi \partial^\delta_x I_{\alpha,\beta}(x,\xi)\right| \\ &\leq \int_0^1 \int_{\mathbb{R}^{2d}} |\eta|^{|\alpha|} |y|^{|\beta|} \left|\partial^{\alpha+\gamma}_\xi \partial^{\beta+\delta}_x a\,(x+ty, \xi+t\eta)\right| e^{-|y|^2-|\eta|^2} dy d\eta dt \\ &\leq C_1' \int_0^1 \int_{\mathbb{R}^{2d}} |\eta|^{|\alpha|} |y|^{|\beta|} \frac{h^{|\gamma|+|\delta|+2n+2} A_{\alpha+\gamma} A_{\beta+\delta} e^{M(m|\xi+t\eta|)} e^{M(m|x+ty|)}}{\langle (x+ty, \xi+t\eta)\rangle^{\rho|\gamma|+\rho|\delta|+(2n+2)\rho} e^{|y|^2+|\eta|^2}} dy d\eta dt \\ &\leq C_1' \int_0^1 \int_{\mathbb{R}^{2d}} \frac{h^{|\gamma|+|\delta|+2n+2} A_{\gamma+\delta+2n+2} \langle (y,\eta)\rangle^{2n+2} e^{M(m|\xi+t\eta|)} e^{M(m|x+ty|)}}{\langle (x+ty, \xi+t\eta)\rangle^{(2n+2)\rho} e^{|y|^2+|\eta|^2}} dy d\eta dt \\ &\leq C_1'' \frac{(2hL)^{|\gamma|+|\delta|+2n+2} M^\rho_{\gamma+\delta+2n+2}}{\langle (x,\xi)\rangle^{(2n+2)\rho}} \\ &\qquad \cdot \int_0^1 \int_{\mathbb{R}^{2d}} \frac{\langle (y,\eta)\rangle^{4n+4} e^{M(m|\xi+t\eta|)} e^{M(m|x+ty|)}}{e^{|y|^2+|\eta|^2}} dy d\eta dt \\ &\leq C_1 \frac{\sqrt{(4n+4)!}(8hLH)^{|\gamma|+|\delta|+2n+2} M_{\gamma+\delta} M^\rho_{2n+2}}{\langle (x,\xi)\rangle^{(2n+2)\rho}} \end{aligned}$$

$$\cdot \int_0^1 \int_{\mathbb{R}^{2d}} \frac{e^{M(m|\xi+t\eta|)}e^{M(m|x+ty|)}}{e^{|y|^2/2+|\eta|^2/2}} dyd\eta dt,$$

where, in the last inequality, we used (5.3). For shorter notations, we will denote the last integral by $\tilde{I}(x,\xi)$. Note that $\langle (x,\xi)\rangle \geq Rm_n$ on the support of $\chi_{n+1}-\chi_n$. For the derivatives of $(\chi_{n+1}-\chi_n)(x,\xi)I_{\alpha,\beta}(x,\xi)$, we have

$$\begin{aligned}
&\left|\partial_\xi^\gamma \partial_x^\delta \left((\chi_{n+1}-\chi_n)(x,\xi)I_{\alpha,\beta}(x,\xi)\right)\right| \\
&\leq \sum_{\substack{\gamma'\leq\gamma\\ \delta'\leq\delta}} \binom{\gamma}{\gamma'}\binom{\delta}{\delta'} \left|\partial_\xi^{\gamma-\gamma'}\partial_x^{\delta-\delta'}\left((\chi_{n+1}-\chi_n)(x,\xi)\right)\right| \left|\partial_\xi^{\gamma'}\partial_x^{\delta'} I_{\alpha,\beta}(x,\xi)\right| \\
&\leq C_2 \sum_{\substack{\gamma'\leq\gamma\\ \delta'\leq\delta}} \binom{\gamma}{\gamma'}\binom{\delta}{\delta'} \frac{h_1^{|\gamma|-|\gamma'|+|\delta|-|\delta'|}A_{\gamma-\gamma'}A_{\delta-\delta'}}{(Rm_n)^{|\gamma|-|\gamma'|+|\delta|-|\delta'|}} \\
&\qquad \cdot \frac{\sqrt{(4n+4)!}(8hLH)^{|\gamma'|+|\delta'|+2n+2}M_{\gamma'+\delta'}M_{2n+2}^\rho}{(Rm_n)^{(2n+2)\rho}} \cdot \tilde{I}(x,\xi) \\
&\leq C_3 \sum_{\substack{\gamma'\leq\gamma\\ \delta'\leq\delta}} \binom{\gamma}{\gamma'}\binom{\delta}{\delta'} \frac{(h_1L)^{|\gamma|-|\gamma'|+|\delta|-|\delta'|}}{(RM_1)^{|\gamma|-|\gamma'|+|\delta|-|\delta'|}} \\
&\qquad \cdot \frac{\sqrt{(4n+4)!}(8hLH)^{|\gamma'|+|\delta'|+2n+2}H^{2n+2}M_{\gamma+\delta}M_{n+1}^{2\rho}}{(Rm_n)^{(2n+2)\rho}} \cdot \tilde{I}(x,\xi) \\
&\leq C_4 \left(\frac{h_1L}{RM_1}+8hLH\right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!}(8hLH^3)^{2n+2}M_{\gamma+\delta}M_n^{2\rho}}{R^{(2n+2)\rho}m_n^{(2n+2)\rho}} \cdot \tilde{I}(x,\xi) \\
&\leq C_4' \left(\frac{h_1L}{RM_1}+8hLH\right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!}(8hLH^3)^{2n+2}M_{\gamma+\delta}}{R^{(2n+2)\rho}} \cdot \tilde{I}(x,\xi),
\end{aligned}$$

where, in the last inequality, we used that

$$m_n^{n+1} \geq m_n \cdot \ldots \cdot m_2 \cdot m_1 \cdot m_1 = M_n M_1.$$

Let $m'>0$ be arbitrary but fixed. Then one easily proves that $e^{M(m'|(x,\xi)|)} \leq e^{M(m'(|x|+|\xi|))} \leq 2e^{M(2m'|x|)}e^{M(2m'|\xi|)}$ (one easily proves that $e^{M(\lambda+\nu)} \leq 2e^{M(2\lambda)}e^{M(2\nu)}$, $\lambda,\nu>0$). Then we have

$$\begin{aligned}
e^{M(m|\xi+t\eta|)} &= e^{-M(2m'|\xi|)}e^{M(2m'|\xi|)}e^{M(m|\xi+t\eta|)} \\
&\leq 2e^{-M(2m'|\xi|)}e^{M(4m'|t\eta|)}e^{M(4m'|\xi+t\eta|)}e^{M(m|\xi+t\eta|)} \\
&\leq c_1 e^{-M(2m'|\xi|)}e^{M(4m'|\eta|)}e^{M((m+4m')H|\xi+t\eta|)},
\end{aligned}$$

where, in the last inequality, we used proposition 1.2.1. Similarly

$$e^{M(m|x+ty|)} \leq c_1 e^{-M(2m'|x|)}e^{M(4m'|y|)}e^{M((m+4m')H|x+ty|)}.$$

Obviously $e^{M(4m'|\eta|)} \leq c_2 e^{|\eta|^2/4}$ and $e^{M(4m'|y|)} \leq c_2 e^{|y|^2/4}$ for some $c_2>0$ which depends only on M_p and m'. We obtain

$$\tilde{I}(x,\xi) \leq \frac{c_3}{e^{M(m'|(x,\xi)|)}} \int_0^1 \left(\int_{\mathbb{R}^d} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}}dy \cdot \int_{\mathbb{R}^d} \frac{e^{M((m+4m')H|\xi+t\eta|)}}{e^{|\eta|^2/4}}d\eta\right) dt.$$

Note that, when $|y| \leq |x|$ we have

$$e^{M((m+4m')H|x+ty|)} \leq e^{M(2(m+4m')H|x|)} \leq e^{M(6(m+4m')HRm_{n+1})},$$

on the support of $\chi_{n+1} - \chi_n$ (where $|x| \leq 3Rm_{n+1}$). When $|y| > |x|$,

$$e^{M((m+4m')H|x+ty|)} \leq e^{M(2(m+4m')H|y|)} \leq c_4 e^{|y|^2/8},$$

for some $c_4 > 0$. We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}} dy \\
&\quad = \int_{|y|\leq|x|} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}} dy + \int_{|y|>|x|} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}} dy \\
&\quad \leq e^{M(6(m+4m')HRm_{n+1})} \int_{|y|\leq|x|} e^{-|y|^2/4} dy + c_4 \int_{|y|>|x|} e^{-|y|^2/8} dy \\
&\quad \leq c_5 e^{M(6(m+4m')HRm_{n+1})}.
\end{aligned}$$

We can obtain similar estimate for the other integral. By lemma 4.1.3, we have

$$e^{M(6(m+4m')HRm_{n+1})} \leq c_0 H^{2(n+1)(6c_0(m+4m')HR+2)}.$$

Hence

$$\begin{aligned}
\tilde{I}(x,\xi) &\leq c_6 e^{-M(m'|(x,\xi)|)} e^{2M(6(m+4m')HRm_{n+1})} \\
&\leq c_7 e^{-M(m'|(x,\xi)|)} H^{4(n+1)(6c_0(m+4m')HR+2)}
\end{aligned}$$

on the support of $\chi_{n+1} - \chi_n$. If we insert this in the estimates for the derivatives of the terms $(\chi_{n+1} - \chi_n)(x,\xi) I_{\alpha,\beta}(x,\xi)$, we obtain

$$\begin{aligned}
&\left|\partial_\xi^\gamma \partial_x^\delta \left((\chi_{n+1} - \chi_n)(x,\xi) I_{\alpha,\beta}(x,\xi) \right)\right| \\
&\quad \leq C_5 \left(\frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!}(8hLH^3)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho}} \\
&\qquad \cdot e^{-M(m'|(x,\xi)|)} H^{4(n+1)(6c_0(m+4m')HR+2)}.
\end{aligned}$$

First, we consider the (M_p) case. Take R such that $RM_1 \geq L$ and $32d/R^\rho \leq 1/2$. Choose h_1 such that $h_1 \leq 1/(2m')$ and h such that $8hLH^{3+2(6c_0(m+4m')HR+2)} \leq 1$ and $8hLH \leq 1/(2m')$. Note that, the choice of R (and hence χ_j) doesn't depend on m', only on A_p, M_p and a. For $|\alpha + \beta| = 2n+2$, we have

$$\alpha!\beta! \geq \frac{|\alpha|!|\beta|!}{d^{2n+2}} \geq \frac{(2n+2)!}{(2d)^{2n+2}}.$$

Also, $\sqrt{(4n+4)!} \leq 2^{2n+2}(2n+2)!$. Now we obtain

$$\begin{aligned}
&\sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha!\beta!} \left|\partial_\xi^\gamma \partial_x^\delta \left((\chi_{n+1} - \chi_n)(x,\xi) I_{\alpha,\beta}(x,\xi) \right)\right| \\
&\quad \leq C_5 \sum_{|\alpha+\beta|=2n+2} \frac{2^{2n+2}(2d)^{2n+2}}{(2n+2)!} \left(\frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{2^{2n+2}(2n+2)! M_{\gamma+\delta}}{R^{(2n+2)\rho} e^{M(m'|(x,\xi)|)}}
\end{aligned}$$

$$\leq \quad C_5 e^{-M(m'|(x,\xi)|)} \frac{M_{\gamma+\delta}}{m'^{|\gamma|+|\delta|}} \cdot \left(\frac{8d}{R^\rho}\right)^{2n+2} \cdot 2^{2n+2+2d-1}$$
$$\leq \quad C_6 \frac{M_{\gamma+\delta}}{m'^{|\gamma|+|\delta|}} e^{-M(m'|(x,\xi)|)} \cdot \frac{1}{4^{2n+2}},$$

where, in the last inequality, we put $C_6 = 2^{2d-1}C_5$. Hence, for the derivatives of

$$\sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x,\xi) \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha!\beta!} I_{\alpha,\beta}(x,\xi),$$

we obtain the estimate $C\frac{M_{\gamma+\delta}}{m'^{|\gamma|+|\delta|}} e^{-M(m'|(x,\xi)|)}$ and by the arbitrariness of m', it follows that it is a $\mathcal{S}^{(M_p)}$ function. Let us consider the $\{M_p\}$ case. Take R such that $\frac{256dhLH^9}{R^\rho} \leq \frac{1}{2}$. Then, choose m and m' such that $6c_0(m+4m')HR \leq 1$. Then we have

$$\left|\partial_\xi^\gamma \partial_x^\delta \left((\chi_{n+1} - \chi_n)(x,\xi) I_{\alpha,\beta}(x,\xi)\right)\right|$$
$$\leq \quad C_5 \left(\frac{h_1 L}{RM_1} + 8hLH\right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!}(8hLH^3)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho}} \cdot e^{-M(m'|(x,\xi)|)} H^{4(n+1)(6c_0(m+4m')HR+2)}$$
$$\leq \quad C_5 \left(\frac{h_1 L}{RM_1} + 8hLH\right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!}(8hLH^3)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho} e^{M(m'|(x,\xi)|)}} \cdot H^{12(n+1)}.$$

So

$$\sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha!\beta!} \left|\partial_\xi^\gamma \partial_x^\delta \left((\chi_{n+1} - \chi_n)(x,\xi) I_{\alpha,\beta}(x,\xi)\right)\right|$$
$$\leq \quad C_5 \sum_{|\alpha+\beta|=2n+2} \left(\frac{h_1 L}{RM_1} + 8hLH\right)^{|\gamma|+|\delta|} \frac{(8d)^{2n+2}(8hLH^9)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho} e^{M(m'|(x,\xi)|)}}$$
$$\leq \quad C_5 e^{-M(m'|(x,\xi)|)} M_{\gamma+\delta} \left(\frac{h_1 L}{RM_1} + 8hLH\right)^{|\gamma|+|\delta|} \cdot \left(\frac{64dhLH^9}{R^\rho}\right)^{2n+2} \cdot 2^{2n+2+2d-1}$$
$$\leq \quad C_6 M_{\gamma+\delta} \left(\frac{h_1 L}{RM_1} + 8hLH\right)^{|\gamma|+|\delta|} e^{-M(m'|(x,\xi)|)} \cdot \frac{1}{4^{2n+2}}.$$

Hence, for the derivatives of

$$\sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x,\xi) \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha!\beta!} I_{\alpha,\beta}(x,\xi),$$

we obtain the estimate $CM_{\gamma+\delta}\frac{1}{m''^{|\gamma|+|\delta|}} e^{-M(m'|(x,\xi)|)}$, where we put $\frac{1}{m''} = \frac{h_1 L}{RM_1} + 8hLH$, i.e. it is a $\mathcal{S}^{\{M_p\}}$ function. In both cases we obtain that $b - \tilde{b} \in \mathcal{S}^*$, which completes the proof. □

Now we want to represent the Weyl quantization of $b \in \Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ by an Anti-Wick operator A_a, for some $a \in \Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. First we will prove the following technical lemma.

Lemma 5.1.1. *Let* $\sum_k q_k^{(j)} \in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ *for all* $j \in \mathbb{N}$*, such that* $q_0^{(j)} = ... = q_{j-1}^{(j)} = 0$*. Assume that there exist* $m > 0$ *and* $B > 0$*, resp.* $h > 0$ *and* $B > 0$*, such that* $\sum_k q_k^{(j)} \in FS^{(M_p),\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}; B, m\right)$ *for all* $j \in \mathbb{N}$*, resp.* $\sum_k q_k^{(j)} \in FS^{\{M_p\},\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}; B, h\right)$ *for all* $j \in \mathbb{N}$*. Moreover, assume that the constants* $C_{j,h}$*, resp.* $C_{j,m}$*, in*

$$\sup_{k\in\mathbb{N}}\sup_{\alpha,\beta}\sup_{(x,\xi)\in Q^c_{Bm_k}} \frac{\left|D^\alpha_\xi D^\beta_x q_k^{(j)}(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2k\rho} e^{-M(m|\xi|)}e^{-M(m|x|)}}{h^{|\alpha|+|\beta|+2k}A_\alpha A_\beta A_k A_k} = C_{j,h}$$

resp. the same with $C_{j,m}$ *in place of* $C_{j,h}$ *in the* $\{M_p\}$ *case, are bounded for all* j*, i.e.* $\sup_j C_{j,h} = C_h < \infty$*, resp.* $\sup_j C_{j,m} = C_m < \infty$*. Then, there exist* $p_j \in \mathcal{C}^\infty\left(\mathbb{R}^{2d}\right)$ *such that* $p_j \sim \sum_k q_k^{(j)}$*, for all* $j \in \mathbb{N}$ *and* $\sum_j p_j \in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$*. Moreover,* $\sum_{j=0}^{\infty} p_j \sim \sum_{j=0}^{\infty}\sum_{l=0}^{j} q_j^{(l)}$ *in* $FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$*.*

Remark 5.1.1. $p_j \sim \sum_k q_k^{(j)}$ should be understand as equivalence of the sums $\underbrace{0+...+0}_{j} + p_j + 0 + ...$ and $\sum_k q_k^{(j)}$.

Proof. Let $R \geq 2B$ and take p_j as in the proof of theorem 4.2.2, i.e. $p_j = \sum_{k=j}^{\infty}(1-\chi_k)q_k^{(j)}$, for χ_k constructed there. First, we consider the (M_p) case. We will prove that $\sum_j p_j \in FS^{(M_p),\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}; B, m\right)$, for sufficiently large R. Let $h > 0$ be arbitrary but fixed. Obviously, without losing generality, we can assume that $h \leq 1$. For simplicity, denote C_h by C. Using the fact that $1 - \chi_k(x,\xi) = 0$ for $(x,\xi) \in Q_{Rm_k}$, we have the estimate

$$\begin{aligned}
&\frac{\left|D^\alpha_\xi D^\beta_x p_j(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho j} e^{-M(m|\xi|)}e^{-M(m|x|)}}{(8hH)^{|\alpha|+|\beta|+2j}A_\alpha A_\beta A_j A_j}\\
&\leq \sum_{k=j}^{\infty}\sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}} \binom{\alpha}{\gamma}\binom{\beta}{\delta}\left|D^{\alpha-\gamma}_\xi D^{\beta-\delta}_x q_k^{(j)}(x,\xi)\right| e^{-M(m|\xi|)}e^{-M(m|x|)}\\
&\qquad\cdot\frac{\left|D^\gamma_\xi D^\delta_x\left(1-\chi_k(x,\xi)\right)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho j}}{(8hH)^{|\alpha|+|\beta|+2j}A_\alpha A_\beta A_j A_j}\\
&\leq C\sum_{k=j}^{\infty}\sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}} \binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{h^{|\alpha|-|\gamma|+|\beta|-|\delta|+2k}A_{\alpha-\gamma}A_{\beta-\delta}A_k A_k}{(8hH)^{|\alpha|+|\beta|+2j}A_\alpha A_\beta A_j A_j}\\
&\qquad\cdot\langle (x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho j-2\rho k}\left|D^\gamma_\xi D^\delta_x\left(1-\chi_k(x,\xi)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq (c_0c_0')^2C\sum_{k=j}^{\infty}\frac{1}{8^{|\alpha|+|\beta|+2j}H^{2j}}h^{2(k-j)}H^{2k}L^{2(k-j)}M_{k-j}^{2\rho}\left|1-\chi_k(x,\xi)\right|\langle(x,\xi)\rangle^{2\rho(j-k)}\\
&\quad+(c_0c_0')^2C\sum_{k=j}^{\infty}\frac{1}{8^{|\alpha|+|\beta|+2j}H^{2j}}\sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\\
&\qquad\cdot\frac{h^{2(k-j)}H^{2k}L^{2(k-j)}M_{k-j}^{2\rho}\left|D_\xi^\gamma D_x^\delta\left(1-\chi_k(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho j-2\rho k}}{h^{|\gamma|+|\delta|}A_\gamma A_\delta}\\
&= S_1+S_2,
\end{aligned}
$$

where S_1 and S_2 are the first and the second sum, correspondingly. To estimate S_1 note that, on the support of $1-\chi_k$, the inequality $\langle(x,\xi)\rangle\geq Rm_k$ holds. One obtains

$$
S_1\leq(c_0c_0')^2C\sum_{k=j}^{\infty}\frac{(hLH)^{2(k-j)}M_{k-j}^{2\rho}}{R^{2\rho(k-j)}m_k^{2\rho(k-j)}}\leq(c_0c_0')^2C\sum_{k=0}^{\infty}\frac{(hLH)^{2k}}{R^{2\rho k}}<\infty,
$$

for $R^\rho\geq 2LH\geq 2hLH$ (in the second inequality we use the fact that $m_j^j\geq M_j$). For the estimate of S_2, note that $D_\xi^\gamma D_x^\delta\left(1-\chi_k(x,\xi)\right)=0$ when $(x,\xi)\in Q_{3Rm_k}^c$, because $(\delta,\gamma)\neq(0,0)$ and $\chi_k(x,\xi)=0$ on $Q_{3Rm_k}^c$. So, for $(x,\xi)\in Q_{3Rm_k}$, we have that $\langle(x,\xi)\rangle\leq\langle x\rangle+\langle\xi\rangle\leq 6Rm_k$. Moreover, from the construction of χ, we have that for the chosen h, there exists $C_1>0$ such that $\left|D_\xi^\alpha D_x^\beta\chi(x,\xi)\right|\leq C_1h^{|\alpha|+|\beta|}A_\alpha A_\beta$. By using $m_k^k\geq M_k$, one obtains

$$
\begin{aligned}
S_2 &\leq (c_0c_0')^2CC_1\sum_{k=j}^{\infty}\frac{1}{8^{|\alpha|+|\beta|+2j}}\sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\\
&\qquad\cdot\frac{(hLH)^{2(k-j)}6^{\rho|\gamma|+\rho|\delta|}M_{k-j}^{2\rho}(Rm_k)^{\rho|\gamma|+\rho|\delta|}}{R^{2\rho(k-j)}m_k^{2\rho(k-j)}(Rm_k)^{|\gamma|+|\delta|}}\\
&\leq (c_0c_0')^2CC_1\sum_{k=0}^{\infty}\frac{(hLH)^{2k}}{R^{2\rho k}},
\end{aligned}
$$

which is convergent for $R^\rho\geq 2LH\geq 2hLH$. Moreover, note that the choice of R for these sums to be convergent does not depend on j, hence χ_k can be chosen to be the same for all p_j. So, these estimates does not depend on j and from this it follows that $\sum_j p_j\in FS_{A_p,A_p,\rho}^{(M_p),\infty}\left(\mathbb{R}^{2d}\right)$ (actually, to be precise, $\sum_j p_j\in FS_{A_p,A_p,\rho}^{(M_p),\infty}\left(\mathbb{R}^{2d};B,m\right)$, i.e. the same space as for $\sum_k q_k^{(j)}$).

In the $\{M_p\}$ case, there exist $h_1,C_1>0$ such that

$$
\left|D_\xi^\alpha D_x^\beta\chi(x,\xi)\right|\leq C_1h_1^{|\alpha|+|\beta|}A_\alpha A_\beta.
$$

Arguing in similar fashion, one proves that $\sum_j p_j\in FS_{A_p,A_p,\rho}^{\{M_p\},\infty}\left(\mathbb{R}^{2d};B,8\tilde{h}H\right)$, where $\tilde{h}=\max\{h,h_1\}$, i.e. $\sum_j p_j\in FS_{A_p,A_p,\rho}^{\{M_p\},\infty}\left(\mathbb{R}^{2d}\right)$.

It remains to prove the second part of the lemma. One easily proves that

$\sum_{j=0}^{\infty}\sum_{l=0}^{j} q_j^{(l)} \in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Note that $\sum_{j=0}^{N-1} p_j - \sum_{j=0}^{N-1}\sum_{l=0}^{j} q_j^{(l)} = \sum_{j=0}^{N-1}\left(p_j - \sum_{k=j}^{N-1} q_k^{(j)}\right)$. Moreover, for $(x,\xi)\in Q^c_{3Rm_N}$ and $N>j$, $p_j - \sum_{k=j}^{N-1} q_k^{(j)} = \sum_{k=N}^{\infty}(1-\chi_k)\,q_k^{(j)}$. This easily follows from the definition of χ_k and the fact that m_n is monotonically increasing. We will consider first the (M_p) case. For arbitrary but fixed $0<h\leq 1$ and $(x,\xi)\in Q^c_{3Rm_N}$, we estimate as follows

$$
\begin{aligned}
&\frac{\left|D_\xi^\alpha D_x^\beta \sum_{k=N}^{\infty}(1-\chi_k(x,\xi))\,q_k^{(j)}(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N} e^{-M(m|\xi|)}e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha|+|\beta|+2N}A_\alpha A_\beta A_N A_N}\\
&\leq \sum_{k=N}^{\infty}\frac{(1-\chi_k(x,\xi))\left|D_\xi^\alpha D_x^\beta q_k^{(j)}(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N} e^{-M(m|\xi|)}e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha|+|\beta|+2N}A_\alpha A_\beta A_N A_N}\\
&\quad +\sum_{k=N}^{\infty}\sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\left|D_\xi^{\alpha-\gamma}D_x^{\beta-\delta}q_k^{(j)}(x,\xi)\right| e^{-M(m|\xi|)}e^{-M(m|x|)}\\
&\qquad\cdot\frac{\left|D_\xi^\gamma D_x^\delta(1-\chi_k(x,\xi))\right|\langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N}}{(8(1+H)h)^{|\alpha|+|\beta|+2N}A_\alpha A_\beta A_N A_N}\\
&\leq \frac{C}{64^N}\sum_{k=N}^{\infty}\frac{(1-\chi_k(x,\xi))\,h^{2k-2N}A_kA_k}{(1+H)^{2N}\langle (x,\xi)\rangle^{2\rho k-2\rho N}A_NA_N}\\
&\quad +\frac{C}{64^N}\sum_{k=N}^{\infty}\frac{1}{8^{|\alpha|+|\beta|}}\sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\\
&\qquad\cdot\frac{h^{2k-2N}\left|D_\xi^\gamma D_x^\delta(1-\chi_k(x,\xi))\right|\langle (x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|}A_kA_k}{(1+H)^{2N}h^{|\gamma|+|\delta|}\langle (x,\xi)\rangle^{2\rho k-2\rho N}A_\gamma A_\delta A_NA_N}\\
&= S_1+S_2,
\end{aligned}
$$

where S_1 and S_2 are the first and the second sum, correspondingly. To estimate S_1, observe that on the support of $1-\chi_k$ the inequality $\langle (x,\xi)\rangle \geq Rm_k$ holds. Using the monotone increasingness of m_n and $(M.2)$ for A_p, one obtains

$$
\begin{aligned}
S_1 &\leq \frac{c_0^2C}{64^N}\sum_{k=N}^{\infty}\frac{h^{2k-2N}H^{2k}A_{k-N}A_{k-N}}{(1+H)^{2N}R^{2\rho k-2\rho N}m_k^{2\rho k-2\rho N}}\\
&\leq \frac{(c_0c_0')^2C}{64^N}\sum_{k=N}^{\infty}\frac{h^{2k-2N}H^{2k}L^{2k-2N}M_{k-N}^{2\rho}}{(1+H)^{2N}R^{2\rho k-2\rho N}m_{k-N}^{2\rho k-2\rho N}}\\
&= \frac{(c_0c_0')^2CH^{2N}}{64^N(1+H)^{2N}}\sum_{k=0}^{\infty}\left(\frac{hHL}{R^\rho}\right)^{2k}\leq\frac{(c_0c_0')^2C}{64^N}\sum_{k=0}^{\infty}\left(\frac{HL}{R^\rho}\right)^{2k}=\frac{(c_0c_0')^2C\tilde{C}}{64^N},
\end{aligned}
$$

where we put $\tilde{C}=\sum_{k=0}^{\infty}\left(\frac{HL}{R^\rho}\right)^{2k}$, for some fixed $R^\rho\geq 2HL$. For the sum S_2, observe that $D_\xi^\gamma D_x^\delta(1-\chi_k(x,\xi))=0$ when $(x,\xi)\in Q^c_{3Rm_k}$, because $(\delta,\gamma)\neq(0,0)$

and $\chi_k(x,\xi)=0$ on $Q^c_{3Rm_k}$. Moreover, from the construction of χ, we have that for the chosen h, there exists $C_1>1$ such that $\left|D^\alpha_\xi D^\beta_x \chi(x,\xi)\right| \leq C_1 h^{|\alpha|+|\beta|} A_\alpha B_\beta$. Now

$$\begin{aligned}
S_2 &\leq \frac{c_0^2 C C_1}{64^N}\sum_{k=N}^{\infty}\frac{1}{8^{|\alpha|+|\beta|}}\sum_{\substack{\gamma\leq\alpha,\delta\leq\beta\\(\delta,\gamma)\neq(0,0)}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\frac{h^{2k-2N}6^{|\gamma|+|\delta|}H^{2k}A_{k-N}A_{k-N}}{(1+H)^{2N}R^{2\rho k-2\rho N}m_k^{2\rho k-2\rho N}}\\
&\leq \frac{c_0^2 C C_1}{64^N}\sum_{k=N}^{\infty}\frac{h^{2k-2N}H^{2k}A_{k-N}A_{k-N}}{(1+H)^{2N}R^{2\rho k-2\rho N}m_k^{2\rho k-2\rho N}} \leq \frac{(c_0c_0')^2 C C_1\tilde{C}}{64^N},
\end{aligned}$$

where we used the above estimate for the last sum. So, we have

$$\begin{aligned}
&\frac{\left|D^\alpha_\xi D^\beta_x \sum_{j=0}^{N-1}\left(p_j(x,\xi)-\sum_{k=j}^{N-1} q_k^{(j)}(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N}}{(8(1+H)h)^{|\alpha|+|\beta|+2N}A_\alpha A_\beta A_N A_N e^{M(m|\xi|)}e^{M(m|x|)}}\\
&\leq \sum_{j=0}^{N-1}\frac{\left|D^\alpha_\xi D^\beta_x\left(p_j(x,\xi)-\sum_{k=j}^{N-1} q_k^{(j)}(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N}}{(8(1+H)h)^{|\alpha|+|\beta|+2N}A_\alpha A_\beta A_N A_N e^{M(m|\xi|)}e^{M(m|x|)}}\\
&\leq \sum_{j=0}^{N-1}\frac{2(c_0c_0')^2 C C_1\tilde{C}}{64^N}=\frac{2N(c_0c_0')^2 C C_1\tilde{C}}{64^N},
\end{aligned}$$

which is bounded uniformly for all $N\in\mathbb{Z}_+$, for $(x,\xi)\in Q^c_{3Rm_N}$, $\alpha,\beta\in\mathbb{N}^d$. The proof for the $\{M_p\}$ case is similar. □

Theorem 5.1.2. *Let $b\in\Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. There exist $a\in\Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ and *-regularizing operator T such that $b^w=A_a+T$.*

Proof. Put $p'_{0,0}=b$ and $p'_{k,0}=0$ for all $k\in\mathbb{Z}_+$. For $j\in\mathbb{Z}_+$, define $p'_{0,j}=...=p'_{j-1,j}=0$ and

$$\begin{aligned}
p'_{k,j}(x,\xi) \;=\; &\sum_{\substack{l_1+l_2+...+l_j=k\\ l_1\geq1,...,l_j\geq1}}\;\sum_{|\alpha^{(1)}+\beta^{(1)}|=2l_1,...,|\alpha^{(j)}+\beta^{(j)}|=2l_j}\frac{c_{\alpha^{(1)},\beta^{(1)}}\cdot\ldots\cdot c_{\alpha^{(j)},\beta^{(j)}}}{\alpha^{(1)}!\beta^{(1)}!\cdot\ldots\cdot\alpha^{(j)}!\beta^{(j)}!}\\
&\cdot\partial_\xi^{\alpha^{(1)}+...+\alpha^{(j)}}\partial_x^{\beta^{(1)}+...+\beta^{(j)}}b(x,\xi),
\end{aligned}$$

for $k\geq j$, $k\in\mathbb{Z}_+$. We will prove that $\sum_k p'_{k,j}$ is an element of $FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. To do this note that, for $k\geq j$,

$$\begin{aligned}
&\left|\partial_\xi^{\gamma+\alpha^{(1)}+...+\alpha^{(j)}}\partial_x^{\delta+\beta^{(1)}+...+\beta^{(j)}}b(x,\xi)\right|\\
&\leq c_0^2\|b\|_{h,m,\Gamma}\frac{h^{|\gamma|+|\delta|+2k}H^{|\gamma|+|\delta|+2k}A_\gamma A_\delta A_{2k}e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho k}}\\
&\leq c_0^3\|b\|_{h,m,\Gamma}\frac{(hH^2)^{|\gamma|+|\delta|+2k}A_\gamma A_\delta A_k A_k e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho k}}.
\end{aligned}$$

If we use the same estimates as in the beginning of the proof of theorem 5.1.1, we have

$$\frac{|c_{\alpha^{(s)},\beta^{(s)}}|}{\alpha^{(s)}!\beta^{(s)}!} \leq \frac{c'd^{2l_s}}{\sqrt{|\alpha^{(s)}|!|\beta^{(s)}|!}} \leq c'd^{2l_s}, \tag{5.4}$$

for all $s \in \{1, ..., j\}$, where $c' = \frac{1}{\pi^d}\int_{\mathbb{R}^{2d}} e^{-|y|^2/2-|\eta|^2/2}dyd\eta$. Hence

$$\frac{|c_{\alpha^{(1)},\beta^{(1)}}| \cdot ... \cdot |c_{\alpha^{(j)},\beta^{(j)}}|}{\alpha^{(1)}!\beta^{(1)}! \cdot ... \cdot \alpha^{(j)}!\beta^{(j)}!} \leq c'^j d^{2k} \leq (c'd^2)^k.$$

The number of ways we can choose the positive integers $l_1, ..., l_j$ such that $l_1 + ... + l_j = k$ is $\binom{k-1}{j-1}$. For every fixed $l_1, ..., l_j$, we have

$$\sum_{|\alpha^{(s)}+\beta^{(s)}|=2l_s} 1 = \binom{2l_s+2d-1}{2d-1} \leq 2^{2l_s+2d-1} = 2^{2d-1}4^{l_s},$$

for $s \in \{1, ..., j\}$. So, if we use that $k \geq j$, we have

$$\sum_{\substack{l_1+l_2+...+l_j=k\\ l_1\geq 1,...,l_j\geq 1}} \sum_{|\alpha^{(1)}+\beta^{(1)}|=2l_1,...,|\alpha^{(j)}+\beta^{(j)}|=2l_j} 1 \leq 2^{j(2d-1)}4^k\binom{k-1}{j-1} \leq 2^{k(2d+2)}.$$

We obtain

$$\sum_{\substack{l_1+l_2+...+l_j=k\\ l_1\geq 1,...,l_j\geq 1}} \sum_{|\alpha^{(1)}+\beta^{(1)}|=2l_1,...,|\alpha^{(j)}+\beta^{(j)}|=2l_j} \frac{|c_{\alpha^{(1)},\beta^{(1)}}| \cdot ... \cdot |c_{\alpha^{(j)},\beta^{(j)}}|}{\alpha^{(1)}!\beta^{(1)}! \cdot ... \cdot \alpha^{(j)}!\beta^{(j)}!} \leq \left(c'2^{2d+2}d^2\right)^k,$$

i.e.

$$\frac{\left|D^\gamma_\xi D^\delta_x p'_{k,j}(x,\xi)\right| \langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2\rho k}e^{-M(m|\xi|)}e^{-M(m|x|)}}{(c'2^{2d+2}d^2hH^2)^{|\gamma|+|\delta|+2k}\,A_\gamma A_\delta A_k A_k} \leq c_0^3\|b\|_{h,m,\Gamma},$$

for all $(x,\xi) \in \mathbb{R}^{2d}$, $\gamma,\delta \in \mathbb{N}^d$, $k \in \mathbb{N}$ (for $k < j$, $p'_{k,j} = 0$). So $\sum_k p'_{k,j} \in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Note that $c_0^3\|b\|_{h,m}$ does not depend on j, i.e. the estimates are uniform in j. By the above lemma, there exist $\mathcal{C}^\infty$ functions b_j such that $b_j \sim \sum_k p'_{k,j}$, for $j \in \mathbb{N}$ and $\sum_j b_j \in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Note that, by the construction in the lemma and the way we define $p'_{k,j}$, $b_0 = p'_{0,0} = b$. By theorem 4.2.2, there exists $a \in \Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ such that $a \sim \sum_j(-1)^j b_j$. We will prove that this a satisfies the conditions in the theorem. By theorem 5.1.1, there exist $c \in \Gamma^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ and *-regularizing operator T_1 such that $A_a = c^w + T_1$ and $c \sim \sum_{j=0}^{\infty}\sum_{|\alpha+\beta|=2j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial^\alpha_\xi\partial^\beta_x a(x,\xi)$. One obtains

$$c \sim \sum_{j=0}^{\infty}\sum_{l+k=j}\sum_{|\alpha+\beta|=2l}(-1)^k\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial^\alpha_\xi\partial^\beta_x b_k(x,\xi).$$

To prove this, first, by using $\sum_j (-1)^j b_j \in FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$ and (5.4), one easily verifies that the sum is an element of $FS^{*,\infty}_{A_p,A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Note that

$$\sum_{j=0}^{N-1}\sum_{|\alpha+\beta|=2j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta a(x,\xi)-\sum_{j=0}^{N-1}\sum_{l=0}^{j}\sum_{|\alpha+\beta|=2l}(-1)^{j-l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_{j-l}(x,\xi)$$

$$\begin{aligned}
&=\sum_{j=0}^{N-1}\sum_{|\alpha+\beta|=2j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta a(x,\xi)-\sum_{l=0}^{N-1}\sum_{j=l}^{N-1}\sum_{|\alpha+\beta|=2l}(-1)^{j-l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_{j-l}(x,\xi)\\
&=\sum_{j=0}^{N-1}\sum_{|\alpha+\beta|=2j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta a(x,\xi)-\sum_{j=0}^{N-1}\sum_{l=j}^{N-1}\sum_{|\alpha+\beta|=2j}(-1)^{l-j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_{l-j}(x,\xi)\\
&=\sum_{j=0}^{N-1}\sum_{|\alpha+\beta|=2j}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta\left(a(x,\xi)-\sum_{s=0}^{N-j-1}(-1)^s b_s(x,\xi)\right).
\end{aligned}$$

By using that $a\sim\sum_j(-1)^j b_j$ and the inequality (5.4), one easily proves the desired equivalence. Now, observe that, if we prove the equivalence

$$b\sim\sum_{j=0}^{\infty}\sum_{l+k=j}\sum_{|\alpha+\beta|=2l}(-1)^k\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_k(x,\xi),$$

the claim of the theorem will follow. Observe that

$$\sum_{j=0}^{N-1}\sum_{l+k=j}\sum_{|\alpha+\beta|=2l}(-1)^k\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_k(x,\xi)-b(x,\xi)$$

$$=\sum_{j=1}^{N-1}\sum_{l+k=j}\sum_{|\alpha+\beta|=2l}(-1)^k\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_k(x,\xi) \tag{5.5}$$

$$=\sum_{k=1}^{N-1}(-1)^{k-1}\left(\sum_{j=k}^{N-1}\sum_{|\alpha+\beta|=2(j-k+1)}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_{k-1}(x,\xi)-b_k(x,\xi)\right). \tag{5.6}$$

Because of the way we defined $p'_{s,k}$, for $s\geq k\geq 2$, we have

$$\begin{aligned}
p'_{s,k}(x,\xi)&=\sum_{l=1}^{s-k+1}\sum_{|\alpha+\beta|=2l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta\sum_{\substack{l_1+\ldots+l_{k-1}=s-l\\ l_1\geq1,\ldots,l_{k-1}\geq1}}\sum_{|\alpha^{(1)}+\beta^{(1)}|=2l_1,\ldots,|\alpha^{(k-1)}+\beta^{(k-1)}|=2l_{k-1}}\\
&\qquad\frac{c_{\alpha^{(1)},\beta^{(1)}}\cdot\ldots\cdot c_{\alpha^{(k-1)},\beta^{(k-1)}}}{\alpha^{(1)}!\beta^{(1)}!\cdot\ldots\cdot\alpha^{(k-1)}!\beta^{(k-1)}!}\partial_\xi^{\alpha^{(1)}+\ldots+\alpha^{(k-1)}}\partial_x^{\beta^{(1)}+\ldots+\beta^{(k-1)}}b(x,\xi)\\
&=\sum_{l=1}^{s-k+1}\sum_{|\alpha+\beta|=2l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta p'_{s-l,k-1}(x,\xi).
\end{aligned}$$

For $k=1$ one easily checks that the same formula holds for $p'_{s,1}$ (by definition,

$p'_{s-l,0}=0$ when $s>l$ and $p'_{0,0}=b$). Hence

$$\begin{aligned}\sum_{s=k}^{N-1} p'_{s,k}(x,\xi) &= \sum_{l=1}^{N-k}\sum_{|\alpha+\beta|=2l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta\left(\sum_{s=l+k-1}^{N-1} p'_{s-l,k-1}(x,\xi)\right)\\ &= \sum_{l=1}^{N-k}\sum_{|\alpha+\beta|=2l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta\left(\sum_{s=k-1}^{N-l-1} p'_{s,k-1}(x,\xi)\right).\end{aligned}$$

Now, we obtain

$$\begin{aligned}&\sum_{j=k}^{N-1}\sum_{|\alpha+\beta|=2(j-k+1)}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_{k-1}(x,\xi)-b_k(x,\xi)\\ &= \sum_{l=1}^{N-k}\sum_{|\alpha+\beta|=2l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta b_{k-1}(x,\xi)-\sum_{s=k}^{N-1}p'_{s,k}(x,\xi)+\sum_{s=k}^{N-1}p'_{s,k}(x,\xi)-b_k(x,\xi)\\ &= \sum_{l=1}^{N-k}\sum_{|\alpha+\beta|=2l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta\left(b_{k-1}(x,\xi)-\sum_{s=k-1}^{N-l-1}p'_{s,k-1}(x,\xi)\right)\\ &\qquad +\sum_{s=k}^{N-1}p'_{s,k}(x,\xi)-b_k(x,\xi).\end{aligned}$$

By construction, $b_{k-1}\sim\underbrace{0+...+0}_{k-1}+\sum_{s=k-1}^{\infty}p'_{s,k-1}$. Moreover, by the above estimates for the derivatives of $p'_{s,k}$, the above lemma and its proof, it follows that there exist $B>0$, $m>0$ and $\tilde{C}_h>0$ in the (M_p) case, resp. there exist $B>0$, $h>0$ and $\tilde{C}_m>0$ in the $\{M_p\}$ case, such that for every $h>0$

$$\frac{\left|D_\xi^\alpha D_x^\beta\left(b_k(x,\xi)-\sum_{s<N}p'_{s,k}(x,\xi)\right)\right|\langle(x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2N\rho}e^{-M(m|\xi|)}e^{-M(m|x|)}}{h^{|\alpha|+|\beta|+2N}A_\alpha A_\beta A_N A_N}\leq\tilde{C}_h,$$

for all $(x,\xi)\in Q^c_{Bm_N}$, $\alpha,\beta\in\mathbb{N}^d$ and $k,N\in\mathbb{N}$, $N>k$, in the (M_p) case, resp. the same as above but for some h and every m with $\tilde{C}_m$ in place of $\tilde{C}_h$, in the $\{M_p\}$ case. Now, if we use the estimate (5.4), we get that

$$\begin{aligned}&\left|\sum_{|\alpha+\beta|=2l}\frac{c_{\alpha,\beta}}{\alpha!\beta!}\partial_\xi^{\alpha+\gamma}\partial_x^{\beta+\delta}\left(b_{k-1}(x,\xi)-\sum_{s=k-1}^{N-l-1}p'_{s,k-1}(x,\xi)\right)\right|\\ &\qquad\leq\tilde{C}\sum_{|\alpha+\beta|=2l}\frac{|c_{\alpha,\beta}|}{\alpha!\beta!}\frac{h^{|\gamma|+|\delta|+2N}A_{\alpha+\gamma}A_{\beta+\delta}A_{N-l}A_{N-l}e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}}\\ &\qquad\leq c_0^3\tilde{C}c'd^{2l}\sum_{|\alpha+\beta|=2l}\frac{(hH^2)^{|\gamma|+|\delta|+2N}A_\gamma A_\delta A_N A_N e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}}\\ &\qquad\leq c_0^3c'\tilde{C}2^{2d-1}\frac{(2hdH^2)^{|\gamma|+|\delta|+2N}A_\gamma A_\delta A_N A_N e^{M(m|\xi|)}e^{M(m|x|)}}{\langle(x,\xi)\rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}},\end{aligned}$$

for all $(x,\xi) \in Q^c_{Bm_N}$, $\gamma, \delta \in \mathbb{N}^d$, $N \geq l+1$ (in the last inequality we used $\sum_{|\alpha+\beta|=2l} 1 \leq 2^{2l+2d-1}$), where we put $\tilde{C} = \tilde{C}_h$ in the (M_p) case, resp. $\tilde{C} = \tilde{C}_m$ in the $\{M_p\}$ case. Note that the estimates are uniform in l and k. One obtains

$$\left| \partial^\gamma_\xi \partial^\delta_x \left(\sum_{l=1}^{N-k} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha!\beta!} \partial^\alpha_\xi \partial^\beta_x \left(b_{k-1}(x,\xi) - \sum_{s=k-1}^{N-l-1} p'_{s,k-1}(x,\xi) \right) \right) \right|$$

$$\leq c_0^3 c' \tilde{C} 2^{2d-1} \frac{(4hdH^2)^{|\gamma|+|\delta|+2N} A_\gamma A_\delta A_N A_N e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x,\xi) \rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}},$$

for all $(x,\xi) \in Q^c_{Bm_N}$, $\gamma, \delta \in \mathbb{N}^d$, $N > k$, with uniform estimates in k. Similar estimates hold for $\sum_{s=k}^{N-1} p'_{s,k}(x,\xi) - b_k(x,\xi)$ (by the definition of b_k). By using the equality (5.6), we obtain the desired result. □

The importance in the study of the Anti-Wick quantization lies in the following results. The proofs are similar to the case of Schwartz distributions and we omit them (see for example [36]).

Proposition 5.1.4. *Let a be a locally integrable function with *-ultrapolynomial growth (for example, an element of $\Gamma^{*,\infty}_{A_p,B_p,\rho}\left(\mathbb{R}^{2d}\right)$). If $a(x,\xi) \geq 0$ for almost every $(x,\xi) \in \mathbb{R}^{2d}$, then $(A_a u, u)_{L^2} \geq 0$, $\forall u \in \mathcal{S}^*$. Moreover, if $a(x,\xi) > 0$ for almost every $(x,\xi) \in \mathbb{R}^{2d}$, then $(A_a u, u)_{L^2} > 0$, $\forall u \in \mathcal{S}^*$, $u \neq 0$.*

Nontrivial symbols a that satisfy the conditions of this proposition, for example, are the ultrapolynomials of the form $\sum_\alpha c_{2\alpha}\xi^{2\alpha}$, where $c_{2\alpha} > 0$ satisfy the necessary conditions for this to be an ultrapolynomial, i.e. there exist $C > 0$ and $\tilde{L} > 0$, resp. for every $\tilde{L} > 0$ there exists $C > 0$, such that $|c_{2\alpha}| \leq C\tilde{L}^{2|\alpha|}/M_{2\alpha}$, for all $\alpha \in \mathbb{N}^d$.

Proposition 5.1.5. *Let $a \in L^\infty\left(\mathbb{R}^{2d}\right)$. Then A_a extends to a bounded operator on L^2, with the following estimate of its norm $\|A_a\|_{\mathcal{L}_b\left(L^2\left(\mathbb{R}^d\right)\right)} \leq \|a\|_{L^\infty\left(\mathbb{R}^{2d}\right)}$.*

5.2 Convolution with the Gaussian Kernel

Our goal in this section is to find the largest subspace of $\mathcal{D}'^*$ such that the convolution of each element of that subspace with $e^{s|\cdot|^2}$ exists, where $s \in \mathbb{R}$, $s \neq 0$ is fixed. The general idea is similar to that in [58], where the case of Schwartz distributions is considered.

Put $B^* = \{S \in \mathcal{D}'^* |\ \cosh(k|x|)S \in \mathcal{S}'^*, \forall k \geq 0\}$ and for $s \in \mathbb{R}\backslash\{0\}$, put $B^*_s = e^{-s|x|^2} B^*$. Obviously $B^* \subseteq \mathcal{S}'^*$ and $B^*_s \subseteq \mathcal{D}'^*$. Define

$$A^* = \left\{ f \in \mathcal{O}\left(\mathbb{C}^d\right) | \forall K \subset\subset \mathbb{R}^d_\xi, \exists h, C > 0, \text{ resp. } \forall h > 0, \exists C > 0, \text{ such that } |f(\xi + i\eta)| \leq Ce^{M(h|\eta|)}, \forall \xi \in K, \forall \eta \in \mathbb{R}^d \right\},$$

$A^*_{\text{real}} = \{f_{|\mathbb{R}^d} | \, f \in A^*\}$ and $A^*_s = e^{s|x|^2} A^*_{\text{real}}$. Assume that $k > 0$. First we will prove that $\cosh(k|x|) \in \mathcal{C}^\infty(\mathbb{R}^d)$. For $\rho \geq 0$, we have

$$\cosh(k\rho) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{k^n \rho^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n k^n \rho^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{k^{2n}\rho^{2n}}{(2n)!},$$

hence $\cosh(k|x|) = \sum_{n=0}^{\infty} \frac{k^{2n}|x|^{2n}}{(2n)!}$ and the function $\sum_{n=0}^{\infty} \frac{k^{2n}|x|^{2n}}{(2n)!}$ is obviously in $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$. We will give another two equivalent definitions of B^*. We need the following lemmas.

Lemma 5.2.1. *Let $k > 0$. The function $\frac{\cosh(k|x|)}{\cosh(2k|x|)}$ is an element of $\mathcal{S}^*$.*

Proof. Consider the function $g_k(z) = \sum_{n=0}^{\infty} \frac{k^{2n}(z^2)^n}{(2n)!}$. Obviously $g_k(z)$ is an entire function. Put $W = \{z = x + iy \in \mathbb{C}^d | \, |x| > 2|y|\}$ and consider the set $W_r = W \backslash \overline{B(0,r)}$, where $B(0,r)$ is the ball in $\mathbb{C}^d$ with centre at 0 and radius $r > 0$. Then $\frac{e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}}{2}$ is analytic and single valued function on W_r, where we take the principal branch of the square root which is analytic on $\mathbb{C}\backslash(-\infty, 0]$. Also, for $z \in W_r$, put $\rho = \sqrt{\left(|x|^2 - |y|^2\right)^2 + 4(xy)^2}$, $\cos\theta = \frac{|x|^2 - |y|^2}{\sqrt{\left(|x|^2 - |y|^2\right)^2 + 4(xy)^2}}$ and $\sin\theta = \frac{2xy}{\sqrt{\left(|x|^2 - |y|^2\right)^2 + 4(xy)^2}}$, where $\theta \in (-\pi, \pi)$, from what it follows $\theta \in (-\pi/2, \pi/2)$ (because $\cos\theta > 0$ and $\theta \in (-\pi, \pi)$). We will need sharper estimate for $\cos\theta$.

$$\begin{aligned}
\cos\theta &= \frac{|x|^2 - |y|^2}{\sqrt{\left(|x|^2 - |y|^2\right)^2 + 4(xy)^2}} = \left(1 + \left(\frac{2|xy|}{|x|^2 - |y|^2}\right)^2\right)^{-1/2} \\
&\geq \left(1 + \left(\frac{|x|^2 + |y|^2}{|x|^2 - |y|^2}\right)^2\right)^{-1/2} \geq \left(1 + \left(\frac{\frac{5}{4}|x|^2}{\frac{3}{4}|x|^2}\right)^2\right)^{-1/2} = \frac{3}{\sqrt{34}}.
\end{aligned}$$

Then

$$\begin{aligned}
\left|e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right| &\geq \left|e^{k\sqrt{z^2}}\right| - \left|e^{-k\sqrt{z^2}}\right| = e^{k\operatorname{Re}\sqrt{\rho(\cos\theta + i\sin\theta)}} - e^{-k\operatorname{Re}\sqrt{\rho(\cos\theta + i\sin\theta)}} \\
&= e^{k\operatorname{Re}\sqrt{\rho}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)} - e^{-k\operatorname{Re}\sqrt{\rho}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)} \geq e^{k\sqrt{\rho}\cos\frac{\theta}{2}} - 1
\end{aligned}$$

where the second equality follows from the fact that we take the principal branch of the square root. Now, using the above estimate for $\cos\theta$, we have

$$\sqrt{\rho}\cos\frac{\theta}{2} = \sqrt{\rho}\sqrt{\frac{\cos\theta + 1}{2}} \geq \sqrt{\rho}\sqrt{\frac{3 + \sqrt{34}}{2\sqrt{34}}}.$$

So, if we put $c_1 = \sqrt{\dfrac{3+\sqrt{34}}{2\sqrt{34}}}$, we obtain

$$\left|e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right| \geq e^{c_1 k \sqrt[4]{(|x|^2-|y|^2)^2+4(xy)^2}} - 1 \geq e^{c_1 k\sqrt{|x|^2-|y|^2}} - 1 > 0. \tag{5.7}$$

Hence $e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}$ doesn't have zeroes in W_r. Now, $f(z) = \dfrac{e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}}{e^{2k\sqrt{z^2}} + e^{-2k\sqrt{z^2}}}$ is an analytic function on W_r. Moreover, because $\left(e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right)/2 = g_k(z)$, for $z \in W_r \cap \mathbb{R}^d_x$ and from the uniqueness of analytic continuation, it follows $\left(e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right)/2 = g_k(z)$ on W_r. Hence $f(z) = g_k(z)/g_{2k}(z)$ on W_r and this holds for all $r > 0$, hence on W. Note that $g_{2k}(0) = 1$, so, there exists $r_0 > 0$ such that $|g_{2k}(z)| > 0$ on $B(0, 2r_0)$ and hence $g_k(z)/g_{2k}(z)$ is analytic function on $W \cup B(0, 2r_0)$. Let $C_{r_0} > 0$ be a constant such that $|g_k(z)/g_{2k}(z)| \leq C_{r_0}$ on $\overline{B(0, r_0)}$. Take $r_1 > 0$ such that $\overline{B(x, 2dr_1)} \subseteq \left(\mathbb{C}^d \backslash \overline{B(0, r_0/16)}\right) \cap W$, for all $x \in W_{\frac{r_0}{4}} \cap \mathbb{R}^d_x$. Then, for such x, from Cauchy integral formula, we have

$$|\partial_z^\alpha f(x)| \leq \frac{\alpha!}{r_1^{|\alpha|}} \sup_{|w_1-x_1|\leq r_1,\dots,|w_d-x_d|\leq r_1} |f(w)|.$$

Now, for $w = u + iv \in \mathbb{C}^d$ such that $|w_j - x_j| \leq r_1$, for all $j = 1, \dots, d$, using the estimate (5.7) but with $2k$ instead of k and the fact $\operatorname{Re}\sqrt{z^2} > 0$, for $z \in W$, which we proved above, we get

$$\begin{aligned} |f(w)| &= \left|\frac{e^{k\sqrt{w^2}} + e^{-k\sqrt{w^2}}}{e^{2k\sqrt{w^2}} + e^{-2k\sqrt{w^2}}}\right| \leq \frac{e^{k\sqrt[4]{(|u|^2-|v|^2)^2+4(uv)^2}} + 1}{e^{2c_1 k\sqrt{|u|^2-|v|^2}} - 1} \\ &\leq \frac{2e^{k\sqrt{|u|^2-|v|^2+2|uv|}}}{e^{2c_1 k\sqrt{|u|^2-|v|^2}} - 1} \leq \frac{2e^{\sqrt{2}k|u|}}{e^{\sqrt{3}c_1 k|u|} - 1} \leq C_1 e^{(\sqrt{2}-\sqrt{3}c_1)k|u|} \end{aligned}$$

and it is easy to check that $\sqrt{2} - \sqrt{3}c_1 < 0$. If we put $c = \sqrt{3}c_1 - \sqrt{2}$, we get

$$|f(w)| \leq C_1 e^{-ck|u|} \leq C_1 e^{-ck(|x|-|u-x|)} \leq C_1 e^{ckr_1\sqrt{d}} e^{-ck|x|} = C_2 e^{-ck|x|}.$$

Hence $|\partial_x^\alpha f(x)| \leq C_2 \dfrac{\alpha!}{r_1^{|\alpha|}} e^{-ck|x|}$. For $x \in (B(0, r_0/2) \cap \mathbb{R}^d_x)\backslash\{0\}$, if we take $r_2 > 0$ small enough such that $\overline{B(x, 2dr_2)} \subseteq B(0, r_0)$ we have (from Cauchy integral formula)

$$|\partial_x^\alpha f(x)| = \left|\partial_z^\alpha \left(\frac{g_k(x)}{g_{2k}(x)}\right)\right| \leq C_{r_0} \frac{\alpha!}{r_2^\alpha} \leq C_3 \frac{\alpha!}{r_2^\alpha} e^{-ck|x|}.$$

Because $f(x)$ is in $\mathcal{C}^\infty(\mathbb{R}^d)$ the same inequality will hold for the derivatives at $x = 0$. If we take $r = \min\{r_1, r_2\}$ we get that, for $x \in \mathbb{R}^d$,

$$|\partial_x^\alpha f(x)| \leq C \frac{\alpha!}{r^\alpha} e^{-ck|x|}, \tag{5.8}$$

for some $C > 0$. From this it easily follows that $f(x) = \dfrac{\cosh(k|x|)}{\cosh(2k|x|)} \in \mathcal{S}^*$. $\square$

Lemma 5.2.2. *If $\psi \in \mathcal{S}^*$ and $T \in \mathcal{S}'^*$ then $\psi T \in \mathcal{O}'^*_C$.*

Proof. The Fourier transform is a bijection between $\mathcal{O}'^*_C$ and $\mathcal{O}^*_M$ (see proposition 8 of [17]) and $\mathcal{F}(\psi T) = \mathcal{F}\psi * \mathcal{F}T$. Hence, it is enough to prove that $\psi * T \in \mathcal{O}^*_M$ for all $\psi \in \mathcal{S}^*$ and $T \in \mathcal{S}'^*$. From the representation theorem of ultradistributions in $\mathcal{S}'^*$ (theorem 2 of [42]), there exists locally integrable function $F(x)$ (in fact it can be taken to be continuous) such that there exist $m, C > 0$, resp. for every $m > 0$ there exists $C > 0$, such that $\left\|F(x)e^{-M(m|x|)}\right\|_{L^\infty} \leq C$ and an ultradifferential operator $P(D)$ of class $*$ such that $T = P(D)F$. Because

$$\psi * T = \psi * P(D)F = P(D)(\psi * F) = P(D)\psi * F$$

and $P(D)\psi \in \mathcal{S}^*$ it is enough to prove that for every $\psi \in \mathcal{S}^*$ and every such F, $\psi * F \in \mathcal{O}^*_M$. We will give the proof only in the $\{M_p\}$ case, the (M_p) case is similar. Let ψ and F are such function. There exists $h > 0$ such that

$$\sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d} \frac{h^{|\alpha|}e^{M(h|x|)}\left|D^\alpha\psi(x)\right|}{M_\alpha} < \infty.$$

Take m such that $\displaystyle\int_{\mathbb{R}^d} e^{-M(h|t|)}e^{M(2m|t|)}dt$ is finite. Later on we will impose another condition on m. Then $\left\|F(x)e^{-M(m|x|)}\right\|_{L^\infty} \leq C_m$. Note that $e^{M(m|x-t|)} \leq 2e^{M(2m|x|)}e^{M(2m|t|)}$ (one easily proves that for $\lambda, \nu > 0$, $e^{M(\lambda+\nu)} \leq 2e^{M(2\lambda)}e^{M(2\nu)}$), so we have

$$\begin{aligned}
\left|D^\alpha(\psi * F)(x)\right| &\leq \int_{\mathbb{R}^d}\left|D^\alpha\psi(t)\right|\left|F(x-t)\right|dt \leq C'C_m\int_{\mathbb{R}^d}\frac{e^{-M(h|t|)}M_\alpha}{h^{|\alpha|}}e^{M(m|x-t|)}dt\\
&\leq C'C_mC''\frac{e^{M(2m|x|)}M_\alpha}{h^{|\alpha|}}\int_{\mathbb{R}^d}e^{-M(h|t|)}e^{M(2m|t|)}dt \leq C\frac{e^{M(2m|x|)}M_\alpha}{h^{|\alpha|}}.
\end{aligned}$$

We will use the equivalent condition given in proposition 7 of [17] for a $\mathcal{C}^\infty$ function to be a multiplier for $\mathcal{S}'^{\{M_p\}}$. Let $k > 0$ be arbitrary but fixed. Take m small enough such that $2m \leq k$. Choose $h_1 < h$. Then, by the previous estimates, we obtain

$$\frac{h_1^{|\alpha|}e^{-M(k|x|)}\left|D^\alpha(\psi * F)(x)\right|}{M_\alpha} \leq C\frac{h_1^{|\alpha|}e^{-M(k|x|)}e^{M(2m|x|)}M_\alpha}{h^{|\alpha|}M_\alpha} \leq C,$$

hence $\psi * F$ is a multiplier for $\mathcal{S}'^{\{M_p\}}$ and the proof is complete. □

For $S \in B^*$, by lemma 5.2.1, for $k > 0$, $\dfrac{\cosh(k|x|)}{\cosh(2k|x|)} \in \mathcal{S}^*$ and by lemma 5.2.2 we have

$$\cosh(k|x|)S = \frac{\cosh(k|x|)}{\cosh(2k|x|)}\cosh(2k|x|)S \in \mathcal{O}'^*_C.$$

Similarly as in the proof of lemma 5.2.1 one can prove that $(\cosh(k|x|))^{-1} \in \mathcal{S}^*$, for $k > 0$. So, for $S \in B^*$, we also have $S = (\cosh(k|x|))^{-1}\cosh(k|x|)S \in \mathcal{O}'^*_C$. Using this, we get

$$B^* = \left\{S \in \mathcal{D}'^* \middle|\ \cosh(k|x|)S \in \mathcal{O}'^*_C,\ \forall k \geq 0\right\}. \tag{5.9}$$

Lemma 5.2.3. *$\mathcal{O}'^{(M_p)}_C \subseteq \mathcal{D}'^{(M_p)}_{L^1}$ and $\mathcal{O}'^{\{M_p\}}_C \subseteq \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$.*

Proof. We will give the proof only in the $\{M_p\}$ case, the (M_p) case is similar. Let $S \in \mathcal{O}'^{\{M_p\}}_C$. From proposition 2 of [17], there exist $k > 0$ and $\{M_p\}$ - ultradifferential operator $P(D)$ such that $S = P(D)F_1 + F_2$ where

$$\left\| e^{M(k|x|)} \left(|F_1(x)| + |F_2(x)|\right)\right\|_{L^\infty} < \infty.$$

We will assume that $F_2 = 0$ and put $F = F_1$. The general case is proved analogously. Let $\varphi \in \mathcal{D}^{\{M_p\}}$. We have

$$|\langle S, \varphi\rangle| = |\langle F, P(-D)\psi\rangle| \leq \left\|e^{M(k|\cdot|)}F\right\|_{L^\infty} \left\|e^{-M(k|\cdot|)}\right\|_{L^1} \|P(-D)\varphi\|_{L^\infty} \leq C\|\varphi\|_{(t_j)},$$

for some $C > 0$ and $(t_j) \in \mathfrak{R}$, where, the last inequality follows from the fact that $P(D) : \dot{\tilde{\mathcal{B}}}^{\{M_p\}} \to \dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ is continuous. Because $\mathcal{D}^{\{M_p\}}$ is dense in $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$, the claim in the lemma follows. □

If we use the previous lemma in (5.9), we get

$$B^{(M_p)} = \left\{S \in \mathcal{D}'^{(M_p)} | \cosh(k|x|)S \in \mathcal{D}'^{(M_p)}_{L^1}, \forall k \geq 0\right\}, \tag{5.10}$$

$$B^{\{M_p\}} = \left\{S \in \mathcal{D}'^{\{M_p\}} | \cosh(k|x|)S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}, \forall k \geq 0\right\}. \tag{5.11}$$

Now we will give the theorem that characterises the elements of $\mathcal{D}'^*$ for which the convolution with $e^{s|x|^2}$ exists as an element of $\mathcal{D}'^*$.

Theorem 5.2.1. *Let $s \in \mathbb{R}$, $s \neq 0$. Then*

a) The convolution of $S \in \mathcal{D}'^*$ and $e^{s|x|^2}$ exists if and only if $S \in B^*_s$.

b) $\mathcal{L} : B^* \to A^*$ is well defined and bijective mapping. For $S \in B^*$ and $\xi, \eta \in \mathbb{R}^d$, $e^{-(\xi+i\eta)x}S(x) \in \mathcal{D}'^{(M_p)}_{L^1}\left(\mathbb{R}^d_x\right)$, resp. $e^{-(\xi+i\eta)x}S(x) \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}\left(\mathbb{R}^d_x\right)$ and the Laplace transform of S is given by $\mathcal{L}(S)(\xi + i\eta) = \left\langle e^{-(\xi+i\eta)x}S(x), 1_x\right\rangle$.

c) The mapping $B^*_s \to A^*_s$, $S \mapsto S*e^{s|x|^2}$ is bijective and for $S \in B^*_s$, $\left(S * e^{s|\cdot|^2}\right)(x) = e^{s|x|^2}\mathcal{L}\left(e^{s|\cdot|^2}S\right)(2sx)$.

Proof. First we will prove a). Let $S \in B^*_s$. Let $\varphi \in \mathcal{D}^*$ is fixed and $K \subset\subset \mathbb{R}^d$, such that $\operatorname{supp}\varphi \subseteq K$. Note that

$$\left(\varphi * e^{s|\cdot|^2}\right)(x) = e^{s|x|^2}\int_{\mathbb{R}^d} \varphi(y)e^{s|y|^2-2sxy}dy$$

and define $f(x) = (\cosh(k|x|))^{-1}\displaystyle\int_{\mathbb{R}^d} \varphi(y)e^{s|y|^2-2sxy}dy$ where k will be chosen later. Put $l = \sup\{|y|\,|y \in K\}$ to simplify notations. We will prove that $f \in \mathcal{D}^*_{L^\infty}$, for large enough k. For $w \in \mathbb{C}^d$, put $g(w) = \displaystyle\int_{\mathbb{R}^d} \varphi(y)e^{s|y|^2-2swy}dy$. Then $g(w)$ is an entire function. To estimate its derivatives we use the Cauchy integral formula and obtain

$$|\partial^\alpha g(x)| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1-x_1|\leq r,\ldots,|w_d-x_d|\leq r} |g(w)|.$$

Take $r < 1/(2dl|s|)$. We put $w = \xi + i\eta$ and estimate

$$\begin{aligned} |g(w)| &\leq \int_{\mathbb{R}^d} |\varphi(y)| e^{s|y|^2 - 2s\xi y} dy \leq e^{2|s||\xi|l} \|\varphi\|_{L^\infty} \int_K e^{s|y|^2} dy = c'' \|\varphi\|_{L^\infty} e^{2l|s||\xi|} \\ &\leq c'' \|\varphi\|_{L^\infty} e^{2l|s|(|x|+|\xi-x|)} = c'' \|\varphi\|_{L^\infty} e^{2l|s||\xi-x|} e^{2l|s||x|} \leq 3c'' \|\varphi\|_{L^\infty} e^{2l|s||x|}, \end{aligned}$$

where we denote $c'' = \int_K e^{s|y|^2} dy$. Hence, we get

$$|\partial_x^\alpha g(x)| \leq \frac{3c'' \|\varphi\|_{L^\infty} \alpha!}{r^{|\alpha|}} e^{2l|s||x|}. \tag{5.12}$$

We can use the same methods as in the proof of lemma 5.2.1 to prove that

$$\left| D^\alpha \left(\frac{1}{\cosh(k|x|)} \right) \right| \leq C \frac{\alpha!}{r^{|\alpha|}} e^{-c'k|x|}$$

for some $C > 0$, $c' > 0$ and c' doesn't depend on k. If we take $r > 0$ small enough we can make it the same for (5.12) and the above estimate. Now take k large enough such that $2l|s| < c'k$. Then, for $h > 0$ fixed, we have

$$\begin{aligned} \frac{h^{|\alpha|} |D^\alpha f(x)|}{M_\alpha} &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{h^{|\alpha|} \left|D^{\alpha-\beta} g(x)\right| \left|D^\beta \left((\cosh(k|x|))^{-1}\right)\right|}{M_\alpha} \\ &\leq 3c'' C \|\varphi\|_{L^\infty} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{(2h)^{|\alpha|} (\alpha-\beta)! e^{2l|s||x|} \beta! e^{-c'k|x|}}{2^{|\alpha|} r^{|\alpha-\beta|} r^{|\beta|} M_\alpha} \\ &\leq \frac{3c'' C \|\varphi\|_{L^\infty}}{2^{|\alpha|}} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \left(\frac{2h}{r}\right)^{|\alpha|} \frac{\alpha!}{M_\alpha} e^{(2l|s|-c'k)|x|} \leq c'' C' \|\varphi\|_{L^\infty}, \end{aligned}$$

where we use the fact $\frac{k^p p!}{M_p} \to 0$ when $p \to \infty$. From the arbitrariness of h we have $f \in \mathcal{D}^*_{L^\infty}$. Because $\mathcal{D}^{\{M_p\}}_{L^\infty} = \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$ as a set, $f \in \tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty}$. Now, we obtain

$$\left(\varphi * e^{s|\cdot|^2}\right)(x) S = f(x) \cosh(k|x|) e^{s|x|^2} S.$$

$e^{s|x|^2} S \in B^*$ (because $S \in B^*_s$), hence, by (5.10), resp. (5.11), $\cosh(k|x|) e^{s|x|^2} S \in \mathcal{D}'^{(M_p)}_{L^1}$, resp. $\cosh(k|x|) e^{s|x|^2} S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. Hence $\left(\varphi * e^{s|x|^2}\right) S \in \mathcal{D}'^{(M_p)}_{L^1}$, resp. $\left(\varphi * e^{s|x|^2}\right) S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. Theorem 3.1.1 implies that the convolution of S and $e^{s|x|^2}$ exists, in the (M_p) case. Let us consider the $\{M_p\}$ case. If we prove that for arbitrary compact subset K of $\mathbb{R}^d$, the bilinear mapping $(\varphi, \chi) \mapsto \left\langle \left(\varphi * e^{s|\cdot|^2}\right) S, \chi \right\rangle$,

$\mathcal{D}_K^{\{M_p\}} \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}} \to \mathbb{C}$, is continuous, theorem 3.3.1 will imply the existence of convolution of S and $e^{s|x|^2}$. Let $K \subset\subset \mathbb{R}^d$ be fixed. By the above consideration, we have

$$\left|\left\langle \left(\varphi * e^{s|\cdot|^2}\right) S, \chi \right\rangle\right| = \left|\left\langle \cosh(k|x|) e^{s|x|^2} S(x), f(x)\chi(x) \right\rangle\right| \leq C_1 p_{(t_j)}(f\chi),$$

for some $C_1 > 0$ and $(t_j) \in \mathfrak{R}$, where, in the last inequality, we used that $\cosh(k|x|)e^{s|x|^2} S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. For brevity, denote $T_\alpha = \prod_{j=1}^{|\alpha|} t_j$ and $T_0 = 1$. Observe that

$$\begin{aligned} \frac{|D^\alpha (f(x)\chi(x))|}{T_\alpha M_\alpha} &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{\left|D^\beta f(x)\right| \left|D^{\alpha-\beta}\chi(x)\right|}{T_\beta M_\beta T_{\alpha-\beta} M_{\alpha-\beta}} \leq \tilde{C} c'' \|\varphi\|_{L^\infty} p_{(t_j/2)}(\chi) \\ &\leq \tilde{C} c'' p_{(t_j/2),K}(\varphi) p_{(t_j/2)}(\chi), \end{aligned}$$

where we used the above estimates for the derivatives of f. Note that c'' does not depend on φ, only on K. From this, the continuity of the bilinear mapping in consideration follows.

For the other direction, let the convolution of S and $e^{s|x|^2}$ exists. Then, by theorem 3.1.1, resp. theorem 3.3.1, for every $\varphi \in \mathcal{D}^*$, $\left(\varphi * e^{s|\cdot|^2}\right) S \in \mathcal{D}'^{(M_p)}_{L^1}$, resp. $\left(\varphi * e^{s|\cdot|^2}\right) S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. Let $\varphi \in \mathcal{D}^*$, such that $\varphi(y) \geq 0$. Put $U = \{y \in \mathbb{R}^d | \, \varphi(y) \neq 0\}$ and $t = \sup\{|y| | \, y \in \operatorname{supp} \varphi\}$. Then we have

$$\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy \geq c e^{\inf_{y \in U}(-2sxy)},$$

where $c = \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2} dy$. Let $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ be fixed. There exists $\varphi \in \mathcal{D}^*$, such that $U \subseteq B(x_0, \varepsilon)$ ($B(x_0,\varepsilon)$ is the ball in $\mathbb{R}^d$ with centre at x_0 and radius ε). Then

$$\begin{aligned} \inf_{y \in U} (-2sxy) &\geq \inf_{y \in B(x_0,\varepsilon)} (-2sxy) = -2sxx_0 + \inf_{y \in B(x_0,\varepsilon)} (-2sx(y - x_0)) \\ &\geq -2sxx_0 - 2\varepsilon|s||x|. \end{aligned}$$

We get

$$\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy \geq c e^{-2sxx_0 - 2\varepsilon|s||x|}.$$

Define $f(x) = e^{-2sxx_0 - 2\varepsilon|s|\sqrt{1+|x|^2}} \left(\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy\right)^{-1}$. We will prove that $f \in \mathcal{D}^*_{L^\infty}$. $g(w) = \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2swy} dy$ is an entire function. Put $w = \xi + i\eta$. Then, for w in the strip $\mathbb{R}^d_\xi + i\{\eta \in \mathbb{R}^d | \, |\eta| < 1/(8|s|t)\}$ and $y \in \operatorname{supp}\varphi$, we have $|2s\eta y| \leq 2|s||\eta||y| \leq 1/4 < \pi/4$, hence

$$\left|\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2swy} dy\right| \geq \left|\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2s\xi y} \cos(2s\eta y) dy\right|$$

$$\geq \frac{1}{\sqrt{2}}\int_{\mathbb{R}^d}\varphi(y)e^{s|y|^2-2s\xi y}dy > 0.$$

Moreover, $e^{-2swx_0-2\varepsilon|s|\sqrt{1+w^2}}$ is analytic on the strip $\mathbb{R}^d_\xi + i\{\eta \in \mathbb{R}^d|\,|\eta| < 1/4\}$, where we take the principal branch of the square root which is single valued and analytic on $\mathbb{C}\backslash(-\infty,0]$. So, for $r_0 = \min\{1/4, 1/(8|s|t)\}$, $f(w)$ is analytic on the strip $\mathbb{R}^d + i\{\eta \in \mathbb{R}^d|\,|\eta| < r_0\}$. To estimate the derivatives of f, we use Cauchy integral formula and obtain

$$|\partial^\alpha f(x)| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1-x_1|\leq r,\dots,|w_d-x_d|\leq r} |f(w)|, \tag{5.13}$$

where $r < r_0/(2d)$. Put

$$\rho = \sqrt{\left(1+|\xi|^2-|\eta|^2\right)^2 + 4(\xi\eta)^2},$$

$$\cos\theta = \frac{1+|\xi|^2-|\eta|^2}{\sqrt{\left(1+|\xi|^2-|\eta|^2\right)^2+4(\xi\eta)^2}} \text{ and } \sin\theta = \frac{2\xi\eta}{\sqrt{\left(1+|\xi|^2-|\eta|^2\right)^2+4(\xi\eta)^2}},$$

where $\theta \in (-\pi,\pi)$, from what it follows that $\theta \in (-\pi/2,\pi/2)$ (because $\cos\theta > 0$ and $\theta \in (-\pi,\pi)$). Then

$$\begin{aligned}
\operatorname{Re}\left(\sqrt{1+w^2}\right) &= \operatorname{Re}\left(\sqrt{\rho}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right) = \sqrt{\rho}\cos\frac{\theta}{2}\\
&= \sqrt{\rho}\sqrt{\frac{\cos\theta+1}{2}} = \frac{\sqrt{\rho\cos\theta+\rho}}{\sqrt{2}}\\
&= \frac{1}{\sqrt{2}}\sqrt{1+|\xi|^2-|\eta|^2+\sqrt{\left(1+|\xi|^2-|\eta|^2\right)^2+4(\xi\eta)^2}}\\
&\geq \frac{1}{\sqrt{2}}\sqrt{1+|\xi|^2-|\eta|^2+1+|\xi|^2-|\eta|^2} = \sqrt{1+|\xi|^2-|\eta|^2},
\end{aligned}$$

where the first equality follows from the fact that we take the principal branch of the square root. We obtain

$$\begin{aligned}
|f(w)| &= \frac{\left|e^{-2swx_0-2\varepsilon|s|\sqrt{1+w^2}}\right|}{\left|\displaystyle\int_{\mathbb{R}^d}\varphi(y)e^{s|y|^2-2swy}dy\right|} \leq \frac{\sqrt{2}e^{-2s\xi x_0}e^{-2\varepsilon|s|\operatorname{Re}\left(\sqrt{1+w^2}\right)}}{\displaystyle\int_{\mathbb{R}^d}\varphi(y)e^{s|y|^2-2s\xi y}dy}\\
&\leq \frac{\sqrt{2}e^{-2s\xi x_0}e^{-2\varepsilon|s|\sqrt{1+|\xi|^2-|\eta|^2}}}{\displaystyle\int_{\mathbb{R}^d}\varphi(y)e^{s|y|^2-2s\xi y}dy} \leq \frac{\sqrt{2}e^{-2s\xi x_0}e^{-2\varepsilon|s||\xi|}}{ce^{-2s\xi x_0-2\varepsilon|s||\xi|}} \leq C_0''.
\end{aligned}$$

So, from (5.13), we have $|\partial^\alpha_x f(x)| \leq C_0\alpha!/r^{|\alpha|}$, for some $C_0 > 0$. From this it easily follows that $f \in \mathcal{D}^*_{L^\infty}$. Now we have

$$e^{-2sxx_0-2\varepsilon|s|\sqrt{1+|x|^2}}e^{s|x|^2}S = f(x)\left(\varphi * e^{s|\cdot|^2}\right)(x)S \in \mathcal{D}'^{(M_p)}_{L^1}, \text{ resp.} \tag{5.14}$$

$$e^{-2sxx_0-2\varepsilon|s|\sqrt{1+|x|^2}}e^{s|x|^2}S \;=\; f(x)\left(\varphi * e^{s|\cdot|^2}\right)(x)S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}, \tag{5.15}$$

where we used the fact that $\left(\varphi * e^{s|\cdot|^2}\right)S \in \mathcal{D}'^{(M_p)}_{L^1}$, resp. $\left(\varphi * e^{s|\cdot|^2}\right)S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ (which, as noted before, follows from the existence of the convolution of S and $e^{s|x|^2}$) and these hold for every $x_0 \in \mathbb{R}^d$ and every $\varepsilon > 0$. Now, put $x_0' = 2sx_0$, $x_0'' = -2sx_0$ and $\varepsilon' = 2|s|\varepsilon$. Then, from (5.14), resp. (5.15), we have

$$\begin{aligned}
&e^{-xx_0'-\varepsilon'\sqrt{1+|x|^2}}e^{s|x|^2}S \in \mathcal{D}'^{(M_p)}_{L^1} \quad , \quad e^{xx_0''-\varepsilon'\sqrt{1+|x|^2}}e^{s|x|^2}S \in \mathcal{D}'^{(M_p)}_{L^1}, \text{ resp.}\\
&e^{-xx_0'-\varepsilon'\sqrt{1+|x|^2}}e^{s|x|^2}S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1} \quad , \quad e^{xx_0''-\varepsilon'\sqrt{1+|x|^2}}e^{s|x|^2}S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}
\end{aligned}$$

and from arbitrariness of x_0 and $\varepsilon > 0$ it follows

$$\frac{\cosh(xx_0)}{e^{\varepsilon\sqrt{1+|x|^2}}}e^{s|x|^2}S \in \mathcal{D}'^{(M_p)}_{L^1}, \text{ resp. } \frac{\cosh(xx_0)}{e^{\varepsilon\sqrt{1+|x|^2}}}e^{s|x|^2}S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1} \tag{5.16}$$

for all $x_0 \in \mathbb{R}^d$ and all $\varepsilon > 0$. Let $l > 0$. Take $x^{(j)} \in \mathbb{R}^d$, $j = 1, ..., d$, to be such that $x_q^{(j)} = 0$, for $j \neq q$ and $x_j^{(j)} = ld$. Then

$$\cosh(l|x|) \leq \prod_{j=1}^{d} e^{l|x_j|} \leq \left(\sum_{j=1}^{d} \frac{1}{d} e^{l|x_j|}\right)^d \leq \sum_{j=1}^{d} e^{ld|x_j|} \leq 2\sum_{j=1}^{d} \cosh\left(x^{(j)}x\right). \tag{5.17}$$

We will prove that $\cosh(l|x|)\left(\sum_{j=1}^{d}\cosh\left(2x^{(j)}x\right)\right)^{-1} \in \mathcal{D}^*_{L^\infty}$. Observe that the function $\sum_{j=1}^{d}\cosh(2ldw_j)$ is an entire function of $w = \xi + i\eta$. Moreover, for $w \in U = \mathbb{R}^d_\xi + i\{\eta \in \mathbb{R}^d |\, |\eta| < 1/(4ld^2)\}$, we have

$$\begin{aligned}
&\left|\sum_{j=1}^{d}\cosh(2ldw_j)\right|\\
&\quad= \frac{1}{2}\left|\sum_{j=1}^{d}\left(e^{2ld\xi_j}+e^{-2ld\xi_j}\right)\cos(2ld\eta_j) + i\sum_{j=1}^{d}\left(e^{2ld\xi_j}-e^{-2ld\xi_j}\right)\sin(2ld\eta_j)\right|\\
&\quad\geq \frac{1}{2}\left|\sum_{j=1}^{d}\left(e^{2ld\xi_j}+e^{-2ld\xi_j}\right)\cos(2ld\eta_j)\right| \geq \frac{\sqrt{2}}{4}\sum_{j=1}^{d}\left(e^{2ld\xi_j}+e^{-2ld\xi_j}\right)\\
&\quad\geq \frac{\sqrt{2}}{4}\sum_{j=1}^{d} e^{2ld|\xi_j|},
\end{aligned}$$

hence

$$\left|\sum_{j=1}^{d}\cosh(2ldw_j)\right| \geq \frac{\sqrt{2}}{4}\sum_{j=1}^{d} e^{2ld|\xi_j|} > 0, \text{ for all } w = \xi + i\eta \in U. \tag{5.18}$$

For $\cosh(l|x|)$, we already proved that is the restriction to $\mathbb{R}^d\backslash\{0\}$ of the function $\cosh(l\sqrt{w^2})$ which is analytic on $W=\{w=\xi+i\eta\in\mathbb{C}^d|\,|\xi|>2|\eta|\}$ (see the proof of lemma 5.2.1). Hence $\cosh(l\sqrt{w^2})\left(\sum_{j=1}^{d}\cosh(2ldw_j)\right)^{-1}$ is analytic on $W\cap U$. We will use the same notations that were used in the proof of lemma 5.2.1. Similarly as there, put $g_k(w)=\sum_{n=0}^{\infty}\frac{k^{2n}(w^2)^n}{(2n)!}$. Then $g_k(w)=\left(e^{k\sqrt{w^2}}+e^{-k\sqrt{w^2}}\right)/2$, for $w\in W_r\cap\mathbb{R}^d_\xi$ and from the uniqueness of analytic continuation and arbitrariness of $r>0$ it follows $g_k(w)=\left(e^{k\sqrt{w^2}}+e^{-k\sqrt{w^2}}\right)/2$ on W. Fix $0<r_0<1/(8ld^3)$. Then, for $w\in\overline{B(0,r_0)}$, by (5.18), we have $\left|g_l(w)\left(\sum_{j=1}^{d}\cosh(2ldw_j)\right)^{-1}\right|\leq C_{r_0}$. Take $r_1>0$ such that $\overline{B(x,2dr_1)}\subseteq\left(\mathbb{C}^d\backslash\overline{B(0,r_0/16)}\right)\cap W\cap U$, for all $x\in W_{\frac{r_0}{4}}\cap\mathbb{R}^d_x$. For such x, we use Cauchy integral formula to estimate

$$\left|\partial^\alpha\left(\frac{\cosh(l\sqrt{x^2})}{\sum_{j=1}^{d}\cosh(2ldx_j)}\right)\right|\leq\frac{\alpha!}{r_1^\alpha}\sup_{|w_1-x_1|\leq r_1,\dots,|w_d-x_d|\leq r_1}\left|\frac{\cosh(l\sqrt{w^2})}{\sum_{j=1}^{d}\cosh(2ldw_j)}\right|.$$

Now, using (5.18), we have

$$\begin{aligned}\left|\frac{\cosh(l\sqrt{w^2})}{\sum_{j=1}^{d}\cosh(2ldw_j)}\right| &\leq \frac{2}{\sqrt{2}}\frac{e^{l\mathrm{Re}\sqrt{w^2}}+e^{-l\mathrm{Re}\sqrt{w^2}}}{\sum_{j=1}^{d}e^{2ld|\xi_j|}}\leq\frac{4e^{l\sqrt[4]{(|\xi|^2-|\eta|^2)^2+4(\xi\eta)^2}}}{\sum_{j=1}^{d}e^{2ld|\xi_j|}}\\ &\leq\frac{4e^{l\sqrt{|\xi|^2-|\eta|^2+2|\xi\eta|}}}{\sum_{j=1}^{d}e^{2ld|\xi_j|}}\leq\frac{4e^{2l|\xi|}}{\sum_{j=1}^{d}e^{2ld|\xi_j|}}\leq\frac{8\cosh(2l|\xi|)}{\sum_{j=1}^{d}e^{2ld|\xi_j|}}\leq C',\end{aligned}$$

where the last inequality follows from (5.17). Hence, for $x\in W_{\frac{r_0}{4}}\cap\mathbb{R}^d_x$ we get

$$\left|\partial^\alpha\left(\frac{\cosh(l|x|)}{\sum_{j=1}^{d}\cosh(2ldx_j)}\right)\right|\leq C'\frac{\alpha!}{r_1^\alpha}.$$

For $x\in(B(0,r_0/2)\cap\mathbb{R}^d_x)\backslash\{0\}$, if we take $r_2>0$ small enough such that $\overline{B(x,2dr_2)}\subseteq B(0,r_0)$ we have (from Cauchy integral formula)

$$\left|\partial^\alpha\left(\frac{\cosh(l\sqrt{x^2})}{\sum_{j=1}^{d}\cosh(2ldx_j)}\right)\right|=\left|\partial^\alpha\left(\frac{g_l(x)}{\sum_{j=1}^{d}\cosh(2ldx_j)}\right)\right|\leq C_{r_0}\frac{\alpha!}{r_2^\alpha}.$$

Because $\cosh(l|x|)\left(\sum_{j=1}^{d}\cosh\left(2x^{(j)}x\right)\right)^{-1}$ is in $\mathcal{C}^\infty(\mathbb{R}^d)$ the same inequality will hold for the derivatives at $x=0$. If we take $r=\min\{r_1,r_2\}$ we get that, for $x\in\mathbb{R}^d$,

$$\left|\partial_x^\alpha\left(\frac{\cosh(l|x|)}{\sum_{j=1}^{d}\cosh\left(2x^{(j)}x\right)}\right)\right|\leq C\frac{\alpha!}{r^\alpha}.$$

Now, it easily follows that $\cosh(l|x|)\left(\sum_{j=1}^{d}\cosh\left(2x^{(j)}x\right)\right)^{-1} \in \mathcal{D}^*_{L^\infty}$. From (5.16), we have

$$\frac{\cosh(l|x|)}{e^{\varepsilon\sqrt{1+|x|^2}}}e^{s|x|^2}S \in \mathcal{D}'^{(M_p)}_{L^1}, \text{ resp. } \frac{\cosh(l|x|)}{e^{\varepsilon\sqrt{1+|x|^2}}}e^{s|x|^2}S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}, \tag{5.19}$$

for every $l > 0$ and every $\varepsilon > 0$. Let $l > 0$ be fixed. By considering the function $e^{\varepsilon\sqrt{1+z^2}}$, which is analytic on the strip $\mathbb{R}^d + i\{y \in \mathbb{R}^d | \, |y| < 1/4\}$, we obtain the estimates $\left|\partial^\alpha e^{\varepsilon\sqrt{1+|x|^2}}\right| \leq \tilde{C}\frac{\alpha!}{\tilde{r}^{|\alpha|}}e^{2\varepsilon\sqrt{1+|x|^2}}$, for $\tilde{r} < 1/(8d)$ and some $\tilde{C} > 0$. By this and (5.8), for small enough $r > 0$, we have

$$\begin{aligned}
\left|D^\alpha\left(\frac{\cosh\left(\frac{l|x|}{2}\right)}{\cosh(l|x|)}e^{\varepsilon\sqrt{1+|x|^2}}\right)\right| &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left|D^\beta\left(\frac{\cosh\left(\frac{l|x|}{2}\right)}{\cosh(l|x|)}\right)\right|\left|D^{\alpha-\beta}e^{\varepsilon\sqrt{1+|x|^2}}\right|\\
&\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}C'\frac{\beta!}{r^{|\beta|}}e^{-\frac{c}{2}l|x|}\frac{(\alpha-\beta)!}{r^{|\alpha|-|\beta|}}e^{2\varepsilon\sqrt{1+|x|^2}}\\
&\leq C'\frac{\alpha!}{r^{|\alpha|}}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}e^{-\frac{c}{2}l|x|}e^{2\varepsilon\sqrt{1+|x|^2}} \leq C''\alpha!\left(\frac{2}{r}\right)^{|\alpha|},
\end{aligned}$$

where the last inequality will hold if we take $\varepsilon < cl/4$ and c is the one defined in the proof of lemma 5.2.1. We get that $\frac{\cosh\left(\frac{l}{2}|x|\right)}{\cosh(l|x|)}e^{\varepsilon\sqrt{1+|x|^2}} \in \mathcal{D}^*_{L^\infty}$. From this and (5.19) we get $\cosh\left(\frac{l}{2}|x|\right)e^{s|x|^2}S \in \mathcal{D}'^{(M_p)}_{L^1}$, resp. $\cosh\left(\frac{l}{2}|x|\right)e^{s|x|^2}S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$. From the arbitrariness of $l > 0$, we obtain

$$\cosh(l|x|)e^{s|x|^2}S \in \mathcal{D}'^{(M_p)}_{L^1}, \text{ resp. } \cosh(l|x|)e^{s|x|^2}S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$$

for all $l > 0$. By (5.10), resp. (5.11), we have that $e^{s|x|^2}S \in B^*$. Hence $S \in B^*_s$.

Let us prove b). Let $S \in B^*$. Similarly as in the proof of lemma 5.2.1, we can prove that for each fixed $\xi \in \mathbb{R}^d$ there exists $k_\xi > 0$ (k depends on ξ) such that $\frac{e^{-x\xi}}{\cosh(k_\xi|x|)} \in \mathcal{S}^*\left(\mathbb{R}^d_x\right)$. Then, for fixed $\xi \in \mathbb{R}^d$, we have

$$e^{-x\xi}S = \frac{e^{-x\xi}}{\cosh(k_\xi|x|)}\cosh(k_\xi|x|)S \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right).$$

Hence, by theorem 2.1.1, the Laplace transform of S exists and belongs to A^*. Analogously, for $\varepsilon > 0$ and $\xi + i\eta$ fixed, we can find $k > 0$ (k depends on ε and $\xi + i\eta$) such that $\frac{e^{-(\xi+i\eta)x}e^{\varepsilon\sqrt{1+|x|^2}}}{\cosh(k|x|)} \in \mathcal{S}^*\left(\mathbb{R}^d_x\right)$. Then

$$e^{-(\xi+i\eta)x}e^{\varepsilon\sqrt{1+|x|^2}}S = \frac{e^{-(\xi+i\eta)x}e^{\varepsilon\sqrt{1+|x|^2}}}{\cosh(k|x|)}\cosh(k|x|)S \in \mathcal{D}'^{(M_p)}_{L^1}\left(\mathbb{R}^d_x\right),$$

in the (M_p) case and resp. $e^{-(\xi+i\eta)x}e^{\varepsilon\sqrt{1+|x|^2}}S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$ in the $\{M_p\}$ case. By (2.11) (see remark 2.1.1), we have

$$\mathcal{L}(S)(\xi+i\eta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-(\xi+i\eta)x}S(x), e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle = \left\langle e^{-(\xi+i\eta)x}S(x), 1_x\right\rangle.$$

The injectivity is obvious. Let us prove the surjectivity. By theorem 2.1.2, for $f \in A^*$ there exists $T \in \mathcal{D}'^*$ such that $e^{-x\xi}T(x) \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right)$, for all $\xi \in \mathbb{R}^d_\xi$ and $\mathcal{L}(T)(\xi+i\eta) = f(\xi+i\eta)$. Because $e^{-x\xi}T(x) \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right)$, for all $\xi \in \mathbb{R}^d$ we obtain that $\cosh(x\xi)T(x) \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right)$ for all $\xi \in \mathbb{R}^d$. Let $k > 0$. By the considerations in the proof of a), if take $x^{(j)} \in \mathbb{R}^d$, $j = 1, ..., d$, such that $x^{(j)}_q = 0$, for $j \neq q$ and $x^{(j)}_j = kd$, we obtain that $\cosh(k|x|)\left(\sum_{j=1}^{d}\cosh\left(2x^{(j)}x\right)\right)^{-1} \in \mathcal{D}^*_{L^\infty}$. Obviously $\mathcal{D}^*_{L^\infty} \subseteq \mathcal{O}^*_M$. Hence

$$\cosh(k|x|)T(x)$$
$$= \cosh(k|x|)\left(\sum_{j=1}^{d}\cosh\left(2x^{(j)}x\right)\right)^{-1}\sum_{j=1}^{d}\cosh\left(2x^{(j)}x\right)T(x) \in \mathcal{S}'^*(\mathbb{R}^d).$$

We obtain $T \in B^*$ and the surjectivity is proved.

Now we will prove c). By a), $S * e^{s|\cdot|^2}$ is well defined for $S \in B^*_s$. Let $\psi \in \mathcal{D}^*$ is such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ when $|x| \leq 1$ and $\psi(x) = 0$ when $|x| > 2$. Put $\psi_j(x) = \psi(x/j)$ for $j \in \mathbb{Z}_+$. Because the convolution of S and $e^{s|x|^2}$ exists,

$$\left\langle S * e^{s|\cdot|^2}, \varphi\right\rangle = \left\langle\left(\varphi * e^{s|\cdot|^2}\right)S, 1\right\rangle = \lim_{j\to\infty}\left\langle\left(\varphi * e^{s|\cdot|^2}\right)S, \psi_j\right\rangle, \tag{5.20}$$

for all $\varphi \in \mathcal{D}^*$. Fix $j \in \mathbb{Z}_+$ and observe that $\left\langle\left(\varphi * e^{s|\cdot|^2}\right)S, \psi_j\right\rangle = \left\langle(\psi_j S) * e^{s|\cdot|^2}, \varphi\right\rangle$. Let $l \in \mathbb{N}$ be so large such that $\operatorname{supp}\psi_j \subseteq \{x \in \mathbb{R}^d |\, \psi_l(x) = 1\}$. We have

$$\begin{aligned}
\left\langle\left(\varphi * e^{s|\cdot|^2}\right)S, \psi_j\right\rangle &= \left\langle\left(\varphi * e^{s|\cdot|^2}\right)(\xi)(\psi_j S)(\xi), \psi_l(\xi)\right\rangle\\
&= \left\langle e^{s|\xi|^2}\int_{\mathbb{R}^d}\varphi(x)e^{s|x|^2-2sx\xi}dx(\psi_j S)(\xi), \psi_l(\xi)\right\rangle\\
&= \left\langle e^{s|\xi|^2}e^{s|x|^2-2sx\xi}(\psi_j S)(\xi), \psi_l(\xi)\varphi(x)\right\rangle\\
&= \left\langle e^{s|x|^2}\left\langle e^{s|\xi|^2}e^{-2sx\xi}(\psi_j S)(\xi), \psi_l(\xi)\right\rangle, \varphi(x)\right\rangle\\
&= \left\langle e^{s|x|^2}\left\langle e^{s|\xi|^2}e^{-2sx\xi}S(\xi), \psi_j(\xi)\right\rangle, \varphi(x)\right\rangle,
\end{aligned}$$

where the third and the fourth equality follow from theorem 1.2.9. We obtain $\left\langle(\psi_j S) * e^{s|\cdot|^2}, \varphi\right\rangle = \left\langle e^{s|x|^2}\left\langle e^{s|\xi|^2}e^{-2sx\xi}S(\xi), \psi_j(\xi)\right\rangle, \varphi(x)\right\rangle$, for all $\varphi \in \mathcal{D}^*$ and all $j \in \mathbb{Z}_+$. Hence

$$e^{s|x|^2}\left\langle e^{s|\xi|^2}e^{-2sx\xi}S(\xi), \psi_j(\xi)\right\rangle = \left((\psi_j S) * e^{s|\cdot|^2}\right)(x) \tag{5.21}$$

in $\mathcal{D}'^*\left(\mathbb{R}^d_x\right)$, for all $j \in \mathbb{Z}_+$. Because

$$\left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle = \left\langle \psi_j(\xi) S(\xi), e^{s|\xi|^2} e^{-2sx\xi} \right\rangle,$$

for each fixed $x \in \mathbb{R}^d$, theorem 3.10 of [28] implies that the left hand side of (5.21) is an element of $\mathcal{E}^*\left(\mathbb{R}^d_x\right)$. By (5.20), the right hand side of (5.21) tends to $S * e^{s|\cdot|^2}$ in $\mathcal{D}'^*$. Because $S \in B^*_s$, $e^{s|\cdot|^2} S \in B^*$ and by *b*), for each fixed $x, y \in \mathbb{R}^d$, $e^{-(x+iy)\cdot} e^{s|\cdot|^2} S \in \mathcal{D}'^{(M_p)}_{L^1}$, resp. $e^{-(x+iy)\cdot} e^{s|\cdot|^2} S \in \tilde{\mathcal{D}}'^{\{M_p\}}_{L^1}$, the Laplace transform of $e^{s|\cdot|^2} S$ exists and $\mathcal{L}\left(e^{s|\cdot|^2} S\right)(2sx) = \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), 1_\xi \right\rangle$, for every fixed $x \in \mathbb{R}^d$. So, the right hand side of (5.21) tends to $e^{s|x|^2} \mathcal{L}\left(e^{s|\cdot|^2} S\right)(2sx)$ pointwise. We will prove that the convergence holds in $\mathcal{D}'^*$. Let K be a fixed compact subset of $\mathbb{R}^d$. With similar technic as in the proof of lemma 5.2.1, we can find large enough $k > 0$ (k depends on K) such that $e^{-2sx\xi}\left(\cosh(k|\xi|)\right)^{-1} \in \mathcal{S}^*\left(\mathbb{R}^d_\xi\right)$, for each $x \in K$ and the set $\left\{e^{-2sx\cdot}\left(\cosh(k|\cdot|)\right)^{-1} \in \mathcal{S}^*\left(\mathbb{R}^d_\xi\right) \middle| x \in K\right\}$ is bounded subset of $\mathcal{S}^*\left(\mathbb{R}^d_\xi\right)$. Because $S \in B^*_s$, $\cosh(k|\cdot|) e^{s|\cdot|^2} S \in \mathcal{S}'^*$. Hence

$$\begin{aligned} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle &= \left\langle e^{s|\xi|^2} e^{-2sx\xi}\left(\cosh(k|\xi|)\right)^{-1} \cosh(k|\xi|) S(\xi), \psi_j(\xi) \right\rangle \\ &= \left\langle e^{s|\xi|^2} \cosh(k|\xi|) S(\xi), e^{-2sx\xi}\left(\cosh(k|\xi|)\right)^{-1} \psi_j(\xi) \right\rangle. \end{aligned}$$

By the way we defined ψ_j, one easily verifies that

$$\left\{e^{-2sx\cdot}\left(\cosh(k|\cdot|)\right)^{-1} \psi_j(\cdot) \middle| x \in K,\, j \in \mathbb{Z}_+\right\}$$

is a bounded subset of $\mathcal{S}^*\left(\mathbb{R}^d_\xi\right)$. From this it follows that there exists $C_K > 0$ (C_K depends on K) such that $\left| e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle \right| \leq C_K$, for all $x \in K$, $j \in \mathbb{Z}_+$. Because $e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle$ tends to $e^{s|x|^2} \mathcal{L}\left(e^{s|\cdot|^2} S\right)(2sx)$ pointwise, by the above, the convergence also holds in $\mathcal{D}'^*\left(\mathbb{R}^d_x\right)$. Hence, we obtain $e^{s|x|^2} \mathcal{L}\left(e^{s|\cdot|^2} S\right)(2sx) = \left(S * e^{s|\cdot|^2}\right)(x)$. Now, *b*) implies $S * e^{s|\cdot|^2} \in A^*_s$. The bijectivity of $S \mapsto S * e^{s|\cdot|^2}$ follows from the bijectivity of $\mathcal{L} : B^* \to A^*$. □

5.3 A New Class of Anti-Wick Operators

Theorem 5.2.1, along with (5.2), allows us to define Anti-Wick operators $A_a : \mathcal{D}^*\left(\mathbb{R}^d\right) \to \mathcal{D}'^*\left(\mathbb{R}^d\right)$, when a is not necessary in $\mathcal{S}'^*\left(\mathbb{R}^{2d}\right)$. If $a \in B^*_{-1}$ (and only then) $b(x,\xi) = \pi^{-d}\left(a(\cdot,\cdot) * e^{-|\cdot|^2 - |\cdot|^2}\right)(x,\xi)$ exists and is an element of A^*_{-1}. If this b is such that, for every $\chi \in \mathcal{D}^*\left(\mathbb{R}^{2d}\right)$ the integral

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} b\left(\frac{x+y}{2}, \xi\right) \chi(x,y)\, dx dy d\xi \tag{5.22}$$

is well defined as oscillatory integral and $\langle K_b, \chi \rangle$ defined as the above integral is well defined ultradistributions, then the operator associated to that kernel (see theorem 1.2.9) $\varphi \mapsto \langle K_b(x,y), \varphi(y)\rangle$, $\mathcal{D}^*\left(\mathbb{R}^d\right) \to \mathcal{D}'^*\left(\mathbb{R}^d\right)$, can be called the Anti-Wick operator with symbol a (because of proposition 5.1.3, this is appropriate generalisation of Anti-Wick operators). The next theorem gives an example of such b.

Theorem 5.3.1. *Let $a \in B^*_{-1}$ is such that b, given by (5.2), satisfies the following condition: for every $K \subset\subset \mathbb{R}^d_x$ there exists $\tilde{r} > 0$ such that there exist $m, C_1 > 0$, resp. there exist $C_1 > 0$ and $(k_p) \in \mathfrak{R}$, (in both cases C_1 and m, resp. C_1 and (k_p) depend on K) such that*

$$|b(x+i\eta,\xi)| \leq C_1 e^{M(m|\xi|)}, \text{ resp. } |b(x+i\eta,\xi)| \leq C_1 e^{N_{k_p}(|\xi|)}, \tag{5.23}$$

for all $x \in K$, $|\eta| < \tilde{r}$, $\xi \in \mathbb{R}^d$. Then (5.22) is oscillatory integral and K_b, defined by (5.22), is well defined ultradistribution.

Proof. Under the conditions in the theorem, Cauchy integral formula yields

$$|D^\alpha_x b(x,\xi)| \leq C\alpha!/r_1^{|\alpha|} e^{M(m|\xi|)}, \text{ resp. } |D^\alpha_x b(x,\xi)| \leq C\alpha!/r_1^{|\alpha|} e^{N_{k_p}(|\xi|)},$$

for all $x \in K$, $\xi \in \mathbb{R}^d$ (r_1 and C depend on K). Let U be an arbitrary bounded open subset of $\mathbb{R}^{2d}$. Then $V = \left\{t \in \mathbb{R}^d | \, t = (x+y)/2, \, (x,y) \in U\right\}$ is a bounded set in $\mathbb{R}^d$, hence $K = \overline{V}$ is compact set. For this K, let m, resp. (k_p) be as in (5.23). Take P_l, resp. P_{l_p}, as in proposition 2.1.1, such that $|P_l(\xi)| \geq C_2 e^{M(r|\xi|)}$, resp. $\left|P_{l_p}(\xi)\right| \geq C_2 e^{N_{r_p}(\xi)}$, for some $C_2 > 0$, such that $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$, resp. $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p}(|\xi|)} d\xi < \infty$. We can define $K_{b,U}$ as

$$\langle K_{b,U}, \chi\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} \frac{e^{i(x-y)\xi}}{P_l(\xi)} P_l(D_y) \left(b\left(\frac{x+y}{2},\xi\right)\chi(x,y)\right) dxdyd\xi,$$

for $\chi \in \mathcal{D}^{(M_p)}(U)$ in the (M_p) case, resp. the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case and then one easily checks that $K_{b,U} \in \mathcal{D}'^*(U)$. Moreover, if $\psi \in \mathcal{D}^*\left(\mathbb{R}^d\right)$ is such that $\psi(\xi) = 1$ in a neighbourhood of 0, for $\delta > 0$, we can define $K_{b,U,\psi,\delta} \in \mathcal{D}'^*(U)$ as

$$\langle K_{b,U,\psi,\delta}, \chi\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} e^{i(x-y)\xi} \psi(\delta\xi) b\left(\frac{x+y}{2},\xi\right) \chi(x,y) dxdyd\xi.$$

Then $K_{b,U,\psi,\delta} \to K_{b,U}$, when $\delta \to 0^+$, in $\mathcal{D}'^*(U)$. Combining these results, we obtain that the definition of $K_{b,U}$ does not depend on P_l resp. P_{l_p}, when these are appropriately chosen (see the above discussion) and on the choice of ψ with the above properties. Moreover, when U_1 and U_2 are two bounded open sets in $\mathbb{R}^{2d}$ with nonempty intersection, it follows that $K_{b,U_1} = K_{b,U_1\cup U_1} = K_{b,U_2}$ in $\mathcal{D}'^*(U_1 \cap U_2)$. Because $\mathcal{D}'^*$ is a sheaf, K_b can be defined as an element of $\mathcal{D}'^*\left(\mathbb{R}^{2d}\right)$ as the oscillatory integral (5.22). □

Example 5.3.1. Interesting such symbols a are given by $e^{l|x|^2}P(\xi)$, where $l < 1$ and $P(\xi)$ is an ultrapolynomial of class *. In this case, obviously $a \in B^*_{-1}$. Moreover

$$\begin{aligned} b(x,\xi) &= \frac{1}{\pi^d} e^{-|x|^2-|\xi|^2} \mathcal{L}\left(e^{-|\cdot|^2-|\cdot|^2} a(\cdot,\cdot)\right)(-2x,-2\xi) \\ &= \frac{1}{\pi^d}\left(\frac{\pi}{1-l}\right)^{d/2} e^{l|x|^2/(1-l)} \int_{\mathbb{R}^d} e^{-|\eta|^2} P(\xi-\eta) d\eta \end{aligned}$$

In the (M_p) case, there exist $m, C_1 > 0$ such that $|P(\xi-\eta)| \leq C_1 e^{M(m|\xi|)} e^{M(m|\eta|)}$, resp. in the $\{M_p\}$ case, there exist $C_1 > 0$ and $(k_p) \in \mathfrak{R}$, such that $|P(\xi-\eta)| \leq C_1 e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|\eta|)}$ (in the (M_p) case this estimate follows from proposition 4.5 of [26], in the $\{M_p\}$ case the estimate easily follows by combining proposition 4.5 of [26] and lemma 3.4 of [28]). Hence, b satisfies the conditions in the above theorem and b^w can be the defined as the operator corresponding to the kernel K_b defined as the oscillatory integral (5.22).

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Abstract: We investigate the Laplace transform in Komatsu ultradistributions and give conditions under which an analytic function is a Laplace transformation of an ultradistribution. We prove the equivalence of several definitions of convolution of two Roumieu ultradistributions. For that purpose, we consider the ε tensor product of $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ and a locally convex space. We define specific global symbol classes of Shubin type and study the corresponding pseudodifferential operators of infinite order that act continuously on the spaces of tempered ultradistributions of Beurling and Roumieu type. For these classes, we develop symbolic calculus. We investigate the connection between the Anti-Wick and Weyl quantization when the symbols belong to these classes. We find the largest subspace of ultradistributions for which the convolution with the gaussian kernel exists. This gives a way to extend the definition of Anti-Wick quantization for symbols that are not necessarily tempered ultradistributions.

AB

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Извод: Проучавамо Лапласову трансформацију у просторима Коматсуове ултрадистрибуције и дајемо услов под којим аналитичка функција је Лапласова трансформација ултрадистрибуције. Доказујемо еквивалентност неколико дефиниција о конволуцији две Румие ултрадистрибуције. За ову сврху разматрамо ε тензорски производ $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ и локално конвексни простор. Дефинирамо специфичне глобалне симбол класе Шубиновог типа и проучавамо одговарајуће псевдо диференцијалне операторе бесконачног реда који непрекидно делују на просторима темперираних ултрадистрибуција Берлинеовог и Румиеовог типа. За ове класе градимо симболички калкулус. Проучавамо везу између Anti-Wick-ове и Weyl-ове квантизације кад симболи припадају ове симбол класе. Налазимо највећи подпростор ултрадистрибуција за које конволуција са гаусовог језгра постоји. То пружа могућност да проширимо дефиницију Anti-Wick квантизације за симболе које неморају да су темпериране ултрадистрибуције.

ИЗ

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